**1.** Show that $\oint_C \frac{e^{tz}}{(z^2+1)^2} dz = \pi i\big(dz \sin t - t\cos t\big)$ where $C$ is the circle $|z| = 3$ and $t > 0$.

**Solution:** Here the circle is $|z| = 3$.

The poles of $\frac{e^{tz}}{(z^2+1)^2}$ are obtained by solving the equation $(z^2+1)^2 = 0$

$\Rightarrow \{(z+i)(z-i)\}^2 = 0 \quad \Rightarrow z = i, -i$

Both of are double poles and lie inside $C$, since $|i| = |-i| = 1 < 3$.

Residue at $z = i$ is

$$\lim_{z\to i} \frac{1}{1!}\frac{d}{dz}\left\{(z-i)^2 \cdot \frac{e^{tz}}{(z-i)^2(z+i)^2}\right\}$$

$$= \lim_{z\to i} \frac{d}{dz}\left\{\frac{e^{tz}}{(z+i)^2}\right\}$$

$$= \lim_{z\to i} \frac{(z+i)^2 \cdot te^{tz} - e^{tz} \cdot 2(z+i)}{(z+i)^4}$$

$$= \lim_{z\to i} \frac{(z+i) \cdot te^{tz} - 2e^{tz}}{(z+i)^3}$$

$$= \frac{2it\, e^{it} - 2e^{it}}{8i^3} = \frac{(it-1)e^{it}}{-4i}$$

$$= \frac{-(t+i)e^{it}}{4}$$

Similarly residue at $z = -i$ is $\frac{-(t-i)e^{-it}}{4}$

Therefore by Cauchy's residue theorem we have

$$\oint_C \frac{e^{tz}}{(z^2+1)^2} dz = 2\pi i\ [\text{sum of the residues}]$$

$$= -2\pi i\left[\frac{(t+i)e^{it}}{4} + \frac{(t-i)e^{-it}}{4}\right]$$

$$= -2\pi i\left[\frac{t}{4}\cdot\left(e^{it} + e^{-it}\right) + \frac{i}{4}\cdot\left(e^{it} - e^{-it}\right)\right]$$

$$= -2\pi i\left[\frac{t}{4}\cdot 2\cos t + \frac{i}{4}\cdot 2i\sin t\right]$$

$$= -\pi i\ (t\cos t - \sin t)$$

$$= \pi i\ (\sin t - t\cos t) \qquad \text{(Showed)}$$

**2.** Show that $\oint_C \frac{e^z}{(z^2+\pi^2)^2} dz = \frac{1}{\pi}$ where $C$ is the circle $|z| = 4$.

**Solution:** Let $\mathrm{f}(z) = \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2\,(z-\pi i)^2}$

The poles of $\mathrm{f}(z)$ are obtained by solving the equation $(z+\pi i)^2\,(z-\pi i)^2 = 0$

$\Rightarrow z = \pi i, \pi i$ and $z = -\pi i, -\pi i$

$\Rightarrow z = \pi i$ and $-\pi i$ are two poles of $\mathrm{f}(z)$ each of double poles.

Residue at $z = \pi i$ is

$$\lim_{z\to\pi i} \frac{1}{1!}\frac{d}{dz}\left\{(z-\pi i)^2 \cdot \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}\right\}$$

$$= \lim_{z\to\pi i} \frac{d}{dz}\left\{\frac{e^z}{(z+\pi i)^2}\right\}$$

$$= \lim_{z\to\pi i} \frac{(z+\pi i)^2 \cdot e^z - e^z \cdot 2(z+\pi i)}{(z+\pi i)^4}$$

$$= \lim_{z\to\pi i} \frac{e^z(z+\pi i-2)}{(z+i)^3}$$

$$= \frac{e^{i\pi}(2\pi i-2)}{(2\pi i)^3} = \frac{2(\pi i-1)e^{i\pi}}{-8\pi^3 i}$$

$$= \frac{-(\pi+i)e^{i\pi}}{4\pi^3}$$

Similarly residue at $z = -\pi i$ is $\frac{-(\pi-i)e^{-i\pi}}{4\pi^3}$

Therefore by Cauchy's residue theorem we have

$$\oint_C \frac{e^{tz}}{(z^2+1)^2} dz = 2\pi i \text{ [sum of the residues]}$$

$$= 2\pi i \left[\frac{-(\pi+i)e^{i\pi}}{4\pi^3} + \frac{-(\pi-i)e^{-i\pi}}{4\pi^3}\right]$$

$$= \frac{i}{2\pi^2}\left[-\pi\cdot\left(e^{i\pi}+e^{-i\pi}\right) - \pi\cdot\left(e^{i\pi}-e^{-i\pi}\right)\right]$$

$$= \frac{i}{2\pi^2}\left[-\pi\cdot 2\cos\pi - i\cdot 2i\sin\pi\right]$$

$$= \frac{i}{2\pi^2}\left[-2\pi\cdot(-1) - 2i^2\times 0\right]$$

$$= \frac{i}{\pi}$$

**3.** Show that $I = \frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \frac{1}{2}(t-1) + \frac{1}{2}e^{-t}\cos t$ where $C$ is the circle with equation $|z| = 3$.

**Solution:** Poles of $\frac{e^{zt}}{z^2(z^2+2z+2)}$ are obtained by solving the equation $z^2(z^2+2z+2) = 0$

$\Rightarrow z^2 = 0$ and $z^2(z^2+2z+2) = 0$

$\Rightarrow z = 0, 0$ and $(z+1)^2 = -1 = i^2$

$$\Rightarrow z + 1 = \pm\, i$$

$$\Rightarrow z = -1 \pm\ i$$

$$|-1 +\ i| = \sqrt{1+1} = \sqrt{2} < 3$$

$$|-1 -\ i| = \sqrt{1+1} = \sqrt{2} < 3$$

$\therefore$ $z = 0$ is a pole of order 2 inside $C$.

$z = -1 +\ i$ is a pole of order 1 inside $C$.

$z = -1 -\ i$ is a pole of order 1 inside $C$.

Residue at $z = 0$ is

$$\lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\left\{z^2\cdot\frac{e^{zt}}{z^2(z^2+2z+2)}\right\}$$

$$= \lim_{z\to 0}\frac{d}{dz}\left\{\frac{e^{zt}}{z^2+2z+2}\right\}$$

$$= \lim_{z\to 0}\frac{(z^2+2z+2)\cdot te^{zt} - e^{zt}\cdot(2z+2)}{(z^2+2z+2)^2}$$

$$= \lim_{z\to 0}\frac{(0+0+2)\cdot te^{0} - e^{0}\cdot(0+2)}{(0+0+2)^2}$$

$$= \frac{2t-2}{4} = \frac{t-1}{2}$$

Residue at $z = -1 + i$ is

$$\lim_{z\to -1+i}\left\{(z+1-i)^2\cdot\frac{e^{tz}}{z^2(z^2+2z+2)}\right\}$$

$$= \lim_{z\to -1+i}\left\{(z+1-i)^2\cdot\frac{e^{tz}}{z^2(z+1-i)(z+1+i)}\right\}$$

$$= \lim_{z\to -1+i}\left\{\frac{e^{tz}}{z^2(z+1+i)}\right\}$$

$$= \frac{e^{(-1+i)t}}{(-1+i)^2(-1+i+1+i)}$$

$$= \frac{e^{(-1+i)t}}{(1-2i+i^2)(2i)}$$

$$= \frac{e^{(-1+i)t}}{(1-2i-1)(2i)}$$

$$= \frac{e^{(-1+i)t}}{-4i^2} = \frac{e^{-t}}{4}\, e^{it}$$

Similarly residue at $z = -1 - i$ is $\frac{e^{-t}}{4}\, e^{-it}$

Therefore by Cauchy's residue theorem we have

$$\frac{1}{2\pi i}\oint_C \frac{e^{tz}}{(z^2+1)^2}dz = \frac{1}{2}(t-1) + \frac{e^{-t}}{4}\, e^{it} + \frac{e^{-t}}{4}\, e^{-it}$$

$$= \frac{1}{2}(t-1) + \frac{e^{-t}}{4}\left(e^{it} + e^{-it}\right)$$

$$= \frac{1}{2}(t-1) + \frac{e^{-t}}{4} \cdot 2\cos t$$

$$= \frac{t-1}{2} + \frac{1}{2}\, e^{-t}\cos t$$

(Showed)

**4.** Evaluate $\oint_C \frac{e^{3z}}{z+\pi i}\, dz,$ where $C$ is the circle $|z+1| = 4$.

**Solution:** The given circle is $|z+1| = 4$. Let

$$\oint_C \frac{e^{3z}}{z+\pi i}\, dz = \oint_C f(z)\, dz$$

where $f(z) = \frac{e^{3z}}{z+\pi i}$

The poles of $f(z)$ are given by the equation $z + \pi i = 0 \Rightarrow z = -\pi i$

$$|z| = |-\pi i| = \pi < 4$$

$\therefore$ The pole $z = -\pi i$ lies inside $C$.

Residue at $z = -\pi i$ is

$$\lim_{z \to -\pi i}(z + i\pi)\, f(z)$$

$$= \lim_{z \to -\pi i} (z + i\pi) \frac{e^{3z}}{z + i\pi}$$

$$= \lim_{z \to -\pi i} e^{3z}$$

$$= e^{-i3\pi}$$

$$= \cos 3\pi - i \sin 3\pi$$

$$= (-1) - 0 = -1$$

Therefore, by Cauchy's residue theorem we have

$$\oint_C \frac{e^{3z}}{z + \pi i} dz = 2\pi i \ (-1)$$

$$= -2\pi i \qquad \text{Ans.}$$

**5.** Evaluate the integral $\oint_C \frac{e^{-iz}}{(z+3)(z-i)^2} dz$**,** $C = \{z: z = 1 + 2e^{i\theta}, 0 \leq \theta \leq 2\pi\}$ using Cauchy's residue theorem.

**Solution:** Equation of the given curve is

$$z = 1 + 2e^{i\theta}$$

$$\Rightarrow z - 1 = 2e^{i\theta}$$

$$\Rightarrow |z - 1| = |2e^{i\theta}|$$

$$\Rightarrow |z - 1| = 2 \qquad \because |e^{i\theta}| = 1$$

$\therefore$ The given curve is a circle whose centre is $(1,0)$ and the radius is 2. Let

$$\oint_C \frac{e^{-iz}}{(z + 3)(z - i)^2} dz = \oint_C f(z)\ dz$$

where $f(z) = \dfrac{e^{-iz}}{(z + 3)(z - i)^2}$

The poles of $f(z)$ are given by $(z + 3)(z - i)^2 = 0$

$$\Rightarrow z = -3 \ and \ z = i, i$$

$|-3| = 3 > 2$ and $|i| = 1 < 2$

$\therefore$ The pole $z = i$ lies inside the circle which is a double pole.

Residue at $z = i$ is

$$\lim_{z \to i} \frac{1}{(2 - 1)!} \frac{d}{dz} [(z - i)^2 \cdot f(z)]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \cdot \frac{e^{-iz}}{(z+3)(z-i)^2}\right]$$

$$= \lim_{z\to i} \frac{d}{dz}\left[\frac{e^{-iz}}{(z+3)}\right]$$

$$= \lim_{z\to i} \left[\frac{(z+3)\left(-ie^{-iz}\right) - e^{-iz}\cdot 1}{(z+3)^2}\right]$$

$$= \frac{(i+3)(-ie) - e}{(i+3)^2}$$

$$= \frac{(1-3i-1)e}{i^2+6i+9}$$

$$= \frac{-3ie}{8+6i}$$

$$= \frac{-3ie}{2(4+3i)} \times \frac{4-3i}{4-3i}$$

$$= \frac{-12ie-9e}{2(16+9)}$$

$$= \frac{-12ie-9e}{50}$$

Therefore, by Cauchy's residue theorem we have

$$\oint_C \frac{e^{-iz}}{(z+3)(z-i)^2}\,dz = 2\pi i \cdot (\text{Residue at } z = i)$$

$$= 2\pi i \cdot \frac{-12ie-9e}{50}$$

$$= \frac{(12-9i)\pi e}{25}$$

Ans.

**6.** Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$, where $C$ is the a curve $|z-1| = 4$.

**Solution:** The given curve is $|z-1| = 4$. Let

$$\oint_C \frac{e^{3z}}{z-\pi i}\,dz = \oint_C f(z)\,dz$$

where $f(z) = \frac{e^{3z}}{z-\pi i}$

The poles of $\mathrm{f}(z)$ are given by the equation $z - \pi i = 0 \Rightarrow z = \pi i$

$$|z| = |\pi i| = \pi < 4$$

$\therefore$ The pole $z = \pi i$ lies inside $C$.

Residue at $z = \pi i$ is

$$\lim_{z \to \pi i} (z - i\pi)\ \mathrm{f(z)}$$

$$= \lim_{z \to \pi i} (z - i\pi) \frac{\mathrm{e}^{3\mathrm{z}}}{z - i\pi}$$

$$= \lim_{z \to \pi i} e^{3z} = e^{i3\pi}$$

$$= \cos 3\pi + i \sin 3\pi$$

$$= -1 + i(0) = -1$$

Therefore, by Cauchy's residue theorem we have

$$\oint_C \frac{e^{3z}}{z - \pi i}\, dz = 2\pi i\ (-1)$$

$$= -2\pi i \qquad \text{Ans.}$$

**7.** Evaluate $\oint_C \frac{e^{3z}}{z - \pi i}\, dz$, where $C$ is the a curve $|z - 2| + |z + 2| = 6$.

**Solution:** The given curve is $|z - 2| + |z + 2| = 6$

which is the equation of an ellipse whose foci are $(2, 0)$ and $(-2, 0)$

and length of the major axis is 6.

$$\oint_C \frac{e^{3z}}{z - \pi i}\, dz = \oint_C f(z)\, dz$$

where $f(z) = \frac{e^{3z}}{z - \pi i}$

The poles of $\mathrm{f}(z)$ are given by the equation $z - \pi i = 0 \Rightarrow z = \pi i$

$$|z| = |\pi i| = \pi = 3.14 > 3$$

$\therefore$ The pole $z = \pi i$ lies outside the ellipse.

Therefore, by Cauchy's residue theorem we have

$$\oint_C f(z)\, dz = 0$$

$$\Rightarrow \oint_C \frac{e^{3z}}{z - \pi i}\, dz = 0$$

Ans.

**8.** Let $C$ denote the square whose side lie along the lines $x = \pm 2, y = \pm 2$ described in the positive sense. Determine $\oint_C \frac{1}{z^2+9}\, dz.$

**Solution:** Poles are obtained from the equation $z^2 + 9 = 0$

$$\Rightarrow z^2 = -9 = (3i)^2$$

$$\Rightarrow z = \pm 3i$$

$\therefore$ Each pole lie outside the square $C$.

Therefore, by Cauchy's residue theorem we have

$$\oint_C \frac{1}{z^2+9} dz = 0$$

Ans.

**9.** Let $C$ denote the square whose side lie along the lines $x = \pm 2, y = \pm 2$ described in the positive sense. Determine $\oint_C \frac{1}{z(z^2+9)}\, dz.$

**Solution:** Poles are obtained from the equation $z(z^2 + 9) = 0$

$$\Rightarrow z(z + 3i)(z - 3i) = 0$$

$$\Rightarrow z = 0, z = 3i, z = -3i$$

$\therefore$ Only the pole $z = 0$ lies inside the square $C$.

Residue at $z = 0$ is

$$\lim_{z \to 0} \mathrm{z} \cdot \frac{1}{z(z^2+9)}$$

$$= \lim_{z \to 0} \frac{1}{z^2+9}$$

$$= \frac{1}{9}$$

Therefore, by Cauchy's residue theorem we have

$$\oint_C \frac{1}{z(z^2+9)} dz = 2\pi i \left(\frac{1}{9}\right)$$

$$= \frac{2\pi i}{9}$$

Ans.

**10.** Let $C$ denote the square whose side lie along the lines $x = \pm 2, y = \pm 2$ described in the positive sense. Determine $\oint_C \frac{1}{(z^2+1)(z^2+9)}\, dz$.

**Solution:** Poles are obtained from the equation $(z^2 + 1)(z^2 + 9) = 0$

$\Rightarrow (z + i)(z - i)(z + 3i)(z - 3i) = 0$

$\Rightarrow z = i, -i, 3i, -3i$

$\therefore$ Only the poles $i, -i$ lie inside the square $C$ which are simple poles.

Residue at $z = i$ is

$$\lim_{z \to i} \left\{ (\mathrm{z} - \mathrm{i}) \cdot \frac{1}{(z^2 + 1)(z^2 + 9)} \right\}$$

$$= \lim_{z \to i} \frac{1}{(z + i)(z^2 + 9)}$$

$$= \frac{1}{2i(i^2 + 9)}$$

$$= \frac{1}{16i}$$

$$= \frac{-i}{16}$$

Similarly, residue at $z = -i$ is $\frac{i}{16}$

Therefore, by Cauchy's residue theorem we have

$$\oint_C \frac{1}{z(z^2 + 9)}\, dz = 2\pi i \text{ (sum of the residues)}$$

$$= 2\pi i \left( \frac{-i}{16} + \frac{i}{16} \right)$$

$$= 0$$

Ans.