1. Show that $\oint_C \frac{\sin 3z}{z+\frac{\pi}{2}} dz = 2\pi i$ where C is the circle $|z| = 5$.

Solution: Let $f(z) = \sin 3z$

Then $f(z) = \sin 3z$ is analytic inside and on the circle $|z| = 5$.

Also here, $z = -\frac{\pi}{2}$

$$\Rightarrow |z| = \left|-\frac{\pi}{2}\right| = \frac{\pi}{2} = \frac{3.14}{2} = 1.57 < 5$$

$\therefore z = -\frac{\pi}{2}$ lies in the circle $|z| = 5$.

Hence by Cauchy's integral formula we have

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a} dz = f(a)$$

$$\Rightarrow \oint_C \frac{f(z)}{z-a} dz = 2\pi i\, f(a)$$

$$\Rightarrow \oint_C \frac{\sin 3z}{z+\frac{\pi}{2}} dz = 2\pi i\, f(\frac{-\pi}{2})$$

$$= 2\pi i\ \sin(\frac{-3\pi}{2})$$

$$= 2\pi i\ \cos 0°$$

$$= 2\pi i \times 1$$

$$= 2\pi i$$ (Showed)

2. Show that $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz = \begin{cases} e^2, \ if\ C\ is\ the\ circle\ |z| = 3 \\ 0\ ,\ if\ C\ is\ the\ circle\ |z| = 1 \end{cases}$

Solution: Let $f(z) = e^z$

Then $f(z) = e^z$ is analytic inside and on the circle $|z| = 3$.

Also here, $z = a = 2$

$$\Rightarrow |z| = |2| = 2 < 3$$

$\therefore z = 2$ lies inside the circle $|z| = 3$.

Hence by Cauchy's integral formula we have

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz = f(a)$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz = f(2)$$

$$= e^2 \qquad \text{(Showed)}$$

Again, $f(z) = \frac{e^z}{z-2}$ is analytic inside and on the circle $|z| = 1$.

Also here, $z = a = 2$

$$\Rightarrow |z| = |2| = 2 > 1$$

$\therefore z = 2$ lies outside the circle $|z| = 1$.

Hence by Cauchy's integral theorem we have

$$\oint_C f(z)dz = 0$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz = 0$$

(Showed)

3. Show that $\oint_C \frac{e^{3z}}{z-\pi i} dz = \begin{cases} -2\pi i, \; if\ C\ is\ the\ circle\ |z-1| = 4 \\ \quad 0, \; if\ C\ is\ the\ ellipse\ |z-2| + |z+2| = 6 \end{cases}$

Solution: Let $\mathrm{f}(z) = e^{3z}$

Then $\mathrm{f}(z) = e^{z}$ is analytic inside and on the circle $|z-1| = 4$.

Also here, $z = a = \pi i$

$$\Rightarrow |z| = |\pi i| = \pi = 3.14 < 4$$

$\therefore z = \pi i$ lies inside the circle $|z-1| = 4$.

Hence by Cauchy's integral formula we have

$$\frac{1}{2\pi i}\oint_C \frac{\mathrm{f}(z)}{z-a} dz = \mathrm{f}(a)$$

$$\Rightarrow \oint_C \frac{\mathrm{f}(z)}{z-a} dz = 2\pi i\, \mathrm{f}(a)$$

$$\Rightarrow \oint_C \frac{\mathrm{e}^{3\mathrm{z}}}{z-\pi i} dz = 2\pi i\, \mathrm{f}(\pi i)$$

$$= 2\pi i\, e^{3\pi i}$$

$$= 2\pi i\, (\cos 3\pi + i \sin 3\pi)$$

$$= 2\pi i\, (-\cos 0 + i \sin 0)$$

$$= 2\pi i\, (-1 + 0)$$

$$= -2\pi i$$ (Showed)

Again, $\mathrm{f}(z) = \frac{e^{3z}}{z-\pi i}$

Also here, $|z-2| + |z+2| = 6$ is the equation of an ellipse whose foci are $(2,0)$ and $(-2,0)$ and length of the major axis is 6.

$\therefore z = \pi i$ lies outside the ellipse.

Hence by Cauchy's integral theorem we have

$$\oint_C \mathrm{f}(z) dz = 0 \quad \Rightarrow \oint_C \frac{\mathrm{e}^{3\mathrm{z}}}{z-\pi i} dz = 0$$

(Showed)

4. Show that $\oint_C \frac{e^{tz}}{z^2+1} dz = 2\pi i \sin t$ where C is the circle $|z| = 3$ and $t > 0$.

Solution: Let $\mathrm{f}(z) = e^{tz}$

Then $\mathrm{f}(z) = e^{tz}$ is analytic inside and on the circle $|z| = 3$.

Again $z^2 + 1 = z^2 - (-1)$

$$= z^2 - i^2$$

$$= (z+i)(z-i)$$

We have $\frac{1}{(z+i)(z-i)} = \frac{-\frac{1}{2i}}{z+i} + \frac{\frac{1}{2i}}{z-i} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$

$$\therefore \oint_C \frac{e^{tz}}{z^2+1} dz = \frac{1}{2i}\oint_C \frac{e^{tz}}{z-i} dz - \frac{1}{2i}\oint_C \frac{e^{tz}}{z+i} dz \cdots\cdots (1)$$

Also, $z = i \Rightarrow |z| = |i| = 1 < 3$ and $z = -i \Rightarrow |z| = |-i| = 1 < 3$

$\therefore z = i$ and $z = -i$ lie inside the circle $|z| = 3$.

Hence by Cauchy's integral formula we have

$$\oint_C \frac{e^{tz}}{z-i} dz = 2\pi i\, \mathrm{f}(i) = 2\pi i\, e^{it}$$

$$and \oint_C \frac{e^{tz}}{z+i} dz = 2\pi i\, \mathrm{f}(-i) = 2\pi i\, e^{-it}$$

Putting these values in (1) we get

$$\Rightarrow \oint_C \frac{e^{tz}}{z^2+1} dz = \frac{1}{2i} 2\pi i\, e^{it} - \frac{1}{2i} 2\pi i\, e^{-it}$$

$$= \pi i\, e^{it} - \pi i\, e^{-it}$$

$$= 2\pi i\, \frac{e^{it} - e^{-it}}{2i}$$

$$= 2\pi i \sin t \qquad \text{(Showed)}$$

5. Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle $|z| = 3$.

Solution: We have $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

Let $\mathrm{f}(z) = \sin \pi z^2 + \cos \pi z^2$. Then $\mathrm{f}(z)$ is analytic inside C and $z = 1, z = 2$ lie in the circle $|z| = 3$. Therefore by Cauchy's integral formula we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i\{\sin \pi (2)^2 + \cos \pi (2)^2\}$$

$$-2\pi i\{\sin \pi (1)^2 + \cos \pi (1)^2\}$$

$$= 2\pi i\{0 + 1 - 0 + 1\}$$

$$= 2\pi i \times 2$$

$$= 4\pi i$$

6. Show that $\oint_C \frac{e^z}{z(z+1)} dz = 2\pi i(1 - e^{-1})$ where C is the circle $|z - 1| = 3$.

Solution: We have $\frac{1}{z(z+1)} = \frac{1}{z} - \frac{1}{z+1}$

$$\oint_C \frac{e^z}{z(z+1)} dz = \oint_C \frac{e^z}{z} dz - \oint_C \frac{e^z}{z+1} dz$$

Let $\mathrm{f}(z) = e^z$. Then $\mathrm{f}(z)$ is analytic inside C and $z = 0, z = -1$ lie in the circle $|z - 1| = 3$. Therefore by Cauchy's integral formula we have

$$\oint_C \frac{e^z}{z(z+1)} dz = 2\pi i\, e^0 - 2\pi i\, e^{-1}$$

$$= 2\pi i\{1 - e^{-1}\} \qquad \text{(Showed)}$$

7. Show that $\frac{1}{2\pi i}\oint_C \frac{ze^{tz}}{(z+1)^3}dz = \left(t - \frac{1}{2}t^2\right)e^{-t}$ where C is any simple closed curve enclosing $z = -1$ and $t > 0$.

Solution: From Cauchy's integral formula for higher derivatives we know that,

$$\mathrm{f}^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{\mathrm{f}(z)}{(z-a)^{n+1}}dz$$

$$\Rightarrow \frac{1}{2\pi i}\oint_C \frac{\mathrm{f}(z)}{(z-a)^{n+1}}dz = \frac{1}{n!}\ \mathrm{f}^{(n)}(a)$$

$$= \frac{1}{n!}\left[\frac{d^n}{dz^n}\ \mathrm{f}(z)\right]_{z=a} \quad \cdots\cdots (1)$$

Let $\mathrm{f}(z) = ze^{tz}$ and $a = -1, n = 2$

Then $\mathrm{f}(z) = ze^{tz}$ is analytic inside and on C

Thus, from (1) we get

$$\frac{1}{2\pi i}\oint_C \frac{\mathrm{ze^{tz}}}{(z+1)^3}dz = \frac{1}{2!}\left[\frac{d^2}{dz^2}\ (ze^{tz})\right]_{z=-1}$$

$$= \frac{1}{2}\left[\frac{d}{dz}\ (e^{tz} + tze^{tz})\right]_{z=-1}$$

$$= \frac{1}{2}\left[te^{tz} + te^{tz} + t^2ze^{tz}\right]_{z=-1}$$

$$= \frac{1}{2}\left[te^{-t} + te^{-t} + t^2(-1)e^{-t}\right]$$

$$= \frac{1}{2}\left(2te^{-t} - t^2e^{-t}\right)$$

$$= \frac{1}{2}\left(2t - t^2\right)e^{-t}$$

$$= \left(t - \frac{1}{2}t^2\right)e^{-t}$$

8. Show that $\oint_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi i e^{-2}}{3}$ where C is the circle $|z| = 3$.

Solution: From Cauchy's integral formula for higher derivatives we know that,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

$$\Rightarrow \oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

$$= \frac{2\pi i}{n!} \left[\frac{d^n}{dz^n} f(z)\right]_{z=a} \quad \cdots\cdots (1)$$

Let $f(z) = e^{2z}$ and $a = -1, n = 3$

Then $f(z) = e^{2z}$ is analytic inside and on the circle $|z| = 3$.

Also, $z = a = -1 \quad \Rightarrow |z| = |-1| = 1 < 3$

$\therefore z = -1$ lies inside the circle $|z| = 3$

Thus, from (1) we get

$$\oint_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} \left[\frac{d^3}{dz^3} e^{2z}\right]_{z=-1}$$

$$= \frac{2\pi i}{6} [8e^{2x}]_{z=-1}$$

$$= \frac{\pi i}{3} 8e^{-2}$$

$$= \frac{8\pi i e^{-2}}{3}$$

9. Show that $\oint_C \frac{e^{iz}}{z^3}\,dz = -\pi i$ where C is the circle $|z| = 2$.

Solution: From Cauchy's integral formula for higher derivatives we know that,

$$\mathrm{f}^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{\mathrm{f}(z)}{(z-a)^{n+1}}\,dz$$

$$\Rightarrow \oint_C \frac{\mathrm{f}(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!}\,\mathrm{f}^{(n)}(a)$$

$$= \frac{2\pi i}{n!}\left[\frac{d^n}{dz^n}\,\mathrm{f}(z)\right]_{z=a} \quad \cdots\cdots (1)$$

Let $\mathrm{f}(z) = e^{iz}$ and $a = 0, n = 2$

Then $\mathrm{f}(z) = e^{iz}$ is analytic inside and on the circle $|z| = 2$.

Also, $z = a = 0 \quad \Rightarrow |z| = |0\,| = 0 < 2$

$\therefore z = 0$ lies inside the circle $|z| = 2$

Thus, from (1) we get

$$\oint_C \frac{\mathrm{e}^{\mathrm{iz}}}{z^3}\,dz = \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}\,\mathrm{e}^{\mathrm{iz}}\right]_{z=0}$$

$$= \frac{2\pi i}{2}\left[\frac{d}{dz}\,i\mathrm{e}^{\mathrm{iz}}\right]_{z=0}$$

$$= \pi i\left[i^2\mathrm{e}^{\mathrm{iz}}\right]_{z=0}$$

$$= \pi i\, i^2\,\mathrm{e}^{\mathrm{i.0}}$$

$$= -\pi i \qquad \text{(Showed)}$$

10. Show that $\oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \frac{21\pi i}{6}$ where C is the circle $|z| = 1$.

Solution: From Cauchy's integral formula for higher derivatives we know that,

$$\mathrm{f}^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{\mathrm{f}(z)}{(z-a)^{n+1}} dz$$

$$\Rightarrow \oint_C \frac{\mathrm{f}(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} \mathrm{f}^{(n)}(a)$$

$$= \frac{2\pi i}{n!} \left[\frac{d^n}{dz^n} \mathrm{f}(z)\right]_{z=a} \quad \cdots\cdots (1)$$

Let $\mathrm{f}(z) = \sin^6 z$ and $a = \frac{\pi}{6}, n = 2$

Then $\mathrm{f}(z) = \sin^6 z$ is analytic inside and on the circle $|z| = 1$.

Also, $z = a = \frac{\pi}{6} \quad \Rightarrow |z| = \left|\frac{\pi}{6}\right| = 0.52 < 2$

$\therefore z = \frac{\pi}{6}$ lies inside the circle $|z| = 1$

Thus, from (1) we get

$$\oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!} \left[\frac{d^2}{dz^2} \sin^6 z\right]_{z=\frac{\pi}{6}}$$

$$= \frac{2\pi i}{2} \left[\frac{d}{dz} (6\sin^5 z \cos z)\right]_{z=\frac{\pi}{6}}$$

$$= \pi i \left[-6\sin^6 z + 30\sin^4 z \cos^2 z\right]_{z=\frac{\pi}{6}}$$

$$= \pi i \left[-6\left(\frac{1}{2}\right)^6 + 30\left(\frac{1}{2}\right)^6 \left(\frac{\sqrt{3}}{2}\right)^2\right] = \frac{(-6+90)\pi i}{64}$$

$$= \frac{21\pi i}{16}$$

(Showed)