Complex Variable

Theorem:

The continuous single valued function is $f(z) = u(x,y) + iv(x,y)$ will be analytic in a region R if and only if four partial derivatives $u_x,\ v_x, u_y,\ v_y,$ exists, continuous and Cauchy Riemann equations $u_x = v_y, u_y = -v_x$ are satisfied.

Proof: If $f(z) = u(x, y) + iv(x, y)$ is analytic then

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x + i\Delta y}$$

Exists and unique. Two cases arise.

Case 1. Along x-axis $\Delta y = 0$ and $\Delta x \to 0$

$$f'(z) = \lim_{\Delta x \to 0} \frac{\{u(x+\Delta x, y) - u(x, y)\} + i\{v(x+\Delta x, y) - v(x, y)\}}{\Delta x}$$

$$f'(z) = u_x + iv_x \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Case 2. Along y-axis $\Delta x = 0$ and $\Delta y \to 0$

$$f'(z) = \lim_{\Delta y \to 0} \frac{\{u(x, y+\Delta y) - u(x, y)\} + i\{v(x, y+\Delta y) - v(x, y)\}}{i\Delta y}$$

$$f'(z) = \frac{1}{i} u_y + v_y$$

$$f'(z) = -iu_y + v_y \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

From equation (1) and (2) we get

$$u_x + iv_x = -iu_y + v_y$$

Therefore $u_x = v_y,\ u_y = -v_x$

Since u_x and u_y are continuous then

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$$

$$\Delta u = \{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)\} + \{u(x, y+\Delta y) - u(x,y)\}$$

$$= (\frac{\partial u}{\partial x} + \varepsilon_1)\Delta x + (\frac{\partial u}{\partial y} + \eta_1)\Delta y$$

$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1 \Delta x + \eta_1 \Delta y \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

Where $\varepsilon_1 \to 0$ and $\eta_1 \to 0$ Since $\Delta x \to 0$ and $\Delta y \to 0$

Similarly v_x and v_y are continuous then

$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_2 \Delta x + \eta_2 \Delta y$$

Where $\varepsilon_2 \to 0$ and $\eta_2 \to 0$ Since $\Delta x \to 0$ and $\Delta y \to 0$

Now $\Delta w = \Delta u + i\Delta v = (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})\Delta x + (\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y})\Delta y + \varepsilon \Delta x + \eta \Delta y$

Where $\varepsilon = \varepsilon_1 + i\varepsilon_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ Since $\Delta x \to 0$ and $\Delta y \to 0$

Now using Cauchy Riemann equations

$$\Delta w = \Delta u + i\Delta v = (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})\Delta x + (-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x})\Delta y + \varepsilon \Delta x + \eta \Delta y$$

$$\Delta w = (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})\Delta x + (i^2\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x})\Delta y + \varepsilon \Delta x + \eta \Delta y$$

$$\Delta w = (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})\Delta x + i(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})\Delta y + \varepsilon \Delta x + \eta \Delta y$$

$$\Delta w = (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})(\Delta x + i\Delta y) + \varepsilon \Delta x + \eta \Delta y$$

$$\Delta w = (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})\Delta z + \varepsilon\Delta x + \eta\Delta y$$

Dividing by Δz and Taking limit $\Delta z \to 0$

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = (\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x})$$

Hence $f(z)$ is analytic.

Problem: Show that the function $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. And find the conjugate harmonic function v.

Problem: Show that the function $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. And find the conjugate harmonic function of u

Solution: Given that $u = x^2 - y^2 - 2xy - 2x + 3y$

$$\frac{\partial u}{\partial x} = 2x - 2y - 2$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -2$$

Therefore $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Implies that u is harmonic

From CR equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x - 2y - 2 \qquad \text{……………………. (1)}$$

$$\text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2y - 2x + 3 \qquad \text{……………………..(2)}$$

Integrating (1) on both sides $v = \int (2x - 2y - 2)dy$

$$\Rightarrow v = 2xy - y^2 - 2y + F(x) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Differentiate

$$\frac{\partial v}{\partial x} = 2y + F'(x)$$

$$\Rightarrow 2y + 2x - 3 = 2y + F'(x)$$

$$\Rightarrow F'(x) = 2x - 3$$

$$\Rightarrow F(x) = \int (2x - 3)dx = x^2 - 3x + c$$

Putting this value in equation (3)

$$\Rightarrow v = 2xy - y^2 - 2y + x^2 - 3x + c$$

Conjugate harmonic function

Problem: Show that $u = e^x(x\cos y - y\sin y)$ is harmonic. Also find the conjugate harmonic function of u

Problem: Find the conjugate harmonic function of $u = e^{-x}(x\sin y - y\cos y)$

Cauchy's Integral formula:

If $f(z)$ is analytic for all points inside of C and connected a simple closed curve C. a is any point inside C. Then $f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz$.

Proof: Since $f(z)$ is analytic for all points inside of C

$f(z) = f(a) + (z-a)f'(a) + (z-a)\eta$ where $z \to a$, then $\eta \to 0$

Now $\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\oint_c \frac{f(a) + [f(z) - f(a)]}{z-a}dz$.

$$= \frac{f(a)}{2\pi i}\oint_c \frac{dz}{z-a} + \frac{1}{2\pi i}\oint_c \frac{[f(z)-f(a)]}{z-a}dz\,.$$

$$= \frac{f(a)}{2\pi i}\oint_c \frac{dz}{z-a} + \frac{1}{2\pi i}\oint_c \frac{[f(a)+(z-a)f'(a)+(z-a)\eta - f(a)]}{z-a}dz\,.$$

$$= \frac{f(a)}{2\pi i}2\pi i + \frac{1}{2\pi i}\oint_c \frac{(z-a)[f'(a)+\eta]}{z-a}dz\,.$$

$$= f(a) + \frac{f'(a)}{2\pi i}\oint_c dz + \frac{1}{2\pi i}\oint_c \eta dz\,.$$

$$= f(a) + 0 + \frac{1}{2\pi i}\oint_c \eta dz\,.$$

$$\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz - f(a) = \frac{1}{2\pi i}\oint_c \eta dz\,.$$

Considering C is very small so that $|\eta| < \varepsilon$ for all points on C.

$$\left|\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz - f(a)\right| = \left|\frac{1}{2\pi i}\oint_c \eta dz\right| < \frac{\varepsilon}{2\pi}L\,.$$

$$\left|\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz - f(a)\right| = 0\,.$$

$$\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz - f(a) = 0\,.$$

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz\,.$$

Cauchy's Integral formula for nth order derivative:

$$f^{n}(a) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-a)^{n+1}}dz\,.$$

Evaluate:

1. $\oint_c \frac{z}{z-1} dz$ where C is the circle $|z| = 2$

2. $\oint_c \frac{z^2}{(z-i)} dz$ where C is the circle $|z| = 2$

3. $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle $|z| = 3$

4. $\oint_c \frac{\sin 3z}{(z+\frac{\pi}{2})} dz$ where C is the circle $|z| = 5$

5. $\oint_c \frac{e^{tz}}{(z^2+1)} dz$ where C is the circle $|z| = 3$

6. $\oint_c \frac{\sin^6 z}{(z-\frac{\pi}{6})^3} dz$ where C is the circle $|z| = 1$

Solution 1:

We know

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz .$$

Here $f(z) = z$ and $a = 1$

then $f(1) = 1$

Now $\oint_c \frac{z}{z-1} dz = 2\pi i f(1) = 2\pi i \times 1 = 2\pi i$

Solution 2:

We know

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz .$$

Here $f(z) = z^2$ and $a = i$

then $f(i) = i^2$

Now $\oint_c \frac{z^2}{z-i}dz = 2\pi i f(i) = 2\pi i \times i^2 = 2\pi i(-1) = -2\pi i$

Solution 3:

We know

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz .$$

Now

$\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$ Here $f(z) = \sin \pi z^2 + \cos \pi z^2$

$$\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$$

$$= \int_c \frac{f(z)}{(z-1)(z-2)}dz$$

$$= \oint_c \left(\frac{1}{z-2} - \frac{1}{z-1}\right) f(z)dz$$

$$= \oint_c \frac{f(z)}{z-2}dz - \oint_c \frac{f(z)}{z-1}dz$$

$$= 2\pi i f(2) - 2\pi i f(1)$$

$$= 2\pi i(\sin \pi 2^2 + \cos \pi 2^2) - 2\pi i(\sin \pi 1^2 + \cos \pi 1^2)$$

$$= 2\pi i(0+1) - 2\pi i(0-1)$$

$$= 2\pi i + 2\pi i$$

$$= 4\pi i$$

Solution 4. $\oint_c \frac{\sin 3z}{(z+\frac{\pi}{2})}dz$ where C is the circle $|z| = 5$

We know

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz .$$

$f(z) = \sin 3z$ and $a = -\frac{\pi}{2}$

then $f(-\frac{\pi}{2}) = \sin 3(-\frac{\pi}{2})$

$$\oint_c \frac{\sin 3z}{(z+\frac{\pi}{2})} dz = 2\pi i \times f(-\frac{\pi}{2}) = 2\pi i \times \sin 3(-\frac{\pi}{2}) = 2\pi i$$

Solution 5. $\oint_c \frac{e^{tz}}{(z^2+1)} dz$ where C is the circle $|z| = 3$

We know

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz$$

$$f(z) = e^{tz}$$

Now

$$\oint_c \frac{e^{tz}}{(z^2+1)} dz = \oint_c \frac{f(z)}{(z+i)(z-i)} dz$$

$$= \frac{1}{2i} \oint_c \left(\frac{1}{z-i} - \frac{1}{z+i}\right) f(z) dz$$

$$= \frac{1}{2i} \left(\oint_c \frac{f(z)}{z-i} dz - \oint_c \frac{f(z)}{z+i} dz\right)$$

$$= \frac{1}{2i} [2\pi i f(i) - 2\pi i f(-i)]$$

$$= \frac{1}{2i} \times 2\pi i (e^{ti} - e^{-ti})$$

$$= \pi \times 2i \sin t$$

$$= 2\pi i \sin t$$

Solution 6. $\oint_c \frac{\sin^6 z}{(z-\frac{\pi}{6})^3} dz$ where C is the circle $|z| = 1$

We know

$$f^n(a) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{(z-a)^{n+1}} dz \; .$$

$f(z)=\sin^6 z$, $a=\frac{\pi}{6}$ and $n=2$

Now

$$\oint_c \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}dz = \frac{2\pi i}{2!}\,f''(\frac{\pi}{6}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Here $f'(z)=6\sin^5 z\cos z$,

$$f''(z)=6[5\sin^4 z\cos z\cos z+\sin^5 z(-\sin z)]$$

$$f''(\frac{\pi}{6})=\frac{21}{16}$$

From (1)

$$\oint_c \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}dz$$

$$=\frac{2\pi i}{2!}\,f''(\frac{\pi}{6})$$

$$=\pi i\times\frac{21}{16}$$

Problem 7. $\oint_c \frac{e^{tz}}{(z+1)^3}dz$

Problem 8. $\oint_c \frac{ze^{tz}}{(z+1)^3}dz$

Theorem: if $f(z)$ is analytic inside and on a simple closed curve C except at the pole $z=a$ of order m then the residue of $f(z)$ at $z=a$ is

$$\operatorname{Re}s(a)=\lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]$$

If $z=a$ is a simple pole then $\operatorname{Re}s(a)=\lim_{z\to a}(z-a)f(z)$

Cauchy's Residue Theorem:

Let $f(z)$ be analytic inside and on a simple closed curve C except at a finite number of singular points $a_1,a_2,a_3,..................,a_n$, then

$$\oint_c f(z)dz=2\pi i[\operatorname{Re}s(a_1)+\operatorname{Re}s(a_2)+..................+\operatorname{Re}s(a_n)]$$

Proof: Let $a_1,a_2,a_3,..................,a_n$ be the center of the circle $C_1,C_2,C_3,..................,C_n$ respectively. $f(z)$ be analytic inside and on a simple closed curve C.

$$\oint_C f(z)dz=\oint_{C_1} f(z)dz+\oint_{C_2} f(z)dz+.......................+\oint_{C_n} f(z)dz \qquad (1)$$

But $\operatorname{Re}s(a_1)=\dfrac{1}{2\pi i}\displaystyle\oint_{C_1} f(z)dz\Rightarrow\oint_{C_1} f(z)dz=2\pi i\operatorname{Re}s(a_1)$

$$\operatorname{Re}s(a_2)=\frac{1}{2\pi i}\oint_{C_2} f(z)dz\Rightarrow\oint_{C_2} f(z)dz=2\pi i\operatorname{Re}s(a_2)$$

....

....

$$\operatorname{Re}s(a_n)=\frac{1}{2\pi i}\oint_{C_n} f(z)dz\Rightarrow\oint_{C_n} f(z)dz=2\pi i\operatorname{Re}s(a_n)$$

From (1)

$$\oint_c f(z)dz=2\pi i\operatorname{Re}s(a_1)+2\pi i\operatorname{Re}s(a_2)+..................+2\pi i\operatorname{Re}s(a_n)$$

$$\oint_c f(z)dz=2\pi i[\operatorname{Re}s(a_1)+\operatorname{Re}s(a_2)+..................+\operatorname{Re}s(a_n)]$$

Ex: Show that $\oint_c \frac{e^z}{(z^2+\pi^2)^2} dz = \frac{1}{\pi}$

Solution: let $f(z) = \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2 (z-\pi i)^2}$

To find the pole, $(z+\pi i)^2 (z-\pi i)^2 = 0$

$z = \pi i, -\pi i$ is a pole of order 2.

Now

$$\operatorname{Re} s(\pi i) = \lim_{z\to\pi i} \frac{1}{(2-1)!} \frac{d}{dz}[(z-\pi i)^2 f(z)]$$

$$= \lim_{z\to\pi i} \frac{1}{(2-1)!} \frac{d}{dz}[(z-\pi i)^2 \frac{e^z}{(z+\pi i)^2 (z-\pi i)^2}]$$

$$= \lim_{z\to\pi i} \frac{d}{dz}[\frac{e^z}{(z+\pi i)^2}]$$

$$= \lim_{z\to\pi i} \frac{(z+\pi i)^2 e^z - e^z.2(z+\pi i)}{(z+\pi i)^4}$$

$$= \lim_{z\to\pi i} \frac{(z+\pi i)e^z - e^z.2}{(z+\pi i)^3}$$

$$= \frac{2(\pi i - 1)e^{\pi i}}{-8i\pi^3}$$

$$= \frac{2(\pi i + i^2)e^{\pi i}}{-8i\pi^3}$$

$$= \frac{-(\pi + i)e^{\pi i}}{4\pi^3}$$

$$\operatorname{Re} s(-\pi i) = \lim_{z\to -\pi i} \frac{1}{(2-1)!} \frac{d}{dz}[(z+\pi i)^2 f(z)]$$

$$= \lim_{z \to -\pi i} \frac{1}{(2-1)!} \frac{d}{dz}[(z+\pi i)^2 \frac{e^z}{(z+\pi i)^2 (z-\pi i)^2}]$$

$$= \lim_{z \to -\pi i} \frac{d}{dz}[\frac{e^z}{(z-\pi i)^2}]$$

$$= \lim_{z \to -\pi i} \frac{(z-\pi i)^2 e^z - e^z .2(z-\pi i)}{(z-\pi i)^4}$$

$$= \lim_{z \to -\pi i} \frac{(z-\pi i)e^z - e^z .2}{(z-\pi i)^3}$$

$$= \frac{2(-\pi i - 1)e^{-\pi i}}{8i\pi^3}$$

$$= \frac{2(-\pi i + i^2)e^{-\pi i}}{8i\pi^3}$$

$$= \frac{-(\pi - i)e^{-\pi i}}{4\pi^3}$$

By Cauchy residue theorem

$$\oint_c \frac{e^z}{(z^2+\pi^2)^2} dz = 2\pi i[\operatorname{Re} s(\pi i) + \operatorname{Re} s(-\pi i)]$$

$$= 2\pi i[\frac{-(\pi + i)e^{\pi i}}{4\pi^3} + \frac{-(\pi - i)e^{-\pi i}}{4\pi^3}$$

$$= \frac{1}{2\pi^2}[-\pi(e^{\pi i} + e^{-\pi i}) - i(e^{i\pi} - e^{-i\pi})]$$

$$= \frac{1}{2\pi^2}[-2\pi \cos\pi - i.2i \sin\pi]$$

$$= \frac{1}{2\pi^2}.2\pi$$

$$= \frac{1}{\pi}$$