ICS Answer Sheet #4

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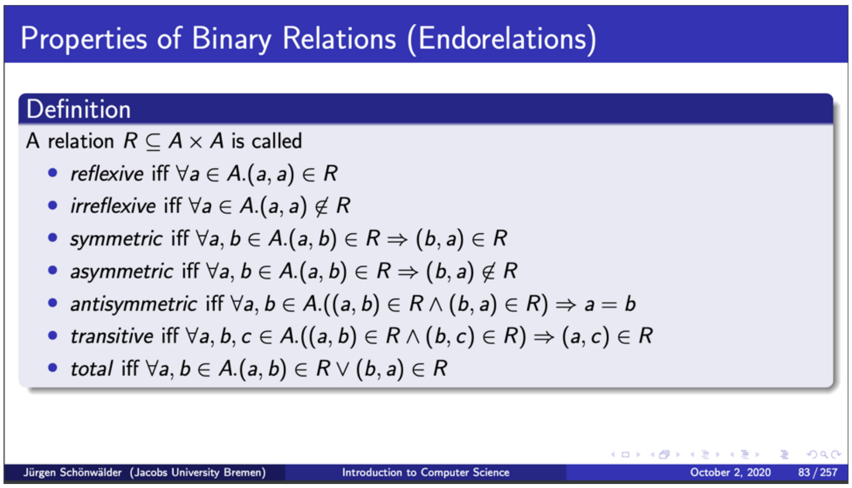
Problem 4.1: prefix order relations



a. Let $\preccurlyeq \subseteq \Sigma^* \times \Sigma^*$ be a relation such that p $\preccurlyeq$ w for p, w $\in \Sigma^*$ if p is a prefix of w. Show that $\preccurlyeq$ is a partial order.

For $\preccurlyeq$ to be a partial order, it must have the following properties: reflexive, antisymmetric and transitive.

I. Let (p, p) $\in \preccurlyeq$ such that (p, p) $\in \Sigma^*$

which implies that,
p $\preccurlyeq$ p, and
p $\preccurlyeq$ p.

That means, (p $\preccurlyeq$ p)

w = pq when p $\preccurlyeq$ w

Let **q** be an empty word, we have,

(p, p) $\in \preccurlyeq$ -> p = p -> p = pq -> p = pε. -> p = p

The word p can be a prefix of itself.
$w = pp$ is also possible,

Hence, $p \preccurlyeq p$ holds $\forall\, p \in \Sigma^*$

This means that the relation is **reflexive.**

II. Let (p, w) and $(w, p) \in \Sigma^*$
having $(p, w) \preccurlyeq (w, p)$ and $(w, p) \preccurlyeq (p, w)$.

which implies that,
$p \preccurlyeq w$ & $w \preccurlyeq p$, and
$w \preccurlyeq p$ & $p \preccurlyeq w$.

Combining those two relations, we get,
$p \preccurlyeq w \preccurlyeq p$, so $p = w$, and
$w \preccurlyeq p \preccurlyeq w$, so $w = p$.

$((p \preccurlyeq w) \wedge (w \preccurlyeq p))$ implies, $(p = w)$

Therefore, $(p, w) = (w, p)$, which means it is **antisymmetric.**

III. Let p, w and $z \in \Sigma^*$
having $p \preccurlyeq w$, $w \preccurlyeq z$,

which implies that,
$p \preccurlyeq w \preccurlyeq z$

$((p \preccurlyeq w) \wedge (w \preccurlyeq z))$ implies, $(p \preccurlyeq z)$

If p is a prefix of w and w is a prefix of z, then it implies that p is also the prefix of z.

Therefore, $(p \preccurlyeq z)$ which means it is **transitive.**

Since, the relation is **reflexive, antisymmetric and transitive,** it means that ‘$\preccurlyeq$’ is **a partial order.**

b. Let $\prec \subset \Sigma^* \times \Sigma^*$ be a relation such that for $p \prec w$ for $p, w \in \Sigma^*$ if p is a proper prefix of w. Show that $\prec$ is a strict partial order.

For $\prec$ to be a strict partial order, it must have the following properties: irreflexive, asymmetric and transitive.

p is a proper prefix means that $p = wq$ where $p \neq w$

I. Let $(p, p) \in \prec$ such that $(p, p) \in \Sigma^*$
$\prec \subset \Sigma^* \times \Sigma^*$

which implies that,
$p \prec p$, and
$p \prec p$.

That means, $(p \prec p)$

It doesn't make sense as a word 'p' cannot be a proper prefix of itself.
$w = pp$ is not possible.

To prove this,

$(p, p) \in \prec$ -> $p \prec p$ -> $p = pq$ -> **proper prefix of w** -> $p \neq w$

Hence, $p \prec p$ does not hold $\forall\, p \in \Sigma^*$

This means that the relation is **irreflexive.**

II. Let $(p, w) \in \prec$ and $(w, p) \in \prec$
having $(p, w) \prec (w, p)$ and $(w, p) \prec (p, w)$.

which implies that,
$p \prec w$ & $w \prec p$, and
$w \prec p$ & $p \prec w$.

Combining those two relations, we get,
$p \prec w \prec p$, so $p = w$, and
$w \prec p \prec w$, so $w = p$.

BUT,
Given is that p is a proper prefix of w, which means $p \neq w$ (irreflexive proof) so for this, it would mean $w \neq p$ i.e. $w \prec p$ does not hold.

This means if (p, w) is in Σ^*, then (w, p) is not in Σ^*.

Therefore, (p, w) $\neq$ (w, p), which means it is **asymmetric.**

III. Let p, w and z $\in \Sigma^*$
having $p \prec w$, $w \prec z$,

which implies that,
$p \prec w \prec z$

$((p \prec w) \wedge (w \prec z))$ implies, $(p \prec z)$

If p is a proper prefix of w and w is a proper prefix of z, then it implies that p is also the proper prefix of z.

Since, the relation is **irreflexive, asymmetric and transitive,** it means that the relation is **a strict partial order.**

c. Are the two order relations $\preccurlyeq$ and $\prec$ total?

Σ^* = set of all words from Σ

A word $p \in \Sigma^*$ is called a prefix of a word $w \in \Sigma^*$ if there is a word $q \in \Sigma^*$ such that $w = pq$

I. **$\preccurlyeq$ total?**

From the proofs of *Problem 4.1a* above, we can say that,

the p can be prefix of any word

For it to be total, either pair (p, w) or (w, p) should be part of the relation (Σ^*).

We know, that (p, w) exists in the list which means p is a prefix of w. In an OR statement, one clause satisfied means that the entire statement is true and hence, $\forall p, w \in \Sigma^*$, **the order relation $\preccurlyeq$ is total.**

II. **$\prec$ total?**

The proofs from *Problem 4.1b* suggests that w $\neq$ p i.e. w $\prec$ p does not hold.
Also, p is a proper prefix of w.

Which means, p cannot be a proper prefix of itself, which excludes the possibility of w = pp => (p, p) existing in the set.

Even if we switch para 1 with para 2 from the above tuple, it's the same and hence doesn't exist.

In an OR statement, the only time it is false is when both conditions fail to be true. In our case, it is that and hence $\forall$p, w $\notin \Sigma^*$, **the order relation $\prec$ is not total.**

Problem 4.2: function composition

Definitions:

Injective function:

"Let $f : X \to Y$ be a function. Then f is **injective** if distinct elements of X are mapped to distinct elements of Y." *(Brilliant, 2020)* [1]

Surjective function:

"Let $f : X \to Y$ be a function. Then f is **surjective** if every element of Y is the image of at least one element of X." *(Brilliant, 2020)* [2]

Bijective function:

"Let $f : X \to Y$ be a function. Then f is **bijective** if it is injective and surjective; that is, every element of $y \in Y$ is the image of exactly one element of $x \in X$." *(Brilliant, 2020)* [3]

Let A, B and C be sets and let $f : A \to B$ and $g : B \to C$ be two functions.

a. **Prove the following statement: If $g \circ f$ is bijective, then f is injective and g is surjective**

   Given,
   $g \circ f$: $A \to C$ is bijective

   To show,
   f is injective and **g** is surjective

   We know that a bijective function is the combination of injective and surjective function. So, we can let a new function, **h** be defined such that **$h = g \circ f$** is injective as well as surjective.

[1] That is, if x_1 and x_2 are in X such that $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

[2] $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

[3] Having 1 and 2

I. **f** is injective

Given,
h = g ∘ f is injective function

To show,
f : A → B is injective.

Proof:
Let's suppose f(x) = f(y) for some x, y ∈ A

Taking the function **g** to both sides,

g(f(x)) = g(f(y))

h(x) = h(y).

Since, h = g ∘ f is injective function,

x = y

Therefore, f is injective.

II. **g** is surjective

Given,
h = g ∘ f is surjective function

To show,
g : B → C is surjective.

Proof:
Let's take any **y ∈ C** (co-domain of G)
For **g** to be surjective, **g(b) = y**. Find x.

Since, h = g ∘ f is surjective function,

∃a ∈ A such that, h(a) = y,

g(f(a)) = y,

According to the question, f : A → B

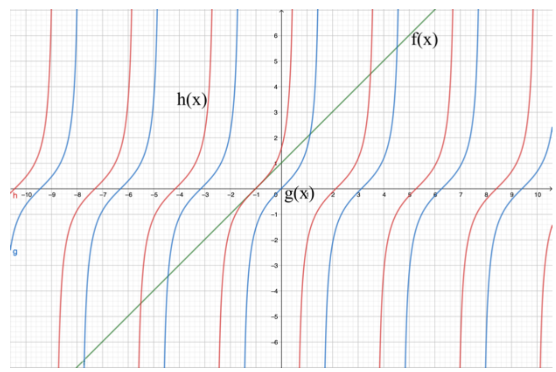
So, set x = f(a) ∈ B, then

g(b) = g(f(a)) = y

Therefore, g(b) = y and **g** is surjective.

b. Find an example demonstrating that g ◦ f is not bijective even though f is injective and g is surjective.

Let h(x) = g(f(x))



For the function h to not be bijective, it should not be both injective and bijective.

For f to be injective, a simple linear function would work.

Let **f(x) = x + 1.**
There is a one-to-one relation and there is only one x for each y.

For g to be surjective,

Let **g(x) = tan(x)** for all possible values of x. The function g only has vertical asymptotes which means the range is all real numbers.

The composite function, h would be,

h(x) = tan(x+1)

It is basically the function g, shifted 1 unit to the left.

From the graph, we can see that, h is indeed a surjective function but it fails to be injective. The function h has multiple roots (granted x is not limited) which means for the same value of y, there exists infinite x’s.

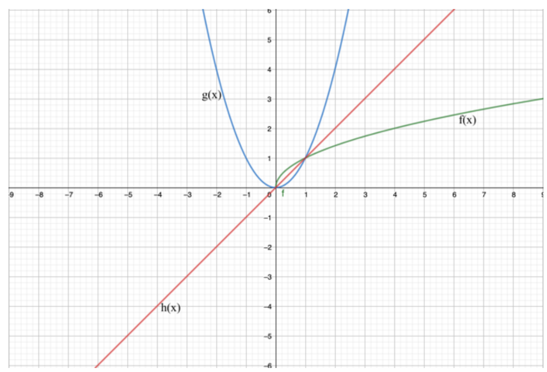
Therefore with,

f(x) = x + 1,
g(x) = tan(x),
h(x) = tan (x + 1),

The function **h is not bijective,** even if **f** is **injective** and **g** is **surjective**.

c. Find an example demonstrating that g ◦ f is bijective even though f is not surjective and g is not injective.

Let h(x) = g(f(x))



For f to be not surjective,
Let **f(x) =** $\sqrt{x}$

On the graph alongside, we can see that y is only defined in the positive side and doesn't cover all R.
None of the values of x covers all the possible y values.
Therefore, **f is not surjective.**

For g to be not injective,
Let **g(x) =** $\mathbf{x^2}$

A quadratic function is not injective as for one value of y, there are two x. In the case of x^2, y = 4 can be produced by x = $\pm$ 2.
Therefore, **g is not injective.**

The composite function, h would be,

h(x) = g ($\sqrt{x}$) = $\sqrt{x^2}$ = x

h(x) = x

The classic linear function, or any linear function, would be a bijective function as every possible value of y is covered by a unique value of x, as shown by the graph alongside.

Therefore with,

f(x) = $\sqrt{x}$,
g(x) = x^2,
h(x) = x,

The function **h is bijective,** even if **f** is **not surjective** and **g** is **not injective**.

Problem 4.3: prime numbers with a fixed prime gap (haskell)

(primes_gappies.txt present in the same .zip file)

primes_gappies.hs — Assignment 4

primes_gappies.hs

```haskell
isPrime :: Integer -> Bool
isPrime n = if n > 1 then null [ x | x <- [2..n `div` 2], n `mod` x == 0] else False
--the if...then...else... statement ignores the 1 from the input, because 1 is not a prime.


primes :: Integer -> Integer -> [Integer]
primes a b  = filter isPrime [a..b]
--uses isPrime to filter primes between ranges


gappies :: Integer -> Integer -> Integer -> [(Integer, Integer)]
gappies g a b = [(x, y) | x <- [a..b], y <- [x..b], isPrime x, isPrime y, y - x == g]
--g = gap between a and b.
--uses isPrime again.


twins = gappies 2
cousins = gappies 4
sexies = gappies 6
-- ^^^ Given ^^^
```

References

1. *Properties of Binary Relations taken from https://cnds.jacobs-university.de/courses/ics-2020/ics-slides.pdf*

2. *Bijection, Injection, And Surjection | Brilliant Math & Science Wiki. (2020). Retrieved 6 October 2020, from https://brilliant.org/wiki/bijection-injection-and-surjection/*

3. *Graphs generated from https://www.geogebra.org/calculator*