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Applied Robotics Technologies:

Manipulability Ellipses in Force

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Contents

Abstract

Jacobian Matrix plays an important role in the case of finding the manipulability ellispoids of a robot. So it is tried to bring all the basic calculations of the scara robot .

1. Introduction

The manipulability ellipses are synthetic instruments through which one can represent the Kinetostatic performance of the manipulators. On the other hand, for a robot, manipulability is a way to analyze its kinematic properties, based on the analysis of its Jacobian matrix [Yoshikawa-1985].

It will be beneficial for design and control of robots and for task planning to have a quantitative measure of manipulating ability of robot arms in positioning and orienting the end-effectors.

All the calculation and results in this report are based on a 2D planer SCARA robot and there will be the calculations of the Direct and Inverse Kinematic that mentioned during the lectures.

2. Kinematics Analysis

All the calculation and results in this report are based on a 2D planer SCARA robot and there will be the calculations of the Direct and Inverse Kinematic that mentioned during the lectures.

Kinematics analysis in robotics involves the study of the motion of robot mechanisms without considering the forces that cause the motion. It is primarily concerned with understanding the relationship between the joint angles and positions, as well as the end-effector position and orientation. There are two main aspects of kinematics analysis: Forward (Direct) kinematics and Inverse kinematics.

Please noticed that we define the working space S, embodying the spatial position and orientation of the end-effector, and the joint space Q, encapsulating the configurations specified by the joint angles.

$$S = [x\ y]^T$$

$$Q = [\alpha\ \beta]^T$$

3. Direct Kinematics

With reference to the figure of the robot and being aware of the coordinates of the End-Effector, the solution of the direct kinematic is quite simple to achieve.

Based on the actuator positions the Direct solution will be:

$$x_{p=}\ l_1 \cos\alpha + l_2 \cos(\beta + \alpha)$$

$$y_{p=}\ l_1 \sin\alpha + l_2 \sin(\beta + \alpha)$$

such that l_1 , l_2 , α and β will be the length of the first and the second link and their angles respectively.

4. Inverse Kinematics

And based on the coordinates of Tool Center Point of the End-Effector the Inverse solutions will be:

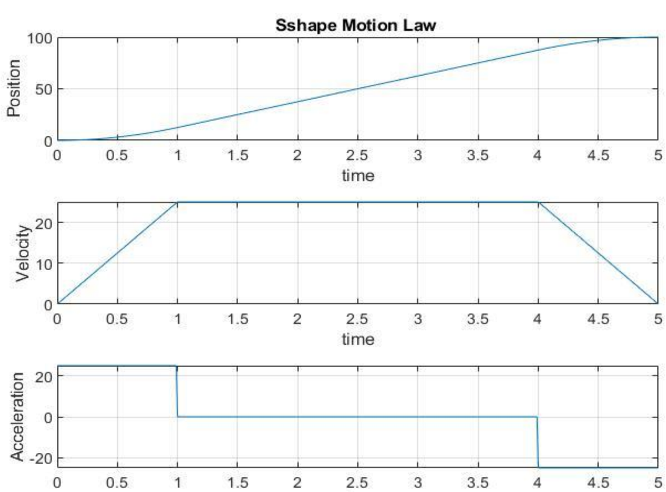
$$\beta = \pm \arccos\left(\frac{{x_p}^2 + {y_p}^2 - {l_1}^2 - {l_2}^2}{2 * {l_1}^2 * {l_2}^2}\right)$$

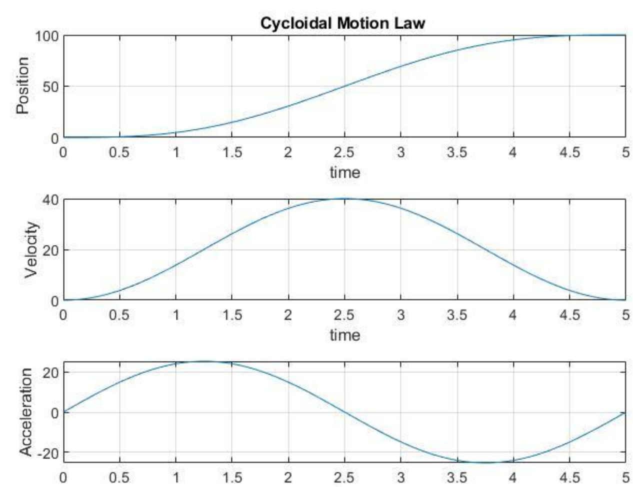
$$\alpha = \operatorname{arctan2}(y_p, x_p) - \arctan(2l_2 \sin(\beta), l_1 + l_2 \cos(\beta))$$

As it is obvious, we will have two 2 solutions for the Inverse kinematic of the Scara robot.

5. Motion Law

In this report Cycloidal and S-shape motion curved used to evaluate the results, achieved based o the motion of SCARA robot from an initial solution to the final position.





6. Jacobian Matrix

Indeed, understanding the relationship between velocities and accelerations in both the joint space and the workingspace is crucial in kinematics analysis. To achieve these relationships, it is required to drive the function of $F(Q)$ with regard the time. Actually, The Jacobian matrix in the context of robotics is a mathematical tool used to describe the relationship between the joint velocities and the end-effector velocities of a robot. It plays a crucial role in velocity and force analyses, as well as in solving the inverse kinematics problem.

The Jacobian matrix is typically denoted as $\boldsymbol{J}$ and relates the joint velocities **($\dot{\boldsymbol{Q}}$)** to the end-effector velocities **($\dot{\boldsymbol{S}}$)** through the equation:

$$S = F(Q)$$

$$\dot{\boldsymbol{S}} = \frac{\partial F}{\partial Q}\dot{\boldsymbol{Q}} = J\,\dot{\boldsymbol{Q}}$$

so

$$J = \frac{\partial F}{\partial Q}$$

therefore, based on the direct solution which is found; the final J for this 2D Scara robot will be:

$$J = \begin{bmatrix} -l_1 \sin\alpha - l_2 \sin(\beta + \alpha) & -l_2 \sin(\beta + \alpha) \\ l_1 \cos\alpha + l_2 \cos(\beta + \alpha) & l_2 \cos(\beta + \alpha) \end{bmatrix}$$

The most important point is the determinant of Jacobian matrix which represents that whenever the robot is in the singularity, this matrix will not be invertible and determinant will be equal to Zero.

The Jacobian matrix is a function of the current joint configuration and is derived from the partial derivatives of the forward kinematics equations. Its structure depends on the type of robot and the number of degrees of freedom.

7. Manipulability Ellipsoid in Velocity and Force

The concept involves employing a set of joint velocities with a constant norm, specifically a unitary norm. The objective is to compute the transformation produced by the Jacobian at every location within the working space.

A unitary norm matrix typically refers to a square matrix that has a special property related to its norm, often associated with linear algebra and complex numbers. A unitary matrix is a complex square matrix whose conjugate transpose (also known as the adjoint or Hermitian transpose) is its inverse. Mathematically, for a unitary matrix Q:

$Q^\dagger Q = QQ^\dagger = I$

Here, Q† represents the conjugate transpose of matrix Q, and I is the identity matrix. The property Q†Q = QQ† = I imply that the product of a unitary matrix and its conjugate transpose is equal to the identity matrix.

The expression $|\ (\dot{q}_{1,}\dot{q}_{2,})\ |= 1$ indicates that the vector $\dot{q}_{1,}\dot{q}_{2,}$ in a two-dimensional space has a magnitude (or norm) equal to 1. Here $\dot{q_1}$ *and* $\dot{q}_2$ are referred to as the set of joint velocities.

This expression is often associated with the idea of unit velocity vectors or normalized velocities. In the context of robotics or kinematics, it could mean that the joint velocities are constrained to have a constant norm of 1.

Mathematically, the expression $|\ (\dot{q}_{1,}\dot{q}_{2,})\ |= 1$ implies:

$$\sqrt{\dot{q}_1{}^2 + \dot{q}_2{}^2} = 1$$

The solutions to this equation form a circle in the $\dot{q}_{1,}\dot{q}_2$ plane with a radius of 1. Each point on this circle represents a pair of joint velocities ($q1$,$q2$) that satisfy the constraint of having a magnitude of 1.

.

$$\dot{q}^{T}\dot{q} \quad = 1$$

$$(J^{-1}\dot{S})^T * J^{-1}\dot{S} \quad = 1$$
$$\dot{S}^T(J\ J^T)^{-1} * \dot{S} \quad = 1$$

Here the solution of join velocities with a unitary norm that is always a circle led to the equation related to the work space that will give a circle or an Ellipse in the working space. The final shape in the work space depends on the Jacobian matrix.

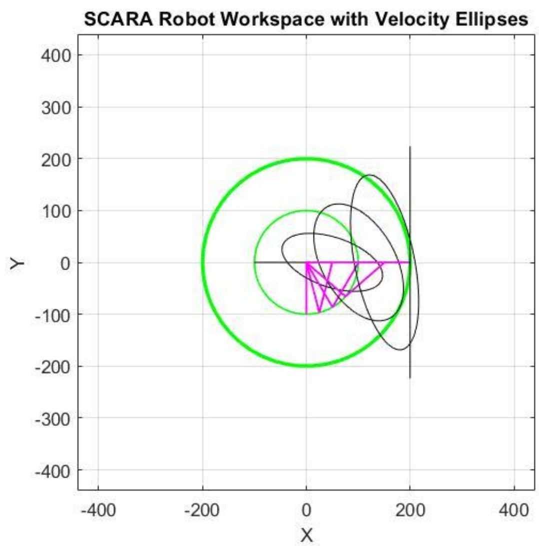
With respect to the kinetostatic duality the solution for the forces can be achieved:

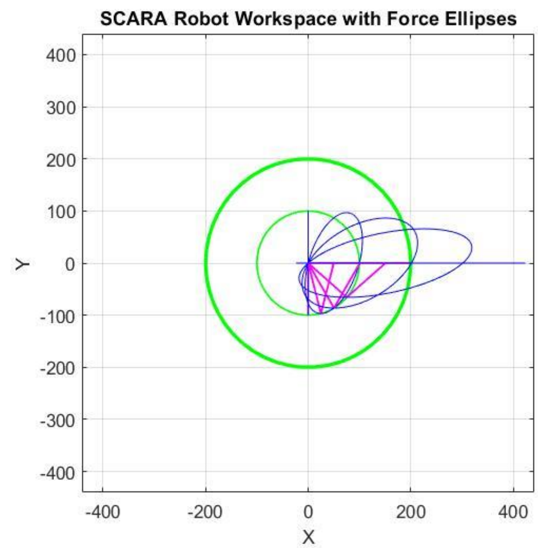
$$\mathrm{F}_S{}^T\ \mathrm{JJ}^T\ \mathrm{F}_S = 1$$

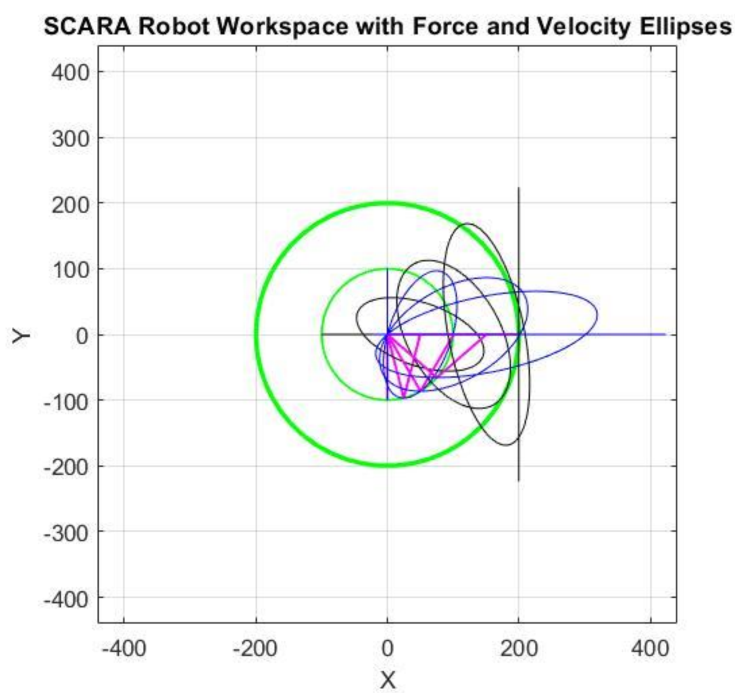
The ellipses under consideration are associated with the manipulation matrix, M$= JJ^T$, which is defined as the product of the Jacobian matrix (J) and its transpose (J^T).

These ellipses represent the manipulability of a robotic system. In the context of force and velocity, manipulability describes how effectively the system can generate forces and velocities in different directions. The statement highlights that the manipulability ellipses for force and velocity are orthogonal. Orthogonality implies that the directions in which the system is most effective in generating force and velocity are independent of each other.

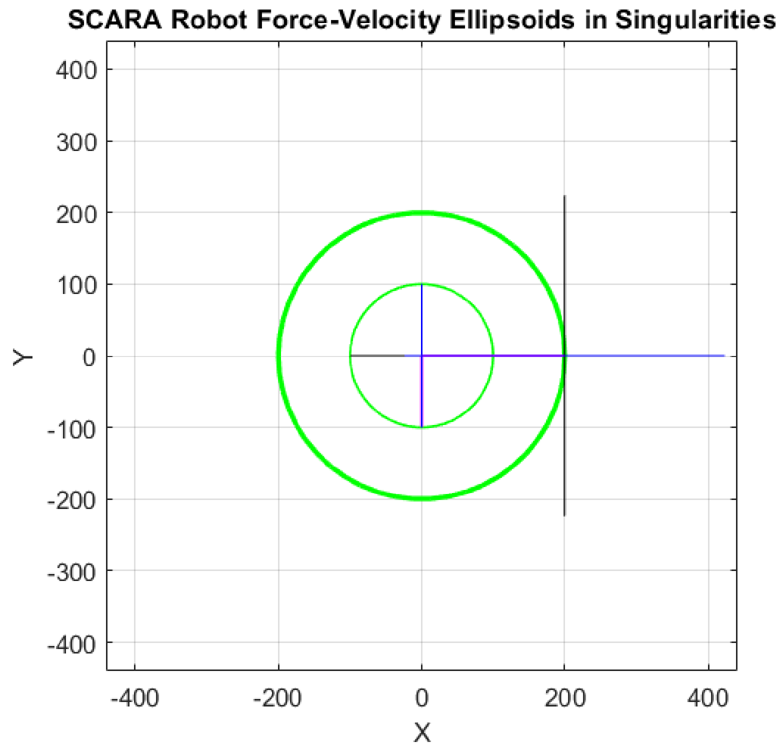
The eigenvectors of matrix M and its inverse M^{-1} are coincident. Eigenvectors are special vectors associated with a matrix, and in this case, their coincidence suggests a relationship between the manipulability of the system and its inverse manipulability. The eigenvalues of matrix M and M^{-1} are inversely related.







In singularity configurations, the ellipses or ellipsoids associated with manipulability undergo a degeneration into segments. Conversely, when the ellipsoid approaches a spherical shape (unit eccentricity), it signifies enhanced isotropic movement of the end-effector across all directions within the operational space.



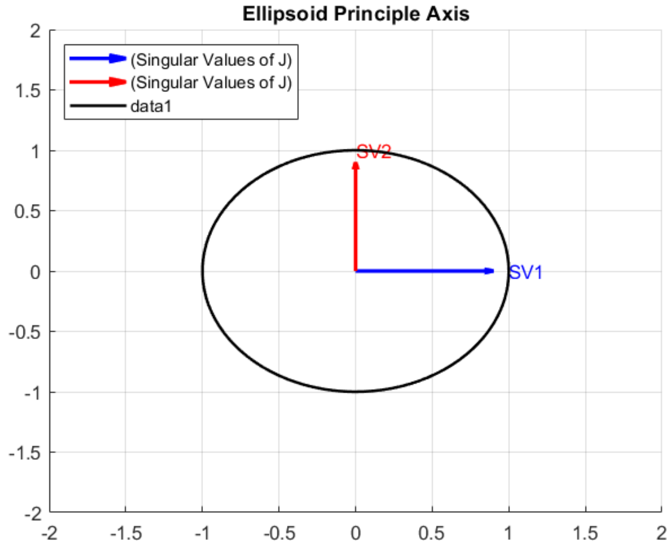
1. **Singular Values of the Jacobian Matrix:** The Jacobian matrix (J) represents the mapping between joint velocities and end-effector velocities in a robotic system. The singular values of J are obtained through singular value decomposition (SVD).

2. **Singular Value Decomposition (SVD):** If J is an $m \times n$ matrix, SVD decomposes it into three matrices U, Σ, and V^T such that $J = U\Sigma V^T$. The diagonal elements of Σ are the singular values.

3. **Relation to Ellipsoid Axes:** The singular values of J are related to the lengths of the semi-axes of the ellipsoid associated with manipulability.

4. **Principal Axes of the Ellipsoid:** The principal axes of the manipulability ellipsoid align with the directions of the singular vectors. The lengths of these axes are determined by the

singular values, with larger singular values indicating greater manipulability along the corresponding axes.



5. **Connection to the Manipulability Matrix (*M*):** The manipulability matrix $M=JJ^T$ is related to the Jacobian matrix, and its eigenvalues and eigenvectors also provide information about the ellipsoid. The square roots of the eigenvalues of M correspond to the lengths of the principal axes of the manipulability ellipsoid.

8. Manipulability Indices

The manipulability index (Area of the Ellipse in XY plane) was proposed as a kinematic performance measure by Yoshikawa. The Yoshikawa manipulability index happens to be the most widely accepted and used measure for kinematic manipulability. Like most kinematic indices the manipulability index is based on the manipulator's Jacobian matrix. For a redundant manipulator the manipulability index is defined as the square root of the determinant of the product of the Jacobian matrix and its transpose.

$$w = \sqrt{\det(J * J^T)}$$

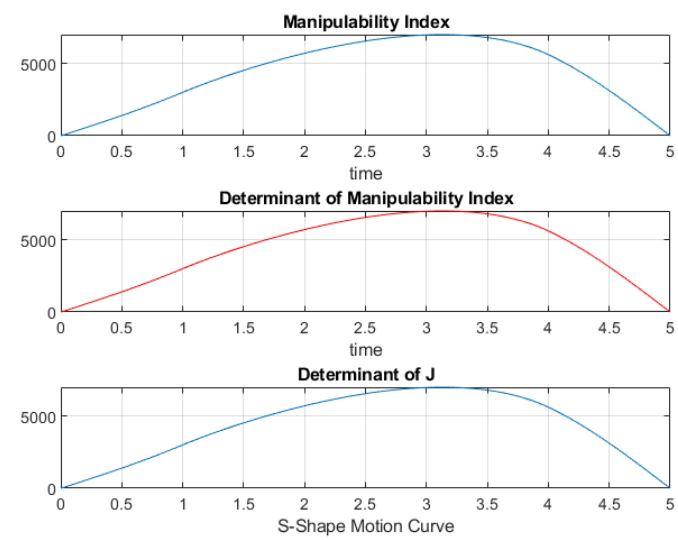
The manipulability index can be applied for both redundant and nonredundant manipulators. When nonredundant manipulators are considered, the manipulability measure shrinks to:

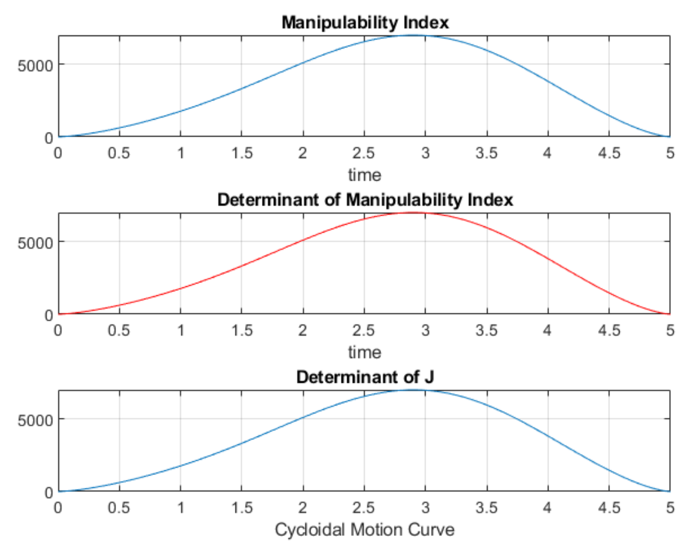
$$w = |det\ (J\ )|$$

The Jacobian manipulability can also be expressed as:

$w = \sqrt{\lambda_1\lambda_2\lambda_3 \ldots . \lambda_i} = \sigma_1\sigma_2\sigma_3, \ldots, \sigma_i$ where λ_i is the Eigenvalue of $J * J^T$ matrix and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_i$ and σ_i is the singular value of (J) matrix.

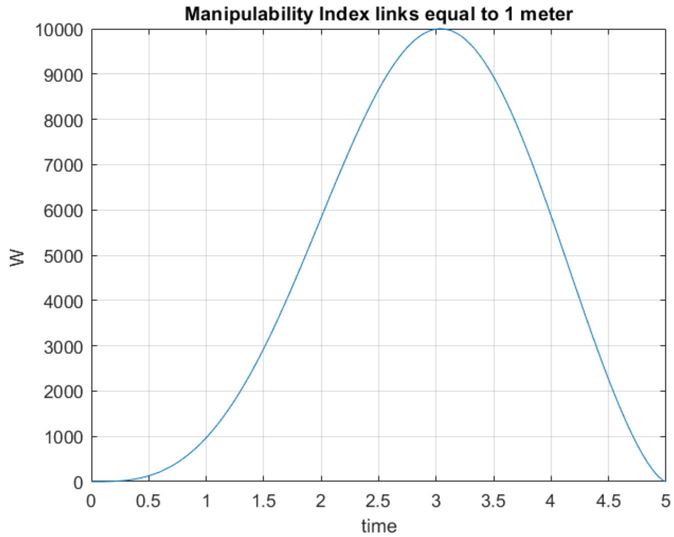
Here all of these statements are plotted by using the S-shape and Cycloidal motion curves.

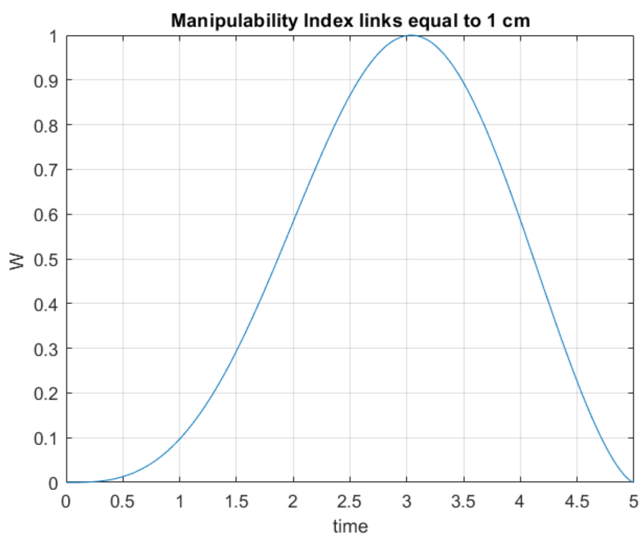




The assessment of the ellipsoid volume (Area of the Ellipse in XY plane) does not serve as a universally comparable or reference area near the singularity configuration due to its reliance on machine size (dimensions). To illustrate, let's examine two SCARA robots with matching arms, but one has links measuring 1 meter while the other has links of 1 centimeter. When positioned identically, the manipulability values differ significantly, with a **staggering four-order** magnitude gap! This discrepancy underscores the impact of the robots' physical dimensions on manipulability, emphasizing the need for cautious consideration when evaluating and comparing these metrics.

Here by using the Cycloidal motion curve for each link, considering the length of links once equal to 1 and in the second time 100, when the robot links are going from the initial point to the final point, this **four-order in magnitude** can be observed.



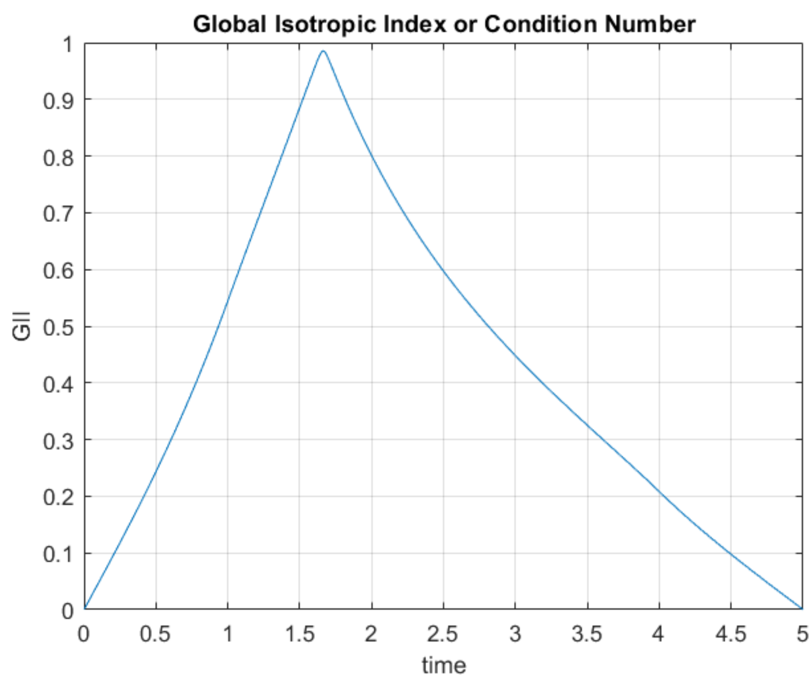


To address this issue, a better approach involves utilizing the ratio between the smallest and largest singular values of the Jacobian matrix, known as the Matrix's Conditioning Number or Global Isotropic Index. This value remains at 1 when the robot's pose is isotropic, indicating a balanced maneuverability in all directions. Typically, the reciprocal of this conditioning number is employed, resulting in a range of values between 0 and 1. Conditions are deemed acceptable when this value surpasses 0.2, signifying a reasonably stable and manageable state.

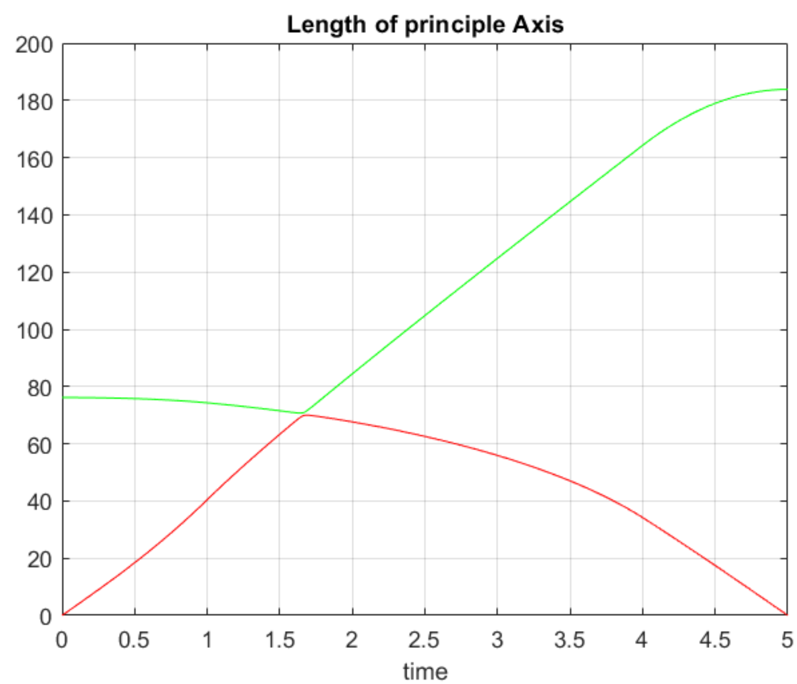
The Global Isotropy Index (GII), offers a perspective on a manipulator's performance by examining the relationship between the minimum (σ_{min}) and maximum (σ_{max}) singular values across its entire workspace. The formula for GII is expressed as

$$GII = \frac{\sigma_{min}}{\sigma_{max}}$$

Unlike the manipulability index, which focuses on specific points in the workspace, GII takes into account the extremities of performance throughout the entire reachable area. Essentially, GII serves as a metric to identify the manipulator's weakest performance, offering insights into its overall isotropic behavior across diverse poses and configurations.



To enhance the clarity of this index, the length of the principal axis those are visualized. When examining the plot, it becomes evident that in the vicinity of the isotropic position, the lengths of the links closely align with each other. Consequently, this alignment results in a semi-circle shape for the force and velocity ellipsoids. This graphical representation helps illustrate how, in positions nearing isotropy, the lengths of the links contribute to a more uniform and symmetric distribution of forces and velocities, reflected in the characteristic semi-circular shape.



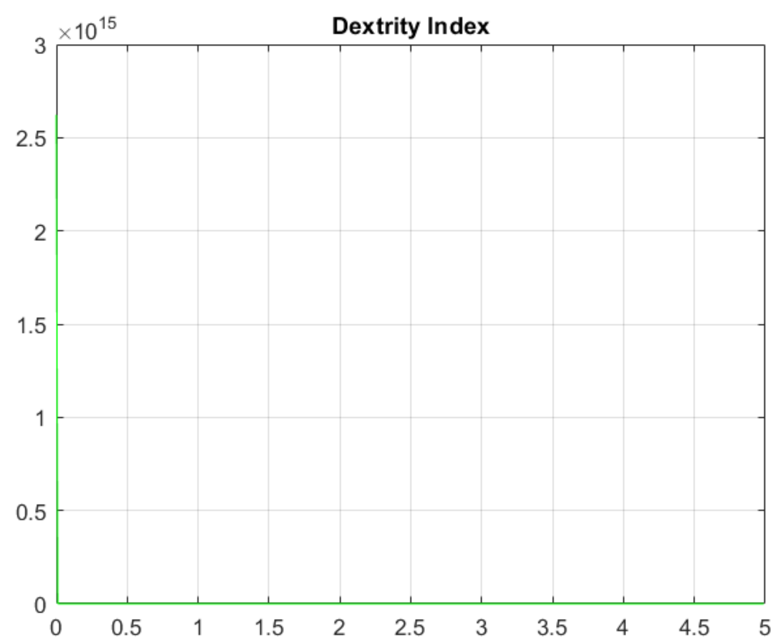
Please notice that the length of the first and second links of the robot were 100 and 70 respectively.

Also, to see all the changes of the different values and indices, initial and final position for the end effector considered once the links are folded ($\begin{bmatrix}30\\0\end{bmatrix}$) and once the links of the robot are completely extended ($\begin{bmatrix}170\\0\end{bmatrix}$) in the other words from one of the singularities to another one. As far as motion curved used to achieve these graphs, so all the graphs are plotted with respect to the time.

Another common index used to detect and avoid singularities is the dexterity index:

$$\kappa = \frac{\sigma_{max}}{\sigma_{min}}$$

where σ_{max} and σ_{min} are the maximal and minimal singular values of the Jacobian, also known as the dexterity index. The value of the dexterity index for a given configuration is associated with the distortion of task space sensitivity. This value can be interpreted geometrically as a ratio between the longest and shortest axis lengths of the manipulability ellipsoid. When the Jacobian loses its full rank, the minimum singular value σ_{min} is equal to zero and the dexterity index becomes infinity. In other words, the dexterity index does not have an upper bound it will be $K = [0 \;\; \infty)$.



9. Conclusion

In conclusion, Manipulability ellipsoids are vital in robotics, offering a quantitative assessment of a robot's motion capabilities. These ellipsoids inform optimal path planning, singularity avoidance, and redundancy resolution. They guide design decisions by revealing insights into a robot's efficiency across its workspace.

10. MATLAB function

Here the MATLAB function related to the calculations the values related of the Velocity and Force Ellipsoid are as follows:

```
function [MAP_Axis, MIP_Axis ,M_I, M_I_d, D_I,G_I,detJ] =
plotEllipsoid(L, S)
%[MAP_Axis, MIP_Axis ,M_I, M_I_d, D_I,G_I,detJ] these values are
taken to
%show the indices with respect to the time based on the motion
curve.

%% Calculation of the Jacobian Matrix
Q = SCARAinv(S, L, 1);
J = SCARAjac(Q, L);
detJ = abs(det(J));

%% Compute Singular Values of J that are equal to Dimensions of
the Axes
S_V = svd(J);
Maximum_SV = max(S_V);
Minimum_SV= min(S_V);

%% Compute JJT and its Eigenvalues and Eigenvectors
M = (J* J');
[EV, EI] = eig(M);

%% Plot the ellipsoid centered at the TCP position
ponit_angles = 0 : 0.01 : 2*pi ;
Ellipsoid_points_Force = EV * [Maximum_SV * cos(ponit_angles);
Minimum_SV * sin(ponit_angles)];
Ellipsoid_points_Velocity = EV * [Minimum_SV *
cos(ponit_angles);Maximum_SV * sin(ponit_angles)];

%% All the points that we achieved for ellipsoids must be located
in the TCP(S = [X Y])
Ellipsoid_Velocity = Ellipsoid_points_Velocity + S;
Force_Ellipsoid= Ellipsoid_points_Force + S;

hold on;
```

```
plot(Ellipsoid_Velocity(1, :), Ellipsoid_Velocity(2, :), 'k',
'LineWidth', 0.1);
plot(Force_Ellipsoid(1, :), Force_Ellipsoid(2, :), 'b',
'LineWidth', 0.1);

%% Plot eigenvectors with annotations
S_f = 50;  % Adjust the scale factor for better visualization
quiver(S(1,1), S(2,1), EV(1, 1) * S_f, EV(2, 1) * S_f, 'r',
'LineWidth', 1);
quiver(S(1,1), S(2,1), EV(1, 2) * S_f, EV(2, 2) * S_f, 'b',
'LineWidth', 1);

%% Calculating the indices of the Manipulability Ellipsoids
M_I = (sqrt((max(eig(M)))*(min(eig(M))))); %Manipulability Index
= (sqrt(det(M))) Area of the ellipse
M_I_d = (sqrt(det(J*J')));% W in the time that the robot is in
nonredundent situation( area of the ellipse)
D_I = (Maximum_SV/Minimum_SV);%Dextrity Index
G_I = min(S_V) / max(S_V);%Global Isotropy Index or Condition
Number
MAP_Axis=Maximum_SV;
MIP_Axis=Minimum_SV;

end
```

11. References

- Manipulability of Robotic Mechanisms, Tsuneo Yoshikawa, The International Journal of Robotics Research 1985 4: 3, DOI: 10.1177/027836498500400201

- Manipulator Performance Measures - A Comprehensive, Literature Survey, Sarosh Patel · Tarek Sobh, DOI 10.1007/s10846-014-0024-y

- Robot Dynamics and Control Second Edition Mark W. Spong, Seth Hutchinson, and M. Vidyasagar, January 28, 2004

- Kinematic Isotropy and Robot Design Optimization, A genetic Algorithm Method by Shabnam Khatami, The university of British Columbia, October 2021

- A Geometric Approach for Generating Feasible Configurations of Robotic Manipulators by Filip Marić, University of Toronto 2023

- Robotics Modelling, Planning and Control, Bruno Siciliano , Lorenzo Sciavicco Luigi Villani , Giuseppe Oriolo

- Slides of the course By Prof. Hermes Giberti