Name:

USC ID:

Notes:

- Write your name and ID number in the spaces above.
- No books, cell phones or other notes are permitted. Only one letter size cheat sheet (back and front) and a calculator are allowed.
- Problems are not sorted in terms of difficulty. Please avoid guess work and long and irrelevant answers.
- Show all your work and your final answer. Simplify your answer as much as you can.
- Open your exam only when you are instructed to do so.
- The exam has 5 questions, 11 pages, and 20 points extra credit. However, your grade cannot exceed 100/100.

Problem	Score	Earned
1	30	
2	25	
3	25	
4	20	
5	20	
Total	120	

1. The following least squares liner regression model was fitted to a sample of 25 students using data obtained at the end of their senior year in college. The aim was to explain students' weight gains:

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_3 x_3$$
$$\hat{y} = 7.35 + 0.653 x_1 - 1.345 x_2 + 0.613 x_3$$

The standard errors are:

$$\mathrm{SE}(\hat{\beta}_1) = 0.189$$
$$\mathrm{SE}(\hat{\beta}_2) = 0.565$$
$$\mathrm{SE}(\hat{\beta}_3) = 0.243$$

- $\hat{y}$: weight gained, in pounds, during senior year
- x_1: average number of meals eaten per week
- x_2: average number of hours of exercise per week
- x_3: average number of beers consumed per week

(a) Interpret the estimated coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$

(b) Test, at the 2% level, the null hypothesis that the true coefficient on the variable x_3 is 0 against the alternative that it is not 0.

(c) Find and interpret a 99.8% confidence interval for the parameter β_1.

(d) If for the model, SSR=79.2 (Regression Sum of Squares) and SSE = 45.9 (Residual Sum of Squares), test the hypothesis that all the coefficients of the model are 0 (test overall significance of the model) using $\alpha = 1\%$. .

a) $\hat{\beta}_0$: The amount of weight gained not explained by the predictors (i.e. even when $x_1 = x_2 = x_3 = 0$, $\hat{y} = \hat{\beta}_0$)

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3 $\hat{\beta}_1$: the average change in weight

Solution:

gained, when the average number of meals per week is increased by one unit.

(b) $t = \dfrac{\hat{\beta}_3}{SE(\hat{\beta}_3)} \approx \dfrac{0.613}{.243} = 2.523$

$t_{n-p-1,\alpha/2} = t_{25-3-1,\,.01} = t_{21,0.01} = 2.518$

$\Rightarrow$ Reject the null. x_3 is statistically significant

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(c) $\hat{\beta}_1 \pm t_{n-p-1, \alpha/2} SE(\hat{\beta}_1)$

$\alpha = 1 - .998 = .002$

$t_{21, .001}$ $= 3.527$

$.653 \pm 3.527 (.189)$

$= [-.014 , 1.320]$

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-0.0136 1.3196

$$F = \frac{\frac{SSR}{p}}{\frac{SSE}{n-p-1}} = \frac{\frac{79.2}{3}}{\frac{45.9}{21}} = 12.078$$

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$F_{3,21,.01} = 4.874$

We reject the null.

2. Consider the following dataset:

Index	X_1	X_2	Y
1	0	-1	0
2	1	0	0
3	-1	1	1
4	1	-1	1

Using the Naïve Bayes' assumption for continuous features X_1 and X_2, determine the class of $(X_1, X_2) = (0, 0)$. Note: use the unbiased estimate of variance for samples $x_1, \ldots, x_n$, which is $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

$f_{X_1|Y=0}$: $\bar{X} = \frac{1}{2}$ $\hat{\sigma} = \sqrt{(0-\frac{1}{2})^2 + (1-\frac{1}{2})^2} = \sqrt{\frac{1}{2}}$

$$f_{X_1|Y=0}(x_1|0) = \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2}}} e^{-\frac{(x_1-\frac{1}{2})^2}{2\times 1/2}}$$

$f_{X_2|Y=0}$: $\bar{X} = -\frac{1}{2}$ $\hat{\sigma} = \sqrt{(-1+\frac{1}{2})^2 + (0+\frac{1}{2})^2} = \sqrt{\frac{1}{2}}$

$$f_{X_2|Y=0}(x_2|0) = \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2}}} e^{-\frac{(x_1+\frac{1}{2})^2}{2\times\frac{1}{2}}}$$

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$f_{X_1|Y=1}$: $\bar{X} = 0$ $\hat{\sigma} = \sqrt{(-1-0)^2 + (1-0)^2} = \sqrt{2}$

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$f_{X_2|Y=1}$: $\bar{X} = 0$ $\hat{\sigma} = \sqrt{2}$

$$f_{X_1|Y=1}(x_1|1) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{(x_1)^2}{4}} \qquad f_{X_2|Y=1}(x_2|1) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{x_2^2}{4}}$$

$$\Pr\left(Y=0 \mid (X_1,X_2)=(0,0)\right) \propto \frac{1}{\sqrt{2\pi}\sqrt{1/2}}\, e^{-1/4}\, \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2}}}\, e^{-1/4} \quad 3$$

$$= e^{-1/2}/\pi$$

$$\Pr\left(Y=1 \mid (X_1,X_2)=(0,0)\right) \propto \frac{1}{\sqrt{2\pi}\sqrt{2}}\, e^{0}\, \frac{1}{\sqrt{2\pi}\sqrt{2}}\, e^{0} \quad 3$$

$$= \frac{1}{4\pi}$$

$$\Pr(Y=0 \mid X_1, X_2) > \Pr(Y=1 \mid X_1, X_2)$$

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$$\Rightarrow \quad Y=0$$

3. Consider the following dataset:

Index	X_1	X_2	Y
1	0	0	0
2	1	0	0
3	0	1	1
4	1	1	1

We wish to fit a standard logistic regression model to this dataset.

(a) Write down the *negative log likelihood* loss function for this dataset.

(b) Is the iterative algorithm for determining the regression coefficients stable? Explain your answer.

(a) $$NLL = -\sum_{i=1}^{4}\left[y_i \log p(\underline{x_i}) + (1-y_i)\log(1-p(x_i))\right]$$

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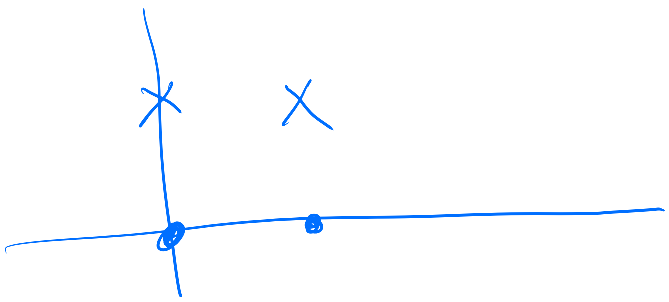
$$= -\left\{\log\left(\frac{1}{1+e^{\beta_0+\beta_1 \times 0+\beta_2 \times 0}}\right) + \log\left(\frac{1}{1+e^{\beta_0+\beta_1 \times 1+\beta_2 \times 0}}\right) + \log\left(\frac{e^{\beta_0+\beta_1 \times 0+\beta_2 \times 1}}{1+e^{\beta_0+\beta_1 \times 0+\beta_2 \times 1}}\right) + \log\left(\frac{e^{\beta_0+\beta_1 \times 1+\beta_2 \times 1}}{1+e^{\beta_0+\beta_1 \times 1+\beta_2 \times 1}}\right)\right\}$$

(Simplification not needed)

$$= \log(1+e^{\beta_0}) + \log(1+e^{\beta_0+\beta_1})$$

$$-\log\left(\frac{e^{\beta_0+\beta_2}}{1+e^{\beta_0+\beta_2}}\right) - \log\left(\frac{e^{\beta_0+\beta_1+\beta_2}}{1+e^{\beta_0+\beta_1+\beta_2}}\right)$$

(b)



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The data are linearly separable, so the iterative algorithm is unstable.

4. Consider the following dataset:

Index	X_1	X_2	Y
1	0	-1	1
2	1	0	-1
3	-1	1	2
4	1	-1	-2
5	-1	-1	0

Determine the leave-one-out cross validation estimate of the MSE $\frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$ of the null model $\hat{y}_i = \hat{\beta}_0$ on this dataset.

Index 1 : $\bar{y} = -\frac{1}{4}$ $\widehat{MSE} = \left(1+\frac{1}{4}\right)^2 = \frac{25}{16}$

Index 2 : $\bar{y} = \frac{1}{4}$ $\widehat{MSE} = \left(1+\frac{1}{4}\right)^2 = \frac{25}{16}$

Index 3 : $\bar{y} = -\frac{2}{4}$ $\widehat{MSE} = \left(2+\frac{2}{4}\right)^2 = \frac{25}{4}$

Index 4 : $\bar{y} = \frac{2}{4}$ $\widehat{MSE} = \left(-2-\frac{2}{4}\right)^2 = \frac{25}{4}$

Index 5 : $\bar{y} = 0$ $\widehat{MSE} = (0-0)^2 = 0$

$\widehat{MSE}_{LOOCV} = \frac{1}{5}\left(\frac{25}{8} + \frac{25}{2}\right) = \frac{25}{8}$

5. Assume that we have a Ridge regression problem with only one predictor, and the true model is linear *without an intercept*, i.e. $Y = \beta_1 X + \epsilon$. Assume that we have n samples, $(x_i, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ and we want to find the $\mathscr{L}_2$ regularized least squares estimate $\hat{\beta}_1$ from the data.

   (a) Formulate the objective function in terms of a candidate $\hat{\beta}_1$ and x_i's and y_i's, which are known. Assume that the regularization parameter is λ.

   (b) Find $\hat{\beta}_1$ in terms of λ and the data,.

(a) $J = \sum_{i=1}^{n} (y_i - \beta_1 x_i)^2 + \lambda \beta_1^2$

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(b) $\frac{\partial J}{\partial \beta_1} = 0 \Rightarrow \sum_{i=1}^{n} (-2x_i)(y_i - \beta_1 x_i) + 2\lambda\beta_1 = 0$

$-\sum x_i y_i + \beta_1 \sum x_i^2 + \lambda \beta_1 = 0$

$\Rightarrow \hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum x_i^2 + \lambda}$

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Scratch paper

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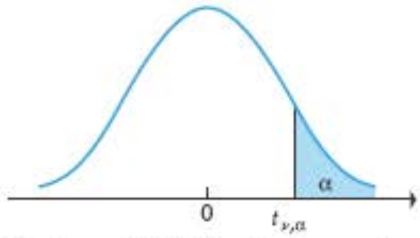
Upper Critical Values of Student's t Distribution with ν Degrees of Freedom



For selected probabilities, α, the table shows the values $t_{\nu,\alpha}$ such that $P(t_\nu > t_{\nu,\alpha}) = \alpha$, where t_ν is a Student's t random variable with ν degress of freedom. For example, the probability is .10 that a Student's t random variable with 10 degrees of freedom exceeds 1.372.

	Probability of Exceeding the Critical Value					
ν	0.10	0.05	0.025	0.01	0.005	0.001
1	3.078	6.314	12.706	31.821	63.657	318.313
2	1.886	2.920	4.303	6.965	9.925	22.327
3	1.638	2.353	3.182	4.541	5.841	10.215
4	1.533	2.132	2.776	3.747	4.604	7.173
5	1.476	2.015	2.571	3.365	4.032	5.893
6	1.440	1.943	2.447	3.143	3.707	5.208
7	1.415	1.895	2.365	2.998	3.499	4.782
8	1.397	1.860	2.306	2.896	3.355	4.499
9	1.383	1.833	2.262	2.821	3.250	4.296
10	1.372	1.812	2.228	2.764	3.169	4.143
11	1.363	1.796	2.201	2.718	3.106	4.024
12	1.356	1.782	2.179	2.681	3.055	3.929
13	1.350	1.771	2.160	2.650	3.012	3.852
14	1.345	1.761	2.145	2.624	2.977	3.787
15	1.341	1.753	2.131	2.602	2.947	3.733
16	1.337	1.746	2.120	2.583	2.921	3.686
17	1.333	1.740	2.110	2.567	2.898	3.646
18	1.330	1.734	2.101	2.552	2.878	3.610
19	1.328	1.729	2.093	2.539	2.861	3.579
20	1.325	1.725	2.086	2.528	2.845	3.552
21	1.323	1.721	2.080	2.518	2.831	3.527
22	1.321	1.717	2.074	2.508	2.819	3.505
23	1.319	1.714	2.069	2.500	2.807	3.485
24	1.318	1.711	2.064	2.492	2.797	3.467
25	1.316	1.708	2.060	2.485	2.787	3.450
26	1.315	1.706	2.056	2.479	2.779	3.435
27	1.314	1.703	2.052	2.473	2.771	3.421
28	1.313	1.701	2.048	2.467	2.763	3.408
29	1.311	1.699	2.045	2.462	2.756	3.396
30	1.310	1.697	2.042	2.457	2.750	3.385
40	1.303	1.684	2.021	2.423	2.704	3.307
60	1.296	1.671	2.000	2.390	2.660	3.232
100	1.290	1.660	1.984	2.364	2.626	3.174
∞	1.282	1.645	1.960	2.326	2.576	3.090
ν	0.10	0.05	0.025	0.01	0.005	0.001

F Table for α = 0.01

	DF1	α = 0.01																	
DF2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	Inf
1	4052.2	4999.5	5403.4	5624.6	5763.7	5859	5928.4	5981.1	6022.5	6055.8	6106.3	6157.3	6208.7	6234.6	6260.6	6286.8	6313	6339.4	6365.9
2	98.503	99	99.166	99.249	99.299	99.333	99.356	99.374	99.388	99.399	99.416	99.433	99.449	99.458	99.466	99.474	99.482	99.491	99.499
3	34.116	30.817	29.457	28.71	28.237	27.911	27.672	27.489	27.345	27.229	27.052	26.872	26.69	26.598	26.505	26.411	26.316	26.221	26.125
4	21.198	18	16.694	15.977	15.522	15.207	14.976	14.799	14.659	14.546	14.374	14.198	14.02	13.929	13.838	13.745	13.652	13.558	13.463
5	16.258	13.274	12.06	11.392	10.967	10.672	10.456	10.289	10.158	10.051	9.888	9.722	9.553	9.466	9.379	9.291	9.202	9.112	9.02
6	13.745	10.925	9.78	9.148	8.746	8.466	8.26	8.102	7.976	7.874	7.718	7.559	7.396	7.313	7.229	7.143	7.057	6.969	6.88
7	12.246	9.547	8.451	7.847	7.46	7.191	6.993	6.84	6.719	6.62	6.469	6.314	6.155	6.074	5.992	5.908	5.824	5.737	5.65
8	11.259	8.649	7.591	7.006	6.632	6.371	6.178	6.029	5.911	5.814	5.667	5.515	5.359	5.279	5.198	5.116	5.032	4.946	4.859
9	10.561	8.022	6.992	6.422	6.057	5.802	5.613	5.467	5.351	5.257	5.111	4.962	4.808	4.729	4.649	4.567	4.483	4.398	4.311
10	10.044	7.559	6.552	5.994	5.636	5.386	5.2	5.057	4.942	4.849	4.706	4.558	4.405	4.327	4.247	4.165	4.082	3.996	3.909
11	9.646	7.206	6.217	5.668	5.316	5.069	4.886	4.744	4.632	4.539	4.397	4.251	4.099	4.021	3.941	3.86	3.776	3.69	3.602
12	9.33	6.927	5.953	5.412	5.064	4.821	4.64	4.499	4.388	4.296	4.155	4.01	3.858	3.78	3.701	3.619	3.535	3.449	3.361
13	9.074	6.701	5.739	5.205	4.862	4.62	4.441	4.302	4.191	4.1	3.96	3.815	3.665	3.587	3.507	3.425	3.341	3.255	3.165
14	8.862	6.515	5.564	5.035	4.695	4.456	4.278	4.14	4.03	3.939	3.8	3.656	3.505	3.427	3.348	3.266	3.181	3.094	3.004
15	8.683	6.359	5.417	4.893	4.556	4.318	4.142	4.004	3.895	3.805	3.666	3.522	3.372	3.294	3.214	3.132	3.047	2.959	2.868
16	8.531	6.226	5.292	4.773	4.437	4.202	4.026	3.89	3.78	3.691	3.553	3.409	3.259	3.181	3.101	3.018	2.933	2.845	2.753
17	8.4	6.112	5.185	4.669	4.336	4.102	3.927	3.791	3.682	3.593	3.455	3.312	3.162	3.084	3.003	2.92	2.835	2.746	2.653
18	8.285	6.013	5.092	4.579	4.248	4.015	3.841	3.705	3.597	3.508	3.371	3.227	3.077	2.999	2.919	2.835	2.749	2.66	2.566
19	8.185	5.926	5.01	4.5	4.171	3.939	3.765	3.631	3.523	3.434	3.297	3.153	3.003	2.925	2.844	2.761	2.674	2.584	2.489
20	8.096	5.849	4.938	4.431	4.103	3.871	3.699	3.564	3.457	3.368	3.231	3.088	2.938	2.859	2.778	2.695	2.608	2.517	2.421
21	8.017	5.78	4.874	4.369	4.042	3.812	3.64	3.506	3.398	3.31	3.173	3.03	2.88	2.801	2.72	2.636	2.548	2.457	2.36
22	7.945	5.719	4.817	4.313	3.988	3.758	3.587	3.453	3.346	3.258	3.121	2.978	2.827	2.749	2.667	2.583	2.495	2.403	2.305
23	7.881	5.664	4.765	4.264	3.939	3.71	3.539	3.406	3.299	3.211	3.074	2.931	2.781	2.702	2.62	2.535	2.447	2.354	2.256
24	7.823	5.614	4.718	4.218	3.895	3.667	3.496	3.363	3.256	3.168	3.032	2.889	2.738	2.659	2.577	2.492	2.403	2.31	2.211
25	7.77	5.568	4.675	4.177	3.855	3.627	3.457	3.324	3.217	3.129	2.993	2.85	2.699	2.62	2.538	2.453	2.364	2.27	2.169
26	7.721	5.526	4.637	4.14	3.818	3.591	3.421	3.288	3.182	3.094	2.958	2.815	2.664	2.585	2.503	2.417	2.327	2.233	2.131
27	7.677	5.488	4.601	4.106	3.785	3.558	3.388	3.256	3.149	3.062	2.926	2.783	2.632	2.552	2.47	2.384	2.294	2.198	2.097
28	7.636	5.453	4.568	4.074	3.754	3.528	3.358	3.226	3.12	3.032	2.896	2.753	2.602	2.522	2.44	2.354	2.263	2.167	2.064
29	7.598	5.42	4.538	4.045	3.725	3.499	3.33	3.198	3.092	3.005	2.868	2.726	2.574	2.495	2.412	2.325	2.234	2.138	2.034
30	7.562	5.39	4.51	4.018	3.699	3.473	3.304	3.173	3.067	2.979	2.843	2.7	2.549	2.469	2.386	2.299	2.208	2.111	2.006
40	7.314	5.179	4.313	3.828	3.514	3.291	3.124	2.993	2.888	2.801	2.665	2.522	2.369	2.288	2.203	2.114	2.019	1.917	1.805
60	7.077	4.977	4.126	3.649	3.339	3.119	2.953	2.823	2.718	2.632	2.496	2.352	2.198	2.115	2.028	1.936	1.836	1.726	1.601
120	6.851	4.787	3.949	3.48	3.174	2.956	2.792	2.663	2.559	2.472	2.336	2.192	2.035	1.95	1.86	1.763	1.656	1.533	1.381
Inf	6.635	4.605	3.782	3.319	3.017	2.802	2.639	2.511	2.407	2.321	2.185	2.039	1.878	1.791	1.696	1.592	1.473	1.325	1