1. Prove that the function $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find its harmonic conjugate v and express $u + iv$ as an analytic function of z.

Solution: Given that, $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = 6xy + 4x \;\cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y \;\cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = 6y + 4 \;\cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -6y - 4 \;\cdots\cdots (4)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6y + 4 - 6y - 4 = 0$$

$\Rightarrow u$ satisfied Laplace equation. Hence u is harmonic.

By Cauchy-Riemann equations we have,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -3x^2 + 3y^2 + 4y \quad [\text{by } (2)]$$

By integrating this with respect to x keeping y as constant,

$$v = \int (-3x^2 + 3y^2 + 4y)\, dx$$

$$\Rightarrow v = -x^3 + 3xy^2 + 4xy + F(y) \;\cdots\cdots (5)$$

$$\Rightarrow \frac{\partial v}{\partial y} = 6xy + 4x + F'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 6xy + 4x + F'(y)\,, \quad \text{by CR equation}$$

$$\Rightarrow 6xy + 4x = 6xy + 4x + F'(y) \quad [\text{by } (1)]$$

$$\Rightarrow 0 = F'(y)$$

$$\therefore F(y) = c_1\,, \quad \text{by integrating}$$

Putting this value in (5) we get

$$v = -x^3 + 3xy^2 + 4xy + c_1$$

Let $\mathrm{f}(z) = u + iv$

$$= 3x^2y + 2x^2 - y^3 - 2y^2 + i(-x^3 + 3xy^2 + 4xy + c_1)$$

$$= (-\mathrm{i}x^3 + 3ixy^2 + 3x^2y - y^3) + 2(x^2 - y^2 + 2ixy) + ic_1$$

$= -\mathrm{i}(\mathrm{x}^3 + 3i^2xy^2 + 3ix^2y + i^3y^3) + 2(x^2 + 2ixy + i^2y^2) + ic_1$

$= -\mathrm{i}(\mathrm{x} + iy)^3 + 2(x + iy)^2 + c\,,\quad \text{where } c = ic_1$

$\Rightarrow \mathrm{f}(z) = u + iv = iz^3 + 2z^2 + c\,.$ (Ans)

2. Prove that the function $u = 2x - x^3 + 3xy^2$ is harmonic. Find its harmonic conjugate v if $\mathrm{f(z)} = u + iv$ is analytic.

Solution: Given that, $u = 2x - x^3 + 3xy^2$

$$\therefore \frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 \;\cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = 6xy \;\cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = -6x \;\cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = 6x \;\cdots\cdots (4)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$$

$\Rightarrow u$ satisfied Laplace equation. Hence u is harmonic.

By Cauchy-Riemann equations we have,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -6xy \quad [\text{by } (2)]$$

By integrating this with respect to x keeping y as constant,

$$v = \int (-6xy\,)\,dx$$

$$\Rightarrow v = -3x^2y + F(y) \;\cdots\cdots (5)$$

$$\Rightarrow \frac{\partial v}{\partial y} = -3x^2 + F'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = -3x^2 + F'(y)\,,\quad \text{by CR equation}$$

$$\Rightarrow 2 - 3x^2 + 3y^2 = -3x^2 + F'(y) \quad [\text{by } (1)]$$

$$\Rightarrow 2 + 3y^2 = F'(y)$$

$$\therefore F(y) = y^3 + 2y + c_1\,,\quad \text{by integrating}$$

Putting this value in (5) we get

$$v = -3x^2y + y^3 + 2y + c_1$$

3. Prove that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic. Find its harmonic conjugate v if $\mathrm{f(z)} = u + iv$ is analytic.

Solution: Given that, $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \;\cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = -6xy - 6y \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6 \;\cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -6x - 6 \;\cdots\cdots (4)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6 = 0$$

$\Rightarrow u$ satisfied Laplace equation. Hence u is harmonic.

By Cauchy-Riemann equations we have,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy + 6y \quad [\text{by } (2)]$$

By integrating this with respect to x keeping y as constant,

$$v = \int (6xy + 6y)\, dx$$

$$\Rightarrow v = 3x^2y + 6xy + F(y) \;\cdots\cdots (5)$$

$$\Rightarrow \frac{\partial v}{\partial y} = 3x^2 + 6x + F'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 + 6x + F'(y)\,, \quad \text{by CR equation}$$

$$\Rightarrow 3x^2 - 3y^2 + 6x = 3x^2 + 6x + F'(y) \quad [\text{by } (1)]$$

$$\Rightarrow -3y^2 = F'(y)$$

$$\therefore F(y) = -y^3 + c_1\,, \quad \text{by integrating}$$

Putting this value in (5) we get

$$v = 3x^2y + 6xy - y^3 + c_1$$

4. Prove that the function $u = e^{-x}(x \sin y - y \cos y)$ is harmonic. Find its harmonic conjugate v if $\mathrm{f(z)} = u + iv$ is analytic.

Solution: Given that, $u = e^{-x}(x \sin y - y \cos y)$

$$\therefore \frac{\partial u}{\partial x} = \sin y \, \frac{\partial}{\partial x} e^{-x}.x - y \cos y \, \frac{\partial}{\partial x} e^{-x}$$

$$= e^{-x} \sin y - x e^{-x} \sin y + y e^{-x} \cos y \;\cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = x e^{-x} \, \frac{\partial}{\partial y} \sin y - e^{-x} \, \frac{\partial}{\partial y} y.\cos y$$

$$= x e^{-x} \cos y + y e^{-x} \sin y - e^{-x} \cos y \;\cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = \sin y \, \frac{\partial}{\partial x} e^{-x} - \sin y \, \frac{\partial}{\partial x} x.e^{-x} + y \cos y \, \frac{\partial}{\partial x} e^{-x}$$

$$= -2e^{-x} \sin y + x e^{-x} \sin y - y e^{-x} \cos y \;\cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = x e^{-x} \, \frac{\partial}{\partial y} \cos y + e^{-x} \, \frac{\partial}{\partial y} y \sin y - e^{-x} \, \frac{\partial}{\partial y} \cos y$$

$$= -x e^{-x} \sin y + y e^{-x} \cos y + 2e^{-x} \sin y \;\cdots\cdots (4)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2e^{-x} \sin y + x e^{-x} \sin y - y e^{-x} \cos y - x e^{-x} \sin y + y e^{-x} \cos y$$
$$+ 2e^{-x} \sin y = 0$$

$\Rightarrow u$ satisfied Laplace equation. Hence u is harmonic.

By Cauchy-Riemann equations we have,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -x e^{-x} \cos y - y e^{-x} \sin y + e^{-x} \cos y \quad [\text{by } (2)]$$

By integrating this with respect to x keeping y as constant,

$$v = \int (-x e^{-x} \cos y - y e^{-x} \sin y + e^{-x} \cos y) \, dx$$

$$= -\cos y \int x.e^{-x} \, dx - y \sin y \int e^{-x} \, dx + \cos y \int e^{-x} \, dx$$

$$= -\cos y \left\{ x \int e^{-x} \, dx - \int \left(\frac{d}{dx} x \int e^{-x} \, dx \right) dx \right\} - y \sin y \int e^{-x} \, dx + \cos y \int e^{-x} \, dx$$

$$= -\cos y \left\{ x \, \frac{e^{-x}}{-1} - \frac{e^{-x}}{(-1)(-1)} \right\} - y \sin y \, \frac{e^{-x}}{-1} + \cos y \, \frac{e^{-x}}{-1} + F(y)$$

$$= x e^{-x} \cos y + e^{-x} \cos y + y e^{-x} \sin y - e^{-x} \cos y + F(y)$$

$$\Rightarrow v = x e^{-x} \cos y + y e^{-x} \sin y + F(y) \;\cdots\cdots (5)$$

$$\Rightarrow \frac{\partial v}{\partial y} = xe^{-x}\ \frac{\partial}{\partial y}\cos y + e^{-x}\ \frac{\partial}{\partial y}\ y \sin y + \frac{\partial}{\partial y}F(y)$$

$$= -xe^{-x}\sin y + ye^{-x}\cos y + e^{-x}\sin y + F'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = -xe^{-x}\sin y + ye^{-x}\cos y + e^{-x}\sin y + F'(y), \quad \text{by CR equation}$$

$$\Rightarrow e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y = -xe^{-x}\sin y + ye^{-x}\cos y + e^{-x}\sin y + F'(y)$$

$$\Rightarrow 0 = F'(y)$$

$$\therefore F(y) = c_1\,, \quad \text{by integrating}$$

Putting this value in (5) we get

$$v = xe^{-x}\cos y + ye^{-x}\sin y + c_1 = e^{-x}(x\cos y + y\sin y) + c_1$$

5. Prove that the function $u = x^3 + 6x^2y - 3xy^2 - 2y^3$ is harmonic. Find its harmonic conjugate v if $\mathrm{f(z)} = u + iv$ is analytic.

Solution: Given that, $u = x^3 + 6x^2y - 3xy^2 - 2y^3$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 + 12xy - 3y^2 \quad \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = 6x^2 - 6xy - 6y^2 \quad \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 12y \quad \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -6x - 12y \quad \cdots\cdots (4)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 12y - 6x - 12y = 0$$

$\Rightarrow u$ satisfied Laplace equation. Hence u is harmonic.

By Cauchy-Riemann equations we have,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -6x^2 + 6xy + 6y^2 \quad [\text{by } (2)]$$

By integrating this with respect to x keeping y as constant,

$$v = \int (-6x^2 + 6xy + 6y^2)\, dx$$

$$\Rightarrow v = -2x^3 + 3x^2y + 6xy^2 + F(y) \quad \cdots\cdots (5)$$

$$\Rightarrow \frac{\partial v}{\partial y} = 3x^2 + 12xy + F'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 + 12xy + F'(y)\,, \quad \text{by CR equation}$$

$$\Rightarrow 3x^2 + 12xy - 3y^2 = 3x^2 + 12xy + F'(y) \quad [\text{by } (1)]$$

$$\Rightarrow -3y^2 = F'(y)$$

$$\therefore F(y) = -y^3 + c_1\,, \quad \text{by integrating}$$

Putting this value in (5) we get

$$v = -2x^3 + 3x^2y + 6xy^2 - y^3 + c_1$$

6. Prove that the function $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find its harmonic conjugate v if $\mathrm{f(z)} = u + iv$ is analytic.

Solution: Given that, $u = x^2 - y^2 - 2xy - 2x + 3y$

$$\therefore \frac{\partial u}{\partial x} = 2x - 2y - 2 \quad \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3 \quad \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -2 \quad \cdots\cdots (4)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$\Rightarrow u$ satisfied Laplace equation. Hence u is harmonic.

By Cauchy-Riemann equations we have,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y + 2x - 3 \quad [\text{by } (2)]$$

By integrating this with respect to x keeping y as constant,

$$v = \int (2y + 2x - 3)\, dx$$

$$\Rightarrow v = 2xy + x^2 - 3x + F(y) \quad \cdots\cdots (5)$$

$$\Rightarrow \frac{\partial v}{\partial y} = 2x + F'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x + F'(y)\,, \quad \text{by CR equation}$$

$$\Rightarrow 2x - 2y - 2 = 2x + F'(y) \quad [\text{by } (1)]$$

$\Rightarrow -2y - 2 = F'(y)$

$\therefore F(y) = -y^2 - 2y + c_1$, by integrating

Putting this value in (5) we get

$v = 2xy + x^2 - 3x - y^2 - 2y + c_1$

7. Prove that the function $u = 2x(1-y)$ is harmonic. Find its harmonic conjugate v if $f(z) = u + iv$ is analytic.

Solution: Given that, $u = 2x(1-y)$

$$\therefore \frac{\partial u}{\partial x} = 2 - 2y \quad \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = -2x \ \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \ \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = 0 \ \cdots\cdots (4)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 + 0 = 0$$

$\Rightarrow u$ satisfied Laplace equation. Hence u is harmonic.

By Cauchy-Riemann equations we have,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2x \quad [\text{by } (2)]$$

By integrating this with respect to x keeping y as constant,

$$v = \int (2x)\,dx$$

$$\Rightarrow v = x^2 + F(y) \ \cdots\cdots (5)$$

$$\Rightarrow \frac{\partial v}{\partial y} = F'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = F'(y)\,, \quad \text{by CR equation}$$

$$\Rightarrow 2 - 2y = F'(y) \quad [\text{by } (1)]$$

$\therefore F(y) = 2y - y^2 + c_1$, by integrating

Putting this value in (5) we get

$v = x^2 + 2y - y^2 + c_1$

8. Prove that the function $u = y^3 - 3x^2y$ is harmonic. Find its harmonic conjugate v if $\mathrm{f(z)} = u + iv$ is analytic.

Solution: Given that, $u = y^3 - 3x^2y$

$$\therefore \frac{\partial u}{\partial x} = -6xy \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = 3y^2 - 3\mathrm{x}^2 \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = -6y \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = 6y \cdots\cdots (4)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$$

$\Rightarrow u$ satisfied Laplace equation. Hence u is harmonic.

By Cauchy-Riemann equations we have,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -3y^2 + 3\mathrm{x}^2 \quad [\text{by } (2)]$$

By integrating this with respect to x keeping y as constant,

$$v = \int (-3y^2 + 3\mathrm{x}^2)\, dx$$

$$\Rightarrow v = -3xy^2 + x^3 + F(y) \cdots\cdots (5)$$

$$\Rightarrow \frac{\partial v}{\partial y} = -6xy + F'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = -6xy + F'(y), \quad \text{by CR equation}$$

$$\Rightarrow -6xy = -6xy + F'(y) \quad [\text{by } (1)]$$

$$\Rightarrow 0 = F'(y)$$

$$\therefore F(y) = c_1, \quad \text{by integrating}$$

Putting this value in (5) we get

$$v = -3xy^2 + x^3 + c_1$$

9. Prove that the function $u = x^2 - y^2 + 2y$ is harmonic. Find its harmonic conjugate v if $\mathrm{f(z)} = u + iv$ is analytic.

Solution: Given that, $u = x^2 - y^2 + 2y$

$$\therefore \frac{\partial u}{\partial x} = 2x \quad \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = -2y + 2 \quad \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -2 \quad \cdots\cdots (4)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$\Rightarrow u$ satisfied Laplace equation. Hence u is harmonic.

By Cauchy-Riemann equations we have,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y - 2 \quad [\text{by } (2)]$$

By integrating this with respect to x keeping y as constant,

$$v = \int (2y - 2)\, dx$$

$$\Rightarrow v = 2xy - 2x + F(y) \quad \cdots\cdots (5)$$

$$\Rightarrow \frac{\partial v}{\partial y} = 2x + F'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x + F'(y)\,, \quad \text{by CR equation}$$

$$\Rightarrow 2x = 2x + F'(y) \quad [\text{by } (1)]$$

$$\Rightarrow 0 = F'(y)$$

$$\therefore F(y) = c_1\,, \quad \text{by integrating}$$

Putting this value in (5) we get

$$v = 2xy - 2x + c_1$$

10. Prove that the function $u = 2xy + y^3 - 3x^2y$ is harmonic. Find its harmonic conjugate v if $\mathrm{f(z)} = u + iv$ is analytic.

Solution: Given that, $u = 2xy + y^3 - 3x^2y$

$$\therefore \frac{\partial u}{\partial x} = 2y - 6xy \quad \cdots\cdots (1)$$

$$\frac{\partial u}{\partial y} = 2x + 3y^2 - 3x^2 \quad \cdots\cdots (2)$$

$$\frac{\partial^2 u}{\partial x^2} = -6y \quad \cdots\cdots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = 6y \cdots\cdots (4)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$$

$\Rightarrow u$ satisfied Laplace equation. Hence u is harmonic.

By Cauchy-Riemann equations we have,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -2x - 3y^2 + 3x^2 \quad [\text{by } (2)]$$

By integrating this with respect to x keeping y as constant,

$$v = \int (-2x - 3y^2 + 3x^2)\, dx$$

$$\Rightarrow v = -x^2 - 3xy^2 + x^3 + F(y) \quad \cdots\cdots (5)$$

$$\Rightarrow \frac{\partial v}{\partial y} = -6xy + F'(y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = -6xy + F'(y)\,, \quad \text{by CR equation}$$

$$\Rightarrow 2y - 6xy = -6xy + F'(y) \quad [\text{by } (1)]$$

$$\Rightarrow 2y = F'(y)$$

$$\therefore F(y) = y^2 + c_1\,, \quad \text{by integrating}$$

Putting this value in (5) we get

$$v = -x^2 - 3xy^2 + x^3 + y^2 + c_1$$