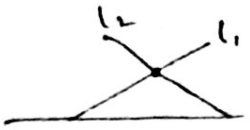
MAT-216

System of linear equation

* power must be 1 ($\sqrt{x}$, xy are not linear)

* If linear equations are inconsistent then solution is not possible for that. Like, $0 \neq 15$ if, equation comes like this

* Consistent → 2 types. (i) unique (ii) infinitely many



one solution is unique.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

a_{mn}

row column

coefficient matrix:
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right)$$

matrix: A rectanglelar array of numbers of the form:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Row matrix: $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ [row vector]

coloumn matrix: $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ [coloumn vector]

Square matrix: $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & -5 \\ 3 & 2 & 4 \end{pmatrix}$

Diagonal matrix: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ $a_{ij} = 0$ but $i \neq j$

Identity / Unit matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Zero / null matrix: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

idempotent matrix: $A = \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix}$ | $A^2 = A.A$ row x column

transpose matrix: $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

$A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

trace of A: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$Tr(A) = a_{11} + a_{22} + a_{33}$

symmetric matrix: If $A = A^T$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 0 \end{pmatrix}$ $\therefore$ $A^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 0 \end{pmatrix}$

Triangular matrix :

lower triangular :

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

upper triangular :

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

System of Linear equation solve by,

(i) Gaussian elimination method / Row-echelon form

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

→ have to make 1 diagonally & make the lower part zero.

(ii) Gauss Jordan Elimination method / Reduced-row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right)$$ → make diagonally 1 and convert lower and upper value into zero

Slide-16
Book 11th
Ex: 05

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \quad\quad + 2x_5 \quad\quad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 \quad\quad + 15x_6 &= 5 \\ 2x_1 + 6x_2 \quad\quad + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

Solving by G.J.E method

Sol: The augmented matrix is,

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right)$$

$$= \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right) \begin{array}{l} \\ ;[r_2' = -2r_1 + r_2] \\ ;[r_4' = -2r_1 + r_4] \\ \\ \end{array}$$

$$
= \left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
0 & 0 & 4 & 8 & 0 & 18 & 6
\end{array}\right) \therefore [r_2' = (-1) r_1]
$$

$$
= \left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 2
\end{array}\right) \begin{array}{l} ;[r_3' = -5r_2 + r_3] \\ ;[r_4' = -4r_2 + r_4] \end{array}
$$

$$
= \left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \begin{array}{l} ;[r_3 \longleftrightarrow r_4] \\ ;[r_3' = (1/6) r_3] \end{array}
$$

The corresponding system of linear equation is,

$$
= \left(\begin{array}{cccccc|c}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) ;[r_2' = -3r_3 + r_2]
$$

$$
= \left(\begin{array}{cccccc|c}
1 & 3 & 0 & 4 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) ;[r_1' = 2r_2 + r_1]
$$

The corresponding system of Linear equation is,

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$

$$x_3 + 2x_4 = 0$$

$$x_6 = 1/3$$

$\therefore x_1 = -3x_2 - 4x_4 - 2x_5$

$\therefore x_3 = -2x_4$

$\therefore x_6 = 1/3$

Let, $x_2 = a$

$x_4 = b$

$x_5 = c$

$\therefore x_1 = -(3a + 4b + 2c)$

$\therefore x_3 = -2b$

$\therefore x_6 = 1/3$ (Ans)

Practice sheet-1 doc file Page-3

(G.E)

3 a

$$x_1 + x_2 + 2x_3 = 8$$

$$-x_1 - 2x_2 + 3x_3 = 1$$

$$3x_1 - 7x_2 + 4x_3 = 10$$

or

book Slide-23 Exercise-5

The augmented matrix is,

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right) \quad ; [r_2' = r_1 + r_2] \quad ; [r_3' = -3r_1 + r_3]$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right) \quad ; [r_2' = (-1) r_2]$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right) \quad ; [r_3' = 10 r_2 + r_3]$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad ; [r_3' = \left(-\tfrac{1}{52}\right) r_3]$$

The corresponding system of linear equation is,

$$x_1 + x_2 + 2x_3 = 8$$
$$x_2 - 5x_3 = -9$$
$$x_3 = 2$$

$x_3 = 2$ (Ans)

$x_2 = 1$ (Ans)

$x_1 = 3$ (Ans)

Book Slide-23 Exercise-6

$$2x_1 + 2x_2 + 2x_3 = 0$$
$$-2x_1 + 5x_2 + 2x_3 = 1$$
$$8x_1 + x_2 + 4x_3 = -1$$

(G.J.E)

The augmented matrix,

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right) ; [r_1' = \tfrac{1}{2} r_1]$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right) \begin{array}{l} ; [r_2' = 2r_1 + r_2] \\ ; [r_3' = -8r_1 + r_3] \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & -7 & -4 & -1 \end{array}\right) ; [r_2' = \tfrac{1}{7} r_2]$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 3/7 & -1/7 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} ; [r_1' = r_2 - r_1] \\ ; [r_3' = 7r_2 + r_3] \end{array}$$

The corresponding system of linear equations,

$$x_1 + \frac{3}{7} x_3 = -\frac{1}{7}$$

$$x_2 + \frac{4}{7} x_3 = \frac{1}{7}$$

$$\therefore x_1 = -\frac{1}{7} - \frac{3}{7} x_3$$

$$\therefore x_2 = \frac{1}{7} - \frac{4}{7} x_3$$

Let, $x_3 = a$

$$\therefore x_1 = -\frac{1}{7}(1 + 3a)$$

$$\therefore x_2 = \frac{1}{7}(1 - 4a) \quad \text{(Ans)}$$

Homogeneous $\rightarrow$ $\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$ $\leftarrow$ All must be (0) zero

Non-homogeneous $\rightarrow$

$$x_1 + 2x_2 + 3x_3 = 4$$
$$4x_1 + 5x_2 + 6x_3 = 2$$
$$7x_1 + 8x_2 + 9x_3 = 5$$

$\leftarrow$ if there is a single non zero value then it is non-homogeneous.

(7) slide - 23
book

$$x - y + 2z - \omega = -1$$
$$2x + y - 2z - 2\omega = -2$$
$$-x + 2y - 4z + \omega = 1$$
$$3x \qquad - 3\omega = -3$$

(G.J.F)

The augmented matrix,

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right) ; \left[\begin{array}{l} r_2' = -2r_1 + r_2 \\ r_3' = r_1 + r_3 \\ r_4' = -3r_1 + r_4 \end{array}\right]$$

$$= \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right) ; \left[r_2' = -\tfrac{1}{3} r_2\right]$$

$$= \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) ; \left[\begin{array}{l} r_1' = -r_2 + r_1 \\ r_3' = r_2 + r_3 \\ r_4' = 3r_2 + r_4 \end{array}\right]$$

$\therefore$ The corresponding system of linear equations,

$$x_1 - x_4 = -1 \qquad x_1 = -1 + x_4$$

$$-x_2 + 2x_3 = 0 \qquad x_2 = 2x_3$$

Let, $x_3 = a$

$x_4 = b$

$\therefore \; x_1 = b - 1$

$\therefore \; x_2 = 2a$ (Ans)

Slide-23
book
Exercise
8

$$-2b + 3c = 1$$
$$3a + 6b - 3c = -2$$
$$6a + 6b + 3c = 5$$

The augmented matrix is,

$$\left(\begin{array}{ccc|c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right) ; [r_1 \longleftrightarrow r_2]$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right) \; ; [r_1' = \tfrac{1}{3} r_1]$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2/3 \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right) \; ; [r_3' = -6r_1 + r_3]$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & -6 & 9 & 9 \end{array}\right) \; ; [r_2' = (-1/2) r_2]$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2/3 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & 0 & 6 \end{array}\right) \; ; [r_3' = 6r_2 + r_3]$$

Here, $0 = 6$

but, $0 \neq 6$.

$\therefore$ The system is inconsistent.

so, Solution cannot possible.

Book slide 33
Ex: 13

$$2x + 2y - 4z = 0$$
$$w - y - 3z = 0$$
$$2w + 3x + y + z = 0$$
$$-2w + x + 3y - 2z = 0$$

(G.J.E)

The augmented matrix is,

$$\left(\begin{array}{cccc|c} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right) ; [r_1 \leftrightarrow r_2]$$

$$= \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{array}\right) \left[\begin{array}{l} r_3' = -2r_1 + r_3 \\ r_4 = 2r_1 + r_4 \end{array}\right]$$

$$= \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{array}\right) [r_2' = \tfrac{1}{2} r_2]$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{array}\right) \quad \begin{array}{l} [r_3' = -3r_2 + r_3] \\ [r_4' = -r_2 + r_4] \end{array}$$

$$\cancel{\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & & 0 \end{array}\right)} \quad \cancel{[r_3' = \tfrac{1}{\,}\, r_3]}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \left[\begin{array}{l} r_1' = 3r_3 + r_1 \\ r_2' = -2r_3 + r_2 \\ r_4' = 10r_3 + r_4 \end{array}\right]$$

The corresponding system of linear equations,

$$\begin{aligned} w - y &= 0 \\ x + y &= 0 \\ z &= 0 \end{aligned} \qquad \Bigg| \qquad \text{let, } y = t$$

$\therefore\ w = t$

$\therefore\ x = -t$

$\therefore\ z = 0$ (Ans)

Practice Sheet-1 doc. file page-3

3(b) $2x_1 + 2x_2 - x_3 + x_5 = 0$

$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$

$x_1 + x_2 - 2x_3 - x_5 = 0$

$x_3 + x_4 + x_5 = 0$

The augmented matrix is,

$$\left(\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \quad [r_1 \leftrightarrow r_3]$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \quad \begin{bmatrix} r_2' = r_1 + r_2 \\ r_3' = -2r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \quad [r_2 \leftrightarrow r_3]$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \quad \begin{array}{l} [r_2' = \frac{1}{3} r_2] \\ \cancel{[\because r_4' = r_2 + r_1]} \\ \cancel{[r_4' = -r_2 + r_4]} \end{array}$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \quad \begin{array}{l} [\because r_1' = r_2 + r_1] \\ [\because r_4' = -r_2 + r_4] \end{array}$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{array}\right) \quad [r_3 \leftrightarrow r_4]$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad [r_4' = 3r_3 + r_4]$$

The corresponding system of linear equations,

$$\begin{array}{l|l} x_1 + x_2 - x_3 = 0 & \therefore x_1 = x_3 - x_2 \\ x_3 + x_5 = 0 & \therefore x_3 = -x_5 \\ x_4 = 0 & \therefore x_4 = 0 \end{array}$$

Let, $x_5 = b$, $x_2 = a$

$$\left.\begin{array}{l} \therefore x_1 = -b - a \\ \therefore x_3 = -b \\ \therefore x_4 = 0 \end{array}\right\} \text{(Ans)}$$

Practice sheet- Page-3 doc file

① $x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

The augmented matrix is,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right) \quad \begin{bmatrix} r_2' = -r_1 + r_2 \\ r_3' = -r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right) \quad [r_3' = -r_2 + r_3]$$

$\therefore$ The corresponding system of linear equation is,

$x + y + z = 6 \quad \ldots (i)$

$y + 2z = 4 \quad \ldots (ii)$

$(\lambda - 3) z = \mu - 10 \quad \ldots (iii)$

From equation (iii),

(a) If $\lambda \neq 3$ then it is unique solution

(b) If $\lambda = 3$, $\mu \neq 10$ then it has no solution

(c) If $\lambda = 3$, $\mu = 10$ then it has more than one solution. (Ans)

ractice
heet
age 3
oc file

② (a)

$$\lambda x + y + z = 1$$
$$x + \lambda y + z = 1$$
$$x + y + \lambda z = 1$$

$$\# \; -\frac{(1-\lambda)}{1+\lambda} + \lambda - 1 = \frac{-1+\lambda+\lambda+\lambda^2-1-\lambda}{1+\lambda} = \frac{\lambda^2+\lambda-2}{1+\lambda} = \frac{\lambda^2+2\lambda-\lambda-2}{1+\lambda} = \frac{\lambda(\lambda+2)-1(\lambda+2)}{1+\lambda} = \frac{(\lambda+2)(\lambda-1)}{1+\lambda}$$

The augmented matrix is,

$$\left(\begin{array}{ccc|c} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ \lambda & 1 & 1 & 1 \\ 1 & 1 & \lambda & 1 \end{array}\right) \; ; [r_1 \leftrightarrow r_2]$$

$$= \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1-\lambda^2 & 1-\lambda & 1-\lambda \\ 0 & 1-\lambda & \lambda-1 & 0 \end{array}\right) \; \begin{array}{l} ; [r_2' = (-1)\lambda r_1 + r_2] \\ ; [r_3' = -r_1 + r_3] \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1 & \frac{1}{1+\lambda} & \frac{1}{1+\lambda} \\ 0 & 1-\lambda & \lambda-1 & 0 \end{array}\right) \; ; \left[r_2' = \left(\frac{1}{1-\lambda^2}\right) r_2\right]$$

$$= \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1 & \frac{1}{1+\lambda} & \frac{1}{1+\lambda} \\ 0 & 0 & \frac{(\lambda+2)(\lambda-1)}{1+\lambda} & \frac{-1+\lambda}{1+\lambda} \end{array}\right) \; ; \left[r_3' = (-1)(1-\lambda) r_2 + r_3\right]$$

The corresponding system of linear equation is,

$$x + \lambda y + z = 1 \quad \cdots\cdots\cdots (i)$$
$$y + \left(\frac{1}{1+\lambda}\right) z = \frac{1}{1+\lambda} \quad \cdots\cdots (ii)$$
$$\frac{(\lambda+2)(\lambda-1)}{(1+\lambda)} z = \frac{-1+\lambda}{1+\lambda} \quad \cdots\cdots (iii)$$

(a) if $\lambda \neq -2$, unique solution

(b) if $\lambda = -2$, $\lambda \neq 1$; no solution

(c) if $\lambda = 1$; more than one solution

2(b)

$$x + y - z = 1$$

$$2x + 3y + \lambda z = 3$$

$$x + \lambda y + 3z = 2$$

The augmented matrix is,

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & \lambda & 3 \\ 1 & \lambda & 3 & 2 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & \lambda-1 & 4 & 1 \end{array}\right) \quad \left[\begin{array}{l} r_2' = -2r_1 + r_2 \\ r_3' = -r_1 + r_3 \end{array}\right]$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & 0 & -(\lambda+3)(\lambda-2) & 2-\lambda \end{array}\right) \quad \left[r_3' = (-1)(\lambda-1)r_2 + r_3\right]$$

$$\text{(H)} \; -1\,(\lambda-1)(\lambda+2)+4$$

$$= (-\lambda+1)(\lambda+2)+4$$

$$= -\lambda^2+\lambda-2\lambda+2+4$$

$$= -(\lambda^2+\lambda-6)$$

$$= -(\lambda^2+3\lambda-2\lambda-6)$$

$$= -(\lambda+3)(\lambda-2)$$

The corresponding system of linear equation, ~~i.e.~~,

$$x + y - z = 1 \quad \cdots\cdots (i)$$

$$y + (\lambda+2)z = 1 \quad \cdots\cdots (ii)$$

$$-(\lambda+3)(\lambda-2)z = 2-\lambda \quad \cdots\cdots (iii)$$

From equation (iii),

$$(\lambda+3)(\lambda-2)z = \lambda - 2$$

(a) if $\lambda \neq 2$, $\lambda \neq -3$; unique solution

(b) if $\lambda = -3$, $\lambda \neq 2$; no solution

(c) if $\lambda = 2$; many solution. (Ans)

2(c)

$$x + y + Kz = 2$$

$$3x + 4y + 2z = K$$

$$2x + 3y - z = 1$$

The augmented matrix is,

$$\left(\begin{array}{ccc|c} 1 & 1 & K & 2 \\ 3 & 4 & 2 & K \\ 2 & 3 & -1 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & K & 2 \\ 0 & 1 & 2-3K & K-6 \\ 0 & 1 & -1-2K & -3 \end{array}\right) \quad \begin{bmatrix} r_2' = -3r_1 + r_2 \\ r_3' = -2r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & k & 2 \\ 0 & 1 & 2-3k & k-6 \\ 0 & 0 & k-3 & 3-k \end{array}\right) \quad [r_3' = -r_2 + r_3]$$

The corresponding system of linear equation,

$$x + y + kz = 2 \quad \cdots (i)$$

$$y + (2-3k)z = k-6 \quad \cdots (ii)$$

$$(k-3)z = 3-k \quad \cdots (iii)$$

From equation (iii),

ⓐ $k \neq 3$; unique solution

~~ⓑ $k = 3$, $k \neq 3$, no solution~~

ⓒ $k = 3$; many solution

2(d)

$$x \quad - 3z = -3$$

$$2x + \lambda y - z = -2$$

$$x + 2y + \lambda z = 1$$

The augmented matrix is,

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 2 & \lambda & -1 & -2 \\ 1 & 2 & \lambda & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & \lambda & 5 & 4 \\ 0 & 2 & 3+\lambda & 4 \end{array}\right) \quad \begin{bmatrix} r_2' = -2r_1 + r_2 \\ r_3' = -r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & 5/\lambda & 4/\lambda \\ 0 & 2 & 3+\lambda & 4 \end{array}\right) \quad \left[r_2' = \left(\tfrac{1}{\lambda}\right) r_2\right]$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & 5/\lambda & 4/\lambda \\ 0 & 0 & \frac{(\lambda+5)(\lambda-2)}{\lambda} & \frac{4\lambda-8}{\lambda} \end{array}\right) \quad \left[r_3' = -2r_2 + r_3\right]$$

$$(-2)\frac{5}{\lambda} + 3 + \lambda =$$

$$= -\frac{10}{\lambda} + 3 + \lambda$$

$$= \frac{-10 + 3\lambda + \lambda^2}{\lambda}$$

$$= \frac{\lambda^2 + 3\lambda - 10}{\lambda}$$

$$= \frac{\lambda^2 + 5\lambda - 2\lambda - 10}{\lambda}$$

$$= \frac{\lambda(\lambda+5) - 2(\lambda+5)}{\lambda}$$

$$= \frac{(\lambda+5)(\lambda-2)}{\lambda}$$

$$(-2)\frac{4}{\lambda} + 4$$

$$= \frac{-8}{\lambda} + 4$$

$$= \frac{4\lambda - 8}{\lambda}$$

The corresponding system of linear equations,

$$x \qquad - 3z = -3 \quad \cdots\cdots (i)$$

$$y + \left(\frac{5}{\lambda}\right)z = \frac{4}{\lambda} \quad \cdots (ii)$$

$$\left\{\frac{(\lambda+5)(\lambda-2)}{\lambda}\right\} z = \frac{4\lambda - 8}{\lambda} \quad \cdots (iii)$$

From (iii),

(a) if $\lambda \neq -5$; unique solution

(b) if, $\lambda = -5$, $\lambda \neq 2$; no solution

(c) If, $\lambda = 2$; many solution.

2(e)

$$x + y + \lambda z = 1$$

$$x + \lambda y + z = \lambda$$

$$\lambda x + y + z = \lambda^2$$

The augmented matrix is,

$$\left(\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 1 & \lambda & 1 & \lambda \\ \lambda & 1 & 1 & \lambda^2 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 1-\lambda & \lambda-1 \\ 0 & 1-\lambda & 1-\lambda^2 & \lambda^2-\lambda \end{array}\right) \begin{bmatrix} r_2' = -r_1 + r_2 \\ r_3' = (-1)\lambda r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1-\lambda & 1-\lambda^2 & \lambda(\lambda-1) \end{array}\right) \left[r_2' = \left(\frac{1}{\lambda-1}\right) r_2\right] \left[\frac{-(1-\lambda)}{\lambda-1} = -1\right]$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & (\lambda+2)(1-\lambda) & (\lambda+1)(\lambda-1) \end{array}\right) \left[r_3' = (-1)(1-\lambda) r_2 + r_3\right]$$

$$(-1)(-1)(1-\lambda) + 1 - \lambda^2$$

$$\Rightarrow 1-\lambda+1-\lambda^2$$

$$= -\lambda^2 - \lambda + 2$$

$$= -\lambda^2 - 2\lambda + \lambda + 2$$

$$= -\lambda(\lambda+2) + 1(\lambda+2)$$

$$= (\lambda+2)(1-\lambda)$$

$$(-1)(1)(1-\lambda) + \lambda^2 - \lambda$$

$$= -1 + \lambda + \lambda^2 - \lambda$$

$$= \lambda^2 - 1$$

$$= (\lambda+1)(\lambda-1)$$

The corresponding system of linear equations,

$$x + y + \lambda z = 1 \quad \cdots\cdots (i)$$

$$y - z = 1 \quad \cdots\cdots (ii)$$

$$(\lambda+2)(1-\lambda) z = (\lambda+1)(\lambda-1) \quad \cdots (iii)$$

From (iii),

(a) if $\lambda \neq -2$; unique solution

(b) if $\lambda = -2$, $\lambda \neq 1$; no solution.

(c) if $\lambda = 1$; many solution

Inverse matrix : A matrix is inversable if and only if it has a non-zero determinant.

if $D = 0 \rightarrow$ singular

$D \neq 0 \rightarrow$ non-singular

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

if A and B are inverntable matrices of the same side then AB is invartable and $(AB)^{-1} = B^{-1} \cdot A^{-1}$

(i) Row - cononical method

(ii) Adjoint

Row-Canonical Process

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \begin{bmatrix} R_2' = -2R_1 + R_2 \\ R_3' = -R_1 + R_2 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \begin{bmatrix} R_1' = -2R_2 + R_3 \\ R_3' = 2R_2 + R_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \left[R_3' = (-1)R_3\right]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \begin{bmatrix} R_1' = (-9)R_3 + R_1 \\ R_2' = 3R_3 + R_2 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

$A = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$ row canonical

Sol^n: $(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right)$

$= \left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right)$ $[r_2' = -2r_1 + r_2]$

$= \left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & 1 & 9/8 & 1/4 & -1/8 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right)$ $[r_2' = (-1/8)\, r_2]$

$= \left(\begin{array}{ccc|ccc} 1 & 0 & -11/4 & -1/2 & 3/4 & 0 \\ 0 & 1 & 9/8 & 1/4 & -1/8 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right)$ $\begin{bmatrix} r_1' = -6r_2 + r_1 \\ r_3' = -8r_2 + r_3 \end{bmatrix}$

$\therefore$ A is not inwertable.

Practice sheet Doc file P-2

7 (a)

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 2 & 5 & 5 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 5/2 & 5/2 & 1/2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right) \quad [R_1' = (1/2) R_1]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 5/2 & 5/2 & 1/2 & 0 & 0 \\ 0 & 3/2 & 5/2 & 1/2 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \quad \left[\begin{array}{l} R_2' = R_1 + R_2 \\ R_3' = (-2)R_1 + R_3 \end{array}\right]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 5/2 & 5/2 & 1/2 & 0 & 0 \\ 0 & 1 & 5/3 & 1/3 & 2/3 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \quad [R_2' = (2/3) R_2]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -5/3 & -1/3 & -5/3 & 0 \\ 0 & 1 & 5/3 & 1/3 & 2/3 & 0 \\ 0 & 0 & -1/3 & -2/3 & 2/3 & 1 \end{array}\right) \quad \left[\begin{array}{l} R_1' = (-5/2) R_2 + R_1 \\ R_3' = R_2 + R_3 \end{array}\right]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -5/3 & -1/3 & -5/3 & 0 \\ 0 & 1 & 5/3 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 2 & -2 & -3 \end{array}\right) \quad [R_3' = (-3) R_3]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5 & -5 \\ 0 & 1 & 0 & -3 & 4 & 5 \\ 0 & 0 & 1 & 2 & -2 & -3 \end{array}\right) \quad \left[\begin{array}{l} R_2' = (-5/3) R_3 + R_2 \\ R_1' = (5/3) R_3 + R_1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{pmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{pmatrix} \text{ (Ans)}$$

2(c)

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 2 & -3 & 5 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & -3/2 & 5/2 & 1/2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \quad [r_1' = (1/2)\, r_1]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1/2 & 3/2 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \quad [r_1' = (3/2)\, r_2 + r_1]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1/2 & 3/2 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) \quad [r_3' = (1/2)\, r_3]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 3/2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 3/2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right) \quad \left[\begin{array}{l} r_2' = 3r_3 + r_2 \\ r_1' = 2r_3 + r_1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{pmatrix} 1/2 & 3/2 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1/2 \end{pmatrix} \quad \text{(Ans)}$$

2(d)

$$(A|I) = \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right) \quad [r_1' = (-1) r_1]$$

$$= \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right) \quad \begin{bmatrix} r_2' = (-2) r_1 + r_2 \\ r_3' = (-4) r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right) \quad [r_2' = (1/5) r_2]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 3/5 & -1/5 & 2/5 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 8/5 & -6/5 & 1 \end{array}\right) \quad \begin{bmatrix} r_1' = 2r_2 + r_1 \\ r_3' = -6 r_2 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 3/5 & -1/5 & 2/5 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & -3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right) \quad \begin{bmatrix} r_2' = (6/5) r_3 + r_2 \\ r_1' = (-3/5) r_3 + r_1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{pmatrix}$$ (Ans)

2 (i)

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right) \quad \left[\begin{array}{l} R_2' = (-2)R_1 + R_2 \\ R_3' = -R_1 + R_3 \end{array}\right]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \quad \left[\begin{array}{l} R_1' = R_2 + R_1 \\ R_3' = -R_2 + R_3 \end{array}\right]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \quad \left[\begin{array}{l} R_2' = 2R_3 + R_2 \\ R_1' = R_3 + R_1 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

2(f)

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 4/3 & -1/3 & 1/3 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right) \quad [r_1' = (1/3)\, r_1]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 4/3 & -1/3 & 1/3 & 0 & 0 \\ 0 & -4/3 & 10/3 & -1/3 & 1 & 0 \\ 0 & 7/3 & -10/3 & -2/3 & 0 & 1 \end{array}\right) \quad \begin{bmatrix} r_2' = -r_1 + r_2 \\ r_3' = (-2) r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 4/3 & -1/3 & 1/3 & 0 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 7/3 & -10/3 & -2/3 & 0 & 1 \end{array}\right) \quad \left[r_2' = \left(-\tfrac{3}{4}\right) r_2\right]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 5/2 & -5/4 & 7/4 & 1 \end{array}\right) \quad \begin{bmatrix} r_1' = (-4/3) r_2 + r_1 \\ r_3 = (-7/3) r_2 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right) \quad [r_3' = (2/5)\, r_3]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -11/10 & -6/5 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right) \quad \begin{bmatrix} r_2' = (5/2) r_3 + r_2 \\ r_1' = (-3) r_3 + r_1 \end{bmatrix}$$

$$A^{-1} = \begin{pmatrix} 3/2 & -11/10 & -6/5 \\ -1 & 1 & 1 \\ -1/2 & 7/10 & 2/5 \end{pmatrix} \quad \text{(Ans)}$$

Adjoint matrix: If A is an invertable matrix, then

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

Adjoint of a matrix: if A is any $n \times n$ matrix and C_{ij} is the co-factor of a_{ij}, then the matrix is,

C_{ij} | i = row, j = coloum

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

→ is called the ~~most~~ matrix of co-factors, the transpose of this matrix is called the adjoint of A and is denoted by $\operatorname{adj}(A)$.

cofactor: The number $(-1)^{i+j} M_{ij}$ is denoted by C_{ij} and is called the co-factor of entry a_{ij}.

$$C_{ij} = (-1)^{i+j} M_{ij}$$ | Minor = M_{ij}

Minor: if A is a square matrix, then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the i-row and j-column are deleted form A.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{23} & a_{32} & a_{33} \end{pmatrix}$$ submatrix, $M_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Find the adj of A, & det of A.

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$$

$\therefore C_{11} = (-1)^{1+1} M_{11}$

$= 1 \cdot \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix}$

$= [6 \cdot 0 - \{(-4) \cdot 3\}]$

$= 12$

$\therefore C_{12} = (-1)^{1+2} M_{12}$

$= - \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}$

$= -(1 \cdot 0 - 2 \cdot 3)$

$= 6$

$\therefore C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix}$

$= 1 \cdot (-4) - 6 \cdot 2$

$= -16$

$\therefore C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix}$

$= \cdot 4$

$\therefore C_{22} = (-1)^{2+2} \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$

$= 2$

$\therefore C_{23} = -\{(-4) - 12\}$

$= 16$

$\therefore C_{31} = 6 - (-1) \cdot 6 = 12$

$\therefore C_{32} = -(9+1) = -10$

$\therefore C_{33} = -\{4 - (6 \cdot 2)\}$

$= \cdot 16$

The matrix of co-factor is, $\begin{pmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{pmatrix}$

The adj (A) is, $Adj(A) = \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$ (Ans)

If $\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$\det(A) = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{vmatrix}$

$= 3(0+12) - 2(0-6) + (-1)(-4-12)$

$= 36 + 12 + 16$

$= 64$

$\therefore A^{-1} = \frac{1}{64} \begin{pmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{pmatrix}$

$= \begin{pmatrix} 3/16 & 1/16 & 3/16 \\ 3/32 & 1/32 & -5/32 \\ -1/4 & 1/4 & 1/4 \end{pmatrix}$ (Ans)

ab 11th
Page 121

(2.0) $A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{pmatrix}$

$C_{11} = -12$	$C_{21} = 0$	$C_{31} = -9$
$C_{12} = -4$	$C_{22} = -2$	$C_{32} = -4$
$C_{13} = 6$	$C_{23} = 0$	$C_{33} = 6$

$\therefore$ The matrix of co-factor is, $\begin{pmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{pmatrix}$

~~Adj A = Adj(A)~~

$\therefore \text{Adj}(A) = \begin{pmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{pmatrix}$

$\therefore \det(A) = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{vmatrix}$

$= 2(-12-0) + 3(0+6)$

$= -24 + 18 = -6$

$\therefore A^{-1} = \frac{1}{-6} \begin{pmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 & 3/2 \\ 2/3 & 1/3 & 2/3 \\ -1 & 0 & -1 \end{pmatrix}$ (Ans)

Book- 11th
Page-127
(24)

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{pmatrix}$$

$\therefore c_{11} = 6$ | $c_{21} = 0$ | $c_{31} = 0$

$\therefore c_{12} = -48$ | $c_{22} = 12$ | $c_{32} = 0$

$\therefore c_{13} = 29$ | $c_{23} = -6$ | $c_{33} = 2$

The matrix of co-factors $= \begin{pmatrix} 6 & -48 & 29 \\ 0 & 12 & -6 \\ 0 & 0 & 2 \end{pmatrix}$

$$\therefore Adj(A) = \begin{pmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{pmatrix}$$

$$\therefore det(A) = \begin{vmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{vmatrix}$$

$$= 2(6-0)$$

$$= 12$$

$$\therefore A^{-1} = \frac{1}{12}\begin{pmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 & 0 \\ -4 & 1 & 0 \\ 29/12 & -1/2 & 1/6 \end{pmatrix}$$

Book-11th
Page-127
(12)

$$A = \begin{pmatrix} 4 & 2 & 8 \\ -2 & 1 & -4 \\ 3 & 1 & 6 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{pmatrix}$$

$C_{11} = -35$	$C_{21} = -9$	$C_{31} = -1$
$C_{12} = 41$	$C_{22} = 7$	$C_{32} = -13$
$C_{13} = 89$	$C_{23} = -9$	$C_{33} = -1$

$\therefore$ The matrix of co-factor $= \begin{pmatrix} -35 & 41 & 89 \\ -9 & 7 & -9 \\ -1 & -13 & -1 \end{pmatrix}$

$$\therefore \ Adj(A) = \begin{pmatrix} -35 & -9 & -1 \\ 41 & 7 & -13 \\ 89 & -9 & -1 \end{pmatrix}$$

$$det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{vmatrix}$$

$$= 1(1-36) + (-1)(81+8)$$

$$= -35 - 89$$

$$= -124$$

$$\therefore \ A^{-1} = \frac{1}{-124}\begin{pmatrix} -35 & -9 & -1 \\ 41 & 7 & -13 \\ 89 & -9 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 35/124 & 9/124 & 1/124 \\ -41/124 & -7/124 & 13/124 \\ -89/124 & 9/124 & 1/124 \end{pmatrix} \text{(Ans)}$$

Solve the following system of linear equation by using $X = A^{-1}b$

$$x + y + z = 9$$

$$2x + 5y + 7z = 52$$

$$2x + y - z = 0$$

Sol:

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 5 & -2 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right) \begin{bmatrix} r_2' = (-2) r_1 + r_2 \\ r_3' = (-2) r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5/3 & -2/3 & 1/3 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right) \left[r_2' = \left(\tfrac{1}{3}\right) r_2 \right]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -2/3 & 5/3 & -1/3 & 0 \\ 0 & 1 & 5/3 & -2/3 & 1/3 & 0 \\ 0 & 0 & -4/3 & -8/3 & 1/3 & 1 \end{array}\right) \begin{bmatrix} r_1' = (-1) r_2 + r_1 \\ r_3' = r_2 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -2/3 & 5/3 & -1/3 & 0 \\ 0 & 1 & 5/3 & -2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 2 & -1/4 & -3/4 \end{array}\right) \left[r_3' = \left(-\tfrac{3}{4}\right) r_3 \right]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1/2 & -1/2 \\ 0 & 1 & 0 & -4 & 3/4 & 5/4 \\ 0 & 0 & 1 & 2 & -1/4 & -3/4 \end{array}\right) \begin{bmatrix} r_2' = (-5/3) r_3 + r_2 \\ r_1' = 2/3 \; r_3 + r_1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 3 & -1/2 & -1/2 \\ -4 & 3/4 & 5/4 \\ 2 & -1/4 & -3/4 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad b = \begin{pmatrix} 9 \\ 52 \\ 0 \end{pmatrix}$$

$$X = A^{-1} b$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1/2 & -1/2 \\ -4 & 3/4 & 5/4 \\ 2 & -1/4 & -3/4 \end{pmatrix} \begin{pmatrix} 9 \\ 52 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \cdot 9 + (-1/2) \cdot 52 + (-1/2) \cdot 0 \\ (-4) \cdot 9 + 3/4 \cdot 52 + 5/4 \cdot 0 \\ 2 \cdot 9 + (-1/4) \cdot 52 + (-3/4) \cdot 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$\therefore x = 1$

$\therefore y = 3$

$\therefore z = 5$

Practice sheet - page-4 (doc) - 4 (a)

$$x_1 + x_2 + 2x_3 = 8$$

$$-x_1 - 2x_2 + 3x_3 = 1$$

$$3x_1 - 7x_2 + 4x_3 = 10$$

Sol: $(A|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & -2 & 3 & 0 & 1 & 0 \\ 3 & -7 & 4 & 0 & 0 & 1 \end{array}\right)$

$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 5 & 1 & 1 & 0 \\ 0 & -10 & -2 & -3 & 0 & 1 \end{array}\right) \begin{bmatrix} R_2' = R_1 + R_2 \\ R_3' = (-3)R_1 + R_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & -1 & 0 \\ 0 & -10 & -2 & -3 & 0 & 1 \end{array}\right) \left[R_2' = (-1)R_2\right]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 7 & 2 & 1 & 0 \\ 0 & 1 & -5 & -1 & -1 & 0 \\ 0 & 0 & -52 & -13 & -10 & 1 \end{array}\right) \begin{bmatrix} R_1' = (-1)R_2 + R_1 \\ R_3' = 10R_2 + R_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 7 & 2 & 1 & 0 \\ 0 & 1 & -5 & -1 & -1 & 0 \\ 0 & 0 & 1 & 13/52 & 10/52 & -1/52 \end{array}\right) \left[R_3' = \left(-1/52\right)R_3\right]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -9/26 & 7/52 \\ 0 & 1 & 0 & 1/4 & -1/26 & -5/52 \\ 0 & 0 & 1 & 13/52 & 10/52 & -1/52 \end{array}\right) \begin{bmatrix} R_2' = 5R_3 + R_2 \\ R_1' = (-7)R_3 + R_1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 1/4 & -9/26 & 7/52 \\ 1/4 & -1/26 & -5/52 \\ 1/4 & 5/26 & -1/52 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/4 & -9/26 & 7/52 \\ 1/4 & -1/26 & -5/52 \\ 1/4 & 5/26 & -1/52 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \\ 10 \end{pmatrix}$$

$$= \begin{pmatrix} 1/4 \cdot 8 + (-9/26) \cdot 1 + 7/52 \cdot 10 \\ 1/4 \cdot 8 + (-1/26) \cdot 1 + (-5/52) \cdot 10 \\ 1/4 \cdot 8 + 5/26 \cdot 1 + (-1/52) \cdot 10 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$\therefore x = 3, \quad y = 1, \quad z = 2$

4(d) $2x_1 + 2x_2 + 2x_3 = 0$

$-2x_1 + 5x_2 + 2x_3 = 1$

$8x_1 + x_2 + 4x_3 = -1$

Sol: $(A|I) = \left(\begin{array}{ccc|ccc} 2 & 2 & 2 & 1 & 0 & 0 \\ -2 & 5 & 2 & 0 & 1 & 0 \\ 8 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$

$= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1/2 & 0 & 0 \\ -2 & 5 & 2 & 0 & 1 & 0 \\ 8 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$

$= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1/2 & 0 & 0 \\ 0 & 7 & 4 & 1 & 1 & 0 \\ 0 & -7 & -4 & -4 & 0 & 1 \end{array}\right) \left[\begin{array}{l} r_2' = 2r_1 + r_2 \\ r_3' = (-8)r_1 + r_3 \end{array}\right]$

$= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 4/7 & 1/7 & 1/7 & 0 \\ 0 & -7 & -4 & -4 & 0 & 1 \end{array}\right) \left[r_2' = (1/7)\, r_2\right]$

$= \left(\begin{array}{ccc|ccc} 1 & 0 & 3/7 & 5/14 & -1/7 & 0 \\ 0 & 1 & 4/7 & 1/7 & 1/7 & 0 \\ 0 & 0 & 0 & -3 & 1 & 1 \end{array}\right) \left[\begin{array}{l} r_1' = (-1)r_2 + r_1 \\ r_3' = 7r_2 + r_3 \end{array}\right]$

$\therefore$ This matrix is not inversable.

4(b) $x_1 + 3x_2 + x_3 = 4$

$2x_1 + 2x_2 + x_3 = -1$

$2x_1 + 3x_2 + x_3 = 3$

Sol: $(A|I) = \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right)$

$= \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right) \quad \begin{bmatrix} r_2' = (-2)r_1 + r_2 \\ r_3' = (-2)r_1 + r_3 \end{bmatrix}$

$= \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/4 & 1/2 & -1/4 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right) \quad \left[r_2' = \left(-\tfrac{1}{4}\right) r_2 \right]$

$= \left(\begin{array}{ccc|ccc} 1 & 0 & 1/4 & -1/2 & 3/4 & 0 \\ 0 & 1 & 1/4 & 1/2 & -1/4 & 0 \\ 0 & 0 & -1/4 & -1/2 & -3/4 & 1 \end{array}\right) \quad \begin{bmatrix} r_1' = (-3)r_2 + r_1 \\ r_3' = 3r_2 + r_3 \end{bmatrix}$

$= \left(\begin{array}{ccc|ccc} 1 & 0 & 1/4 & -1/2 & 3/4 & 0 \\ 0 & 1 & 1/4 & 1/2 & -1/4 & 0 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right) \quad \left[r_3' = (-4) r_3 \right]$

$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 3 & -4 \end{array}\right) \quad \begin{bmatrix} r_2' = \left(-\tfrac{1}{4}\right) r_3 + r_2 \\ r_1' = \left(-\tfrac{1}{4}\right) r_3 + r_1 \end{bmatrix}$

$\therefore A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} (-1).4 + 0(-1) + 1.3 \\ 0.4 + (-1)(-1) + 1.3 \\ 2.4 + 3(-1) + (-4).3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -7 \end{pmatrix}$$

$\therefore x = -1$

$\therefore y = 4$ (Ans)

$\therefore z = -7$

4(h)

$$x + y + z = 5$$

$$x + y - 4z = 10$$

$$-4x + y + z = 0$$

$$\therefore (A|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -4 & 0 & 1 & 0 \\ -4 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & -1 & 1 & 0 \\ 0 & 5 & 5 & 4 & 0 & 1 \end{array}\right) \begin{bmatrix} r_2' = (-1) r_1 + r_2 \\ r_3' = 4r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 4/5 & 0 & 1/5 \\ 0 & 0 & -5 & -1 & 1 & 0 \end{array}\right) \begin{bmatrix} r_2 \longleftrightarrow r_3 \\ r_2' = (1/5) r_2 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & -1/5 \\ 0 & 1 & 1 & 4/5 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right) \begin{bmatrix} r_1' = (-1)r_2 + r_1 \\ r_3' = (-1/5)\, r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 0 & -1/5 \\ 0 & 1 & 0 & 3/5 & 1/5 & 1/5 \\ 0 & 0 & 1 & 1/5 & -1/5 & 0 \end{array}\right) \left[r_2' = (-1)\, r_3 + r_2 \right]$$

$$\therefore A^{-1} = \begin{pmatrix} 1/5 & 0 & -1/5 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & -1/5 & 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/5 & 0 & -1/5 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & -1/5 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/5 \cdot 5 + 0 \cdot 10 + (-1/5) \cdot 0 \\ 3/5 \cdot 5 + 1/5 \cdot 10 + 1/5 \cdot 0 \\ 1/5 \cdot 5 + (-1/5) \cdot 10 + 0 \cdot 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$$

4(g) $5x_1 + 3x_2 + 2x_3 = 4$

$3x_1 + 3x_2 + 2x_3 = 2$

$x_2 + x_3 = 5$

Tanjil 08.07.17 MAT-216

Sol:

$$(A|I) = \left(\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 3/5 & 2/5 & 1/5 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad [r_1' = (1/5) r_1]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 3/5 & 2/5 & 1/5 & 0 & 0 \\ 0 & 6/5 & 4/5 & -3/5 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad [r_2' = (-3) r_1 + r_2]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 3/5 & 2/5 & 1/5 & 0 & 0 \\ 0 & 1 & 2/3 & -1/2 & 5/6 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad [r_2' = 5/6\, r_2]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 2/3 & -1/2 & 5/6 & 0 \\ 0 & 0 & 1/3 & 1/2 & -5/6 & 1 \end{array}\right) \quad \begin{bmatrix} r_1' = (-3/5) r_2 + r_1 \\ r_3' = (-1) r_2 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 2/3 & -1/2 & 5/6 & 0 \\ 0 & 0 & 1 & 3/2 & -5/2 & 3 \end{array}\right) \quad [r_3' = 3 r_3]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & -3/2 & 5/2 & -2 \\ 0 & 0 & 1 & 3/2 & -5/2 & 3 \end{array}\right) \quad [R_2' = (-\tfrac{2}{3}) R_3 + R_2]$$

$$\therefore A^{-1} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \cdot 4 + (-1/2) \cdot 2 + 0 \cdot 5 \\ (-3/2) \cdot 4 + 5/2 \cdot 2 + (-2) \cdot 5 \\ 3/2 \cdot 4 + (-5/2) \cdot 2 + 3 \cdot 5 \end{pmatrix}$$

Tanji
08.07.17
MAT-216

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -11 \\ 16 \end{pmatrix}$$

$\therefore x = 1, \quad y = -11, \quad z = 16$ (Ans)

~~Solve the~~ following Vector Space

A vector space over an arbitrary field F is a non-empty set V, whose elements are called vector for which two operations are prescribed. The first operation, called vector addition and the second operation called multiplication. The two operations are required to statisfy the following conditions:

① (i) For all vectors $u, v \in V \Rightarrow u+v = v+u$

(ii) For all vectors $u, v, w \in V \Rightarrow (u+v)+w = u+(v+w)$

(iii) There exists a vector $0 \in V$ such that $\forall v \in V$,

($\forall \rightarrow$ For all sign)

$v + 0 = 0 + v = v$

[sometimes Vector spaces are also called linear spaces]

(iv) For each $v \in V$, There is a vector $(-v) \in V$ for which $v + (-v) = (-v) + v = 0$

② (i) For any scalar $\alpha \in F$ and vectors $u, v \in V$,

$\alpha(u+v) = \alpha u + \alpha v$

(ii) For any scalars $\alpha, \beta \in F$ and any vector $v \in V$,

$(\alpha+\beta)v = \alpha v + \beta v$

(iii) For any scalars $\alpha, \beta \in F$ and any vector $v \in V$

$(\alpha\beta)v = \alpha(\beta v)$

(iv) For each $v \in V$, $1v = v$, where 1 is the unit scalars and $1 \in F$.

Subspace: A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem: W is a subspace of V if and only if

(i) W is non-empty.

(ii) W is closed under vector addition, i.e: $u, \omega \in W$ implied that $u + \omega \in W$

(iii) W is closed under scalar multiplication, i.e: $u \in W$ implied that $\alpha, u \in W$ for every $\alpha \in F$.

Vector space.

Abdur rahman → Slide-64 → ex : (1-3)

Linear combination: Let V be a vector space over the field F and let $v_1, v_2, \dots, v_n \in V$ then any vector $v \in V$ is called a linear combination of $v_1, v_2, \dots, v_n$ if and only if there exist scalar $\alpha_1, \alpha_2, \dots, \alpha_n$ in F such that,

$$V = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

$$= \sum_{i=1}^{n} \alpha_i v_i$$

5. Consider the vectors $v_1 = (2,1,4)$, $v_2 = (1,-1,3)$ & $v_3 = (3,2,5)$ in $\mathbb{R}^3$. Show that $v = (5,9,5)$ is a linear combination of v_1, v_2 and v_3.

Sol: Set v as a linear combination of v_1, v_2 and v_3 using the unknowns $\alpha_1, \alpha_2, \alpha_3$:

$$V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$$

i.e. $(5,9,5) = \alpha_1 (2,1,4) + \alpha_2 (1,-1,3) + \alpha_3 (3,2,5)$

$= (2\alpha_1, \alpha_1, 4\alpha_1) + (\alpha_2, -\alpha_2, 3\alpha_2) + (3\alpha_3, 2\alpha_3, 5\alpha_3)$

$= (2\alpha_1 + \alpha_2 + 3\alpha_3,\ \alpha_1 - \alpha_2 + 2\alpha_3,\ 4\alpha_1 + 3\alpha_2 + 5\alpha_3)$

corresponding ~~system of linear equation~~ components,

$$\begin{aligned} 2\alpha_1 + \alpha_2 + 3\alpha_3 &= 5 \\ \alpha_1 - \alpha_2 + 2\alpha_3 &= 9 \\ 4\alpha_1 + 3\alpha_2 + 5\alpha_3 &= 5 \end{aligned}$$

The augmented matrix is,

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 5 \\ 1 & -1 & 2 & 9 \\ 4 & 3 & 5 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 2 & 1 & 3 & 5 \\ 4 & 3 & 5 & 5 \end{array}\right) \quad [r_1 \leftrightarrow r_2]$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -1 & -13 \\ 0 & 7 & -3 & -31 \end{array}\right) \quad \begin{bmatrix} r_2' = (-2) r_1 + r_2 \\ r_3' = (-4) r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & -1/3 & -13/3 \\ 0 & 7 & -3 & -31 \end{array}\right) \quad \left[r_2' = \left(\tfrac{1}{3}\right) r_2\right]$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 5/3 & 14/3 \\ 0 & 1 & -1/3 & -13/3 \\ 0 & 0 & -2/3 & -2/3 \end{array}\right) \quad \left[\begin{array}{l} r_1' = r_2 + r_1 \\ r_3' = (-7) r_2 + r_3 \end{array}\right]$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 5/3 & 14/3 \\ 0 & 1 & -1/3 & -13/3 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \left[r_3' = \left(-\tfrac{3}{2}\right) r_3\right]$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \left[\begin{array}{l} r_2' = \left(\tfrac{1}{3}\right) r_3 + r_2 \\ r' = \left(-\tfrac{5}{3}\right) r_3 + r_1 \end{array}\right]$$

$\therefore$ The corresponding system of linear equation,

$$\alpha_1 = 3, \quad \alpha_2 = -4, \quad \alpha_3 = 1$$

Hence, $v = 3v_1 - 4v_2 + v_3$ (Ans)

Therefore, v is a linear combination of v_1, v_2, v_3

In the vector $v = (2,-5,3)$ in $\mathbb{R}^3$ is a linear combination of the vectors $v_1 = (1,-3,2)$, $v_2 = (2,-4,-1)$ and $v_3 = (1,-5,7)$

Sol: Set v as a linear combination of v_1, v_2, v_3 using the unknowns $\alpha_1, \alpha_2, \alpha_3$;

i.e. $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$$(2,-5,3) = \alpha_1(1,-3,2) + \alpha_2(2,-4,-1) + \alpha_3(1,-5,7)$$

$$= (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -4\alpha_2, -\alpha_2) + (\alpha_3, -5\alpha_3, 7\alpha_3)$$

$$= (\alpha_1 + 2\alpha_2 + \alpha_3,\ -3\alpha_1 - 4\alpha_2 - 5\alpha_3,\ 2\alpha_1 - \alpha_2 + 7\alpha_3)$$

$\therefore$ The corresponding ~~system of linear equation~~ components,

$$\alpha_1 + 2\alpha_2 + \alpha_3 = 2$$
$$-3\alpha_1 - 4\alpha_2 - 5\alpha_3 = -5$$
$$2\alpha_1 - \alpha_2 + 7\alpha_3 = 3$$

$\therefore$ The augmented matrix,

Tanjil 08.07.17 MAT-216

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{array}\right) \begin{bmatrix} r_2' = 3r_1 + r_2 \\ r_3' = (-2)r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 1/2 \\ 0 & -5 & 5 & -1 \end{array}\right) \left[r_2' = (1/2)\, r_2 \right]$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 3/2 \end{array}\right) \begin{bmatrix} r_1' = (-2) r_2 + r_1 \\ r_3' = 5 r_2 + r_3 \end{bmatrix}$$

$\because \; 0 \neq 3/2$

$\therefore$ The system is inconsistent. It has no solution.

There fore v is not a linear combination of v_1, v_2, v_3

\# For which value of λ will be the vector $v = (1, \lambda, 5)$ in $\mathbb{R}^3$ is a linear combination of the vectors $v_1 = (1, -3, 2)$ $v_2 = (2, -1, 1)$.

Sol: Set v as a linear combination of v_1, v_2 using the unknowns α_1 & α_2.

$$v = \alpha_1 v_1 + \alpha_2 v_2$$

$$\text{i.e. } (1, \lambda, 5) = \alpha_1 (1, -3, 2) + \alpha_2 (2, -1, 1)$$

$$= (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -\alpha_2, \alpha_2)$$

$$= (\alpha_1 + 2\alpha_2, -3\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2)$$

$\therefore$ The corresponding ~~system of linear equations~~, components,

$$\alpha_1 + 2\alpha_2 = 1$$
$$-3\alpha_1 - \alpha_2 = \lambda$$
$$2\alpha_1 + \alpha_2 = 5$$

The augmented matrix,

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ -3 & -1 & \lambda \\ 2 & 1 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 5 & \lambda+3 \\ 0 & -3 & 3 \end{array}\right) \quad \begin{bmatrix} r_2' = 3r_1 + r_2 \\ r_3' = -2r_1 + r_3 \end{bmatrix}$$

∴ The corresponding system of linear equation

$$\alpha_1 + 2\alpha_2 = 1$$

$$5\alpha_2 = \lambda + 3 \quad \cdots \text{(ii)}$$

$$-3\alpha_2 = 3 \quad \cdots \text{(iii)}$$

Tanjil
06.07.17
MAT-216

eq (iii), $\alpha_2 = -1$

eq (ii), $5\alpha_2 = \lambda + 3$ ---

If the value of $\lambda = -8$ then $\alpha_1 = 3$, $\alpha_2 = -1$

∴ The system has a solution if $\lambda = -8$

d) Write the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ as a linear combination of the matrices $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

Sol: Set A as a linear combination of A_1, A_2 and A_3 using the unknowns $\alpha_1, \alpha_2, \alpha_3$:

$$A = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$$

i.e $$\begin{pmatrix} 3 & -1 \\ 1 & -2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 & \alpha_1 \\ 0 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} \alpha_2 & \alpha_2 \\ -\alpha_2 & 0 \end{pmatrix} + \begin{pmatrix} \alpha_3 & -\alpha_3 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 - \alpha_3 \\ -\alpha_2 & -\alpha_1 \end{pmatrix}$$

The corresponding components,

$\alpha_1 + \alpha_2 + \alpha_3 = 3$ | $\alpha_1 = 2$

$\alpha_1 + \alpha_2 - \alpha_3 = -1$ | $\alpha_2 = -1$

$-\alpha_2 = 1$ | $\alpha_3 = 2$

$-\alpha_1 = -2$

Tanjil
08.07.17
MAT-216

$\therefore A = 2A_1 - A_2 + 2A_3$ (Ans)

Linear Independence and dependence:

Definition: Let V be a vector space over the field F. The vectors $v_1, v_2, \ldots, v_m \in V$ are said to be linearly dependent over F or simply dependent if there exists a non-trivial linear combination of them equal to the zero vector 0.

i.e. $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0$

where, $\alpha_i \neq 0$ for at least one i,

On the otherhand, the vectors $v_1, v_2, \ldots, v_m$ in V are said to be linearly independent over F or simply independent if only linear combination of them equal to 0 is the trivial one.

i.e. $\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_m V_m = 0$

if & only if, $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0 \longrightarrow$ trivial solution.

Prove that the set of vectors $\{(2,1,2), (0,1,-1), (4,3,3)\}$ is linearly dependent.

Form the matrix where rows are the given vectors.

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 3 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 1 & -1 \\ 4 & 3 & 3 \end{pmatrix} \left[r_1' = \left(\tfrac{1}{2}\right) r_1\right]$$

$$= \begin{pmatrix} 1 & 1/2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \quad [r_3' = (-4) r_1 + r_3]$$

$$= \begin{pmatrix} 1 & 1/2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad [r_3' = (-1) r_2 + r_3]$$

Since the row echelon matrix has a zero row, so the vectors are dependent. [Proved]

\# Show that the set of vectors $\{(3,0,1,-1), (2,-1,0,1), (1,1,1,-2)\}$ is linearly dependent.

Sol: Form the matrix where rows are the given vectors:

$$\begin{pmatrix} 3 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1/3 & -1/3 \\ 2 & -1 & 0 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix} \quad [r_1' = (1/3) r_1]$$

$$= \begin{pmatrix} 1 & 0 & 1/3 & -1/3 \\ 0 & -1 & -2/3 & 5/3 \\ 0 & 1 & 2/3 & -5/3 \end{pmatrix} \quad \begin{bmatrix} r_2' = (-2) r_1 + r_2 \\ r_3' = (-1) r_1 + r_3 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1/3 & -1/3 \\ 0 & 1 & 2/3 & -5/3 \\ 0 & 1 & 2/3 & -5/3 \end{pmatrix} \quad [r_2' = (-1) r_2]$$

$$= \begin{pmatrix} 1 & 0 & 1/3 & -1/3 \\ 0 & 1 & 2/3 & -5/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad [r_3' = (-1) r_2 + r_3]$$

$\therefore$ Since the row echelon matrix has a zero row, so the vectors are dependent.

Show that the vectors (2, -1, 4) (3, 6, 2) and (2, 10, -4) are linearly independent.

Form the matrix where rows are the given vectors

$$\begin{pmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1/2 & 2 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{pmatrix} \quad [r_1' = (1/2)\, r_1]$$

$$= \begin{pmatrix} 1 & -1/2 & 2 \\ 0 & 15/2 & -4 \\ 0 & 11 & -8 \end{pmatrix} \quad \begin{bmatrix} r_2' = (-3)\, r_1 + r_2 \\ r_3' = (-2)\, r_1 + r_3 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & -1/2 & 2 \\ 0 & 1 & -8/15 \\ 0 & 11 & -8 \end{pmatrix} \quad [r_2' = (2/15)\, r_2]$$

$$= \begin{pmatrix} 1 & 0 & 26/15 \\ 0 & 1 & -8/15 \\ 0 & 0 & -32/15 \end{pmatrix} \quad \begin{bmatrix} r_1' = (1/2)\, r_2' + r_1 \\ r_3' = (-11)\, r_2 + r_3 \end{bmatrix}$$

$\therefore$ Since the row echelon matrix has no zero row, so the vectors are linearly independent.

Tanjil
08.07.17
MAT-216

Test the dependency of the following sets:

(i) $\{(1,2,-3),\ (2,0,-1),\ (7,6,-11)\}$

(ii) $\{(2,0,-1),\ (1,1,0),\ (0,-1,1)\}$

(i) From the matrix where rows are the given vectors,

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & -1 \\ 7 & 6 & -11 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & 2 & -3 \\ 0 & -4 & 5 \\ 0 & -8 & 10 \end{pmatrix} \quad \begin{bmatrix} r_2' = (-2)r_1 + r_2 \\ r_3' = (-7)r_1 + r_3 \end{bmatrix}$$

$$\approx \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -5/4 \\ 0 & -8 & 10 \end{pmatrix} \quad \left[r_2' = \left(-\tfrac{1}{4}\right) r_2 \right]$$

$$\approx \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -5/4 \\ 0 & 0 & 0 \end{pmatrix} \quad \left[r_3' = 8r_2 + r_3 \right]$$

Since the row echelon has a zero row, so the vectors are dependent.

(ii) Form the matrix where rows are the given matrix,

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1/2 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad [r_1' = (1/2) r_1]$$

$$= \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & -1 & 1 \end{pmatrix} \quad [r_2' = (-1) r_1 + r_2]$$

$$= \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 3/2 \end{pmatrix} \quad [r_3' = r_2 + r_3]$$

$\therefore$ Since the row echelon matrix has no zero row, so the vectors are independent.

Basis & Dimension

Basis:

Let V be the vector space and $\{v_1, v_2, \dots, v_n\}$ finite set of a vectors in V. We call $\{v_1, v_2, \dots, v_n\}$ a basis for if and only if

(i) $\{v_1, v_2, \dots, v_n\}$ is linearly independent.

(ii) $\{v_1, v_2, \dots, v_n\}$ spans V.

Dimension: The dimension of a finite vector space is the number of vectors in any basis of it.

Prove that the vectors $(1,2,0)$, $(0,5,7)$ and $(-1,1,3)$ form a basis for $\mathbb{R}^3$.

$\Rightarrow$ Firstly we have to show that the given vectors are span.

Let us consider an arbitrary vector, $b = (b_1, b_2, b_3)$

The linear combination:

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = b$$

$$\Rightarrow \alpha_1(1,2,0) + \alpha_2(0,5,7) + \alpha_3(-1,1,3) = (b_1, b_2, b_3)$$

$$\Rightarrow (\alpha_1 - \alpha_3,\ 2\alpha_1 + 5\alpha_2 + \alpha_3,\ 7\alpha_2 + 3\alpha_3) = (b_1, b_2, b_3)$$

The corresponding components,

$$\begin{aligned} \alpha_1 \qquad\quad - \alpha_3 &= b_1 \\ 2\alpha_1 + 5\alpha_2 + \alpha_3 &= b_2 \\ 7\alpha_2 + 3\alpha_3 &= b_3 \end{aligned}$$

∴ The augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 2 & 5 & 1 & b_2 \\ 0 & 7 & 3 & b_3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 5 & 3 & b_2-2b_1 \\ 0 & 7 & 3 & b_3 \end{array}\right) \quad [r_2' = (-2)r_1 + r_2]$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 1 & 3/5 & \frac{b_2-2b_1}{5} \\ 0 & 7 & 3 & b_3 \end{array}\right) \quad [r_2' = (1/5)r_2]$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 1 & 3/5 & \frac{b_2-2b_1}{5} \\ 0 & 0 & -6/5 & \frac{14b_1-7b_2+5b_3}{5} \end{array}\right) \quad [r_3' = (-7)r_2 + r_3]$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 1 & 3/5 & \frac{b_2-2b_1}{5} \\ 0 & 0 & 1 & \frac{7b_2-14b_1-5b_3}{6} \end{array}\right) \quad [r_3' = (-5/6)r_3]$$

∴ The corresponding system of linear equations,

$$\alpha_1 - \alpha_3 = b_1$$

$$\alpha_2 + (3/5)\alpha_3 = \frac{b_2-2b_1}{5}$$

$$\alpha_3 = \frac{7b_2-14b_1-5b_3}{6}$$

∴ The set of vectors are consistent, so the given vectors are span.

$\therefore$ Now we wanto prove that the given vectors are linearly independent.

Let us consider, $b = (0,0,0)$.

$$\therefore \alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 0$$

So the given vectors are linearly independent, therefore the given vectors ~~are~~ form a basis for $\mathbb{R}^3$.

II Let U be the subspace of $\mathbb{R}^3$ spand (generated) by the vectors $(1,2,1)$, $(0,-1,0)$ & $(2,0,2)$. Find a basis ~~of~~ and dimension of U.

Sol: Form the matrix whose rows are the given vectors,

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -4 & 0 \end{pmatrix} \quad [r_3' = (-2) r_1 + r_3]$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -4 & 0 \end{pmatrix} \quad [r_2' = (-1) r_2]$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [r_3' = (4) r_2 + r_3]$$

Tanjil

08.07.17

MAT-216

This matrix is in row-echelon form and the non-zero rows in the matrix are (1,2,1) & (0,1,0)

These non-zero form a basis of the row space.

Basis of $U = \{(1,2,1), (0,1,0)\}$

and $\text{Dim}(U) = 2$.

Q) Let W be the subspace of $\mathbb{R}^5$ spanned by the vectors

$(1,-2,0,0,3)$, $(2,-5,-3,-2,6)$, $(0,5,15,10,0)$, $(2,6,18,8,6)$.

Find the basis and the dimension of W.

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 5 & 15 & 10 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{pmatrix} \begin{bmatrix} r_2' = (-2)r_1 + r_2 \\ r_4' = (-2)r_1 + r_4 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 5 & 15 & 10 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -12 & 0 \end{pmatrix} \begin{bmatrix} r_3' = (-5)r_2 + r_3 \\ r_4' = (-10)r_2 + r_4 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad [R_4' = (-1/_{12})R_4]$$

$\therefore$ Basis of $W = \{(1,-2,0,0,3), (0,1,3,2,0), (0,0,1,1,0)\}$

$\therefore$ $\dim(W) = 3$

Determine a basis and the dimension of the solution space of the homogeneous system.

$$\begin{aligned} 2x_1 + 2x_2 - x_3 \qquad\qquad + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 \qquad\qquad - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

Tanjil
08.07.17
MAT-216

The augmented matrix,

$$\left(\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \quad [r_1 \leftrightarrow r_3]$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \quad \begin{bmatrix} r_2' = r_1 + r_2 \\ r_3' = (-2) r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \quad [r_2 \leftrightarrow r_3]$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \quad [r_2' = (1/3) r_2]$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \quad [r_4' = (-1) r_2 + r_4]$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \quad [r_3' = (-1/3) r_3]$$

$$= \left(\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad [r_4' = (-1)r_3 + r_4]$$

$\therefore$ corresponding system of linear equations,

$$\begin{aligned} x_1 + x_2 - 2x_3 + x_5 &= 0 \\ x_3 + x_5 &= 0 \\ x_4 &= 0 \end{aligned} \quad \left| \quad \begin{aligned} &\therefore x_1 = 2x_3 - x_2 - x_5 \quad \text{--- (i)} \\ &\therefore x_3 = -x_5 \\ &\therefore x_4 = 0 \end{aligned} \right.$$

Let us consider,

$x_2 = s$

$x_5 = t$

$$\therefore \text{(i)} \Rightarrow x_1 = -2t - s - t = -3t - s = -s - 3t \quad \left| \quad \begin{aligned} x_3 &= -t \\ x_4 &= 0 \end{aligned} \right.$$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -s-3t \\ s \\ -t \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} -s \\ s \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3t \\ 0 \\ -t \\ 0 \\ t \end{pmatrix}$$

$$= s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ Basis $= \{(-1, 1, 0, 0, 0), (-3, 0, -1, 0, 1)\}$

$\therefore$ Dim $= 2$

Rank & Nullity

Rank:

(i) The rank of a matrix A is the maximum number of linearly independent rows or columns in the matrix.

(ii) Let A be an $m \times n$ matrix and let A_R be the row-echelon form of A. Then the Rank of the matrix A is the number of non-zero rows of A_R.

The Rank of the matrix is denoted by rank (A) or $\rho(A)$.

⇒ **Row-Rank:** The maximum number of linearly independent rows of matrix A is called the row rank of A.

⇒ **Column Rank:** The maximum number of linearly independent columns of a matrix A is called the column rank of A.

Nullity: Let A be the matrix, The nullity of A is the dimension of the solution space of linear system $Ax = 0$

Tanjil
06.07.17
MAT-216

dor file
Practice sheet
Q6 (c)
Page 5

$$A = \begin{bmatrix} 6 & 2 & 0 & 4 \\ -2 & -1 & 3 & 4 \\ -1 & -1 & 6 & 10 \end{bmatrix}$$

$$= \begin{pmatrix} -1 & -1 & 6 & 10 \\ -2 & -1 & 3 & 4 \\ 6 & 2 & 0 & 4 \end{pmatrix} \quad [R_1 \longleftrightarrow R_3]$$

$$= \begin{pmatrix} 1 & 1 & -6 & -10 \\ -2 & -1 & 3 & 4 \\ 6 & 2 & 0 & 4 \end{pmatrix} \quad [R_1' = (-1) R_1]$$

$$= \begin{pmatrix} 1 & 1 & -6 & -10 \\ 0 & 1 & -9 & -16 \\ 0 & -4 & 36 & 64 \end{pmatrix} \quad \begin{bmatrix} R_2' = 2R_1 + R_2 \\ R_3' = (-6)R_1 + R_3 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -6 & -10 \\ 0 & 1 & -9 & -16 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad [R_3' = (4) R_2 + R_3]$$

$\therefore$ Rank $(A) = 2$ (Ans)

(iii) $A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & -2 & 1 & 5 \\ 0 & 7 & 2 & 3 \end{pmatrix} \quad \begin{bmatrix} r_2' = (-3)r_1 + r_2 \\ r_3' = 2r_1 + r_3 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & -1/2 & -5/2 \\ 0 & 7 & 2 & 3 \end{pmatrix} \quad \left[r_2' = \left(-1/2\right) r_2\right]$$

$$= \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & -1/2 & -5/2 \\ 0 & 0 & 11/2 & 41/2 \end{pmatrix} \quad \left[r_3' = (-7) r_2 + r_3\right]$$

$$= \cancel{\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & -1/2 & -5/2 \\ 0 & 0 & 11/2 & 41/2 \end{pmatrix}} \quad \cancel{\left[r_3' = (2/11)\, r_3\right]}$$

$\therefore$ rank$(A) = 3$

(iv) $A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix} \quad \left[r_1 \longleftrightarrow r_2\right]$$

$$= \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{pmatrix} \quad \begin{bmatrix} r_3' = (-3)r_1 + r_3 \\ r_4' = (-1)r_1 + r_4 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ Rank (A) = 2 (Ans)

$\Rightarrow$ Find the nullity of the matrix,

$$A = \begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{pmatrix} \quad \begin{bmatrix} r_2' = (-3)r_1 + r_2 \\ r_3' = r_1 + r_3 \\ r_4' = (-2)r_1 + r_4 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{pmatrix} \quad \left[r_2' = \left(-\tfrac{1}{14}\right) r_2 \right]$$

$$= \begin{pmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ rank (A) = 2 (Ans)

$\therefore$ The corresponding system of linear equations:

$$x_1 + 4x_2 + 5x_3 + 6x_4 + 9x_5 = 0$$
$$x_2 + x_3 + x_4 + 2x_5 = 0$$

So,

$$x_1 = -(4x_2 + 5x_3 + 6x_4 + 9x_5)$$
$$x_2 = -(x_3 + x_4 + 2x_5) = 0$$

Let us consider,

$$x_3 = r$$
$$x_4 = s$$
$$x_5 = t$$

Tanjil 08.07.17 MAT-216

$$\therefore x_2 = -r - s - 2t$$

$$\therefore x_1 = 4r + 4s + 8t - 5r - 6s - 9t$$
$$= -r - 2s - t$$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -r-s-2t \\ -r-2s-t \\ r \\ s \\ t \end{pmatrix}$$

[nullity sometimes called solution space.]

$$= r\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ Solution space $= \{v_1, v_2, v_3\}$

$\therefore$ nullity $= 3$ (Ans)

[rank + nullity = number of columns]

Find the nullity

$$A = \begin{pmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & 6 & 0 & -3 \end{pmatrix} \quad \begin{bmatrix} r_3' = (-2) r_1 + r_3 \\ r_4' = (-3) r_1 + r_4 \\ r_5' = (2) r_1 + r_5 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & 6 & 0 & -3 \end{pmatrix} \quad \left[r_2' = (1/3) r_2 \right]$$

$$= \begin{pmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & -12 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{bmatrix} r_3' = (-3) r_2 + r_3 \\ r_4' = (-3) r_4 + r_2 \\ r_5' = (-3) r_5 + r_2 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -5/12 \\ 0 & 0 & -12 & 0 & 5 \end{pmatrix} \quad \left[r_3' = (-1/12) r_3 \right]$$

$$= \begin{pmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -5/12 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \left[r_4' = 12 r_3 + r_4 \right]$$

$\therefore$ rank $(A) = 3$

The corresponding system of linear equations,

$$x_1 - 3x_2 + 2x_3 + 2x_4 + x_5 = 0$$
$$x_2 + 2x_3 - x_5 = 0$$
$$3x_3 - \tfrac{5}{12}x_5 = 0$$

Let us consider,

$x_4 = r$

$x_5 = s$

$\therefore x_1 = 3x_2 - 2x_3 - 2x_4 - x_5$

$\therefore x_2 = x_5 - 2x_3$

$\therefore x_3 = \tfrac{5}{36} x_5$

So, $x_3 = \tfrac{5}{36} s$

$$\therefore x_2 = s - \left(2 \times \tfrac{5}{36}\right) s$$
$$= s - \tfrac{5}{18} s$$
$$= \tfrac{13}{18} s$$

$$\therefore x_1 = \left(3 \times \tfrac{13}{18}\right) s - \left(2 \times \tfrac{5}{36}\right) s - 2r - s$$
$$= \tfrac{13}{6} s - \tfrac{5}{18} s - s - 2r$$
$$= \tfrac{8}{9} s - 2r$$

Tanjil
08.07.17
MAT-216

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{8}{9}s - 2r \\ \frac{13}{18}s \\ \frac{5}{36}s \\ r \\ s \end{pmatrix}$$

$$= s \begin{pmatrix} 8/9 \\ 13/18 \\ 5/36 \\ 0 \\ 1 \end{pmatrix} + r \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore v_1 = \begin{pmatrix} 8/9 \\ 13/18 \\ 5/36 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ Solution Space $= \{v_1, v_2\}$

$\therefore$ nullity $= 2$ (Ans)

#

$$A = \begin{pmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -5 & 6 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \end{pmatrix} \quad \begin{bmatrix} r_2' = (-3)r_1 + r_2 \\ r_3' = (-5)r_1 + r_3 \\ r_4' = (-4)r_1 + r_4 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -5 & 6 \\ 0 & 1 & -8 & 8 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \end{pmatrix} \quad \left[r_2' = \left(-\tfrac{1}{2}\right) r_2 \right]$$

$$= \begin{pmatrix} 1 & 0 & -5 & 6 \\ 0 & 1 & -8 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{bmatrix} r_3' = 2r_2 + r_3 \\ r_4' = 2r_2 + r_4 \end{bmatrix}$$

$\therefore$ rank $(A) = \underline{2}$

$\therefore$ The corresponding system of linear equations,

$\therefore x_1 - 5x_3 + 6x_4 = 0$

$\therefore x_2 - 8x_3 + 8x_4 = 0$

Let us consider,

$x_3 = r$

$x_4 = s$

So,

$x_1 = 5r - 6s$

$x_2 = 8r - 8s$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5r - 6s \\ 8r - 8s \\ r \\ s \end{pmatrix}$$

$$= r \begin{pmatrix} 5 \\ 8 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -6 \\ -8 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore V_1 = \begin{pmatrix} 5 \\ 8 \\ 1 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} -6 \\ -8 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ Solution space $= \{V_1, V_2\}$

$\therefore$ nullity $= 2$

Tanjil
08.07.17
MAT-216

Linear Transformation

If T such that $V \to W$ is a function from a vector space V into a vector space W, T is called a linear transformation from V to W if for all vectors u_1, u_2 in V and every scalar c such that,

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

(ii) $T(cu_1) = cT(u_1)$

Practice sheet
doc file, Page-7

8(i) $T(x, y, z) = (x-y, x-z)$

$\therefore$ Let,

$u_1 (x_1, y_1, z_1)$

$u_2 (x_2, y_2, z_2)$

$\therefore u_1 + u_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$T(u_1) = T(x_1, y_1, z_1)$

$= (x_1 - y_1, x_1 - z_1)$

$T(u_2) = T(x_2, y_2, z_2)$

$= (x_2 - y_2, x_2 - z_2)$

$\therefore T(u_1 + u_2) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (x_1 + x_2 - y_1 - y_2, x_1 + x_2 - z_1 - z_2)$

$\therefore T(u_1) + T(u_2) = (x_1 - y_1, x_1 - z_1) + (x_2 - y_2, x_2 - z_2)$

$= (x_1 - y_1 + x_2 - y_2, x_1 - z_1 + x_2 - z_2)$

$= (x_1 + x_2 - y_1 - y_2, x_1 + x_2 - z_1 - z_2)$

$\therefore T(u_1 + u_2) = T(u_1) + T(u_2)$

$\therefore T(cu_1) = T(cx_1, cy_1, cz_1)$

$= (cx_1 - cy_1, cx_1 - cz_1)$

$= c(x_1 - y_1, x_1 - z_1)$

$= cT(u_1)$

(ii) $T(x, y, z) = (3x - 2y + z, x - 3y - 2z)$

Let, $u_1(x_1, y_1, z_1) \mid u_2(x_2, y_2, z_2)$

$\therefore u_1 + u_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$\therefore T(u_1) = T(x_1, y_1, z_1)$

$= (3x_1 - 2y_1 + z_1, x_1 - 3y_1 - 2z_1)$

$\therefore T(u_2) = T(x_2, y_2, z_2)$

$= (3x_2 - 2y_2 + z_2, x_2 - 3y_2 - 2z_2)$

$\therefore T(u_1 + u_2) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (3x_1 + 3x_2 - 2y_1 - 2y_2 + z_1 + z_2, x_1 + x_2 - 3y_1 - 3y_2 - 2z_1 - 2z_2)$

$T(u_1) + T(u_2) = (3x_1 - 2y_1 + z_1, x_1 - 3y_1 - 2z_1)$
$+ (3x_2 - 2y_2 + z_2, x_2 - 3y_2 - 2z_2)$

$= (3x_1 + 3x_2 - 2y_1 - 2y_2 + z_1 + z_2, x_1 + x_2 - 3y_1 - 3y_2 - 2z_1 - 2z_2)$

$\therefore T(u_1 + u_2) = T(u_1) + T(u_2)$

$\therefore T(cu_1) = T(cx_1, cy_1, cz_1)$

$= (3cx_1 - 2cy_1 + cz_1,\ cx_1 - 3cy_1 - 2cz_1)$

$= c\,(3x_1 - 2y_1 + z_1,\ x_1 - 3y_1 - z_1)$

$\therefore T(cu_1) = cT(u_1)$

(iii) $T(x, y, z) = (x+1, y+z)$

let, $u_1 (x_1, y_1, z_1)$; $u_2 (x_2, y_2, z_2)$

$\therefore u_1 + u_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$

$\therefore T(u_1) = T(x_1, y_1, z_1)$
$= (x_1 + 1,\ y_1 + z_1)$

$\therefore T(u_2) = T(x_2, y_2, z_2)$
$= (x_2 + 1,\ y_2 + z_2)$

$\therefore T(u_1 + u_2) = T(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$

$= (x_1 + x_2 + 1,\ y_1 + y_2 + z_1 + z_2)$

$\therefore T(u_1) + T(u_2) = (x_1 + 1,\ y_1 + z_1) + (x_2 + 1,\ y_2 + z_2)$

$= (x_1 + x_2 + 2,\ y_1 + y_2 + z_1 + z_2)$

$\therefore T(u_1 + u_2) \neq T(u_1) + T(u_2)$

$\therefore$ This can't be linearly transformed

Linear Transformation

Range: The range of T is the subset of W consisting of all $y \in W$ such that $T(x) = y$ for all $x \in V$. It is generally denoted by $R(T)$.

Rank of a linear transformation: If $T : V \to W$ is a linear transformation, the dimension of the range of T is called the rank of T and is denoted by rank(T).

Kernel or null space: Let T be a linear transformation of V into W. Then the kernel of T is the subset of V, consisting of all $x \in V$ for which $T(x) = 0$, when $0 \in W$. The kernel of T is generally denoted by ker(T).

Nullity: If T such that $V \to W$ is a linear transformation then the dimension of the kernel is called the nullity of T and is denoted by nullity(T).

Tanjil
08.07.17
MAT-216

Prac. sheet p. 6 → doc

Q9. Given that,

$$T(x, y, z, t) = (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t)$$

∴ The image of the generators $\mathbb{R}^4$ generate the $Im(T)$:

$$T(1, 0, 0, 0) = (1, 1, 1)$$

$$T(0, 1, 0, 0) = (-1, 0, 1)$$

$$T(0, 0, 1, 0) = (1, 2, 3)$$

$$T(0, 0, 0, 1) = (1, -1, -3)$$

∴ Form the matrix whose rows are the generators of $Im(T)$:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{pmatrix} \quad \begin{bmatrix} r_2' = r_1 + r_2 \\ r_3' = (-1)r_1 + r_3 \\ r_4' = (-1)r_1 + r_4 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{bmatrix} r_3' = (-1)r_2 + r_3 \\ r_4' = 2r_2 + r_4 \end{bmatrix}$$

∴ This matrix is in row-echelon form and it has two non-zero rows so, $\{(1,1,1), (0,1,2)\}$ this form a basis

$\therefore$ Dim $= 2$ (Ans)

For nullity we have to consider, $T(x) = 0$

$$T(x, y, z) = (0, 0, 0)$$

$$\Rightarrow (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t) = (0, 0, 0)$$

The corresponding components,

$$x - y + z + t = 0$$
$$x + 2z - t = 0$$
$$x + y + 3z - 3t = 0$$

$\therefore$ The augmented matrix,

Tanjil 08.07.17 MAT-216

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & 2 & -1 & 0 \\ 1 & 1 & 3 & -3 & 0 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 2 & 2 & -4 & 0 \end{array}\right) \quad \left[\begin{array}{l} r_2' = (-1) r_1 + r_2 \\ r_3' = (-1) r_1 + r_3 \end{array}\right]$$

$$= \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad [r_3' = (-2) r_2 + r_3]$$

$\therefore$ The corresponding system of linear equations,

$$x - y + z + t = 0$$
$$y + z - 2t = 0$$

Let us consider,

$$z = p$$
$$t = q$$

sheet

$\therefore\ y = -p + 2q$

$\therefore\ x = -p + 2q - p - q$

$= -2p + q$

$$\therefore\ x = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -2p + q \\ -p + 2q \\ p \\ q \end{pmatrix}$$

$$= p \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + q \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore\ v_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ Null space $= \{v_1, v_2\}$

$\therefore$ nullity $= 2$

Tanjil
08.07.17
MAT-216

Q.10 Given that,

$$T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$$

The images of the generators of R^3 generate the Im T:

$$T(1,0,0) = (1,0,1)$$

$$T(0,1,0) = (2,1,1)$$

$$T(0,0,1) = (-1,1,-2)$$

Form the matrix whose rows are the generators of Im T;

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{bmatrix} r_2' = (-2) r_1 + r_2 \\ r_3' = r_1 + r_3 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} [r_3' = r_2 + r_3]$$

$\therefore$ This matrix is in row-echelon form and it has 2 non-zero rows.

So, $\{(1,0,1), (0,1,-1)\}$ form a basis,

$\therefore$ Dim = 2 Ans

$\therefore$ For nullity we have to consider $T(v) = 0$

$$\therefore T(x, y, z) = (0, 0, 0)$$

$$\Rightarrow (x+2y-z, y+z, x+y-2z) = (0, 0, 0)$$

$\therefore$ The corresponding components,

$$x + 2y - z = 0$$
$$y + z = 0$$
$$x + y - 2z = 0$$

$\therefore$ The augmented matrix,

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) \left[r_3' = (-1)r_1 + r_3\right]$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \left[r_3' = r_2 + r_3\right]$$

$\therefore$ The corresponding system of linear equations,

$$x + 2y - z = 0$$
$$y + z = 0$$

Let us consider,
$z = p$

$\therefore\ y = -p$

$\therefore\ x = -2y + z$

$= 2p + p$

$= 3p$

$$\therefore\ x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3p \\ -p \\ p \end{pmatrix}$$

$$= p \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$\therefore$ Solution space $= \{(3, -1, 1)\}$

$\therefore$ nullity $= 1$ (Ans)

Q11 Given that,

$$T(x, y, z) = (3x - y,\ y - z,\ 3x - 2y + z)$$

The image of generator of R^3 generate the $Im(T)$:

$T(1, 0, 0) = (3, 0, 3)$

$T(0, 1, 0) = (-1, 1, -2)$

$T(0, 0, 1) = (0, -1, 1)$

Form the matrix whose rows are the generator of $Im(T)$.

$$\begin{pmatrix} 3 & 0 & 3 \\ -1 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix} \quad [R_1' = (1/3)R_1]$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad [R_2' = R_1 + R_2]$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad [R_3' = R_2 + R_3]$$

$\therefore$ This is in row-echelon form and has two non-zero rows

So, $\{(1,0,1), (0,1,-1)\}$ form a basis

Dim = 2 — (Ans)

$\therefore$ For nullity we have to consider $T(u) = 0$

$$\therefore T(x, y, z) = (0, 0, 0)$$

$$\Rightarrow (3x - y,\ y - z,\ 3x - 2y + z) = (0, 0, 0)$$

$\therefore$ The corresponding components,

$$3x - y = 0$$
$$y - z = 0$$
$$3x - 2y + z = 0$$

$\therefore$ The augmented matrix,

$$\left(\begin{array}{ccc|c} 3 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & -2 & 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & -2 & 1 & 0 \end{array}\right) \left[r_1' = \left(\frac{1}{3}\right) r_1\right]$$

$$= \left(\begin{array}{ccc|c} 1 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \left[r_3' = (-3) r_1 + r_3\right]$$

$$= \left(\begin{array}{ccc|c} 1 & -1/3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \left[r_3' = r_2 + r_3\right]$$

$\therefore$ The corresponding system of linear equations,

$$x - \tfrac{1}{3} y = 0$$
$$y - z = 0$$

Let us consider,
$z = p$

$\therefore y = p$

$\therefore x = \tfrac{1}{3} p$

$$\therefore X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{3} p \\ p \\ p \end{pmatrix} = p \begin{pmatrix} \frac{1}{3} \\ 1 \\ 1 \end{pmatrix}$$

$\therefore$ null space $= \{(\tfrac{1}{3}, 1, 1)\}$

$\therefore$ ker(T) = 1 (Ans)

Tanjil
08.07.17
MAT.216

Eigenvalues & Eigenvectors

Def: Let A be an $n \times n$ matrix and x is a non-zero vector in $\mathbb{R}^n$ is called an eigenvector of A if Ax is a scaler multiple of x that is,

$$Ax = \lambda x$$

$$\Rightarrow (\lambda I - A)x = 0 \ ; \text{ for some scaler } \lambda.$$

The scalar is called an eigenvalue of A & x is ~~the~~ said to be an eigenvector of A corresponding to λ.

$$(\lambda I - A)x = 0$$

(i) $\lambda I - A \rightarrow$ is called the characteristic matrix

(ii) $\det(\lambda I - A) \rightarrow$ is called the characteristic polynominal

(iii) $\det(\lambda I - A) = 0$, characteristic matrix

\# Find the eigen values and corresponding eigenvectors of the following matrix:

(i) $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{pmatrix}$

(ii) $A = \begin{pmatrix} -2 & 2 & -3 \\ 0 & 1 & -6 \\ 0 & -2 & 0 \end{pmatrix}$

24
(i)
Sol. The characteristic matrix is,

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{pmatrix} \quad \cdots\cdots (i)$$

The characteristic equation,

$$\det(\lambda I - A) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{vmatrix} = 0$$

Tanji
08.07.17
MAT-216

$$\Rightarrow (\lambda-1)(\lambda+2)(\lambda-2) = 0$$

$$\therefore \lambda = 1, \ \lambda = -2, \ \lambda = 2$$

These are the eigenvalues (Ans)

Now, let us consider,

$$(\lambda I - A)x = 0$$

$$\left(\begin{array}{ccc|c} \lambda-1 & -2 & 1 & 0 \\ 0 & \lambda+2 & 0 & 0 \\ 0 & 5 & \lambda-2 & 0 \end{array}\right) \quad \cdots\cdots (ii)$$

if $\lambda = 1$. eq (ii),

$$\left(\begin{array}{ccc|c} 0 & -2 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 5 & -1 & 0 \end{array}\right)$$

corresponding system of linear equations

$$-2x_2 + x_3 = 0$$

$$3x_2 = 0$$

$$5x_2 - x_3 = 0$$

$\therefore x_2 = 0$ | Let us consider, $x_1 = a$

$\therefore x_3 = 0$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$$

$$= a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore P_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$: is the eigenvector corresponding to $\lambda = 1$

(Ans)

if $\lambda = -2$, eq (ii),

$$\left(\begin{array}{ccc|c} -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & -4 & 0 \end{array}\right)$$

corresponding system of linear equations,

$$-3x_1 - 2x_2 + x_3 = 0$$
$$5x_2 - 4x_3 = 0$$

Let us consider, $x_3 = b$

$$\therefore x_2 = 4b/5$$

$$\therefore x_1 = -1/3 \left(2 \cdot \frac{4b}{5} - b\right)$$

$$= -1/3 \left(\frac{8b}{5} - b\right)$$

$$= -\frac{3b}{15}$$

$$= -b/5$$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -b/5 \\ 4b/5 \\ b \end{pmatrix}$$

$$= b/5 \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$$

$P_2 = \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$; is the eigenvector corresponding to $\lambda = -2$

(Ans)

$\therefore$ if $\lambda = 2$, eq (ii),

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right)$$

$\therefore$ corresponding system of linear equations,

$$x_1 - 2x_2 + x_3 = 0$$
$$4x_2 = 0$$
$$5x_2 = 0$$

Let us consider,

$$x_3 = a$$

$\therefore x_2 = 0$

$\therefore x_1 = -a$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -a \\ 0 \\ a \end{pmatrix}$$

$$= a \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore P_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$; is the eigenvector corresponding to $\lambda = 2$

(Ans)

(ii) Sol: The characteristic matrix is,

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -2 & 2 & -3 \\ 0 & 1 & -6 \\ 0 & -2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda+2 & -2 & 3 \\ 0 & \lambda-1 & 6 \\ 0 & 2 & \lambda \end{pmatrix} \cdots\cdots (i)$$

The characteristic equation,

$$\det(\lambda I - A) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda+2 & -2 & 3 \\ 0 & \lambda-1 & 6 \\ 0 & 2 & \lambda \end{vmatrix} = 0$$

$$\cancel{\Rightarrow 3(-4-\lambda+1) - 6(2\lambda+4+2) + \lambda\{(\lambda+2)(\lambda-1)-4\}} = 0$$

$$\cancel{) + \lambda(\lambda^2 - \lambda + 2\lambda - 2 - 4)} = 0$$

$$\cancel{+ \lambda(\lambda^2 + \lambda - 6) = 0}$$

$$\lambda^3 + \lambda^2 - 6\lambda = 0$$

Tanjil
08.07.17
MAT-216

$$\cancel{) + (\lambda \ \ \ )) = 0}$$

$$\Rightarrow (\lambda+2)\ \{(\lambda-1)\lambda - 12\} = 0$$

$$\Rightarrow (\lambda+2)\ (\lambda^2 - \lambda - 12) = 0$$

$$\Rightarrow (\lambda+2)\ (\lambda^2 - 4\lambda + 3\lambda - 12) = 0$$

$$\Rightarrow (\lambda+2)\{\lambda(\lambda-4) + 3(\lambda-4)\} = 0$$

$$\Rightarrow (\lambda+2)\ (\lambda-4)\ (\lambda+3) = 0$$

$\therefore \lambda = -2,\ \lambda = 4,\ \lambda = -3$; are the eigenvalues (Ans)

If $\cancel{\lambda = -2 \text{, eq (ii)}}$,

Now let us consider,

$$(\lambda I - A)\, x = 0$$

$$\left(\begin{array}{ccc|c} \lambda+2 & -2 & 3 & 0 \\ 0 & \lambda-1 & 6 & 0 \\ 0 & 2 & \lambda & 0 \end{array}\right) \quad \cdots\text{(ii)}$$

$\therefore$ If $\lambda = -2$ then eq (ii),

$$\left(\begin{array}{ccc|c} 0 & -2 & 3 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 0 & 2 & -2 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & -2 & 3 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & -2 & 3 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad \begin{bmatrix} r_2' = 3r_1 + r_2 \\ r_3' = 2r_1 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

∴ corresponding system of linear equations,

$$x_2 - x_3 = 0$$

$$x_3 = 0$$

Let us consider,

$x_1 = a$

$\therefore\ x_2 = 0, \quad x_3 = 0$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore P_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the eigenvector corresponding to $\lambda = -2$ (Ans)

∴ If $\lambda = 4$, then eq. (ii),

$$\left(\begin{array}{ccc|c} 6 & -2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right)$$

∴ The corresponding system of linear equation.

$$6x_1 - 2x_2 + 3x_3 = 0$$

$$3x_2 + 6x_3 = 0$$

$$2x_2 + 4x_3 = 0$$

Let us consider,

$x_3 = b$

$\therefore\ x_2 = 2b$

$\therefore\ 6x_1 - 4b + 3b = 0$

$\Rightarrow x_1 = b/6$

$$\therefore x \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b/6 \\ 2b \\ b \end{pmatrix} = b/6 \begin{pmatrix} 1 \\ 12 \\ 6 \end{pmatrix}$$

$\therefore P_2 = \begin{pmatrix} 1 \\ 12 \\ 6 \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 4$ (Ans)

If $\lambda = -3$ then eq (ii),

$$\left(\begin{array}{ccc|c} -1 & -2 & 3 & 0 \\ 0 & -4 & 6 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The corresponding system of linear equations,

$$x_1 + 2x_2 + 3x_3 = 0$$
$$x_2 - 3/2\, x_3 = 0$$

Let us consider,
$x_3 = c$

$\therefore x_2 = 3/2\, c$

$\therefore x_1 = \left(-2 \times 3/2\right)c - 3c$

$= -3c - 3c$

$= -6c$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6c \\ 3/2\, c \\ c \end{pmatrix} = c/2 \begin{pmatrix} -12 \\ 3 \\ 2 \end{pmatrix}$$

$\therefore P_3 = \begin{pmatrix} -12 \\ 3 \\ 2 \end{pmatrix}$ is the eigenvector corresponding to $\lambda = -3$

(Ans)

26

Find the eigen values and eigenspaces.

$$A = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$\therefore$ The characteristic matrix,

$$(\lambda I - A) = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda+1 & 2 & 2 \\ -1 & \lambda-2 & -1 \\ 1 & 1 & \lambda \end{pmatrix} \quad \cdots\cdots (i)$$

$\therefore$ The characteristic equation,

$$\det(\lambda I - A) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda+1 & 2 & 2 \\ -1 & \lambda-2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

Tanjil
08.07.17
MAT-216

$$\Rightarrow (\lambda+1)(\lambda^2-2\lambda+1) + 1(2\lambda-2) + 1(-2-2\lambda+4) = 0$$

$$\Rightarrow \lambda^3 - 2\lambda^2 + \lambda + \lambda^2 - 2\lambda + 1 + 2\lambda - 2 - 2\lambda + 2 = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0$$

$$\Rightarrow \lambda^2(\lambda-1) - 1(\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2-1) = 0$$

$\therefore \lambda = 1, \ \lambda = 1, -1$; These are eigenvalues (Ans)

∴ ~~if $\lambda \neq$~~ now let us consider,

$$(\lambda I - A) X = 0$$

$$\therefore \left(\begin{array}{ccc|c} \lambda+1 & 2 & 2 & 0 \\ -1 & \lambda-2 & -1 & 0 \\ 1 & 1 & \lambda & 0 \end{array}\right) \quad \cdots (ii)$$

∴ If $\lambda = 1$, then eq (ii)

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

corresponding system of linear equations,

$$x_1 + x_2 + x_3 = 0$$

let us consider $x_2 = a$, $x_3 = b$

$$\therefore x_1 = -a - b$$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -a-b \\ a \\ b \end{pmatrix}$$

$$= a \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore P_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \ \& \ P_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \text{eigenspace} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$ (Ans)

$\therefore$ If $\lambda = -1$, then eq (ii),

$$\left(\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ -1 & -3 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & -3 & -1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

corresponding system of linear equations,

$$x_1 + x_2 - x_3 = 0$$
$$x_2 + x_3 = 0$$

let us consider $x_3 = c$

$\therefore x_2 = -c$

$\therefore x_1 - c - c = 0$

$\Rightarrow x_1 = 2c$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2c \\ -c \\ c \end{pmatrix} = c \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore P_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\therefore \text{eigen-space} = \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Def:
A square matrix A is called diagonalizable if there is an invertable matrix P such that $P^{-1}AP$ is a diagonal matrix P is said to be diagonalize of A.

* working rule:

(i) At first find n linearly independent eigenvectors of A say $P_1, P_2, \ldots, P_n$.

(ii) Secondly, form the matrix P having $P_1, P_2, \ldots, P_n$ as its column vectors.

(iii) Finally, the matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal matrix where λ is the eigenvalue corresponding to P_i for $i = 1, 2, \ldots, n$

\# If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), det (A) is the product of the entries on the main diagonal of the matrix.

\# If A is an $n \times n$ triangular matrix (upper, lower or diagonal) the eigenvalues of A are the entries on the main diagonal of A is $\lambda_1 = a_{11}$, $\lambda_2 = a_{22}$, $\lambda_3 = a_{33}$

27

From the eigen vectors,

$$P = \begin{pmatrix} -1 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\therefore (P|I) = \left(\begin{array}{ccc|ccc} -1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad [r_1' = (-1)\, r_1]$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad [r_2' = r_1 + r_2]$$

Tanjil
08.07.17
MAT-216

$$= \begin{pmatrix} 1 \\ 0 \\ 0 & 0 & 4 & 1 & -1 & 1 \end{pmatrix} \quad \begin{bmatrix} r_1' = (-1)\, r_2 + r_1 \\ r_3' = (-1)\, r_2 + r_3 \end{bmatrix}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad [r_1 \leftrightarrow r_2]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad [r_2' = r_1 + r_2]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad [r_2' = (-1)\, r_2]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right) \quad [r_3' = (-1)\, r_2 + r_3]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right) \quad [r_3' = (1/2)\, r_3]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & 1/2 \end{array}\right) \quad \begin{bmatrix} r_2' = r_3 + r_2 \\ r_1' = r_3 + r_1 \end{bmatrix}$$

$$\therefore \; P^{-1} = \begin{pmatrix} 1/2 & 3/2 & 1/2 \\ -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{pmatrix}$$

$$\therefore \; P^{-1}AP = \begin{pmatrix} 1/2 & 3/2 & 1/2 \\ -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\therefore$ P is diagonaligable.

Double Integral

Interrated (or repeated) integration: A partial definite integral with respect to x is a function of y can be integrated with respect to y.

Similarly, partial definite integral with respect to y can be integrated with respect to x.

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

③ $\int_2^4 \int_0^1 x^2 y\,dx\,dy$

$= \int_2^4 y\left(\int_0^1 x^2\,dx\right) dy$

$= \int_2^4 y \left[\frac{x^3}{3}\right]_0^1 dy$

$= \frac{1}{3}\int_2^4 y\,dy$

$= \frac{1}{3}\left[\frac{y^2}{2}\right]_2^4$

$= \frac{1}{3} \times 6$

$= 2$ (Ans)

④ $\int_{-2}^0 \int_{-1}^2 (x^2+y^2)\,dx\,dy$

$= \int_{-2}^0 \left[\int_{-1}^2 (x^2+y^2)\,dx\right] dy$

$= \int_{-2}^0 \left[\int_{-1}^2 x^2\,dx + \int_{-1}^2 y^2\,dx\right] dy$

$= \int_{-2}^0 \left[y^2(2+1) + \left[\frac{x^3}{3}\right]_{-1}^2\right] dy$

$= \int_{-2}^0 (3y^2+3)\,dy$

$= \int_{-2}^0 3y^2 + \int_{-2}^0 3\,dy$

$= 3\left[\frac{y^3}{3}\right]_{-2}^0 + 3[y]_{-2}^0$

$= 8 + 6$

$= 14$ (Ans)

⑤ $\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y}\,dy\,dx$

$= \int_0^{\ln 3}\left[\int_0^{\ln 2} e^{x+y}\,dy\right]dx$

$= \int_0^{\ln 3}\left[\int_0^{\ln 2} e^{x}.\,e^{y}\,dy\right]dx$

$= \int_0^{\ln 3} e^{x}\left[e^{y}\right]_0^{\ln 2}\,dx$

$=$

$= \int_0^{\ln 3} e^{x}\left[e^{\ln 2} - e^{0}\right]dx$

$= \int_0^{\ln 3} e^{x}\,dx$

$= e^{\ln 3} - e^{0}$

$= 3-1$

$= 2$ (Ans)

⑥ $\int_0^{2}\int_0^{1} y\sin x\,dy\,dx$

$= \int_0^{2}\sin x\left[\int_0^{1} y\,dy\right]dx$

$= \int_0^{2}\sin x\left[\frac{y^2}{2}\right]_0^{1}dx$

$= \frac{1}{2}\int_0^{2}\sin x\,dx$

$= \frac{1}{2}\left[-\cos x\right]_0^{2}$

$= \frac{1}{2}\left[-\cos 2 + \cos 0\right]$

$= \frac{1-\cos 2}{2}$

$= \frac{2\sin^2 1}{2} = \sin^2 1$ (Ans)

⑦ $\int_0^{1}\int_0^{1}\frac{x}{(xy+1)^2}\,dy\,dx$

$u = xy+1$
$du = x\,dy$
if, $y=1$, $u = x+1$
if, $y=0$, $u=1$

$= \int_0^{1}\left[\int_1^{x+1}\frac{1}{u^2}\,du\right]dx$

$= \int_0^{1}\left[-\frac{1}{u}\right]_1^{x+1}dx$

$= -\int_0^{1}\left[\frac{1}{x+1} - 1\right]dx$

$= -\int_0^{1}\left(\frac{1}{x+1} - 1\right)dx$

$= -\left[\ln(x+1)\right]_0^{1} + \left[x\right]_0^{1}$

$= -(\ln 2 - \ln 1) + (1-0)$

$= 1 - \ln 2$ (Ans)

(10) $\int_{\pi/2}^{\pi}\int_{1}^{2} x\cos xy \, dy\, dx$

$u = xy$

$du = x\, dy$

$y = 2 \; ; u = 2x$

$y = 1 \; ; u = x$

$= \int_{\pi/2}^{\pi}\left[\int_{x}^{2x}\cos u \, du\right] dx$

$= \int_{\pi/2}^{\pi}\left[\sin u\right]_{x}^{2x} dx$

$= \int_{\pi/2}^{\pi}\left[\sin(2x) - \sin x\right] dx$

$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_{\pi/2}^{\pi}$

$= \left(-\frac{1}{2}\cos 2\pi + \cos\pi\right) + \left(\frac{1}{2}\cos\pi + \cos\frac{\pi}{2}\right)$

$= \left(-\frac{1}{2} - 1\right) + \left(-\frac{1}{2} + 0\right)$

$= -2$ (Ans)

(11) $\int_{0}^{\ln 2}\int_{0}^{1} xy e^{y^2 x} \, dy\, dx$

$u = y^2 x$

$\Rightarrow du = 2yx\, dy$

$\therefore \frac{1}{2} du = xy\, dy$

$\therefore y = 1 \; ; u = x$

$\therefore y = 0 \; ; u = 0$

$= \int_{0}^{\ln 2}\left[\frac{1}{2}\int_{0}^{x} e^{u} du\right] dx$

$= \frac{1}{2}\int_{0}^{\ln 2}\left[e^{u}\right]_{0}^{x} dx$

$= \frac{1}{2}\int_{0}^{\ln 2}\left(e^{x} - e^{0}\right) dx$

$= \frac{1}{2}\int_{0}^{\ln 2}\left(e^{x} - 1\right) dx$

$= \frac{1}{2}\left[e^{x} - x\right]_{0}^{\ln 2}$

$= \frac{1}{2}\left(e^{\ln 2} - e^{0} - \ln 2 + 0\right)$

$= \frac{1}{2}\left(2 - 1 - \ln 2\right)$

$= \frac{1}{2}\left(1 - \ln 2\right)$ (Ans)

(12) $$\int_3^4 \int_1^2 \frac{1}{(x+y)^2}\, dy\, dx$$

$u = x+y$

$du = dy$

$y = 2 \; ; \; u = x+2$

$y = 1 \; ; \; u = x+1$

$$= \int_3^4 \left[\int_{x+1}^{x+2} \frac{1}{u^2}\, du\right] dx$$

$$= \int_3^4 \left[-\frac{1}{u}\right]_{x+1}^{x+2} dx$$

$$= \int_3^4 \left[-\frac{1}{x+2} + \frac{1}{x+1}\right] dx$$

$$= \left[\ln(x+1) - \ln(x+2)\right]_3^4$$

$$= \left[\ln \frac{x+1}{x+2}\right]_3^4$$

$$= \ln \frac{4+1}{4+2} - \ln \frac{3+1}{3+2}$$

$$= \ln \frac{5/6}{4/5}$$

$$= \ln\left(25/24\right) \text{ (Ans)}$$

Practice sheet calculus

(1)(a) $$\int_{\pi/2}^{\pi} \int_0^{x^2} \frac{1}{x} \cos \frac{y}{x}\, dy\, dx$$

$u = y/x$

$\Rightarrow x\,du = dy.$

$\therefore y = 0 \; ; \; u = 0$

$\therefore y = x^2 \; ; \; u = x$

$$= \int_{\pi/2}^{\pi} \left[\int_0^{x} \frac{1}{x} \cdot x \cos u\, du\right] dx$$

$$= \int_{\pi/2}^{\pi} \left[\sin u\right]_0^x dx$$

$$= \int_{\pi/2}^{\pi} (\sin x - \sin 0)\, dx$$

$$= \int_{\pi/2}^{\pi} \sin x\, dx$$

$$= \left[-\cos x\right]_{\pi/2}^{\pi}$$

$$= -\cos \pi + \cos \pi/2$$

$$= -(-1+0)$$

$$= 1 \text{ (Ans)}$$

1(b)

$$\int_0^1 \int_0^1 \frac{x}{(xy+1)^2}\, dy\, dx$$

$$= \int_0^1 \left[\int_1^{x+1} \frac{1}{u^2}\, du\right] dx$$

$$= \int_0^1 \left[-\frac{1}{u}\right]_1^{x+1} dx$$

$$= \int_0^1 \left[-\frac{1}{x+1} + \frac{1}{1}\right] dx$$

$$= \int_0^1 \left[-(x+1)^{-1} + 1\right] dx$$

$$= -\int_1^2 \frac{1}{v}\, dv + \int_0^1 dx$$

$u = xy + 1$

$du = x\, dy$

$y = 0,\ u = 1$

$y = 1,\ u = x+1$

$$= -\left[\ln v\right]_1^2 + \left[x\right]_0^1$$

$$= -\ln 2 + \ln 1 + 1$$

$$= 1 - \ln 2 \text{ (Ans)}$$

$v = x+1$

$\Rightarrow dv = dx$

$x = 0,\ v = 1$

$x = 1,\ v = 2$

1(c)

$$\int_1^2 \int_0^{y^2} e^{x/y^2}\, dx\, dy$$

$$= \int_1^2 y^2 \left[\int_0^1 e^u\, du\right] dy$$

$$= \int_1^2 y^2 \left[e^u\right]_0^1 dy$$

$$= \int_1^2 y^2 \left[e^1 - e^0\right]$$

$$= \int_1^2 y^2 (e-1)\, dy$$

$$= (e-1)\left[\frac{y^3}{3}\right]_1^2$$

$u = x/y^2$

$\Rightarrow y^2 du = dx$

$x = 0,\ u = 0$

$x = y^2,\ u = 1$

$$= (e-1)\left[\frac{2^3}{3} - \frac{1^3}{3}\right]$$

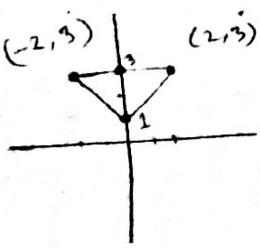
$$= \frac{7}{3}(e-1) \text{ (Ans)}$$

evaluate $\iint_R (2x-y^2)\,dA$ over the triangular region R enclosed between the lines $y=-x+1$, $y=x+1$ and $y=3$

$$\iint_R (2x-y^2)\,dA = \int_1^3 \int_{1-y}^{y-1} (2x-y^2)\,dx\,dy$$

$y=-x+1$
$\Rightarrow x = 1-y = -2$
& $y = x+1$
$\therefore x = y-1 = 2$
if $x=0$, $y=1$

$$= \int_1^3 \left\{ 2\left[\frac{x^2}{2}\right]_{1-y}^{y-1} - y^2\left[x\right]_{1-y}^{y-1} \right\} dy$$



$$= \int_1^3 \left\{ 2\left[\frac{(y-1)^2}{2} - \frac{(1-y)^2}{2}\right] - y^2(y-1-1+y) \right\} dy$$

$$= \int_1^3 \left\{ y^2-2y+1-1+2y-y^2-y^2(2y-2) \right\} dy$$

$$= \int_1^3 (2y^2-2y^3)\,dy$$

$$= 2\cdot\left[\frac{y^3}{3}\right]_1^3 - 2\left[\frac{y^4}{4}\right]_1^3$$

$$= 2\left(\frac{27}{3}-\frac{1}{3}\right) - 2\left(\frac{81}{4}-\frac{1}{4}\right)$$

$$= \frac{52}{3} - 40$$

$$= -\frac{68}{3} \text{ (Ans)}$$

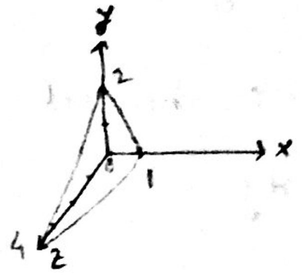
Tanjil
08.07.17
MAT-216

Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$

if, $y = 0$ & $z = 0$ then, $x = 1$

if, $x = 0$ & $z = 0$ then, $y = 2$

if, $x = 0$ & $y = 0$ then, $z = 4$



$$\therefore V = \iint_R (4 - 4x - 2y)\, dA$$

if, $z = 0$

$0 = 4 - 4x - 2y$

$\Rightarrow y = 2 - 2x$

$$= \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)\, dy\, dx$$

$$= \int_0^1 \left\{ \left[4y\right]_0^{2-2x} - \left[4xy\right]_0^{2-2x} - \left[2 \cdot \frac{y^2}{2}\right]_0^{2-2x} \right\} dx$$

$$= \int_0^1 \left\{ 4(2-2x) - 4x(2-2x) - (2-2x)^2 \right\} dx$$

$$= \int_0^1 \left(8 - 8x - 8x + 8x^2 - 4 + 8x - 4x^2\right) dx$$

$$= \int_0^1 \left(4 - 8x + 4x^2\right) dx$$

$$= \left[4x\right]_0^1 - 8\left[\frac{x^2}{2}\right]_0^1 + 4\left[\frac{x^3}{3}\right]_0^1$$

$$= 4 - 4 + \frac{4}{3}$$

$$= \frac{4}{3} \text{ (Ans)}$$

* # Find the volume of the solid bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$

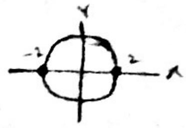
Solve:

$$V = \iint_R (4-y)\, dA$$

$z = 4 - y$

As, $x^2+y^2=2^2$

so,



$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\, dy\, dx$$

$x^2+y^2=4$

$\therefore y = \pm\sqrt{4-x^2}$

$$= \int_{-2}^{2} \left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left[4\left(\sqrt{4-x^2}+\sqrt{4-x^2}\right) - \left(\frac{4-x^2}{2} - \frac{4-x^2}{2}\right)\right] dx$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\, dx$$

trigonometric substitution

$\sin^2\theta + \cos^2\theta = 1$

$\therefore \cos^2\theta = 1 - \sin^2\theta$

$x = 2\sin\theta$

$\Rightarrow dx = 2\cos\theta$

if, $x = 2$, $\theta = \frac{\pi}{2}$

if, $x = -2$, $\theta = -\frac{\pi}{2}$

$$= 8\int_{-\pi/2}^{\pi/2} \sqrt{4-(2\sin\theta)^2}\; 2\cos\theta\, d\theta$$

$$= 8\int_{-\pi/2}^{\pi/2} 2\sqrt{1-\sin^2\theta}\cdot 2\cos\theta\, d\theta$$

$$= 32\int_{-\pi/2}^{\pi/2} \cos\theta \cdot \cos\theta\, d\theta$$

$$= 32\int_{-\pi/2}^{\pi/2} \tfrac{1}{2}(2\cos^2\theta)\, d\theta$$

$$= 16\int_{-\pi/2}^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= 16\left\{[\theta]_{-\pi/2}^{\pi/2} + \tfrac{1}{2}[\sin 2\theta]_{-\pi/2}^{\pi/2}\right\}$$

$$= 16\left\{\left(\tfrac{\pi}{2}+\tfrac{\pi}{2}\right) + \tfrac{1}{2}(\sin\pi + \sin\pi)\right\}$$

$$= 16\left\{\pi + \tfrac{1}{2}(1+1)\right\}$$

$$= 16\pi \text{ (Ans)}$$

Use double integral to find the volume of the solid that is bounded by the plane $z = 4 - x - y$ and below the rectangle $R : \{ (x,y) : 0 \le x \le 1 , 0 \le y \le 2 \}$

Sol:

$$V = \iint_R (4 - x - y)\, dA$$

$$= \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy$$

$$= \int_0^2 \left\{ [4x]_0^1 - \left[\frac{x^2}{2}\right]_0^1 - y[x]_0^1 \right\} dy$$

$$= \int_0^2 \left[4 - \frac{1}{2} - y \right] dy$$

$$= [4y]_0^2 - \frac{1}{2}[y]_0^2 - \left[\frac{y^2}{2}\right]_0^2$$

$$= 4(2-0) - 1 - 2$$

$$= 8 - 1 - 2$$

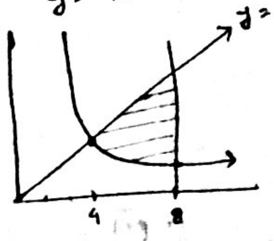
$$= 5 \text{ (Ans)}$$

Tanjil 08.07.17 MAT-216

30

$\iint\limits_R x^2 \, dA$; R is the region bounded by $y = 16/x$, $y = x$ and $x = 8$.

Evaluate the iterated integral.



Sol:

$y = 16/x$, $y = x$

$\therefore x = 4$; $x = 8$
[from graph]

$$\therefore \iint\limits_R x^2 \, dA = \int_4^8 \int_{16/x}^{x} x^2 \, dy \, dx$$

$$= \int_4^8 \left\{ x^2 \left[y \right]_{16/x}^{x} \right\} dx$$

$$= \int_4^8 x^2 \left(x - \frac{16}{x} \right) dx$$

$$= \int_4^8 (x^3 - 16x) \, dx$$

$$= \left[\frac{x^4}{4} - 8x^2 \right]_4^8$$

$$= \left(\frac{8^4}{4} - \frac{4^4}{4} \right) - 8 (8^2 - 4^2)$$

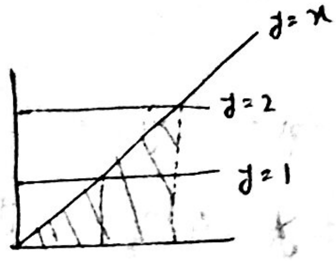
$$= 960 - 384$$

$$= 576 \text{ (Ans)}$$

Tanjil
08.07.17
MAT-216

$\iint_R xy^2 \, dA$; R is enclosed by $y=1$, $y=2$, $x=0$ and $y=x$

Evaluate the iterated integral.



Sol:

$$\therefore \iint_R xy^2 \, dA = \int_1^2 \int_0^y xy^2 \, dx \, dy$$

$$= \int_1^2 \left[\frac{x^2}{2}\right]_0^y y^2 \, dy$$

$$= \int_1^2 \frac{y^2}{2} y^2 \, dy$$

$$= \frac{1}{2} \int_1^2 y^4 \, dy$$

$$= \frac{1}{2} \left[\frac{y^5}{5}\right]_1^2$$

$$= \frac{1}{2} \left(\frac{2^5}{5} - \frac{1^5}{5}\right)$$

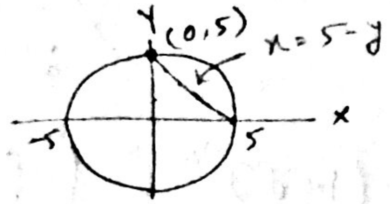
$$= \frac{31}{10}$$

Tanjil
08.07.17
MAT-216

11 $\iint_R y\,dA$; R is the region in the first quadrant enclosed between the circle $x^2+y^2=25$ and the line $x+y=5$; Evaluate the iterated integral

sol:

$\therefore x=\sqrt{25-y^2}$



$$\therefore \iint_R y\,dA = \int_0^5\int_{5-y}^{\sqrt{25-y^2}} y\;dx\,dy$$

$u=25-y^2$

$\Rightarrow du=-2y\,dy$

$\Rightarrow -\frac{1}{2}du = y\,dy$

$y=0,\ u=25$

$y=5,\ u=0$

$$=\int_0^5 y\,[x]_{5-y}^{\sqrt{25-y^2}}\,dy$$

$$=\int_0^5 y\left(\sqrt{25-y^2}-5+y\right)dy$$

$$=\int_0^5 y\sqrt{25-y^2}\,dy-\int_0^5 y(5-y)\,dy$$

$$=-\frac{1}{2}\int_{25}^{0}u^{1/2}\,du-\int_0^5(5y-y^2)\,dy$$

$$=\frac{1}{2}\int_0^{25}u^{1/2}\,du-\left\{5\left[\frac{y^2}{2}\right]_0^5-\left[\frac{y^3}{3}\right]_0^5\right\}$$

$$=\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_0^{25}-\left\{5\left(\frac{5^2}{2}-\frac{0^2}{2}\right)-\frac{5^3}{3}\right\}$$

$$=\frac{1}{2}\times\frac{2}{3}\left[u^{3/2}\right]_0^{25}-\frac{125}{6}$$

$$=\frac{1}{3}\left(25^{3/2}\right)-\frac{125}{6}$$

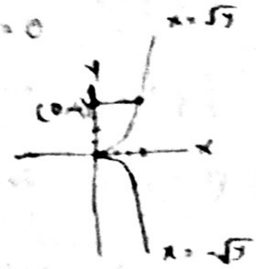
$$=\frac{125}{3}-\frac{125}{6}$$

$$=\frac{125}{6}\ \text{(Ans)}$$

3. Find the volume of the solid that is bounded by the cylinder $y = x^2$ and by the planes $y + z = 4$ and $z = 0$

Sol:

$z = 4 - y \quad \Big| \quad x = \pm\sqrt{y}$



$y + z = 4$
$\Rightarrow y = 4 \; [z = 0]$

$$V = \iint_R (4 - y)\, dA$$

$$= \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} (4 - y)\, dx\, dy$$

$$= \int_0^4 \left[4x - yx\right]_{-\sqrt{y}}^{\sqrt{y}} dy$$

$$= \int_0^4 \left\{(4\sqrt{y} + 4\sqrt{y}) - (y\sqrt{y} + y\sqrt{y})\right\} dy$$

$$= \int_0^4 \left(8\sqrt{y} - 2y\sqrt{y}\right) dy$$

$$= \int_0^4 \left(8\sqrt{y} - 2y^1 y^{1/2}\right) dy$$

$$= \int_0^4 \left(8y^{1/2} - 2y^{3/2}\right) dy$$

$$= 8\left[\frac{y^{3/2}}{3/2}\right]_0^4 - 2\left[\frac{y^{5/2}}{5/2}\right]_0^4$$

$$= 16/3\,(4^{3/2}) - 4/5\,(4)^{5/2}$$

$$= (16/3 \times 8) - (4/5 \times 32)$$

$$= \frac{256}{15} \text{ (Ans)}$$

Tanjil
08.07.17
MAT-216

1

Polar Co-ordinates

If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ & $\theta = \beta$ and the curves $r = r_1(\theta)$ & $r = r_2(\theta)$ and if $f(r,\theta)$ is continuous on R, then

$$\iint_R f(r,\theta)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta .$$

$$x = r\cos\theta \quad\Big|\quad r = \sqrt{x^2+y^2} \quad ; \quad 0 \le \theta \le 2\pi$$

$$y = r\sin\theta \quad\Big|\quad \theta = \tan^{-1}\left(y/x\right) \quad ; \quad r \geq 0$$

Evaluate the iterated integral by polar co-ordinate.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$$

Sol: reconize r & θ

$y = \sqrt{1-x^2}$

$\Rightarrow y^2 = 1-x^2$

$\Rightarrow x^2+y^2 = 1$

$\Rightarrow r = 1; [r \geq 0]$

$x = 1$

$\Rightarrow r\cos\theta = 1$

$\Rightarrow \cos\theta = \cos(0)$

$\Rightarrow \theta = 0$

$x = 0$

$\Rightarrow r\cos\theta = \cos 90$

$\Rightarrow \cos\theta = \cos 90$

$\therefore \theta = \pi/2$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^1 r^2 . r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} d\theta$$

$$= \frac{1}{4} \left[\theta\right]_0^{\pi/2}$$

$$= \pi/8 \text{ (Ans)}$$

Polar coordinates:

$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$

$= \int_0^{\pi/2} \int_0^1 r^2 r\, dr\, d\theta$

$= \int_0^{\pi/2} \int_0^1 r^3\, dr\, d\theta$

$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^1 d\theta$

$= \frac{1}{4} \int_0^{\pi/2} d\theta$

$= \frac{1}{4} \left[\theta\right]_0^{\pi/2}$

$= \frac{1}{4} \cdot \frac{\pi}{2}$

$= \frac{\pi}{8}$

$y = \sqrt{1-x^2}$

$\Rightarrow y^2 = 1 - x^2$

$\Rightarrow r = 1$

$x = 1$

$\Rightarrow r\cos\theta = 1$

$\Rightarrow \cos\theta = 1$

$\therefore \theta = 0$

$x = 0$

$\Rightarrow r\cos\theta = 0$

$\Rightarrow \cos\theta = 0$

$\therefore \theta = \pi/2$

$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

$= \int_0^{\pi/2} \int_0^{2\cos\theta} \sqrt{r^2}\; r\, dr\, d\theta$

$= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\, dr\, d\theta$

$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$

$= \frac{1}{3} \int_0^{\pi/2} (2\cos\theta)^3\, d\theta$

$= \frac{8}{3} \int_0^{\pi/2} \cos^3\theta\, d\theta$

$= \frac{8}{3} \int_0^{\pi/2} \left\{\frac{1}{4}(\cos 3\theta + 3\cos\theta)\right\} d\theta$

$= \frac{2}{3} \int_0^{\pi/2} (\cos 3\theta + 3\cos\theta)\, d\theta$

$= \frac{2}{3} \left[\frac{1}{3}\sin 3\theta + 3\sin\theta\right]_0^{\pi/2}$

$= \frac{2}{3}\left(-\frac{1}{3} + 3\right)$

$= \frac{2}{3} \times \frac{8}{3}$

$= \frac{16}{9}$ (Ans)

$y = \sqrt{2x - x^2}$

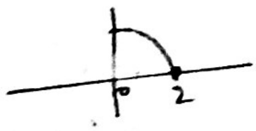
$\Rightarrow y^2 = 2x - x^2$

$\Rightarrow r^2 = 2x$

$\Rightarrow r^2 = 2r\cos\theta$

$\Rightarrow r = 2\cos\theta$

$x = \pi/2$



\# $\int_0^1 \int_0^{\sqrt{x^2+y^2}} e^{\sqrt{x^2+y^2}}$

solve:

$\therefore y = \sqrt{x}$

$\Rightarrow r =$

Tanjil

08.07.17

MAT-216

ordinate

$x = 0$

$\cancel{r\cos\theta} = \cos 90$

$\therefore \theta = \pi/2$

\# $\iint_R \sin(x^2+y^2)\, dA$, where R is the region enclosed by the circle $x^2+y^2=9$ [Evaluate by polar co-ordinate]

solve:

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sin(x^2+y^2)\, dy\, dx$$

$y^2 = 9 - x^2$

$\Rightarrow x^2+y^2 = 9$

$\therefore r = \pm 3 \quad [r \geq 0]$

$$= \int_0^{2\pi}\int_0^{3} \sin(r^2)\, r\, dr\, d\theta$$

[enclosed by the circle $x^2+y^2=9$; so $0 \leq r \leq 2\pi$]

$$= \int_0^{2\pi}\int_0^{9} \tfrac{1}{2} \sin(u)\, du\, d\theta$$

$u = r^2$

$\Rightarrow du = 2r\, dr \Rightarrow \frac{1}{2} du = r\, dr$

$r = 0,\ u = 0$

$r = 3,\ u = 9$

$$= \tfrac{1}{2}\int_0^{2\pi} \left[-\cos u\right]_0^9 d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} \left(-\cos 9 + \cos 0\right) d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} \left(1 - \cos 9\right) d\theta$$

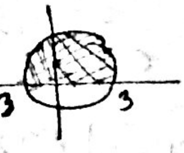
$$= \tfrac{1}{2}(1-\cos 9)\int_0^{2\pi} d\theta$$

$$= \tfrac{1}{2}(1-\cos 9)\left[\theta\right]_0^{2\pi}$$

$$= \tfrac{1}{2}(1-\cos 9)\, 2\pi$$

$$= \pi(1-\cos 9) \text{ Ans.}$$

$\iint \sqrt{9-x^2-y^2}\, dA$, where R is the region in the first quadrant within the circle $x^2+y^2=9$



solve;
$$\int_0^3 \int_{\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{9-x^2-y^2}\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^3 \sqrt{9-r^2}\; r\, dr\, d\theta$$

$u = 9 - r^2$
$\Rightarrow du = -2r\, dr$
$\Rightarrow -\frac{1}{2} du = r\, dr$
$r = 0,\ u = 9$
$r = 3,\ u = 0$

$$= -\frac{1}{2} \int_0^{\pi/2} \int_9^0 \sqrt{u}\, du\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \int_0^9 u^{1/2}\, du\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left[\frac{u^{3/2}}{3/2}\right]_0^9 d\theta$$

$y = \sqrt{9-x^2}$
$\Rightarrow x^2+y^2 = 9$
$\Rightarrow r = 3$

$$= \left(\frac{1}{2} \times \frac{2}{3}\right) \int_0^{\pi/2} \left(9^{3/2}\right) d\theta$$

$$= \frac{27}{3} \int_0^{\pi/2} d\theta$$

$$= 9\, [\theta]_0^{\pi/2}$$

$$= \frac{9\pi}{2} \text{ (Ans)}$$

$x = 3$
$\Rightarrow r\cos\theta = 3$
$\Rightarrow 3\cos\theta = 3$
$\therefore \theta = 0$

$x = 0$
$\Rightarrow r\cos\theta = 0$
$\therefore \theta = \pi/2$

$\iint_R \frac{1}{1+x^2+y^2}\, dA$, where R is the sector in the first quadrant bounded by $y=0$, $y=x$ and $x^2+y^2=4$

Solve:

$$\therefore \iint_R \frac{1}{1+x^2+y^2}\, dA = \int_0^2 \int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\, dx\, dy$$

$$= \int_0^{\pi/4} \int_0^2 \frac{1}{1+r^2}\, r\, dr\, d\theta$$

- $u = r^2+1$
- $\Rightarrow du = 2r\, dr$
- $\Rightarrow \frac{1}{2} du = r\, dr$
- $r=0,\ u=1$
- $r=2,\ u=5$

$$= \frac{1}{2} \int_0^{\pi/4} \left[\int_1^5 \frac{1}{u}\, du \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} \left[\ln u \right]_1^5\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} (\ln 5)\, d\theta$$

$$= \frac{1}{2} (\ln 5) \left[\theta \right]_0^{\pi/4} \cdot d\theta$$

$$= \frac{1}{2} \ln 5 \left[\pi/4 - 0 \right]$$

$$= \frac{\pi \ln 5}{8} \text{ (Ans)}$$

$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\,dx\,dy$; evaluate the iterated integral

Solve:

$$\therefore \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_{\pi/2}^{\pi/2}\int_{0}^{2} e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\int_{0}^{-4} -\tfrac{1}{2}\, e^{u}\,du\right] d\theta$$

$$= \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}\left[\int_{-4}^{0} e^{u}\,du\right] d\theta$$

$$= \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}\left[e^{u}\right]_{-4}^{0} d\theta$$

$$= \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}\left(e^{0}-e^{-4}\right) d\theta$$

$$= \tfrac{1}{2}\int_{-\pi/2}^{\pi/2}\left(1-e^{-4}\right) d\theta$$

$$= \tfrac{1}{2}\left(1-e^{-4}\right)\left[\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \tfrac{1}{2}\left(1-e^{-4}\right)\left(\tfrac{\pi}{2}+\tfrac{\pi}{2}\right)$$

$$= \tfrac{\pi}{2}\left(1-e^{4}\right) \text{ (Ans)}$$

$x = \sqrt{4-y^2}$

$\Rightarrow x^2+y^2 = 4$

$\Rightarrow r = \pm 2 \; [r \geq 0]$

$y = 2$

$\Rightarrow r\sin\theta = 2$

$\Rightarrow 2\sin\theta = 2$

$\therefore \theta = \pi/2$

$y = -2$

$\Rightarrow r\sin\theta = -2$

$\therefore \theta = -\pi/2$

$u = -r^2$

$\Rightarrow du = -2r\,dr$

$\Rightarrow -\tfrac{1}{2}\,du = r\,dr$

$r = 0,\; u = 0$

$r = 2,\; u = -4$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dy\,dx}{(1+x^2+y^2)^{3/2}}$$

Solve:

$$\therefore \int_0^{\pi/2}\int_0^a \frac{1}{(1+r^2)^{3/2}}\, r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\int_1^{a^2+1} \frac{1}{u^{3/2}}\,du\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[\frac{u^{-1/2}}{-1/2}\right]_1^{a^2+1} d\theta$$

$$= \frac{1}{2}(-2)\int_0^{\pi/2} \left\{(a^2+1)^{-1/2} - (1)^{-1/2}\right\} d\theta$$

$$= -\int_0^{\pi/2} \left\{(a^2+1)^{-1/2} - 1\right\} d\theta$$

$$= \left(1 - \frac{1}{\sqrt{a^2+1}}\right)\left[\theta\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}\left(1 - \frac{1}{\sqrt{a^2+1}}\right) \text{ (Ans)}$$

$y = \sqrt{a^2-x^2}$

$\Rightarrow x^2+y^2 = a^2$

$\Rightarrow r = a$

$x = a$

$\Rightarrow r\cos\theta = a$

$\Rightarrow a\cos\theta = a$

$\therefore \theta = 0$

$x = 0$

$\Rightarrow r\cos\theta = 0$

$\therefore \theta = \pi/2$

$u = 1+r^2$

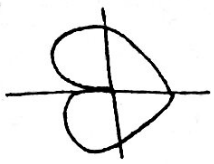
$\Rightarrow du = 2r\,dr$

$\therefore \frac{1}{2}du = r\,dr$

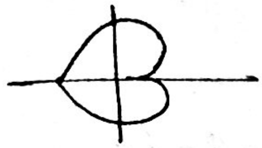
$r = 0,\ u = 1$

$r = a,\ u = 1+a^2$

cardioid:

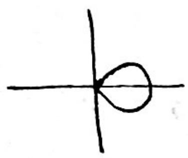


$r = 1 + a\cos\theta$



$r = 1 - a\sin\theta$

Rose petals:



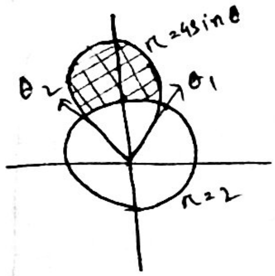
$r = \cos\theta$

$r = \sin\theta$

$r = a\sin n\theta$ | n = number of petals | a = radius

Find the area of the region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$

solve:



$r = 4\sin\theta$; $r = 2$

$\therefore 4\sin\theta = 2$

$\Rightarrow \sin\theta = \frac{1}{2}$

$\therefore \theta = 30$

$= \frac{\pi}{6}$

$\therefore \sin\theta = \sin(\pi - \frac{\pi}{6})$

$= \sin\frac{5\pi}{6}$

$$\text{Area}, A = \iint_R dA$$

$$= \int_{\pi/6}^{5\pi/6}\int_{2}^{4\sin\theta} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \left[\frac{r^2}{2}\right]_2^{4\sin\theta} d\theta$$

$$= \int_{\pi/6}^{5\pi/6} \frac{1}{2}(16\sin^2\theta - 4)\, d\theta$$

$$= 2\int_{\pi/6}^{5\pi/6} (4\sin^2\theta - 1)\, d\theta$$

$$= 2\int_{\pi/6}^{5\pi/6} \{2(1-\cos 2\theta) - 1\}\, d\theta \quad [\because 2\sin^2\theta = 1-\cos 2\theta]$$

$$= 2\int_{\pi/6}^{5\pi/6} (2 - 2\cos 2\theta - 1)\, d\theta$$

$$= 2\int_{\pi/6}^{5\pi/6} (1 - 2\cos 2\theta)\, d\theta$$

$$= 2 \times \left[\theta - 2\cdot\frac{1}{2}\sin 2\theta\right]_{\pi/6}^{5\pi/6}$$

$$= 2\left\{\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) - \left(\sin\frac{5\pi}{} - \sin\frac{\pi}{}\right)\right.$$

$$= 2\left[\frac{4\pi}{6} - \left\{\left(-\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}\right\}\right]$$

$$= 2\left(\frac{4\pi}{6} + \sqrt{3}\right)$$

$$= 2\left(\frac{2\pi}{3} + \sqrt{3}\right) \text{ (Ans)}$$

Tanji

08.07.17

MAT-216

Find the area of the region enclosed by the side of the circle $r = \sin\theta$ and outside of the cardioid $r = 1-\cos\theta$

Solve:

$$A = \iint_R dA$$

$$\sin\theta = 1-\cos\theta$$

$$\Rightarrow \sin^2\theta = (1-\cos\theta)^2$$

$$\Rightarrow 1-\cos^2\theta = 1-2\cos\theta+\cos^2\theta$$

$$\Rightarrow 2\cos\theta - 2\cos^2\theta = 0$$

$$\Rightarrow 2\cos^2\theta - 2\cos\theta = 0$$

$$\Rightarrow 2\cos\theta(\cos\theta - 1) = 0$$

$\therefore 2\cos\theta = 0 \Rightarrow \theta = \pi/2$ | $\cos\theta - 1 = 0 \Rightarrow \theta = 0$

$$= \int_0^{\pi/2}\int_{1-\cos\theta}^{\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{1-\cos\theta}^{\sin\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left\{\sin^2 - (1-\cos\theta)^2\right\} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left(1-\cos^2\theta - 1 + 2\cos\theta - \cos^2\theta\right) d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left(2\cos\theta - 2\cos^2\theta\right) d\theta$$

$$= \frac{1}{2}\left[\int_0^{\pi/2}(2\cos\theta)\,d\theta - \int_0^{\pi/2} 2\cos^2\theta\,d\theta\right]$$

$$= \frac{1}{2}\times 2\left[\sin\theta\right]_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta \;;\left[2\cos^2\theta = 1+\cos 2\theta\right]$$

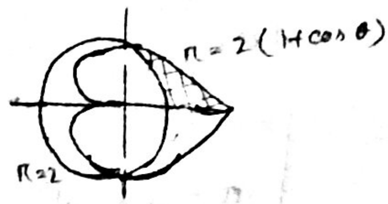
$$= 1 - \frac{1}{2}\left\{[\theta]_0^{\pi/2} + \frac{1}{2}\left[\sin 2\theta\right]_0^{\pi/2}\right\}$$

$$= 1 - \left[\frac{\pi}{4} + \frac{1}{2}\sin\left(2\times\frac{\pi}{2}\right)\right]$$

$$= 1 - \frac{\pi}{4} - 0 = \left(1 - \frac{\pi}{4}\right) \text{(Ans)}$$

$\iint_R \sin\theta \, dA$; where R is the region in the first quadrant that is outside the circle $r=2$ and inside the cardioid $r=2(1+\cos\theta)$

Solve:



$$2(1+\cos\theta)=2$$
$$\Rightarrow 1+\cos\theta = 1$$
$$\Rightarrow \cos\theta = 0$$
$$\therefore \theta = \pi/2$$

$$\therefore \iint_R \sin\theta \, dA = \int_0^{\pi/2}\int_2^{2(1+\cos\theta)} \sin\theta \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left[r^2\right]_2^{2(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left[4(1+\cos\theta)^2 - 4\right] \sin\theta \, d\theta$$

$$= \frac{4}{2}\int_0^{\pi/2} \left[(1+\cos\theta)^2 \sin\theta - \sin\theta\right] d\theta$$

$u = 1+\cos\theta$

$\Rightarrow du = -\sin\theta \, d\theta$

$\therefore -du = \sin\theta \, d\theta$

$\theta = \pi/2, u = 1$

$\theta = 0, u = 2$

$$= 2\left[\int_0^{\pi/2} (1+\cos\theta)^2 \sin\theta \, d\theta - \int_0^{\pi/2} \sin\theta\right] d\theta$$

$$= 2\left[-\int_2^1 u^2 \, du\right] + 2\left[\cos\theta\right]_0^{\pi/2}$$

$$= 2\int_1^2 u^2 \, du + 2\left(\cos\pi/2 - \cos 0\right)$$

$$= 2\left[\frac{u^3}{3}\right]_1^2 - 2$$

$$= \frac{2}{3}\left[2^3 - 1^3\right] - 2$$

$$= \frac{14}{3} - 2$$

$$= \frac{8}{3} \text{ (Ans)}$$

Tanjil

08.07.17

MAT-216

rectangular to polar

$x = r\cos\theta$, $y = r\sin\theta$ | $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

cylindrical coordinates,

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

rec → cylin

$r = \sqrt{x^2+y^2}$ $\quad r \geq 0$

$\theta = \tan^{-1}(y/x)$ $\quad 0 \leq \theta \leq 2\pi$

$z = z$ $\quad z \in (-\alpha, \alpha)$

$$\boxed{dz\,dy\,dx = r\,dz\,dr\,d\theta}$$

Use cylindrical co-ordinate to evaluate the integral:

$$\rightarrow \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2\,dz\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} r^2\cos^2\theta \cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3}\left[z\right]_0^{9-r^2} r^3\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3}(9-r^2)r^3\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\int_0^3 9r^3\,dr - \int_0^3 r^5\,dr\right].\cos^2\theta\,d\theta$$

$$= \int_0^{2\pi}\left\{9\left[\frac{r^4}{4}\right]_0^3 - \left[\frac{r^6}{6}\right]_0^3\right\}\cos^2\theta\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{729}{4} - \frac{243}{2}\right)\cos^2\theta\,d\theta$$

$z = 9 - x^2 - y^2$

$= 9 - r^2$

$y = \pm\sqrt{9-x^2}$

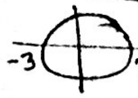
$\Rightarrow y^2 = 9 - x^2$

$\Rightarrow x^2 + y^2 = 9$

$\Rightarrow r^2 = 9$

$\therefore r = 3$

x



$\boxed{2\pi = \theta}$

$$= \int_0^{2\pi} \frac{243}{4}\left(\frac{1}{2}\cdot 2\cos^2\theta\right) d\theta$$

$$= \frac{243}{4} \int_0^{2\pi} \left\{\frac{1}{2}(1+\cos 2\theta)\right\} d\theta$$

$$= \frac{243}{8} \left\{ [\theta]_0^{2\pi} + \frac{1}{2}[\sin 2\theta]_0^{2\pi} \right\}$$

$$= \frac{243}{8}(2\pi + 0)$$

$$= \frac{243\pi}{4} \text{ (Ans)}$$

Tanjil
08.07.17
MAT-216

Use cylindrical co-ordinates to evaluate the integral:

$$\int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2 \, dz \, dx \, dy$$

$$= \int_0^{\pi/2} \int_0^2 \int_r^{\sqrt{8-r^2}} z^2 r \, dz \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^2 \left[\frac{z^3}{3}\right]_r^{\sqrt{8-r^2}} r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^2 \left\{ \frac{(8-r^2)^{3/2}}{3} - \frac{r^3}{3} \right\} r \, dr \, d\theta$$

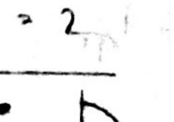
$$= \int_0^{\pi/2} \frac{1}{3} \left[\int_0^2 (8-r^2)^{3/2} r \, dr - \int_0^2 r^4 \, dr \right] d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \left\{ -\frac{1}{5}\left[(8-r^2)^{5/2}\right]_0^2 - \left[\frac{r^5}{5}\right]_0^2 \right\} d\theta$$

$z = \sqrt{8-x^2-y^2}$
$= \sqrt{8-(x^2+y^2)}$
$= \sqrt{8-r^2}$

$z = \sqrt{x^2+y^2}$
$= r$

$x = \sqrt{4-y^2}$
$\Rightarrow x^2+y^2 = 4$
$\Rightarrow r = 2$

$y =$ 

$\therefore \theta = \pi/2$

$u = 8 - r^2$
$\Rightarrow du = -\frac{1}{2} r \, dr$
$\Rightarrow -2\,du = r \, dr$
$\therefore -\frac{1}{2} \int u^{3/2} \, du$
$= -\frac{1}{2} \frac{u^{5/2}}{5/2}$
$= -\frac{1}{5} u^{5/2} = -\frac{1}{5}(8-r^2)^{5/2} + c$

$$= \frac{1}{3}\int_0^{\pi/2}\left\{2\left[\frac{u^{5/2}}{5/2}\right]_4^8 - \frac{32}{5}\right\}d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \quad \} d\theta$$

$$= \frac{1}{3} \times \frac{96}{}$$

$$= \frac{32}{5}$$

$$= \frac{1}{3} \quad - \frac{32}{5}$$

Tanjil

08.07.17

MAT-216

$$= -\frac{1}{15}\int_0^{\pi/2}\left\{(8-4)^{5/2} - 8^{5/2} + 2^5\right\}d\theta$$

$$= -\frac{1}{15}\int_0^{\pi/2}\left[64 - (2\sqrt{2})^5\right]d\theta$$

$$= -\frac{1}{15}\left[64\theta - (2\sqrt{2})^5\theta\right]_0^{\pi/2}$$

$$= -\frac{1}{15}\left\{64\cdot\frac{\pi}{2} - (2\sqrt{2})^5\cdot\frac{\pi}{2}\right\}$$

$$= -\frac{1}{15}\left\{64\frac{\pi}{2} - 32(\sqrt{2})^5\frac{\pi}{2}\right\}$$

$$= -\frac{1}{15}\times\frac{32\pi}{2}\left\{2 - (\sqrt{2})^5\right\}$$

$$= \frac{16\pi}{15}\left\{(\sqrt{2})^5 - 2\right\} \text{ (Ans)}$$

3) .

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{a^2-x^2-y^2} x^2 \, dz \, dy \, dx$$

$$= \int_0^{\pi/2} \int_0^a \int_0^{a^2-r^2} r^2 \cos^2\theta \, r \, dz \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^a \left[z\right]_0^{a^2-r^2} r^2 \cos^2\theta \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^a (a^2-r^2) \, r^2 \cos^2\theta \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^a (a^2 r^3 - r^5) \cos^2\theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left\{ \left[a^2 r^4/4\right]_0^a - \left[r^6/6\right]_0^a \right\} \cos^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \left(a^2/4 \cdot a^4 - a^6/6 \right) \cos^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \left[\left(\frac{a^6}{4} - \frac{a^6}{6} \right) \tfrac{1}{2} \cdot 2\cos^2\theta \right] d\theta$$

$$= a^6/12 \int_0^{\pi/2} \left\{ \tfrac{1}{2}(1+\cos 2\theta) \right\} d\theta$$

$$= a^6/24 \left\{ \left[\theta\right]_0^{\pi/2} + \tfrac{1}{2}\left[\sin 2\theta\right]_0^{\pi/2} \right\}$$

$$= a^6/24 \cdot \pi/2$$

$$= \pi a^6/48 \text{ (Ans)}$$

$z = a^2 - x^2 - y^2$

$= a^2 - r^2$

$y = \sqrt{a^2-x^2}$

$\Rightarrow y^2 = a^2 - x^2$

$\Rightarrow x^2 + y^2 = a^2$

$\Rightarrow r = a$

$x =$  $x = \pi/2$

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} e^{-(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx$$

$$= \int_{0}^{2\pi}\int_{0}^{1}\int_{0}^{\sqrt{1-r^2}} e^{-(r^2+z^2)^{3/2}}\,r\,dz\,dr\,d\theta$$

$z = \sqrt{1-x^2-y^2}$
$= \sqrt{1-r^2}$

$y = \sqrt{1-x^2}$

Find the value of the solid that is bounded by the cylinder $y = x^2$ and by the planes $y+z=4$ & $z=0$

Solve:

$$V = \iiint_G dz\,dy\,dx$$

$z = 4-y$
$z = 0$

$y = 4\ [z=0]$
$y = x^2$

$x^2 = 4\ [y=4]$
$\therefore x = \pm 2$

$$= \int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{4-y} dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{x^2}^{4} [z]_0^{4-y}\,dy\,dx$$

$$= \int_{-2}^{2}\int_{x^2}^{4} (4-y)\,dy\,dx$$

$$= \int_{-2}^{2} \left[4y - \frac{y^2}{2}\right]_{x^2}^{4} dx$$

$$= \int_{-2}^{2} \left\{4(4-x^2) - \left(\frac{16}{2} - \frac{x^4}{2}\right)\right\} dx$$

$$= \int_{-2}^{2} \left\{4(4-x^2) - \left(8 - \frac{x^4}{2}\right)\right\} dx$$

$$= \int_{-2}^{2} \left(16 - 4x^2 - 8 + \frac{x^4}{2}\right) dx$$

$$= \int_{-2}^{2} \left(8 - 4x^2 + \frac{x^4}{2}\right) dx$$

$$= 8[x]_{-2}^{2} - 4\left[\frac{x^3}{3}\right]_{-2}^{2} + \frac{1}{2}\left[\frac{x^5}{5}\right]_{-2}^{2}$$

$$= 32 - \frac{64}{3} + \frac{32}{5}$$

$$= \frac{256}{15}\ \text{(Ans)}$$

Find the volume of the solid enclosed between the paraboloids $z = 5x^2 + 5y^2$ & $z = 6 - 7x^2 - y^2$

Solve:

$$V = \iiint_G dv$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} dz\, dy\, dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} [z]_{5x^2+5y^2}^{6-7x^2-y^2} dy\, dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (6-7x^2-y^2-5x^2-5y^2)\, dy\, dx$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (6-12x^2-6y^2)\, dy\, dx$$

$$5x^2+5y^2 = 6-7x^2-y^2$$

$$\Rightarrow 12x^2+6y^2 = 6$$

$$\Rightarrow 2x^2+y^2 = 1$$

$$\therefore y = \pm\sqrt{1-2x^2}$$

$$1-2x^2 \geq 0$$

$$\Rightarrow -2x^2 \geq 1$$

$$\Rightarrow -2x^2 \leq -1$$

$$\Rightarrow 2x^2 \leq 1$$

$$\Rightarrow x \leq 1/\sqrt{2}$$

$$\therefore -1/\sqrt{2} \leq x \leq 1/\sqrt{2}$$

$$= 6 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (1-2x^2-y^2)\, dy\, dx$$

$$= 6 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[y - 2x^2 y - y^3/3\right]_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= 6 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[(1-2x^2)\left(\sqrt{1-2x^2}+\sqrt{1-2x^2}\right) - \tfrac{1}{3}\left\{\left(\sqrt{1-2x^2}\right)^3 + \left(\sqrt{1-2x^2}\right)^3\right\}\right] dx$$

$$= 6 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left\{(1-2x^2)\left(2\sqrt{1-2x^2}\right) - \tfrac{1}{3}\, 2\left(\sqrt{1-2x^2}\right)^3\right\} dx$$

$$= 6 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left\{2(1-2x^2)(1-2x^2)^{1/2} - \tfrac{2}{3}(1-2x^2)^{3/2}\right\} dx$$

$$= 6\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left\{2\,(1-2x^2)^{3/2} - \tfrac{2}{3}\,(1-2x^2)^{3/2}\right\} dx$$

$$= 6\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \tfrac{4}{3}\,(1-2x^2)^{3/2}\, dx$$

$$= 8\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\, dx$$

trigonometric substitution:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = 1 - \sin^2\theta$$

$$x = \tfrac{1}{\sqrt{2}}\sin\theta$$

$$\Rightarrow dx = \tfrac{1}{\sqrt{2}}\cos\theta\, d\theta$$

$$\therefore x = 1/\sqrt{2},\ \theta = \pi/2$$

$$\therefore x = -1/\sqrt{2},\ \theta = -\pi/2$$

$$= 8\int_{-\pi/2}^{\pi/2} \left\{1 - 2\left(\tfrac{1}{\sqrt{2}}\sin\theta\right)^2\right\}^{3/2} \tfrac{1}{\sqrt{2}}\cos\theta\, d\theta$$

$$= \tfrac{8}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} (1-\sin^2\theta)^{3/2}\cos\theta\, d\theta$$

$$= 8/\sqrt{2}\int_{-\pi/2}^{\pi/2} (\cos^2\theta)^{3/2}\cos\theta\, d\theta$$

$$= 8/\sqrt{2}\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta \quad \ldots\ldots (i)$$

$$\therefore \int \cos^4\theta\, d\theta$$

use $\cos^3 A = 1/4(\cos 3A + 3\cos A)$

$$= \int \cos^3\theta \,.\, \cos\theta\, d\theta$$

$$= \cos^3\theta \int \cos\theta\, d\theta - \int \left(\tfrac{d}{d\theta}\cos^3\theta \int \cos\theta\, d\theta\right) d\theta \; ; \left[\int uv\, dx = u\int v\, dx - \int\left[\tfrac{d}{dx}(u)\int v\, dx\right] dx\right]$$

$$= \cos^3\theta\, \sin\theta - \int \left[\left\{\tfrac{d}{du}(u^3)\right\}\left(\tfrac{d}{d\theta}\cos\theta\right)\sin\theta\right] d\theta$$

$$= \cos^3\theta\, \sin\theta + \int (3u^2 \sin\theta \,.\, \sin\theta)\, d\theta$$

$$= \cos^3\theta\, \sin\theta + 3\left[\int \cos^2\theta\, \sin^2\theta\, d\theta\right] \; ; \left[\cos^2\theta\, \sin^2\theta = \frac{1-\cos(4\theta)}{8}\right]$$

$$= \cos^3\theta\, \sin\theta + 3\int \left(\frac{1-\cos 4\theta}{8}\right) d\theta$$

$$= \cos^3\theta\, \sin\theta + \tfrac{3}{8}\left(\theta - \tfrac{1}{4}\sin 4\theta\right)$$

From (i),

$$\therefore \; 8/\sqrt{2} \left[\cos^3\theta \sin\theta + 3/8 \left(\theta - 1/4 \sin 4\theta\right)\right]_{-\pi/2}^{\pi/2}$$

$$= 8/\sqrt{2} \left[0 + 3/8 \left(\pi/2 + \pi/2\right) - 1/4 \left\{ \sin 4(\pi/2) - \sin 4(-\pi/2) \right\} \right]$$

$$= 8/\sqrt{2} \left[3\pi/8 - 1/4 \, (0 - 0) \right]$$

$$= 3\pi/\sqrt{2} \text{ (Ans)}$$

1) Use the triple integration in cylindrical coordinates to find to find the volume of the solid G that is bounded by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane, and laterally by the the cylinder $x^2 + y^2 = 9$.

Solve:

$$\iiint_G dv = \iint_R \left[\int_0^{\sqrt{25-r^2}} dz \right] dA$$

$$= \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{25-r^2}} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^3 r \, [z]_0^{\sqrt{25-r^2}} \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^3 r \left(\sqrt{25-r^2}\right) dr \, d\theta$$

$$= \int_0^{2\pi} \int_{25}^{16} \left(-1/2 \, u^{1/2}\right) du \, d\theta$$

$$= 1/2 \int_0^{2\pi} \left[\int_{16}^{25} u^{1/2} \, du \right] d\theta$$

$z = \sqrt{25 - x^2 - y^2}$

$= \sqrt{25 - r^2}$

$x^2 + y^2 = 9$

$u = 25 - r^2$

$\Rightarrow du = -2r \, dr$

$\Rightarrow -1/2 \, du = r \, dr$

$r = 3, \; u = 16$

$r = 0, \; u = 25$

$$= \frac{1}{2} \int_0^{2\pi} \left[\frac{u^{3/2}}{3/2} \right]_{16}^{25} d\theta$$

$$= \left(\frac{1}{2} \times \frac{2}{3} \right) \int_0^{2\pi} \left(25^{3/2} - 16^{3/2} \right) d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} 61 \, d\theta$$

$$= \frac{61}{3} \left[\theta \right]_0^{2\pi}$$

$$= \frac{122\pi}{3} \text{ (Ans)}$$

Tanjil
08.07.17
MAT-216

Use the cylindrical coordinate to evaluate, $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x^2 \, dz \, dy \, dx$

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x^2 \, dz \, dy \, dx$$

$$= \int_0^{2\pi} \int \int_0^{9-r^2} r^2 \cos^2\theta \, r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int (r^3 \cos^2\theta) \left[z \right]_0^{9-r^2} dr \, d\theta$$

$$= \int_0^{2\pi} \left[\int \left\{ \cos^2\theta \, (9r^3 - r^5) \right\} dr \right] d\theta$$

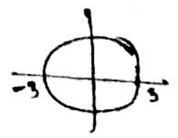
$$= \int_0^{2\pi} \cos^2\theta \left[\frac{9r^4}{4} - \frac{r^6}{6} \right]_0^3$$

$$= \int_0^{2\pi} \cos^2\theta \left(9 \cdot \frac{3^4}{4} - \frac{3^6}{6} \right) d\theta$$

$$= \frac{243}{4} \left[\int_0^{2\pi} \frac{1}{2} (1 + \cos 2\theta) \, d\theta \right]$$

$z = 9 - x^2 - y^2$
$= 9 - r^2$

$y = \sqrt{9 - x^2}$
$\Rightarrow y^2 = 9 - x^2$
$\Rightarrow r = 3$



$x = 2\pi$

$$= \frac{243}{4} \times \frac{1}{2} \left\{ \left[\theta\right]_0^{2\pi} + \left[\frac{1}{2}\sin 2\theta\right]_0^{2\pi} \right\}$$

$$= \frac{243}{8} \times (2\pi + 0)$$

$$= \frac{243\pi}{4} \quad \text{(Ans)}$$

Tanjil
08.07.17
MAT-216

Spherical coordinates:

(Rec to Sphe)

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

(Sphe - rect)

$r = \sqrt{x^2+y^2+z^2}$; $r \geq 0$

$\theta = \tan^{-1}(y/x)$; $0 \leq \theta \leq 2\pi$

$\phi = \cos^{-1}\left(\frac{z}{x^2+y^2+z^2}\right)$; $z \in (-\alpha, \alpha)$

Jacobian in two variables: If T is the transformation from the uv plane to the xy-plane defined by the equation $x = x(u,v)$, $y = y(u,)$, then the jacobian of T is donated by $J(u,v)$ or $\partial(x,y)/\partial(u,v)$ and as defined by $J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$

Change of variable in double integrals: If the transformation $x = x(u,v)$, $y = y(u,v)$ maps the region S in the uv-plane in the region R in the xy-plane and if the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is non-zero then,

$$\iint_R f(x,y)\, dA_{xy} = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$\iint_R \frac{x-y}{x+y}\, dA$, where R is the region enclosed by the lines $x-y=0$, $x-y=1$, $x+y=1$ & $x+y=3$, using the transformation.

Solve: Let, $u = x-y$ ....(i)

$v = x+y$ .....(ii)

(i) + (ii),

$2x = u+v$

$\Rightarrow x = \frac{u+v}{2}$

(ii) − (i),

$2y = v-u$

$\Rightarrow y = \frac{v-u}{2}$

$$\therefore J(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = 1/4 + 1/4 = 1/2$$

Now the given integral becomes,

$$\iint_R \frac{x-y}{x+y}\, dA = \int_1^3 \int_0^1 u/v \, |J(u,v)| \, du\, dv$$

$$= \int_1^3 \int_0^1 u/v \, (1/2)\, du\, dv$$

$$= 1/2 \int_1^3 1/v \left[\int_0^1 u\, du\right] dv$$

$$= 1/2 \int_1^3 1/v \left[\frac{u^2}{2}\right]_0^1 dv$$

$$= 1/4 \int_1^3 1/v \, dv$$

$$= 1/4 \left[\ln v\right]_1^3$$

$$= 1/4 \left[\ln 3 - \ln 1\right] = 1/4 \ln(3) \text{ (Ans)}$$

Tanjil
08.07.17
MAT-216

$\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy$ by using transformation where, $u = \frac{2x-y}{2}$ & $v = y/2$ and integration over an appropriate region in uv-plane.

Solve: $x = y/2 + 1$ | $x = y/2$

$\therefore u = \frac{2x-y}{2}$

$\Rightarrow 2u = 2x - y$

$\Rightarrow 2u = 2x - 2v$

$\Rightarrow x = u + v$

| $v = y/2$

$\Rightarrow y = 2v$

For x:

$y/2 + 1 = u + v$

$\Rightarrow 2v/2 + 1 = u + v$

$\Rightarrow v + 1 = u + v$

$\Rightarrow u = 1$

| $y/2 = u + v$

$\Rightarrow 2v/2 = u + v$

$\Rightarrow v = u + v$

$\therefore u = 0$

For y:

$0 = 2v$

$\therefore v = 0$

| $4 = 2v$

$\therefore v = 2$

$$\therefore J(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

$\therefore$ Now the given integral becomes,

$$\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy = \int_0^2 \int_0^1 u\,|J(u,v)|\,du\,dv$$

$$= \int_0^2 \int_0^1 2u\,du\,dv$$

$$= 2\int_0^2 \left[\frac{u^2}{2}\right]_0^1 dv$$

$$= \int_0^2 dv$$

$$= [v]_0^2$$

$$= 2 - 0$$

$$= 2 \text{ (Ans)}$$

Evaluate $\iint_R e^{xy}\,dA$, where R is the region enclosed by the lines $y = \frac{1}{2}x$, $y = x$, $y = 1/x$, $y = 2/x$

Solve: $y = \frac{1}{2}x \Rightarrow y/x = \frac{1}{2}$ | $y = x \therefore y/x = 1$ | $y = 1/x \Rightarrow xy = 1$ | $y = 2/x \Rightarrow xy = 2$

Let us consider,

$u = y/x$ | $v = xy$

$\therefore u = \frac{1}{2}, \; u = 1$ | $v = 1, \; v = 2$

$\therefore u = y/x$

$\Rightarrow u = y \Big/ \frac{v}{y} \quad \left[x = \frac{v}{y}\right]$

$\Rightarrow u = y^2/v$

$\Rightarrow y^2 = vu$

$\therefore y = \sqrt{uv}$

$\therefore v = xy$

$\Rightarrow v = x\sqrt{uv}$

$\Rightarrow x = v\,(uv)^{-1/2}$

$\Rightarrow x = v^{1-1/2} \cdot u^{-1/2}$

$\therefore x = \sqrt{v/u}.$

$$\therefore J(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{1}{2u}\sqrt{v/u} & \frac{1}{2}\cdot\frac{1}{\sqrt{uv}} \\ \frac{1}{2}\cdot\sqrt{v/u} & \frac{1}{2}\sqrt{u/v} \end{vmatrix}$$

$$= -\frac{1}{4u} - \frac{1}{4u} = -\frac{1}{2u}$$

$\frac{\partial}{\partial u}(x) = \frac{\partial}{\partial u}\left(\sqrt{v/u}\right)$

$= \frac{\partial}{\partial p}\sqrt{p}\cdot\frac{\partial}{\partial u}\left(\frac{v}{u}\right)$

$= \frac{1}{2\sqrt{p}}\,(-1)\,\frac{v}{u^2}$

$= -\frac{1}{2\sqrt{v/u}}\cdot\frac{v}{u^2}$

$= -\frac{1}{2u}\sqrt{\frac{v}{u}}$

$$\frac{\partial}{\partial v}(x) = \frac{\partial}{\partial v}\sqrt{v/u}$$

$$= \frac{\partial}{\partial p}(\sqrt{p}) \quad \frac{\partial}{\partial v}(v/u)$$

$$= \frac{1}{2\sqrt{p}} \cdot \frac{1}{u}$$

$$= \frac{1}{2\sqrt{v/u}} \cdot \frac{1}{u}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{uv}}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(\sqrt{uv})$$

$$= \frac{\partial}{\partial p}(\sqrt{p}) \cdot \frac{\partial}{\partial u}(uv)$$

$$= \frac{1}{2\sqrt{p}} \cdot v$$

$$= \frac{v}{2\sqrt{uv}}$$

$$= \frac{1}{2} \cdot \sqrt{v/u}$$

$$\frac{\partial y}{\partial v} = \frac{1}{2}\sqrt{u/v}$$

$\therefore$ Now the given integral becomes,

$$\therefore \iint_R e^{xy}\, dA = \int_1^2 \int_{1/2}^1 \left|-\frac{1}{2u}\right| e^v \, du\, dv$$

$$xy = \sqrt{v/u} \cdot \sqrt{uv}$$

$$= \frac{v^{1/2}}{u^{1/2}} \cdot u^{1/2} \cdot v^{1/2}$$

$$= v$$

$$= \frac{1}{2}\int_1^2 e^v \left[\int_{1/2}^1 \frac{1}{u}\, du\right] dv$$

$$= \frac{1}{2} \int_1^2 e^v \left[\ln u\right]_{1/2}^1 dv$$

$$= \frac{1}{2}\left\{\ln(1) - \ln(1/2)\right\} \int_1^2 e^v\, dv$$

$$= \frac{1}{2}\left(-\ln \frac{1}{2}\right)\left[e^v\right]_1^2$$

$$= -\frac{1}{2}\left(\ln 1 - \ln 2\right)\left(e^2 - e^1\right)$$

$$= \frac{\ln 2}{2}\left(e^2 - e\right) \text{ (Ans)}$$

4) Use the transformation $u = x - 2y$, $v = 2x + y$ to find $\iint_R \frac{x-2y}{2x+y}\, dA$ where R is the rectangular region enclosed by the lines $x-2y=1$, $x-2y=4$, $2x+y=1$, $2x+y=3$

Solve:

$u = 1$ & $u = 4$ | $v = 1$, $v = 3$

$u = x - 2y$

$\Rightarrow u = x - 2(v - 2x) \quad [y = v - 2x]$

$\Rightarrow u = x - 2v + 4x$

$\Rightarrow 5x = u + 2v$

$\therefore x = \frac{u+2v}{5}$

$\therefore y = v - 2x$

$\Rightarrow y = v - \frac{2u+4v}{5}$

$= \frac{5v - 2u - 4v}{5}$

$= \frac{v - 2u}{5}$

$\frac{x-2y}{2x+y}$

$= \dfrac{\frac{u+2v}{5} - \frac{2v-4u}{5}}{\frac{2u+4v}{5} + \frac{v-2u}{5}}$

$= \dfrac{\frac{u+2v-2v+4u}{5}}{\frac{2u+4v+v-2u}{5}}$

$= \frac{u}{v}$

$\therefore J(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$

$= \begin{vmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{vmatrix}$

$= 1/25 + 4/25$

$= 1/5$

Tanji
08.07.17
MAT-216

Now the given integral becomes,

$$\iint_R \frac{x-2y}{2x+y}\, dA = \int_1^3 \int_1^4 (1/5)\, u/v \; du\, dv$$

$$= 1/5 \int_1^3 1/v \left[u^2/2\right]_1^4 dv$$

$$= 1/10 \int_1^3 1/v \,(16-1)\, dv$$

$$= 15/10 \left[\ln v\right]_1^3$$

$$= 3/2 \left\{\ln(3) - \ln(1)\right\}$$

$$= 3/2 \cdot \ln(3) \text{ (Ans)}$$

Tanjil
08.07.17
MAT-216

Use the transformation $u = x+y$, $v = x-y$ to find $\iint_R (x-y)\, e^{x^2-y^2} dA$ over the rectangular region R enclosed by the lines $x+y=0$, $x+y=1$, $x-y=1$, $x-y=4$

Solve: $u = 0$, $u = 1$ & $v = 1$, $v = 4$

$u = x + y$

$\Rightarrow u = x + x - v$ $[y = x - v]$

$\Rightarrow x = \dfrac{u+v}{2}$

$y = x - v$

$\Rightarrow y = \dfrac{u+v}{2} - v$

$\Rightarrow y = \dfrac{u-v}{2}$

$$\therefore J(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/4 - 1/4$$

$$= -1/2$$

$$\therefore x^2 - y^2 = \left(\frac{u+v}{2}\right)^2 - \left(\frac{u-v}{2}\right)^2$$

$$= \frac{1}{4}\left\{u^2 + 2uv + v^2 - u^2 + 2uv - v^2\right\}$$

$$= \frac{1}{4} \cdot 4uv$$

$$= uv$$

$$(x-y) = \frac{u+v}{2} - \frac{u-v}{2}$$

$$= \frac{u+v-u+v}{2}$$

$$= 2v/2$$

$$= v$$

Now the given integral becomes,

$$\iint_R (x-y)\, e^{x^2-y^2}\, dA$$

$$= \int_1^4 \int_0^1 |-1/2|\, e^{uv} \cdot v\, du\, dv$$

$$= 1/2 \int_1^4 \left[e^{uv}\right]_0^1 dv$$

$$= 1/2 \int_1^4 (e^v - 1)\, dv$$

$$= 1/2 \left\{\left[e^v\right]_1^4 - \left[v\right]_1^4\right\}$$

$$= 1/2 \left\{(e^4 - e^1) - (4-1)\right\}$$

$$= 1/2 (e^4 - e - 3) \text{ (Ans)}$$

$p = uv$

$dp = v\, du$

$$\int v\, e^{uv}\, du$$

$$= \int e^p\, dp$$

$$= e^p + c$$

$$= e^{uv} + c$$

Tanjil
08.07.17
MAT-216

Use the transformation $u = y/x$, $v = xy$ to find $\iint_R xy^3\,dA$ over the region R in the first quadrant enclosed by $y = x$, $y = 3x$, $xy = 1$, $xy = 4$.

Solve:

$u = 1$, $u = 3$ | $v = 1$, $v = 4$

$u = y/x$

$\Rightarrow u = \dfrac{v/x}{x} \quad [y = v/x]$

$\Rightarrow u = v/x^2$

$\therefore x = \sqrt{v/u}$

$\therefore y = v/x$

$\Rightarrow y = \dfrac{v}{\sqrt{v/u}}$

$\Rightarrow y = \dfrac{v}{v^{1/2}/u^{1/2}}$

$\Rightarrow y = v \times \dfrac{u^{1/2}}{v^{1/2}} = \sqrt{vu}$

$$\therefore J(u,v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{1}{2u}\sqrt{v/u} & \frac{1}{2}\cdot\frac{1}{\sqrt{uv}} \\ \frac{1}{2}\sqrt{v/u} & \frac{1}{2}\sqrt{u/v} \end{vmatrix}$$

$$= -\frac{1}{2u}$$

$\dfrac{\partial x}{\partial u} = \dfrac{\partial}{\partial u}\sqrt{v/u}$

$= \dfrac{\partial}{\partial p}(\sqrt{p})\ \dfrac{\partial}{\partial u} v/u$

$= \dfrac{-1}{2\sqrt{p}} \cdot v/u^2$

$= -\dfrac{1}{2\sqrt{v/u}} \cdot \dfrac{v}{u^2}$

$= -\dfrac{1}{2u} \cdot \sqrt{\dfrac{v}{u}}$

$\therefore xy^3 = \sqrt{v/u}\,.\left(\sqrt{vu}\right)^3$

$= \sqrt{v/u}\ .\ (vu)^{3/2}$

$= v^{1/2}.u^{-1/2}.\ v^{3/2}.u^{3/2}$

$= v^2 u$

Now the integral becomes,

$$\iint_R xy^3\, dA = \int_1^4 \int_1^3 v^2 u \left| -\frac{1}{2u} \right| du\, dv$$

$$= \frac{1}{2} \int_1^4 v^2 \left[u \right]_1^3 dv$$

$$= \frac{1}{2} [3-1] \int_1^4 v^2 dv.$$

$$= \left[\frac{v^3}{3} \right]_1^4$$

$$= \frac{1}{3} \left[4^3 - 1^3 \right]$$

$$= 21 \text{ (Ans)}$$

Tanjil
08.07.17
MAT-216

Line integral

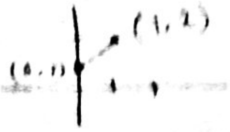
Evaluate $\int_C [(x^2-y)\,dx + (y^2+x)\,dy]$

(i) along the straight line C from $(0,1)$ to $(1,2)$

(ii) along the straight line C from $(0,1)$ to $(1,1)$ and then from $(1,1)$ to $(1,2)$

(iii) along the parabola $C: x = t,\ y = t^2+1$ from $(0,1)$ to $(1,2)$

(ii) Solve:



$(x_1, y_1) = (0,1)$

$(x_2, y_2) = (1,2)$

$$\therefore \quad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow y - y_1 = \frac{2-1}{1-0} \times (x - 0)$$

$$\Rightarrow y = x + y_1$$

$$\Rightarrow y = x + 1$$

$$\therefore \quad dy = dx$$

$$\therefore \int_c \left[(x^2 - y)\,dx + (y^2 + x)\,dy\right]$$

$$= \int_0^1 \left[\{x^2 - (x+1)\}dx + \{(x+1)^2 + x\}dx\right]$$

$$= \int_0^1 (x^2 - x - 1 + x^2 + 2x + 1 + x)\,dx$$

$$= \int_0^1 (2x^2 + 2x)\,dx$$

$$= 2\int_0^1 (x^2 + x)\,dx$$

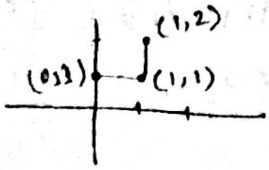
$$= 2\left\{\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1\right\}$$

$$= 2\left(\frac{1}{3} + \frac{1}{2}\right)$$

$$= 5/3 \text{ (Ans)}$$

Tanjil
08.03.17
MAT-216

(ii).



C_1: $y = 1 \Rightarrow dy = 0$ | $x = (0, 1)$

$$\therefore \int_{C_1} \left[(x^2-y)\,dx + (y^2+x)\,dy\right] = \int_0^1 \left[(x^2-)\,dx + (y^2+x)\cdot 0\right]$$

$$= \int_0^1 (x^2-1)\,dx$$

$$= \left[\frac{x^3}{3} - x\right]_0^1$$

$$= \left(\frac{1}{3} - 1\right)$$

$$= -\frac{2}{3}$$

For C_2: $x = 1$, $dx = 0$ | $y = (1, 2)$

$$\therefore \int_{C_2} \left[(x^2-y)\,dx + (y^2+x)\,dy\right] = \int_1^2 (y^2+1)\,dy \; ; [dx = 0]$$

$$= \left[\frac{y^3}{3} + y\right]_1^2$$

$$= \left(\frac{8}{3} - \frac{1}{3}\right) + (2-1)$$

$$= \frac{10}{3}$$

$$\therefore \int_{C_1} \left[(x^2-y)\,dx + (y^2+x)\,dy\right] + \int_{C_2} \left[(x^2-y)\,dx + (y^2+x)\,dy\right]$$

$$= -\frac{2}{3} + \frac{10}{3}$$

$$= \frac{8}{3} \text{ (Ans)}$$

(iii) $x = t$ | $y = t^2+1$

$\Rightarrow dx = dt$ | $\Rightarrow dy = 2t\,dt$

$\therefore (0,1) \quad x = 0, \quad t = 0 \quad ; [x = t]$

$\therefore (1,2) \quad x = 1, \quad t = 1 \quad ; [x = t]$

$$\therefore \int_c [(x^2-y)\,dx + (y^2+x)\,dy]$$

$$= \int_0^1 \left[(t^2-t^2-1)\,dt + \{(t^2+1)^2+t\}\,2t\,dt\right]$$

$$= \int_0^1 (-1)\,dt + \int_0^1 \{(t^4+2t^2+1+t)\,2t\,dt\}$$

$$= -[t]_0^1 + \int_0^1 (2t^5+4t^3+2t+2t^2)\,dt$$

$$= -1 + \left\{2\left[\frac{t^6}{6}\right]_0^1\right\} + \left\{4\left[\frac{t^4}{4}\right]_0^1\right\} + \left\{2\left[\frac{t^2}{2}\right]_0^1\right\} + \left\{2\left[\frac{t^3}{3}\right]_0^1\right\}$$

$$= -1 + \frac{1}{3} + 1 + 1 + \frac{2}{3}$$

$$= 2 \text{ (Ans)}$$

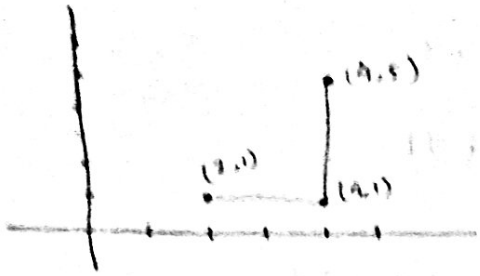
Evaluate $\int_c (xy\,dx + x^2\,dy)$ if

(a) C consists of line segments from $(2,1)$ to $(4,1)$ and from $(4,1)$ to $(4,5)$

(b) C is the line segment from $(2,1)$ and $(4,5)$

(c) Parametric equation for C are $x = 3t-1$, $y = 3t^2-2t$; $1 \le t \le 5/3$

(A)



C_1:

$y = 1$

$\Rightarrow dy = 0 \quad [x = (2,4)]$

$$\int_{C_1} (xy\,dx + x^2 dy) = \int_2^4 x\,dx \;;\; [dy = 0]$$

$$= \left[\frac{x^2}{2}\right]_2^4$$

$$= \frac{4^2}{2} - \frac{2^2}{2}$$

$$= 6$$

C_2:

$x = 4$

$\Rightarrow dx = 0 \quad [y = (1,5)]$

$$\int_{C_2} (xy\,dx + x^2 dy) = \int_1^5 (4)^2\,dy$$

$$= 16 \int_1^5 dy$$

$$= 16\,[y]_1^5$$

$$= 64$$

$$\therefore \int_{C_1} (xy\,dx + x^2 dy) + \int_{C_2} (xy\,dx + x^2 dy)$$

$$= 6 + 64$$

$$= 70 \text{ (Ans)}$$

Tanji
08.07.17
MAT-216

(b) $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1} \times (x - x_1)$ | $(x_1, y_1) = (2, 1)$

$(x_2, y_2) = (4, 5)$

$\Rightarrow y - 1 = \dfrac{5 - 1}{4 - 2} (x - 2)$

$\Rightarrow y - 1 = 2x - 4$

$\Rightarrow y = 2x - 3$

$\Rightarrow dy = 2\,dx$

$\therefore \int_C xy\,dx + x^2 dy = \int_2^4 \left[\{x(2x-3)\}dx + 2x^2 dx\right]$

$= \int_2^4 \{(2x^2 - 3x)\,dx + 2x^2 dx\}$

$= 2\left[\frac{x^3}{3}\right]_2^4 - 3\left[\frac{x^2}{2}\right]_2^4 + 2\left[\frac{x^3}{3}\right]_2^4$

$= \dfrac{112}{3} + \dfrac{112}{3} - 18$

$= 170/3$ (Ans)

(c) $x = 3t - 1$ | $y = 3t^2 - 2t$

$\Rightarrow dx = 3\,dt$ | $\Rightarrow dy = (6t - 2)\,dt$

$\therefore \int_C xy\,dx + x^2 dy$

$= \int_1^{5/3} \left[\{(3t-1)(3t^2-2t)\,3\,dt\} + (3t-1)^2 (6t-2)\,dt\right]$

$= \int_1^{5/3} \left[\{(9t^3 - 3t^2 - 6t^2 + 2t)\,3\,dt\} + \{(9t^2 - 6t + 1)(6t-2)\,dt\}\right]$

$$= \int_1^{5/3} \left[\{(9t^3 - 9t^2 + 2t)\, 3\,dt\} + (54t^3 - 36t^2 + 6t - 18t^2 + 12t - 2)\,dt \right]$$

$$= \int_1^{5/3} (27t^3 - 27t^2 + 6t + 54t^3 - 54t^2 + 18t - 2)\,dt$$

$$= \int_1^{5/3} (81t^3 - 81t^2 + 24t - 2)\,dt$$

$$= 81\left[\frac{t^4}{4}\right]_1^{5/3} - 81\left[\frac{t^3}{3}\right]_1^{5/3} + 24\left[\frac{t^2}{2}\right]_1^{5/3} - 2\left[t\right]_1^{5/3}$$

$$= 136 - 98 + \frac{64}{3} - \frac{4}{3}$$

$$= 58 \text{ (Ans)}$$

Independent path

The line integral $\int_c (p\,dx + Q\,dy)$ is independent of the path c joing any two points in a region R is that in R,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Show that (a) $\int_C [(6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy]$ is independent of the path joining the points (1,2) to (3,2) and (3,2) to (3,4) (b) hence evaluate the integral

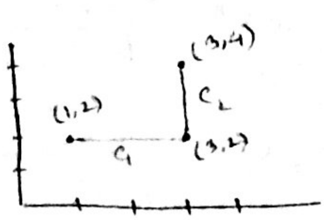
Solve:

(a) $P = 6xy^2 - y^3$ | $Q = 6x^2y - 3xy^2$

$\partial P/\partial y = 12xy - 3y^2$ | $\partial Q/\partial x = 12xy - 3y^2$

$\therefore \partial P/\partial y = \partial Q/\partial x$; So the line integral is independent of path.

(b)



C_1: $y = 2$

$\Rightarrow dy = 0$ $[x = (1,3)]$

$\int_{C_1} [(6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy]$

$= \int_1^3 (6x.4 - 8)\,dx$

$= 24\left[\frac{x^2}{2}\right]_1^3 - 8[x]_1^3$

$= 12(3^2 - 1) - 8(3-1)$

$= 96 - 16$

$= 80$

C_2: $x = 3$

$\Rightarrow dx = 0$ $[y = (2,4)]$

$\int_{C_2} [(6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy]$

$= \int_2^4 (6.9y - 3.3.y^2)\,dy$

$= 54\left[\frac{y^2}{2}\right]_2^4 - 9\left[\frac{y^3}{3}\right]_2^4$

$= \frac{54}{2}(4^2 - 2^2) - \frac{9}{3}(4^3 - 2^3)$

$= 156$

$\therefore \int_C [(6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy]$

$= 80 + 156$

$= 236$ (Ans)

Show that (a) $\int \{(6x^2y - 3xy^2)\,dy + (6xy^2 - y^3)\,dx\}$ is independent of the joining the points (1,2) and (3,4), (b) hence evaluate the integral.

(a) $Q = 6x^2y - 3xy^2$

$\partial Q/\partial x = 12xy - 3y^2$

$P = 6xy^2 - y^3$

$\partial P/\partial y = 12xy - 3y^2$

$\therefore \partial Q/\partial x = \partial P/\partial y$; independent of path

(b) $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$(x_1, y_1) = (1,2)$

$(x_2, y_2) = (3,4)$

$\Rightarrow y - 2 = \dfrac{4-2}{3-1}(x-1)$

$\Rightarrow y - 2 = x - 1$

$\Rightarrow y = x + 1$

$\Rightarrow dy = dx \quad [x = (1,3)]$

$$\int [(6x^2y - 3xy^2)\,dy + (6xy^2 - y^3)\,dx]$$

$$= \int_1^3 \left[\{6x^2(x+1) - 3x(x+1)^2\}\,dx + \{6x(x+1)^2 - (x+1)^3\}\,dx\right]$$

$$= \int_1^3 (6x^3 + 6x^2 - 3x^3 - 6x^2 - 3x + 6x^3 + 12x^2 + 6x - x^3 - 3x^2 - 3x - 1)\,dx$$

$$= \int_1^3 (8x^3 + 9x^2 - 1)\,dx$$

$$= 8\left[\frac{x^4}{4}\right]_1^3 + 9\left[\frac{x^3}{3}\right]_1^3 - [x]_1^3$$

$= 160 + 78 - 2$

$= 236$ (Ans)

Tanji 08.07.17 MAT-216

Fourier Series

Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. Then the fourier series or fourier expansion corresponding to $f(x)$ is defined as:

$$f(x) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)}_{\text{even}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right)}_{\text{odd}}$$

Fourier Coefficients:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots$$

even : $f(-x) = f(x)$ | odd : $f(-x) = -f(x)$

\# Find the ⓐ fourier coefficients for, $f(x) = \begin{cases} 0, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases}$ period 10

ⓑ Fourier series

ⓒ Graph the function

ⓐ $a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots$ | Here, $2L = 10$ $\therefore L = 5$

$$= \frac{1}{5}\int_{-5}^{5} f(x) \cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{5}\left[\int_{-5}^{0} f(x) \cos\frac{n\pi x}{L}\,dx + \int_{0}^{5} f(x) \cos\frac{n\pi x}{5}\,dx\right]$$

$$= \frac{1}{5}\left[\int_{-5}^{0} 0 \cos\frac{n\pi x}{5}\,dx + \int_{0}^{5} 3 \cos\frac{n\pi x}{5}\,dx\right]$$

$$= \frac{1}{5}\left[\frac{15}{n\pi} \sin \frac{n\pi x}{5}\right]_0^5$$

$$= \frac{3}{n\pi}\left[\sin \frac{n\pi(5-0)}{5}\right]$$

$$= \frac{3}{n\pi} \sin n\pi \quad [n=1,2,3,\dots]$$

$$= \frac{3}{n\pi} \cdot 0$$

$$= 0 \text{ (Ans)}$$

$\therefore$ if $n=0$,

$$a_0 = \frac{1}{5}\int_0^5 3 \cdot \cos \frac{n\pi x}{5}\, dx$$

$$= \frac{3}{5}\int_0^5 \cos(0)\, dx$$

$$= \frac{3}{5}\int_0^5 dx$$

$$= \frac{3}{5}[x]_0^5$$

$$= \frac{3}{5}[5-0]$$

$$= 3 \text{ (Ans)}$$

Tanjil
08.07.17
MAT-216

$$\therefore b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \quad [n=1,2,3,\dots]$$

$$= \frac{1}{5}\int_{-5}^{5} f(x) \sin \frac{n\pi x}{5}\, dx$$

$$= \frac{1}{5}\int_0^5 3 \sin \frac{n\pi x}{5}\, dx$$

$$= \frac{3}{5} \cdot \frac{5}{n\pi}\left[-\cos \frac{n\pi x}{5}\right]_0^5$$

$$= \frac{-3}{n\pi}\left[\cos \frac{n\pi \cdot 5}{5} - \cos \frac{n\pi \cdot 0}{5}\right]$$

$$= \frac{3}{n\pi}(1-\cos n\pi) \text{ (Ans)}$$

(b) Fourier Series,

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$= 3/2 + 0 + \sum_{n=1}^{\infty} \frac{3}{n\pi}(1-\cos n\pi)\sin\frac{n\pi x}{5}$$

$$= 3/2 + \sum_{n=1}^{\infty} 3/n\pi \,(1-\cos n\pi)\sin\frac{n\pi x}{5} \quad [n = 1, 2, 3, \dots]$$

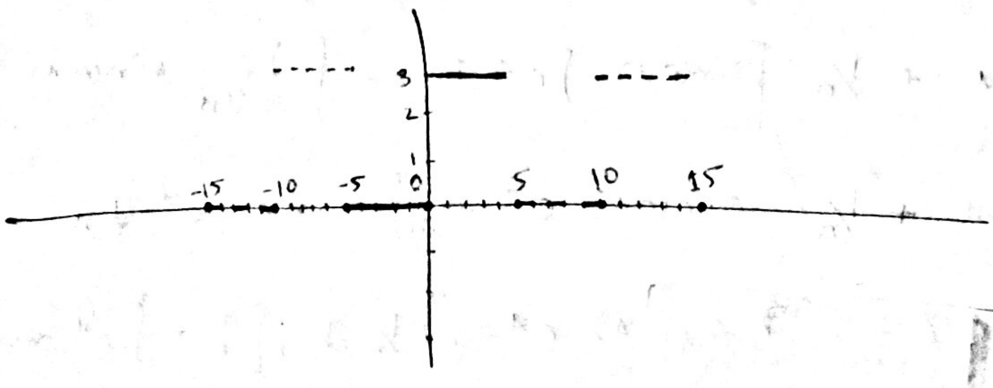
(Ans)

(c)

$(-5,5) = (-5,0) \cup (0,5)$

$(5,15) = (5,10) \cup (10,15)$

$(-15,-5) = (-15,-10) \cup (-10,-5)$



Tanjil
08.07.17
MAT-216

Q) Determine the Fourier Series for

$$f(x) = \begin{cases} -x, & -4 \le x \le 0 \\ x, & 0 < x < 4 \end{cases} \quad \text{period} = 8$$

$$\therefore a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$2L = 8$
$\Rightarrow L = 4$

$$= \frac{1}{4}\int_{-4}^{4} f(x)\cos\frac{n\pi x}{4}\,dx$$

$$= \frac{1}{4}\left\{\int_{-4}^{0} f(x)\cos\frac{n\pi x}{4}\,dx + \int_{0}^{4} f(x)\cos\frac{n x\pi}{4}\,dx\right]$$

$$= \frac{1}{4}\left\{\int_{-4}^{0} (-x)\cos\frac{n\pi x}{4}\,dx + \int_{0}^{4} x\cos\frac{n\pi x}{4}\,dx\right]$$

$$= \frac{1}{4}\left\{\int_{0}^{4} x\cos\frac{n\pi x}{4}\,dx - \int_{-4}^{0} x\cos\frac{n\pi x}{4}\,dx\right\} \quad \ldots (i)$$

$$\therefore \int x\cos\frac{n\pi x}{4}\,dx = x\int\cos\frac{n\pi x}{4}\,dx - \int\left\{\frac{d}{dx}(x)\int\cos\frac{n\pi x}{4}\right\}dx$$

$$= \frac{4x}{n\pi}\cdot\sin\frac{n\pi x}{4} - \frac{4}{n\pi}\int\sin\frac{n\pi x}{4}\,dx$$

$$= \frac{4x}{n\pi}\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4} + c$$

(i) $$a_n = \frac{1}{4}\left(\left[\frac{4x}{n\pi}\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_0^4 - \left[\frac{4x}{n\pi}\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_{-4}^{0}\right)$$

$$= \frac{1}{4}\left[\left(\frac{16}{n^2\pi^2}\cos n\pi - \frac{16}{n^2\pi^2}\right) - \left\{\frac{16}{n^2\pi^2} - \frac{16}{n^2\pi^2}\cos(-n\pi)\right\}\right]$$

$; [\sin\pi = \sin 2\pi = \sin n\pi = 0]$

$$= \frac{1}{4}\left(\frac{16}{n^2\pi^2}\cos n\pi - \frac{16}{n^2\pi^2} - \frac{16}{n^2\pi^2} + \frac{16}{n^2\pi^2}\cos n\pi\right)$$

$$= \frac{1}{4} \times \frac{16}{n^2\pi^2}\left(\cos n\pi - 1 - 1 + \cos n\pi\right)$$

$$= \frac{4}{n^2\pi^2}\left(2\cos n\pi - 2\right)$$

$$= \frac{8}{n^2\pi^2}\left(\cos n\pi - 1\right)$$

$$\therefore a_0 = \frac{1}{4}\left\{\int_0^4 x \cos\frac{n\pi x}{4}\,dx - \int_{-4}^0 x\cos\frac{n\pi x}{4}\,dx\right\}$$

$$= \frac{1}{4}\left[\int_0^4 x\,dx - \int_{-4}^0 x\,dx\right]$$

$$= \frac{1}{4}\left(\left[\frac{x^2}{2}\right]_0^4 - \left[\frac{x^2}{2}\right]_{-4}^0\right)$$

$$= \frac{1}{4}\left\{\frac{4^2}{2} + \frac{(-4)^2}{2}\right\}$$

$$= \frac{1}{4}\left(\frac{16}{2} + \frac{16}{2}\right)$$

$$= 4$$

Tanjil
08.07.17
MAT-216

$$\therefore b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{4}\left(\int_0^4 x\sin\frac{n\pi x}{L} - \int_{-4}^0 x\sin\frac{n\pi x}{4}\,dx\right) \quad \cdots \text{(ii)}$$

$$\therefore \int x \sin\frac{n\pi x}{4}\,dx = x\int \sin\frac{n\pi x}{4}\,dx - \int\left\{ \frac{d}{dx}(x) \int \sin\frac{n\pi x}{4} \right\} dx$$

$$= -x\cdot\frac{4}{n\pi}\cos\frac{n\pi x}{4} + \int \frac{4}{n\pi}\cdot\cos\frac{n\pi x}{4}\,dx$$

$$= -\frac{4x}{n\pi}\cos\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4} + c$$

(ii),

$$b_n = \frac{1}{4}\left(\left[-\frac{4x}{n\pi}\cos\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right]_0^4 - \left[\frac{-4x}{n\pi}\cos\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right]_{-4}^0\right)$$

$$= \frac{1}{4}\left[-\frac{16}{n\pi}\cos(n\pi) + 16/n\pi + 16/n\pi - 16/n\pi \cos(n\pi)\right]$$

$$= \frac{1}{4}\cdot\frac{16}{n\pi}\left(-\cos n\pi + 1 + 1 - \cos n\pi\right)$$

$$= \frac{4}{n\pi}(2 - 2\cos n\pi) = \frac{8}{n\pi}(1-\cos n\pi)$$

$\therefore$ Fourier Series,

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$= 4/2 + \sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{4} + \sum_{n=1}^{\infty}\frac{8}{n\pi}(1-\cos n\pi)\sin\frac{n\pi x}{L}$$

$$= 2 + \sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}(\cos n\pi - 1)\cdot\cos\frac{n\pi x}{4}$$

$$+ \sum_{n=1}^{\infty}\frac{8}{n\pi}(1-\cos n\pi)\sin\frac{n\pi x}{L}$$

(Ans)

4) Determine the complex form of the fourier series of the function

$$f(x) = \begin{cases} 0 , & -\pi \leq x \leq 0 \\ 1 , & 0 \leq x \leq \pi \end{cases}$$

Solve:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} \, dx$$

$2L = 2\pi$

$\therefore L = \pi$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \frac{\cos n\pi x}{\pi} \, dx$$

$$= \frac{1}{\pi} \left[\int_{0}^{\pi} \cos nx \, dx + \int_{-\pi}^{0} \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \cos nx \, dx$$

$$= \frac{1}{n\pi} \left[\sin nx \right]_{0}^{\pi}$$

$$= 0$$

$$a_0 = \frac{1}{\pi} \int_{0}^{\pi} \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} dx$$

$$= \frac{1}{\pi} \left[x \right]_{0}^{\pi}$$

$$= \frac{1}{\pi} \left[\pi - 0 \right]$$

$$= 1$$

Tanjil

08.07.17

MAT-216

$$\therefore b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin\frac{n\pi x}{\pi}\,dx + \int_{-\pi}^{0} f(x)\sin\frac{n\pi x}{\pi}\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\cos nx\right]_0^{\pi}$$

$$= -\frac{1}{\pi}(\cos n\pi - 1)$$

$$= \frac{1}{\pi}(1-\cos n\pi)$$

$\therefore$ Fourier Series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} 0\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty}\frac{1}{\pi}(1-\cos n\pi)\sin\frac{n\pi x}{L}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{\pi}(1-\cos n\pi)\sin\frac{n\pi x}{L}$$ (Ans)

Fourier Series: $f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$

Half range Fourier Sine series or cosine series: Half range Fourier sine or cosine series is a series in which only sine term or only cosine terms are present.

(i) $a_n = 0$, $b_n = 2/L \int_0^L f(x) \sin \frac{n\pi x}{L} dx$ (only odd)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \text{ (Half range sine series)}$$

(ii) $b_n = 0$, $a_n = 2/L \int_0^L f(x) \cos \frac{n\pi x}{L} dx$ (only even)

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \text{ (Half rang cosine series)}$$

Expand $f(x) = x$, $0 < x < 2$ in half range

(i) Sine series (ii) Cosine series

(i) For sine series $a_n = 0$, Here, $L = 2$

$$\therefore b_n = 2/L \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$= 2/2 \int_0^2 x \sin \frac{n\pi x}{2} dx \quad \ldots (a)$$

$$\therefore \int x \sin\frac{n\pi x}{2}\,dx = x\int \sin\frac{n\pi x}{2}\,dx - \int\left\{\frac{d}{dx}(x)\int \sin\frac{n\pi x}{2}\,dx\right\}dx$$

$$= \frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \int \frac{2}{n\pi}\cos\frac{n\pi x}{2}\,dx$$

$$= -\frac{2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2} + c$$

From (a),

$$b_n = \left[-\frac{2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= -\frac{4}{n\pi}(\cos n\pi - 1)$$

$$= 4/n\pi\,(1-\cos n\pi)$$

$$\therefore f(x) = \sum_{n=1}^{\infty} 4/n\pi\,(1-\cos n\pi)\sin\frac{n\pi x}{L} \quad \text{(Ans)}$$

(ii) Cosine series, $b_n = 0$

$$a_n = 2/L\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= 2/2\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^2 x\cos\frac{n\pi x}{2}\,dx \quad \ldots\ldots (b)$$

$$\therefore \int x\cos\frac{n\pi x}{2}\,dx = \frac{2x}{n\pi}\sin\frac{n\pi x}{2} - \int\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2} + c$$

From (x),

$$a_n = \left[\frac{2x}{n\pi} \sin \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \cos \frac{n\pi x}{2} \right]_0^2$$

$$= \frac{4}{n^2\pi^2} (\cos n\pi - 1)$$

$$\therefore a_0 = \int_0^2 x \cos \frac{n\pi . 0}{2} \, dx$$

$$= \left[x^2/2 \right]_0^2$$

$$= 2$$

Tanjil
08.07.17
MAT-216

$$\therefore f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$= 2/2 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} (\cos n\pi - 1) \cos \frac{n\pi x}{L}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} (\cos n\pi - 1) \cos \frac{n\pi x}{L}$$ (Ans)