ICS Answer Sheet #5

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Problem 6.1: completeness of → and ¬

Prove that the two elementary boolean functions → (implication) and ¬ (negation) are universal, i.e., they are sufficient to express all possible boolean functions.

Proof:

→ means statements of the form "if X then Y " (where X is called the precondition and Y the consequence). That means, only if X is true, then Y can be true. Y cannot be true while X is false.

I. An implication truth table looks like:

A	B	A → B
0	0	1
0	1	1
1	0	0
1	1	1

¬ means the statement provided is reversed, i.e. 0 is changed to 1 and 1 is changed to 0.

II. A negation truth table looks like:

A	¬A
0	1
1	0

To say they are universal means they are sufficient to express all possible Boolean expressions. To prove this, we can use these two Boolean functions and prove that it can produce a NAND gate or a NOR gate.

III. A NAND truth table looks like:

A	B	¬ (A ∧ B)
0	0	1
0	1	1
1	0	1
1	1	0

IV. A NOR truth table looks like:

A	B	¬ (A ∨ B)
0	0	1
0	1	0
1	0	0
1	1	0

V. Using implication and negation to prove that it can be a NAND gate:

A	B	A → ¬B
0	0	1
0	1	1
1	0	1
1	1	0

VI. Using implication and negation to prove that it can be a NOR gate:

A	B	¬A → B	¬ (¬A → B)
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

As we can see, III and V & IV and VI match which proves that the two elementary boolean functions → (implication) and ¬ (negation) are universal, i.e., they are sufficient to express all possible boolean functions.

Problem 6.2: boolean equivalence laws

Simplify the following Boolean formulas by repeatedly applying Boolean equivalence laws. (The simplified formulas contain at most one ∧ or ∨.) Indicate in every step which law you have applied. You obtain points for the derivation, not for the result alone.

Boolean Equivalence laws

Proposition (equivalence laws)

For any Boolean formulas φ, ψ, χ, the following equivalences hold:

1. $\varphi \land 1 \equiv \varphi$, $\varphi \lor 0 \equiv \varphi$ *(identity)*
2. $\varphi \lor 1 \equiv 1$, $\varphi \land 0 \equiv 0$ *(domination)*
3. $(\varphi \land \varphi) \equiv \varphi$, $(\varphi \lor \varphi) \equiv \varphi)$ *(idempotency)*
4. $(\varphi \land \psi) \equiv (\psi \land \varphi)$, $(\varphi \lor \psi) \equiv (\psi \lor \varphi)$ *(commutativity)*
5. $((\varphi \land \psi) \land \chi) \equiv (\varphi \land (\psi \land \chi))$, $((\varphi \lor \psi) \lor \chi) \equiv (\varphi \lor (\psi \lor \chi))$ *(associativity)*
6. $\varphi \land (\psi \lor \chi) \equiv (\varphi \land \psi) \lor (\varphi \land \chi)$, $\varphi \lor (\psi \land \chi) \equiv (\varphi \lor \psi) \land (\varphi \lor \chi)$ *(distributivity)*
7. $\neg\neg\varphi \equiv \varphi$, $\varphi \land \neg\varphi \equiv 0$, $\varphi \lor \neg\varphi \equiv 1$ *(double negation, complementation)*
8. $\neg(\varphi \land \psi) \equiv (\neg\varphi \lor \neg\psi)$, $\neg(\varphi \lor \psi) \equiv (\neg\varphi \land \neg\psi)$ *(de Morgan's laws)*
9. $\varphi \land (\varphi \lor \psi) \equiv \varphi$, $\varphi \lor (\varphi \land \psi) \equiv \varphi$ *(absorption laws)*

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a. $\mathbf{\Phi(A, B) = (\neg A \lor \neg B) \land (\neg A \lor B) \land (A \lor \neg B)}$

$= (\neg A \land (\neg B \lor B)) \land (A \lor \neg B)$ (Distributive law)

$= \neg A \land (A \lor \neg B)$ (Complement and Identity law)

$= 0 \lor (\neg A \land \neg B)$ (Complement law)

$= \neg (A \lor B)$ (de Morgan's law)

b. Φ (A, B, C) = (A ∧ ¬B) ∨ (A ∧ ¬B ∧ C)

$= (A \wedge \neg B) \wedge (1 \vee C)$ (Distributive law)

$= (A \wedge \neg B) \wedge 1$ (Annulment)

$= (A \wedge \neg B)$ (Identity law)

c. Φ (A, B, C, D) = (A ∨ ¬ (B ∧ A)) ∧ (C ∨ (D ∨ C))

$= (A \vee (\neg A \vee \neg B)) \wedge (C \vee C \vee D)$ (de Morgan's law)

$= ((\neg A \vee \neg B) \vee A) \wedge (C \vee D)$ (commutativity and idempotency)

$= ((\neg A \vee A) \vee \neg B) \wedge (C \vee D)$ (Associativity law)

$= (1 \vee \neg B) \wedge (C \vee D)$ (Complement)

$= (1) \wedge (C \vee D)$ (Annulment)

$= (C \vee D)$ (Identity law)

d. Φ (A, B, C) = (¬(A ∧ B) ∨ ¬C) ∧ (¬A ∨ B ∨ ¬C)

$= (\neg A \vee \neg B \vee \neg C) \wedge (\neg A \vee B \vee \neg C)$ (de Morgan's law)

$= (\neg A \vee \neg C) \wedge (\neg B \vee B)$ (Distributive law)

$= (\neg A \vee \neg C) \wedge 1$ (Complement law)

$= \neg (A \wedge C)$ (Identity and de Morgan's law)

e. **Φ (A, B) = (A ∨ B) ∧ (¬A ∨ B) ∧ (A ∨ ¬B) ∧ (¬A ∨ ¬B)**

= (A ∨ B) ∧ (¬A ∨ ¬B) ∧ (¬A ∨ B) ∧ (A ∨ ¬B) (Rearranging)

= (A⊕B) ∧ ¬ (A⊕B) (XOR and XNOR gates)

= 0 (Complement law)

Problem 6.3: conjunctive and disjunctive normal form

Consider the following boolean formula:

Φ (P, Q, R, S) = (¬ P ∨ Q) ∧ (¬ Q ∨ R) ∧ (¬ R ∨ S) ∧ (¬ S ∨ P)

a. How many interpretations of the variables P, Q, R, S satisfy ϕ? Provide a proof for your answer (e.g., by providing a truth table).

P	Q	R	S	(¬ P ∨ Q)	(¬ Q ∨ R)	(¬ R ∨ S)	(¬ S ∨ P)
0	0	0	0	1	1	1	1
0	0	0	1	1	1	1	0
0	0	1	0	1	1	0	1
0	0	1	1	1	1	1	0
0	1	0	0	1	0	1	1
0	1	0	1	1	0	1	0
0	1	1	0	1	1	0	1
0	1	1	1	1	1	1	0
1	0	0	0	0	1	1	1
1	0	0	1	0	1	1	1
1	0	1	0	0	1	0	1
1	0	1	1	0	1	1	1
1	1	0	0	1	0	1	1
1	1	0	1	1	0	1	1
1	1	1	0	1	1	0	1
1	1	1	1	1	1	1	1

$(\neg P \lor Q) \land (\neg Q \lor R) \land (\neg R \lor S) \land (\neg S \lor P)$
1
0
0
0
0
0
0
0
0
0
0
0
0
0
0
1

As we can see, there are only two combinations which output a true value in the final form i.e. either when all PQRS are false or all PQRS are true.

Therefore, 2 interpretations of the variables P, Q, R, S satisfy ϕ.

b. Given the interpretations that satisfy ϕ, write the formula for ϕ in disjunctive normal form (DNF).

Taking only the interpretations that satisfied ϕ, i.e. final result as true, we get,

$$(\neg P \land \neg Q \land \neg R \land \neg S) \lor (P \land Q \land R \land S)$$

where **not** represents 0 (False) and **no not** represents 1 (True)