تمرین سری اول

سجاد کاظم بیگی (98107077)

1 احتمال توزیع حاشیه‌ای و شرطی

$$\rho = \frac{Cov(y_1, y_2)}{\sigma_1 \sigma_2} = \frac{\Sigma_{12}}{\Sigma_{11}^{\frac{1}{2}} \Sigma_{22}^{\frac{1}{2}}}$$

الف)

$$P(y_2) = \mathcal{N}(\mu_2, \Sigma_{22})$$

$$p(y_2) = \int p(y_2, y_1)\, dy_1 = \int \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2}\right)\right]$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_2} \exp\left[-\frac{1}{2}\frac{(y_2-\mu_2)^2}{\sigma_2^2}\right]$$

$$\Rightarrow p(y_2) = \frac{1}{\sqrt{2\pi}\, \Sigma_{22}^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\frac{(y_2-\mu_2)^2}{\Sigma_{22}}\right)$$

$$= \mathcal{N}(\mu_2, \Sigma_{22})$$

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1 ب) z را به صورت $z = y_1 + A y_2$ تعریف می‌کنیم تا z و y_2

uncorrelated شوند $\qquad z = y_1 + A y_2 \quad , \quad A = -\Sigma_{12}\Sigma_{22}^{-1}$

$$Cov(z, y_2) = Cov(y_1, y_2) + Cov(A y_2, y_2)$$

$$= \Sigma_{12} + A\Sigma_{22} = \Sigma_{12} - \Sigma_{12}\cancel{\Sigma_{22}^{-1}\Sigma_{22}} = 0$$

(z و y_2 مستقل نیز هستند)

$$E(z) = E[y_1] + AE[y_2] = \mu_1 + A\mu_2$$

$$\Rightarrow \underline{E(y_1 | y_2)} = E(z - Ay_2 | y_2) = E(z | y_2) - AE(y_2 | y_2)$$

$$= E(z) - Ay_2 = \mu_1 + A(\mu_2 - y_2)$$

$$\underline{= \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(y_2 - \mu_2)} \ \star$$

$$\underline{Var(y_1 | y_2)} = Var(z - Ay_2 | y_2) = \underline{Var(z | y_2) + Var(Ay_2 | y_2)}$$

$\rightarrow$ z, y_2 are uncorrelated

$$= Var(z | y_2) = Var(z)$$

$$Var(z) = Var(y_1 + Ay_2) = \Sigma_{11} + \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22}\Sigma_{22}^{-1}\Sigma_{21} - 2\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$\underline{= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}} \ \star$$

$$\Rightarrow P(y_1 | y_2) = N\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(y_2 - \mu_2) \,,\, \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)$$

2 ـ ب) با توجه به روابط قسمت الف داریم :

$$p(z,y) = \mathcal{N}(\mu, \Sigma)$$

$$\longrightarrow \log p(z,y) = -\frac{1}{2}(z-\mu_z)^T \Sigma_z^{-1}(z-\mu_z) - \frac{1}{2}(y-Wz-b)^T \Sigma_y^{-1}(y-Wz-b)$$

$$\Rightarrow Q = -\frac{1}{2} z^T \Sigma_z^{-1} z - \frac{1}{2} y^T \Sigma_y^{-1} y - \frac{1}{2}(Wz)^T \Sigma_y^{-1}(Wz) + y^T \Sigma_y^{-1} W z$$

$$= \frac{1}{2}\begin{pmatrix} z \\ y \end{pmatrix}^T \begin{pmatrix} \Sigma_z^{-1} + W^T \Sigma_y^{-1} W & -W^T \Sigma_y^{-1} \\ -\Sigma_y^{-1} W & \Sigma_y^{-1} \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} z \\ y \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} z \\ y \end{pmatrix}$$

با استفاده از Σ در قسمت الف، Σ^{-1} به این صورت است:

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_z^{-1} + W^T \Sigma_y^{-1} W & -W^T \Sigma_y^{-1} \\ -\Sigma_y^{-1} W & -\Sigma_y^{-1} \end{pmatrix} \triangleq \Lambda = \begin{pmatrix} \Lambda_{xx} & \Lambda_{xy} \\ \Lambda_{yx} & \Lambda_{yy} \end{pmatrix}$$

$\Longleftarrow$ با توجه به روابط شرطی بدست آمده در قسمت 1 و ($\mu_y = W\mu_z + b$ داریم :

$$\Sigma_{z|y} = \Lambda^{-1}_{xx} = \left(\Sigma^{-1} + W^T \Sigma_y^{-1} W\right)^{-1}$$

$$\mu_{z|y} = \Sigma_{z|y}\left(\Lambda_{xx}\mu_z - \Lambda_{xy}(y - \mu_y)\right)$$

$$= \Sigma_{z|y}\left(\Sigma_z^{-1}\mu_z + W^T \Sigma_y^{-1}\left(W\mu_z + y - \mu_y\right)\right)$$

$$= \Sigma_{z|y}\left(\Sigma_z^{-1}\mu_z + W^T \Sigma_y^{-1}(y - b)\right)$$

2 ـ سایر احتمال :

$X \sim N(0, 1)$

الف) 1

$Y \sim N(-1, 4)$

$X+Y \sim N\left(-1, 1+4+2\left(\left(-\frac{1}{2}\right)\times 1 \times 2\right)\right)$

$Z = X+Y \sim N(-1, 3)$

$P(Z>0) = 1-P(Z\leqslant 0)$

$= 1-\Phi\left(\frac{0-(-1)}{\sqrt{3}}\right)$

$= 1-\Phi(0.588)$

$= 1-0.72 = 0.28$

$\underbrace{aX+Y}_{W}$, $\underbrace{X+2Y}_{V}$ are independent $\underline{2}$ (ب

$\Rightarrow Cov(W,V) = 0$ $\quad Cov(X,Y) = \rho\sigma_X\sigma_Y = -\frac{1}{2}\times1\times2 = -1$

$Cov(aX+Y, X+2Y) = 0$

$= Cov(aX,X) + Cov(aX,2Y) + Cov(X,Y) + Cov(Y,2Y)$

$= aVar(X) + 2aCov(X,Y) + Cov(X,Y) + 2Var(Y)$

$-2a-1$

$= a + \cancel{(2a+1)(-1)} + 2\cancel{(4)}$ 8

$= -a+7 = 0 \Rightarrow \underline{a=7}$

$P(X+Y>0 \mid 2X-Y=0) \Rightarrow P(Z>0 \mid W=0)$ $\underline{3}$

$X+Y=Z$ Z & W are jointly normal

$2X-Y=W$

$$\Rightarrow \begin{cases} E[Z \mid W=w] = \mu_Z + \rho \sigma_Z \dfrac{w-\mu_W}{\sigma_W} \\ Var(Z \mid W=w) = (1-\rho^2)\sigma_Z^2 \end{cases}$$

$\boxed{\sigma_Z^2 \longrightarrow 3}$

$\boxed{\sigma_W^2 \longrightarrow Var(W) = Var(2X-Y) = \underset{4}{4\cancel{Var(X)}} + \underset{4}{\cancel{Var(Y)}} \underbrace{-4\cancel{Cov(X,Y)}}_{4} = 12}$

$\mu_Z \longrightarrow -1$

$\mu_W \longrightarrow 1$

$$\rho_{ZW} \longrightarrow \rho_{ZW} = \frac{Cov(Z,W)}{\sigma_Z \sigma_W} = \frac{\overset{2}{2\cancel{Var(X)}} - \overset{-1}{\cancel{Cov(X,Y)}} + \overset{-4}{\cancel{2Cov(X,Y)} - \cancel{Var(Y)}}}{\sqrt{3 \times 12}}$$

$$\rho_{ZW} = \frac{-3}{6} = -0.5$$

$$\Rightarrow E[Z \mid W=0] = -1 + (-0.5)(\sqrt{3})\frac{-1}{\sqrt{12}} = -0.75$$

$$Var(Z \mid W=0) = (\underset{\frac{3}{4}}{\cancel{1-0.25}})3 = 2.25$$

$p(z|w=0) \longrightarrow$ normal distributed with $\mu = -0.75$ & $Var = 2.25$ so we have:

$$p(z>0|w=1) = 1 - p(z<0|w=1)$$

$$= 1 - \phi\left(\frac{0-\mu}{\sqrt{Var}}\right)$$

$$= 1 - \phi\left(\frac{0.75}{1.5}\right) = 1 - \phi(0.5) = 1 - 0.69$$

$$= 0.31$$

$$Var(X) = E(\underbrace{Var(X|Y)}_{W}) + Var(\underbrace{E(X|Y)}_{Z})$$

$$E(X|Y) = Z \Longrightarrow \left(E(E[X|Y]) = E[x] = E[Z]\right)$$

$$Var(X|Y) = W$$

$$W = E[X^2|Y] - (E[X|Y])^2$$

$$= E[X^2|Y] - Z^2 \xrightarrow{E} EW = E[E[X^2|Y]] - E[Z^2]$$

$$= E[X^2] - E[Z^2] \quad ①$$

$$Var(Z) = E[Z^2] - (E[Z])^2 = E[Z^2] - (E[X])^2 \quad ②$$

$$\xrightarrow{1,2} Var(X) = E[W] + Var(Z)$$

$$\Longrightarrow Var(X) = E(Var(X|Y)) + Var(E(X|Y))$$