Theory Assignment: vector spaces and Geometry of Data.

1) Explain the Concept of a vector in R^n. How does the mathmatical defination of a vector relate to features representation in machine learning?

Solution: A vector in R^n is an ordered n-tuple of real number written as

$$X = (x_1, x_2, x_3 \dots x_n)^T$$

Each Component represent a coordinate along a specific axis in n-dimensional space. In machine Learning, each coordinate corresponds to a features of a data point. Thus a data set with n features is mathmatically represented as vectors in R^n.

2) Define a vector Space over R. Discuss why closure under addition and scalar multipliartion are essential properties?.

Solutions: A vector space over $\mathbb{R}$ is a set V equipped with vector addition and scalar multiplication such that for all $u, v \in V$ and $c \in \mathbb{R}$:

$$u+v \in V \; ; \; cu \in V.$$

Closure under addition ensures combining two elements remains within space, closure under scalar multiplication ensures scaling a vectors doesn't leave the space. These properties guarantee algebric consistency.

3) What is a subspace? State and explain the three nessary conditions required to prove that a subset is a subspace.

Solutions: A subspace H of a vector space V is a subset that is itself a vector space. It must satisfy:

a) The zero vector $0 \in H$

b) closure under addition:

if $u, v \in H$, then $u+v \in H$.

c) closure under scalar multiplication:

if $u \in H$ and $c \in R$ then $cu \in H$.

4) Explain the concept of linear combination. why is linear combination fundamental in understanding span & basis?

__Solutions__: A linear combination of vectors $v_1, \ldots v_p$ is

$$c_1v_1 + c_2v_2 \cdots + c_pv_p.$$

Linear combinations describes all possible weighted sum of vectors and form the foundation of span and basis.

5) Define the span of a set of vectors. Explain its geometric meaning in R^2 and R^3.

__Solution__: The span of vectors $\{v_1 \ldots v_p\}$ is the set of all Linear combinations of those vectors:

$\text{Span}\{v_1 \cdots v_p\}$.

In R^2, the span of non zero vector forms a line through the origin. Two Linearly independent vectors Span the entire plane. In R^3, one vector spans a line, two independent vectors span a plane, and three independent vectors span the entire space.

6. What is Linear independence? Explain the diff Betn Linear independence and linear dependence with a clear mathmatical condition.

Solutions:

A Set of vector is linearly independent if

$$c_1v_1 + \cdots c_pv_p = 0.$$

implies $c_1 = \cdots = c_p = 0$. If there exists a non-trivial solution (Some $c_i \neq 0$), the vectors are Linearly dependent.

7. Discuss the problem of redundant features in ML from the prespeertive of linear dependence.

Solutions:

In ML, redundant features occur when one feature can be expressed as a linear combination of other. This creats Linear dependence, leading to multicollinearity, unstable parameter estimates, and reduced model interpretability.

8. Define a basis of a vector space. Why must a basis satisfy both spanning and Linear independence Conditions?

Solutions:

A basis of a vector space V is a set of vectors that:

a) Span V

b) Are linearly independent

Spanning ensure coverage of the entire space, while independence ensures no redundancy.

9) What is the ~~max~~ dimension of a vector space? explain its relationships with basis and matrix rank.

Solutions:

The dimension of a vector space is the number of vectors in any basis of the space:

$$\dim(V) = \text{number of basis space.}$$

It is equal to the rank of a matrix whose rows (or columns) from a basis, since rank represent the number of linearly independent vectors.

10) Explain the importance of understanding vectors space in ML, especially in dimensionality reduction technique such as PCA.

Solution.

Understanding vector space allows us to analyze data geometrically. Dimensionality reduction methods.

such as PCA. identify a lower-dimensional subspace that captures maximum variance. This improve computational efficiency and reduces redundancy while preserving essential structure.

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