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Linear quadratic bumpless transfer☆

Matthew C. Turner*, Daniel J. Walker

Control Systems Research, Department of Engineering, University of Leicester, Leicester LE1 7RH, UK

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A method for bumpless transfer using ideas from LQ theory is presented and shown to reduce to the Hanus conditioning scheme under certain conditions.

Abstract

The problem of bumpless transfer is posed in the linear quadratic context, and formulae derived for a full-information gain which minimises a certain cost function. These formulae incorporate two weighting matrices to add flexibility to the design, and they can also be used when the controllers in question are non-invertible. Results are given for both continuous and discrete time cases, and both 1 and 2 DOF bumpless transfer is considered. When extended to the infinite horizon, the formulae depend only on known matrices and the solution to a single Riccati equation. Finally, it is shown that the formulae given here reduce to the Hanus Conditioned Transfer formulae under certain circumstances; that is, the results here could be considered as a generalisation of the Hanus technique. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Switching; Linear quadratic regulators; Linear control systems; Riccati equations; Bumpless transfer

1. Introduction

The success and widespread use of linear design techniques in control system design can, in part, be attributed to the relative ease of synthesis and implementation of linear controllers, and to the powerful, intuitive and convenient mathematics associated with linear systems theory. However, the strengths of these techniques have to be balanced against the fact that all real-world systems are, to some varying degrees, inherently nonlinear. This has the consequence that most linear controllers have to be designed around a specific operating point. Variation around this operating point can cause degradation of the performance of the controlled system, even when the engineer employs robust methods of design.

It is thus common practice to design more than one linear controller, each at a different operating point, and to switch between them; this enables the system to be controlled satisfactorily within the whole of its operating range. The process of switching introduces a nonlinearity into the loop (Campo, Morari & Nett, 1989) which may cause transients which affect the system's response adversely, or even dangerously. The suppression of these transient effects is referred to as 'bumpless transfer' between controllers.

To date, various schemes have been proposed to enable bumpless transfer. Classically, a high gain approach has been used, whereby a large gain is placed in a feedback loop around the off-line controller (see Green & Limebeer, 1995, pp. 433–437) in an attempt to force the off-line control signal to be identical to the on-line control signal. However, by considering elementary root-locus ideas, it is obvious that if the system in question has right half plane zeros, including those at infinity, the gain in the loop cannot be increased indefinitely: this may render this approach ineffective.

One of the most celebrated and widely used bumpless transfer schemes was proposed by Hanus, Kinnaert and Henrotte (1987), and has successfully been applied to many real-life projects; for example a VSTOL aircraft (Hyde, 1991). The Hanus conditioning scheme attempts to initialise the states of the off-line controller to those of

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* Corresponding author. Tel.: + 44-(0)116-252-2567/2529; fax: + 44-(0)116-252-2619.

E-mail address: mct6@sun.engg.le.ac.uk (M.C. Turner).

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the on-line controller by partially inverting the off-line controller to synthesise a 'realisable reference'. In the linear setting this requires the controller to be bi-proper and minimum phase. The former condition is sometimes restrictive, as many controllers lack a direct feedthrough term. Various remedies have been suggested to overcome this difficulty, most with a certain degree of success: for example in Hyde (1991), the Q-parametrisation of all H^{∞} controllers is used to select a bi-proper controller. However, no way in which to choose the direct feedthrough term (the D matrix) for those controllers lacking one has been universally accepted. Furthermore, if the controller is non-square, the controller cannot be inverted, and this renders the Hanus method inapplicable.

Since Hanus introduced the conditioning technique there have been a number of attempts to modify and generalise it by various authors. A good summary of the present methods available, and a general framework for bumpless transfer can be found in Campo et al. (1989). A more recent summary of various results and a new method of bumpless transfer utilising H^{∞} optimisation can be found in Edwards and Postlethwaite (1996).

In this paper we consider a methodical procedure which, apart from assuming that the controllers are finite dimensional, linear, and time-invariant (FDLTI), makes few additional assumptions on the controller. In this way, our proposed method of bumpless transfer is applicable to most general types of problems, and requires no ad hoc procedures for choosing D-matrices, etc. Furthermore, our method does give the designer some flexibility, by allowing him to choose two weighting matrices, so he can tailor his design suitably.

The results are derived for both continuous and discrete time domains, and both one degree of freedom (1 DOF) and two degrees of freedom (2 DOF) controller switching formulae are developed, although we note that the 1 DOF can be considered as a special case of the 2 DOF case. The most detailed derivation is given for the 1 DOF continuous time case, to convey the main ideas of the scheme. The other cases are derived in a similar manner, although their treatment is somewhat briefer. Due to the similarity between the discrete and continuous derivations, the discrete formulae are derived in the appendix. It is then shown that, under the assumptions made in Hanus et al. (1987), the formulae presented here reduce to those of Hanus et al. (1987) in certain circumstances.

We note that the concept of optimisation has been used before in the context of controller substitution/limitation, although this has mainly been directed towards anti-windup. Specifically, Walagama and Sternby (1993) suggest the minimisation of a quadratic Kuhn–Tucker function to reflect the control constraints, resulting in a quadratic convex programming problem. We suggest the minimisation in terms of LQ theory.

To illustrate the potential of the LQ bumpless transfer scheme presented here, an example is given. Finally, some concluding remarks and closing comments are made.

2. Notation and assumptions

The notation used throughout this paper is, in the main, standard. In the interests of clarity and simplicity we shall, wherever possible, omit the dimensions of vectors and matrices, unless it is important to draw the reader's attention to them.

We shall assume that all of the controllers are FDLTI, and that their states are available to the designer: a modest assumption since most modern controllers will be realised in software form, so the states will be computer variables. Importantly, we make the assumption that all controllers' realisations are completely controllable and observable and locally stabilise the plant in question. We shall furthermore assume that all signals available to, and produced by, the on-line controller are available. Effectively, this means the off-line controller has access to the nominal on-line control signal, the nominal plant output, the error signal and the reference signal. Note that, for a general linear controller controlling a nonlinear plant, the actual plant input and output will be different from those in the linearisation. Therefore, signals have to be added to the plant input and output to ensure that the linear controller gives the correct performance.

The Lebesgue space of all continuous square-integrable functions is given by

$$\mathscr{L}_2 = \left\{ x\colon \int_0^{\infty} x'x \, \mathrm{d}t < \infty \right\}, \tag{1}$$

where $x = x(t) \in \Re^n$.

The Lebesgue space of all absolutely square-summable sequences is given by

$$l_2 = \left\{ x\colon \sum_{k=0}^{\infty} x_k' x_k < \infty \right\}, \tag{2}$$

where $x_k \in \Re^n$ and $k \in \{0, 1, 2 \ldots\}$.

The extended Lebesgue space of continuous square integrable functions, $\mathscr{L}_{2e}$ is identical to the Lebesgue space, $\mathscr{L}_2$, except that the upper limit of the integral is taken as some finite time, so that functions which have an infinite $\mathscr{L}_2$ norm may still belong to $\mathscr{L}_{2e}$. The extended Lebesgue space of discrete square summable sequences, l_{2e} is defined similarly. In this paper it is assumed that all continuous signals belong to $\mathscr{L}_{2e}$, and that all discrete signals belong to l_{2e}, as we are often concerned with minimising certain functionals which amounts to minimising weighted combinations of these norms. Recall that the majority of signals encountered have finite $\mathscr{L}_{\infty}$ (resp. l_{∞}) norms if they have finite $\mathscr{L}_{2e}$ (resp. l_{2e}) norms.

This paper considers both 1 DOF and 2 DOF controllers which are assigned the following state-space realisations, respectively:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C & D_1 & D_2 \end{array}\right]$$

In the continuous-time case we shall assume that the state equation of the controller is of the form

$$\dot{x} = f(x, w, t) \tag{5}$$

and in the discrete-time case we shall assume that the next state is determined from

$$x_{k+1} = f(x_k, w_k, k). \tag{6}$$

Unless otherwise stated, the state vector is $x \in \Re^n$, the control signal is $u \in \Re^m$, the error vector (the difference between the reference and the plant output) is $\tilde{e} \in \Re^p$, the output of the plant is $y \in \Re^{p1}$, and the reference signal is $r \in \Re^{p2}$. An arbitrary exogenous vector is sometime used and is represented as $w \in \Re^q$. All other vectors and matrices shall be assumed to be of compatible dimensions.

3. The continuous time case

3.1. One-degree-of-freedom bumpless transfer

The aim of this section is to derive a static feedback matrix, F, which can be used with a view to providing bumpless transfer for 1 DOF controllers. During the course of the derivation of F, we will draw heavily on some concepts from linear quadratic theory. A highly readable discussion of much of this material can be found in Lewis (1986).

We will now describe the basic idea of our proposal. As previously stated, we assume that there is access to the on-line controller's states, together with all signals fed into, and produced by, the on-line controller, i.e. we have full information. With reference to Fig. 1, we propose to synthesise a static feedback gain, F, which can be used to drive the off-line controller in such a way that, at the time of transfer between on and off-line controllers the transients produced by this switching are minimal.

To achieve a 'minimal' amount of transient behaviour during switching we shall minimise a quadratic cost function. The minimisation of a weighted combination of two signals is proposed. Firstly, at the time of switching, we would like the on and off-line controllers to be producing control signals which are as close to each other as possible: this would reduce the magnitude of the discontinuity which occurs during transfer. Secondly, we must also take into account the signals driving the controllers. We note that the off-line control signal is being driven by the signal produced by the feedback gain and that, in the 1 DOF case, the on-line controller is being driven by the error signal. It is desirable to avoid a large difference in these signals because, in order to maintain good tracking, the signal driving the off-line controller will be switched to the error signal. That is, after switching the off-line controller becomes the current on-line controller. This situation is described diagramatically in Fig. 1.

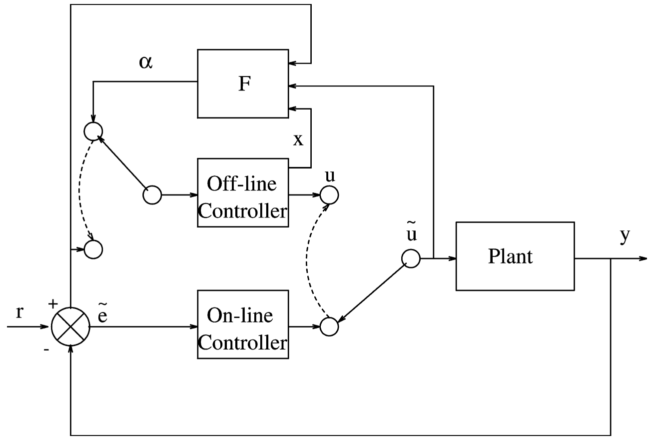


Fig. 1. Bumpless transfer scheme.

To recapitulate, we want to minimise the difference between two sets of signals: the difference between the two control signals, and the difference between the signals driving the two controllers. To pose this problem in the LQ context, we minimise the following functional;

$$J(u, \alpha, T) = \frac{1}{2}\int_0^T z_u(t)' W_u z_u(t) + z_e(t)' W_e z_e(t)\,\mathrm{d}t + \frac{1}{2} z_u(T)' P z_u(T), \tag{7}$$

where

$$z_u(t) = u(t) - \tilde{u}(t), \tag{8}$$

$$z_e(t) = \alpha(t) - \tilde{e}(t), \tag{9}$$

and where $\tilde{u}(t)$ and $\tilde{e}(t)$ are the on-line control signal and error signal, respectively; $u(t)$ is the off-line control signal; $\alpha(t)$ is the signal produced by the feedback gain which drives the off-line controller. W_u and W_e are constant positive-definite weighting matrices of appropriate dimensions which are used to tailor the design as required. Finally, $z_u(T) = u(T) - \tilde{u}(T)$ is the difference between the two control signals at the terminal time T (which will most commonly be taken as infinity), and P is the positive semi-definite terminal weighting matrix; although may well be set to zero and is introduced only for the benefit of derivation, and, indeed, generality.

To synthesise a feedback matrix, F, we can therefore solve the problem of minimising this quadratic performance index. The signal $\alpha(t)$ produced by F, is a function of the off-line controller states, the error signal, and the on-line control signal. The gain F can be regarded as a full-information 'sub-controller', which temporarily controls the off-line controller.

To derive F we invoke standard LQ procedures procedures, a more complete discussion of which can be found in Lewis (1986), for example . If the off-line controller is being driven by the signal $\alpha(t)$, then its state-space equations are

$$\dot{x} = Ax + B\alpha, \tag{10}$$

$$u = Cx + D\alpha. \tag{11}$$

Substituting for u, in the performance index (7), we obtain

$$J = \frac{1}{2}\int_0^T (Cx + D\alpha - \tilde{u})' W_u (Cx + D\alpha - \tilde{u}) + (\alpha - \tilde{e})' W_e (\alpha - \tilde{e})\,\mathrm{d}t + \frac{1}{2} z_u(T)' P z_u(T). \tag{12}$$

Note that Eq. (10) is a constraint on the minimisation of the performance index (12), which can be adjoined to it via the introduction of a dynamic Lagrange multiplier, $\lambda(t) \in \Re^n$. Thus we can write

$$\tilde{J} = \frac{1}{2}\int_0^T [H(t) - \lambda(t)'\dot{x}]\,\mathrm{d}t + \phi(T) \tag{13}$$

where $\phi(T) := \frac{1}{2} z_u(T)' P z_u(T)$ and the *Hamiltonian*, $H(t)$ is given by

$$H = \tfrac{1}{2}[(Cx + D\alpha - \tilde{u})' W_u (Cx + D\alpha - \tilde{u}) + (\alpha - \tilde{e})' W_e (\alpha - \tilde{e})] + \lambda'(Ax + B\alpha). \tag{14}$$

First-order necessary conditions for a minimum of the cost (13) (and, equivalently, the cost (12)) are given by (see Lewis, 1986)

$$\dot{x} = \frac{\partial H}{\partial \lambda} = f(x, w, t), \tag{15}$$

$$-\dot{\lambda} = \frac{\partial H}{\partial x}, \tag{16}$$

$$0 = \frac{\partial H}{\partial w}, \tag{17}$$

$$\lambda(T) = \frac{\partial \phi}{\partial x(T)}. \tag{18}$$

Note that (15) and (16) constitute the *state* and *co-state* equations respectively, where the co-state equation is a differential equation that has to be solved from the terminal condition, (18), and describes the system's adjoint. Eq. (17) is often referred to (Lewis, 1986) as the *stationarity condition*. Evaluating the Eqs. (15)–(17) leads to

$$\frac{\partial H}{\partial \lambda} = Ax + B\alpha, \tag{19}$$

$$\frac{\partial H}{\partial x} = A'\lambda + C'W_u Cx - C'W_u \tilde{u} + C'W_u D\alpha, \tag{20}$$

$$\frac{\partial H}{\partial \alpha} = (D'W_u D + W_e)\alpha + D'W_u Cx + B'\lambda - D'W_u \tilde{u} - W_e \tilde{e}, \tag{21}$$

which yields α as

$$\alpha = (D'W_u D + W_e)\alpha + D'W_u Cx + B'\lambda - D'W_u \tilde{u} - W_e \tilde{e}, \tag{22}$$

where $\Delta = -(D'W_u D + W_e)^{-1}$.

If we now use this expression for α in the state and co-state Eqs. (19) and (20) respectively, we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ -\tilde{C} & -\tilde{A}' \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} -B\Delta W_e \\ C'W_u D\Delta W_e \end{bmatrix} \tilde{e} + \begin{bmatrix} -B\Delta D'W_u \\ C'W_u(I + D\Delta D'W_u) \end{bmatrix} \tilde{u}, \tag{23}$$

where

$$\tilde{A} = A + B\Delta D'W_u C, \tag{24}$$

$$\tilde{B} = B\Delta B', \tag{25}$$

$$\tilde{C} = C'W_u(I + D\Delta D'W_u)C. \tag{26}$$

The above non-homogenous differential equation is of the form which often arises in LQ minimisation and can be solved by the method of sweep (see Lewis, 1986). Therefore we have

$$\lambda(t) = \Pi(t)x(t) - g(t). \tag{27}$$

Differentiating,

$$\dot{\lambda}(t) = \dot{\Pi}(t)x(t) + \Pi(t)\dot{x}(t) - \dot{g}(t). \tag{28}$$

If we now combine (27), (28) and (23) we can derive two expressions for $\dot{\lambda}$:

$$\dot{\lambda} = (\dot{\Pi} + \Pi\tilde{A} + \Pi\tilde{B}\Pi)x - \Pi\tilde{B}g - \Pi B\Delta D'W_u \tilde{u} - \Pi B\Delta W_e \tilde{e} - \dot{g} \tag{29}$$

and

$$\dot{\lambda} = -(\tilde{C} + \tilde{A}'\Pi)x + C'(W_u + W_u D\Delta D' W_u)\tilde{u} + C'W_u D\Delta W_e \tilde{e} + \tilde{A}'g. \tag{30}$$

Equating coefficients of x we obtain the following differential Riccati equation:

$$-\dot{\Pi} = \Pi\tilde{A} + \tilde{A}'\Pi + \Pi\tilde{B}\Pi + \tilde{C}. \tag{31}$$

We can also equate the other coefficients to obtain

$$-\dot{g} = (C'W_u + C'W_u D\Delta D'W_u + \Pi B\Delta D'W_u)\tilde{u} + (C'W_u D\Delta W_e + \Pi B\Delta W_e)\tilde{e} + (\tilde{A}' + \Pi\tilde{B})g. \tag{32}$$

If we define

$$B_g = \begin{bmatrix} (C'W_u + C'W_u D\Delta D'W_u + \Pi B\Delta D'W_u)' \\ (C'W_u D\Delta W_e + \Pi B\Delta W_e)' \end{bmatrix}' \tag{33}$$

and

$$\tilde{w} = \begin{bmatrix} \tilde{u} \\ \tilde{e} \end{bmatrix}, \tag{34}$$

then we obtain

$$-\dot{g} = (\tilde{A}' + \Pi\tilde{B})g + B_g\tilde{w}. \tag{35}$$

From the boundary condition (18) we have that

$$g(T) = -(I - C'PD\Delta B')^{-1}D_g w(\tilde{T}) \tag{36}$$

and

$$\Pi(T) = (I - C'PD\Delta B')^{-1}(C'PC + C'PD\Delta D'W_u C), \tag{37}$$

where

$$D_g = \begin{bmatrix} -(C'PD\Delta D'W_u + C'P)' \\ -(C'PD\Delta W_e)' \end{bmatrix}'. \tag{38}$$

We use these terminal conditions in (35) and (31) to determine the time-varying matrix Π and vector g. In turn, this can be used in the adjoint equation (27) to determine the co-state vector, λ. Recalling the definition of α (Eq. (22)), we obtain the following expression for the minimising feedback:

$$\alpha = \Delta \begin{bmatrix} (D'W_u C)' \\ B \\ -(D'W_u)' \\ -W_e \end{bmatrix}' \begin{bmatrix} x \\ \lambda \\ \tilde{u} \\ \tilde{e} \end{bmatrix}. \tag{39}$$

A problem now exists; for implementation, we require the co-state vector λ, which is obtained from a differential equation which develops backward in time from the terminal values of $\tilde{u}$ and $\tilde{e}$: we require future knowledge in order to arrive at a solution.

3.1.1. Extension to infinite horizon

Although we have derived conditions for a finite horizon bumpless transfer, there are two important points to note about this general case: the output of the closed-loop system adjoint, λ, can only be computed on a finite horizon if the reference signals ($\tilde{e}$ and $\tilde{u}$) are known a priori; moreover, it is normally the case that these signals are *not* known a priori. In the case that these signals *are* known beforehand, we would have to determine either the values of $\lambda(t)$ off-line before implementation; or solve (27) for the initial condition $\lambda(0)$ off-line, using (36) and (37), and then solve the adjoint equation (27) on-line to obtain the values of $\lambda(t)$ needed by F. These two points limit the usefulness of the finite horizon results for many real-life applications.

However, if the results are extended to an infinite time support, these problems are not applicable; practical implementation then becomes possible. Here we develop arguments to extend the finite horizon results, and state conditions which must apply for this to be achieved. To aid us we present a theorem which gives conditions for the Riccati equation (31) to have a solution that tends to the positive semi-definite solution of the corresponding algebraic Riccati equation (ARE):

$$\Pi\tilde{A} + \tilde{A}'\Pi + \Pi\tilde{B}\Pi + \tilde{C} = 0 \tag{40}$$

as T tends to infinity in the cost function (12).

The theorem is essentially from Lewis (1986), but is a well-known fact.

Theorem 1. *Assume* $(\tilde{A}, \tilde{B})$ *is stablisable,* $(\tilde{A}, \sqrt{\tilde{C}})$ *is detectable and the terminal solution to the Riccati differential equation* (31) *is positive semi-definite; then*

- *the Riccati equation* (31) *has a steady-state solution, independent of the terminal value, such that* $\Pi = \Pi(\infty) = \lim_{T\to\infty}\Pi(t) \geq 0$,
- *the solution* $\Pi(\infty) = \Pi$ *is positive semi-definite and stabilising.*

Our assumption on the controllability of (A, B) implies that the pair $(\tilde{A}, \tilde{B})$ are also controllable. To see this note that due to the full rank assumption on Δ, controllability of $(\tilde{A}, \tilde{B})$ is equivalent to controllabilty of $(\tilde{A}, B)$. Also as $\Delta D'W_u C$ is equivalent to a state feedback term in $\tilde{A}$, and that state feedback does not alter controllability, then controllability of $(\tilde{A}, \tilde{B})$ is ensured. By a similar argument observability of (A, C) can be seen to imply observability of $(\tilde{A}, \sqrt{\tilde{C}})$ if the term $W_u + W_u D\Delta D'W_u$ is positive definite. This is not ensured, as $\Delta < 0$, but can be made so by appropriate choices of weighting matrices.

Furthermore, if $D = 0$, controllability and observability of $(\tilde{A}, \tilde{B}, \sqrt{\tilde{C}})$ is unconditionally ensured. This can be seen by noting that, when $D = 0$, we have the following expressions:

$$\tilde{A} = A, \tag{41}$$

$$\tilde{B} = -BW_e^{-1}B', \tag{42}$$

$$\tilde{C} = C'W_u C. \tag{43}$$

As W_u and W_e have both been chosen positive definite, their ranks are full and therefore neither $\tilde{B}$ nor $\tilde{C}$ lose rank. This implies the controllabilty and observability statement made above. Hence from Theorem 1, we can conclude that the steady-state solution of Eq. (31) converges to that of the ARE (40); thus in the infinite horizon we need only solve an algebraic equation and moreover this solution is stabilising.

Now consider the differential equation (35). It is well known from optimal control that such a differential equation does not converge to a constant solution for an arbitrary exogenous vector $\tilde{w}$. Thus, even in the infinite horizon we must still solve a differential equation, using future information, to obtain an exact solution.

We suggest using an approximation for g, as used in LQ tracking, as described in Athans and Falb (1966). To this end, we make the additional assumption that the exogenous signals $\tilde{u}$ and $\tilde{e}$ are constant. Then, noting that $A_g := (\tilde{A} + \tilde{B}\Pi)'$ is Hurwitz (by the stabilising property of the ARE solution), it can be shown that

$$\lim_{T\to\infty} g(t) = g = -A_g^{-1}B_g\tilde{w}, \tag{44}$$

where g is a constant as a consequence of $\tilde{w}$ being a constant. Using the expressions for A_g and B_g this can be written as a linear equation:

$$g = -(\tilde{A}' + \Pi\tilde{B})^{-1}[(C'W_u + C'W_uD\Delta D'W_u + \Pi B\Delta D'W_u)\tilde{u} + (C'W_uD\Delta W_e + \Pi B\Delta W_e)\tilde{e}] \tag{45}$$

Eq. (45) no longer needs the exogenous signals to be known a priori as it is also purely algebraic. Hence both Π and g can be computed for substitution into (27) (in turn this can be inserted into (22)). Furthermore, note that the ARE, (40), has a positive definite stablising solution, which always exists provided the stabilisability and detectability conditions are satisfied. Thus in the infinite horizon α can be computed as

$$\alpha = F\begin{bmatrix} x \\ \tilde{u} \\ \tilde{e} \end{bmatrix}, \tag{46}$$

where F is given by

$$F = \Delta\begin{bmatrix} (B'\Pi + D'W_uC)' \\ (-D'W_u + B'M(C'W_u + C'W_uD\Delta D' + \Pi B\Delta D'W_u))' \\ (-W_e + B'M(C'W_uD\Delta W_e + \Pi B\Delta W_e))' \end{bmatrix}' \tag{47}$$

and $M = (\tilde{A}' + \Pi\tilde{B})^{-1}$.

Note that if $D = 0$, then the formula for F simplifies considerably.

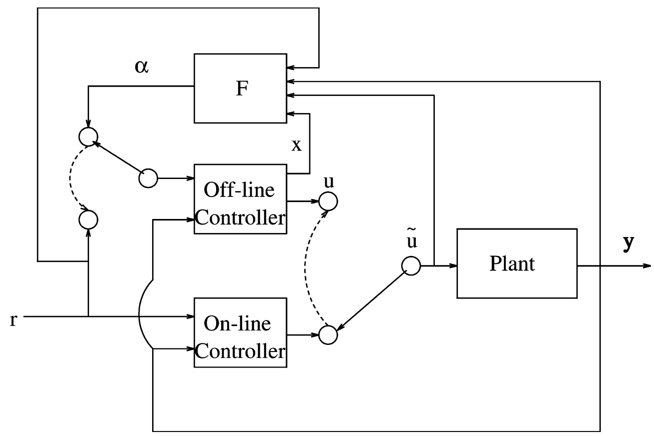


Fig. 2. 2 DOF configuration.

Remark. Our infinite horizon solution relies heavily upon our approximation for g, which assumes the exogenous signals are constant. However, many signals in real systems, such as steps, can be considered constant over a certain period of time. For these signals, the signal α is optimal, but for others it is suboptimal. Thus if the system responds quickly enough (large poles), the approximation may serve as quite satisfactory and these infinite horizon results may be applied. Furthermore, simulation has indicated that in many cases the infinite horizon solution is quite adequate.

3.2. Two-degrees-of-freedom design

The ideas developed can also be applied to the 2 DOF configuration[1] where the on-line controller is driven by the reference signal, r, and the plant output, y; and the off-line controller is driven by the plant output and the output from the feedback matrix, α — see Fig. 2. The main objective will remain the same in the 2 DOF configuration: to minimise the difference between the off-line and on-line control signals. However, as neither controller is driven by the error signal, and both controllers are partly driven by the plant output signal, for our subsidiary minimisation we must minimise the difference between the other signals that are driving the two controllers: the reference, and the feedback matrix output. Hence our performance index becomes

$$J(u,\alpha,T) = \frac{1}{2}\int_0^T z_u'(t)W_uz_u(t) + z_e'(t)W_ez_e(t)\,dt + \frac{1}{2}z_u(T)'Pz_u(T), \tag{48}$$

[1] We note, that 1 DOF controllers may be considered a special case of the 2 DOF configuration. Hence, this bumpless transfer scheme may easily be adapted to switch between 1 DOF and 2 DOF controllers.

where

$$z_u(t) = u(t) - \tilde{u}(t), \tag{49}$$

$$z_e(t) = \alpha(t) - r(t), \tag{50}$$

and the state-space description of the off-line controller is now

$$\dot{x} = Ax + B_1\alpha + B_2 y, \tag{51}$$

$$u = Cx + D_1\alpha + D_2 y. \tag{52}$$

Following a similar argument as before, in the finite horizon case we get the expression for F as

$$\alpha = \Delta \begin{bmatrix} (D_1' W_u C)' \\ B_1 \\ (D_1' W_u D_2)' \\ -(D_1' W_u)' \\ -W_e \end{bmatrix}' \begin{bmatrix} x \\ \lambda \\ y \\ \tilde{u} \\ r \end{bmatrix}, \tag{53}$$

where the solution to the adjoint is determined from

$$\lambda = \Pi x - g \tag{54}$$

and Π and g are the solutions to the differential equations

$$-\dot{\Pi} = \Pi\tilde{A} + \tilde{A}'\Pi + \Pi\tilde{B}\Pi + \tilde{C}, \tag{55}$$

$$-\dot{g} = (\tilde{A}' + \Pi\tilde{B})g + \begin{bmatrix} -(C'W_uD_1\Delta D_1'W_uD_2 + C'W_uD_2 + \Pi(B_2 + B_1\Delta D_1'W_uD_2))' \\ (C'W_uD_1\Delta D_1'W_u + C'W_u + \Pi B_1\Delta D_1'W_u)' \\ (C'W_uD_1\Delta W_e + \Pi B_1\Delta W_e)' \end{bmatrix}' \begin{bmatrix} y \\ \tilde{u} \\ r \end{bmatrix}, \tag{56}$$

which are solved subject to the terminal conditions

$$\Pi(T) = (I - C'PD_1\Delta B_1')^{-1}(C'PC + C'PD_1\Delta D_1'W_uC), \tag{57}$$

$$g(T) = -(I - C'PD_1\Delta B_1')^{-1} \times \begin{bmatrix} (C'PD_2 + C'PD_1\Delta D_1 W_u C)' \\ -(C'P + C'PD_1\Delta D_1' W_u)' \\ -(C'PD_1\Delta W_e)' \end{bmatrix}' \begin{bmatrix} y(T) \\ \tilde{u}(T) \\ r(T) \end{bmatrix}, \tag{58}$$

where

$$\tilde{A} = A + B_1\Delta D_1' W_u C, \tag{59}$$

$$\tilde{B} = B_1\Delta B_1', \tag{60}$$

$$\tilde{C} = C'(W_u + W_u D_1\Delta D_1' W_u)C. \tag{61}$$

If we make the same assumptions as in the 1 DOF case, in the infinite horizon the matrix F is given by:

$$\alpha = \Delta \begin{bmatrix} (D_1' W_u C + B_1'\Pi)' \\ (D_1' W_u D_2 - B_1' M\hat{Y})' \\ (-D_1' W_u + B_1' M\hat{U})' \\ (-W_e + B_1' M\hat{R})' \end{bmatrix}' \begin{bmatrix} x \\ y \\ \tilde{u} \\ r \end{bmatrix}, \tag{62}$$

where

$$\Delta = -(D_1' W_u D_1 + W_e)^{-1}, \tag{63}$$

$$M = (\tilde{A}' + \Pi\tilde{B})^{-1}, \tag{64}$$

$$\hat{Y} = (C'W_uD_1 + \Pi B_1)\Delta D_1' W_u D_2 + \Pi B_2 + C'W_u D_2, \tag{65}$$

$$\hat{U} = C'W_u + (C'W_uD_1 + \Pi B_1)\Delta D_1' W_u, \tag{66}$$

$$\hat{R} = (C'W_uD_1 + \Pi B_1)\Delta W_e, \tag{67}$$

and Π is the positive-semi-definite stabilising solution to the ARE:

$$\Pi\tilde{A} + \tilde{A}'\Pi + \Pi\tilde{B}\Pi + \tilde{C} = 0. \tag{68}$$

Note that the infinite horizon formula requires no prior knowledge of the external inputs to determine F, as with the 1 DOF case. Again, in the infinite horizon, stability of the off-line loop is ensured as the solution to the Riccati equation converges to a constant, positive-definite solution, provided that $(\tilde{A}, \tilde{B})$ and $(\tilde{A}, \sqrt{\tilde{C}})$ are stabilisable and detectable respectively, and that $\Pi(T) \geq 0$.

As with the 1 DOF case the formula simplifies significantly in the case that $D = 0$.

4. An extension to the Hanus conditioning scheme

Thus far, we have derived formulae for a full-information static gain which can be used to drive the off-line controller in such a way that when this controller is switched on-line, the transient effects are small. We emphasise that no assumptions have been made concerning the properness of the conditioned controllers. Nor have any minimum-phase conditions been imposed, as the stability of the off-line control loop is guaranteed by the stabilising property of the solution to the algebraic Riccati equation.

It is interesting to note, however, that if we make the following assumptions:

- the controllers are bi-proper, i.e. a D-matrix exists and is square,
- the controller is minimum phase,
- W_e is zero,

then the formulae derived here reduce to the Hanus scheme. This follows from some basic matrix algebra, and the fact that the solutions to the respective Riccati equations are singular when these conditions are enforced.

For both continuous and discrete 1 DOF configurations, the expression for F becomes:

$$F = [\,-D^{-1}C \quad D^{-1} \quad 0], \tag{69}$$

while in the 2 DOF cases,

$$F = [\,-D_1^{-1}C \quad -D_1^{-1}D_2 \quad -D_1^{-1}\;0]. \tag{70}$$

These expressions are exactly the Hanus conditioning scheme for the respective cases. Note that Hanus requires the controllers to be minimum phase to ensure stability of the off-line control loop.

However, although the feedback portion of the schemes are identical, the implementation is different: the Hanus still maintains an advantage in this respect, as the controller is constantly driven by the error or reference signal, and thus no switching is required (see Hanus et al., 1987; Campo et al., 1989 for details).

5. Stability

In the infinite horizon all the various formulae for F ensure the stability of the off-line control loop (i.e. the loop consisting of the off-line controller and the 'subcontroller', F) by virtue of the fact that the solution of the corresponding ARE is stabilising. In the finite horizon, stability is not an issue, and the solutions to the Riccati equations depend upon the terminal weights and boundary conditions.

Our assumption that both on- and off-line controllers are stabilising for the plant in question, around a certain operating point, guarantees the stability of the closed-loop system around this operating point, without switching occurring. However, nothing can be concluded in general about the stability of the overall system when arbitrary switching occurs. Recent results based on passivity theory have appeared in Campo and Morari (1997), which guarantee stability for both anti-windup and bumpless transfer. That analysis stems from an earlier paper by Campo et al. (1989), where a unification of many existing schemes is developed. The results arguably lend themselves more conveniently to anti-windup schemes, than to bumpless transfer, however, and require assumptions to be made which are not satisfied by the LQ scheme proposed here.

It is also worth noting that most results concerning stability of anti-windup and bumpless transfer schemes pertain to the stability of linear systems with a single nonlinearity between the controller and plant. This is a realistic representation for anti-windup configurations, but in the case of bumpless transfer, the reason for switching between two controllers is usually due to the actual nonlinearity of the plant. Hence, it would be more desirable to prove stability of bumpless transfer schemes taking into account the nonlinear nature of the plant.

From a practical point of view, however, our experience based on nonlinear simulations is that stability, as defined by any practical measure, is maintained provided both controllers are stabilising, and the off-line control loop is also stable. This is the case with the LQ scheme, and thus we can conjecture that a stable closed loop will result. Notwithstanding, no concrete conclusions may be drawn about stability, although the LQ method is as likely to result in a stable system as most other methods.

6. Simulation results

We now present simulation results which illustrate how the methods derived in this paper fare in practice. The plant to be controlled is a helicopter model: the Westland Lynx, a nonlinear model which was provided by GKN Westland. This models the nonlinear variations that occur in the dynamics over the flight envelope. A high degree of inter-axis coupling and open-loop instability make this a challenging control problem.

The controllers were designed using H^∞ 2 DOF mixed sensitivity techniques (the high- and low-speed controllers were designed using 15-state linearisations taken at 70 and 30 m/s, respectively), which allow time-domain requirements to be indirectly enforced alongside robustness considerations. An interesting discussion of this technique can be found in Tucker and Walker (1997) and details of the theory in Doyle, Glover, Khargonekar and Francis (1989).

The following outputs were chosen for control:

- roll attitude,
- pitch attitude,
- yaw rate,
- vertical velocity,

and, additionally the two rate outputs, were chosen to enhance the design:

- roll rate,
- pitch rate.

The control inputs to the helicopter were:

- lateral cyclic,
- longitudinal cyclic,

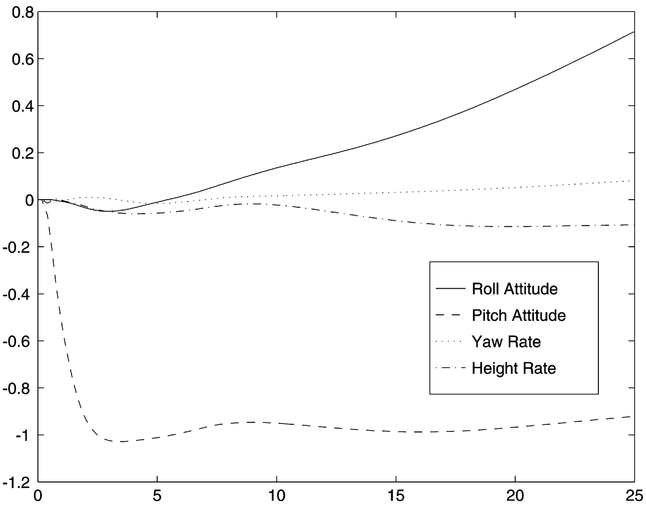


Fig. 3. Helicopter response to step in pitch attitude using low-speed controller.

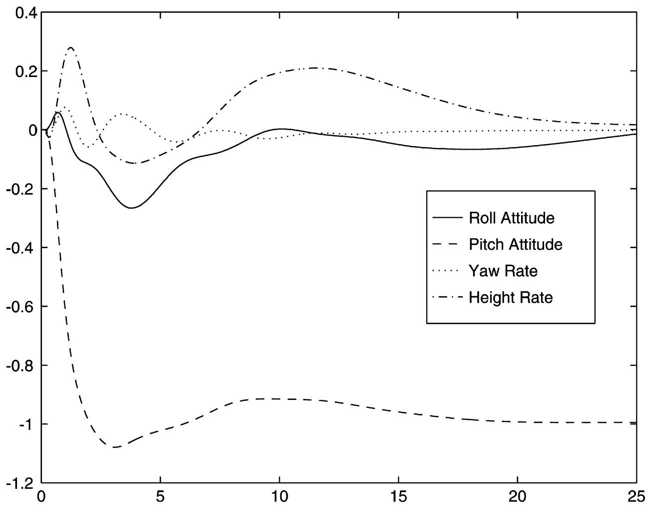


Fig. 4. Helicopter response to step in pitch attitude using high-speed controller.

- tail rotor collective,
- main rotor collective.

The main influence of lateral cyclic is to change the roll angle; likewise the primary effect of longitudinal cyclic is a change in pitch attitude. The height rate is primarily determined by main rotor collective, and yaw rate is controlled by tail rotor collective. The outputs were scaled by 3.3 m/s (vertical velocity), 0.2 rad (pitch attitude), 0.2 rad (roll attitude), and 0.2 rad/s (yaw rate).

We consider the situation where the helicopter is making the transition from low speed to high speed in forward flight. First observe the effects of not switching between controllers, as shown in Figs. 3 and 4. Both plots show a step demand in pitch attitude: the first using a low-speed controller and the second using a high speed controller. A step in pitch attitude will cause the helicopter's velocity to increase and hence it can be seen that the low speed controller's performance degrades considerably in the roll channel (Fig. 3) as the helicopter attains high velocities. Similarly, from Fig. 4, it can be seen that using the high-speed controller at low air speeds causes significant coupling between channels and consequently a loss of performance.

To overcome the degradation of performance which occurs as the helicopter increases speed, a controller switching strategy is introduced. The inclusion of rate feedback to enhance performance and stability, lead to a non-square controller, which, combined with the strict properness of the resulting H^{∞} controllers renders the Hanus technique inapplicable in this instance. However, the LQ scheme can cope with both strictly proper and non-square controllers.

We implement the infinite horizon formulae as they require no prior knowledge of the external signals. There is some practical justification for this as the inputs to controllers of this type will often be steps and pulses, which can be appproximated as constant inputs over short periods.

Initial tests have indicated that the scheme works well when it 'tends to' the Hanus Scheme. This implies that the control signals should be weighted heavily in comparison to the other signals. Therefore, the weights were chosen as

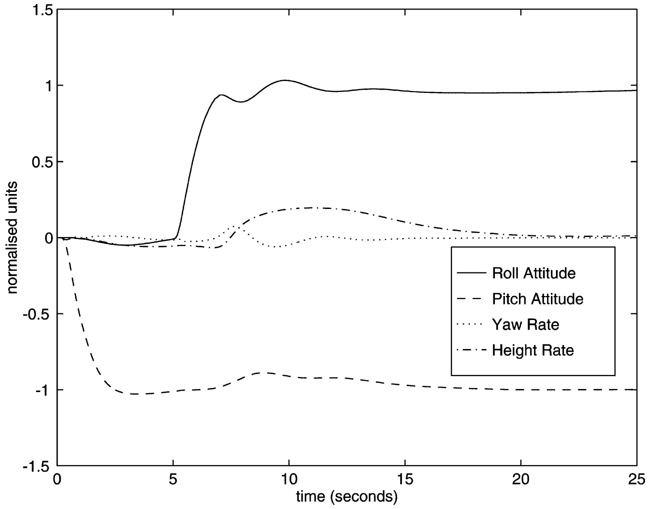
$$W_u = diag(10^3, 10^3, 10^3, 10^3),$$

$$W_e = diag(0.1, 0.1, 0.1, 0.1, 0.1, 0.1).$$

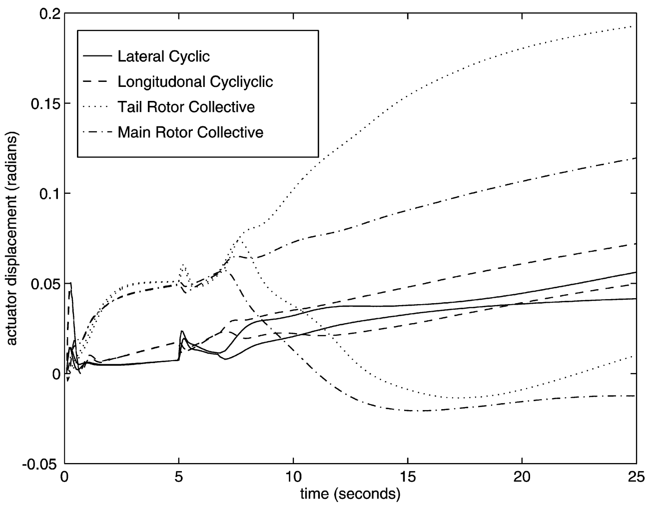
This reflects the diagonal nature of the plant which we attempted to preserve, as well as the relative importance of each channel during transfer. It is also possible to use non-diagonal, positive-definite weighting matrices. This flexibility in the scheme could prove useful in practice, although more research is required on the selection of these matrices. These matrices were used in the continuous-time 2 DOF formulae derived earlier, and a static feedback gain, F was determined.

It was decided to switch from the high-speed to the low-speed controller at 40 m/s. For purposes of illustration, only the high speed controller was 'conditioned' off-line before it was switched in. When the low speed controller was switched off-line, it was left open loop so the difference between the control signals could be clearly seen. It is noted that in practice, as soon as a controller is switched off-line it would be conditioned immediately, so that it could be switched on-line again rapidly.

Fig. 5 shows the responses of the helicopter using the LQ bumpless transfer scheme. First a step in pitch attitude is applied to increase the helicopter's speed; then a step in roll is instigated to change the vehicle's direction. A very smooth transition in the pitch and yaw channels results. In the height channel, a small 'bump'



Response to Steps in Pitch and Roll



Actuator Reponses to Steps in Pitch and Roll

Fig. 5. Helicopter responses due to steps in pitch and roll using LQ bumpless transfer scheme.

occurs at the time of switching which gradually decays; similarly in the roll channel there is a slight transient degradation. Neither of these effects were considered excessive.

From inspecting the actuator signals in Fig. 5, we can see that the control signals produced by the two controllers are virtually identical until switching occurs. This is very close to what we might expect in the Hanus Scheme, and reinforces the idea that it is a special case of the LQ scheme presented here.

7. Conclusion

In this paper a new method aimed at achieving bumpless transfer has been introduced and simulation results have been presented. We have seen that the new LQ method of bumpless transfer is applicable to most linear controller types, whereas the Hanus technique is not always able to be directly applied, particularly in the case of controllers lacking a non-singular D or D_1 matrix. Moreover, the LQ method gives the designer scope for 'shaping' the bumpless transfer through the use of two weighting matrices. In a real application this could be a useful option. As only a static gain is required for the scheme, it requires little on-line computation.

Furthermore, it has also been shown that the formulae derived here can, under certain assumptions, simplifiy to those given by Hanus. This is both theoretically interesting, and practically encouraging, considering the success of the Hanus scheme in industrial applications. In a certain sense this relationship makes our results a generalisation of Hanus' technique; it also shows that, in a linear quadratic context, Hanus' results can constitute an optimal bumpless transfer strategy.

It is well known that many bumpless transfer schemes also work well as anti-windup compensators: this raises the possiblity of the LQ bumpless transfer scheme also being used in this capacity. Future research also needs to discover if any particular choices of the weighting matrices leads to a minimal 'bump' during transfer.

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Appendix A. Discrete-time bumpless transfer

The discrete time versions of the formulae for bumpless transfer are similar to their continuous time counterparts, and are derived here for completeness.

A.1. Discrete-time 1 DOF bumpless transfer

We consider the same configuration as that shown in Fig. 1 and assume that the off-line controller is described by the difference equations:

$$x_{k+1} = Ax_k + B\alpha_k, \tag{A.1}$$

$$u_k = Cx_k + D\alpha_k. \tag{A.2}$$

We aim to minimise the difference between the on- and off-line control signals and also the difference between the signals driving the controllers: α, the signal produced by the 'subcontroller', and the control error. Hence, the corresponding discrete quadratic performance index which we seek to minimise is

$$J = \frac{1}{2}\sum_{0}^{T-1} [z_u(k)'W_u z_u(k) + z_e(k)'W_e z_e(k)] + \frac{1}{2}z_u(T)'Pz_u(T), \tag{A.3}$$

in which

$$z_u(k) = u_k - \tilde{u}_k, \tag{A.4}$$

$$z_e(k) = \alpha_k - \tilde{e}_k, \tag{A.5}$$

$$z_u(T) = u_T - \tilde{u}_T. \tag{A.6}$$

Substituting the control signal into (A.3), we can write

$$J = \frac{1}{2}\sum_{0}^{T-1} [(Cx_k + D\alpha_k - \tilde{u}_k)'W_u(Cx_k + D\alpha_k - \tilde{u}_k) + (\alpha_k - \tilde{e}_k)W_e(\alpha_k - \tilde{e}_k)] + \frac{1}{2}z_u(T)'P(T)z_u(T). \tag{A.7}$$

Applying a Lagrange multiplier, $\lambda_k \in \Re^n$, to (A.7) and the discrete state equation gives the augmented performance index,

$$\tilde{J} = \frac{1}{2}\sum_{0}^{T-1} H_k - \lambda'_{k+1}x_{k+1} + \phi(T), \tag{A.8}$$

where $\phi(T) := \frac{1}{2}z'_u(T)Pz_u(T)$ and the Hamiltonian is defined as

$$H = \tfrac{1}{2}\{(Cx_k + D\alpha_k - \tilde{u}_k)'W_u(Cx_k + D\alpha_k - \tilde{u}_k) + (\alpha_k - \tilde{e}_k)'W_e(\alpha_k - \tilde{e}_k)\} + \lambda'_{k+1}(Ax_k + B\alpha_k). \tag{A.9}$$

The first-order necessary conditions for a minimum (see Lewis, 1986) are given as

$$x_{k+1} = \frac{\partial H_k}{\partial \lambda_{k+1}}, \tag{A.10}$$

$$\lambda k = \frac{\partial H_k}{\partial x_k}, \tag{A.11}$$

$$0 = \frac{\partial H_k}{\partial \alpha_k}, \tag{A.12}$$

$$\lambda_N = \frac{\partial \phi}{\partial x_N}. \tag{A.13}$$

These are similar to the continuous-time first-order necessary conditions and, as before constitute the *state equation*, *co-state equation*, *stationarity condition*, and *boundary condition*, respectively. Again we note that the co-state equation evolves backward in time. Solving these yields the following expression for F:

$$\alpha_k = \Delta \begin{bmatrix} (D'W_uC)' \\ B \\ -(D'W_u)' \\ -W_e \end{bmatrix}' \begin{bmatrix} x_k \\ \lambda_{k+1} \\ \tilde{u}_k \\ \tilde{e}_k \end{bmatrix}, \tag{A.14}$$

where the adjoint equation is given by

$$\lambda_{k+1} = \Pi_{k+1}x_{k+1} - g_{k+1}. \tag{A.15}$$

Π_{k+1} is the solution to the discrete-time Ricatti equation:

$$\tilde{A}'(I - \Pi_{k+1}\tilde{B})^{-1}\Pi_{k+1}\tilde{A} - \Pi_k + \tilde{C} = 0 \tag{A.16}$$

and g_{k+1} is the solution to the difference equation

$$-g_k = -\tilde{A}'(I - \Pi_{k+1}\tilde{B})^{-1}g_{k+1} - \begin{bmatrix} (C'W_u(I + D\Delta D'W_u) + \tilde{A}'(I - \Pi_{k+1}\tilde{B})^{-1}B\Delta D'W_u)' \\ (C'W_uD\Delta W_e + \tilde{A}'(I - \Pi_{k+1}\tilde{B})^{-1}B\Delta W_e)' \end{bmatrix}' \begin{bmatrix} \tilde{u}_k \\ \tilde{e}_k \end{bmatrix}, \tag{A.17}$$

which are solved subject to the boundary conditions

$$\Pi_T = (I - C'PD\Delta B')^{-1}(C'PC + C'PD\Delta D'W_uC), \tag{A.18}$$

$$g_T = (I - C'PD\Delta B')^{-1}\begin{bmatrix} -(C'PD\Delta D'W_u + C'P)' \\ -(C'PD\Delta W_e)' \end{bmatrix}\begin{bmatrix} \tilde{u}_T \\ \tilde{e}_T \end{bmatrix}. \tag{A.19}$$

As with the continuous-time case, we note that the determination of F relies on us knowing a priori the on-line error and control sequences as the equations for g_k and Π_k develop backwards in time. In most situations this information is unavailable so we can extend the results to the infinite horizon, provided we make the assumption of the external inputs being constant.

It is well known that, following Theorem 1, providing that $\tilde{A}$, $\tilde{B}$, and $\sqrt{\tilde{C}}$ are stablisable and detectable, and that $\Pi(T) \geq 0$, then the solution of the Riccati equation converges to a constant value in the infinte horizon: a value identical to the solution of the discrete algebraic Ricatti equation. Importantly, this solution is stabilising for the off-line control loop. Thus in the infinite horizon, F is given by

$$\alpha_k = F\begin{bmatrix} x_k \\ \tilde{u}_k \\ \tilde{e}_k \end{bmatrix}, \tag{A.20}$$

$$F = (I - \Delta B'\Pi B)^{-1}\Delta \begin{bmatrix} (D'W_uC + B'\Pi A)' \\ -(D'W_u + B'(I - M)^{-1}\hat{U})' \\ -(W_e + B'(I - M)^{-1}\hat{E})' \end{bmatrix}', \tag{A.21}$$

where

$$\Delta := -(D'W_uD + W_e)^{-1}, \tag{A.22}$$

$$\tilde{A} := A + B\Delta D'W_uC, \tag{A.23}$$

$$\tilde{B} := B\Delta B', \tag{A.24}$$

$$\tilde{C} := C'W_uC + C'W_uD\Delta D'W_uC, \tag{A.25}$$

$$M := \tilde{A}'(I - \Pi\tilde{B})^{-1}, \tag{A.26}$$

$$\hat{U} := M\Pi B\Delta D'W_u + C'W_u + C'W_uD\Delta D'W_u, \tag{A.27}$$

$$\hat{E} := M\Pi B\Delta W_e + C'W_uD\Delta W_e, \tag{A.28}$$

and Π is the stabilising solution to the discrete-time algebraic Riccati equation:

$$\tilde{A}'(I - \Pi\tilde{B})^{-1}\Pi\tilde{A} - \Pi + \tilde{C} = 0. \tag{A.29}$$

which can also be written as

$$-\Pi + \tilde{A}'\Pi\tilde{A} - \tilde{A}'\Pi B(B'\Pi B + \Delta^{-1})^{-1}B'\Pi\tilde{A} + \tilde{C}. \tag{A.30}$$

It is important to realise that, as with the continuous time case, to arrive at this expression we have extended the horizon to infinity so we write F in purely algebraic terms. Otherwise, F would be written in terms of difference equations and would require a priori knowledge of the reference signal to be able to be computed off-line.

Once again, note that if $D = 0$, the expression for F reduces somewhat.

A.2. Discrete-time 2 DOF bumpless transfer

The discrete-time first-order necessary conditions are now applied to the discrete time 2 DOF case. As in the continuous time case, the feedback matrix, F, has access to the off-line controller states, the on-line control signal, the reference input, and the plant output. Proceeding in exactly the same manner as before, we derive the static feedback gain which will enable bumpless transfer.

The discrete 2 DOF off-line controller has the following state-space representation:

$$x_{k+1} = Ax_k + B_1\alpha_k + B_2y_k, \tag{A.31}$$

$$u_k = Cx_k + D_1\alpha_k + D_2y_k \tag{A.32}$$

and the cost function to be minimised is

$$J = \frac{1}{2}\sum_0^{T-1} [(u_k - \tilde{u}_k)'W_u(u_k - \tilde{u}_k) + (\alpha_k - r_k)W_e(\alpha_k - r_k)] + \frac{1}{2}z_u(T)'Pz_u(T). \tag{A.33}$$

Applying the first-order necessary conditions

$$\alpha = \Delta\begin{bmatrix}(D_2'W_uC)' \\ B_1 \\ (D_1'W_uD_2)' \\ -(D_1'W_u)' \\ -W_e\end{bmatrix}'\begin{bmatrix}x_k \\ \lambda_{k+1} \\ y_k \\ u_k \\ r_k\end{bmatrix}, \tag{A.34}$$

and the co-state, λ is obtained from

$$\lambda_{k+1} = \Pi_{k+1}x_{k+1} - g_{k+1}, \tag{A.35}$$

where Π_k and g_k are the solutions to

$$\tilde{A}'(I - \Pi_{k+1}\tilde{B})^{-1}\Pi_{k+1}\tilde{A} - \Pi_k + \tilde{C} = 0, \tag{A.36}$$

$$-\hat{A}g_{k+1} = -g_k + \begin{bmatrix}[C'W_uD_2 + C'W_uD_1\Delta D_1'W_uD_2 + \tilde{A}'(I - \Pi_{k+1}\tilde{B})^{-1}\Pi_{k+1}(B_2 + B_1\Delta D_1'W_uD_2)]' \\ -(C'W_u + C'W_uD_1\Delta D_1'W_u + \tilde{A}'(I - \Pi_{k+1}\tilde{B})^{-1}\Pi_{k+1}(B_1\Delta D_1'W_u)' \\ -(C'W_uD_1\Delta W_e + \tilde{A}'(I - \Pi_{k+1}\tilde{B})^{-1}\Pi_{k+1}(B_1\Delta W_e)'\end{bmatrix}'\begin{bmatrix}y_k \\ \tilde{u}_k \\ r_k\end{bmatrix}. \tag{A.37}$$

The terminal conditions are

$$\Pi_T = (I - C'PD_1\Delta B_1')^{-1}(C'PC + C'PD_1\Delta D_1W_uC), \tag{A.38}$$

$$g_T = (I - C'PD_1\Delta B_1')^{-1}\begin{bmatrix}(C'PD_2 + C'PD_1\Delta D_1'W_uD_2)' \\ -(C'P + C'PD_1\Delta D_1'W_u)' \\ -(C'PD_1\Delta W_e)'\end{bmatrix}\begin{bmatrix}y_N \\ \tilde{u}_N \\ r_N\end{bmatrix}. \tag{A.39}$$

In the infinite horizon, α is given by

$$\alpha = F\begin{bmatrix}x_k \\ y_k \\ \tilde{u}_k \\ r_k\end{bmatrix}, \tag{A.40}$$

where the constant matrix F is given by

$$F = (I - \Delta B_1' \Pi B_1)^{-1} \Delta \begin{bmatrix} (D_1' W_u C + B_1' \Pi A)' \\ (D_1' W_u D_2 + B_1' \Pi B_2 + B_1'(I-M)^{-1} \hat{Y})' \\ -(D_1' W_u + B_1'(I-M)^{-1} \hat{U})' \\ -(W_e + B_1'(I-M)^{-1} \hat{R})' \end{bmatrix}' \tag{A.41}$$

and

$$\Delta := -(D_1' W_u D_1 + W_e)^{-1}, \tag{A.42}$$

$$\tilde{A} := A + B_1 \Delta D_1' W_u C, \tag{A.43}$$

$$\tilde{B} := B_1 \Delta B_1', \tag{A.44}$$

$$\tilde{C} := C' W_u C + C' W_u D_1 \Delta D_1' W_u C, \tag{A.45}$$

$$M := \tilde{A}'(I - \Pi \tilde{B})^{-1}, \tag{A.46}$$

$$\hat{Y} := C' W_u D_2 + C' W_u D_1 \Delta D_1' W_u D_2 + M \Pi (B_2 + B_1 \Delta D_1' W_u D_2), \tag{A.47}$$

$$\hat{U} := C' W_u + C' W_u D_1 \Delta D_1' W_u + M \Pi B_1 \Delta D_1' W_u, \tag{A.48}$$

$$\hat{R} := C' W_u D_1 \Delta W_e + M \Pi B \Delta D_1' W_u, \tag{A.49}$$

$$\tag{A.50}$$

where Π is the stablising solution to the discrete-time ARE:

$$\tilde{A}(I - \Pi \tilde{B})^{-1} \Pi \tilde{A} - \Pi + \tilde{C} = 0. \tag{A.51}$$

As before, the expression for the subcontroller, F, simplifies somewhat if the direct feedthrough term is zero.

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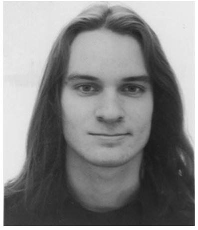
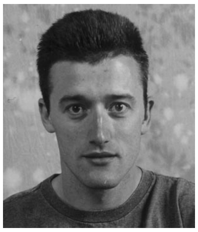


Matthew Turner was born in 1975, Corby, England. After receiving the B.Eng. degree in Electrical and Electronic Engineering from the University of Surrey in 1996, he joined the Control Systems Research Group at the University of Leicester, where he is working towards a Ph.D. in Control Engineering. During the course of his Ph.D. research he has designed, and implemented successfully, advanced control systems on a helicopter. Other topics of his research include robust control, flight control and linear systems with input nonlinearities. In 1996 he became an Associate Member of the Institution of Electrical Engineers. He currently resides in Desborough, which is a small town near Leicester.



Daniel Walker is a lecturer in Engineering at Leicester University. His main research interest is control theory and its application to a variety of electrical, mechanical and aerospace engineering problems. In 1985 he received a 1st Class B.A. degree at Oxford University, having in 1981 won an Open Scholarship to read Engineering Science. He was awarded the Schuftan prize for top graduating Mechanical Engineer. He received the degree of Ph.D. in Electrical Engineering (Control) at Imperial College, where from 1985 to 1989 he pursued research into discrete-time H_∞ optimization. In 1987 he was a visiting scholar at the VMK (Computational Mathematics and Cybernetics) Department of Moscow State University, under the terms of the Anglo-Soviet Cultural Agreement. From 1989 and 1992 he was a Research Associate at Leicester University, working on robust control of helicopters. This is still a major interest of his; several H_∞ controllers have recently been tested successfully on a fly-by-wire helicopter.