How Linear Algebra is Used in…

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Abstract

This paper explores various applications of linear algebra. We have detailed the use of linear algebra in sixteen different areas: computer graphics, digital signal processing, space flight and control systems, oil exploration, population modeling using Leslie matrices, analytic geometry, multichannel image processing, linear programming, migration (applications related to Markov chains), long-term behavior or evolution, predator-prey system, survival of the spotted owls, readjusting the North American datum, statistical analysis of scientific and engineering data using multiple regression, cryptology, and Leontief input-ouput models.

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Computer Graphics

Linear algebra is used in computer graphics to represent images in a vector format. This is in contrast to images being represents as a collection of pixels, such as a bitmap. As an image stored as a bitmap is scaled it will lose definition past a certain resolution, but an image stored as a vector will not. An image stored in vector format can be manipulated far more easily than one in bitmap form because it can be transformed as described below.

Using linear algebra, images are represented in vectors using homogeneous coordinates. For example in $\mathbb{R}^2$ the point (x,y) is represented as (x,y,1). In $\mathbb{R}^3$ the point (x,y,z) is represented as (x,y,z,1). This is done so that the transformations done will all be linear transformations via matrix multiplication. The image is composed by collecting these points as vectors and forming rules about how they are connected to each other. Once these rules are established the image can be stored in matrix form.

Transformations on the images are done by matrix multiplication using 3x3 matrices for 2-D graphics and using 4x4 matrices for 3-D graphics. Multiplying a coordinate vector by a transformation matrix will yield the new coordinator vector. Possible transformations include translations, rotations, scales, shears and reflections. Composite transformations can be done in one step by multiplying the matrices for each of the individual transformations into one composite matrix.

In order to display color images on a screen vectors used to store the correct information. Then matrix multiplication is used to convert colors between different color standards. One such conversion is from an international CIE standard for color to the standard for commercial television broadcasts.

Numerous applications have been found for vector graphics. One field is in molecular modeling. Using images generated by matrices, biologists can examine the structure of complex molecules. Another application is in the entertainment industry. Vector graphics are used in Sony’s PlayStation III, Microsoft’s Xbox 360, and in the film industry’s special effects.

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Digital Signal Processing

Digital signal processing (DSP) involves vector spaces of discrete-time signals. The digital in DSP means the signals must be in a numeric form. Most signals that are processed, such as voice or video data are analog and before using DSP these signals must be converted to digital format. After being processed, many signals must be converted back to an analog format. The most common uses of DSP are in audio and video compression, audio signal processing, digital image processing, speech processing, speech recognition and digital communications.

The most common use of DSP is to filter a signal. Filtering can be done using matrices. The matrices transform surrounding samples around the current sample of the signal. One type of filtering matrix simply multiplies the input vector by a scalar. Another type of matrix will delay the signal by a certain amount. Another type will output the difference of two input vectors. Another type will output the average of the previous x number of inputs.

A linear filter is one that is described by a difference equation, which can be homogeneous or nonhomogeneous. A difference equation of order n will give a solution space that is n-dimensional. A set of n linearly independent vectors will be a basis for this solution space. Auxiliary equations are used to find these vectors.

Another method to solve a homogeneous linear difference equation is to replace it by a system of first-order difference equations. Then the nth-order difference equation can be written as $\mathbf{x}_{k+1}=A\mathbf{x}_k$, where the vectors are in $\mathbb{R}^n$ and A is an nxn matrix.

A difference equation is realistic to model many real-life situations because our world is dynamic. Future events are dependent on the events of the previous time period. Functions, which are static, are limited in their ability to model real world behavior.

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Space Flight and Control Systems

Space flight is a system that is very unstable and so linear algebra is used to help control the path of the space shuttle. The space shuttle has control surfaces and thruster jets that must be given the correct instructions to make the space flight stable and smooth. With something as complicated as space flight it is simply not possible to program all instructions before flight. Tiny fluctuations in variables would completely change the flight of the shuttle resulting in a crash. Control systems are used to monitor the flight path and give feedback that is used to direct the path of the shuttle.

Different sensors measure the flight of the shuttle and provide a stream of data in the form of vectors. Input vectors give information about the acceleration of the space shuttle, the pitch rate and other variables. These input vectors form a vector space. Functions are performed using the addition and scaling of input vectors. Then the output vectors provide the necessary information to guide the shuttle.

Control engineering is the branch of engineering that studies the best methods of controlling a complex system, like a space flight. Control engineering is used in almost every branch of engineering. Examples of applications are in space and commercial flight, automobiles, washing machines, and heating systems.

There are two types of control systems, those with feedback and those without. They are called closed loop and open loop systems, respectively. The space shuttle flight is a closed loop system.

Initially control systems simply analyzed input data and judged whether it fit into certain predetermined criteria, but they were unable to make decisions about how to optimize the system they controlled. Modern control theory is a new branch of control engineering that uses linear algebra to optimize the system that is being controlled.

Oil Exploration

Oil is formed by the decaying of organic matter from millions of years ago. Given the right conditions, oil will form reservoirs in the ground. Geologists look for oil by searching for the right conditions for a reservoir to form. They use gravity meters, magnetometers, sniffers, and seismology to search.

Each of these methods involves sending out a signal which travels through land or water and then is reflected back. By measuring the variations in the reflections, geologists determine if an oil reservoir, or basin, could exist. Because there is so much data involved in searching the earth for oil, algorithms have been developed to deal with the data. Linear algebra is used to apply these algorithms to the data.

One example of an application is to take the travel times of seismic rays through blocks of earth and to place them in a matrix. Because many rays take the same path, and because some blocks of earth are not sampled by the seismic rays, the matrix can have many zero elements. Interpreting these matrices is an inverse problem. Techniques have been developed to deal with these problems using linear algebra. For example, if the given matrix is not very large, singular value decomposition can be used. Inverse problems are by definition difficult to solve and new methods are constantly being developed.

Another use of linear algebra is in simulating the processes in the earth. Basin analysis studies interactions between the dynamic processes involved in an oil basin. Linear algebra is used to simulate the flow, compaction, migration and accumulation of an oil basin. Different methods are used involving different matrix sizes, time increments, smoothing functions, and techniques of interpolation. Basin analysis helps determine if the conditions exist for an oil basin and how likely a basin is to exist.

Population Modeling Using Leslie Matrices

When modeling populations we have seen that basic matrices can be used. But we do not get a lot of information about the population unless we create more detailed models. For example, with many species, such as fish and trees, age plays a large role in determining the overall growth and change in a population. Matrices can provide information about population structure that logistical and exponential models are unable to provide. One model that provides age structure information is the Leslie Matrix model. The Leslie Matrix gives us a breakdown by age of the population that we are studying. Typically only the female population is considered in this model. Below is a Leslie Matrix:

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ \dots \\ n_m \end{bmatrix}_{t+1} = \begin{bmatrix} F_1 & F_2 & F_3 & \dots & F_m \\ s_1 & 0 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & s_{m-1} & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ \dots \\ n_m \end{bmatrix}_t \quad \Rightarrow \quad \vec{n}_{t+1} = L\vec{n}_t$$

In this matrix:
n_x = the number of individuals in age class "x"
s_x = the survival rate from age class "x" to age class "x+1"
F_x = the fertility rate in age class "x".

We see that the only non-zero entries are along the top row and on the sub-diagonal. A 4x4 example shows how the information flows through the matrix.

$$\begin{bmatrix} F_1 & F_2 & F_3 & F_4 \\ s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix} = \begin{bmatrix} F_1 n_1 + F_2 n_2 + F_3 n_3 + F_4 n_4 \\ s_1 n_1 \\ s_2 n_2 \\ s_3 n_3 \end{bmatrix}$$

We see that our new "n_1" entry, which is the youngest age class, is made up of the births from all the age classes. The new "n_2", "n_3", and "n_4" entries are made up of the individuals that survived and "graduated" on to the next age class.

We can also get information about the general population from the Leslie matrix. The dominant Eigenvalue, λ, (the one with the largest absolute value) will give us the population growth rate. The sensitivities, $s_{ij} = \delta\lambda/\delta a_{ij}$ (partial derivative of λ with respect to a_{ij}), will give us the sensitivity of the growth rate to a change in fertility rate or survival rate. Another measurement that can be made from the matrix is the life expectancy, but the formula is complicated.

Analytic Geometry

Analytic geometry is also called coordinate geometry. It is the study of geometry that uses algebraic methods. It deals with figures, usually in R^2 or R^3, such as planes, lines, circles, spheres, cylinders, parabolas, parabaloids, and prisms. Analytic geometry defines shapes numerically with equations and then we can manipulate the equations to get the information we want.

Common tools used in analytic geometry include: determinants, vector spaces, distance formulas, the dot product (to get the angle of two vectors), the cross product (to get a perpendicular vector of two known vectors) and others.

The determinant of a general *n*-by-*n* matrix is:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}.$$

A vector space is made up of elements called vectors. These vectors must be able to be multiplied by a scalar and also added together so that all linear combinations can be formed. Vectors are commonly represented as ordered n-tuples. There are ten axioms that must be satisfied for a nonempty set of vectors to be a vector space. They are: closure under scalar multiplication and under addition, commutativity, associativity of addition and scalar multiplication, existence of a multiplicative and additive identity and a multiplicative inverse vector, and distributativity of addition and scalar multiplication.

The distance formula in R^n is $\sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$.

The dot product, also called the inner product, of two vectors, **a** and **b** is:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i \quad .$$

The cross product is defined as the vector which is perpendicular to both **a** and **b** with a magnitude equal to the area of the parallelogram they span.

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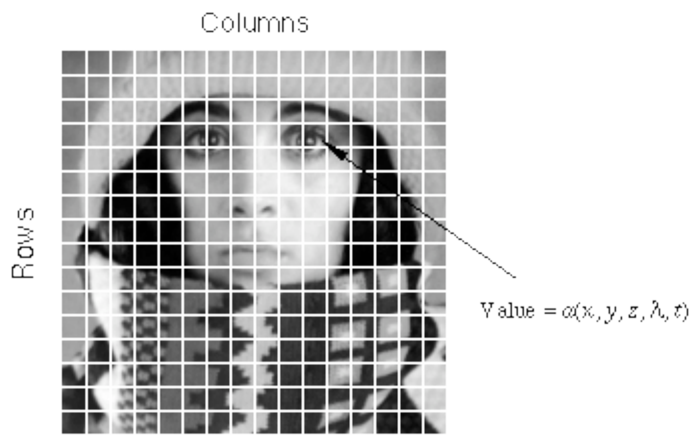
Multichannel Image Processing

Multichannel image processing is a field closely related to computer graphics. The term image in image processing usually refers to the computer description of a "real world" image. Image processing means that the process input is an image and the output is also an image. For the purpose of this paper we will only consider 2D images, but images can also be 3D.

A "real world" image is in continuous 2D analog space. Image processing deals with digital images in 2D discrete space. So the analog image must be converted into a digital image by a sampling process called digitization. A 2D continuous image is divided into rows and columns. The number of rows and columns is called the resolution. The intersection of a row and column is called a pixel. Values are assigned to a pixel by a function that determines the properties of a pixel. The function can measure things such as brightness, contrast, and color content. The function will assign integer values to each pixel based on the amount of these properties. Then typically the information is stored in a matrix.



This image shows a picture that has been divided into rows and columns and a function that measures variables and outputs the corresponding values. This process of assigning values is called quantization.

Image processing, because it is digital, has a lot of advantages over working with analog images. Recently, there have been many advancements in multichannel image processing involving filtering, enhancement, restoration, edge detection, compression, preservation and manipulation of images. All of these advancements have application in law enforcement, security, and entertainment.

Multichannel images are typically color images. Monochromatic images typically have only one channel of information (the amount of gray, or brightness of a pixel). Color images have more channels of information, such as amount of red, blue, and green respectively, brightness, etc.

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Linear Programming

In linear programming, we use matrices to solve a set of linear functions based on a set of constraints. Matrix algebra is used to simplify the equations to make them solvable. Often we are trying to maximize or perhaps minimize our function under certain conditions.

In computer programming, a matrix of information is often too large to store because of limits on a computer's memory. Even if the computer can handle the amount, the processing may take unreasonably long amounts of time. Using linear algebra, we can partition the matrix into smaller submatrices, called blocks. Using this method, the computer will hold fewer amounts of data, but the same information. This way processing can be much more efficient and quicker.

Linear programming is applied in optimization problems and often used in operations research. Some of the applications are for allocating resources such as money. This is often done with large businesses, for example, as trying to maximize their profits. Or sometimes if we have an inconsistent system, we can use a linear program to estimate the solution by replacing the variables with an ε progressively closer to the actual value. Another use of linear programming is for graph theory, like minimizing path lengths or network flows.

After we are given the set of functions and constraints, we need to convert them to matrix form, letting each row represent a function or constraint, and each column a variable. Using row-echelon reduction, a solution is offered, or alternatively we may discover that the system does not have a feasible solution. Using programs to solve this increases the speed it is solved and makes the process very simple. Although linear programming is not necessarily calculated through computers, they are often used for larger calculations to simplify and speed up the results. There may be a system of equations that are practically impossible to solve without the use of a computer.

We can take a look at an example where we want to optimize a given equation with certain constraints applied along with it. Consider the region R in the plane defined by these inequalities and then say we want to find the maximum of $11x + 22y$ over region R:

$$\begin{aligned} &2x + y \leq 20 \\ &x + 3y \leq 52 \\ &4x + 7y \leq 131 \\ &x, y \geq 0 \end{aligned}$$

We can use something called the "Simplex Method" to calculate and optimize. Our matrix is $\begin{bmatrix} 2 & 1 & 20 \\ 1 & 3 & 52 \\ 4 & 7 & 131 \\ -11 & -22 & 0 \end{bmatrix}$. We then add in the identity matrix in the middle and another column at the end which shows ratios between the last column above and one of the other columns that creates the smallest ratio. So, first add in the identity matrix to get what follows: $\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 20 \\ 1 & 3 & 0 & 1 & 0 & 0 & 52 \\ 4 & 7 & 0 & 0 & 1 & 0 & 131 \\ -11 & -22 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Now, add another column at the end which shows the smallest ratio of the last column with one of the first two.

This gives us $\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 20 & 10 \\ 1 & 3 & 0 & 1 & 0 & 0 & 52 & 52/3 \\ 4 & 7 & 0 & 0 & 1 & 0 & 131 & 131/7 \\ -11 & -22 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$. Next we look for the smallest ratio (not counting 0), which here is 10. This row with the smallest ratio must then be eliminated by using it as a "pivot" on column 2 since it is the column with the smallest value (-22).

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 20 & 10 \\ 1 & 3 & 0 & 1 & 0 & 0 & 52 & 52/3 \\ 4 & 7 & 0 & 0 & 1 & 0 & 131 & 131/7 \\ -11 & -22 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 20 & 10 \\ -5 & 0 & -3 & 1 & 0 & 0 & -8 & -38/3 \\ -10 & 0 & -4 & 0 & 1 & 0 & -9 & -359/7 \\ 33 & 0 & 22 & 0 & 0 & 1 & 440 & 220 \end{bmatrix}$$

Thus this tells us that x1 = 20, so we can have a possible solution of (20, 0).

Migration (Applications related to Markov Chains)

If there is a system that has different states, but can only be in one state at a particular time. At a certain observation time, the probability of being in a specific state is related to the k-1st observation time. This is what we call a *Markov Chain*. The Markov matrix (also called a transition matrix) gives probability of movement from one state to another. If we have probability vectors $p_0, p_1, \ldots, p_k$, then the Markov chain is $p_k = A_k p_0$, where A represents the Markov matrix. When Ap = p, p is called the steady-state vector.

Technically the Markov chain is a sequence of random variables, their range is the state space. It is really a conditional distribution since the state at one time depends on previous states. There are a great amount of applications for Markov Chains in the "real world," including biological modeling, statistics, coding, geostatistics, chance, speech and pattern recognition, and data compression to name a few.

For an example of how Markov chains can be used, consider this. Three friends have a money exchange network. Bob gives Brittney 60% of his money and 15% to Bertha. Brittney gives Bob 26% of her money and Bertha gets 53% of it. Bertha only gives 61% to Brittney. This matrix is represented as proportions, so we get $\begin{bmatrix} .25 & .26 & 0 \\ .6 & .21 & .61 \\ .15 & .53 & .39 \end{bmatrix}$, where Bob is the first row/column, Brittney is the second, and Bertha is the third. Let us say that Bob starts with \$2,120, Brittney with \$394, and Bertha with \$5,039. We can take a look at how much money each of them have after any number of changes by taking the power of the original matrix multiplied by the matrix of their starting amounts. For example, we want to know how much money each will have after 4 exchanges.

$$\begin{bmatrix} .25 & .26 & 0 \\ .6 & .21 & .61 \\ .15 & .53 & .39 \end{bmatrix} * \begin{bmatrix} 2120 \\ 394 \\ 5039 \end{bmatrix} = \begin{bmatrix} 632.44 \\ 4378.14 \\ 2492.03 \end{bmatrix}$$

In terms of proportions,

$$\begin{bmatrix} .25 & .26 & 0 \\ .6 & .21 & .61 \\ .15 & .53 & .39 \end{bmatrix} * \begin{bmatrix} .28068 \\ .0522 \\ .66715 \end{bmatrix} = \begin{bmatrix} .08373362 \\ .57965577 \\ .3299391 \end{bmatrix}$$

Markov Chains also have applications in areas of homeland security. In studying the spread of disease, such as anthrax, AIDS, or foot and mouth disease, we can figure out how it will affect areas long-term. For example, looking at foot and mouth disease, each person in a given area is either susceptible, infectious, or recovered (SIR). Everyone starts out susceptible to the disease, and by introducing one that is infections, we can look at the proportion that also becomes infected. After infection, the person recovers (and is immune) for a short recovery span of perhaps two weeks and then they return to being susceptible. By using Markov chains, we can figure out how much of the population is in either stage at a given time and what the long-term proportions are predicted to be.

Long-term behavior or Evolution

Long-term behavior is studied in aspects of Markov Chains, Predator-Prey Systems, and other real-life applications. Markov Chains were already previously discussed. With Markov Chains, we can study behavior of one phase which is based only on the previous stage, no past data required. Integrating linear algebra methods can allow us to figure out the long term behavior of the chains.

We can see this in the Markov Chain example with the money distribution, by figuring out the ending values that each person would have after endless iterations. So, if we want to see what proportions of the total money amount that each person will have after a long time; this is considered a branch of long-term behavior. We have to augment the matrix with 0's after subtracting from a 3x3 identity matrix. This gives us:

$$\begin{bmatrix} -.75 & .26 & 0 & 0 \\ .6 & -.79 & .61 & 0 \\ .15 & .53 & -.61 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.36334479 & 0 \\ 0 & 1 & -1.04811 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then q1 - .36334479q2 = 0 and q2 – 1.04811q3 = 0. This gives a matrix of t* $\begin{bmatrix} .38082531 \\ 1.04811 \\ 1 \end{bmatrix}$. To make this sum to 1 to show proportion, we then get $\begin{bmatrix} .15678693 \\ .43151005 \\ .41170302 \end{bmatrix}$. Thus, in terms of money, the long term distribution of the $7553 will be $1,184.21 to Bob, $3,259.20 to Brittney, and $3,109.59 to Bertha.

Using linear algebra in long-term behavior studies can be very useful. It is a way to simplify and model behavior over long periods of time. We can, for example, look at effects of tourism, stock markets, or populations. Long-term behavior studies are widely utilized in biology to study population behavior and structures. If we know that there is a common reproduction rate, we can use the death and birth rates to calculate long-term population values. Of course there are always other factors that will integrate into such a system, such as disease and epidemics, population capacities from physical, environmental, and sociological pressures, and any other miscellaneous factors. Nonetheless, looking at long-term behaviors through use of matrices can help us predict outcomes from current trends.

Predator-Prey System

Linear algebra can be put to use in modeling predator-prey systems. This is a system involving a predator with a specific prey as a food source, where they generally consume a given proportion of them depending on their own population. Each population also reproduces at a certain rate. Some of the problems with this model is that it assumes the prey has unlimited resources, the prey has only this predator as a threat, the predator only eats this prey, and the growth rate of the predator depends on how much prey it gets. Nonetheless, it is a good place to start to get population growth generalizations and basic estimations.

Predator-prey systems are utilizing discrete dynamical systems, which are described by difference equations. This is also used in basic population dynamic modeling, Markov chains, and other systems that give information by time. These systems are solved using calculations of eigenvalues and eigenvectors and looking at the long term behavior of the sequences of population change. The possible outcomes are exponential growth, equilibrium (maintaining population), or population decay leading to extinction. But, they are only predictions based on current trend.

Let's look at an example. Say that the angry badgers like to gobble up hamsters. Without the hamsters, only 60% of the beavers will survive each year. Without badgers as predators, the hamsters will grow rapidly at a rate of 1.25. With lots of hamsters in the wild, .3*hamsters at time k will make the beaver population grow, and –p*beavers at time k shows deaths of hamsters due to being eaten by the badgers. If we let (predation parameter) p = .21, then we can take a look at the evolution demonstrated by the system. Letting B(k) be the beaver population at time k and H(k) the hamster population at time k, the population equations are B(k+1) = .6*B(k) + .3*H(k) and H(k+1) = -p*B(k) + (1.25)*H(k). With our given (population parameter) p, we can look at how the system evolves over time. The eigenvalues that correspond with the system are approximately 1.1314582 and .71854177. The eigenvectors are then $\begin{bmatrix} -.56448463 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2.5307535 \\ 1 \end{bmatrix}$. Then the behavior x(k) will be x(k) = c1$(1.1314582)^k \begin{bmatrix} -.56448463 \\ 1 \end{bmatrix}$ + c2$(.71854177)^k \begin{bmatrix} -2.5307535 \\ 1 \end{bmatrix}$. Since .71854177 < 1, raising the value to higher powers will send it to 0. So we are left with x(k) = c1$(1.1314582)^k \begin{bmatrix} -.56448463 \\ 1 \end{bmatrix}$. Then, x(k+1) = (1.1314582)*c1$(1.1314582)^k \begin{bmatrix} -.56448463 \\ 1 \end{bmatrix}$. = (1.1314582)*x(k). This tells us that eventually both the badgers and hamsters grow at a rate of about 13% yearly.

Survival of the Spotted Owls

Another use of discrete dynamical systems besides predator-prey systems is the study of the survival of a single population. An example is the survival of the spotted owls in an area of California. The model determines proportions of females at three different age groupings: juvenile, subadult, and adult. The population is modeled by yearly time intervals.

Let's take a look at how the matrix is represented. In the matrix below, the value .25 represents how many female babies each adult female produces yearly on average. The rest of the entries are representative of survival. So, 41% of juvenile females survive to the subadult age, 67% of the subadults continue to the adult age, and 89% of the adults survive every year.

$$\begin{bmatrix} 0 & 0 & .25 \\ .41 & 0 & 0 \\ 0 & .67 & .89 \end{bmatrix}$$

Then, the purpose of this study is to determine the long-term behavior of the population – that is, will it extinguish or grow? First we must determine the eigenvalues of the matrix, which in this case happen to be approximately .9639, -.037-.265i, and -.037+.265i. Since all have magnitude less than one, we unfortunately will be predicting extinction for this species.

Let's try looking at this same problem with some different values. What if instead, 62% of juvenile females survive to be subadults and each female has .31 female babies a year. Now we have a new matrix as shown below with eigenvalues of 1.0149, -.0625 - .35i, and .0625 + .35i. The two complex eigenvalues tend to 0, but the value of 1.0149 shows that as time goes along, the population of the owls increase at a grown rate of able 1.015. The next step is to calculate the eigenvector, which is approximately $\begin{bmatrix} .305 \\ .187 \\ 1 \end{bmatrix}$. This shows the proportion of females in each age group as estimated to be the long-term values. So, for every 100 adults, there will be about 18.7 subadults and 30.5 juveniles. Long-term prediction doesn't always have to lead to extinction!

$$\begin{bmatrix} 0 & 0 & .31 \\ .62 & 0 & 0 \\ 0 & .67 & .89 \end{bmatrix}$$

Readjusting the North American Datum

The North American Datum is a network with marked reference points spanning our continent (above Panama) including Greenland, Hawaii, Virgin Islands, Puerto Rico, and Caribbean Islands. The reference points are tracked by latitudinal and longitudinal coordinates, which need to be quite accurate.

Errors accumulated over time and in 1970, the system was renovated. All of the collected data was converted to the computer and standardized. The system of equations used for this had a "least-squares solution." Since the equations were too large for computers to handle, they were broken down using Helmert blocking, which uses partitioning into gradually smaller blocks. The smallest blocks held 500-2000 reference points now. In 1986, after the computer processed all of this analysis, it was considered to be the largest least-squares problem to be solved.

Let's do an example of least-squares problems. Given the following matrices, we need to find the least-squares solution of A**x** = **b**.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

The first step is to calculate A^TA and $A^T\mathbf{b}$.

$$A^TA = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \text{ and } A^T\mathbf{b} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

The least-squares solution is $\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b}$.

$$(A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We can also determine the least-squares error by finding the distance between **b** and A**x**, which is shown as follows.

$$\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Then, $\|\mathbf{b} - A\mathbf{x}\| = \sqrt{(1)^2 + (1)^2 + (-2)^2} = \sqrt{6}$

Least squares calculations are often used in statistics. Least-squares lines are often calculated to fit lines of data. The least-squares error helps statisticians to determine how closely their fitted line matches with the data. Least squares calculates the average distances from the fitted line with each data point, so a large error value will show a lack of fit between the line and group of data. This way, the analyst can determine whether or not the estimated equation is useful in predictions or not. When the error is low, for example, current trends in stocks or other charts could be more accurately predicted. Clearly, least-squares calculations are very useful as we saw from the example of the North American Datum.

Statistical Analysis of Scientific and Engineering Data using Multiple Regression

In statistics, we maybe want to predict an outcome based on more than one variable. When we create an equation for this prediction, we are performing a multiple regression. One of the applications of this type of analysis is in geology or geography. We can map an area where coordinates give the latitude, longitude, and altitude. When we create a regression equation, we are also involving the least-squares method so that the equation fits as best as possible, minimizing the total distance between all actual and predicted values.

When fitting the data to an equation, we want it to follow $y_n = \beta_0 + \beta_1 u_n + \beta_2 v_n + \varepsilon_n$. The matrix form is $y = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ where y what we want to predict, X is a "design matrix" where the first column consists of ones, the second of all u-values, and the third of v-values. The $\boldsymbol{\beta}$ represents a column matrix of all β values and similarly the $\boldsymbol{\varepsilon}$ matrix is all ε values. The vector of ε's represents all of the associated residual values, which are distances between predicted and observed values.

Let's look at a short example. Suppose we have the following set of data points: (1, 5), (6, 2), (4, 7), (3, 1). Then the design matrix is as follows: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 6 & 36 & 6^3 \\ 1 & 4 & 16 & 64 \\ 1 & 3 & 9 & 27 \end{bmatrix}$. The y-values are also place into a y-matrix. Then we need to solve for the beta-values so that we can create a linear equation for the regression. After row reduction, we get the matrix of beta's to be $\begin{bmatrix} 28.2 \\ -33.57 \\ 11.47 \\ -1.1 \end{bmatrix}$. Thus our linear regression equation ends up being $y = 28.2 - 33.57x + 11.47x^2 - 1.1x^3$. We can solve for an equation this way using any order polynomial that we want for our equation.

Cryptology

Another really interesting use of linear algebra is with the study of cryptanalysis. Cryptology is the encoding of messages, used to pass on secretive messages or to protect private information. It was often used during wars to pass signals without the enemies finding the messages. The tool of cryptanalysis is the processing of decoding these messages to intersect the message of the enemy or to receive messages on your own side. Encrypting is also often used online for information security.

A type of block cipher called the Hill cipher utilizes linear algebra for coding and decoding. A block cipher encrypts blocks of code at the same time. The message is first written as a bunch of row vectors, which we then multiply by the previously created key to encryption, and then mod 26. Then, when we want to decrypt, we find the greatest common denominator of the determinant of the key matrix and 26 to equal 1. Thus there is a matrix N so that key*N = I (mod 26). Continue by multiplying the message by N to get a result. Unfortunately this cipher is not too difficult to decode, so it is not commonly used.

Besides the Hill cipher, linear algebra and matrices are used in many other encryptions of cryptology. Keys can be generated to change a coded message represented as a series of numbers, or translations and transformations are also a good method of encrypting information. Since matrices can be used in many different ways, there are alternatives ways to encrypt a code, whether it be by a series of matrix keys and algebraic manipulation or changing the code into bits rather than numeric (1-26) representations. Things may get slightly more complex when symbols or actual numbers are introduced into the message, but the code can be written accordingly.

Leontief Input-Ouput Models

Wassily Leontief created the Leontief input-output model that predicts the flows between sectors of an economy for use in economic forecasting. His model splits an economy into n sectors. Then the production vector **x** consists of the output of each of the n sectors. Each sector must consume goods from other sectors in order to produce its output. This consumption is called intermediate demand or C**x**. Also in his model is a vector **d** representing the final demand of the open sector which produces goods.

Leontief's model speculated that the production vector would equal the sum of intermediate demand and final demand. This is written as $\mathbf{x} = C\mathbf{x} + \mathbf{d}$, where C is a matrix that represents all the goods demanded by the production sectors. In order to solve this equation for **x** we write it as $\mathbf{x} = (I - C)^{-1}\mathbf{d}$. In practice, this equation is difficult to solve and so $(I - C)^{-1}\mathbf{d}$ is approximated by $I + C + C^2 + C^3 + \ldots + C^m)\mathbf{d}$, which will give an answer sufficiently close to the exact answer by making m as large as necessary.

An example of goods exchange, in terms of millions of dollars, could be as follows. Say there are three industries – agriculture, human services, and manufacturer. The following is a matrix representing the exchange values between them. For example, the value 24.5 is representing the exchange from manufacturer to human services.

$$\begin{array}{c} \begin{array}{ccc} \text{A} & \text{HS} & \text{M} \end{array} \\ \begin{bmatrix} 12.1 & 29.0 & 42 \\ 9.7 & 57.3 & 24.5 \\ 15.8 & 13.3 & 22.2 \end{bmatrix} \end{array}$$

The matrix above is an input-output matrix. What if we have a demand of 100 for each industry? To find the production matrix X, we solve using the equations previously discussed. Thus we get (I – C) = $\begin{bmatrix} -11.1 & 29.0 & 42 \\ 9.7 & -56.3 & 24.5 \\ 15.8 & 13.3 & -21.2 \end{bmatrix}$. The inverse of this matrix is then multiplied by the demand vector containing 100's which gives x = $\begin{bmatrix} 10.16 \\ 1.68 \\ 3.91 \end{bmatrix}$. So, to create a balance with the required demands, the agriculture industry should produce 10.16 million dollars of goods, the human services should produce 1.68 million, and manufacturer should produce about 3.91 million dollars worth.

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