Finite element Modelling of a Cantilever beam

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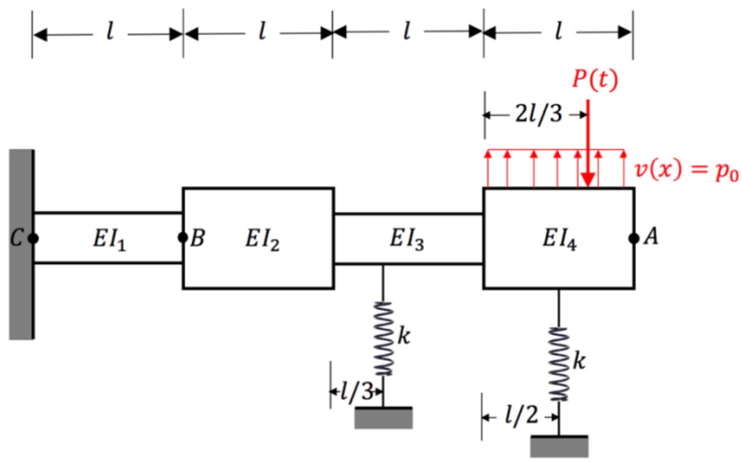


$$\rightarrow m = 100 \text{ lbm/ft} = (100/32.2) \text{ lbf} - \text{s}^2/\text{ft}^2 = 3.1056 \text{ lbf} - \text{s}^2/\text{ft}^2$$

$$\rightarrow EI_1 = EI_3 = 5 \times 10^6 \text{ lbf} - \text{in}^2 = (5 \times 10^6)/144 \text{ lbf} - \text{ft}^2 = 3.47 \times 10^6 \text{ lbf} - \text{ft}^2$$

$$\rightarrow EI_2 = EI_4 = 2(EI_1) \qquad \rightarrow l = 0.25 \text{ ft} \qquad \rightarrow k = 2400 \text{ lbf/ft}$$

1. Plot the displacement of point A as a function of time when $v(x) = p_0 = 0$ i.e., there is no distributed load and

$$\rightarrow \boldsymbol{P(t)} = \begin{cases} \textbf{500 lbf} & \ldots\ 0 \le t \le 0.01 \textbf{ sec} \\ \textbf{0 lbf} & \ldots\ t > 0.01 \textbf{ sec} \end{cases}$$

also plot the bending moment at point C.

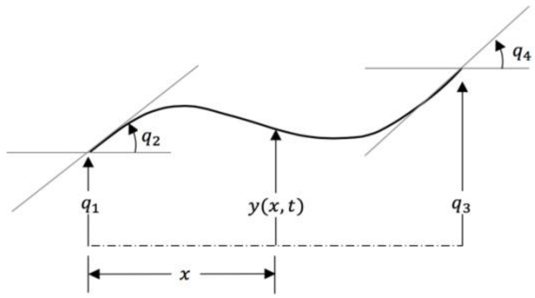
2. Plot the displacement of point A as a function of time when $v(x) = p_0 = 0$ i.e., there is no distributed load and $P(t) = 500 \sin(2.\pi.10.t)$ lbf, also plot the bending moment at point C.

3. Plot the displacement of points A and B when $P(t) = 0$ and

$$\rightarrow \boldsymbol{p_0} = \begin{cases} \textbf{100 lbf/in} = \textbf{1200 lbf/ft} & \ldots\ t \ge 0 \textbf{ sec} \\ \textbf{0 lbf/ft} & \ldots\ t < 0 \textbf{ sec} \end{cases}$$

1. MODEL DERIVATION

2.1:



The generalized displacement vector for the beam element i can be defined as:

$$\rightarrow \boldsymbol{q}^{(i)} = [q_1{}^{(i)}, q_2{}^{(i)}, q_3{}^{(i)}, q_4{}^{(i)}]^T = [y_1{}^{(i)}, \theta_1{}^{(i)}, y_2{}^{(i)}, \theta_2{}^{(i)}]^T$$

2.2:

The 1st step is to derive the element shape functions. The displacement at any point x along the beam can be written in terms of the shape functions and the generalized coordinates as

$$\rightarrow y(x,t) = \phi_1(x)q_1(t) + \phi_2(x)q_2(t) + \phi_3(x)q_3(t) + \phi_4(x)q_4(t) = \boldsymbol{\phi}(x)\boldsymbol{q}(t)$$

Since there are 4 d.o.f, the shape function $\phi(x)$ can be written as

$$\rightarrow \phi(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 \qquad \Longrightarrow \phi(\zeta) = c_1 + c_2\zeta + c_3\zeta^2 + c_4\zeta^3$$

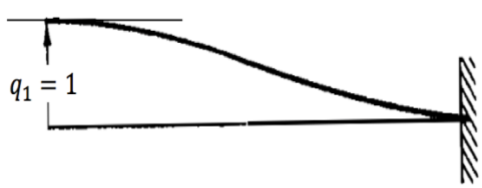
where $\zeta = (x/l)$ represents the non-dimensional position along the length of the beam. The derivation of the 4 shape functions is shown next.

If $q_1 = 1, q_2 = q_3 = q_4 = 0 \implies y(\zeta,t) = \phi_1(\zeta)$

$$\rightarrow \quad y(\zeta,t) = \phi_1(\zeta) = c_1 + c_2\zeta + c_3\zeta^2 + c_4\zeta^3$$

$$\rightarrow y'(\zeta,t) = \frac{dy(\zeta,t)}{dx} = \frac{c_2}{l} + \frac{2c_3\zeta}{l} + \frac{3c_4\zeta^2}{l}$$

Using the 4 BCs to get the 4 constants, we have



(1) $\boldsymbol{y(0,t) = 1}$: $\quad c_1 = 1$

(2) $\boldsymbol{y'(0,t) = 0}$: $\quad c_2 = 0$

(3) $\boldsymbol{y(1,t) = 0}$: $\quad c_3 + c_4 = -1$

(4) $\boldsymbol{y'(1,t) = 0}$: $\quad 2c_3 + 3c_4 = 0$

$$\therefore\ c_1 = 1, c_2 = 0, c_3 = -3, c_4 = 2$$

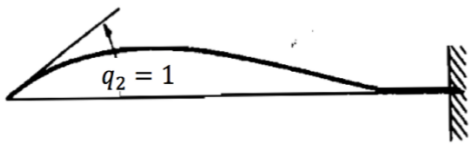
$$\Longrightarrow \boxed{\phi_1(\zeta) = 1 - 3\zeta^2 + 2\zeta^3}$$

If $q_2 = 1, q_1 = q_3 = q_4 = 0 \implies y(\zeta,t) = \phi_2(\zeta)$

$$\rightarrow \; y(\zeta,t) = \phi_2(\zeta) = c_1 + c_2\zeta + c_3\zeta^2 + c_4\zeta^3$$

$$\rightarrow y'(\zeta,t) = \frac{dy(\zeta,t)}{dx} = \frac{c_2}{l} + \frac{2c_3\zeta}{l} + \frac{3c_4\zeta^2}{l}$$

Using the 4 BCs to get the 4 constants, we have



(1) $\boldsymbol{y(0,t) = 0}$**:** $\quad c_1 = 0$

(2) $\boldsymbol{y'(0,t) = 1}$**:** $\quad c_2 = l$

(3) $\boldsymbol{y(1,t) = 0}$**:** $\quad c_3 + c_4 = -l$

(4) $\boldsymbol{y'(1,t) = 0}$**:** $\quad 2c_3 + 3c_4 = -l$

$$\therefore \; c_1 = 0,\, c_2 = l,\, c_3 = -2l,\, c_4 = l$$

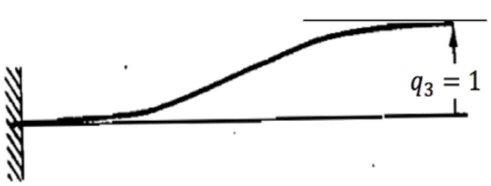
$$\implies \boxed{\phi_2(\zeta) = l\zeta - 2l\zeta^2 + l\zeta^3}$$

If $q_3 = 1, q_1 = q_2 = q_4 = 0 \implies y(\zeta,t) = \phi_3(\zeta)$

$$\rightarrow \; y(\zeta,t) = \phi_3(\zeta) = c_1 + c_2\zeta + c_3\zeta^2 + c_4\zeta^3$$

$$\rightarrow y'(\zeta,t) = \frac{dy(\zeta,t)}{dx} = \frac{c_2}{l} + \frac{2c_3\zeta}{l} + \frac{3c_4\zeta^2}{l}$$

Using the 4 BCs to get the 4 constants, we have



(1) $\boldsymbol{y(0,t) = 0}$**:** $\quad c_1 = 0$

(2) $\boldsymbol{y'(0,t) = 0}$**:** $\quad c_2 = 0$

(3) $\boldsymbol{y(1,t) = 1}$**:** $\quad c_3 + c_4 = 1$

(4) $\boldsymbol{y'(1,t) = 0}$**:** $\quad 2c_3 + 3c_4 = 0$

$$\therefore \; c_1 = 0,\, c_2 = 0,\, c_3 = 3,\, c_4 = -2$$

$$\implies \boxed{\phi_3(\zeta) = 3\zeta^2 - 2\zeta^3}$$

If $q_4 = 1, q_1 = q_2 = q_3 = 0 \implies y(\zeta, t) = \phi_4(\zeta)$

$$\rightarrow \; y(\zeta, t) = \phi_4(\zeta) = c_1 + c_2\zeta + c_3\zeta^2 + c_4\zeta^3$$

$$\rightarrow y'(\zeta, t) = \frac{dy(\zeta, t)}{dx} = \frac{c_2}{l} + \frac{2c_3\zeta}{l} + \frac{3c_4\zeta^2}{l}$$

Using the 4 BCs to get the 4 constants, we have

(1) $\boldsymbol{y(0, t) = 0}$**:** $\quad c_1 = 0$

(2) $\boldsymbol{y'(0, t) = 0}$**:** $\quad c_2 = 0$

(3) $\boldsymbol{y(1, t) = 0}$**:** $\quad c_3 + c_4 = 0$

(4) $\boldsymbol{y'(1, t) = 1}$**:** $\quad 2c_3 + 3c_4 = l$

$$\therefore \; c_1 = 0, c_2 = 0, c_3 = -l, c_4 = l$$

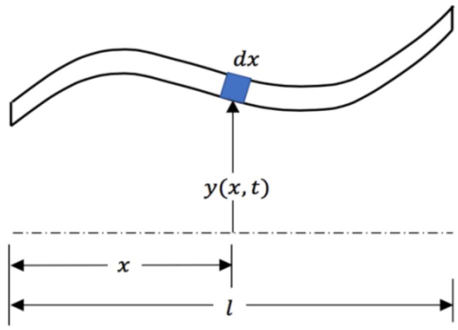
$$\implies \boxed{\phi_4(\zeta) = -l\zeta^2 + l\zeta^3}$$

$$\implies \boxed{\boldsymbol{\phi}(\zeta) = \begin{bmatrix} \phi_1(\zeta) \\ \phi_2(\zeta) \\ \phi_3(\zeta) \\ \phi_4(\zeta) \end{bmatrix}^T = \begin{bmatrix} 1 - 3\zeta^2 + 2\zeta^3 \\ l\zeta - 2l\zeta^2 + l\zeta^3 \\ 3\zeta^2 - 2\zeta^3 \\ -l\zeta^2 + l\zeta^3 \end{bmatrix}^T}$$

2.3:

The element mass matrix $M^{(i)}$ and the stiffness matrix $K^{(i)}$ can be derived from the kinetic and potential energy respectively using the element shape functions.

Element mass matrix using Kinetic energy



$$\rightarrow dT = \frac{1}{2}dm(\dot{y}(x,t))^2 = \frac{1}{2}\rho A dx(\boldsymbol{\phi}(x)\dot{\boldsymbol{q}}(t))^2 = \frac{1}{2}\rho A dx(\boldsymbol{\phi}(x)\dot{\boldsymbol{q}}(t))^T(\boldsymbol{\phi}(x)\dot{\boldsymbol{q}}(t))$$

$$\rightarrow\ T = \frac{1}{2}\ \dot{\boldsymbol{q}}(t)^{\boldsymbol{T}}\ \left[\ \rho A\int_0^l \boldsymbol{\phi}(x)^T\boldsymbol{\phi}(x)\ dx\right]\dot{\boldsymbol{q}}(t)\ =\ \frac{1}{2}\ \dot{\boldsymbol{q}}(t)^{\boldsymbol{T}}\ M^{(i)}\ \dot{\boldsymbol{q}}(t)$$

$$\Longrightarrow \boxed{M^{(i)} = \rho A\int_0^l \boldsymbol{\phi}(x)^T\boldsymbol{\phi}(x)\ dx}$$

$$\therefore M^{(i)} = \rho A\int_0^l \left\{\begin{bmatrix} \phi_1\phi_1 & \phi_1\phi_2 & \phi_1\phi_3 & \phi_1\phi_4 \\ \phi_2\phi_1 & \phi_2\phi_2 & \phi_2\phi_3 & \phi_2\phi_4 \\ \phi_3\phi_1 & \phi_3\phi_2 & \phi_3\phi_3 & \phi_3\phi_4 \\ \phi_4\phi_1 & \phi_4\phi_2 & \phi_4\phi_3 & \phi_4\phi_4 \end{bmatrix}\right\}\ dx$$

This needs to be evaluated to obtain the element mass matrix. As an example the calculation of ${M_{11}}^{(i)}$ is shown next:

$$\rightarrow M_{11} = \rho A\int_0^l \phi_1\phi_1\ dx\ =\ \rho A\int_0^l \left[1 - \frac{6x^2}{l^2} + \frac{4x^3}{l^3} + \frac{9x^4}{l^4} - \frac{12x^5}{l^5} + \frac{4x^6}{l^6}\right] dx$$

$$=\ \rho A\left[l - 2l + l + \frac{9}{5}l - 2l + \frac{4}{7}l\right] = \rho A l\ \frac{13}{35} = \frac{156}{420}\rho A l = \frac{ml}{420}(156)$$

By repeating the same for all the elements, the following element mass matrix can be obtained:

$$\rightarrow\ \boxed{M^{(i)} = \frac{ml}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}}$$

Element stiffness matrix using Potential energy

$$\rightarrow U = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,dx = \frac{1}{2}\int_0^l \frac{[EI(d^2y/dx^2)]^2}{EI}\,dx = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^l \left(\boldsymbol{\phi}_{,xx} q(t)\right)^2 dx$$

$$= \frac{EI}{2}\int_0^l \left(\boldsymbol{\phi}_{,xx} q(t)\right)^T \left(\boldsymbol{\phi}_{,xx} q(t)\right) dx = \frac{EI}{2}\int_0^l q(t)^T \boldsymbol{\phi}_{,xx}{}^{\boldsymbol{T}} \boldsymbol{\phi}_{,xx} q(t)\,dx$$

$$= \frac{1}{2} q(t)^T \left[EI \int_0^l \boldsymbol{\phi}_{,xx}{}^{\boldsymbol{T}} \boldsymbol{\phi}_{,xx}\,dx \right] q(t) = \frac{1}{2} q(t)^T K^{(i)}\, q(t)$$

$$\Longrightarrow \boxed{K^{(i)} = EI \int_0^l \boldsymbol{\phi}_{,xx}{}^{\boldsymbol{T}} \boldsymbol{\phi}_{,xx}\,dx}$$

$$\therefore K^{(i)} = EI \int_0^l \left\{ \begin{bmatrix} \phi_{1,xx}\phi_{1,xx} & \phi_{1,xx}\phi_{2,xx} & \phi_{1,xx}\phi_{3,xx} & \phi_{1,xx}\phi_{4,xx} \\ \phi_{2,xx}\phi_{1,xx} & \phi_{2,xx}\phi_{2,xx} & \phi_{2,xx}\phi_{3,xx} & \phi_{2,xx}\phi_{4,xx} \\ \phi_{3,xx}\phi_{1,xx} & \phi_{3,xx}\phi_{2,xx} & \phi_{3,xx}\phi_{3,xx} & \phi_{3,xx}\phi_{4,xx} \\ \phi_{4,xx}\phi_{1,xx} & \phi_{4,xx}\phi_{2,xx} & \phi_{4,xx}\phi_{3,xx} & \phi_{4,xx}\phi_{4,xx} \end{bmatrix} \right\} dx$$

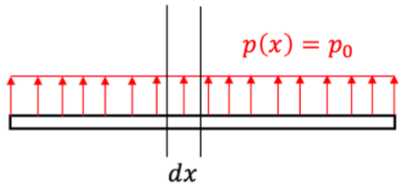
This needs to be evaluated to obtain the element stiffness matrix. As an example the calculation of $K_{11}{}^{(i)}$ is shown next:

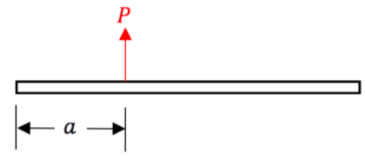
$$\rightarrow K_{11} = EI \int_0^l \phi_{1,xx}\phi_{1,xx}\,dx = EI \int_0^l \left(\frac{-6}{l^2} + \frac{12x}{l^3}\right)^2 dx = \frac{12EI}{l^3}$$

By repeating the same for all the elements, the following element stiffness matrix can be obtained:

$$\rightarrow \boxed{K^{(i)} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}}$$

For this problem, before writing the M and K matrices for the 4 elements, we need to know how to handle distributed loads, off-node point-loads and spring elements. This can be done using work equivalence as follows:





$$\rightarrow \delta W = \int (p_0 dx)\delta y(x,t) = \int p_0 \boldsymbol{\phi}(x)\delta q(t) dx$$

$$= \left(\int p_0 \boldsymbol{\phi}(\boldsymbol{x}) dx\right)\delta q(t) = Q^T \delta q(t)$$

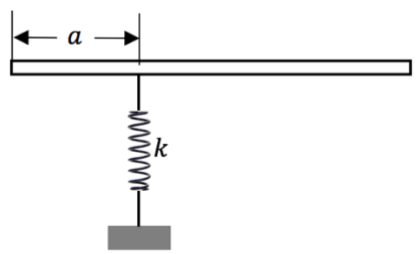
$$\Longrightarrow Q_{p0} = p_0 \int_0^l \boldsymbol{\phi}(\boldsymbol{x}) dx = p_0 \int_0^l \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} dx$$

$$\Longrightarrow \boxed{Q_{p0}{}^{(i)} = \begin{bmatrix} F_{1p0} \\ M_{1p0} \\ F_{2p0} \\ M_{2p0} \end{bmatrix} = p_0 \begin{bmatrix} l/2 \\ l^2/12 \\ l/2 \\ -l^2/12 \end{bmatrix}}$$

$$\rightarrow \delta W = P\delta y(a,t) = \big(P\boldsymbol{\phi}(a)\big)\delta q(t)$$

$$= Q^T \delta q(t)$$

$$\Longrightarrow \boxed{Q_P{}^{(i)} = P\boldsymbol{\phi}(a) = \begin{bmatrix} F_{1P} \\ M_{1P} \\ F_{2P} \\ M_{2P} \end{bmatrix} = P \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \\ \phi_3(a) \\ \phi_4(a) \end{bmatrix}}$$



$y(x,t)$ is + upwards.

$$\rightarrow \delta W = -F_k \delta y(a,t) = -ky(a,t)\delta y(a,t) = -k(\boldsymbol{\phi}(a)q(t))\big(\boldsymbol{\phi}(a)\delta q(t)\big)$$

$$= \big(-k\boldsymbol{\phi}(a)^T \boldsymbol{\phi}(a) q(t)\big)\delta q(t) = Q^T \delta q(t)$$

$$\Longrightarrow \boxed{Q_k{}^{(i)} = -k \begin{bmatrix} \phi_1(a)\phi_1(a) & \phi_1(a)\phi_2(a) & \phi_1(a)\phi_3(a) & \phi_1(a)\phi_4(a) \\ \phi_2(a)\phi_1(a) & \phi_2(a)\phi_2(a) & \phi_2(a)\phi_3(a) & \phi_2(a)\phi_4(a) \\ \phi_3(a)\phi_1(a) & \phi_3(a)\phi_2(a) & \phi_3(a)\phi_3(a) & \phi_3(a)\phi_4(a) \\ \phi_4(a)\phi_1(a) & \phi_4(a)\phi_2(a) & \phi_4(a)\phi_3(a) & \phi_4(a)\phi_4(a) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}}$$

Thus, the spring is handled by adding this matrix to the stiffness matrix of the element to which the spring is attached.

The M, K, Q matrices can now be written for each element which gives the element equations of motion. (calculated using MATLAB)

2.3.1) Element 1:

```
M1 =

    0.2884    0.0102    0.0998   -0.0060
    0.0102    0.0005    0.0060   -0.0003
    0.0998    0.0060    0.2884   -0.0102
   -0.0060   -0.0003   -0.0102    0.0005

K1 =

   1.0e+07 *

    2.6667    0.3333   -2.6667    0.3333
    0.3333    0.0556   -0.3333    0.0278
   -2.6667   -0.3333    2.6667   -0.3333
    0.3333    0.0278   -0.3333    0.0556

Q1 =

     0
     0
     0
     0
```

2.3.2) Element 2:

$\rightarrow M2 = M1$

$\rightarrow K2 = 2 \times K1$

$\rightarrow Q2 = Q1$

2.3.3) Element 3:

$\rightarrow M3 = M1$

$\rightarrow Q3 = Q1$

$\rightarrow K3 = K1 + K3\_\text{spring}$

```
K3_mod =

   1.0e+07 *

    2.6668    0.3333   -2.6666    0.3333
    0.3333    0.0556   -0.3333    0.0278
   -2.6666   -0.3333    2.6667   -0.3333
    0.3333    0.0278   -0.3333    0.0556
```

The effect of the spring is very small which is understandable.

2.3.4) Element 4:

$\rightarrow M4 = M1$

$\rightarrow K4 = (2 \times K1) + K4_{\text{spring}}$

```
K4_mod =

   1.0e+07 *

    5.3334    0.6667   -5.3333    0.6667
    0.6667    0.1111   -0.6667    0.0556
   -5.3333   -0.6667    5.3334   -0.6667
    0.6667    0.0556   -0.6667    0.1111

Q4(p0, P) =

  p0/8 - (7*P)/27
    p0/192 - P/54
 p0/8 - (20*P)/27
    P/27 - p0/192
```

2.4:

The element M, k and Q are then combined using their respective transformation matrices to obtain the global M, k and Q.

$$\rightarrow A^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow A^{(2)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow A^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\rightarrow A^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow M = \sum_{i=1}^{4} A^{(i)^T} M^{(i)} A^{(i)} \qquad \rightarrow K = \sum_{i=1}^{4} A^{(i)^T} K^{(i)} A^{(i)} \qquad \rightarrow Q = \sum_{i=1}^{4} A^{(i)^T} Q^{(i)}$$

```
M =

    0.2884    0.0102    0.0998   -0.0060         0         0         0         0         0         0
    0.0102    0.0005    0.0060   -0.0003         0         0         0         0         0         0
    0.0998    0.0060    0.5768         0    0.0998   -0.0060         0         0         0         0
   -0.0060   -0.0003         0    0.0009    0.0060   -0.0003         0         0         0         0
         0         0    0.0998    0.0060    0.5768         0    0.0998   -0.0060         0         0
         0         0   -0.0060   -0.0003         0    0.0009    0.0060   -0.0003         0         0
         0         0         0         0    0.0998    0.0060    0.5768         0    0.0998   -0.0060
         0         0         0         0   -0.0060   -0.0003         0    0.0009    0.0060   -0.0003
         0         0         0         0         0         0    0.0998    0.0060    0.2884   -0.0102
         0         0         0         0         0         0   -0.0060   -0.0003   -0.0102    0.0005
```

```
K =

   1.0e+07 *

    2.6667    0.3333   -2.6667    0.3333         0         0         0         0         0         0
    0.3333    0.0556   -0.3333    0.0278         0         0         0         0         0         0
   -2.6667   -0.3333    8.0000    0.3333   -5.3333    0.6667         0         0         0         0
    0.3333    0.0278    0.3333    0.1667   -0.6667    0.0556         0         0         0         0
         0         0   -5.3333   -0.6667    8.0001   -0.3333   -2.6666    0.3333         0         0
         0         0    0.6667    0.0556   -0.3333    0.1667   -0.3333    0.0278         0         0
         0         0         0         0   -2.6666   -0.3333    8.0001    0.3333   -5.3333    0.6667
         0         0         0         0    0.3333    0.0278    0.3333    0.1667   -0.6667    0.0556
         0         0         0         0         0         0   -5.3333   -0.6667    5.3334   -0.6667
         0         0         0         0         0         0    0.6667    0.0556   -0.6667    0.1111
```

→ These are 10×10 matrices, however the 1st two degrees of freedom are Zero, so the 1st two rows and columns are deleted to obtain the reduced M and K matrices.

```
Q(p0, P) =

                    0
                    0
                    0
                    0
                    0
                    0
   p0/8 - (7*P)/27
     p0/192 - P/54
  p0/8 - (20*P)/27
     P/27 - p0/192
```

→ This is a 10×1 matrix, however the 1st two degrees of freedom are Zero, so the 1st two rows are deleted to obtain the reduced Q matrix.

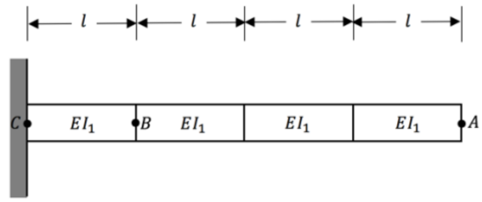
After getting these reduced matrices, we are now ready to solve the system.

$$\rightarrow [M]_{8\times8}[\ddot{q}]_{8\times1} + [K]_{8\times8}[q]_{8\times1} = [Q]_{8\times1}$$

2. SIMULATION RESULTS

3.1:

(a) Verification 1: by taking a simple beam and comparing the analytically calculated mode frequencies and mode shapes with those obtained from the code.



From the data book, the 1st three mode frequencies are equal to:

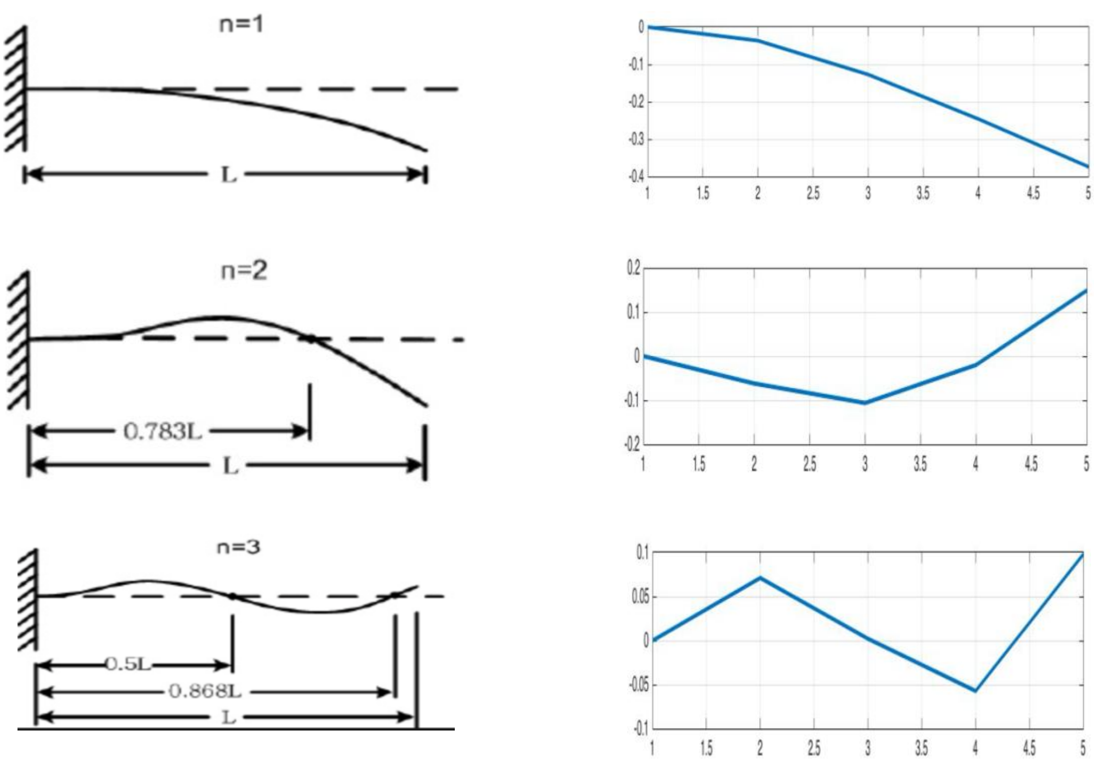
$$\rightarrow \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \sqrt{\frac{EI_1}{m(4l)^4}} \begin{bmatrix} (1.875104)^2 \\ (4.694091)^2 \\ (7.854757)^2 \end{bmatrix} = \begin{bmatrix} 371.7769 \\ 2.3299 \times 10^3 \\ 6.5237 \times 10^3 \end{bmatrix} \text{ rad/s}$$

The values obtained from FEM code are as follows:

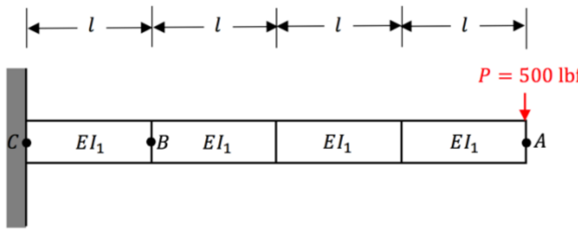
```
Command Window
omega =

   1.0e+05 *

   0.003717890965060
   0.023326010896119
   0.065742578740300
```

<u>which matches the results obtained above.</u>

Comparison of the mode shapes is given next which again looks pretty satisfactory.



(b) Verification 2: by calculating the static deflection under a constant point force at the end analytically and comparing it with that obtained from the code.



$$\rightarrow y(A) = \frac{P(4l)^3}{3EI_1} = 0.005 \text{ ft}$$

```
>> disp_A

disp_A =

   0.005908063267637
```

which looks pretty reasonable considering that the beam is modelled using just 4 elements.

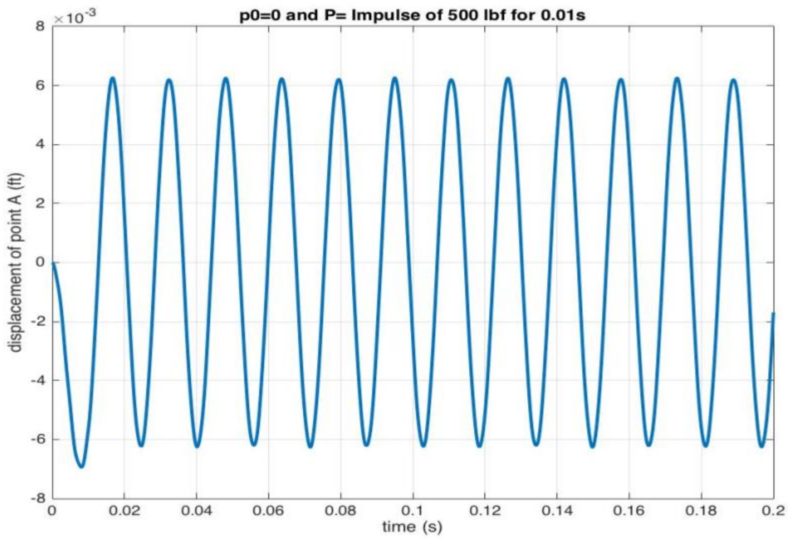
(c) Once the verification is done, we can now perform the simulation for the original beam given in the problem for the three cases:

1. Plot the displacement of point A as a function of time when $v(x) = p_0 = 0$ i.e., there is no distributed load and

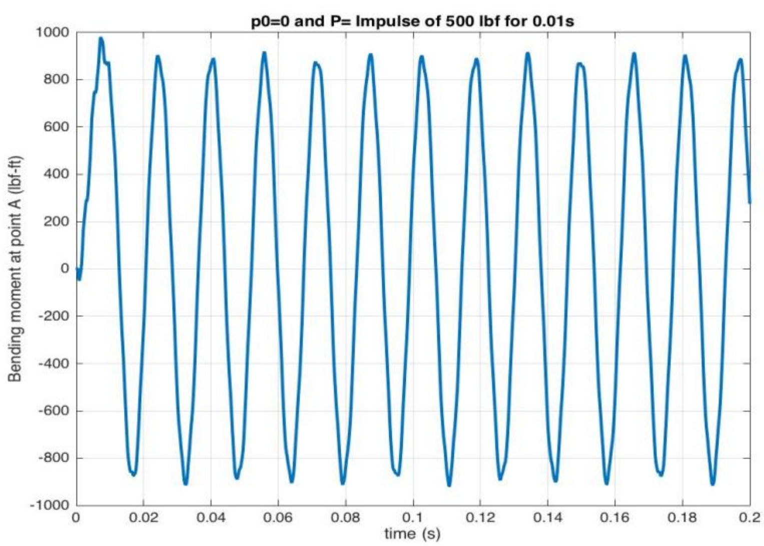
$$\rightarrow P(t) = \begin{cases} 500\ \text{lbf} & \ldots 0 \le t \le 0.01\ \text{sec} \\ 0\ \ \text{lbf} & \ldots \quad t > 0.01\ \text{sec} \end{cases}$$

also plot the bending moment at point C.

<u>Displacement at Point (A)</u>

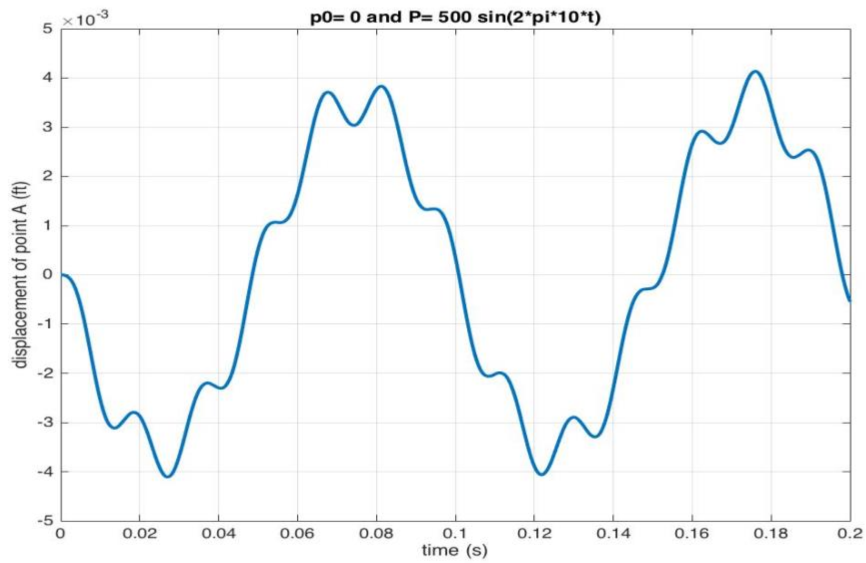


<u>Bending Moment at Point (C)</u>

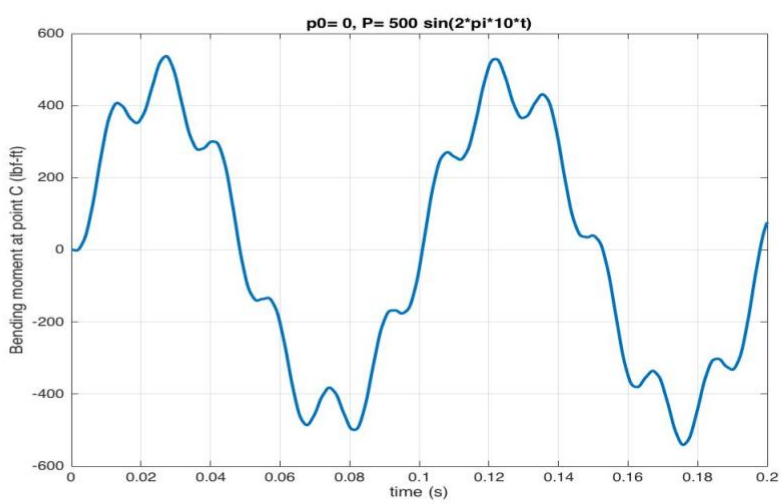


2. Plot the displacement of point A as a function of time when $v(x) = p_0 = 0$ i.e., there is no distributed load and $P(t) = 500\ \sin(2.\pi.10.t)$ lbf, also plot the bending moment at point C.

<u>Displacement at Point (A)</u>



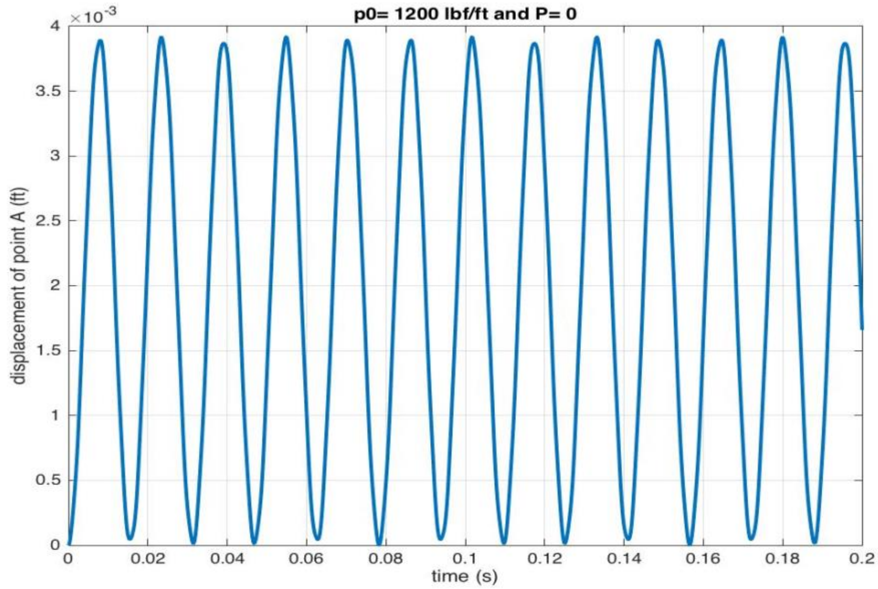
<u>Bending Moment at Point (C)</u>



\3. **Plot the displacement of points A and B when $P(t) = 0$ and**

$$\rightarrow \boldsymbol{p_0} = \begin{cases} \mathbf{100\ lbf/in = 1200\ lbf/ft} & \ldots \boldsymbol{t \geq 0}\ \mathbf{sec} \\ \mathbf{0\ \ lbf/ft} & \ldots\ \boldsymbol{t < 0}\ \mathbf{sec} \end{cases}$$

<u>Point (A)</u>



<u>Point (B)</u>