2022114014 **Assignment 3 : DSA** Saketh Reddy Vemula

Question 1 : Z-transform:

① i) $x[n] = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\}$

the arrow represents the origin at '0'.

then $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$.

$= 2 \cdot z^{2} + 4 \cdot z^{1} + 5 \cdot z^{0} + 7 \cdot z^{-1} + 0 \cdot z^{-2} + 1 \cdot z^{-3}$

$X(z) = 2z^{2} + 4z + 5 + 7z^{-1} + 1 \cdot z^{-3}$.

$$\boxed{\therefore X(z) = 2z^{2} + 4z + 5 + 7z^{-1} + z^{-3}}$$

ROC! values for which $X(z)$ is valid/exists.

ROC include all z expect 0 $\Rightarrow$ $\boxed{z \neq 0}$ ← ROC of $x[n]$.

2) $x(n) = a^n u(n) + b^n u(-n-1)$.

Applying z-transform using the equation $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

$$X(z) = \sum_{n=-\infty}^{\infty} \left(a^n u(n) + b^n u(-n-1) \right) z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} a^n \cdot u(n) z^{-n} + \sum_{n=-\infty}^{\infty} b^n u(-n-1) z^{-n}.$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= \left(a^0 z^{-0} + a^1 z^{-1} + a^2 z^{-2} + \cdots \right) + \left(b^{-1} z + b^{-2} z^2 + \cdots \right).$$

$$= \underbrace{\frac{1}{1 - a z^{-1}}}_{\text{I}} + \underbrace{\frac{b^{-1} z}{1 - b^{-1} z}}_{\text{II}}.$$

I $\Rightarrow$ has ROC $|z| > |a|$.

II $\Rightarrow$ has ROC $|z| < |b|$.

$$\therefore X(z) = \begin{cases} \dfrac{1}{1 - a z^{-1}} + \dfrac{b^{-1} z}{1 - b^{-1} z} & ; \ |a| < |z| < |b| \\ \Phi & ; \ |a| \geq |b|. \end{cases}$$

$\therefore$ ROC of $x(n)$ is $|a| < |z| < |b|$

(2) $x_1(n) = 3\delta(n) + 2\delta(n-1)$

$x_2(n) = 2\delta(n) - \delta(n-1)$.

1) To find z-transform of convolution $X(z) = Z(x_1(n) * x_2(n))$.

we know the property of convolution ~~of~~ in z-transform

$$x_1(n) * x_2(n) \xleftrightarrow{Z.T} X_1(z) \cdot X_2(z)$$

ie, convolution in time domain is ~~fr~~ multiplication in frequency domain

$X_1(z) = ?$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} \quad \text{where } z = re^{j\omega}.$$

$$= \sum_{n=-\infty}^{\infty} (3\delta(n) + 2\delta(n-1)) z^{-n}.$$

$$= (3)(1)(z^{-0}) + (2)(1)(z^{-1}).$$

$$\boxed{X_1(z) = 3 + \frac{2}{z}} \qquad \boxed{|z| > 0}$$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} = \sum_{n=-\infty}^{\infty} (2\delta(n) - \delta(n-1)) z^{-n} = \sum_{n=-\infty}$$

$$= (2)(1)z^{-0} + (-1)(1)(z^{-1})$$

$$= 2 + \frac{-1}{z} = 2 - \frac{1}{z}.$$

~~x_2~~

$$\boxed{X_2(z) = 2 - \frac{1}{z}} \qquad \boxed{|z| > 0}.$$

$\therefore \; X(z) = Z(x_1(n) * x_2(n)) = X_1(z)\cdot X_2(z)$

$= \left(3 + \frac{2}{z}\right)\cdot\left(2 - \frac{1}{z}\right) = 6 - \frac{3}{z} + \frac{4}{z} - \frac{2}{z^2}$

$= 6 + \frac{1}{z} - \frac{2}{z^2} \Rightarrow \frac{6z^2 + z - 2}{z^2}.$

$= \frac{6z^2 + 4z - 3z - 2}{z^2} = \frac{2z(3z+2) - 1(3z+2)}{z^2}$

$$\boxed{X(z) = \frac{(2z-1)(3z+2)}{z^2}.}$$

$\cancel{X(z) = \left(2 - \frac{1}{z}\right)}$

$$\boxed{X(z) = 6 + z^{-1} - 2z^{-2}} \qquad (z \neq 0)$$

b 2) Applying the Inverse - z transform : ~~& using~~

we know that, $\delta(n) \xleftrightarrow{ZT} 1$

$\delta(n-1) \xleftrightarrow{z-1} z^{-1}(1).$

$\delta(n-m) \xleftrightarrow{ZT} z^{-m}(1)$

$\therefore \; X(z) = 6 + z^{-1} - 2z^{-2}$

$\therefore$ Inverse Z-transform of $X(z)$ is $6\delta(n) + \delta(n-1) - 2\delta(n-2).$

$\therefore \; x(n) = x_1(n) * x_2(n) = 6\delta(n) + \delta(n-1) - 2\delta(n-2). \Rightarrow \boxed{x[n] = \{6, 1, -2\}}$

QUESTION-1: Z-transform:

(3) 1).
We know the following Properties of ROCs based on whether signal left/Right/both sided.

Property - 1:
"If the signal $x[n]$ is a left sided sequence and $|z| = r_0$ is in the ROC, (i.e., the signal converges at $|z| = r_0$), then all values of z for which $|z| \in (0, r_0)$ will also be in ROC."

Property - 2:
"If the signal is $x[n]$ is a right sided sequence and $|z| = r_0$ is in the Region of Convergence, ROC, (i.e., the signal converges at $|z| = r_0$), then ~~all~~ ~~for~~ all finite values of z, for which $|z| > r_0$ will also be in ROC"

Property - 3:
"If signal $x[n]$ is a two sided signal, and if the circle $|z| = r_0$ is in the ROC, then ROC will be a ring in the z-plane that includes $|z| = r_0$."

$r_1 < |z| < r_2$

PROOFS:

(i) Property - 1:

let, the left sided, be $x[n]$.

then $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{N} x[n] z^{-n}$.

where, N can be +ve/-ve.

If N is +ve: then transform includes negative powers of z, which will become unbounded as $|z| \to 0$. Consequently, for left sided sequence ROC will ~~be~~ not include $|z| = 0$.

But for Anti-causal system:

$x(n) = 0 ; \forall n > 0$

$\Rightarrow N \leq 0$

then ROC will include $|z| = 0$. : A special case of Property - 1.

(ii) Proof of Property-2 :

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}. \qquad x[n] = \begin{cases} x[n] ; & n \geq N_0 \\ 0 ; & n < N_0 \end{cases}$$

$$= \sum_{n=N_0}^{\infty} x[n] z^{-n}.$$

$\Rightarrow |x[n] \cdot r_0^{-n}| < \infty$ for ROC Convergence

Now, let $r_1 > r_0$ .. $|z| = r_1$ (w.l.o.g).
$r_1^{-n} < r_0^{-n}$.

$\Rightarrow |x[n] r_1^{-n}| < |x[n] r_0^{-n}| < \infty.$

$|x[n] r_1^{-n}| < \infty \Rightarrow$ Converges at 'r_1' also

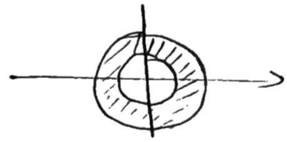
$\therefore |z| > r_0$ will also be in ROC.

for Right sided signals,
$X(z) = \sum_{n=N_1}^{\infty} x[n] \cdot z^{-n}$ where N_1 is finite, may be positive or negative

If N_1 is $-$ve, then the summation will include terms with positive powers of z, which will become unbounded as $|z| \to \infty$.
$\therefore$ In general, ROC will not include '∞'.

★ But for Causal Systems, $n \leq 0 : \Rightarrow x[n] = 0$.
$\therefore N_1$ will be non-negative $\therefore$ ROC will include $z = \infty$

(iii) for $x[n]$ double sided signal, will be Intersection of both constituent ROCs of left sided and Right sided.



1) Given, $H(z)$ is known to converge for $|z| = 1$.
To find ROC and state if $h[n]$ is left / Right / double-sided

from figure, Poles = $\{ 1/3, 2, 3 \}$
Zeroes = $\{ -1 \}$.

$\frac{1}{3}$ and 2 are Poles $\therefore$ ROC doesn't include Poles since, at Poles, the value of $X(z)$ becomes infinite.

$\therefore$ ROC should be $\frac{1}{3} < |z| < 2$.

Using Properties of ROC, $|z| > r_0 \Rightarrow$ Right Sided Signal
$|z| < r_0 \Rightarrow$ left sided Signal
and $|z| \in (r_1, r_2) \Rightarrow$ double Sided Signal.

$\because$ ROC of Signal is $\frac{1}{3} < |z| < 2 \Rightarrow$ $h[n]$ must be a double-sided Signal

ROC $= (\frac{1}{3}, 2)$. Double-Sided

2) Given, $H(z)$ may or maynot converge at $|z| = 1$.

Then Possibles ROCs include.
ROC$_1$: $|z| < \frac{1}{3}$
ROC$_2$: $\frac{1}{3} < |z| < 2$
ROC$_3$: $2 < |z| < 3$
ROC$_4$: $|z| > 3$.

$\therefore$ Four different possibilities are possible

(i) A Stable and Causal System :

Causal is a special case of in Right sided Sequences
Therefore, ROC should be in the form $|z| > r_0$.
$\therefore$ only ROC$_4$ is possible : $|z| > 3$.
But ROC$_4$ doesn't include Unit Circle. Hence it is not stable
therefore None of 4 ROCs satisfies

(ii) Stable but not Casual :

only ROC which unit circle is ROC$_2$ $\therefore$ $\frac{1}{3} < |z| < 2$.
which is also not-causal. $\because$ It is not in form $|z| > r_0$ for some r_0.

$\therefore$ ROC$_2$ satisfies

(iii) Causal but unstable :

$\underbrace{|z| > 3}$ is causal becoz, it is in form $|z| > r_0$ for some r_0.

ROC$_4$:
It doesn't include Unit Circle $\therefore$ ROC$_4$ satisfies

Question - 2 : LTI Analysis : Z-transform.

(1) Given : Impulse Response $h[n] = a^n \cdot u[n]$, $|a| < 1$.

Input $x[n] = u[n]$.

Applying Z-transform on both :

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = z^{-0} + z^{-1} + \cdots$$
$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$H(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = a^0 z^{-0} + a^1 z^{-1} + \cdots$$
$$= \frac{1}{1-az^{-1}}.$$

$\therefore$ ~~Y(z) =~~

we know, $y[n] = x[n] * h[n]$

Also, $Y(z) = X(z) \cdot H(z)$. {$\because$ Convolution in time domain is Multiplication in frequency domain}

$$\Rightarrow Y(z) = \frac{1}{1-z^{-1}} \cdot \frac{1}{1-az^{-1}} = \frac{1}{(1-z^{-1})(1-az^{-1})}.$$

$$Y(z) = \frac{1}{(a-1)} \left[\frac{a}{1-az^{-1}} - \frac{1}{1-z^{-1}} \right]. \longrightarrow (1)$$

~~Y(z) =~~

~~Inverse~~ Taking Inverse Transform, of (1), we get

$$y[n] = \frac{1}{(a-1)} \left[a^{n+1} \cdot u[n] - u[n] \right] = \frac{1}{(a-1)} \left((a^{n+1} - 1) u[n] \right).$$

for stability. ROC : $|az^{-1}| < 1$ & $|z^{-1}| < 1$

$$\Downarrow$$

$|z| > |a|$ $\quad$ $|z| > 1$

$\therefore$ If $|a| < 1$, ~~|z| > |a|~~ $|z| > |a|$ for stability

If $|a| > 1$, $|z| > 1$ for stability

$\therefore$ output : $\boxed{y[n] = \frac{1}{(a-1)} \left((a^{n+1} - 1) u[n] \right)}$

2 Given, Shift-Invariant System:

$$y(n) = 0.1x(n) + 0.2x(n-1) + 0.3x(n-2) + 0.4x(n-4). \longrightarrow (1)$$

By applying z-transform on both sides.. we get,

$$Y(z) = 0.1X(z) + 0.2X(z)\cdot z^{-1} + 0.3z^{-2}X(z) + 0.4z^{-4}X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = 0.1 + 0.2z^{-1} + 0.3z^{-2} + 0.4z^{-4} = H(z).$$

we know,

$$\delta(n) \xleftrightarrow{Z.T} 1.$$

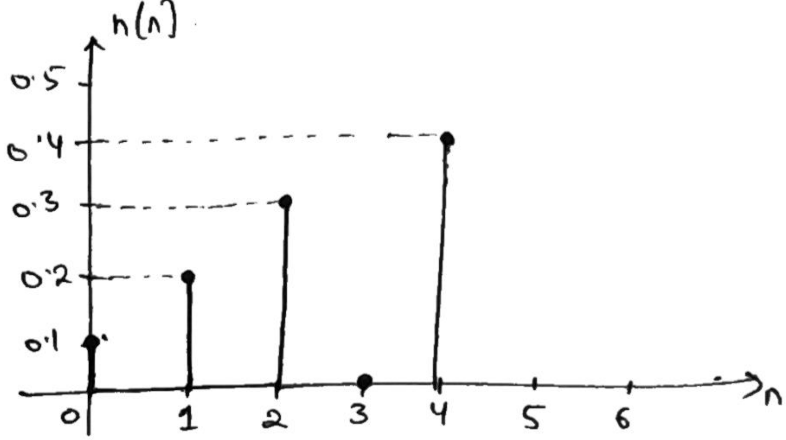
$$\delta(n-n_0) \xleftrightarrow{Z.T} z^{-n_0}(1) = z^{-n_0}$$

$\therefore H(z) = 0.1 + 0.2z^{-1} + 0.3z^{-2} + 0.4z^{-4}$; Applying Inverse Z-Transform, we get,

$$\Rightarrow h(n) = 0.1\delta(n) + 0.2\delta(n-1) + 0.3\delta(n-2) + 0.4\delta(n-4)$$

$$\Rightarrow \boxed{h(n) = \{0.1, 0.2, 0.3, 0, 0.4\}}$$

$\therefore$ Impulse Response of the system is h(n) is:



Impulse Response of the System y(n).

(3) Given difference equation:

$$y(n) = x[n] - 0.5x[n-1] + 0.36x[n-2] \quad —(1)$$

To find, transfer function $H(z)$ and denominator Polynomial $A(z)$, numerator Polynomial $B(z)$.

Applying z-transform on both sides in equation (1):

$$Y(z) = X(z) - 0.5X(z) \cdot z^{-1} + 0.36 \cdot z^{-2} X(z).$$

$$\left\{ \begin{array}{l} \text{Since, } x(n) \xleftrightarrow{ZT} X(z) \\ \text{then } x[n-n_0] \xleftrightarrow{ZT} z^{-n_0} X(z) \end{array} \right\}$$

$$\frac{Y(z)}{X(z)} = 1 - 0.5z^{-1} + 0.36z^{-2}.$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 - 0.5z^{-1} + 0.36z^{-2}.$$

$\therefore$ transfer function $H(z) = 1 - 0.5z^{-1} + 0.36z^{-2}$

$$H(z) = \frac{B(z)}{A(z)} = \frac{1 - 0.5z^{-1} + 0.36z^{-2}}{1}$$

$\therefore$ Numerator Polynomial $B(z) = 1 - 0.5z^{-1} + 0.36z^{-2}$

Denominator Polynomial: $A(z) = 1$.

$$\boxed{\begin{array}{l} \therefore \text{transfer function } H(z) = 1 - 0.5z^{-1} + 0.36z^{-2} \\ \text{Numerator function } B(z) = 1 - 0.5z^{-1} + 0.36z^{-2} \\ \text{Denominator function } A(z) = 1 \end{array}}$$