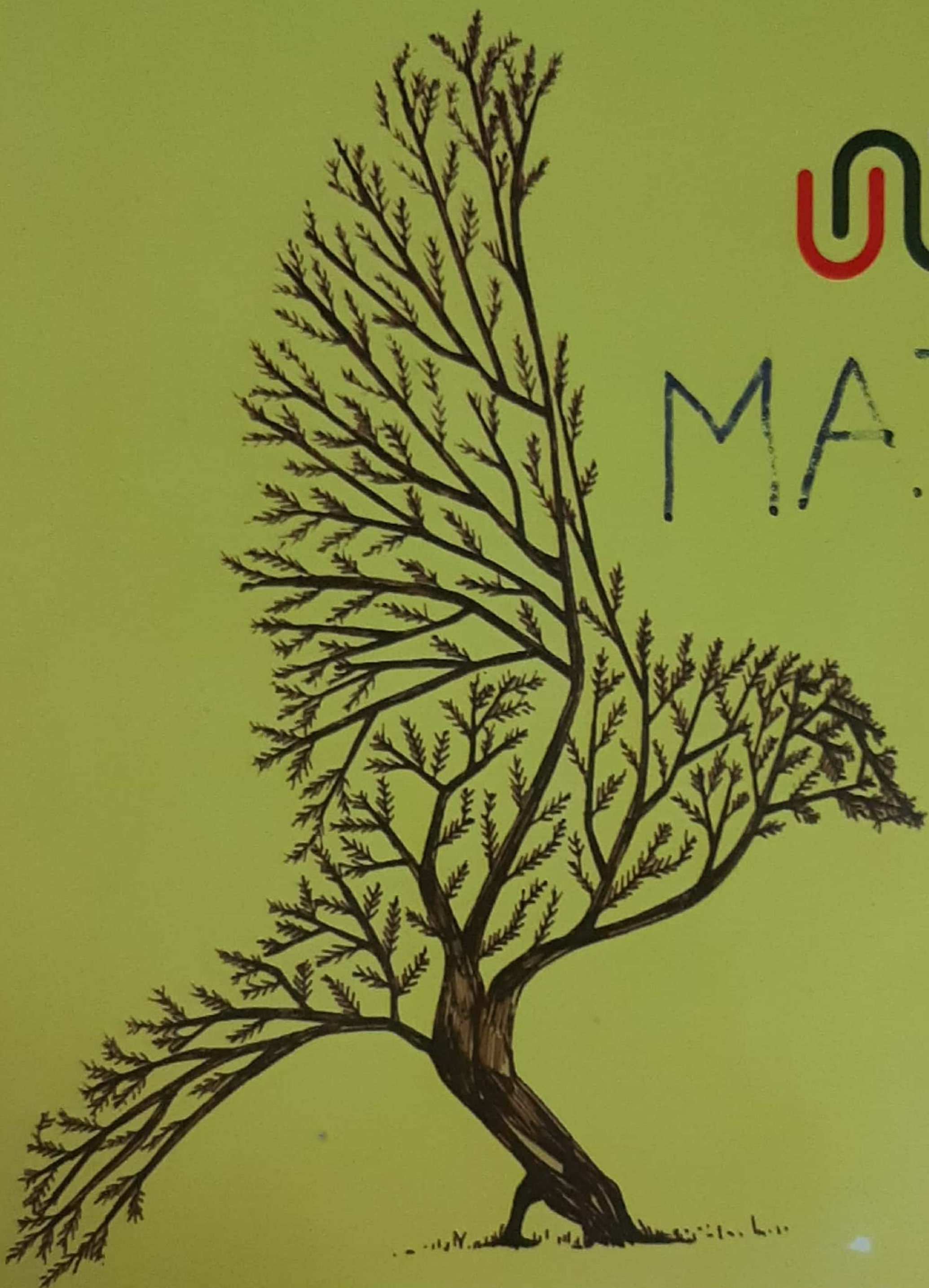

MAT

UNIQUE

WE GIVE PREMIUM QUALITY OVER QUANTITY

Write Your Idea

Name...

Roll/ID .... Course Code.............................

Sub/Course Title.............................. Faculty/Dept..............................

Institute..

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

Reference Books

1. College Mathematical Methods
   Mathematical Physics
   Volume Two

   Integral Transforms and Initial & Boundary Value Problems with Applications
   by Prof. MD. Abdur Rahaman

2. Complex Variables
   by Dewan Abdul Quddus

Plan Title : Online Class – 01

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date : 19/01/2021

Complex Analysis

* Complex Variable

$z = x + iy$

$z \to$ complex variable

Theorem:

The continuous single valued function in $f(z) = u(x,y) + iv(x,y)$ will be analytic in a region R if and only if four partial derivatives u_x, v_x, u_y, v_y exists, are continuous and Cauchy Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$ are satisfied in R.

Proof: If $f(z) = u(x,y) + iv(x,y)$ is analytic then

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)\} - \{u(x,y) + iv(x,y)\}}{\Delta x + i\Delta y}$$

Exists and unique. Two cases arise.

u, v আলাদাভাবে প্রকাশে নিয়ে

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overbrace{\{u(x+\Delta x, y+\Delta y) - u(x,y)\}}^{u} + i\overbrace{\{v(x+\Delta x, y+\Delta y) - v(x,y)\}}^{v}}{\Delta x + i\Delta y}$$

Case-1: Along x-axis $\Delta y = 0$ and $\Delta x \to 0$

$$f'(z) = \lim_{\Delta x \to 0} \frac{\{u(x+\Delta x, y) - u(x,y)\} + i\{v(x+\Delta x, y) - v(x,y)\}}{\Delta x}$$

[Partial Derivatives এর definition অনুযায়ী $u_x = u(x+\Delta x, y) - u(x,y)$] → Same for v_x

$$f'(z) = u_x + iv_x \quad \text{------ (1)}$$

Case-2: Along y-axis $\Delta x = 0$ and $\Delta y \to 0$

$$f'(z) = \lim_{\Delta y \to 0} \frac{\{u(x, y+\Delta y) - u(x,y)\} + i\{v(x, y+\Delta y) - v(x,y)\}}{i\Delta y}$$

$$\left[\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \frac{i\{v(x, y+\Delta y) - v(x,y)\}}{i\Delta y}\right.$$

$$f'(z) = \frac{1}{i} u_y + v_y$$

[Same way Partial derivatives w.r.t y,
$u_y = u(x, y+\Delta y) - u(x,y)$
$v_y = v(x, y+\Delta y) - v(x,y)$]

$$\frac{1\times i}{i \times i} u_y \quad \text{হর, লব কে 'i' দিয়ে গুণ করে}$$
$$\Rightarrow \frac{i}{i^2} u_y$$
$$\Rightarrow \frac{i}{-1} u_y$$
$$\Rightarrow -iu_y$$

$$f'(z) = -iu_y + v_y \quad \text{------ (2)}$$

From equation (1) and (2) we get

$$f'(z) = f'(z)$$
$$u_x + iv_x = -iu_y + v_y$$

[real & imaginary part আলাদা করে]

Therefore $u_x = v_y$, ($u_y = -v_x$ → imaginary part)

[$\therefore$ Cauchy Riemann equation satisfied]

Since u_x and u_y are continuous then

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$$ (দুইটির u এর function এর মধ্যে Difference Δu)

$\frac{\delta u}{\delta x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)}{\Delta x} \cdot \Delta x$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

$$\Delta u = \{u(x+\Delta x, y+\Delta y) - u(x, \Delta y+\Delta y)\} + \{u(x, y+\Delta y) - u(x, y)\}$$

(extra $u(x, y+\Delta y)$ +, - করে)

$$= \left(\frac{\delta u}{\delta x} + \varepsilon_1\right)\Delta x + \left(\frac{\delta u}{\delta y} + \eta_1\right)\Delta y$$

error term

$$\Delta u = \frac{\delta u}{\delta x}\Delta x + \frac{\delta u}{\delta y}\Delta y + \varepsilon_1 \Delta x + \eta_1 \Delta y \quad ----- (3)$$

where $\varepsilon_1 \to 0$ and $\eta_1 \to 0$ since $\Delta x \to 0$ and $\Delta y \to 0$

Similarly v_x and v_y are continuous then

$$\Delta v = \frac{\delta v}{\delta x}\Delta x + \frac{\delta v}{\delta y}\Delta y + \varepsilon_2 \Delta x + \eta_2 \Delta y$$

where $\varepsilon_2 \to 0$ and $\eta_2 \to 0$ since $\Delta x \to 0$ and $\Delta y \to 0$

Now $\Delta w = \Delta u + i\Delta v$

$$= \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)\Delta x + \left(\frac{\delta u}{\delta y} + i\frac{\delta v}{\delta y}\right)\Delta y + \varepsilon \Delta x + \eta \Delta y$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ since $\Delta x \to 0$ and $\Delta y \to 0$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

Now using Cauchy Riemann equations

$$\left[\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}, \quad \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}\right]$$

$$\Delta w = \Delta u + i\Delta v$$

$$\Rightarrow \Delta w = \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)\Delta x + \left(-\frac{\delta v}{\delta x} + i\frac{\delta u}{\delta x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$

$$\Rightarrow \Delta w = \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)\Delta x + \left(i^2\frac{\delta v}{\delta x} + i\frac{\delta u}{\delta x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y$$

$$\Rightarrow \Delta w = \underline{\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)}\Delta x + i\underline{\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)}\Delta y + \varepsilon\Delta x + \eta\Delta y$$

common নিয়ে

$$\Rightarrow \Delta w = \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)(\Delta x + i\Delta y) + \varepsilon\Delta x + \eta\Delta y$$

$$\Rightarrow \Delta w = \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)\Delta z + \varepsilon\Delta x + \eta\Delta y$$

Dividing by Δz and taking limit $\Delta z \to 0$ $(\eta \to 0, \varepsilon \to 0)$

$$\frac{dw}{dz} = f'(z) \Rightarrow \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)$$

Hence $f(z)$ is analytic.

$$\left[\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}\right]$$

* Problem: Show that the function $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic and find the ~~cong~~ conjugate harmonic function v.

[Or]

* Problem: Show that the function $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic and find the conjugate harmonic function of u.

⇒ Solution: Given that $u = x^2 - y^2 - 2xy - 2x + 3y$

$$\frac{\delta u}{\delta x} = 2x - 2y - 2$$

$$\frac{\delta u}{\delta y} = -2y - 2x + 3$$

$$\frac{\delta^2 u}{\delta x^2} = 2 \quad \text{and} \quad \frac{\delta^2 u}{\delta y^2} = -2$$

Therefore $$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$$

Implies that u is harmonic.

From CR equations

$u_x = v_y$

$u_y = -v_x$

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} = 2x - 2y - 2 \quad \text{----- (1)}$$

and $$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} = -2y + 2x \mp 3 \quad \text{----- (2)}$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

Integrating (1) on both sides $v = \int (2x - 2y - 2) dy$

$\Rightarrow V = 2xy - y^2 - 2y + F(x)$ ---- (3)

Differentiate equ^n (iii) $\Rightarrow$

$$\frac{\delta v}{\delta x} = 2y + F'(x)$$

$\Rightarrow 2y + 2x - 3 = 2y + F'(x)$

$\Rightarrow F'(x) = 2x - 3$

$\Rightarrow F(x) = \int (2x-3) dx = x^2 - 3x + c$

Putting this value in equation (3)

$\Rightarrow V = 2xy - y^2 - 2y + x^2 - 3x + c$

conjugate harmonic function

* Problem: Show that $u = e^x(x\cos y - y\sin y)$ is harmonic. Also find the conjugate harmonic function of u.

* Problem: Find the conjugate harmonic function of $u = e^{-x}(x\sin y - y\cos y)$

* Cauchy's Integral Formula:

If $f(z)$ is analytic for all points inside of C and connected a simple closed curve C. a is any point inside C. Then $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$. (C এর ভিতরে z এর একটা point a)

Proof: Since $f(z)$ is analytic for all points inside of C

Taylor series

(derivative এর extra term)

$f(z) = f(a) + (z-a)f'(a) + (z-a)\eta$ where $z \to a$, then $\eta \to 0$

Now $\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz = \frac{1}{2\pi i}\oint_C \frac{f(a) + [f(z) - f(a)]}{z-a}dz$ → [balance করার জন্য f(a) +, − করা হয়েছে]

(partial করে)

$= \frac{f(a)}{2\pi i}\oint_C \frac{dz}{z-a} + \frac{1}{2\pi i}\oint_C \frac{[f(z) - f(a)]}{z-a}dz$

(f(z) এর মান বসিয়ে)

$= \frac{f(a)}{2\pi i}\oint_C \frac{dz}{z-a} + \frac{1}{2\pi i}\oint_C \frac{[f(a) + (z-a)f'(a) + (z-a)\eta - f(a)]}{z-a}dz$

$\oint_C \frac{dz}{z-a} = 2\pi i$ এর মানে বসানো হলো এর কোশি Theorem

[(z-a) common নিয়ে]

$= \frac{f(a)}{2\pi i}2\pi i + \frac{1}{2\pi i}\oint_C \frac{(z-a)[f'(a) + \eta]}{z-a}dz$

$\frac{1}{2\pi i}\oint_C \frac{(z-a)f'(a)}{(z-a)}dz + \frac{1}{2\pi i}\oint_C \frac{(z-a)\eta}{(z-a)}$

$\Rightarrow \frac{f'(a)}{2\pi i}\oint_C dz + \frac{1}{2\pi i}\oint_C \eta\, dz$

$= f(a) + \frac{f'(a)}{2\pi i}\oint_C dz + \frac{1}{2\pi i}\oint_C \eta\, dz$

$f'(a) = 0$ Theorem

$= f(a) + 0 + \frac{1}{2\pi i}\oint_C \eta\, dz$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

$$\Rightarrow \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz - f(a) = \frac{1}{2\pi i}\oint \eta\, dz$$

Considering C is very small so that $|\eta| < \varepsilon$ for all points on C. & $\varepsilon \to 0$

$$\left|\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz - f(a)\right| = \left|\frac{1}{2\pi i}\oint_c \eta\, dz\right| < \frac{\varepsilon}{2\pi}L$$

[Closed curve C এর Length or parametrization]

Curve এর ব্যাসার্ধে length L

$$\left|\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz - f(a)\right| = 0$$ [যখন $\varepsilon \to 0$, $\eta \to 0$]

$$\frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz - f(a) = 0$$

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz$$

[দুইভাব function এ কোনো power থাকে না]

Closed curve এর লিমিট $0 \to 2\pi$

$$\int_0^{2\pi}$$

Plan Title : Online Class - 04

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date : 14 / 02 / 2021

* Cauchy's Integral formula for nth order derivative:

(Theorem লাগবে না; শুধু statement)

$$f^{n}(a) = \frac{n!}{2\pi i} \oint_{c} \frac{f(z)}{(z-a)^{n-1}}\, dz$$

Evaluate:

1. $\oint_c \frac{z}{z-1}\, dz$ where c is the circle $|z| = 2$

2. $\oint_c \frac{z^2}{(z-i)}\, dz$ where c is the circle $|z| = 2$

3. $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz$ where c is the circle $|z| = 3$

4. $\oint_c \frac{\sin 3z}{\left(z + \frac{\pi}{2}\right)}\, dz$ where c is the circle $|z| = 5$

5. $\oint_c \frac{e^{tz}}{(z^2+1)}\, dz$ where c is the circle $|z| = 3$

6. $\oint_c \frac{\sin^6 z}{\left(z - \frac{\pi}{6}\right)^3}\, dz$ where c is the circle $|z| = 1$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

Solution -1 :

$$\oint_c \frac{z}{z-1}\, dz$$

we know

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a}\, dz$$

Here $f(z) = z$ and $a = 1$

then $f(1) = 1$

$2\pi i \times f(a)$

Now $\oint_c \frac{z}{z-1}\, dz = 2\pi i f(1) = 2\pi i \times 1 = 2\pi i$

Solution -2 :

$$\oint_c \frac{z^2}{z-i}\, dz$$

We know

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a}\, dz$$

here $f(z) = z^2$ and $a = i$

then $f(i) = i^2$

Now $\oint_c \frac{z^2}{z-i}\, dz = 2\pi i f(i) = 2\pi i \times i^2 = 2\pi i(-1) = -2\pi i$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

Solution - 3:

We know

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\, dz$$

Now

$$\oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\, dz \quad \text{Here } f(z) = \sin\pi z^2 + \cos\pi z^2$$

$$\oint_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\, dz$$

$$= \oint_c \frac{f(z)}{(z-1)(z-2)}\, dz$$

$$= \oint_c \left(\frac{1}{z-2} - \frac{1}{z-1}\right) f(z)\, dz$$

$$= \oint_c \frac{f(z)}{z-2}\, dz - \oint_c \frac{f(z)}{z-1}\, dz$$

$2\pi i f(a)$ $\quad$ $2\pi i f(a)$

$$= 2\pi i f(2) - 2\pi i f(1)$$

$$= 2\pi i\left(\sin\pi 2^2 + \cos\pi 2^2\right) - 2\pi i\left(\sin\pi 1^2 + \cos\pi 1^2\right)$$

$$= 2\pi i(0+1) - 2\pi i(0-1)$$

$$= 2\pi i + 2\pi i = 4\pi i$$

Plan Title : Online Class-05

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date : 14/03/2021

Solution - 4:

We know,

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a}\, dz$$

$$\oint_c \frac{\sin 3z}{z+\frac{\pi}{2}}\, dz$$

$f(z) = \sin 3z$ and $a = -\frac{\pi}{2}$

then $f(-\frac{\pi}{2}) = \sin 3(-\frac{\pi}{2})$

$$\oint_c \frac{\sin 3z}{(z+\frac{\pi}{2})}\, dz = 2\pi i \times f\left(-\frac{\pi}{2}\right)$$

$$= 2\pi i \times \sin 3\left(-\frac{\pi}{2}\right)$$

$$= 2\pi i$$

Solution - 5:

We know,

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a}\, dz$$

$$f(z) = e^{tz}$$

$$\boxed{\begin{aligned} & z^2 - iz + iz - i^2 \\ & \Rightarrow z^2 - (-1) \\ & \Rightarrow z^2 + 1 \end{aligned}}$$

Now,

$$\oint_c \frac{e^{tz}}{(z^2+1)}\, dz = \oint_c \frac{f(z)}{(z+i)(z-i)}\, dz$$

$$\oint \left\{ \frac{1}{(z+i)(z-i)} \right\} f(z)\, dz$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

$$= \frac{1}{2i} \oint_c \left(\frac{1}{z-i} - \frac{1}{z+i} \right) f(z)\,dz$$

$$= \frac{1}{2i} \left(\oint_c \frac{f(z)}{z-i} dz - \oint_c \frac{f(z)}{z+i} dz \right)$$

$$= \frac{1}{2i} \left[2\pi i f(i) - 2\pi i f(-i) \right]$$

$$= \frac{1}{2i} \times 2\pi i \left(e^{ti} - e^{-ti} \right)$$

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$$= \pi \times \left(e^{it} - e^{-it} \right)$$

$$= \pi \times 2i\sin t$$

$$= 2\pi i \sin t$$

Solution - 6 :

We know

$$f^n(a) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{(z-a)^{n+1}} dz$$

$f^n\left(\frac{\pi}{6}\right) = \frac{2!}{2\pi i} \oint_c \frac{\sin^6 z}{\left(z - \frac{\pi}{6}\right)^{2+1}} dz$

$f(z) = \sin^6 z$, $a = \frac{\pi}{6}$ and $n = 2$

Now

$$\oint_c \frac{\sin^6 z}{\left(z - \frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!} f^n\left(\frac{\pi}{6}\right) \quad \text{------} (1)$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

Here

$$f'(z) = 6\sin^5 z \cos z$$

$$f''(z) = 6\left[5\sin^4 z \cos z \cos z + \sin^5 z(-\sin z)\right]$$

$$f''\left(\frac{\pi}{6}\right) = \frac{21}{16}$$ [$n=2$ তাই ২ বার differentiation করতে হবে]

From (1)

$$\oint_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!} f''\left(\frac{\pi}{6}\right)$$

$$= \pi i \times \frac{21}{16}$$

Problem-7: $\oint_c \frac{e^{tz}}{(z+1)^3} dz$

Problem-8: $\oint_c \frac{ze^{tz}}{(z+1)^3} dz$

Solution-7:

We know

$$f^n(a) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{(z-a)^{n+1}} dz$$

$f(z) = e^{tz}$, $a = -1$, $n = 2$

Now

$$\oint_c \frac{e^{tz}}{(z+1)^3} dz = \frac{2\pi i}{2!} f''(-1) \text{ ------ } (1)$$

Here,

$f'(z) = te^{tz}$

$f''(z) = t^2e^{tz}$

$f''(-1) = t^2e^{-t}$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

from (1)

$$\oint_c \frac{e^{tz}}{(z+1)^3} dz = \frac{2\pi i}{2!} f''(-1)$$

$$= \pi i \times t^2 e^{-t}$$

(Ans)

Solution - 8:

We know

$$f^n(a) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{(z-a)^{n+1}} dz$$

$f(z) = ze^{tz}$, $a = -1$, $n = 2$

Now

$$\oint_c \frac{ze^{tz}}{(z+1)^3} dz = \frac{2\pi i}{n!} f''(-1) \text{ ---- } (1)$$

Here

$$f'(z) = z.\frac{d}{dx}(e^{tz}) + e^{tz}.\frac{d}{dz}(z)$$

$$= e^{tz} tz + e^{tz}$$

$$f''(z) = \frac{d}{dx}(tz\, e^{tz}) + \frac{d}{dx}(e^{tz})$$

$$= t(tz\, e^{tz} + e^{tz}) + t e^{tz} = t^2 z e^{tz} + 2t.e^{tz}.$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

$$f''(-1) = -t^2 e^{-t} + 2t\, e^{-t}$$

From (1)

$$\oint_c \frac{z e^{tz}}{(z+1)^3}\, dz = \frac{2\pi i}{2!} \cdot \left(-t^2 e^{-t} + 2t e^{-t}\right)$$

$$= \pi i \times \left(-t^2 e^{-t} + 2t e^{-t}\right)$$

(Ans)

simple closed

Theorem: If $f(z)$ is analytic inside and on a simple closed curve C except at the pole $z=a$ of order m, then the residue of $f(z)$ at $z=a$ is

$$\operatorname{Res}(a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right]$$

If $z=a$ is a simple pole then,

$$\operatorname{Res}(a) = \lim_{z \to a} (z-a) f(z)$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

* Cauchy's Residue Theorem :

Let $f(z)$ be analytic inside and on a simple closed curve c except at a finite number of singular points $a_1, a_2, a_3, ----, a_n$, then

$$\oint_c f(z)\,dz = 2\pi i\left[\operatorname{Res}(a_1) + \operatorname{Res}(a_2) + --- + \operatorname{Res}(a_n)\right]$$

Proof:

Let $a_1, a_2, a_3 --- a_n$ be the center of the circle $C_1, C_2, C_3, --- C_n$ respectively. $f(z)$ be analytic inside and on a simple closed curve C.

$$\oint_c f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)\,dz + ---- + \oint_{C_n} f(z)\,dz \quad \text{——— (1)}$$

But $\operatorname{Res}(a_1) = \frac{1}{2\pi i}\oint_{C_1} f(z)\,dz$ ⟶ Definition

$$\Rightarrow \oint_{C_1} f(z)\,dz = 2\pi i \operatorname{Res}(a_1)$$

$$\operatorname{Res}(a_2) = \frac{1}{2\pi i}\oint_{C_2} f(z)dz$$

$$\Rightarrow \oint_{C_2} f(z)dz = 2\pi i \operatorname{Res}(a_2) \ldots\ldots$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

$$\operatorname{Res}(a_n) = \frac{1}{2\pi i} \oint_{C_n} f(z)\,dz$$

$$\Rightarrow \oint_{C_n} f(z)\,dz = 2\pi i \operatorname{Res}(a_n)$$

From ①

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz$$

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res}(a_1) + 2\pi i \operatorname{Res}(a_2) + \cdots + 2\pi i \operatorname{Res}(a_n)$$

$$\oint_C f(z)\,dz = 2\pi i \left[\operatorname{Res}(a_1) + \operatorname{Res}(a_2) + \cdots + \operatorname{Res}(a_n)\right]$$

$$\oint_C f(z)\,dz = 2\pi i \sum \operatorname{Res}(a_i) \quad [i = 1, 2, 3, \ldots n]$$

* Cauchy Residue Theorem

Example: Show that $\displaystyle\int_c \frac{e^z}{(z^2+\pi^2)^2}\,dz = \frac{1}{\pi}$

Solution:

$$z^2+\pi^2 \Rightarrow (z+\pi i)(z-\pi i)$$
$$= z^2 - z\pi i + z\pi i - \pi^2 i^2$$
$$= z^2 - \pi^2(-1)$$
$$= z^2+\pi^2$$

Let $f(z) = \dfrac{e^z}{(z^2+\pi^2)^2} = \dfrac{e^z}{(z+\pi i)^2(z-\pi i)^2}$

pole বের করার জন্য হরের সবগুলো term equal to zero ধরতে হবে

To find the pole, $(z+\pi i)^2(z-\pi i)^2 = 0$

$z = \pi i, -\pi i$ is a pole of order $2 \rightarrow m = 2$

$$\left(\operatorname{Res}(a) = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z) \right] \right)$$

Now

$$\operatorname{Res}(\pi i) = \lim_{z\to \pi i} \frac{1}{(2-1)!} \frac{d}{dz}\left[(z-\pi i)^2 f(z) \right]$$

$$= \lim_{z\to \pi i} \frac{1}{(2-1)!} \frac{d}{dz}\left[(z-\pi i)^2 \cdot \frac{e^z}{(z+\pi i)^2 (z-\pi i)^2} \right]$$

$$= \lim_{z\to \pi i} \frac{d}{dz}\left[\frac{e^z}{(z+\pi i)^2} \right]$$

$$\boxed{\frac{d}{dx}\frac{u}{v} = \frac{v\cdot\frac{d}{dx}u - u\cdot\frac{d}{dx}v}{v^2}}$$

$$= \lim_{z\to \pi i} \frac{(z+\pi i)^2 e^z - e^z\cdot 2(z+\pi i)}{(z+\pi i)^4}$$

$$\lim_{z\to\pi i} \frac{\cancel{(z+\pi i)}\left\{(z+\pi i)e^z - 2e^z\right\}}{\cancel{(z+\pi i)}(z+\pi i)^3}$$

$$= \lim_{z\to \pi i} \frac{(z+\pi i)e^z - e^z\cdot 2}{(z+\pi i)^3}$$

$$\frac{(\pi i+\pi i)e^z - e^z 2}{(\pi i+\pi i)^3}$$

$$= \frac{2\pi i e^z - 2e^z}{8\pi^3 i^3} = \frac{2e^z(\pi i-1)}{-8\pi^3 i}$$

$$= \frac{2(\pi i - 1)e^{\pi i}}{-8i\pi^3}$$

$$= \frac{2(\pi i + i^2)e^{\pi i}}{-8i\pi^3}$$

$$= \frac{-(\pi + i)e^{\pi i}}{4\pi^3}$$

$$\operatorname{Res}(-\pi i) = \lim_{z \to \pi i} \frac{1}{(2-1)!} \frac{d}{dz}\left[(z+\pi i)^2 f(z)\right]$$

$m = 2$

$$= \lim_{z \to \pi i} \frac{1}{(2-1)!} \frac{d}{dz}\left[(z+\pi i)^2 \cdot \frac{e^z}{(z+\pi i)^2 \cancel{(z-\pi i)^2}}\right]$$

$$= \lim_{z \to -\pi i} \frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^2}\right]$$

$\frac{d}{dx}\frac{u}{v}$ formula

$$= \lim_{z \to -\pi i} \frac{(z-\pi i)^2 e^z - e^z \cdot 2(z-\pi i)}{(z-\pi i)^4}$$

$$= \lim_{z \to -\pi i} \frac{(z-\pi i)e^z - e^z \cdot 2}{(z-\pi i)^3}$$

$$= \frac{2(-\pi i - 1)e^{-\pi i}}{8i\pi^3}$$

$$= \frac{2(-\pi i + i^2)e^{-\pi i}}{4\cancel{8}i\pi^3} = \frac{-i(\pi - i)e^{-\pi i}}{4i\pi^3}$$

$$= \frac{-(\pi - i)e^{-\pi i}}{4\pi^3}$$

By Cauchy Residue theorem,

$$\oint \frac{e^z}{(z^2+\pi^2)^2}\,dz = 2\pi i\left[\operatorname{Res}(\pi i) + \operatorname{Res}(-\pi i)\right]$$

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Plan Title :

Date :......../......./.........

$$= 2\pi i \left[\frac{-(\pi+i)e^{\pi i}}{4\pi^3} + \frac{-(\pi-i)e^{-\pi i}}{4\pi^3} \right]$$

$$= \frac{1}{2\pi^2}\left[-\pi(e^{\pi i}+e^{-\pi i}) - i(e^{i\pi}-e^{-i\pi})\right]$$

$$= \frac{1}{2\pi^2}\left[-2\pi\cos\pi - i.2i\sin\pi\right]$$

$$= \frac{1}{2\pi^2}\cdot 2\pi$$

$$= \frac{1}{\pi}$$

(showed)

$$2\pi i \cdot \frac{-\pi e^{\pi i} - ie^{\pi i} - \pi e^{-\pi i} + ie^{-\pi i}}{\cancel{2\pi}\,4\pi^3}$$

$$= \cancel{\frac{1}{2\pi}} \cdot \frac{-\pi(e^{\pi i}+e^{-\pi i}) - i(e^{\pi i}-e^{-\pi i})}{2\pi^2}$$

Mid Syllabus - Conjugate Harmonic Functions.

* Find the conjugate harmonic function v of the functions given below:

1. $u = x^2 - y^2 - 2xy - 2x + 3y$

2. $u = e^x (x\cos y - y\sin y)$

3. $u = e^{-x} (x\sin y - y\cos y)$

$\int \cos x\, dx = \sin x$

$\int \sin x\, dx = -\cos x$

$\frac{\delta}{\delta x} \cos x = -\sin x$

$\frac{\delta}{\delta x} \sin x = \cos x$

Solution:

$\left[\int uv\, dx = u\int v\, dx - \int\left\{\frac{d}{dx}u \cdot \int v\, dx\right\}dx\right]$

2# $u = e^x (x\cos y - y\sin y)$ $\quad \left[\frac{\delta}{\delta x} uv = u \cdot \frac{d}{dx} v + v \cdot \frac{d}{dx} u\right]$

$$\frac{\delta u}{\delta x} = e^x (\cos y - 0) + e^x (x\cos y - y\sin y)$$

$$= e^x (\cos y + x\cos y - y\sin y)$$

$$\frac{\delta^2 u}{\delta x^2} = e^x (0 + \cos y - 0) + e^x (\cos y + x\cos y - y\sin y)$$

$$= e^x (\cos y + \cos y + x\cos y - y\sin y)$$

$$= e^x (2\cos y + x\cos y - y\sin y)$$

$$\frac{\delta u}{\delta y} = e^x (-x\sin y - y\cos y - \sin y) + 0$$

$$= e^x (-x\sin y - y\cos y - \sin y)$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......../.........

$$\frac{\delta^2 u}{\delta y^2} = e^x(-x\cos y - \cos y - \cos y + y\sin y) + 0$$

$$= e^x(-x\cos y - 2\cos y + y\sin y)$$

Now,

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = e^x(2\cos y + x\cos y - y\sin y) + e^x(-2\cos y - x\cos y - y\sin y)$$

$$= 0$$

$\therefore$ The function is harmonic.

From CR equations,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} = e^x(\cos y + x\cos y - y\sin y) \quad \text{(i)}$$

$$\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y} = e^x(x\sin y + y\cos y + \sin y) \quad \text{(ii)}$$

Integrating equ^n (i) w.r.t y we get,

$$v = \int e^x(\cos y + x\cos y - y\sin y)\,dy$$

$$= e^x\int(\cos y + x\cos y - y\sin y)\,dy$$

$$= e^x\left[\sin y + x\sin y - \{y(-\cos y) - \int 1\cdot(-\cos y)\,dy\}\right] + F(x)$$

$$= e^x\left[\sin y + x\sin y + y\cos y - \sin y\right] + F(x)$$

CamScanner

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......../.........

$\Rightarrow V = e^x(x\sin y + y\cos y) + F(x)$ ——— (iii)

Differentiating equ^n (iii) w.r.t x we get,

$$\frac{\delta v}{\delta x} = e^x(x\sin y + y\cos y) + e^x \sin y + F'(x)$$

$$\Rightarrow e^x(x\sin y + y\cos y + \sin y) = e^x(x\sin y + y\cos y + \sin y) + F'(x)$$

$$\Rightarrow F'(x) = e^x(x\sin y + y\cos y + \sin y) - e^x(x\sin y + y\cos y + \sin y)$$

$$= 0$$

$\therefore F(x) = 0$

Putting this value in equ^n (iii) we get,

$$V = e^x(x\sin y + y\cos y) + 0$$

$$\therefore V = e^x(x\sin y + y\cos y)$$

This is the conjugate harmonic function of u.

CamScanner

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......../.........

$\frac{\delta}{\delta x}\cos x = -\sin x$

$\frac{\delta}{\delta x}\sin x = \cos x$

$\int \cos x\,dx = \sin x$

$\int \sin x\,dx = -\cos x$

3|| $u = e^{-x}(x\sin y - y\cos y)$

$\left[\frac{d}{dx}uv = u\frac{d}{dx}v + v\frac{d}{dx}u\right]$

$$\frac{\delta u}{\delta x} = e^{-x}\sin y + e^{-x}(x\sin y - y\cos y)\cdot -1$$

$$= e^{-x}(\sin y - x\sin y + y\cos y)$$

$$\frac{\delta^2 u}{\delta x^2} = -e^{-x}\sin y + e^{-x}(x\sin y - y\cos y) - e^{-x}\sin y$$

$$= e^{-x}(-\sin y + x\sin y - y\cos y - \sin y)$$

$$= e^{-x}(x\sin y - y\cos y - 2\sin y)$$

$\left[\int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}u\int v\,dx\right\}dx\right]$

$$\frac{\delta u}{\delta y} = 0 + e^{-x}(x\cos y - \cos y + y\sin y)$$

$$= e^{-x}(x\cos y - \cos y + y\sin y)$$

$$\frac{\delta^2 u}{\delta y^2} = 0 + e^{-x}(-x\sin y + \sin y + y\cos y + \sin y)$$

$$= e^{-x}(-x\sin y + y\cos y + 2\sin y)$$

Now,

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = e^{-x}(x\sin y - y\cos y - 2\sin y) + e^{-x}(-x\sin y - y\cos y - 2\sin y)$$

$$= 0$$

$\therefore$ The function is harmonic.

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

From CR equations,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} = e^{-x}(\sin y - x\sin y + y\cos y) \quad \text{(I)}$$

$$\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y} = e^{x}(\cos y - x\cos y - y\sin y) \quad \text{(II)}$$

Integrating equ^n (I) w.r.t y we get,

$$v = \int e^{-x}(\sin y - x\sin y + y\cos y)\, dy$$

$$= e^{-x}\int (\sin y - x\sin y + y\cos y)\, dy$$

$$= e^{-x}\left[-\cos y + x\cos y + \left\{y\cdot\sin y - \int 1\cdot\sin y\, dy\right\}\right] + F(x)$$

$$\left[\int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}u\cdot\int v\,dx\right\}dx\right] \Rightarrow \text{Formula}$$

$$= e^{-x}(-\cos y + x\cos y + y\sin y + \cos y) + F(x)$$

$$\therefore v = e^{-x}(x\cos y + y\sin y) + F(x) \quad \text{(III)}$$

Differentiating equ^n (III) w.r.t x we get,

$$\frac{\delta v}{\delta x} = e^{-x}(\cos y + 0) - e^{-x}(x\cos y + y\sin y) + F'(x)$$

$$\Rightarrow e^{-x}(\cos y - x\cos y - y\sin y) = e^{x}(\cos y - x\cos y - y\sin y) + F'(x)$$

$$\Rightarrow F'(x) = e^{-x}(\cos y - x\cos y - y\sin y) - e^{x}(\cos y - x\cos y - y\sin y)$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

$\Rightarrow F'(x) = 0$

$\therefore F(x) = 0$

Putting this value in equ^n (11) we get,

$$v = e^{-x}(x\cos y + y\sin y) + 0$$

$$\therefore v = e^{-x}(x\cos y + y\sin y)$$

This is the conjugate harmonic function of u.

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

$\Rightarrow F'(x) = 0$

$\therefore F(x) = 0$

Putting this value in eqn (iii) we get,

$v = e^{-x}(x\cos y + y\sin y) + 0$

$\therefore v = e^{-x}(x\cos y + y\sin y)$

This is the conjugate harmonic

$z = e^{i\theta} = \cos\theta + i\sin\theta$

$z^{-1}/\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$z + \frac{1}{z} = 2\cos\theta$

$\cancel{\sin\theta} \Rightarrow \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

Chapter - 08 (Contoure Integration)

234 (Form 1). Integrals of the form:

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$, where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$:

Let $I = \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ ---- (1) where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

Let $z = e^{i\theta}$, then $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z^2+1}{2z}$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z^2-1}{2iz}$$ and

$$dz = ie^{i\theta} d\theta = iz\, d\theta$$

$$\Rightarrow d\theta = \frac{dz}{iz}.$$

Now using these (1) $\Rightarrow I = \oint_c g(z)\, dz$

where c is the unit circle $|z| = 1$, whose center is at origin and radius is equal to 1.

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

Example - 270: Show that $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ if $a > |b|$

Solution: Suppose $I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ - - - - (i)

Let, $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} = \frac{z^2+1}{2z}$

and $dz = ie^{i\theta}d\theta = iz\,d\theta$ or $d\theta = \frac{dz}{iz}$

Now using these, (i) $\Rightarrow$

$$\oint_c \frac{1}{a+b\frac{z^2+1}{2z}} \cdot \frac{dz}{iz}$$

$$\Rightarrow \frac{2}{i}\oint_c \frac{dz}{bz^2+2az+b} \quad \text{- - - - (ii)}$$

where c is the unit circle $|z|=1$, whose radius is 1 and centre at the origin. The poles of $\frac{1}{bz^2+2az+b}$ are obtained by solving $bz^2+2az+b=0$

If α and β are the poles, then $\left[x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \right]$

Suppose, $z = \alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ and $z = \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

$$b\left(z^2+\frac{2a}{b}z+1\right)=0$$

$$\therefore z=\frac{-\frac{2a}{b}\pm\sqrt{\frac{4a^2}{b^2}-4}}{2}$$

$$\Rightarrow z=\frac{-\frac{2a}{b}\pm\sqrt{\frac{4a^2-4b^2}{b^2}}}{2}$$

$$\Rightarrow z=\frac{-\frac{2a}{b}\pm 2\left(\frac{\sqrt{a^2-b^2}}{b}\right)}{2}$$

$$\Rightarrow z=\frac{2\left\{-\frac{a}{b}\pm\frac{\sqrt{a^2-b^2}}{b}\right\}}{2}$$

$$\therefore z=-\frac{a}{b}\pm\frac{\sqrt{a^2-b^2}}{b}$$

$$\alpha=-\frac{a}{b}+\frac{\sqrt{a^2-b^2}}{b}$$

$$\beta=-\frac{a}{b}-\frac{\sqrt{a^2-b^2}}{b}$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......../.........

of them only the pole α lies inside C since

$$|\alpha| = \left|\frac{-a+\sqrt{a^2-b^2}}{b}\right| = \left|\frac{a-\sqrt{a^2-b^2}}{b}\right| = \left|\frac{b}{a+\sqrt{a^2-b^2}}\right| < 1$$

if $a > |b|$.

Here $\alpha - \beta = \frac{2}{b}\sqrt{a^2-b^2}$ ---- (iii)

Now $\operatorname{Res}(\alpha) = \lim_{z \to \alpha} \left\{ (z-\alpha) \frac{1}{bz^2+2az+b} \right\}$

$$= \lim_{z\to\alpha} \frac{z-\alpha}{b(z-\alpha)(z-\beta)}$$

$$= \frac{1}{b(\alpha-\beta)} = \frac{1}{2\sqrt{a^2-b^2}}, \text{ by (iii)}$$

Now, by the Cauchy's Residue theorem,

$(2) \Rightarrow I = \frac{2}{i} \cdot 2\pi i \cdot \operatorname{Res}(\alpha) = \frac{2\pi}{\sqrt{a^2-b^2}}$ and the required result is obtained.

[Example 271, 272]

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

Example - 271: Show that $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$

if $a > |b|$.

Sol^n: Suppose $I = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ ------ (1)

Let, $z = e^{i\theta}$, then $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i} = \frac{z^2-1}{2iz}$

and $dz = ie^{i\theta}\,d\theta = i\cdot z\cdot d\theta$; $d\theta = \frac{dz}{iz}$

Now using these, (1) $\Rightarrow$

$$I = \oint_C^{2\pi} \frac{1}{a+b\left(\frac{z^2-1}{2iz}\right)} \frac{dz}{iz}$$

$$= \frac{1}{i}\oint_C^{2\pi} \frac{1}{\frac{2iaz+bz^2-b}{2iz}} \cdot \frac{dz}{iz}$$

$$= \frac{1}{i}\oint_C^{2\pi} \frac{2iz}{2iaz+bz^2-b} \cdot \frac{dz}{iz}$$

$$\Rightarrow I = \frac{2}{i} \cdot \oint_C^{2\pi} \frac{dz}{bz^2+2iaz-b} \text{ ------ (11)}$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

~~Now~~ Where C is the unit circle $|z|=1$ whose radius is 1 and centre at origin. The poles of $\frac{1}{bz^2+2aiz-b}$ are obtained by solving $bz^2+2iaz-b=0$.

Now, $bz^2+2aiz-b=0$

$$\Rightarrow b\left(z^2+\frac{2aiz}{b}-1\right)=0$$

$$\Rightarrow z^2+\frac{2aiz}{b}-1=0$$

$$\therefore z=\frac{-\frac{2ai}{b}\pm\sqrt{\frac{4a^2(i)^2}{b^2}-4\cdot 1\cdot(-1)}}{2}$$

$$=\frac{-\frac{2ai}{b}\pm\sqrt{\frac{-4a^2+4b^2}{b^2}}}{2}$$

$$=\frac{2\left\{-\frac{ai}{b}\pm\frac{(\sqrt{a^2-b^2})}{b}\right\}}{2}$$

$$\therefore z=-\frac{ai}{b}\pm\frac{(-\sqrt{a^2-b^2})}{b}$$

If α and β are the poles, then we suppose

$$\alpha=-\frac{ai}{b}+\frac{(-\sqrt{a^2-b^2})}{b}=-\frac{ai}{b}-\frac{\sqrt{a^2-b^2}}{b}$$

$$\beta=-\frac{ai}{b}-\frac{(-\sqrt{a^2-b^2})}{b}=-\frac{ai}{b}+\frac{\sqrt{a^2-b^2}}{b}$$

CamScanner

Among α and β, only pole β lies inside C

Since, $|\beta| = \left| \frac{-ai + \sqrt{a^2-b^2}}{b} \right| = \left| \frac{ai - \sqrt{a^2-b^2}}{b} \right| = \left| \frac{b}{ai + \sqrt{a^2-b^2}} \right| < 1$

if $a > |b|$.

Here, $\beta - \alpha = \left(-\frac{ai}{b} + \frac{\sqrt{a^2-b^2}}{b} \right) - \left(-\frac{ai}{b} - \frac{\sqrt{a^2-b^2}}{b} \right)$

$\therefore \beta - \alpha = 2 \frac{\sqrt{a^2-b^2}}{b}$ ------ (iii)

Now, $\text{Res}(\beta) = \lim_{z \to \beta} \left\{ (z-\beta) \cdot \frac{1}{bz^2 + 2aiz - b} \right\}$

$= \lim_{z \to \beta} \left\{ \frac{\cancel{(z-\beta)}}{b(z-\alpha)\cancel{(z-\beta)}} \right\}$

$= \frac{1}{b(\beta - \alpha)}$

$= \frac{1}{b \cdot \frac{2\sqrt{a^2-b^2}}{b}}$ [From (iii)]

$= \frac{1}{2\sqrt{a^2-b^2}}$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

Now by Cauchy's Residue Theorem, (11) $\Rightarrow$

$$I = \frac{2}{i} \cdot 2\pi i \cdot \text{Res}(\beta)$$

$$= \frac{2}{i} \cdot 2\pi i \cdot \frac{1}{2\sqrt{a^2-b^2}}$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

(showed).

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :....../....../........

Example - 272: Show that. $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = \frac{\pi}{2}$.

Soln: Suppose $I = \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$ - - - - (i)

Let, $z = e^{i\theta}$ then $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2-1}{2iz}$

and $dz = ie^{i\theta} d\theta = i.z.d\theta$; $d\theta = \frac{dz}{iz}$

Now using these, (i) $\Rightarrow$

$$I = \oint_C^{2\pi} \frac{1}{5+3\left(\frac{z^2-1}{2iz}\right)} \cdot \frac{dz}{iz}$$

$$= \frac{1}{i} \oint_C^{2\pi} \frac{2iz}{10iz + 3z^2 - 3} \cdot \frac{dz}{iz}$$

$$\therefore I = \frac{2}{i} \oint_C^{2\pi} \frac{dz}{3z^2 + 10iz - 3} \quad \text{- - - - - - - (ii)}$$

where C is the unit circle $|z| = 1$ whose radius is 1 and centre at origin. The poles α and β can be obtained by solving $3z^2 + 10iz - 3 = 0$

Now, $3z^2 + 10iz - 3 = 0$

$$z = \frac{-10i \pm \sqrt{100(i^2) - 4 \cdot 3 \cdot (-3)}}{2 \cdot 3}$$

$$= \frac{-10i \pm \sqrt{-64}}{6}$$

$$= \frac{-10i \pm (-8)}{6}$$

$$= \frac{-5i \pm (-4)}{3}$$

$$\therefore \alpha = \frac{-5i - 4}{3}$$

$$\beta = \frac{-5i + 4}{3}$$

Among α and β only pole β lies inside c since

$$|\beta| = \left|\frac{-5i + 4}{3}\right| = \left|\frac{5i - 4}{3}\right| = \left|\frac{3}{5i + 4}\right| < 1$$

Here, $\beta - \alpha = 8/3$ ——— (III)

$$\text{Res}(\beta) = \lim_{z \to \beta} \left\{ (z - \beta) \cdot \frac{1}{3z^2 + 10iz - 3} \right\}$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./........

$$= \lim_{z \to \beta} \left\{ \frac{\cancel{(z-\beta)}}{3(z-\alpha)\cancel{(z-\beta)}} \right\}$$

$$= \frac{1}{\beta - \alpha}$$

$$= \frac{1}{3 \cdot 8/3} \quad \left[\text{From (iii)}\right]$$

$$= \frac{1}{8}$$

Now by Cauchy's Residue theorem,

$$I = \frac{2}{i} \cdot 2\pi i \cdot \operatorname{Res}(\beta)$$

$$= \frac{2}{i} \cdot 2\pi i \cdot \frac{1}{8}$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta} = \frac{\pi}{2}$$

(showed)

Plan Title : Online Class 10

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date : 19 / 05 / 2022

Review Class

3 set questions হবে।

থেকে Theorem + Problem

Theorem এর statement

Cauchy's Integral Formula (statement, Proof, Problem)

Cauchy Residue Theorem (statement, Proof, Problem)

Contour Integration (statement, Proof, Problem)

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

Part-Two

Mehetaj Parvine Madam

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

* Syllabus :

- Fourier Series (I)
- Coefficient and Half Range Fourier Series
- Fourier Analysis
- Fourier Integral
- Z- Transformation and Its Application
- Laplace Transforms (I)
- Application of Fourier Series and Laplace Transforms.

3 set questions থাকবে।

(I) Fourier Series

(II) Laplace Transforms

(III) Fourier + Laplace

Fourier Series

* Trigonometric Series :

Any series of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ where the coefficients a_n and b_n are constants, is called the trigonometric series.

* Fourier Series :

The trigonometric series, $f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots$

$+ a_n \cos nx + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{——— (1)}$$

is a fourier series if its coefficients a_0, a_n and b_n are given by the following formulas

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(v)\,dv$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(v) \cos nv\,dv$$

$$(n = 1, 2, 3 \cdots)$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v)\sin nv\,dv$$

$(n = 1, 2, 3 \ldots)$

where $f(x)$ is any single valued function defined on the interval $(-\pi, \pi)$.

* The Fourier cosine and sine series

Definition Even function:

A function $f(x)$ is called even if $f(-x) = f(x)$.

Graphically, an even function is symmetrical about the y-axis.

If $f(x)$ is an even function, then

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx$$

$$= \int_{\pi}^{0} f(-x)\,d(-x) + \int_{0}^{\pi} f(x)\,dx$$

$$= -\int_{\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx$$

$$= \int_{0}^{\pi} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx$$

$$= 2\int_{0}^{\pi} f(x)\,dx$$

Thus if $f(x)$ is even, we have

$$a_0 = \frac{1}{2\pi} \cdot 2\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(v)\,dv$$

Also if $f(x)$ is even i.e. $f(-x) = f(x)$

then $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (n = 1, 2, 3, \ldots)$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)\cos n(-x)\,d(-x)$$

$$= -\frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(v)\cos nv\, dv \qquad \left[\int_{-\pi}^{\pi} f(x)\,dx = 2\int_0^{\pi} f(x)\,dx\right]$$

but $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)\sin n(-x)\, d(-x)$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(x).-\sin nx.-dx \qquad [f(-x) = f(x)]$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= -b_n \qquad \left[b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx\right]$$

$\therefore\ 2b_n = 0$ or, $b_n = 0$

Therefore, if $f(x)$ is even, then we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx \qquad \left[f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} f(v)\,dv + \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\int_0^{\pi} f(v)\cos nv\, dv\right\}\cos nx$$

which represents the function $f(x)$ in a series of cosines and therefore it is known as Fourier Cosine series in the interval $(0, \pi)$.

Definition odd function:

A function $f(x)$ is called odd if $f(-x) = -f(x)$.

Graphically, an odd function is symmetrical about the origin.

When $f(x)$ is odd, we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{\pi}^{-\pi} f(-x)\,d(-x)$$

$$= \frac{1}{2\pi}\int_{\pi}^{-\pi} f(x)\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= -a_0$$

$$\therefore\ 2a_0 = 0 \quad \text{or,}\quad a_0 = 0$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......../.........

Also if $f(x)$ is odd i.e. $f(-x) = -f(x)$ then

$$a_n = \frac{1}{\pi}\int_{\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)\cos n(-x)\,d(-x)$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} -f(x)\cos nx \cdot -dx$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\cos nx\,dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= -a_n \qquad \left[a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\right]$$

$\therefore\ 2a_n = 0$ or, $a_n = 0$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)\sin n(-x)\,d(-x)$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} -f(x) \cdot -\sin nx \cdot -dx$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

$$= -\frac{1}{\pi}\int_{-\pi}^{-\pi} f(x)\cdot -\sin nx \cdot -dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \sin nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\ \sin nx\, dx \quad \left[\int_{-\pi}^{\pi} f(x)dx = 2\int_0^{\pi} f(x)dx\right]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(v)\ \sin nv\, dv$$

Therefore, if $f(x)$ is odd, then we have

$$\cancel{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx\, dx + b_n \sin nx\, dx\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\int_0^{\pi} f(v)\ \sin nv\, dv\right\} \sin nx$$

which represents the function $f(x)$ in a series of sines in the interval $(0, \pi)$ and therefore it is known as Fourier sine series in the interval $(0, \pi)$.

These Fourier cosine and sine ~~serious~~ series are also called half range Fourier cosine series and half range fourier sine series.

Fourier Integral

* Fourier Integral and Fourier Transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \cdot \int_{-\infty}^{\infty} \cos\{u(x-t)\}\,du$$

This double integral is known as Fourier integral and holds if x is a point of continuity of $f(x)$

When $f(x)$ is an even function of x,

$$f(x) = \frac{2}{\pi}\int_0^{\infty} du \int_0^{\infty} f(t) \cos ut \cos ux\,dt$$

$$= \frac{2}{\pi}\int_0^{\infty} f(t)\,dt \int_0^{\infty} \cos ut \cos ux\,du$$

which gives Fourier integral of an even function.

When $f(x)$ is an odd function of x,

$$f(x) = \frac{2}{\pi}\int_0^{\infty} f(t)\,dt \int_0^{\infty} \sin ut \sin ux\,du$$

which is the Fourier Integral of an odd function.

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

* Worked out Examples :

Example-1: The function x^2 is periodic with period $2l$ on the interval $[-l, l]$. Find its Fourier series.

Solution: $f(x) = x^2$

$$\Rightarrow f(-x) = (-x)^2$$
$$= x^2$$
$$= f(x)$$

So $f(x)$ is an even function and hence sine terms will vanish i.e. $b_n = 0$.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + \underbrace{b_n \sin\frac{n\pi x}{l}}_{0}\right)$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx \quad (n=0)$

and $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \quad (n = 1, 2, 3, \ldots)$

since $f(x)$ is even.

$$a_0 = \frac{1}{\ell}\int_0^\ell f(v)\,dv\,.$$

$$a_n = \frac{2}{\ell}\int_0^\ell f(v)\cos\frac{n\pi v}{\ell}\,dv$$

$$a_0 = \frac{1}{\ell}\int_0^\ell x^2\,dx$$

$$= \frac{1}{\ell}\left[\frac{x^3}{3}\right]_0^\ell$$

$$= \frac{1}{3}\ell^2$$

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}u\int v\,dx\right\}dx$$

$$a_n = \frac{2}{\ell}\int_0^\ell x^2\cos\frac{n\pi x}{\ell}\,dx \text{ (integrating by parts)}$$

$$= \frac{2}{\ell}\left[x^2\cdot\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\right]_0^\ell - \frac{2}{\ell}\cdot 2\int_0^\ell x\cdot\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\,dx$$

$$= 0 - \frac{4}{n\pi}\int_0^\ell x\sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{4}{n\pi}\cdot\frac{\ell}{n\pi}\left[x\cos\frac{n\pi x}{\ell}\right]_0^\ell - \frac{4}{n\pi}\cdot\frac{\ell}{n\pi}\int_0^\ell \cos\frac{n\pi x}{\ell}\,dx$$

$$= \frac{4\ell}{n^2\pi^2}\left[\ell\cos n\pi - 0\right] - \frac{4\ell^2}{n^3\pi^3}\left[\sin\frac{n\pi x}{\ell}\right]_0^\ell$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

$$= \frac{4l^2}{n^2\pi^2}(-1)^n - 0$$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n$$

$$\therefore a_n = \frac{4l^2}{n^2\pi^2}(-1)^n$$

There force, $f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{l}$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n \cdot \cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2}\left[-\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{2^2}\cos\frac{2\pi x}{l} - \frac{1}{3^2}\cos\frac{3\pi x}{l} + \cdots\right]$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3}\cos\frac{3\pi x}{l} - \cdots\right]$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

Example-2: Obtain the Fourier series of the function

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 \le x \le \pi \end{cases}$$

and verify the result by assuming the complex form of Fourier series.

Solution:

Def^n: The complex form of the Fourier series can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{c}}, \quad -c < x < c$$

where $c_n = \frac{1}{2c} \int_{-c}^{c} f(x)\, e^{-\frac{in\pi x}{c}}\, dx$

and $n = 0, \pm 1, \pm 2, \ldots$

1st Portion :

By def^n, we have $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ ---(1)

where $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......../.........

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Now $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$$

$$\left[\text{Interval Addition} \quad \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} \cancel{f(x)}\, 0\,dx + \int_{0}^{\pi} 1\,dx\right]$$

$$= \frac{1}{2\pi}[0+\pi] = \frac{1}{2}$$

$\therefore a_0 = \frac{1}{2}$

Again $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (n \neq 0)$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\cos nx\,dx + \int_{0}^{\pi} 1.\cos nx\,dx\right]$$

$$= \frac{1}{\pi}[0] + \frac{1}{\pi n}[\sin nx]_0^{\pi}$$

$$= 0 + 0 = 0$$

$\therefore a_n = 0$

CamScanner

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......../.........

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\sin nx\,dx + \int_{0}^{\pi} 1.\sin nx\,dx\right]$

$= \frac{1}{\pi}[0] + \frac{1}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$

$= 0 - \frac{1}{n\pi}(\cos n\pi - \cos 0)$

$= -\frac{1}{n\pi}\left[(-1)^n - 1\right]$

$= \frac{1}{n\pi}\left[1-(-1)^n\right] \begin{cases} 0 & \text{when } n \text{ is even} \\ \frac{2}{n\pi} & \text{when } n \text{ is odd} \end{cases}$

Now putting the values of a_0, a_n and b_n in (1) we get

$f(x) = \frac{1}{2} + 0 + \sum_{n=1}^{\infty} b_n \sin nx$

$= \frac{1}{2} + b_1 \sin x + \cancel{0} + b_2 \sin 2x + b_3 \sin 3x + \cdots$

$$= \frac{1}{2} + \frac{2}{\pi}\sin x + 0 + \frac{2}{3\pi}\sin 3x + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \cdots\right)$$

Formula → $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] + c$; $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] + c$

Example-3: Show that $\int_0^\infty \frac{\cos ux}{u^2+1}\,du = \frac{\pi}{2}e^{-x}$, $x>0$

Proof: Let $f(x) = \begin{cases} e^{-x}, & x>0 \\ e^{x}, & x<0 \end{cases}$

Then $f(-x) = \begin{cases} e^{x}, & x<0 \\ e^{-x}, & x>0 \end{cases}$

Therefore, $f(x) = f(-x)$. Thus $f(x)$ is an even function.
Now by definition of Fourier integral of an even function for $x>0$, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty du \int_0^\infty f(t)\cos ut \cos ux\,dx \quad ----(1)$$

putting $f(x) = e^{-x}$ in (1), we get

$$e^{-x} = \frac{2}{\pi}\int_0^\infty du \int_0^\infty e^{-t}\cos ut \cos ux\,dt$$

$$= \frac{2}{\pi}\int_0^\infty \cos ux\,du \int_0^\infty e^{-t}\cos ut\,dt$$

$$= \frac{2}{\pi}\int_0^\infty \cos ux\,du \left[\frac{e^{-t}}{1+u^2}(-\cos ut + u\sin ut)\right]_0^\infty$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

$$= \frac{2}{\pi}\int_0^\infty \cos ux \left\{0 + \frac{1}{1+u^2}\right\} du$$

$$\Rightarrow e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\cos ux}{u^2+1}\, du$$

$$\therefore \int_0^\infty \frac{\cos ux}{u^2+1}\, du = \frac{\pi}{2} e^{-x} \;, \; x > 0$$

(Proved)

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

Example-5: Find the Fourier integral of the function

$f(x) = e^{-kx}$ when $x > 0$ and $f(-x) = f(x)$ for $k > 0$

and hence prove that

$$\int_0^\infty \frac{\cos ux}{k^2 + u^2}\, du = \frac{\pi}{2k} e^{-kx}$$

Solution: Since $f(-x) = f(x)$, so $f(x)$ is even and for even function we have the Fourier integral

$$f(x) = \frac{2}{\pi} \int_0^\infty f(t)\, dt \int_0^\infty \cos ut \cos ux\, du$$

$$= \frac{2}{\pi} \int_0^\infty \left[\int_0^\infty f(t) \cos ut\, dt \right] \cos ux\, du \text{ ----(1)}$$

Now $\int_0^\infty f(t) \cos ut\, dt = \int_0^\infty e^{-kt} \cos ut\, dt$

$$= \left[\frac{e^{-kt}}{k^2 + u^2} (-k\cos ut + u\sin ut) \right]_0^\infty$$

$$= 0 + \frac{k}{k^2 + u^2}$$

$$= \frac{k}{k^2 + u^2}$$

CamScanner

Thus from (1), we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{k}{k^2+u^2}\cos ux\,du$$

$$\Rightarrow f(x) = \frac{2k}{\pi}\int_0^\infty \frac{\cos ux}{k^2+u^2}\,du \quad (x>0,\ k>0) \quad \text{----} (2)$$

which is the required Fourier integral of the function $f(x) = e^{-kx}$.

Again putting $f(x) = e^{-kx}$ in (2), we get

$$e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos ux}{k^2+u^2}\,du$$

$$\therefore \int_0^\infty \frac{\cos ux}{k^2+u^2}\,du = \frac{\pi}{2k}e^{-kx}.$$

(Proved)

The Laplace Transforms

* Definition of Laplace transform :

If the kernel $K(s,t)$ is defined as

$$K(s,t) = \begin{cases} 0 & \text{for } t<0 \\ e^{-st} & \text{for } t \geq 0 \end{cases}$$

then $f(s) = \int_0^\infty e^{-st} F(t)\,dt$ ----- (1)

The function $f(s)$ defined by the integral (1) is called the Laplace transforms of the function $F(t)$ and is denoted by $\mathcal{L}\{F(t)\}$. $\mathcal{L}\{F(t)\}$

Alternatively, Let $F(t)$ be a function of t defined for all $t>0$. Then the Laplace transform of $F(t)$, denoted by $\mathcal{L}\{F(t)\}$ or $f(s)$, is defined by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt \text{ ---- (1)}$$

provided that the integral (1) exists. 's' is a parameter which may be real or complex number.

The Laplace transform of $F(t)$ is said to exist if the integral $\int_0^\infty e^{-st} F(t)\,dt$ converges for some value of s; otherwise it does not exist.

Example-1: Find the Laplace transforms of the following elementary functions:

i) $F(t) = 1$, ii) $F(t) = t$, iii) $F(t) = t^n$, $n = 0, 1, 2, 3, \ldots$

iv) $F(t) = e^{at}$, v) $F(t) = \sin at$ vi) $F(t) = \cos at$,

vii) $F(t) = \sinh at$ and viii) $\cosh at$

Solutions: By definition of the Laplace transform of a function $F(t)$, we have

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt \quad \text{----(1)}$$

i) When $F(t) = 1$, equation (1) becomes

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} \cdot 1\,dt$$

$$= \int_0^\infty e^{-st}\,dt$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

$$= \lim_{P\to\infty} \int_0^P e^{-st}\, dt$$

$$= \lim_{P\to\infty} \left[-\frac{1}{s} e^{-st} \right]_0^P$$

$$= \lim_{P\to\infty} \left[-\frac{1}{s} e^{-sP} + \frac{1}{s} \right]$$

$$= -0 + \frac{1}{s}$$

$$\mathcal{L}\{F(t)\} = \frac{1}{s} \quad \text{if } s>0$$

ii) When $F(t) = t$, equation (1) becomes

$$\cancel{\mathcal{L}\{F(t)\} = f(s)} \quad \mathcal{L}(t) = \int_0^\infty e^{-st}.\, t\, dt$$

$$\left[\int e^{ax} = \frac{1}{a} e^{ax} \right]$$

$$\Rightarrow \mathcal{L}(t) = \lim_{P\to\infty} \int_0^P t.\, e^{-st}\, dt$$

[uv formula]

$$\Rightarrow \mathcal{L}(t) = \lim_{P\to\infty} \left[\frac{-t\, e^{-st}}{s} \right]_0^P + \lim_{P\to\infty} \frac{1}{s} \int_0^P e^{-st}\, dt$$

$$\left[\frac{1}{s} \cdot \frac{1}{s} [e^{-st}]_0^P \right]$$

$$\left[e^{-ax} = \frac{1}{e^{ax}} \right]$$

$$= \lim_{P\to\infty} \frac{-P}{s e^{Ps}} + 0 - \lim_{P\to\infty} \frac{1}{s^2} \left[e^{-st} \right]_0^P$$

$$= 0 - \lim_{P\to\infty} \frac{1}{s^2} \left[\frac{1}{e^{sP}} - 1 \right]$$

$$= \frac{1}{s^2} \quad \cancel{\text{is}} \text{ if } s>0.$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

iii) When $F(t) = t^n$, equation (1) becomes

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st}\, t^n dt \quad ---- (2)$$

Putting $st = y$ so that $sdt = dy$

$$\therefore dt = \frac{1}{s}dy$$

Also $t^n = \frac{y^n}{s^n}$. Limits $\left.\begin{matrix} t=0 \\ y=0 \end{matrix}\right\}$ $\left.\begin{matrix} t=\infty \\ y=\infty \end{matrix}\right\}$

$st = y$

$x = y/s$

$\Rightarrow t^n = y^n/s^n$

Thus from (2) we have

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-y} \frac{y^n}{s^n} \cdot \frac{dy}{s} \qquad \left[\frac{1}{s^n}\cdot\frac{1}{s}\int_0^\infty e^{-y} y^n dy\right]$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-y} y^n dy \quad \text{(using gamma function)}$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) \quad \text{if } n > -1 \text{ and } s > 0$$

$$\left[\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt\right]$$

Gamma Function

Again we know that if n be a positive integer, then $\Gamma(n+1) = \lfloor n$

Hence $\mathcal{L}\{t^n\} = \frac{\lfloor n}{s^{n+1}}$, when $n = 0, 1, 2, 3, --$

iv) When $F(t) = e^{at}$, equation (1) becomes

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt \quad \text{[same base হলে তাই power যোগ]}$$

$$= -\frac{1}{(s-a)}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= -\frac{1}{(s-a)}(0-1)$$

$$= \frac{1}{s-a} \quad \text{if } s>a$$

v) When $F(t) = \sin at$, equation (1) becomes,

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt$$

$$= \lim_{P\to\infty} \int_0^P e^{-st} \sin at\, dt$$

$$= \lim_{P\to\infty}\left[\frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2}\right]_0^P$$

$$\left[\begin{array}{l}\sin 0 = 0\\ \cos 0 = 1\end{array}\right]$$

$$= \lim_{P\to\infty}\left[\frac{e^{-sP}(-s\sin aP - a\cos aP)}{s^2+a^2} + \frac{a}{s^2+a^2}\right]$$

$$= 0 + \frac{a}{s^2+a^2} = \frac{a}{s^2+a^2} \quad \text{if } s>0.$$

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right] + C$$

vi) When $F(t) = \cos at$, equation (1) becomes

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \lim_{p\to\infty} \int_0^p e^{-st} \cos at \, dt$$

[$\int e^{ax} \cos bx \, dx$ formula]

$$= \lim_{p\to\infty} \left[\frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2} \right]_0^p$$

$$= \lim_{p\to\infty} \left[\frac{e^{-sp}(-s\cos ap + a\sin ap)}{s^2+a^2} + \frac{s}{s^2+a^2} \right]$$

[$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$]

$$= 0 + \frac{s}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2} \quad \text{if } s > 0$$

vii) When $F(t) = \sinh at$, equation (1) becomes
(sin hyperbola function)

$$\mathcal{L}\{\sinh at\} = \int_0^\infty e^{-st} \sinh at \, dt$$

[$\sinh x = \frac{1}{2}(e^x - e^{-x})$]

$$= \int_0^\infty e^{-st} \cdot \frac{1}{2}(e^{at} - e^{-at}) \, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st} \cdot e^{at} \, dt - \frac{1}{2}\int_0^\infty e^{-st} \cdot e^{-at} \, dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t}\,dt - \frac{1}{2}\int_0^\infty e^{-(s+a)t}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty + \frac{1}{2}\left[\frac{e^{-(s+a)t}}{(s+a)}\right]_0^\infty$$

$$= \frac{1}{-2(s-a)}(0-1) + \frac{1}{2(s+a)}(0-1)$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)}$$

$$= \frac{a}{s^2-a^2} \quad \text{if} \quad s > |a|$$

viii) When $F(t) = \cosh at$, equation (1) becomes (cos hyperbola function)

$$\mathcal{L}\{\cosh at\} = \int_0^\infty e^{-st}\cosh at\,dt$$

$$\left[\cosh x = \frac{1}{2}(e^x + e^{-x})\right] \quad = \int_0^\infty e^{-st}\cdot\frac{1}{2}(e^{at} + e^{-at})\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st}\cdot e^{at}\,dt + \frac{1}{2}\int_0^\infty e^{-st}\cdot e^{-at}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(s+a)t}\,dt$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} - \frac{1}{2}\left[\frac{e^{-(s+a)t}}{(s+a)}\right]_0^{\infty}$$

$$= \frac{1}{-2(s-a)}(0-1) - \frac{1}{2(s+a)}(0-1)$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{s}{s^2-a^2} \quad \text{if} \quad s>|a|.$$

Example-2: Find the Laplace transforms of the following functions:

i) $F(t) = t \sin at$, ii) $F(t) = t \cos at$

Solutions: By definition of the Laplace transform of a function $F(t)$, we have

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt \text{ ---- } (1)$$

i) When $F(t) = t \sin at$, equation (1) becomes

$$\mathcal{L}\{t \sin at\} = \int_0^\infty e^{-st} . t \sin at\, dt$$

$$= \int_0^\infty t . e^{-st} \sin at\, dt$$

Integrating by parts, taking t as first function and $e^{-st} \sin at$ as the second function and applying the result

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right],$$

CamScanner

we get

$$\mathcal{L}\{t\sin at\} = \left[t \cdot \frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$$

$$+\int_0^\infty \frac{e^{-st}}{s^2+a^2}(s\sin at + a\cos at)\,dt$$

$$= 0 + \frac{s}{s^2+a^2}\int_0^\infty e^{-st}\sin at\,dt + \frac{a}{s^2+a^2}\int_0^\infty e^{-st}\cos at\,dt$$

$\sin at = \frac{a}{s^2+a^2}$

$\cos at = \frac{s}{s^2+a^2}$

$$= \frac{s}{s^2+a^2}\cdot\frac{a}{s^2+a^2} + \frac{a}{s^2+a^2}\cdot\frac{s}{s^2+a^2} \quad \text{for } s>0$$

[using ex. 1 (v) & (vi)]

$$= \frac{2as}{(s^2+a^2)^2} \quad \text{for } s>0.$$

ii) When $F(t) = t\cos at$, equation (1) becomes

$$\mathcal{L}\{t\cos at\} = \int_0^\infty e^{-st}\cdot t\cos at\,dt$$

$$= \int_0^\infty \underbrace{t}_{u}\cdot \underbrace{e^{-st}\cos at}_{v}\,dt$$

Integrating by parts, taking t as first function and $e^{-st}\cos at$ as second function and applying the result

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

We get, $\left[\int_0^\infty \underset{u}{t}\cdot \underset{v}{e^{-st}\cos at}\,dt\right]$ $\left[\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx\right]$

$$\mathcal{L}\{t\cos at\} = \left[t\cdot\frac{e^{-st}}{s^2+a^2}(-s\cdot\cos at + a\sin at)\right]_0^\infty$$

$$+ \cancel{s}\int_0^\infty \frac{e^{-st}}{s^2+a^2}(s\cdot\cos at \cancel{+} - a\sin at)\,dt$$

$$= 0 + \int_0^\infty \frac{e^{-st}}{s^2+a^2}\cdot s\cos at\,dt - \int_0^\infty \frac{e^{-st}}{s^2+a^2}\,a\sin at\,dt$$

$$= \frac{\cancel{e^{-st}}\,s}{s^2+a^2}\int_0^\infty e^{-st}\cos at\,dt - \frac{\cancel{e^{-st}}\,a}{s^2+a^2}\int_0^\infty e^{-st}\sin at\,dt$$

$$= \frac{s}{s^2+a^2}\cdot\frac{s}{s^2+a^2} - \frac{a}{s^2+a^2}\cdot\frac{a}{s^2+a^2}$$

$$= \frac{s^2}{(s^2+a^2)^2} - \frac{a^2}{(s^2+a^2)^2} \quad \text{for } s>0$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2} \quad \text{for } s>0$$

Example-3: Find the Laplace transform of the function $F(t)$.

where $F(t) = \begin{cases} t, & 0 < t < 2 \\ 3, & t > 2 \end{cases}$

Solution: Here $F(t)$ is not defined at $t=0$ and $t=2$. By definition of the Laplace transform of a function $F(t)$, we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

$$= \int_0^2 e^{-st} F(t)\,dt + \int_2^\infty e^{-st} F(t)\,dt$$

$$= \int_0^2 e^{-st} \cdot \underset{(0<t<2)}{\underline{t}}\,dt + \int_2^\infty e^{-st} \underset{(t>2)}{\underline{3}}\,dt$$

$$= \int_0^2 t \cdot e^{-st}\,dt + 3\int_2^\infty e^{-st}\,dt$$

$$\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$$

$$= \left[-\frac{t}{s}e^{-st}\right]_0^2 + \int_0^2 \frac{e^{-st}}{s}\,dt - \frac{3}{s}\left[e^{-st}\right]_2^{\infty}$$

$$= -\frac{2}{s}e^{-2s} + 0 - \frac{1}{s^2}\left[e^{-st}\right]_0^2 - \frac{3}{s}\left(0 - e^{-2s}\right)$$

$$= -\frac{2}{s}e^{-2s} - \frac{1}{s^2}e^{-2s} + \frac{1}{s^2} + \frac{3e^{-2s}}{s}$$

$$= \frac{1}{s^2} + \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2}.$$

(Ans)

* Sufficient conditions for existence of Laplace transform.

Theorem - 1: Existance theorem

If the function $F(t)$ is piecewise (sectionally) continuous over any closed interval $0 \leq t \leq N$ and of exponential order γ for $t > N$, then its Laplace transform $f(s)$ exists for all $s > \gamma$. [statement]

Proof:

The Laplace transform of $F(t)$ exists if

$$f(s) = \int_0^{\infty} F(t)\, e^{-st}\, dt \quad \text{exists.}$$

Now for $N > 0$, we have

$$\int_0^{\infty} F(t)\, e^{-st}\, dt = \int_0^{N} e^{-st} F(t)\, dt + \int_N^{\infty} e^{-st} F(t)\, dt \text{ -----(1)}$$

The first integral on the right of (1) is a definite integral and since $F(t)$ is piecewise (sectionally) continues over any closed interval $0 \leq t \leq N$, it exists.

Also the second integral on the right of (1) exists, since $F(t)$ is of exponential order γ for $t>N$.

To see this we have only to observe that in such a case

$$\left|\int_N^\infty e^{-st}F(t)\,dt\right| \le \int_N^\infty \left|e^{-st}F(t)\,\cancel{dt}\right|dt$$

$$\le \int_0^\infty e^{-st}\left|F(t)\right|dt$$

$$\le \int_0^\infty e^{-st}Me^{\gamma t}\,dt \quad \text{since } \left|F(t)\right| \le Me^{\gamma t}$$

$$= \int_0^\infty e^{-(s-\gamma)t}M\,dt$$

$$= \frac{M}{s-\gamma} \text{ for } s>\gamma.$$

Hence the Laplace transform

*** State & Prove the existance theorem.

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

* Some important properties of Laplace transform

(A) Linearity Property

Theorem - 2:

The Laplace transform is a linear transformation i.e. if and λ_1 and λ_2 are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transform $f_1(s)$ and $f_2(s)$ respectively then

$$\mathcal{L}\{\lambda_1 F_1(t) + \lambda_2 F_2(t)\} = \lambda_1 \mathcal{L}\{F_1(t)\} + \lambda_2 \mathcal{L}\{F_2(t)\}.$$

Proof:

Let $\mathcal{L}\{F_1(t)\} = f_1(s) = \int_0^\infty e^{-st} F_1(t)\,dt$

and $\mathcal{L}\{F_2(t)\} = f_2(s) = \int_0^\infty e^{-st} F_2(t)\,dt$

Now if λ_1 and λ_2 are any constants then

$$\mathcal{L}\{\lambda_1 F_1(t) + \lambda_2 F_2(t)\} = \int_0^\infty e^{-st}\{\lambda_1 F_1(t) + \lambda_2 F_2(t)\}\,dt$$

$$= \lambda_1 \int_0^\infty e^{-st} F_1(t)\,dt + \lambda_2 \int_0^\infty e^{-st} F_2(t)\,dt$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

$$= \lambda_1 \mathcal{L}\{F_1(t)\} + \lambda_2 \mathcal{L}\{F_2(t)\}$$

$$= \lambda_1 f_1(s) + \lambda_2 f_2(s)$$

The result may be generalized for any number of functions and for the same number of arbitrary constants. i.e.

$$\mathcal{L}\left\{\sum_{r=1}^{n} \lambda_r F_r(t)\right\} = \sum_{r=1}^{n} \lambda_r \mathcal{L}\{F_r(t)\}.$$

(B) First translation (or shifting) property

Theorem - 3: If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{e^{at} F(t)\} = f(s-a).$$

Proof:

By definition of the Laplace transform we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$$

$$\therefore \mathcal{L}\{e^{at} F(t)\} = \int_0^\infty e^{-st} \{e^{at} F(t)\}\, dt$$

$$= \int_0^\infty e^{-(s-a)t} \cdot F(t)\, dt$$

$$= f(s-a).$$

(c) Second translation (or shifting) property

Theorem-4: If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

then $\mathcal{L}\{G(t)\} = e^{-as} f(s)$

Proof:

By definition of the Laplace transform we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$$

$$\therefore \mathcal{L}\{G(t)\} = \int_0^\infty e^{-st} G(t)\, dt$$

[Integrating by Parts]

$$= \int_0^a e^{-st} G(t)\, dt + \int_a^\infty e^{-st} G(t)\, dt$$

$$= \int_0^{a} e^{-st} (0)\, dt + \int_a^\infty e^{-st} F(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-st} F(t-a)\, dt$$

$$\therefore \mathcal{L}\{G(t)\} = \int_a^\infty e^{-st} F(t-a)\, dt \quad \text{------} \quad (1)$$

Let $t - a = u$

$\therefore dt = du$

Plan Title :

Limits $\left.\begin{matrix} t = a \\ u = 0 \end{matrix}\right\}$ $\left.\begin{matrix} t = \infty \\ u = \infty \end{matrix}\right\}$

Thus from (1) we have

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-s(u+a)} F(u)\,du$$

$$= e^{-sa} \int_0^\infty e^{-su} F(u)\,du$$

$$= e^{-sa} f(s).$$

(D) The change of scale property

Theorem-5:

If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

Proof:

By definition of the Laplace transform, we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt = f(s)$$

$$\therefore \mathcal{L}\{F(at)\} = \int_0^\infty e^{-st} F(at)\,dt \quad ---- (1)$$

Let $at = u$

$\Rightarrow t = \frac{u}{a}$ $\therefore$ $dt = \frac{1}{a} du$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......../.........

Limits $\left.\begin{array}{l} t=a \\ u=0 \end{array}\right\}$ $\left.\begin{array}{l} t=\infty \\ u=\infty \end{array}\right\}$

Thus from (1) we have

$$\therefore \mathcal{L}\{F(at)\} = \int_0^\infty e^{\frac{-su}{a}} F(u)\,du$$

$$= \frac{1}{a}\int_0^\infty e^{\left(\frac{-s}{a}\right)u} F(u)\,du$$

$$= \frac{1}{a} f\left(\frac{s}{a}\right)$$

(E) Laplace transform of derivatives.

Theorem-6: (1st derivative)

If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}\{F'(t)\} = s f(s) - F(0)$.

Proof:

By definition of the Laplace transform we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$$

$$\therefore \mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt \quad \text{(integrating by parts)}$$

$$= \left[e^{-st} F(t)\right]_0^\infty + s\int_0^\infty e^{-st} F(t)\, dt$$

$$= 0 - F(0) + s f(s)$$

$$= s f(s) - F(0).$$

Theorem-7: (2nd derivative)

If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

Proof:

By definition of the Laplace transform, we have

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

$$\therefore \mathcal{L}\{F''(t)\} = \int_0^\infty e^{-st} F''(t)\,dt \quad \text{(integrating by Parts)}$$

$$= \left[e^{-st} F'(t)\right]_0^\infty + \int_0^\infty s e^{-st} F'(t)\,dt$$

$$= 0 - F'(0) + s\int_0^\infty e^{-st} F'(t)\,dt$$

$$= 0 - F'(0) + s\{s f(s) - F(0)\}$$

$$= s^2 f(s) - sF(0) - F'(0).$$

Theorem - 8:

(3rd derivative)

If $\mathcal{L}\{F(t)\} = f(s)$ then

$\therefore \mathcal{L}\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)$ (করে নিতে হবে)

Generalizing it, we find, [Proof নাই]

Theorem - 9:

If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{F^n(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - s^{n-3}F''(0) - \cdots - F^{n-1}(0)$$

$$= s^n f(s) - \sum_{k=0}^{n-1} s^{n-k-1} F^k(0)$$

(F) Laplace Transform of integrals

Theorem -10 :

If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \dfrac{f(s)}{s}$

Proof:

Let $G(t) = \int_0^t F(u)\,du$

Then $Gt(t) = \dfrac{d}{dt}\left[\int_0^t F(u)\,du\right] = F(t)$

$G'(t)$ ও লিখা যাবে

and $G(0) = \int_0^0 F(u)\,du = 0$

Applying theorem 6 we get

$$\mathcal{L}\{\overset{G'(t)}{Gt(t)}\} = s\mathcal{L}\{G(t)\} - G(0)$$

$$\Rightarrow \mathcal{L}\{F(t)\} = s\mathcal{L}\{G(t)\}$$

$$\Rightarrow f(s) = s\mathcal{L}\left\{\int_0^t F(u)\,du\right\}$$

Hence $s\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \dfrac{f(s)}{s}$

* Worked out Examples

Example-4: Find the Laplace transform of

$$e^{4t}+4t^3-2\sin 3t+3\cos 5t$$

Solution: By applying linearity property, we have

$$\mathcal{L}\{e^{4t}+4t^3-2\sin 3t+3\cos 5t\}$$

[Linearly independent]

$$=\mathcal{L}\{e^{4t}\}+4\mathcal{L}\{t^3\}-2\mathcal{L}\{\sin 3t\}+3\mathcal{L}\{\cos 5t\}$$

$$=\frac{1}{s-4}+4\cdot\frac{|\underline{3}}{s^4}-2\cdot\frac{3}{s^2+3^2}+3\cdot\frac{s}{s^2+5^2}$$

$$=\frac{1}{s-4}+\frac{24}{s^4}-\frac{6}{s^2+9}+\frac{3s}{s^2+25}$$

Example-5: Find the Laplace transform of

$$3t^4-2t^3+4e^{-3t}-2\sin 5t+3\cos 2t$$

Solution: By applying linearity property, we have

$$\mathcal{L}\{3t^4-2t^3+4e^{-3t}-2\sin 5t+3\cos 2t\}$$

$$=3\mathcal{L}\{t^4\}-2\mathcal{L}\{t^3\}+4\mathcal{L}\{e^{-3t}\}-2\mathcal{L}\{\sin 5t\}+3\mathcal{L}\{\cos 2t\}$$

$$=3\cdot\frac{|\underline{4}}{s^5}-2\cdot\frac{|\underline{3}}{s^4}+4\cdot\frac{1}{s+3}-2\cdot\frac{5}{s^2+5^2}+3\cdot\frac{s}{s^2+2^2}$$

$$=\frac{24}{s^5}-\frac{12}{s^4}+\frac{4}{s+3}-\frac{10}{s^2+25}+\frac{3s}{s^2+4}$$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......../.........

Example - 6: Find the Laplace transform of $t^3 e^{5t}$

Solution:

We have $\mathcal{L}\{t^3\} = \frac{\lfloor 3}{s^4} = \frac{6}{s^4} = f(s)$ (say)

Then by using first shifting theorem, we get

$\mathcal{L}\{t^3 e^{5t}\} = f(s-5)$ $\quad [\mathcal{L}\{e^{at}F(t)\} = f(s-a)]$

$= \frac{6}{(s-5)^4}$

Example - 7: Find the Laplace transform of

$e^{3t}.(2\cos 5t - 3\sin 5t)$

Solution: we have

$\mathcal{L}\{2\cos 5t - 3\sin 5t\} = 2\mathcal{L}\{\cos 5t\} - 3\mathcal{L}\{\sin 5t\}$

$= 2.\frac{s}{s^2+5^2} - 3.\frac{5}{s^2+5^2}$

$= \frac{2s - 15}{s^2+25} = f(s)$ (say)

Then by applying the first shifting theorem, we have

$\mathcal{L}\{e^{3t}.(2\cos 5t - 3\sin 5t)\}$ $\quad [\mathcal{L}\{e^{at}.F(t)\} = f(s-a)]$

$= f(s-3)$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

$$= \frac{2(s-3)-15}{(s-3)^2+25}$$

$$= \frac{2s-21}{s^2-6s+34}$$

Example-10: Find the Laplace transforms of cos at and sinh at using the change of scale property.

Solution:

(i) We know that $\left[\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)\right]$

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1} = f(s) \text{ (say)}$$

$$\mathcal{L}\{\cos at\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

$$= \frac{1}{a} \cdot \frac{s/a}{(s/a)^2+1}$$

$$= \frac{s}{a^2} \cdot \frac{a^2}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2}$$

(ii) We know that

$$\mathcal{L}\{\sinh at\} = \frac{1}{s^2-1} = f(s) \quad (\text{say})$$

$$\left[\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)\right]$$

$$\therefore \mathcal{L}\{\sinh at\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

$$= \frac{1}{a} \cdot \frac{1}{\left(\frac{s}{a}\right)^2 - 1}$$

$$= \frac{a}{s^2 - a^2}$$

Example-11: Find the Laplace transform of $\cos at$ using theorem 7.

Solution: Let $F(t) = \cos at$

Then
$$\left.\begin{aligned} F'(t) &= -a\sin at \\ F''(t) &= -a^2\cos at \end{aligned}\right\} \qquad \left.\begin{aligned} F(0) &= 1 \\ F'(0) &= 0 \end{aligned}\right\}$$

Now by theorem 7#, we have

$$\mathcal{L}\{F''(t)\} = s^2\mathcal{L}\{F(t)\} - sF(0) - F'(0)$$

$$\Rightarrow \mathcal{L}\{-a^2\cos at\} = s^2\mathcal{L}\{\cos at\} - s\cdot 1 - 0$$

$$\Rightarrow -a^2\mathcal{L}\{\cos at\} = s^2\mathcal{L}\{\cos at\} - s$$

$$\Rightarrow (s^2+a^2)\mathcal{L}\{\cos at\} = s$$

$$\Rightarrow \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

Example-12: Find the Laplace transform of $t^2\cos at$ using theorem 11.

Solution: Since $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}, \; s>0$

$$\therefore \mathcal{L}\{t^2\cos at\} = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\{\cos at\} = \frac{d^2}{ds^2}\left\{\frac{s}{s^2+a^2}\right\}$$

$$= \frac{d}{ds}\left\{ \frac{-2s^2}{(s^2+a^2)^2} + \frac{1}{s^2+a^2} \right\}$$

$$= \frac{8s^3}{(s^2+a^2)^3} - \frac{6s}{(s^2+a^2)^2}$$ [Theorm-11 দেখে নিতে হবে]

$$= \frac{8s^3 - 6s^2 - 6a^2 s}{(s^2+a^2)^3}$$

$$= \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

Example-13: Find the Laplace transform of $\int_0^t \frac{\sin t}{t}\,dt$

Solution: We know that $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = f(s)$ (say)

Now $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(u)\,du$

$$= \int_s^\infty \frac{1}{u^2+1}\,du$$

$$= \left[\tan^{-1} u\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s \quad \left[\text{Since } \tan^{-1}s + \cot^{-1}s = \frac{\pi}{2}\right]$$

$$= \cot^{-1} s$$

$$= \tan^{-1}\frac{1}{s}$$

Now we also know that

$$\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}$$

$$\therefore \mathcal{L}\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s} f(s)$$

$$= \frac{1}{s} \cdot \tan^{-1}\frac{1}{s}.$$

* Table of Laplace transform theorems

No.	Operation	$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$
1.	Linearity Property	$a_1F_1(t) + a_2F_2(t)$	$a_1\mathcal{L}\{F_1(t)\} + a_2\mathcal{L}\{F_2(t)\}$
2.	First Translation or shifting Property	$e^{at}F(t)$	$f(s-a)$
3.	Second Translation or shifting Property	$G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$	$e^{-as}f(s)$
4.	Change of Scale Property	$F(at)$	$\frac{1}{a}f\left(\frac{s}{a}\right)$
5.	Differentiation Theorems	$F'(t)$ $F^n(t)$	$sf(s) - F(0)$ $s^nf(s) - \sum_{r=0}^{n-1} s^{n-r-1} \cdot F^r(0)$
6.	Multiplication Theorems	$tF(t)$ $t^nF(t)$	$-\frac{d}{ds}f(s)$ $(-1)^n \frac{d^n}{ds^n}f(s)$
7.	Division Theorem	$\frac{1}{t}F(t)$	$\int_s^\infty f(u)\,du$
8.	Integral Theorem	$\int_0^t F(u)\,du$	$\frac{1}{s}f(s)$

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

* Table of Laplace Transform of some elementary functions:

No.	$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$
1.	1	$\frac{1}{s}, \; s>0$
2.	t	$\frac{1}{s^2}, \; s>0$
3.	t^n $n = 0, 1, 2, 3, \ldots$	$\frac{\lfloor n}{s^{n+1}}, \; s>0$ where $\lfloor n = 1, 2, 3, \ldots, n$ and $\lfloor 0 = 1$
4.	e^{at}	$\frac{1}{s-a}, \; s>a$
5.	$\sin at$	$\frac{a}{s^2+a^2}, \; s>\cancel{a}0$
6.	$\cos at$	$\frac{s}{s^2+a^2}, \; s>0$
7.	$\sinh at$	$\frac{a}{s^2-a^2}, \; s>\lvert a\rvert$
8.	$\cosh at$	$\frac{s}{s^2-a^2}, \; s>\lvert a\rvert$
9.	$t \sin at$	$\frac{2as}{(s^2+a^2)^2}, \; s>0$
10.	$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}, \; s>0$

Review Class

Fourier ~~Transform~~ Series

Definition

Even Function

Odd Function

Examples - 1, 3, 5, 6, 7, 4

Example-4: Prove that $\int_0^\infty \frac{x \sin mx}{x^2+1} dx = \frac{\pi}{2} e^{-m}, m>0$

Proof: By definition of fourier integral of an odd function, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty du \int_0^\infty f(t) \sin ut \sin ux dt \quad \text{——— (i)}$$

Replacing x by m and u by x in (i), we get

$$f(m) = \frac{2}{\pi}\int_0^\infty dx \int_0^\infty f(t) \sin xt \sin xm dt \quad \text{——— (ii)}$$

Putting $f(m) = e^{-m}$ and $f(t) = e^{-t}$ in (II), we get

$$f(m) = e^{-m} = \frac{2}{\pi}\int_0^\infty \sin xm \, dx \int_0^\infty e^{-t} \sin xt \, dt \quad \text{—— (III)}$$

Since $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

$$\therefore \int_0^\infty e^{-t} \sin xt \, dt = \left[\frac{e^{-t}(-\sin xt - x\cos xt)}{1+x^2}\right]_0^\infty$$

$$= 0 + \frac{x}{1+x^2} \qquad [\cos 0^\circ = 1] \quad [\sin 0^\circ = 0]$$

$$= \frac{x}{1+x^2}$$

Thus from (III), we get

$$e^{-m} = \frac{2}{\pi}\int_0^\infty \sin mx \cdot \frac{x}{1+x^2} \, dx$$

or, $$e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{x \sin mx}{1+x^2} \, dx$$

or, $$\int_0^\infty \frac{x\sin mx}{1+x^2} \, dx = \frac{\pi}{2} e^{-m}, \quad m > 0.$$

[Proved]

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

Example - 6: Find the fourier integral of the function $f(x) = e^{-kx}$ when $x>0$ and $f(-x) = -f(x)$ for $k>0$ and hence prove that

$$\int_0^\infty \frac{u \sin ux}{k^2+u^2}\, du = \frac{\pi}{2} e^{-kx}, \quad k>0$$

Solution: Since $f(-x) = -f(x)$, so $f(x)$ is an odd function for which we have,

$$f(x) = \frac{2}{\pi} \cancel{\int_0^\infty du} \int_0^\infty f(t)\, dt \int_0^\infty \sin ut \sin ux \, du$$

$$= \frac{2}{\pi} \int_0^\infty \left[\int_0^\infty f(t) \sin ut \, dt \right] \sin ux \, du \quad \text{——— (1)}$$

Now, $\int_0^\infty f(t) \sin ut\, dt = \int_0^\infty e^{-kt} \sin ut\, dt$

$\boxed{\sin\infty, \cos\infty = \text{No defined Values}}$

$$= \left[\frac{e^{-kt}(-k\sin ut - u\cos ut)}{k^2+u^2} \right]_0^\infty$$

$$= 0 + \frac{u}{k^2+u^2}$$

$$= \frac{u}{k^2+u^2}$$

CamScanner

Thus from ①, we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{u}{k^2+u^2} \sin ux \, du$$

$$= \frac{2}{\pi}\int_0^\infty \frac{u \sin ux}{k^2+u^2} du \quad (x>0, k>0) \text{ ---- (ii)}$$

which is the required Fourier integral of the function

$$f(x) = e^{-kx}$$

putting $f(x) = e^{-kx}$ in (ii), we get

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{u \sin ux}{k^2+u^2} du$$

$$\text{or, } \int_0^\infty \left[\frac{u \sin ux}{k^2+u^2}\right] du = \frac{\pi}{2} e^{-kx}$$

[Proved]

Example-7: Find the fourier integral of the function

$$f(x) = \begin{cases} 0 & \text{when } x<0 \\ \frac{1}{2} & \text{when } x=0 \\ e^{-x} & \text{when } x>0 \end{cases}$$

Solution: By the definition of Fourier integral in general, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} \cos u(x-t)\,du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{0}^{\infty} \cos u(x-t)\,du \qquad \left[\int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{0}^{\infty} (\cos ux \cos ut + \sin ux \sin ut)\,du$$

$$= \frac{1}{\pi}\left[\int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\cos ut\,dt\right\}\cos ux\,du + \int_{0}^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\sin ut\,dt\right\}\sin ux\,du\right] \quad \text{---} (1)$$

Now, $$\int_{-\infty}^{\infty} f(t)\cos ut\,dt = \int_{-\infty}^{0} f(t)\cos ut\,dt + \int_{0}^{\infty} f(t)\cos ut\,dt$$

$$= \int_{-\infty}^{0} 0\cos ut\,dt + \int_{0}^{\infty} e^{-t}\cos ut\,dt$$

$$= 0 + \left[\frac{e^{-t}}{1+u^2}(-\cos ut + u\sin ut)\right]_0^\infty$$

$$= \frac{1}{1+u^2} + 0 = \frac{1}{1+u^2}$$

Similarly, $\int_{-\infty}^{\infty} f(t)\sin ut\,dt = \int_{-\infty}^{0} f(t)\sin ut\,dt + \int_{0}^{\infty} f(t)\sin ut\,dt$

$$= \int_{-\infty}^{0} 0\sin ut\,dt + \int_{0}^{\infty} e^{-t}\sin ut\,dt$$

$$= 0 + \int_0^\infty e^{-t}\sin ut\,dt$$

$$= \left[\frac{e^{-t}(\sin ut - u\cos ut)}{1+u^2}\right]_0^\infty$$

$$= 0 + \frac{u}{1+u^2} = \frac{u}{1+u^2}$$

Putting these values in (1), we get

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty \frac{\cos ux}{1+u^2}\,du + \int_0^\infty \frac{u\sin ux}{1+u^2}\,du\right]$$

$$= \frac{1}{\pi}\int_0^\infty \left(\frac{\cos ux + u\sin ux}{1+u^2}\right)du \quad \text{—— (11)}$$

Putting $x = 0$ in (11), we get,

$$f(0) = \frac{1}{\pi}\int_0^\infty \frac{du}{1+u^2}$$

$$= \frac{1}{\pi}\left[\tan^{-1}u\right]_0^\infty$$

$$= \frac{1}{\pi}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{1}{2}$$

So, $f(x) = \frac{1}{2}$ for $x = 0$ is satisfied.

Hence $f(x) = \frac{1}{\pi}\int_0^\infty \frac{(\cos ux + u\sin ux)}{1+u^2}\,du$

which is the required fourier integral of the given function.

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Plan Title : Online Class - 08

Date : 10 / 02 / 2022

Review of Fourier Integral

+

Laplace Transform

Examples

Plan Title : Online Class- 09

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date : 13 / 02 / 2022

Laplace Transform

Theorems + Examples

Example - 15, Theorem - 8 বই থেকে দেখতে হবে।

Inverse Property লাগবে না

Page - 45 : Exercise 1 এর i, ii, iii, iv, x - এই গুলো করতে হবে।

3rd derivative

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

Theorem - 08: If $\mathcal{L}\{F(t)\} = f(s)$ then,

$$\mathcal{L}\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0).$$

Proof: By definition of the laplace transform,

we have, $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = f(s)$

$\therefore \mathcal{L}\{F'''(t)\} = \int_0^\infty e^{-st} F'''(t)\, dt$ [Integrating by parts]

$$= \left[e^{-st} F''(t)\right]_0^\infty + \int_0^\infty s \cdot e^{-st} F''(t)\, dt$$

$$= 0 - F''(0) + s\int_0^\infty e^{-st} F''(t)\, dt$$

$$= -F''(0) + s\left\{s^2 f(s) - sF(0) - F'(0)\right\}$$ (From theorem 7- 2nd derivative)

$$= -F''(0) + s^3 f(s) - s^2 F(0) - sF'(0)$$

$$\therefore \mathcal{L}\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)$$

[Proved]

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

Example-15: Prove that

$$\int_0^\infty t\, e^{-st} \cos at\, dt = \frac{s^2-a^2}{(s^2+a^2)^2}$$

Proof: By definition of Laplace transform, we get

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

$$L.H.S = \int_0^\infty t\, e^{-st} \cos at\, dt$$

$$= \int_0^\infty e^{-st}\, t \cos at\, dt$$

$$= \mathcal{L}\{t \cos at\}$$

$$= -\frac{d}{ds}\mathcal{L}\{\cos at\}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= -\left[\frac{(s^2+a^2)1 - s \cdot 2s}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

$$= R.H.S$$

[Proved]

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......./......./.........

Exercises 1: Find the Laplace transform of

i) $t^2 - 3t + 5$ $\left[\text{Ans: } \frac{2}{s^3} - \frac{3}{s^2} + \frac{5}{s} \quad ; \; s>0\right]$

ii) $e^{-4t} + 3e^{-2t}$ $\left[\text{Ans: } \frac{2(2s+7)}{(s+2)(s+4)}, \quad s>-2\right]$

iii) $\cosh at + \sinh at$ $\left[\text{Ans: } \frac{s+a}{s^2-a^2}, \quad s>|a|\right]$

iv) $6\sin 2t - 5\cos 2t$ $\left[\text{Ans: } \frac{12-5s}{s^2+4}\right]$

x) $3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$.

$\left[\text{Ans: } \frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4}\right]$

Solution:

x) $3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t$

⇒ By applying linearity property, we have

$\mathcal{L}\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$

$= 3\mathcal{L}\{t^4\} - 2\mathcal{L}\{t^3\} + 4\mathcal{L}\{e^{-3t}\} - 2\mathcal{L}\{\sin 5t\} + 3\cdot\mathcal{L}\{\cos 2t\}$

$= 3\cdot\frac{4!}{s^5} - 2\cdot\frac{3!}{s^4} + 4\cdot\frac{1}{s+3} - 2\cdot\frac{5}{s^2+5^2} + 3\cdot\frac{s}{s^2+2^2}$

$= \frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4}$

(Ans)

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......./.........

iv) $6\sin 2t - 5\cos 2t$

By applying linearity property, we have

$$\mathcal{L}\{6\sin 2t - 5\cos 2t\}$$

$$= 6\mathcal{L}\{\sin 2t\} - 5\mathcal{L}\{\cos 2t\}$$

$$= 6\cdot\frac{2}{s^2+2^2} - 5\cdot\frac{s}{s^2+2^2}$$

$$= \frac{12}{s^2+4} - \frac{5s}{s^2+4}$$

$$= \frac{12-5s}{s^2+4}$$

(Ans).

iii) $\cosh at + \sinh at$

By applying linearity property, we have

$$\mathcal{L}\{\cosh at + \sinh at\}$$

$$= \mathcal{L}\{\cosh at\} + \mathcal{L}\{\sinh at\}$$

$$= \frac{s}{s^2-a^2} + \frac{a}{s^2-a^2}, \text{ if } s > |a|$$

$$= \frac{s+a}{s^2-a^2}, \text{ if } s > |a|$$

(Ans).

Plan Title :

Sat / Sun / Mon / Tue / Wed / Thu / Fri

Date :......../......../.........

ii) $e^{-4t} + 3e^{-2t}$

By applying linearity property, we have

$\mathcal{L}\{e^{-4t} + 3e^{-2t}\}$

$= \mathcal{L}\{e^{-4t}\} + 3\mathcal{L}\{e^{-2t}\}$

$= \frac{1}{s-(-4)} + 3 \cdot \frac{1}{s-(-2)}$

$= \frac{1}{s+4} + \frac{3}{s+2}$

$= \frac{s+2+3s+12}{(s+4)(s+2)}$

$= \frac{4s+14}{(s+4)(s+2)}$

$= \frac{2(2s+7)}{(s+4)(s+2)}$, $s > -2$

i) $t^2 - 3t + 5$

By applying linearity property, we have

$\mathcal{L}\{t^2 - 3t + 5\}$

$= \mathcal{L}\{t^2\} - 3\mathcal{L}\{t\} + \cancel{\mathcal{L}\{5\}}\ 5\mathcal{L}\{1\}$

$= \frac{2!}{s^3} - 3 \cdot \frac{1}{s^2} + 5 \cdot \frac{1}{s}$

$= \frac{2}{s^3} - \frac{3}{s^2} + \frac{5}{s}$; $s > 0$

(Ans).

Review Class

* Laplace এর property গুলোর statement + proof

Example - 2 | i) t cos at

ii) t sin at

Example - 3 |

* Time scaling property-র math গুলো
* Change of scale, shifting property-র math
* Fourier even & odd series
* All definitions (Laplace + Fourier)
* Applications of Fourier Integral & Laplace transform in CSE (image processing)