DSA Assignment -3

1. Z - Transform

1. z-transform and ROC :-

1. $x(n) = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\}$

Starts at 5, $\Rightarrow$ $x(0) = 5$ $x(1) = 7 \ldots$

$x(-1) = 4$

$x(-2) = 2$

$$\therefore \quad X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= 2z^2 + 4z + 5 + 7z^{-1} + 1 \cdot z^{-3}$$

$$\boxed{X(z) = (2z^2+4z+5) + \frac{(7z^2+1)}{z^3}}$$

Region of Convergence (ROC) :-

$\rightarrow$ Values of z for which $X(z)$ exists.

Only critical point is $z^3 = 0$

$$\therefore \boxed{\text{ROC} \Rightarrow z \neq 0}$$

2. $x(n) = a \cdot u(n) + b \cdot u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty} a \cdot u(n) \cdot z^{-n} + \sum_{n=-\infty}^{\infty} b \cdot u(-n-1) \cdot z^{-n}$$

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$= a \cdot \sum_{n=0}^{\infty} z^{-n} + b \cdot \sum_{n=-\infty}^{-1} z^{-n}$$

$$= a \cdot \frac{1}{1-z^{-1}} + b \cdot \frac{z}{1-z} \rightarrow |z| < 1$$

$$\boxed{X(z) = \frac{a}{1-z^{-1}} + \frac{bz}{1-z}} \rightarrow |z| < 1$$

$$|z^{-1}| < 1$$

$\therefore$ ROC,

~~$1-z^{-1} \neq 0$, $1-z \neq 0$,~~

~~$z \neq 1$ and $\frac{1}{z} \rightarrow z \neq 0$~~

$\therefore$ ~~$\boxed{z \neq \{0, 1\}}$~~

$|z| < 1$ and $|z| > 1$

$\therefore$ $z = \phi$ R.O.C doesn't exist.

2. Given,

$$x_1(n) = 3\,\delta(n) + 2\,\delta(n-1)$$

$$x_2(n) = 2\,\delta(n) - \delta(n-1)$$

$$\delta(n) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$

1. z-transform of:

$$x_1(n) * x_2(n) = y(n)$$

$$y(n) = \sum_{K=-\infty}^{\infty} x_1(K) \cdot x_2(n-K)$$

$$= \sum_{K} \big(\underbrace{3\,\delta(K)}_{K=0} + \underbrace{2\,\delta(K-1)}_{K=1}\big) \cdot \big(2\,\delta(n-K) - \delta(n-K-1)\big)$$

$$= 3 \cdot \big(2\,\delta(n) - \delta(n-1)\big) + 2 \cdot \big(2\,\delta(n-1) - \delta(n-2)\big)$$

$$\underline{y(n) = 6\,\delta(n) + \delta(n-1) - 2\,\delta(n-2)}$$

$$\therefore \quad Y(z) = \sum_{n=-\infty}^{\infty} y(n) \cdot z^{-n}$$

$$= 6 \cdot 1 + z^{-1} - 2 \cdot z^{-2}$$

$$\boxed{Y(z) = 6 + \frac{z-2}{z^2}}, \quad z \neq 0$$

2. Determining convolution sum using z-transform:-

$$x(n) = x_1(n) * x_2(n)$$

$\downarrow$ z-transform

$$Z\{x(n)\} = Z\{x_1(n) * x_2(n)\}$$

$$\boxed{X(z) = X_1(z) \cdot X_2(z)} \quad (1)$$

($\because$ Convolution in one domain is multiplicat in another)

$$x_1(n) = 3\delta(n) + 2\delta(n-1)$$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (3\delta(n) + 2\delta(n-1)) \cdot z^{-n}$$

$$\underline{X_1(z) = 3 + 2z^{-1}}$$

$$x_2(n) = 2\delta(n) - \delta(n-1)$$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) \cdot z^{-n}$$

$$\underline{X_2(z) = 2 - z^{-1}}$$

$\therefore$ From (1),

$$X(z) = X_1(z) \cdot X_2(z)$$

$$= (3 + 2z^{-1}) \cdot (2 - z^{-1})$$

$$= 6 + z^{-1} - 2z^{-2}$$

$$az^{-k} \xleftrightarrow{z} a \cdot \delta(n-k)$$

$$\therefore \quad \boxed{x(n) = 6\delta(n) + \delta(n-1) - 2\delta(n-2)}$$

$\equiv x(n) = \{6, 1, -2\}$

3. Pole-zero Plot :-

Some properties of ROC to know before solving the problem,

Property 1 :- If $x[n]$ is a right sided sequence and if the function converges for $|z| = r_0$, then all finite values of z. for which $|z| > r_0$ will also be in ROC.

Proof :-

Given, converges at $|z| = r_0$.

$$X(z) = \sum_{n=N_0}^{\infty} x(n) \cdot z^{-n} \qquad x(n) = \begin{cases} \text{—}, & n \geq N_0 \\ 0, & n < N_0 \end{cases}$$

$$|x(n) \cdot r_0^{-n}| < \infty \rightarrow \text{Given.}$$

↳ absolutely summable.

Now, for $|z| = r_1$, $r_1 > r_0$,

$$r_1^{-n} < r_0^{-n}$$

$$\Rightarrow |x(n) \cdot r_1^{-n}| < |x(n) \cdot r_0^{-n}| < \infty$$

$$|x(n) \cdot r_1^{-n}| < \infty \Rightarrow \text{converges.}$$

$\therefore$ ~~ROC is~~ $|z| > r_0$ will also be in ROC.

For right-sided signals,

$$X(z) = \sum_{n=N_1}^{\infty} x(n) z^{-n}$$ where N_1 is

finite, may be positive or negative.

If N_1 is negative,

summation includes terms with positive powers of z, which become unbounded, as $|z| \to \infty$. $\therefore$ In general, ROC will not include infinity.

For <u>Causal sequence</u>:- (Special case).

$$n \leq 0 \Rightarrow x[n] = 0$$

$\therefore$ N_1 will be Non-negative

Consequently, ROC will include $z = \infty$.

<u>Property 2</u>:- If $x(n)$ is a left sided sequence, and if circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in ROC

<u>Proof</u>:-

Left sided,

$$X(z) = \sum_{n=-\infty}^{N} x(n) z^{-n}$$

where N can be positive or negative. If N is positive, then transform includes negative powers of z, which will become unbounded as $|z| \to 0$. Consequently, for left-sided sequences, ROC will not include $|z| = 0$.

For Anti-causal system (Special case) :-

$$x(n) = 0, \quad \forall\, n > 0$$

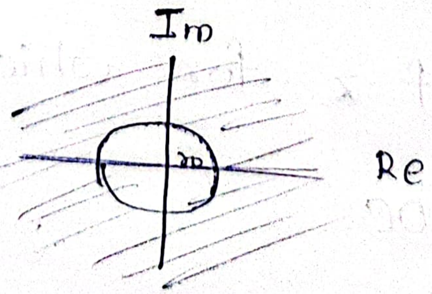
$$\Rightarrow N \leq 0.$$

then ROC will include $|z| = 0$.

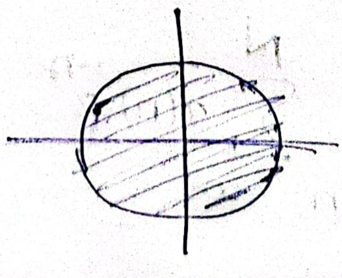
Property-3 :- If ~~ROC~~ $x(n)$ is two sided, and if circle $|z| = r_0$ is in the ROC, then ROC will be a ring in the z-plane that includes $|z| = r_0$.

$x(n)$ = Sum of right-sided + Left-sided.

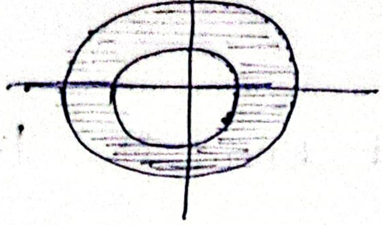
Right-sided $\Rightarrow$



Left-sided $\Rightarrow$



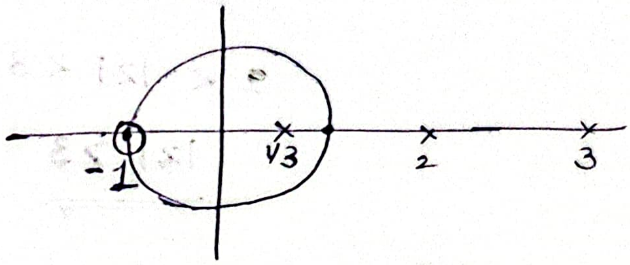
$\therefore$ Two-sided $\Rightarrow$



$r_1 < |z| < r_2$

Coming to the problem statement,

1) Given that converges for $|z| = 1$



Poles $\Rightarrow$ $1/3, 2, 3$

zeroes $\Rightarrow$ -1

$\times \Rightarrow$ Poles

$o \Rightarrow$ zeroes.

Given that ROC includes $|z| = 1$. ~~ROC shoul~~

$\therefore$ ROC should be $\frac{1}{3} < |z| < 2$. $\frac{1}{3}$ and 2 are boundaries for $|z|$, because they are Poles and ROC doesn't include Poles ($\because$ value $= \infty$)

From the properties of ROC,

$|z| > r_0 \Rightarrow$ Right-sided.

$|z| < r_0 \Rightarrow$ Left-sided

$r_1 < |z| < r_2 \Rightarrow$ Two-sided.

$\therefore$ $\frac{1}{3} < |z| < 2$, $h[n]$ is double-sided.

2) Given,

we dont know if $|z|=1$ is in ROC.

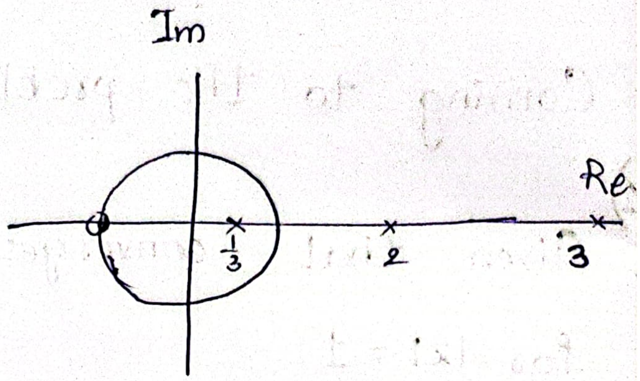
$\therefore$ Possible ROCs are,

$|z| < \frac{1}{3}$

$\frac{1}{3} < |z| < 2$

$2 < |z| < 3$

$|z| > 3$



Stability of system :-

A system is said to be stable if it includes (or) cuts the unit circle, $|z|=1$.

(i) Stable and Causal system:

Causal is a special case of right-sided sequence.

$\therefore$ ROC should be of form

$|z| > r_0$.

Only possibility is $|z|>3$.

But this not stable, as it doesnt include $|z|=1$.

$\therefore$ No-answer.(ROC)

(ii) Stable but not Causal :-

Only Stable ROC is,

$$\frac{1}{3} < |z| < 2$$

$\because$ not of form $|z| > r_0$, not-Causal.

$\therefore$ $\frac{1}{3} < |z| < 2$

(iii) Causal but unstable :-

$$|z| > 3$$

~~Includes~~ Is of form $|z| > r_0$, but

doesn't include unit circle.

2. LTI Analysis : Z-Transform

1) Given, $h[n] = a^n \cdot u[n]$, $|a| < 1$

$x[n] = u[n]$

$y[n] = ?$

Method-1: Using Z-Transform.

$$x(n) \rightarrow \boxed{\begin{matrix} LTI \\ h(n) \end{matrix}} \rightarrow y(n)$$

$$y(n) = x(n) * h(n)$$

$\Big\downarrow$ Z-Transform

$$\boxed{Y(z) = X(z) \cdot H(z)}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} \qquad H(z) = \sum_{n=-\infty}^{\infty} a^n \cdot u[n] \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u[n] \cdot z^{-n} \qquad = \frac{1}{1-az^{-1}}, \ |z|>|a|$$

$$= \frac{1}{1-z^{-1}}, \ |z|>1$$

$$\therefore Y(z) = \frac{1}{1-z^{-1}} \cdot \frac{1}{1-az^{-1}}, \quad |z|>1, \ |z|>|a|$$

$\because |a|<1$, R.O.C = $|z|>1$

$$= \frac{1}{1-(a+1)z^{-1}+az^{-2}} \cdot \frac{z^2}{z^2}$$

$$Y(z) = \frac{z^2}{z^2-(a+1)z+a}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{z}{z^2-(a+1)z+a}$$

$z^2 - (a+1)z + a,$

$$z = \frac{(a+1) \pm \sqrt{(a+1)^2 - 4a}}{2}$$

$a^2 + 2a - 2a$

$$z = \frac{(a+1) \pm (a-1)}{2}$$

$$z = a, 1$$

$$\therefore \frac{Y(z)}{z} = \frac{z}{(z-a)(z-1)}$$

$$= \frac{A}{z-a} + \frac{B}{z-1}$$

$$Az - A + Bz - aB = z$$

$$(A+B) = 1$$

$$A + aB = 0$$

$$B(1-a) = 1 \Rightarrow B = \frac{1}{1-a}$$

$$\therefore A = \frac{-a}{1-a}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{-a}{1-a} \cdot \frac{1}{z-a} + \frac{1}{1-a} \cdot \frac{1}{z-1}$$

$$Y(z) = \frac{-a}{1-a} \cdot \frac{z}{z-a} + \frac{1}{1-a} \cdot \frac{z}{z-1}$$

$$\frac{z}{z-a} \longleftrightarrow a^n \cdot u[n]$$

$$\therefore y[n] = \frac{-a}{1-a} \cdot a^n \cdot u[n] + \frac{1}{1-a} \cdot u[n]$$

$$\boxed{y[n] = u[n] \cdot \left(\frac{1 - a^{n+1}}{1-a} \right)}$$

Method-2 : Using Convolution.

$$y[n] = x[n] * h[n]$$

$$= \sum_{m=-\infty}^{\infty} x[n-m] \cdot h[m]$$

$$= \sum_{m=-\infty}^{\infty} u[n-m] \cdot a^{m} \cdot u[m]$$

$$= \sum_{m=0}^{\infty} u[n-m] \cdot a^{m}$$

$$= \sum_{m=0}^{n} a^{m} \quad , n \geq 0 \qquad = 0 , \; n < 0$$

$$y[n] = 1 + a + a^{2} \cdots a^{n}$$

$$\boxed{y[n] = \frac{1-a^{n+1}}{1-a}} \quad , n \geq 0 \quad , 0 , n < 0$$

$$\equiv \boxed{y[n] = \left(\frac{1-a^{n+1}}{1-a}\right) u[n]}$$

2) Impulse response of:

$$y[n] = 0.1x[n] + 0.2x[n-1] + 0.3x[n-2] + 0.4x[n-4]$$

$\Big\downarrow$ z-transform

$$Y(z) = 0.1 \cdot X(z) + 0.2 \cdot z^{-1} \cdot X(z) + 0.3\, z^{-2} \cdot X(z) + 0.4 \cdot z^{-4} \cdot X(z)$$

$$\frac{Y(z)}{X(z)} = 0.1 + 0.2\, z^{-1} + 0.3\, z^{-2} + 0.4 \cdot z^{-4}$$

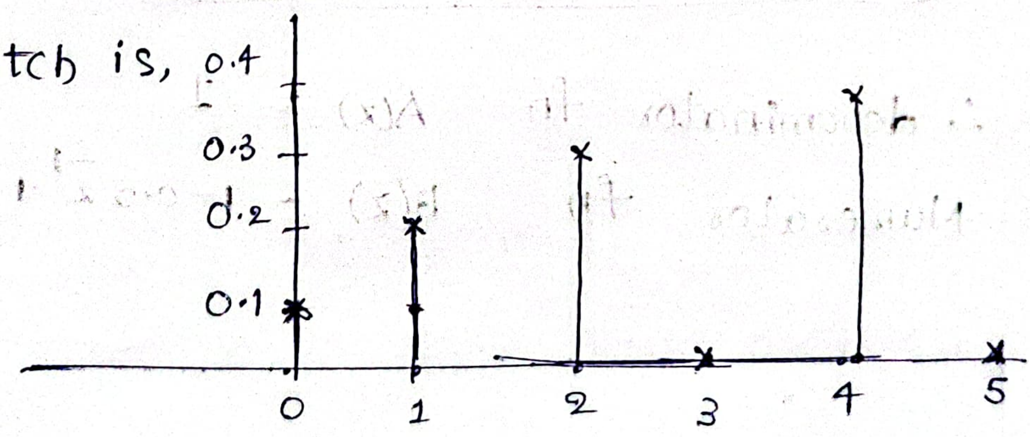
$$H(z) = 0.1 + 0.2\, z^{-1} + 0.3\, z^{-2} + 0.4\, z^{-4}$$

$\Big\downarrow$ Inv. transform

$$\boxed{h(n) = 0.1 \cdot \delta(n) + 0.2\, \delta(n-1) + 0.3\, \delta(n-2) + 0.4 \cdot \delta(n-4)}$$

$$z^{-k} \xleftrightarrow{z} \delta(n-k)$$

$\therefore$ Sketch is,



3. Given,

$$y[n] = x[n] - 0.5\, x[n-1] + 0.36\, x[n-2]$$

Required :- $H(z)$, $A(z)$ (den. fn) $B(z)$ (num. fn)

$\therefore$ Applying z-Transform,

$$Y(z) = X(z) - 0.5\, z^{-1}\, X(z) + 0.36\, z^{-2} X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = 1 - 0.5\, z^{-1} + 0.36\, z^{-2}$$

$$\cancel{H(z) = \frac{z^{2} - 0.5z + 0.36}{z^{2}}}$$

$$\boxed{H(z) = \frac{1 - 0.5\, z^{-1} + 0.36\, z^{-2}}{1 + 0.}}$$

$\therefore$ denominator fn $A(z) = 1$

Numerator fn $B(z) = 1 - 0.5\, z^{-1} + 0.36\, z^{-2}$