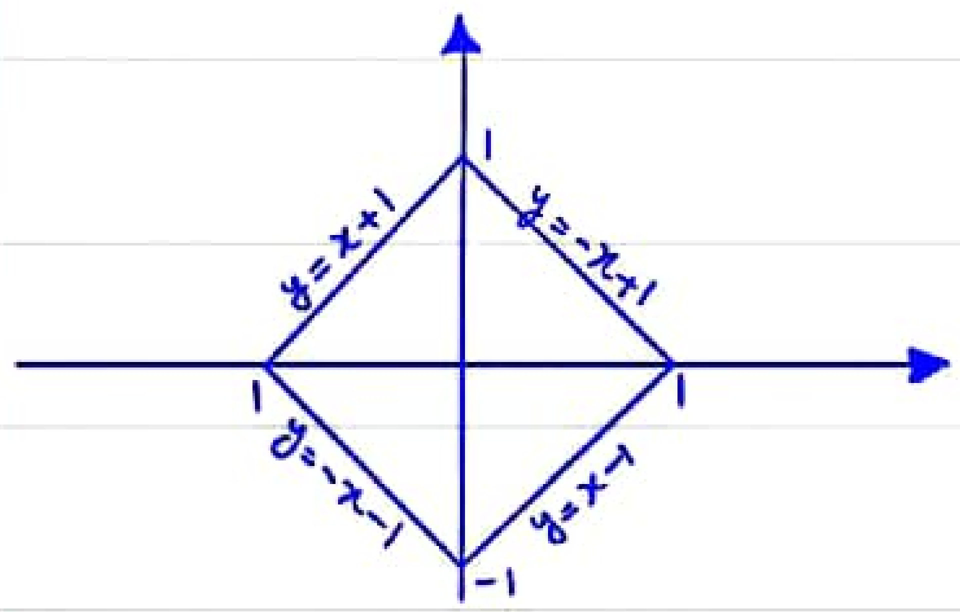


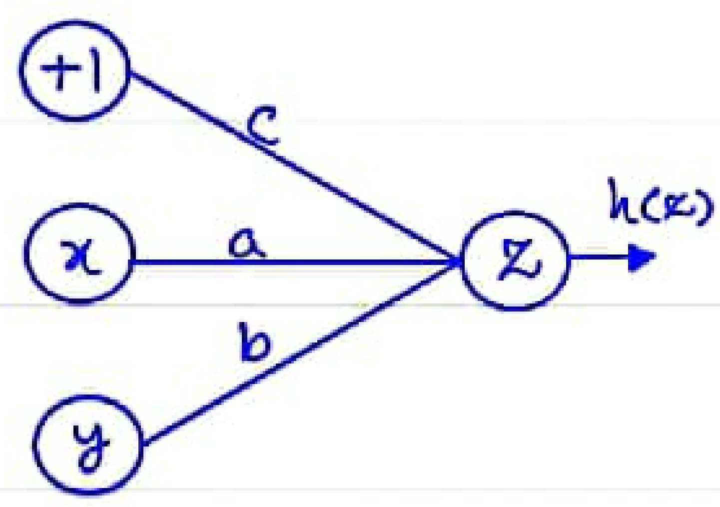
محدوده درونی به صورت زیر است :

(بالای خط $y=-x-1$) $\cap$ (بالای خط $y=x-1$) $\cap$ (پایین خط $y=-x+1$) $\cap$ (پایین خط $y=x+1$)

$$= (y<x+1) \cap (y<-x+1) \cap (y>x-1) \cap (y>-x-1)$$

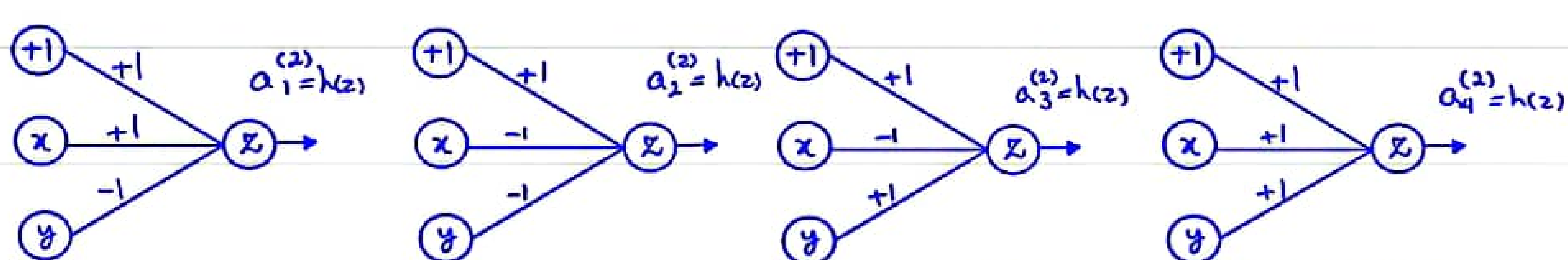
$$= (x+1-y>0) \cap (-x+1-y>0) \cap (y-x+1>0) \cap (y+x+1>0)$$

برای یک عبارت $ax+by+c>0$ ، شبکه نورونی معادل آن برابر است با :



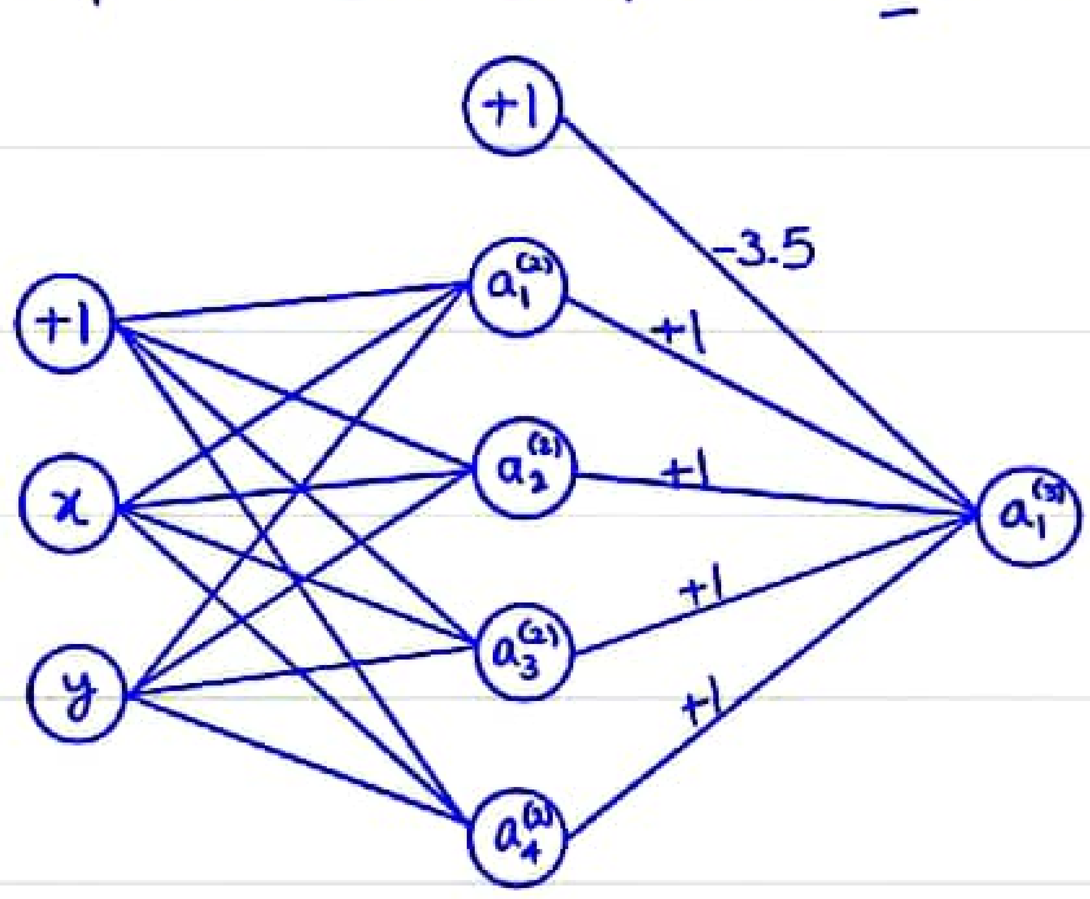
$Z=ax+by+c$, $h(z)=sign(z)$

بنابراین ابتدا برای هر خط ، شبکه متناظر آن را بدست می آوریم :



$x-y+1>0$ $\quad$ $-x-y+1>0$ $\quad$ $-x+y+1>0$ $\quad$ $x+y+1>0$

محدوده مشخص شده اشتراک نواحی بدست آمده توسط چهار خط می باشد. بنابراین زمانی که خروجی هر ۴ نورون $a_1^{(2)}$ تا $a_4^{(2)}$ ، یک باشد ، در ناحیه مشخص شده قرار داریم :



$$a_1^{(2)}+a_2^{(2)}+a_3^{(2)}+a_4^{(2)} \in \{0,1,2,3,4\}$$

بنابراین تنها زمانی خروجی شبکه برابر یک می شود که هر کدام از $a_i^{(2)}$ ها ، یک باشند. $(i\in\{1,\dots,4\})$

②

$$f_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{\pi} x_1 \sin(\pi x_2) \\ e^{x_1 - 1} x_2^2 \\ x_1 x_2 \end{bmatrix} , \quad f_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1^2 + x_2^2 + x_3^2 \end{bmatrix}$$

$$f = f_2 \circ f_1 \Rightarrow f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{\pi} x_1 \sin(\pi x_2) + e^{x_1 - 1} x_2^2 + x_1 x_2 \\ \frac{1}{\pi^2} x_1^2 \sin^2(\pi x_2) + e^{2x_1 - 2} x_2^4 + x_1^2 x_2^2 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1} , \frac{\partial f}{\partial x_2} \right]$$

$$\frac{\partial f}{\partial x_1} = \begin{bmatrix} \frac{1}{\pi} \sin(\pi x_2) + e^{x_1 - 1} x_2^2 + x_2 \\ \frac{2}{\pi^2} x_1 \sin^2(\pi x_2) + 2e^{2x_1 - 2} x_2^4 + 2 x_1 x_2^2 \end{bmatrix} \Rightarrow \left.\frac{\partial f}{\partial x_1}\right|_{x = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]} = \begin{bmatrix} 6 \\ 40 \end{bmatrix}$$

$$\frac{\partial f}{\partial x_2} = \begin{bmatrix} x_1 \cos(\pi x_2) + 2e^{x_1 - 1} x_2 + x_1 \\ \frac{1}{\pi} x_1^2 \sin(2\pi x_2) + 4e^{2x_1 - 2} x_2^3 + 2x_1^2 x_2 \end{bmatrix} \Rightarrow \left.\frac{\partial f}{\partial x_2}\right|_{x = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]} = \begin{bmatrix} 6 \\ 36 \end{bmatrix}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 6 & 6 \\ 40 & 36 \end{bmatrix}$$

③

$$\frac{\partial y}{\partial W^{(1)}} , \frac{\partial y}{\partial w^{(2)}} , \quad z^{(1)} = W^{(1)} x , \; y^{(1)} = f(z^{(1)}) , \; z^{(2)} = w^{(2)T} y^{(1)} , \; y = f(z^{(2)})$$

$$x \in \mathbb{R}^n , \; W^{(1)} \in \mathbb{R}^{m_1 \times n} , \quad w^{(2)} \in \mathbb{R}^{m_1} , \; y \in \mathbb{R}$$

$$z^{(2)} \in \mathbb{R} \Rightarrow \frac{\partial y}{\partial z^{(2)}} = f'(z^{(2)}) , \quad \frac{\partial z_2}{\partial w_i^{(2)}} = y_i^{(1)} \Rightarrow \boxed{\frac{\partial y}{\partial w^{(2)}} = f'(w^{(2)T} y_1) y_1}$$

$$\frac{\partial y}{\partial y_i^{(1)}} = \frac{\partial y}{\partial z^{(2)}} \frac{\partial z^{(2)}}{\partial y_i^{(1)}} = f'(w^{(2)T} y_1) w_i^{(2)}$$

$$\frac{\partial y}{\partial w_{ij}^{(1)}} = \frac{\partial y}{\partial z^{(2)}} \frac{\partial z^{(2)}}{\partial y_i^{(1)}} \frac{\partial y_i^{(1)}}{\partial z_i^{(1)}} \frac{\partial z_i^{(1)}}{\partial w_{ij}^{(1)}} = f'(z^{(2)}) w_i^{(2)} f'(z_i^{(1)}) x_j$$

$$z^{(2)} = w^{(2)T} y_1 , \quad z_i^{(1)} = W_{i,:}^{(1)} x$$

$$\Rightarrow \boxed{\frac{\partial y}{\partial W^{(1)}} = f'(z^{(2)}) \left(w^{(2)} \otimes f'(z^{(1)}) \right) x^T}$$

④

$$\phi_n(x) = \frac{1}{\sqrt{n!}} x^n e^{-\frac{x^2}{2}} \quad , \quad 0 \le n < \infty$$

$$K(x, x') = \langle \phi(x), \phi(x') \rangle$$

$$\Rightarrow K(x, x') = \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{n!}} x^n e^{-x^2/2} \right) \left(\frac{1}{\sqrt{n!}} x'^n e^{-\frac{x'^2}{2}} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} (xx')^n e^{-\frac{x^2+x'^2}{2}}$$

$$= e^{-\frac{x^2+x'^2}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} (xx')^n$$

با توجه به بسط تیلور e^x داریم:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} (xx')^n = e^{xx'}$$

در نتیجه:

$$K(x, x') = e^{-\frac{x^2+x'^2}{2}} e^{xx'} = e^{\frac{-1}{2}(x^2 - 2xx' + x'^2)} = e^{-\frac{(x-x')^2}{2}}$$

$$\xRightarrow{x \in \mathbb{R}} \boxed{K(x, x') = e^{-\frac{|x-x'|^2}{2}}}$$

⑤

$$(x_1 = 0, y_1 = -1) \quad , \quad (x_2 = \sqrt{2}, y_2 = 1) \quad , \quad \phi(x) = [1, \sqrt{2}x, x^2]^T$$

$$\min \|w\|^2 \quad s.t. \quad y_1(w^T\phi(x_1) + w_0) \ge 1 \quad \& \quad y_2(w^T\phi(x_2) + w_0) \ge 1$$

الف)

$$\phi(x_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \phi(x_2) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad , \quad w = (w_1, w_2, w_3)$$

$$w' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \Rightarrow w^* \parallel \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

ب)

$$d = \sqrt{(1-1)^2 + (0-2)^2 + (0-2)^2} = 2\sqrt{2} \Rightarrow margin = \frac{d}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

ج)

$$margin = \frac{1}{\|w\|} = \sqrt{2} \Rightarrow \|w\| = \frac{1}{\sqrt{2}} \quad , \quad \|w'\| = \sqrt{0^2 + 2^2 + 2^2} = 2\sqrt{2}$$

$$\Rightarrow w = \frac{w'}{4} \Rightarrow \boxed{w = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix} \quad (\|w\| = \frac{1}{\sqrt{2}})}$$

(د

$$y_i\,(w^T\phi(x_i)+w_0)\geq 1 \quad \text{for } i=1,2$$

$$\Rightarrow \left\{\begin{array}{l} -\left([0,\ \tfrac{1}{2},\ \tfrac{1}{2}]\begin{bmatrix}1\\0\\0\end{bmatrix}+w_0\right)\geq 1 \Rightarrow -w_0=1 \\ [0,\ \tfrac{1}{2},\ \tfrac{1}{2}]\begin{bmatrix}1\\2\\2\end{bmatrix}+w_0\geq 1 \Rightarrow 2+w_0=1 \end{array}\right\} \boxed{w_0=-1}$$

(ه

$$f(x)=w_0+w^T\phi(x)\quad,\quad w_0=-1\,,\quad w=\begin{bmatrix}0\\ 1/2\\ 1/2\end{bmatrix}\,,\quad \phi(x)=\begin{bmatrix}1\\ \sqrt{2}x\\ x^2\end{bmatrix}$$

$$\Rightarrow w^T\phi(x)=\frac{\sqrt{2}}{2}x+\frac{1}{2}x^2 \Rightarrow \boxed{f(x)=-1+\frac{\sqrt{2}}{2}x+\frac{1}{2}x^2}$$