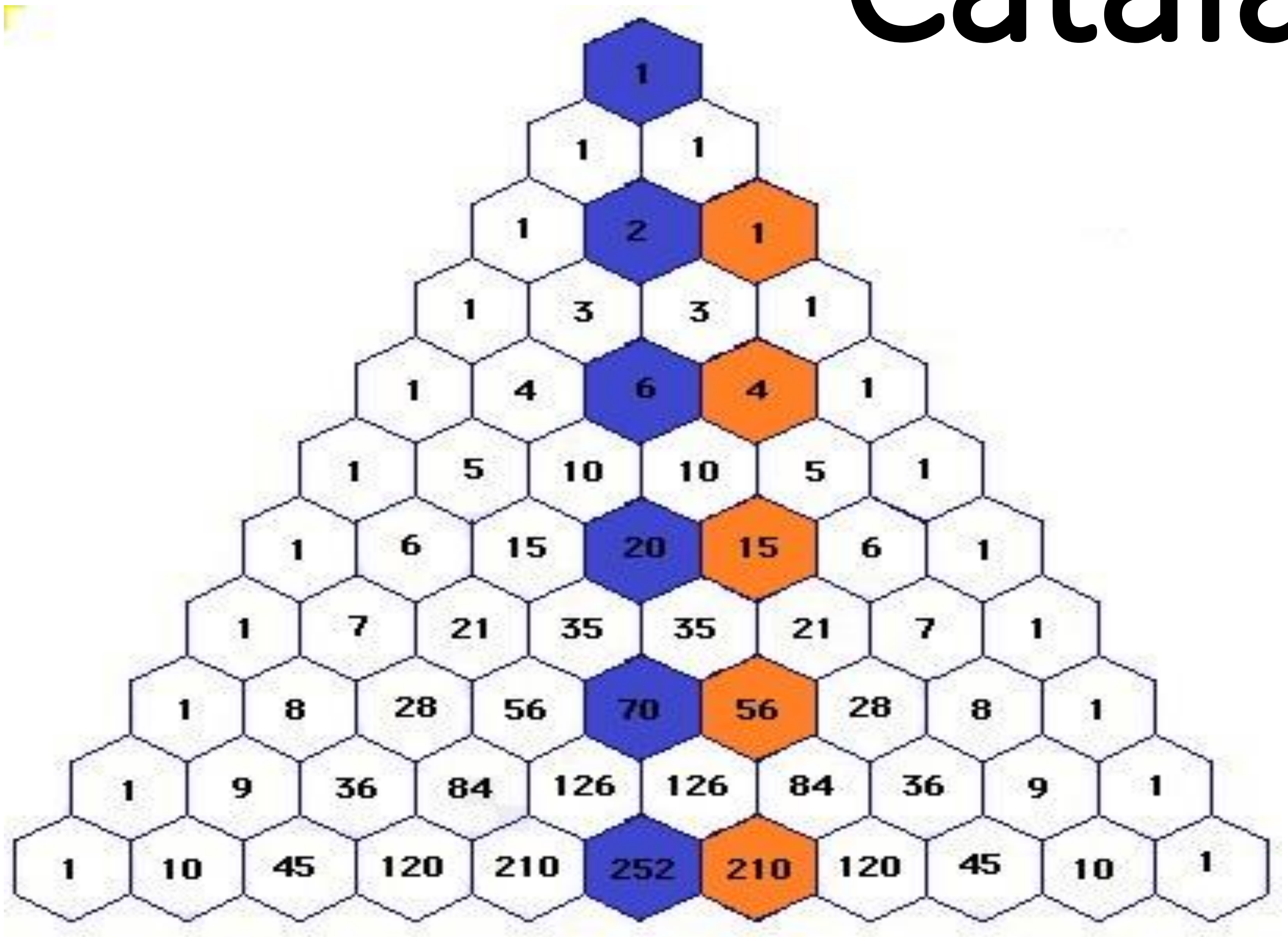
Catalan families

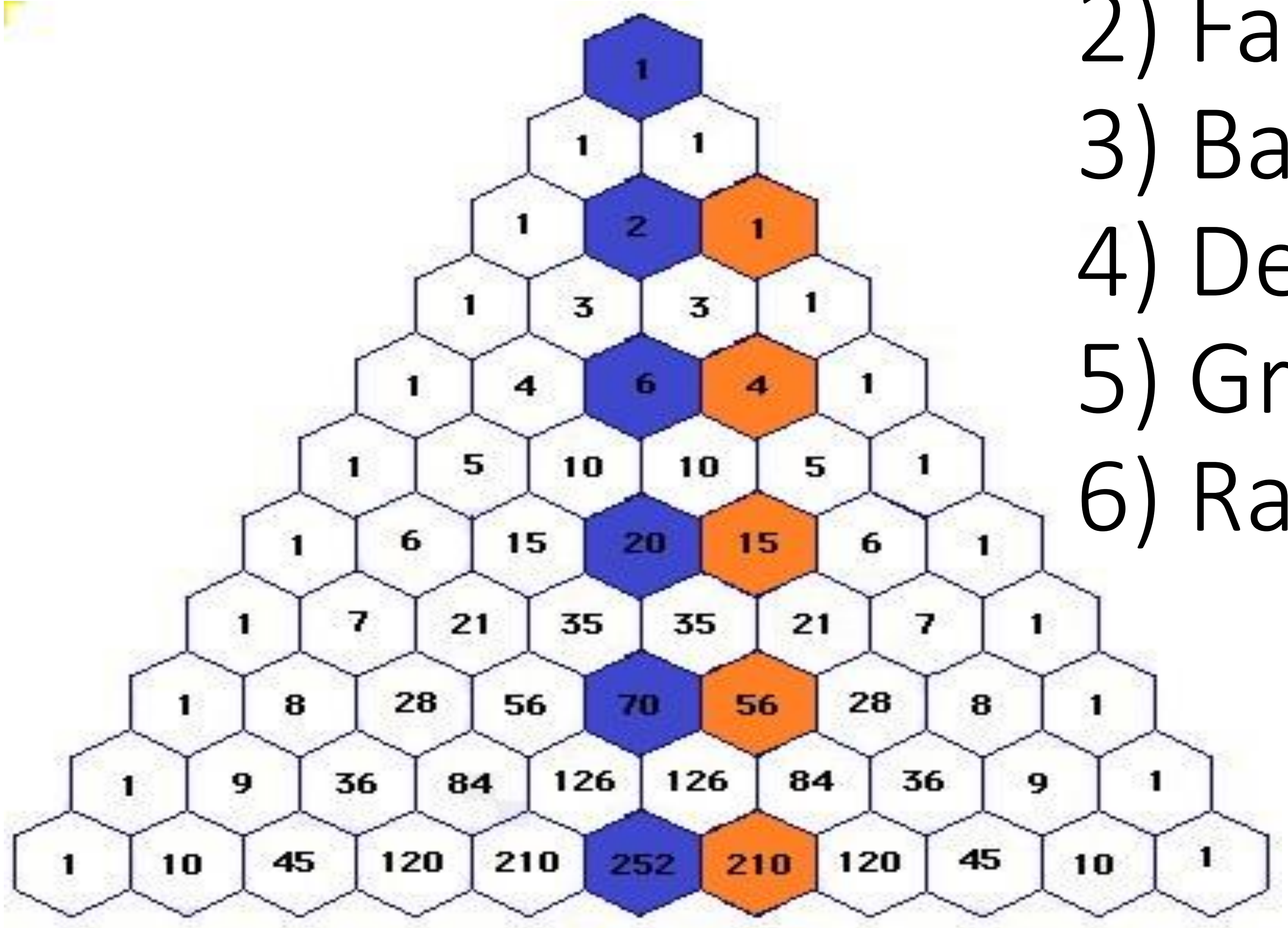


Sajjad Ranjbar

Combinatorial Algorithms

Shahid Beheshti university of Tehran

Fall semester 2024-2025



1) Historical documents
2) Famous problems
3) Balanced sequences
4) Definition of Catalans
5) Graphical path
6) Rank & Unrank

Historical documents

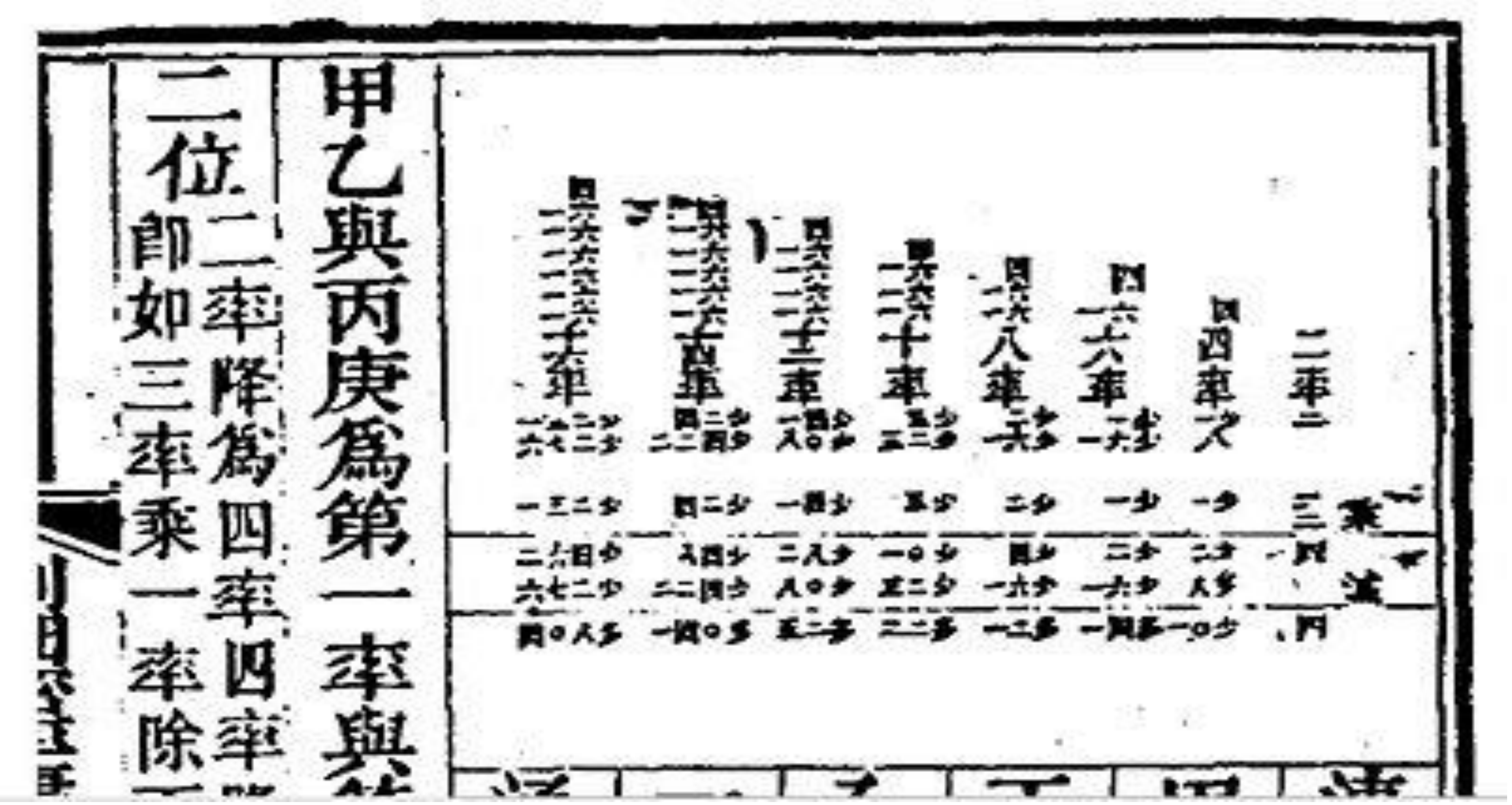
Mingantu 1730

Historical documents

Mingantu 1730

Ge Yuan Mi Lu Jie Fa

The Quick Method for Obtaining the Precise Ratio of Division of a Circle.



甲乙與丙庚爲第一率與
二位
二率降爲四率四率
即如三率乘一率除

Historical documents

Mingantu 1730

Euler 1751

Historical documents

Mingantu 1730

Euler 1751

In 1751, Leonhard Euler (1707–1783) introduced and found a closed formula for what we now call the Catalan numbers. The proof of this result had eluded him, until he was assisted by Christian Goldbach (1690–1764), and more substantially by Johann Segner. By 1759, a complete proof was obtained.

Historical documents

Mingantu 1730

Euler 1751

Désiré André 1887

Historical documents

Mingantu 1730

Euler 1751

Désiré André 1887

He found the reflection counting trick (second proof) for Dyck words.

Historical documents

Mingantu 1730

Euler 1751

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Historical documents

Mingantu 1730

Euler 1751

Désiré André 1887

Eugène Charles Catalan (30 May 1814 – 14 February 1894)

He was a French and Belgian mathematician who worked on continued fractions, descriptive geometry, number theory and combinatorics. stating the famous Catalan's conjecture, which was eventually proved in 2002; and introducing the Catalan numbers to solve a combinatorial problem.

Famous problems

Famous problems

1. Counting Full Binary Trees

Famous problems

1. Counting Full Binary Trees
2. Counting Valid Parentheses Combinations

Famous problems

1. Counting Full Binary Trees
2. Counting Valid Parentheses Combinations
3. Counting Triangulations of an n-gon

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1. **Counting Full Binary Trees**
2. **Counting Valid Parentheses Combinations**
3. **Counting Triangulations of an n-gon**
4. **Counting Lattice Paths**

Famous problems

1. Counting Full Binary Trees
2. Counting Valid Parentheses Combinations
3. Counting Triangulations of an n-gon
4. Counting Lattice Paths
5. Counting Binary Search Trees

Catalan families

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For any n > 0:

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[1, 2, 5, 14, 42, …]

Catalan's definition

Let n be a positive integer

Let a = $[a_1, a_2, a_3, \ldots, a_{2n}] \ni (\mathbb{Z}_2)^{2n}$

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We say that the sequence a is balanced if:

- a contain n 0s and n 1s
- For any $1 \le i \le 2n$ hold that:

$$|\{j: 1 \le j \le i, a_i=0\}| >= |\{j: 1 \le j \le i\ a_i=1\}|$$

Catalan's definition

000001

Catalan's definition

000001 ●

Catalan's definition

000001

010110

Catalan's definition

000001

010110

Catalan's definition

000001

010110

010101

Catalan's definition

000001

010110

010101

Catalan's definition

n=3:

Catalan's definition

n=3:

1. 000111
2. 001011
3. 001101
4. 010011
5. 010101

a

Catalan's definition

Let $\mathbf{C_n}$ denote the set of all balanced sequences in $(\mathbb{Z}_2)^{2n}$

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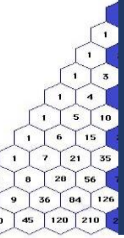
The Catalan number $\mathbf{C}_n$ is defied to be $\mathbf{C}_n = |\ \mathbf{C}_n|$



The few first Catalan numbers

$C_1=1$

$C_2=2$

$C_3=5$

$C_4=14$

$C_5=42$

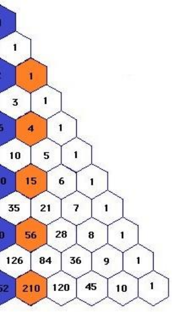
$C_6=132$

$C_7=429$

$C_8=1430$

$C_9=4862$

$C_{10}=16796$



Theorem

For any n > 0:

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Proof

Combinatorial explanation

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Number of string that have exactly n 0s and n 1s is $\binom{2n}{n}$

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$$C_n = \binom{2n}{n} - \quad \textit{squences} \text{ with length 2n } \textit{that do not } \text{satisfy } \textit{second condition of balanced sequences}$$

Proof

Graphical path

$$P = \{(0, 0), (1, y_1), \ldots ,(2n\text{-}1, y_{2n-1}), (2n, y_{2n})\}$$

Proof

Graphical path

$$P = \{(0, 0), (1, y_1), \ldots ,(2n\text{-}1, y_{2n-1}), (2n, y_{2n})\}$$

SequenceToPath([a_1, a_2, … , a_{2n}])

```
P = {(0, 0)}; y = 0;
for i = 1 to 2n
        if a_i == 0
                y += 1
        else  y -= 1
        P = P U (i, y)
end for
```

Proof

Graphical path example

001101

$P = \{(0, 0)\}$

Proof

Graphical path example

001101

P = {(0, 0), (1, 1)}

Proof

Graphical path example

001101

P = {(0, 0), (1, 1), (2, 2)}

Proof

Graphical path example

001101

P = {(0, 0), (1, 1), (2, 2), (3, 1)}

Proof

Graphical path example

001101

$P = \{(0, 0), (1, 1), (2, 2), (3, 1), (4, 0)\}$

Proof

Graphical path example

001101

P = {(0, 0), (1, 1), (2, 2), (3, 1), (4, 0), (5, 1)}

Proof

Graphical path example

001101

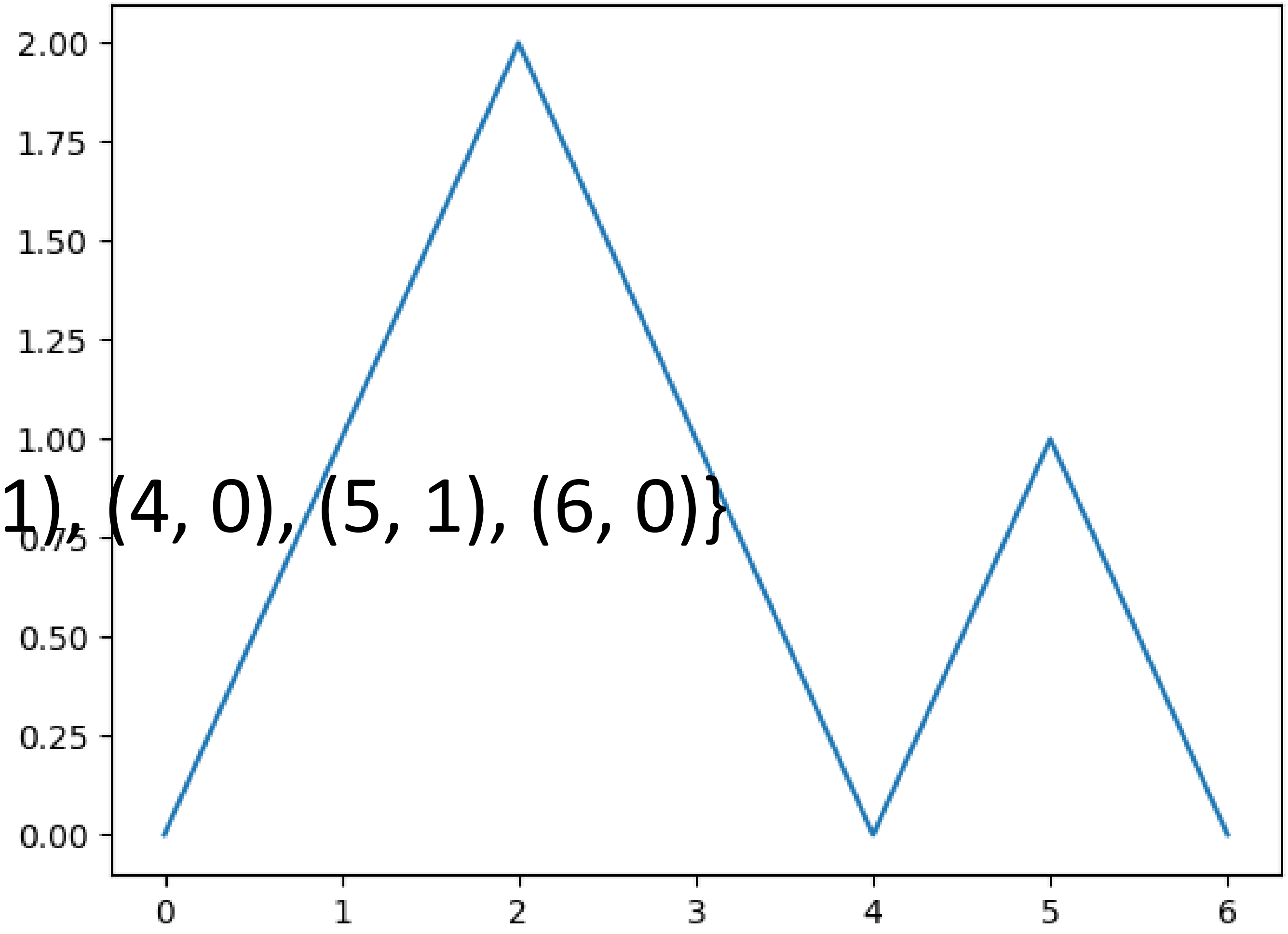
$P = \{(0, 0), (1, 1), (2, 2), (3, 1), (4, 0), (5, 1), (6, 0)\}$

Proof

Graphical path example

001101

P = {(0, 0), (1, 1), (2, 2), (3, 1), (4, 0), (5, 1), (6, 0)}



Proof

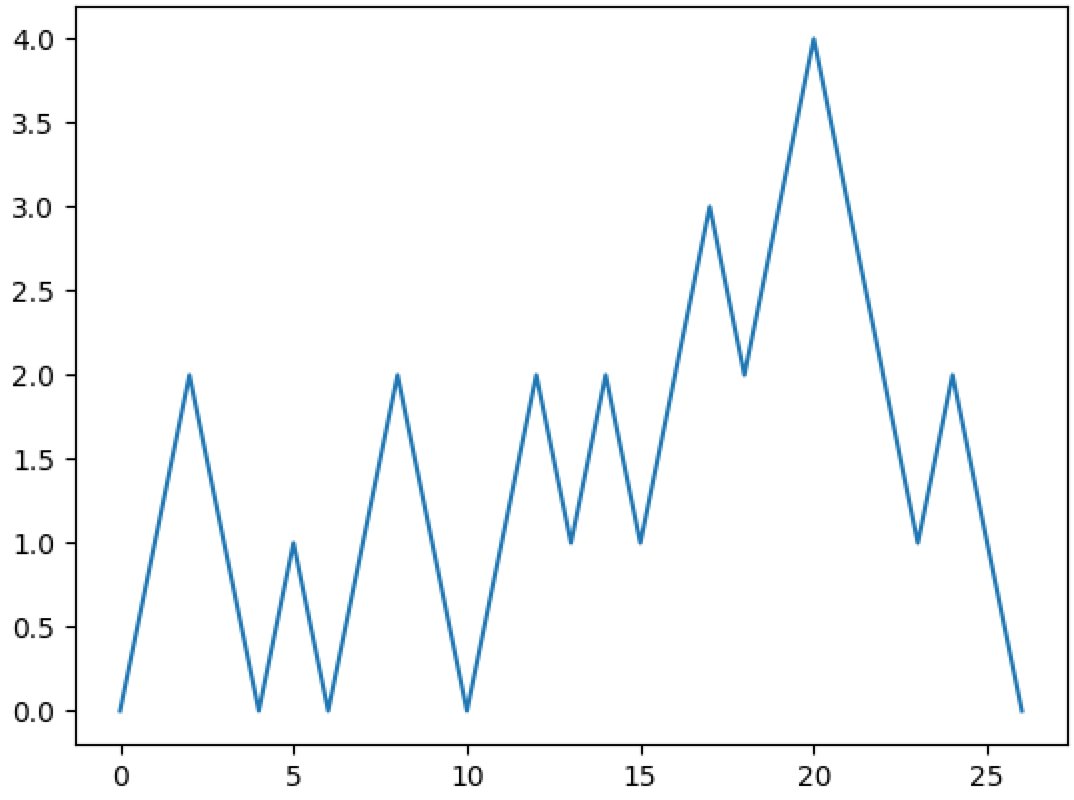
Graphical path

00110100110010100100111011

Proof

Graphical path

00110100110010100100111011



Proof

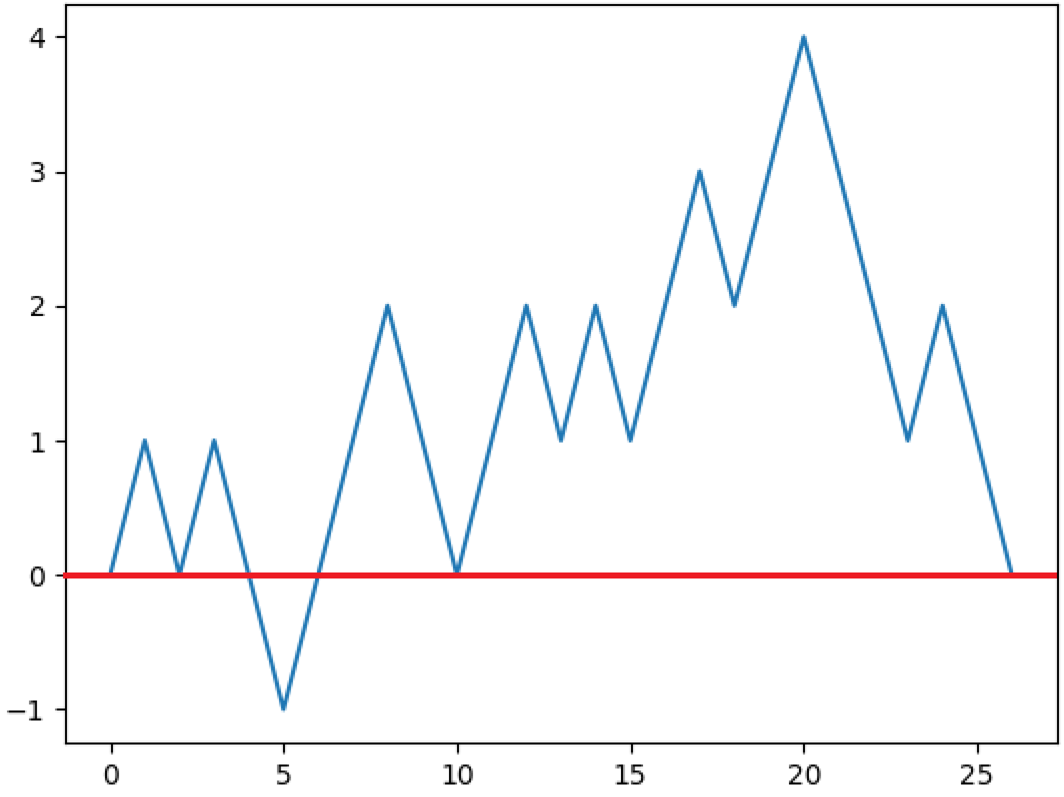
Graphical path

0101100011001010010011011

Proof

Graphical path

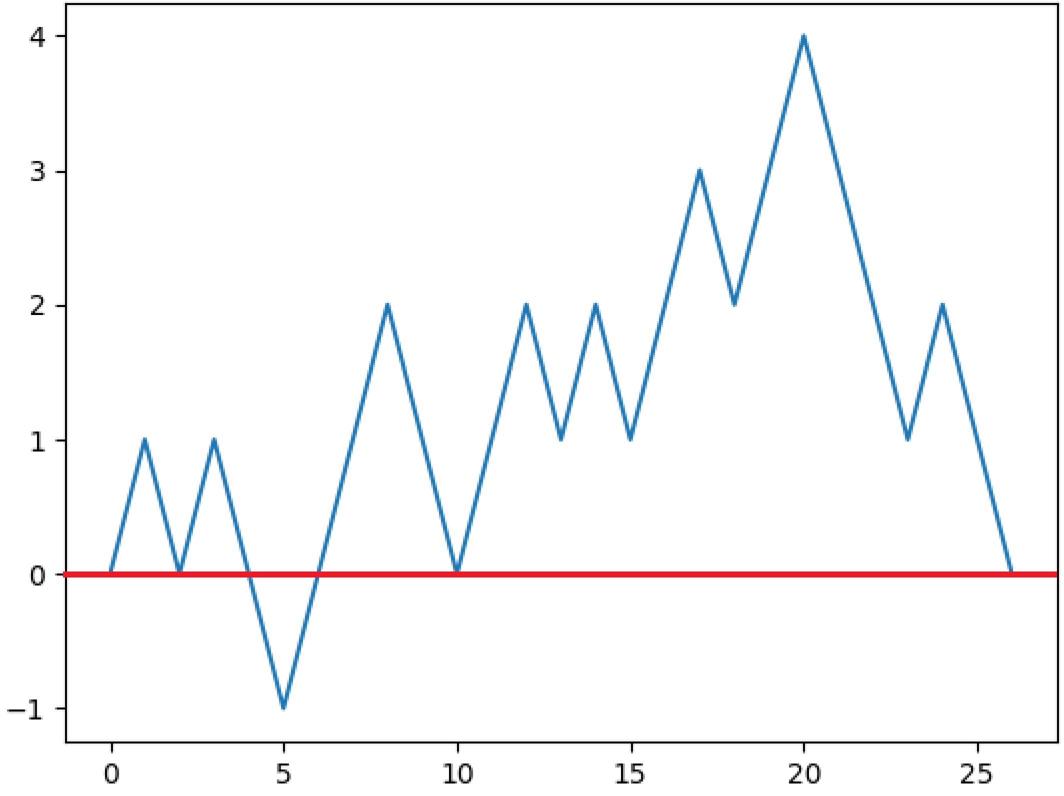
0101100011001010010011011



Proof

Graphical path

0101100011001010010011101 1

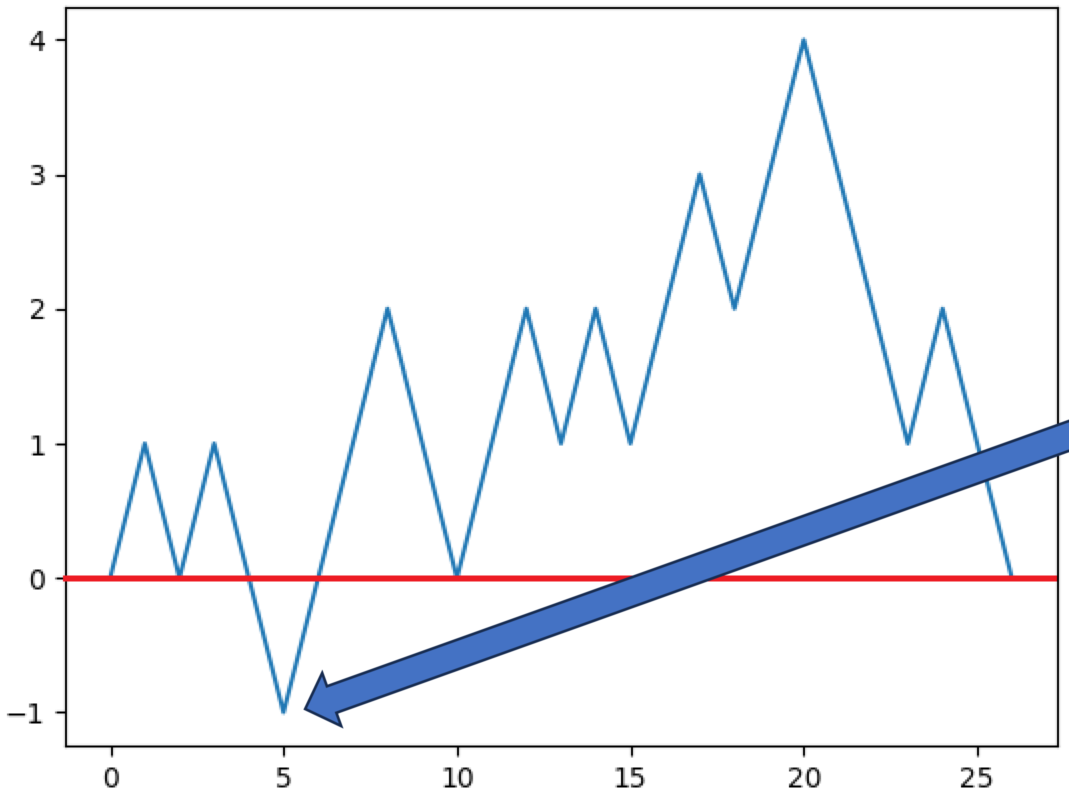


x_0: first point under y=0

Proof

Graphical path

01011000110010100100111011



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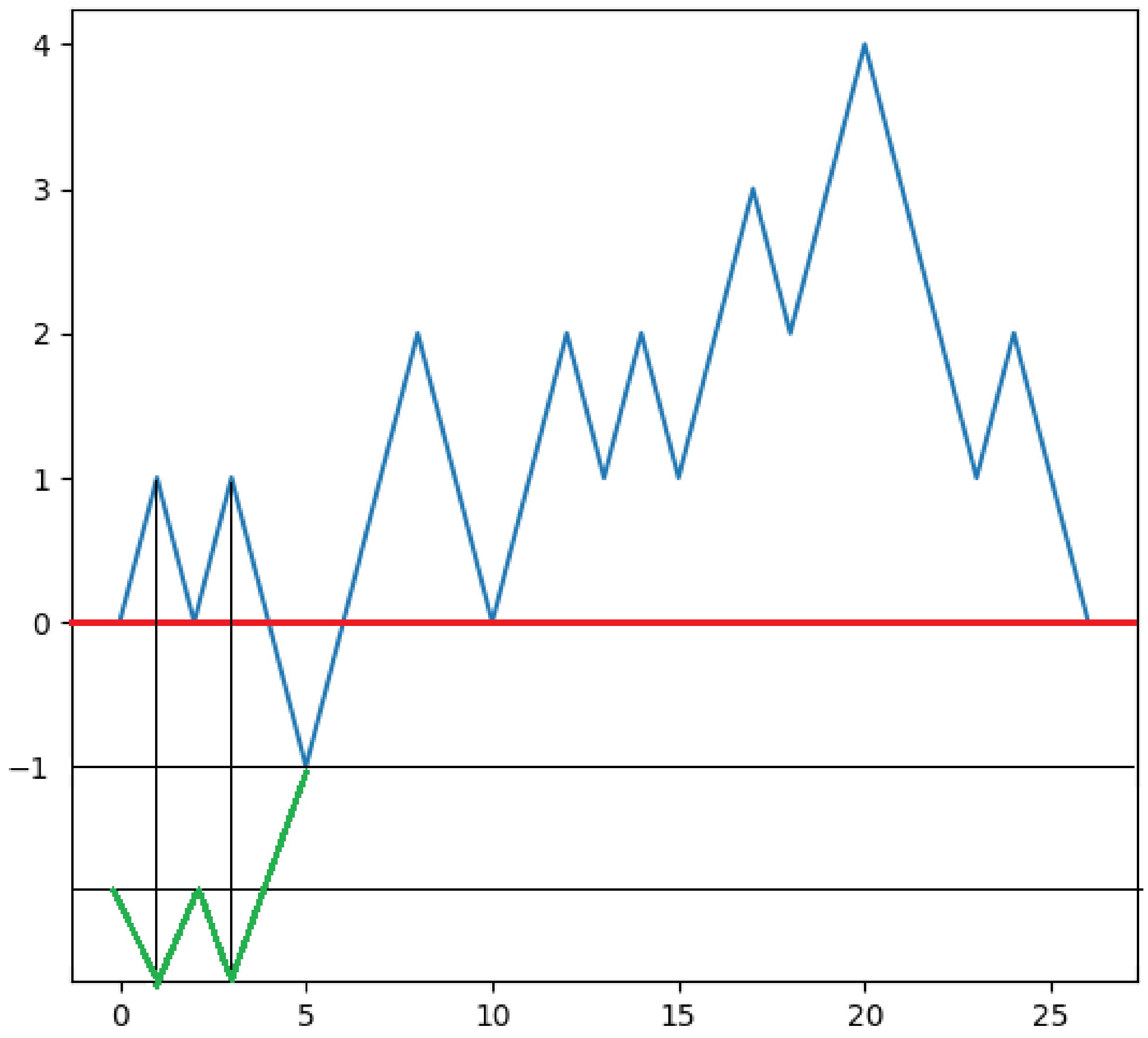
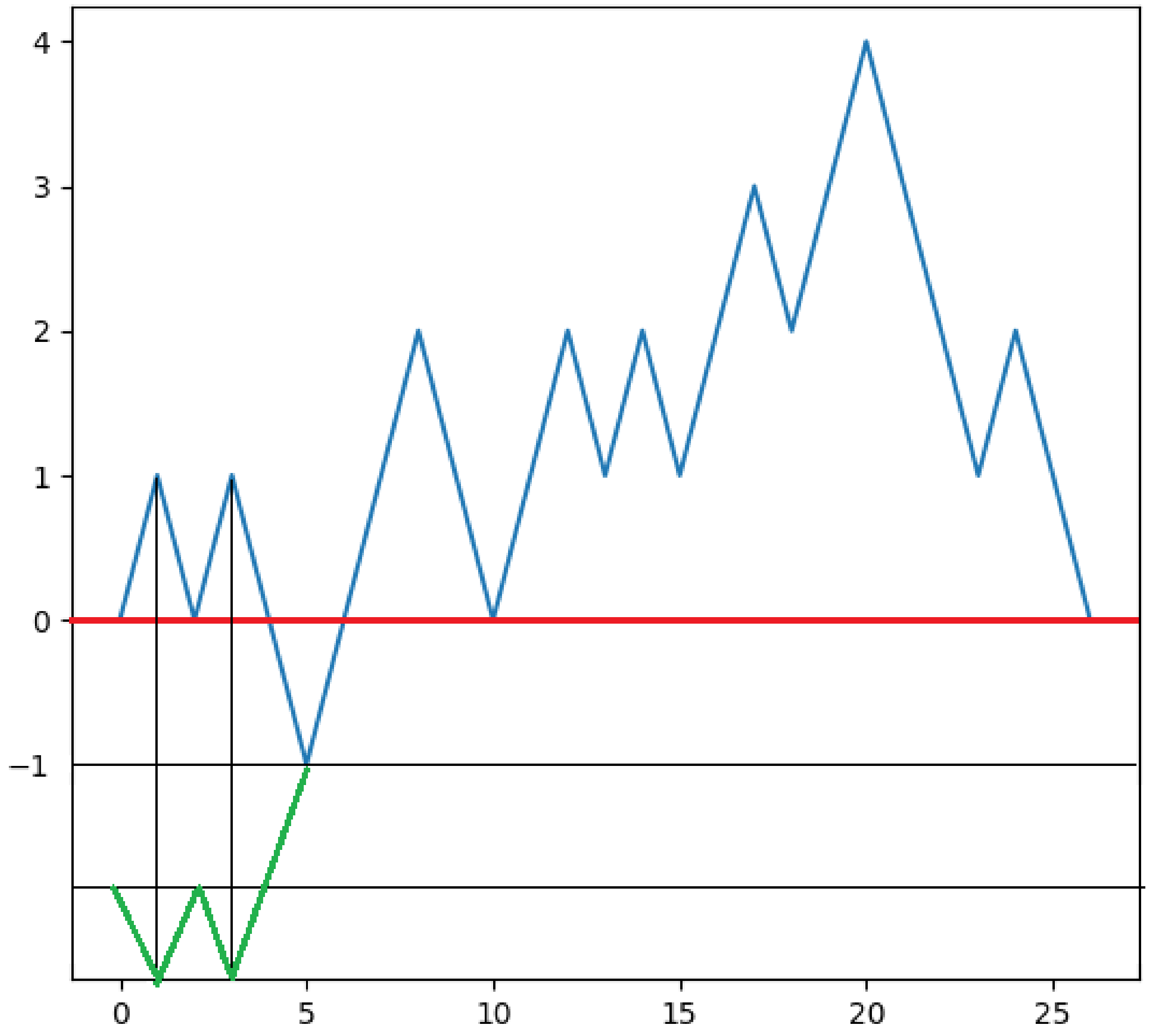


Proof

Graphical path



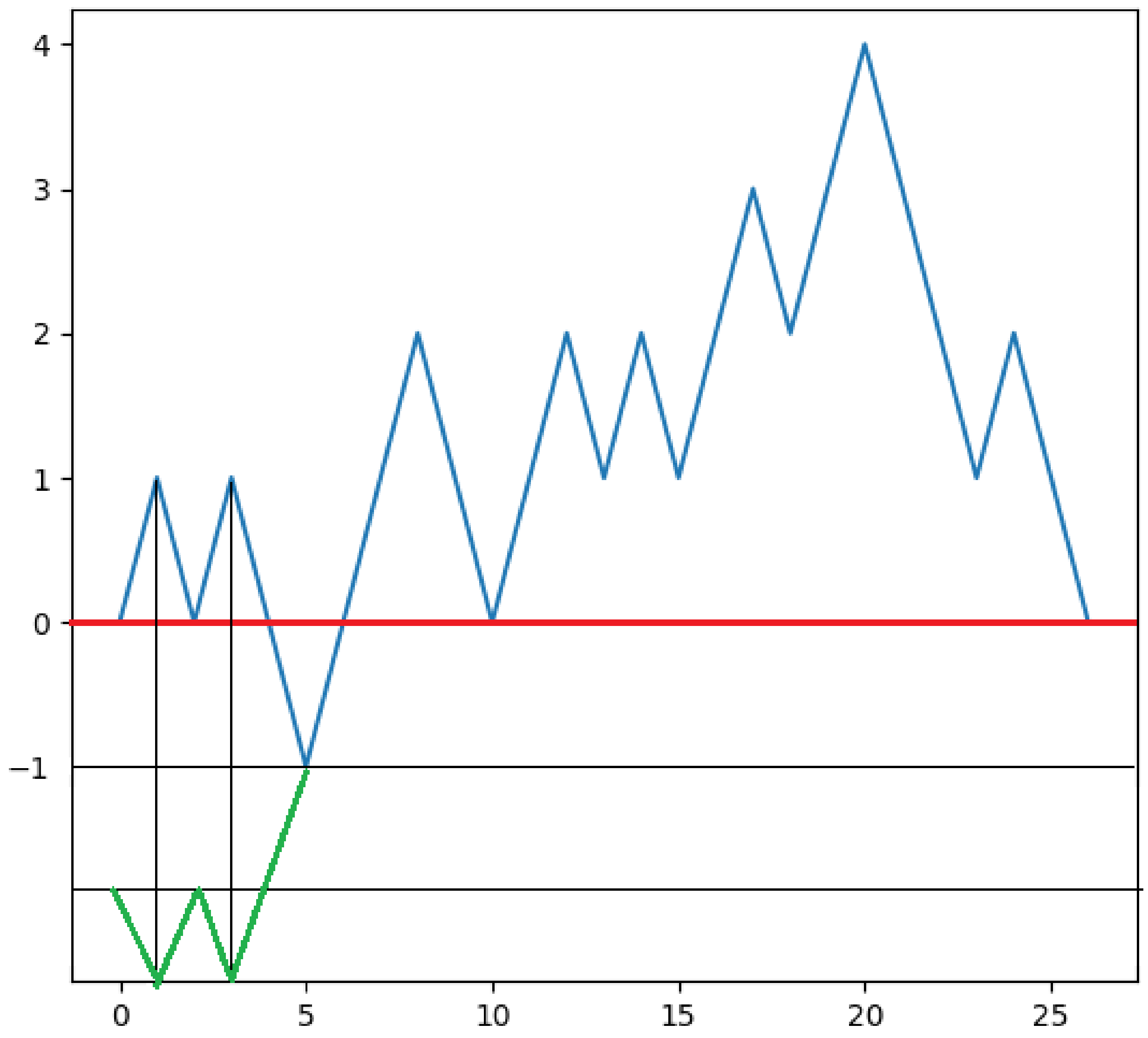
P [(0,0), (2n,0)]

P*[(0,-2), (2n, 0)]

Proof

Graphical path

$P \longrightarrow P^*$

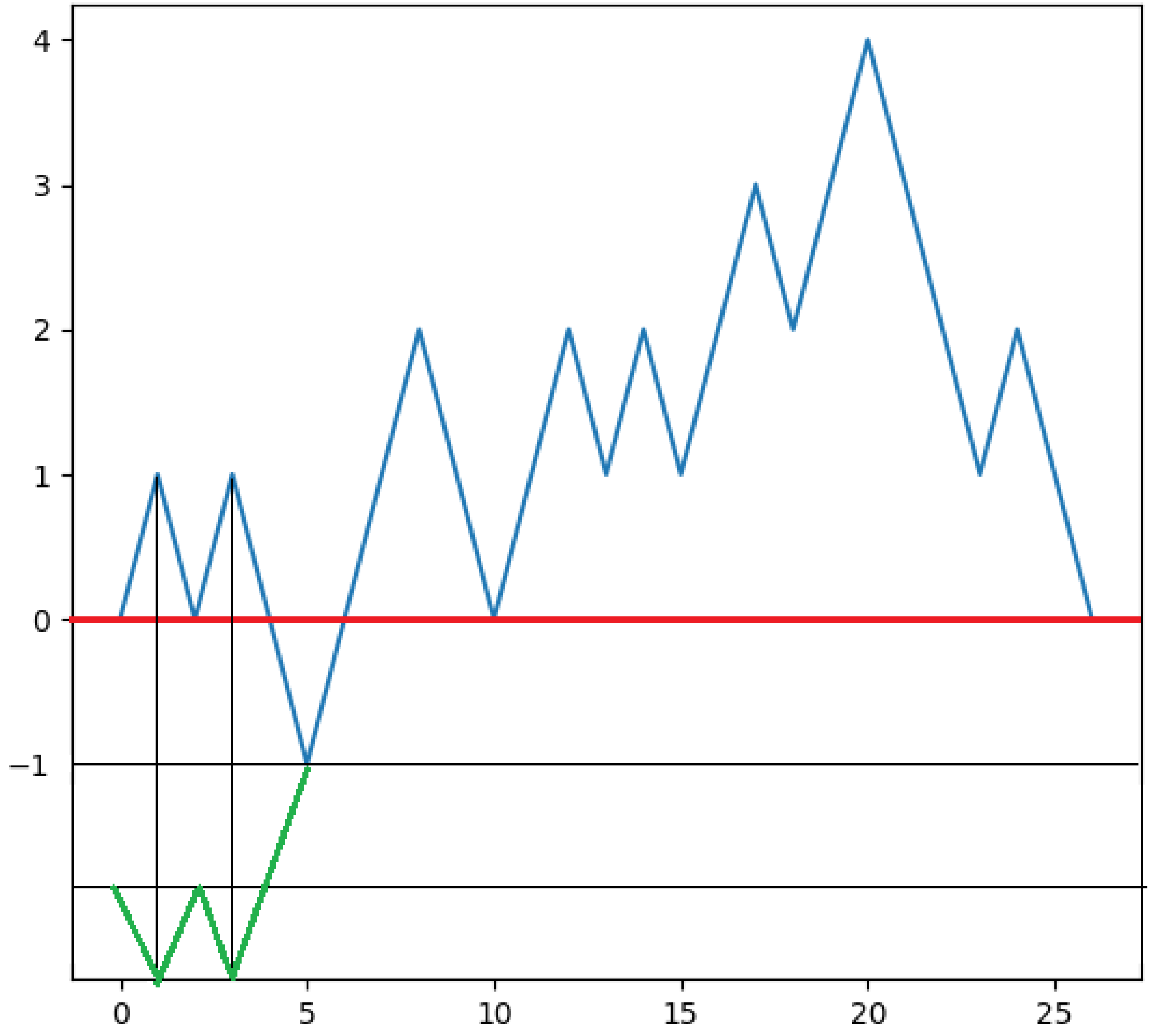


Proof

Graphical path

bijection

$P \longrightarrow P^*$

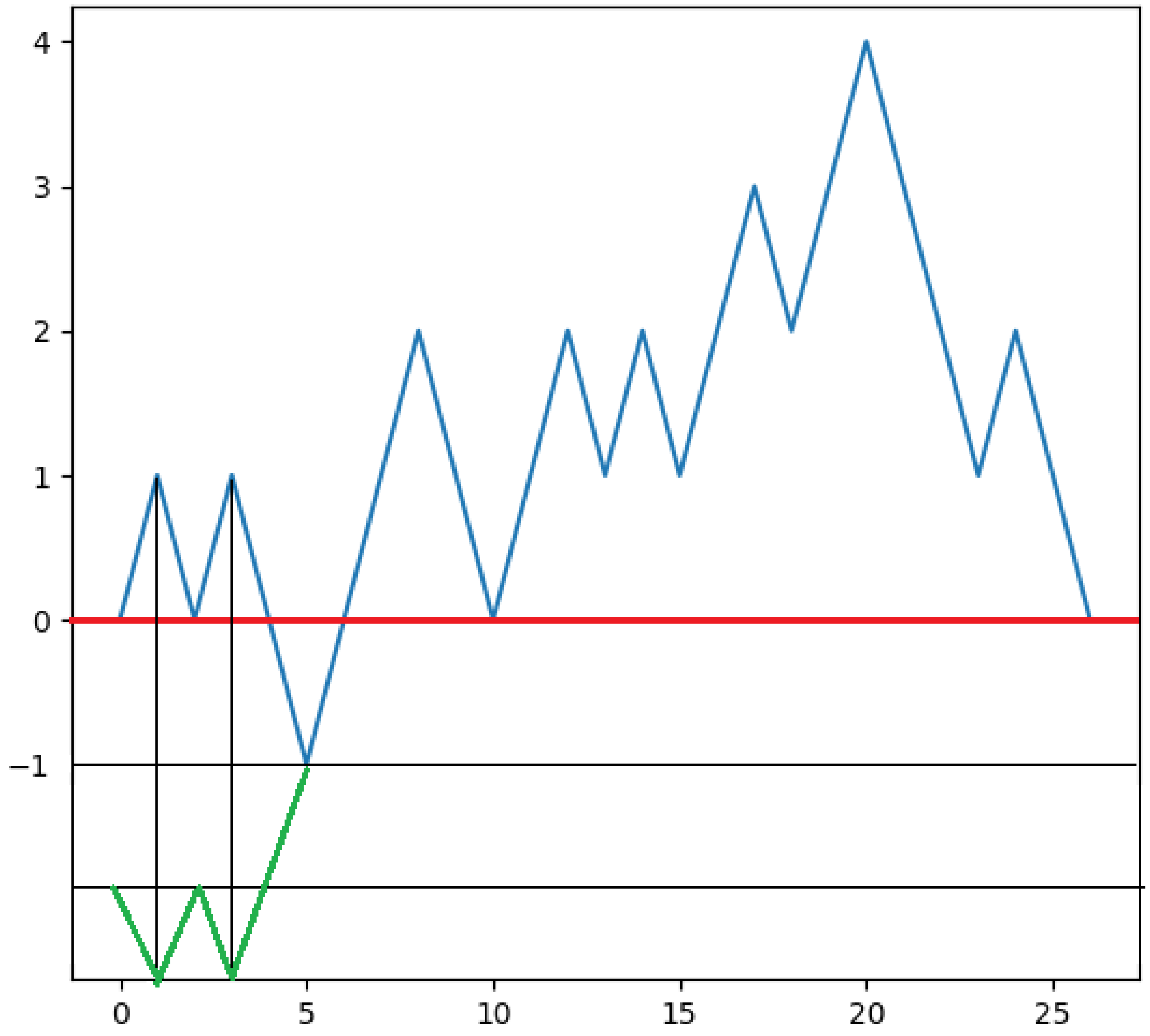


Proof

Graphical path

P^*

Mountain ranges under y=0



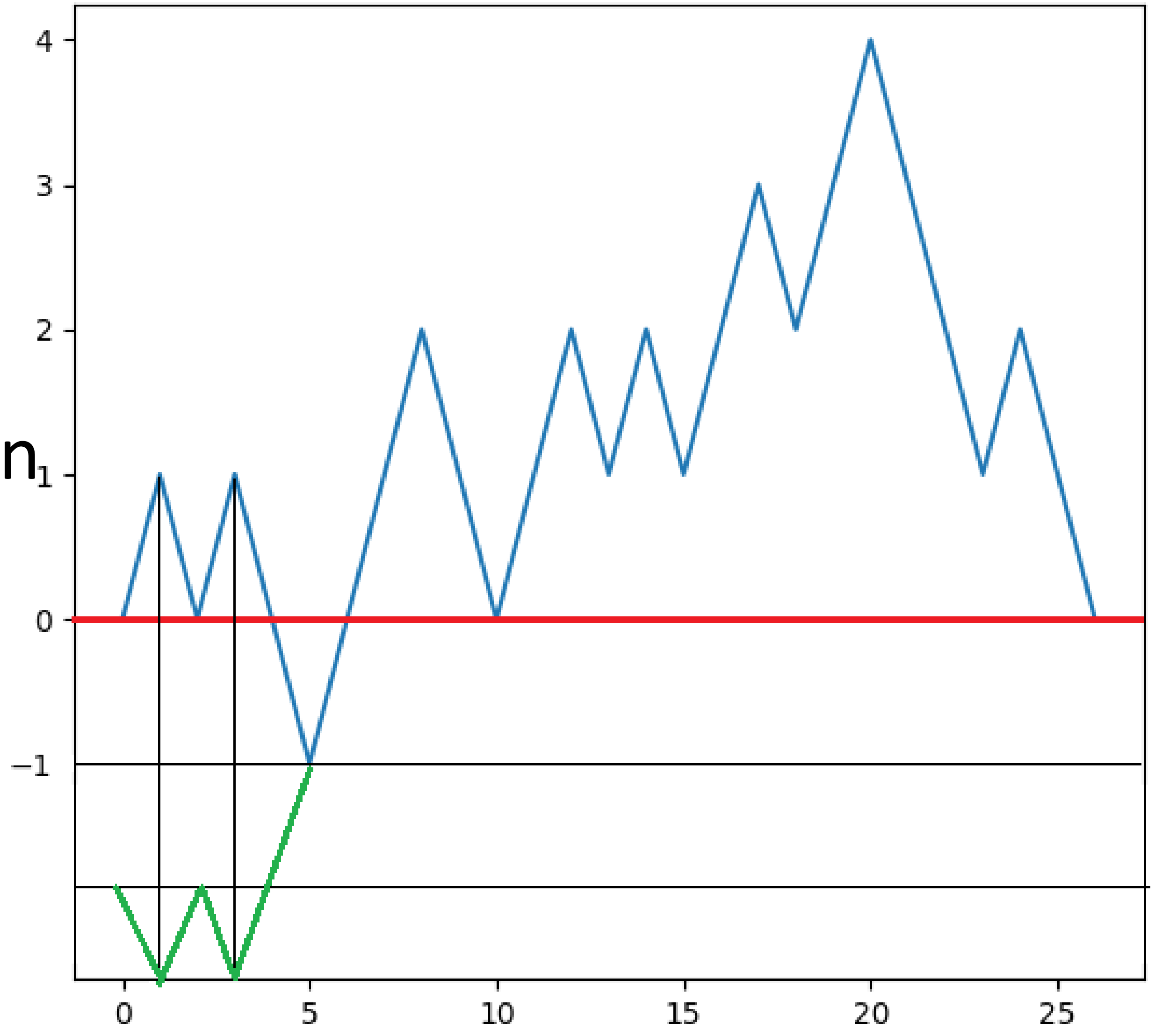
Proof

Graphical path

P*

Mountain ranges under y=0

Don’t satisfy second condition of balanced strings.



Proof

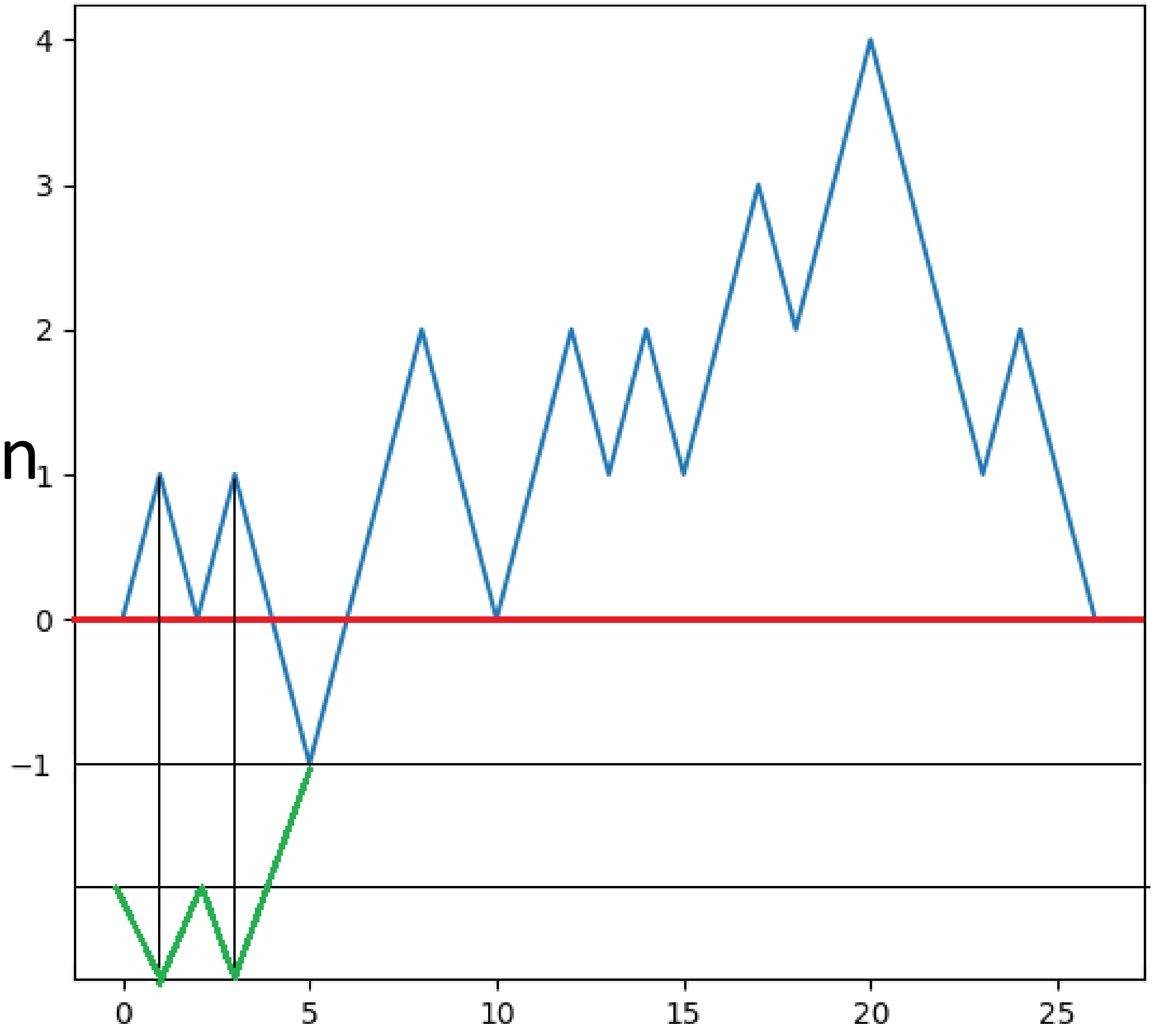
Graphical path

P*

Mountain ranges under y=0

Don't satisfy second condition of balanced strings.

Strings with length 2n that contain n+1 0s and n-1 1s.



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Graphical path

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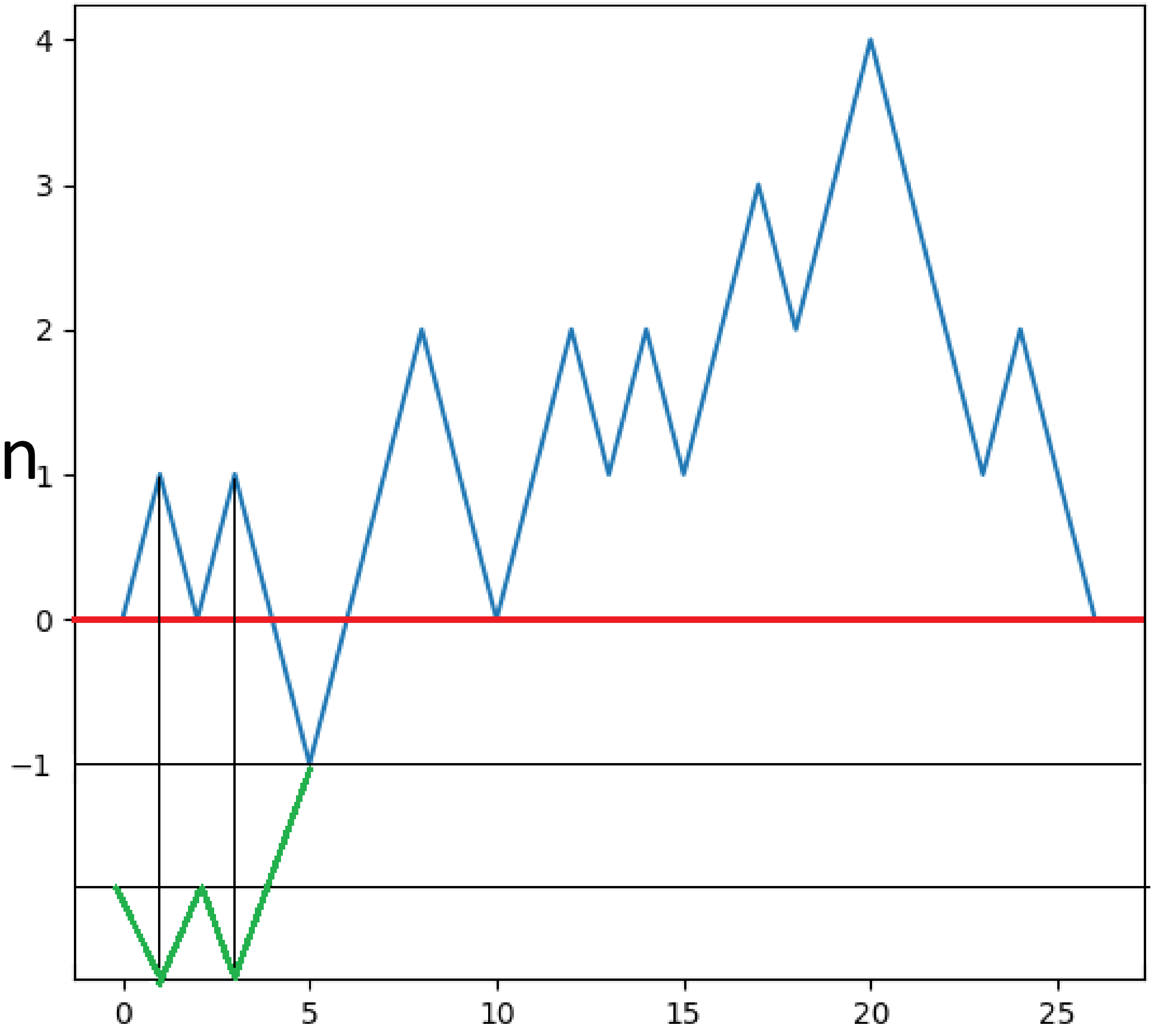
$$\binom{2n}{n+1}$$



Combinatorial explanation

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$$= \frac{2n!(n+1-n)}{n!n!(n+1)}$$

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$$= \frac{2n!(n+1-n)}{n!n!(n+1)} = \frac{1}{1+n}\binom{2n}{n}$$

Theorem

For any n > 0:

$$C_n = \frac{1}{1+n}\binom{2n}{n}$$

Rank & Unrank

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Let n be a positive integer

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Let (x, y)∋ $\mathbb{Z}\times\mathbb{Z}$

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$0 \leq x \leq 2n$

Rank & Unrank

Let n be a positive integer
Let $(x, y) \ni \mathbb{Z} \times \mathbb{Z}$
$0 \leq x \leq 2n$

Mn(x,y) set of all mountain ranges from (x,y) to (2n,0) that do not drop below y=0

Rank & Unrank

Let n be a positive integer
Let $(x, y) \ni \mathbb{Z} \times \mathbb{Z}$
$0 \le x \le 2n$

$\mathbf{M}_n(x,y)$ set of all mountain ranges from (x,y) to $(2n,0)$ that do not drop below $y=0$

$$M_n(x,y) = |\,\mathbf{M}_n(x,y)|$$

Rank & Unrank

$M_n(0,0) = C_n$

Rank & Unrank

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Rank & Unrank

$M_n(0,0) = C_n$
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$M_n(x,y) = 0$ if $y<0$

Rank & Unrank

$M_n(0,0) = C_n$
$M_n(2n,0) = 0$
$M_n(x,y) = 0$ if $y<0$
$M_n(x,y) = 0$ if $x+y>2n$

Lemma

$0 \leq x \leq 2n$

$y \geq 0$

$x+y$ is even

$x+y \leq 2n$

Lemma

$0 \leq x \leq 2n$

$y \geq 0$

$x+y$ is even

$x+y \leq 2n$

$$Mn(x,y) = \binom{2n-x}{n-\frac{x+y}{2}} - \binom{2n-x}{n-1-\frac{x+y}{2}}$$