MAT216

SAKIB

Bashundhara
Exercise Book
Write Your Future

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System of linear equations

07/05/2017

For linear equation, the power of x must be 1.

✓ $x+1=0$

× $\sqrt{x}+2x+3y=5$

× $xy+5x+3=0$

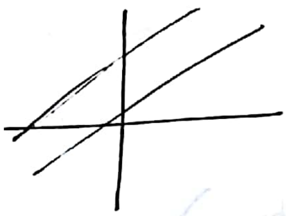
✓ $a_1x+a_2y=b$

✓ $a_1x_1+a_2x_2+\cdots\cdots a_nx_n=b$

$$\left.\begin{aligned} 3x-2y&=4 \\ -6x+4y&=7 \end{aligned}\right\} \quad (i)\times 2+(ii) \longrightarrow \quad \begin{aligned} 6x-4y&=8 \\ -6x+4y&=7 \\ \hline 0&=15 \end{aligned}$$

Inconsistant

অসঙ্গত



Consistant → (i) unique solution/Exactly one sol^n

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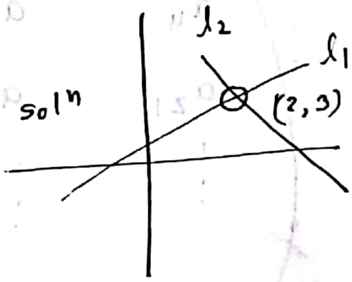
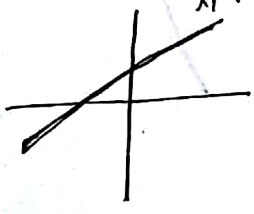
(ii) infinitely many solution





$x=x$

$y=\frac{1}{2}(4-3x)$

$\left(x, \frac{1}{2}(4-3x)\right)$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

For Matrix symbol $\longrightarrow$ ( ), [ ], || ||

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$ Coefficient Matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} & b_m \end{array}\right)$$ Augmented Matrix

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Matrix: A rectangular array of numbers of the form:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

is called a matrix

m rows & n columns

$$a_{11}$$

row column

(1.) Row matrix: $(1 \quad 2 \quad 3)$ / Row Vector

(2.) Column matrix / Column vector: $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$

(3.) Square matrix: $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & -5 \\ 3 & 2 & 4 \end{pmatrix}$

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(4.) Diagonal Matrix: $a_{ij} = 0$ but $i \neq j$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(5.) Indentity matrix:

or, Unit matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(6.) Zero / Null matrix:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(7.) Idempoted matrix: A square matrix A is called an idempoted matrix if $A^2 = A$

$$A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

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$$A^2 = A\cdot A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3-5 & -3-9+15 & -5-15+25 \\ -1-3+5 & 3+9-15 & 5+15-25 \\ 1+3-5 & -3-9+5 & -5-15+25 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} = A$$

Possible: $A_{m\times n} \; B_{n\times p}$ ✓

$(AB)_{m\times p}$

Impossible: $B_{n\times p} \; A_{m\times n}$ ✗

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(8.) <u>Transpose of a Matrix:</u>

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \\ 3 & 5 & 2 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 0 & -1 & 2 \end{bmatrix}$$

<u>Trace of A:</u>

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$Tr(A) = a_{11} + a_{22} + a_{33}$$

(9.) <u>Triangular Matrix</u>

<u>Lower Triangular:</u>

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

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Upper triangular:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

(10.) **Symmetric matrix:**

If $A = A^T$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 0 \end{bmatrix}$$

$A = A^T$

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Inverse Matrix

A matrix is invertible iff it has non-zero determinant

if $D = 0 \rightarrow$ singular

$D \neq 0 \rightarrow$ no singular

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If A and B are invertible matrix of the same size, then AB is invertible and $(AB)^{-1} = B^{-1} A^{-1}$

2 Methods of inverse matrix: $\Rightarrow$

(i) Row- canonical method

(ii) Adjoint

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Adjoint Matrix

If A is a invertible matrix, then $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$

Adjoint of a Matrix

If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}. then the matrix
$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{12} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the matrix of cofactors, the transpose of this matrix is called the adjoint of A and is denoted by adj (A).

Cofactor: The number $(-1)^{i+j} M_{ij}$ is denoted by C_{ij} and is called the cofactor of entry a_{ij}

$$C_{ij} = (-1)^{i+j} M_{ij}$$

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Minor: If A is a square matrix, then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the i row and j column are deleted from A.

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \qquad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{11} = (-1)^{1+1} M_{11}$$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

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Linear Algebra Practice Sheet 1

1. Solve the following matrix equation for a, b, c & d.

$$\begin{bmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$$

$a - b = 8$

$b + c = 1$

$3d + c = 7$

$2a - 4d = 6$

$$\begin{array}{c} \begin{matrix} a & b & c & d & \end{matrix} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 2 & 0 & 0 & -4 & 6 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 2 & 0 & -4 & -10 \end{bmatrix} \quad r_4' = r_4 - 2r_1$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & -2 & -4 & -12 \end{bmatrix} \quad r_4' = r_4 - 2r_2$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \quad r_4' = r_4 + 2r_3$$

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$$= \begin{bmatrix} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad R_4' = \left(\frac{1}{2}\right) \times R_4$$

$\therefore \ d = 1 \ \ldots\ldots \ (i)$

$c + 3d = 7$

$\Rightarrow c = 7 - 3\cdot 1 = 4 \ \ldots \ (ii)$

$b + c = 1$

$\Rightarrow b = 1 - 4 = -3 \ \ldots\ldots \ (iii)$

$a - b = 8$

$\Rightarrow a = 8 + (-3) = 5 \ \ldots\ldots \ (iv)$

Ans: $a = 5, \ b = -3, \ c = 4, \ d = 1$

2. Consider the matrices and compute the following.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

$$E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

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3. (g) $(DA)^T$

$$DA = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-5+2 & 0+10+2 \\ -3+0+1 & 0+0+1 \\ 9-2+4 & 0+4+4 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ -2 & 1 \\ 11 & 8 \end{bmatrix}$$

$$(DA)^T = \begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$$ Ans.

(h) $(C^T B) A^T$

$$= \left(\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}^T \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^T$$

$$= \left(\begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

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$$= \begin{bmatrix} 4+0 & -1+6 \\ 16+0 & -4+2 \\ 8+0 & -2+10 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 \\ 16 & -2 \\ 8 & 8 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12+0 & -4+10 & 4+5 \\ 48+0 & -16-4 & 16-2 \\ 24+0 & -8+16 & 8+8 \end{bmatrix} = \begin{bmatrix} 12 & 6 & 9 \\ 48 & -20 & 14 \\ 24 & 8 & 16 \end{bmatrix}$$ Ans.

(i) $tr(DD^T)$

$$(DD^T) = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+25+4 & -1+0+2 & 3+10+8 \\ -1+0+2 & 1+0+1 & -3+0+4 \\ 3+10+8 & -3+0+4 & 9+4+16 \end{bmatrix} = \begin{bmatrix} 30 & 1 & 21 \\ 1 & 2 & 1 \\ 21 & 1 & 29 \end{bmatrix}$$

$tr(DD^T) = 30+2+29 = 61$ Ans.

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(j) $tr(4E^T - D)$

$$(4E^T - D) = 4\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}^T - \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

$$= 4\begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & -4 & 16 \\ 4 & 4 & 4 \\ 12 & 8 & 12 \end{bmatrix} - \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & -9 & 14 \\ 5 & 4 & 3 \\ 9 & 6 & 8 \end{bmatrix}$$

$tr(4E^T - D) = 23 + 4 + 8 = 35$

Ans.

4. (a) $(2D^T - E)A$

$$= \left(2\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}^T - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\right)\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \left(2\begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\right)\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 2 & -2 & 6 \\ 10 & 0 & 4 \\ 4 & 2 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}\right)\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12+3+3 & 0-6+3 \\ 33+1+2 & 0-2+2 \\ 0-1+5 & 0+2+5 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{bmatrix}$$

Ans.

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4. (b) $(BA^T - 2C)^T$

$$\Rightarrow (BA^T - 2C) = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^T - 2\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 8 & 4 \\ 6 & 2 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 12+0 & -4-2 & 4-1 \\ 0+0 & 0+2 & 0+2 \end{bmatrix} - \begin{bmatrix} 2 & 8 & 4 \\ 6 & 2 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -6 & 3 \\ 0 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 8 & 4 \\ 6 & 2 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -14 & -1 \\ -6 & 0 & -8 \end{bmatrix}$$

Ans.

$$\therefore (BA^T - 2C)^T = \begin{bmatrix} 10 & -6 \\ -14 & 0 \\ -1 & -8 \end{bmatrix}$$

Ans.

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5. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$.

(a) Find all the minors of A

(b) Find all the cofactors

(c) Find adj (A)

(d) Find A^{-1} ; using $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$

Sol: ⇒ (a)

$$M_{11} = \begin{vmatrix} 7 & -1 \\ 1 & 4 \end{vmatrix} = 28+1 = 29$$

$$M_{12} = \begin{vmatrix} 6 & -1 \\ -3 & 4 \end{vmatrix} = 24-3 = 21$$

$$M_{13} = \begin{vmatrix} 6 & 7 \\ -3 & 1 \end{vmatrix} = 6+21 = 27$$

$$M_{21} = \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = -8-3 = -11$$

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$$M_{22} = \begin{vmatrix} 1 & 3 \\ -3 & 4 \end{vmatrix} = 4+9 = 13$$

$$M_{23} = \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix} = 1-6 = -5$$

$$M_{31} = \begin{vmatrix} -2 & 3 \\ 7 & -1 \end{vmatrix} = 2-21 = -19$$

$$M_{32} = \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -1-18 = -19$$

$$M_{33} = \begin{vmatrix} 1 & -2 \\ 6 & 7 \end{vmatrix} = 7+12 = 19$$

(b)

Cofactor, $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \qquad A_{co} = \begin{bmatrix} 29 & -21 & 27 \\ 11 & 13 & 5 \\ -19 & 19 & 19 \end{bmatrix}$$

Ans.

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(c) $Adj(A) = A_{co}^{T}$

$$= \begin{bmatrix} 29 & 11 & -19 \\ -21 & 13 & 19 \\ 27 & 5 & 19 \end{bmatrix}$$

(d)

$det(A) = 1(28+1) + 2(24-3) + 3(6+21)$

$= 29 + (2 \times 21) + (3 \times 27)$

$= 152 \neq 0$

$\therefore A^{-1} = \dfrac{adj(A)}{det(A)}$

$$= \frac{1}{152} \begin{bmatrix} 29 & 11 & -19 \\ -21 & 13 & 19 \\ 27 & 5 & 19 \end{bmatrix}$$

Ans.

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6. Find the inverse of the following matrices:

(a) $\begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

Sol: $(A|I) = \left[\begin{array}{ccc|ccc} 2 & 5 & 5 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right]$

$= \left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 5 & 5 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right]$

$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 2 & 5 & 5 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right]$ $r_1' = (-1) r_1$

$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 3 & 5 & -1 & +2 & 0 \\ 0 & 2 & 1 & 0 & 2 & 1 \end{array}\right]$ $r_2' = r_2 - 2r_1$

$r_3' = r_3 - 2r_1$

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$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 5/3 & 1/3 & 2/3 & 0 \\ 0 & 2 & 3 & 0 & 2 & 1 \end{array}\right] \quad r_2' = \left(\frac{1}{3}\right) r_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 5/3 & 1/3 & 2/3 & 0 \\ 0 & 0 & -1/3 & -2/3 & 2/3 & 1 \end{array}\right] \quad r_3' = r_3 - 2r_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 5/3 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 2 & -2 & -3 \end{array}\right] \quad r_3' = (-3) \times r_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 & 4 & 5 \\ 0 & 0 & 1 & 2 & -2 & -3 \end{array}\right] \quad r_2' = r_2 - \left(\frac{5}{3}\right) r_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5 & -5 \\ 0 & 1 & 0 & -3 & 4 & 5 \\ 0 & 0 & 1 & 2 & -2 & -3 \end{array}\right] \quad r_1' = r_1 - r_2$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -5 & -5 \\ -3 & 4 & 5 \\ 2 & -2 & -3 \end{bmatrix}$$

Ans.

7. (b)

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{bmatrix}.$$

$$= \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 5 & 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 3 & 8 & 9 & 0 & 0 & 1 & 0 \\ 1 & 3 & 2 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - r_1 \\ r_4 = r_4 - r_1 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} r_2' = (-1)\, r_2 \\ r_3 \rightleftharpoons r_4 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 6 & 0 & 1 & -7 \end{array}\right] \qquad r_4' = r_4 - 7r_3 \quad \therefore$$

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7. (c)

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A : I = \left[\begin{array}{ccc|ccc} 2 & -3 & 5 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & -3/2 & 5/2 & 1/2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \quad \begin{array}{l} r_1' = \left(\frac{1}{2}\right) \times r_1 \\ r_3' = \left(\frac{1}{2}\right) \times r_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -3/2 & 0 & 1/2 & 0 & -5/4 \\ 0 & 1 & 0 & 0 & 1 & 3/2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \quad \begin{array}{l} r_1' = r_1 - (5/2)\, r_3 \\ r_2' = r_2 + 3 r_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 3/2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 3/2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \quad r_1' = r_1 + \left(\frac{3}{2}\right) r_2$$

$$\therefore A^{-1} = \begin{bmatrix} 1/2 & 3/2 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

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7. (D) $\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

$$A:I = \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right] \quad r_1' = (-1) r_1$$

$$= \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right] \quad \begin{array}{l} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 4r_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right] \quad r_2' = \left(\frac{1}{5}\right) r_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 8/5 & -6/5 & 1 \end{array}\right] \quad r_3' = r_3 - 6r_2$$

$= -7 + 6 \cdot \frac{6}{5}$ | $4 - 6 \cdot \frac{2}{5}$

$= \frac{36}{5} - 7 = \frac{1}{5}$ | $= 4 - \frac{12}{5} = \frac{8}{5}$

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$$= \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right] \quad r_3' = 5r_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -25 & 18 & -15 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right] \quad \begin{array}{l} r_2' = r_2 + \frac{6}{5} r_3 \\ r_1' = r_1 - 3r_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & -3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right] \quad r_1' = r_1 + 2r_2$$

$$\therefore A^{-1} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$

Ans.

7. (e) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\therefore A \vdots I = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 0 & 0 \end{array}\right] \qquad R_1' \rightleftharpoons R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & -2 \\ 0 & -1 & 1 & 1 & 0 & -1 \end{array}\right] \qquad \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} R_2' = (-1)R_2 \\ R_3' = (-1)R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & -1 & 0 & -1 \end{array}\right] \qquad R_3' = R_3 - R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \qquad R_3' = (-1)R_3$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Ans.

7. (f)

$$\begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

$$A : I = \left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] \quad r_1 \rightleftharpoons r_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right] \quad \begin{aligned} r_2' &= r_2 - 3r_1 \\ r_3' &= r_3 - 2r_1 \end{aligned}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right] \quad r_2' = \left(\frac{1}{4}\right) r_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 5/2 & -5/4 & 7/4 & 1 \end{array}\right] \quad r_3' = r_3 - 5r_2$$

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$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right] \quad r_3' = \left(\frac{2}{5}\right) r_3$$

$$r_2' = r_2 + \left(\frac{5}{2}\right) r_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -11/10 & -6/5 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right] \quad r_1' = r_1 - 3r_3$$

$$\therefore A^{-1} = \begin{bmatrix} 3/2 & -11/10 & -6/5 \\ -1 & 1 & 1 \\ -1/2 & 7/10 & 2/5 \end{bmatrix}$$

Ans.

7. (g) $\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 2 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

$$A : I = \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

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$$
= \left[\begin{array}{cccc|cccc}
1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\
0 & 3 & -4 & -1 & -3 & 1 & 0 & 0 \\
0 & 3 & -5 & -1 & -2 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 & -1 & 0 & 0 & 1
\end{array}\right]
\qquad
\begin{array}{l}
r_2' = r_2 - 3r_1 \\
r_3' = r_3 - 2r_1 \\
r_4' = r_4 - r_1
\end{array}
$$

$$
= \left[\begin{array}{cccc|cccc}
1 & -1 & 2 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \\
0 & 3 & -5 & -1 & -2 & 0 & 1 & 0 \\
0 & 3 & -4 & -1 & -3 & 1 & 0 & 0
\end{array}\right]
\qquad
r_2 \rightleftharpoons r_4
$$

$$
= \left[\begin{array}{cccc|cccc}
1 & -1 & 2 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & -2 & -1 & 1 & 0 & 1 & -3 \\
0 & 0 & -1 & -1 & 0 & 1 & 0 & -3
\end{array}\right]
\qquad
\begin{array}{l}
r_3' = r_3 - 3r_2 \\
r_4' = r_4 - 3r_2
\end{array}
$$

$$
= \left[\begin{array}{cccc|cccc}
1 & -1 & 2 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & -1 & 0 & 3 \\
0 & 0 & 2 & 1 & -1 & 0 & -1 & 3
\end{array}\right]
\qquad
\begin{array}{l}
\cancel{r_3' = r_3 + 2r_2} \\
r_3 \rightleftharpoons r_4
\end{array}
$$

$$
= \left[\begin{array}{cccc|cccc}
1 & -1 & 2 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & -1 & 0 & 3 \\
0 & 0 & 0 & -1 & -1 & 2 & -1 & -3
\end{array}\right]
\qquad
r_4 = r_4 - 2r_3
$$

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$$= \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 0 & 0 & 2 & -1 & -2 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \qquad \begin{array}{l} r_3' = r_3 - r_4 \\ r_1' = r_1 - r_4 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 2 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \qquad \begin{array}{l} r_2' = r_2 + r_3 \\ r_1' = r_1 - 2r_3 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & 3 \end{array}\right] \qquad r_1' = r_1 + r_2$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -2 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 \\ 1 & -2 & 1 & 3 \end{bmatrix}$$

Ans.

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7. (h) $\begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$

$$A : I = \left[\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & -2 & +3 & -1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right] \quad \begin{array}{l} r_1' = (-1)r_1 \\ r_2' = r_2 - 2r_1' \\ r_3' = r_3 - 4r_1' \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 8/5 & -6/5 & 1 \end{array}\right] \quad \begin{array}{l} r_2' = (\frac{1}{5})r_2 \\ r_3' = r_3 - 6r_2' \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right] \quad r_3' = 5r_3$$

$\therefore$

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$$= \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & -25 & 18 & -15 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right]$$

$r_2' = r_2 + \left(\frac{5}{5}\right) r_3$

$r_1' = r_1 - 3r_3$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & -3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right]$$

$r_1' = r_1 + 2r_2$

$$\therefore A^{-1} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$

Ans.

8. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

prove that $(AB)^{-1} = B^{-1}A^{-1}$

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$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3+1 & 5+1+2 & 3+2+1 \\ 2+6+3 & 5+2+6 & 3+4+3 \\ 2+12+9 & 5+4+18 & 3+8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 & 6 \\ 11 & 13 & 10 \\ 23 & 27 & 20 \end{bmatrix}$$

$$(AB)^{-1} = \begin{bmatrix} -5/4 & 1/4 & 1/4 \\ 5/4 & -9/4 & 3/4 \\ -1/4 & 11/4 & -5/4 \end{bmatrix}$$ [By using the calculator]

$$A^{-1} = \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} -3/4 & 1/4 & 7/4 \\ -1/4 & -1/4 & 5/4 \\ 5/4 & 1/4 & -13/4 \end{bmatrix}$$

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$$B^{-1} A^{-1} = \begin{bmatrix} -3/4 & 1/4 & 7/4 \\ -1/4 & -1/4 & 5/4 \\ 5/4 & 1/4 & -13/4 \end{bmatrix} \cdot \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix}$$

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Practice Sheet 02

1. Determine the values of parameters λ & μ, such that the following system has (i) no solution

(ii) a unique solution

(iii) more than one solution

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

Sol: The augmented matrix is

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \qquad \begin{array}{l} r_2' = r_2 - r_1 \\ r_3' = r_3 - r_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] \qquad r_3' = r_3 - r_2$$

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The corresponding system of the linear equation is

$$x + y + z = 6 \quad \text{------} \quad (i)$$

$$y + 2z = 4 \quad \text{------} \quad (ii)$$

$$(\lambda - 3)z = \mu - 10 \quad \text{---} \quad (iii)$$

From equation (iii) $\Rightarrow$

(a) if $\lambda \neq 3$ then unique solution

(b) if $\lambda = 3$ & $\mu \neq 10$ then no solution

(c) if $\lambda = 3$ & $\mu = 10$ then many solution

2. (a) $\lambda x + y + z = 1$

$x + \lambda y + z = 1$

$x + y + \lambda z = 1$

Sol: The augmented matrix is $\left[\begin{array}{ccc|c} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \end{array}\right]$

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$$= \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 1 & \lambda & 1 & 1 \\ \lambda & 1 & 1 & 1 \end{array}\right] \qquad R_1' \rightleftharpoons R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 1-\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda^2 & 1-\lambda \end{array}\right] \qquad \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 - \lambda R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1-\lambda & 1-\lambda^2 & 1-\lambda \end{array}\right] \qquad R_2' = \left\{\frac{1}{(\lambda-1)}\right\} R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1-\lambda^2-\lambda & 1-\lambda \end{array}\right] \qquad R_3' = R_3 + \lambda R_2$$

$$\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \end{array}$$

2. (b) $x + y - z = 1$

$2x + 3y + \lambda z = 3$

$x + \lambda y + 3z = 2$

Sol: The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & \lambda & 3 \\ 1 & \lambda & 3 & 2 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & \lambda-1 & 4 & 1 \end{array}\right] \quad \begin{array}{l} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - r_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \lambda+2 & 1 \\ 0 & 0 & -(\lambda+3)(\lambda-2) & -\lambda+2 \end{array}\right] \quad r_3' = r_3 + (1-\lambda) r_2$$

The corresponding system of the linear equation is

$x + y - z = 1$ .... (i)

$y + (\lambda+2) z = 1$ .... (ii)

$z(\lambda+3)(\lambda-2) = \lambda+2$ .... (iii)

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From equation (iii) $z(\lambda+3)(\lambda-2) = \lambda+2$

(a) If $\lambda \neq -3$ & $\lambda \neq 2$ then unique solution

(b) $\lambda = -3$ & $\lambda \neq 2 \longrightarrow$ no solution

(c) $\lambda = 2 \longrightarrow$ many solution.

2\. (c) $x + y + kz = 2$

$3x + 4y + 2z = k$

$2x + 3y - z = 1$

Sol: The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & k & 2 \\ 3 & 4 & 2 & k \\ 2 & 3 & -1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & k & 2 \\ 0 & 1 & 2-3k & k-6 \\ 0 & 1 & -1-2k & -3 \end{array}\right] \quad \begin{array}{l} r_2' = r_2 - 3r_1 \\ r_3' = r_3 - 2r_1 \end{array}$$

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$$= \left[\begin{array}{ccc|c} 1 & 1 & k & 2 \\ 0 & 1 & 2-3k & k-6 \\ 0 & 0 & k-3 & 3-k \end{array}\right]$$

The corresponding system of the linear equation

$$x + y + kz = 2$$

$$y + (2-3k)z = k-6$$

$$(k-3)z = 3-k$$

From this last equation $(k-3)z = 3-k$ we get,

(a) If $k \neq 3$ then unique solution

(b) If $k = 3$ then many solution

(c) no value for no solution.

2. (D) $x - 3z = -3$

$$2x + \lambda y - z = -2$$

$$x + 2y + \lambda z = 1$$

The augmented matrix is

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$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 2 & \lambda & -1 & -2 \\ 1 & 2 & \lambda & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & \lambda & 5 & 4 \\ 0 & 2 & \lambda+3 & 4 \end{array}\right] \qquad \begin{array}{l} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - r_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & 5/\lambda & 4/\lambda \\ 0 & 0 & \frac{(\lambda+5)(\lambda-2)}{\lambda} & \frac{4\lambda-8}{\lambda} \end{array}\right] \qquad \begin{array}{l} r_2' = \left(\frac{1}{\lambda}\right) r_2 \\ r_3' = r_3 - 2r_2' \end{array}$$

$$\frac{(\lambda+5)(\lambda-2)}{\lambda} z = \frac{4\lambda-8}{\lambda}$$

(i) For no solution ~~(ii) For~~

$$\frac{(\lambda-2)(\lambda+5)}{\lambda} = 0. \qquad \& \qquad \frac{4\lambda-8}{\lambda} \neq 0$$

$$\Rightarrow (\lambda+5)(\lambda-2) = 0 \qquad \Rightarrow 4\lambda - 8 \neq 0$$

$$\lambda \neq \frac{8}{4}$$

$$\therefore \lambda = -5, 2 \qquad \therefore \lambda \neq 2$$

For unique solution

$$\frac{(\lambda-2)(\lambda+5)}{\lambda} \neq 0 \quad \& \quad \frac{4\lambda-8}{\lambda} \neq 0$$

$\therefore \lambda \neq -5, 2$ $\qquad\qquad$ $\therefore \lambda \neq 2$

For many solution

$$\frac{(\lambda-2)(\lambda+5)}{\lambda} = 0 \quad \& \quad \frac{4\lambda-8}{\lambda} = 0$$

$\therefore \lambda = -5, 2$ $\qquad\qquad$ $\therefore \lambda = 2$

2.(e) $x + y + \lambda z = 1$

$x + \lambda y + z = \lambda$

$\lambda x + y + z = \lambda^2$

Sol: The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 1 & \lambda & 1 & 1 \\ \lambda & 1 & 1 & \lambda^2 \end{array}\right]$

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$$= \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 1-\lambda & \lambda-1 \\ 0 & 1-\lambda & 1-\lambda^2 & \lambda^2-\lambda \end{array}\right] \qquad \begin{array}{l} r_2' = r_2 - r_1 \\ r_3' = r_3 - \lambda r_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & (\lambda+2)(1-\lambda) & (\lambda+1)(\lambda-1) \end{array}\right] \qquad \begin{array}{l} r_2' = \left(\frac{1}{\lambda-1}\right) r_2 \\ r_3' = r_3 + \cancel{\lambda r_2'}\,(\lambda-1) r_2' \end{array}$$

$= 1-\lambda^2+1-\lambda$

$= -\lambda^2-\lambda+2$

$= -\lambda^2-2\lambda+\lambda+2$

$= -\lambda(\lambda+2)+1(\lambda+2)$

$= (\lambda+2)(1-\lambda)$

$\lambda^2-\lambda+\lambda-1$

$= \lambda^2-1$

$= (\lambda+1)(\lambda-1)$

The corresponding system of the linear equation is,

$x + y + \lambda z = 1 \; \ldots \; (i)$

$y - z = 1 \; \ldots \; (ii)$

$(\lambda+2)(1-\lambda) z = (\lambda+1)(\lambda-1) \ldots (iii)$

From eq^n (iii) $\rightarrow$ For no solution: $(\lambda+2)(1-\lambda) = 0$. & $(\lambda+1)(\lambda-1) \neq 0$

$\therefore \lambda = -2, 1$ $\qquad \lambda \neq 1, -1$

$\rightarrow \lambda = -2,$ $\lambda \neq 1$

For unique solution, $(\lambda+2)(1-\lambda) \neq 0$ & $(\lambda+1)(\lambda-1) \neq 0$

$\therefore \lambda \neq 1, -2$ $\qquad \therefore \lambda \neq 1, -1$ $\qquad \rightarrow \lambda \neq -2$

For many solution, $(\lambda+2)(1-\lambda) = 0 \rightarrow \lambda = -2, 1$ & $(\lambda+1)(\lambda-1) = 0 \rightarrow \lambda = 1, -1$ $\rightarrow \lambda$

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3. Solve each of the following systems by Gaussian elimnation (Row-Echelon Form) or Gauss-Jordan elimnation (Reduced-row-echelon form)

Sol! $x_1 + x_2 + 2x_3 = 8$

$-x_1 - 2x_2 + 3x_3 = 1$

$3x_1 - 7x_2 + 4x_3 = 10$

The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$ $\quad r_2' = r_2 + r_1$

$\quad r_3' = r_3 - 3r_1$

$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & \cancel{48}52 & -104 \end{array}\right]$ $\quad r_2' = (-1) r_2$

$\quad r_3' = r_3 + 10 r_2$

$= \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$ $\quad r_3' = \left(\frac{-1}{52}\right) r_3$

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The corresponding system of the linear equation is:

$$x_1 + x_2 + 2x_3 = 8 \quad \text{---- (i)}$$

$$x_2 - 5x_3 = -9 \quad \text{--- (ii)}$$

$$x_3 = 2 \quad \text{..... (iii)}$$

From equation (ii) $\longrightarrow x_2 - 5\times 2 = -9$

$\Rightarrow x_2 = 10 - 9 = 1$

From eq^n (i) $\longrightarrow x_1 + 1 + 2\times 2 = 8$

$\Rightarrow x_1 = 8 - 5 = 3$

$\therefore x_1 = 3,\ x_2 = 1\ \&\ x_3 = 2$ Ans.

3. (b)

$$2x_1 + 2x_2 - x_3 + x_5 = 0$$

$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$

$$x_1 + x_2 - 2x_3 - x_5 = 0$$

$$x_3 + x_4 + x_5 = 0$$

Sol: The augmented matrix is:

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$$= \left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \qquad r_1 \rightleftharpoons r_3$$

$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \qquad \begin{aligned} r_2' &= r_2 + r_1 \\ r_3' &= r_3 - 2r_1 \end{aligned}$$

$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{array}\right] \qquad \begin{aligned} r_2 &\rightleftharpoons r_4 \\ r_3' &= \left(\tfrac{1}{3}\right) r_3 \end{aligned}$$

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$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad \begin{array}{l} r_3' = r_3 - r_2 \\ r_4 = \left(\frac{-1}{3}\right) r_4 \end{array}$$

$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} r_3' = (-1) r_3 \\ r_4' = r_4 - r_3' \end{array}$$

The corresponding system of the linear equation is

$x_1 + x_2 - 2x_3 - x_5 = 0 \quad \ldots \text{(i)}$

$x_3 + x_4 + x_5 = 0 \quad \ldots \text{(ii)}$

$x_4 = 0$

Let, $x_5 = t$ & $x_2 = p$

Form equation (ii) $\rightarrow x_3 = -t - 0 = -t$

" " (i) $\rightarrow x_1 + p + 2t - t = 0$

$\therefore x_1 = -p - t$

Ans: $x_1 = -p - t, \; x_2 = p, \; x_3 = -t, \; x_4 = 0, \; x_5 = t$

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3. (c) $2x + 2y + 4z = 0$

$w - y - 3z = 0$

$2w + 3x + y + z = 0$

$-2w + x + 3y - 2z = 0$

Sol: The augmented matrix is $\left[\begin{array}{cccc|c} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right]$

$= \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right]$ $r_1 \rightleftharpoons r_2$

$= \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{array}\right]$ $r_3' = r_3 - 2r_1$

$r_4' = r_4 + 2r_1$

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$$= \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{array}\right] \quad \begin{array}{l} r_3' = r_3 - 3r_2 \\ r_4' = r_4 - r_2 \end{array}$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad r_4' = \left(\frac{-1}{10}\right) r_4$$

The corresponding system of the linear equation is

$$x_1 - x_3 - 3x_4 = 0 \; \ldots\ldots \text{(i)}$$

$$x_2 + x_3 + 2x_4 = 0 \; \ldots\ldots \text{(ii)}$$

$$x_4 = 0 \; \ldots\ldots \text{(iii)}$$

Let, $x_3 = t$

From equation (ii) $\longrightarrow$ $x_2 = -t - 2\times 0$

$= -t$

From equation (i) $\longrightarrow$ $x_1 = t + 3\times 0$

$= t$

Ans: $x_1 = t,\ x_2 = -t,\ x_3 = t,\ x_4 = 0$

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3. (d) $2x_1 + 2x_2 + 2x_3 = 0$

$-2x_1 + 5x_2 + 2x_3 = 1$

$8x_1 + x_2 + 4x_3 = -1$

Sol: The augmented matrix is $\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]$ $\quad r_1' = (\frac{1}{2}) r_1$

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right]$ $\quad r_2' = r_2 + 2r_1$

$r_3' = r_3 - 8r_1$

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 4/7 & 1/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad r_2' = (\frac{1}{7}) r_2$

$r_3' = r_3 + 7r_2'$

The corresponding system of the linear equation is:

$x_1 + x_2 + x_3 = 0$ .... (i)

$x_2 + \frac{4}{7} x_3 = \frac{1}{7}$ .... (ii)

Let, $x_3 = r$

Form equation (ii) $\longrightarrow$ $x_2 = \frac{1}{7} - \frac{4}{7} r$

From eq^n (i) $\longrightarrow$ $x_1 = \frac{4r}{7} - \frac{1}{7} - r = \frac{3r}{7} - \frac{1}{7}$ Ans.

3. (e) $x - y + 2z - w = -1$

$2x + y - 2z - 2w = -2$

$-x + 2y - 4z + w = 1$

$3x - 3w = -3$

Sol: The ~~corres~~ augmented matrix is.
$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right]$$

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$$= \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right] \quad \begin{array}{l} r_2' = r_2 - 2r_1 \\ r_3' = r_3 + r_1 \\ r_4' = r_4 - 3r_1 \end{array}$$

$$= \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right] \quad r_2 \rightleftharpoons r_3$$

$$= \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} r_3' = r_3 - 3r_2 \\ r_4' = r_4 - 3r_2 \end{array}$$

The corresponding system of the linear equation is

$x_1 - x_2 + 2x_3 - x_4 = -1$ ... (i)

$x_2 - 2x_3 = 0$ .... (ii)

Let, $x_4 = p$

$x_3 = q$

From eq^n (ii) $\rightarrow x_2 = 2q$

(i) $- \; x_1 = 2q - 2q + p + 1$

$= p + 1$

Ans.

4. Solve by using $x = A^{-1}b$

(a) $x_1 + x_2 + 2x_3 = 8$

$-x_1 - 2x_2 + 3x_3 = 1$

$3x_1 - 7x_2 + 4x_3 = 10$

Sol:

$$(A|I) = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & -2 & 3 & 0 & 1 & 0 \\ 3 & -7 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 5 & 1 & 1 & 0 \\ 0 & -10 & -2 & -3 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + R_1 \\ R_3' = R_3 - 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & -1 & 0 \\ 0 & 0 & -52 & -13 & -10 & 1 \end{array}\right] \quad \begin{array}{l} R_2' = (-1)R_2 \\ R_3' = R_3 + 10R_2' \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1/4 & 5/26 & -1/52 \end{array}\right] \quad R_3' = \left(\frac{-1}{52}\right) R_3$$

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$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/2 & -5/13 & 1/26 \\ 0 & 1 & 0 & 1/4 & -1/26 & -5/52 \\ 0 & 0 & 1 & 1/4 & 5/26 & -1/52 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + 5R_3 \\ R_1' = R_1 - 2R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -9/26 & 7/52 \\ 0 & 1 & 0 & 1/4 & -1/26 & -5/52 \\ 0 & 0 & 1 & 1/4 & 5/26 & -1/52 \end{array}\right] \quad R_1' = R_1 - R_2$$

$$A^{-1} = \begin{bmatrix} 1/4 & -9/26 & 7/52 \\ 1/4 & -1/26 & -5/52 \\ 1/4 & 5/26 & -1/52 \end{bmatrix}$$

Here, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $b = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$

$\therefore X = A^{-1} b$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/4 & -9/26 & 7/52 \\ 1/4 & -1/26 & -5/52 \\ 1/4 & 5/26 & -1/52 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 - \frac{9}{26} + \frac{70}{52} \\ 2 - \frac{1}{26} - \frac{50}{52} \\ 2 + \frac{5}{26} - \frac{10}{52} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore x = 3, y = 1, z = 2$ Ans.

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4. (b) $x - y + 2z - w = -1$

$2x + y - 2z - 2w = -2$

$-x + 2y - 4z + w = 1$

$3x - 3w = -3$

Sol:

$$(A|I) = \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & -2 & 0 & 1 & 0 & 0 \\ -1 & 2 & -4 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & -3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & -6 & 0 & -3 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 + R_1 \\ R_4' = R_4 - 3R_1 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2/3 & 0 & -2/3 & 1/3 & 0 & 0 \\ 0 & 0 & -4/3 & 0 & 5/3 & -1/3 & 1 & 0 \\ 0 & 0 & -16/3 & 0 & -7/3 & -1/3 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2' = \left(\frac{1}{3}\right) R_2 \\ R_3' = R_3 - R_2' \\ R_4' = R_4 - 3R_2' \end{array}$$

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$$
= \left[\begin{array}{cccc|cccc}
1 & -1 & 2 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & -2/3 & 0 & -2/3 & 1/3 & 0 & 0 \\
0 & 0 & 1 & 0 & -5/4 & 1/4 & -3/4 & 0 \\
0 & 0 & 0 & 0 & \cancel{52/9}\,-9 & 1 & 16/3 & 1
\end{array}\right]
\quad
\begin{array}{l}
R_3' = \left(-\frac{3}{4}\right) R_3 \\[2em]
R_4' = R_4 + \left(\frac{16}{3}\right) R_3'
\end{array}
$$

Vector Space

Defination: A vector space over an arbitrary field F is a non-empty set V, whereas elements are called vectors for which two operators are prescribled. The first operation, called vector addition and the second operation, called sclar multiplication. The two operations are required to satisfy the following condition:

1. (a) For all vectors $u, v \in V$

$$u + v = V + u$$

(b) Far all vector $u, v, w \in V$

$$(u+v) + w = u + (v+w)$$

(c) There exists are vector $0 \in V$

such that for all $v \in V$, $v + 0 = 0 + v = v$

(d) For each $v \in V$ there is a vector $(-v) \in V$

for which $v + (-v) = (-v) + v = 0$.

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(2) (a) For any sclar $\alpha \in F$ and vectors $u, v \in V$;

$$\alpha(u+v) = \alpha u + \alpha v$$

(b) For any sclars $\alpha, \beta \in F$ and any vector $v \in V$,

$$(\alpha+\beta)v = \alpha v + \beta v$$

(c) For any sclars $\alpha, \beta \in F$ and any vector $v \in V$,

$$(\alpha\beta)v = \alpha(\beta v)$$

(d) For each $v \in V$, $1v = v$

when 1 is the unit sclar $1 \in F$

Vector spaces are also sometimes called linear spaces.

<u>Subspace</u>: A subset w of a vector space V is called a subspace of V if w is itself a vector space under the addition and ~~sclar~~ scalar multiplication defined on V.

23/05/17

Theorem: W is a subspace of V iff (if and only if)

(i) W is non-empty

(ii) W is closed under vector addition

i.e. $v, w \in W$ implies that $v+w \in W$

(iii) W is closed under scalar multiplication

i.e. $v \in W$ implies that $\alpha v \in W$ for every $\alpha \in F$

(1.) Show that $S = \{(a,0,c) \mid a,c \in \mathbb{R}\}$ is a subspace of the vector space $\mathbb{R}^3$

Sol: For $0 \in \mathbb{R}^3$, $0 = (0,0,0) \in S$

Since the second component of S is 0, hence S is non-empty

For any vector $u = (a,0,c)$ & $v = (a',0,c')$ in S and scalars α, β, we have

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$\alpha u + \beta v = \alpha(a,0,c) + \beta(a',0,c') = (\alpha a + \beta a'), 0, (\alpha c + \beta c')$

Since second component is zero

So $\alpha u + \beta v \in S$ and S is a subspace of $\mathbb{R}^3$.

(2) Show that, $T = \{(a,b,c,d) \in R^4 : 2a - 3b + 5c - d = 0\}$ is a subspace of R^4.

Sol: From $0 \in R^4$. $0 = (0,0,0,0) \in T$

$\because 2.0 - 3.0 + 5.0 - 0 = 0$ Hence T is non-empty.

Suppose $u = (a,b,c,d)$ & $v = (a',b',c',d') \in T$

then, $2a - 3b + 5c - d = 0$ & $2a' - 3b' + 5c' - d' = 0$

For any $\alpha, \beta \in R$, then we have

$\alpha u + \beta v = (\alpha a + \beta a', \alpha b + \beta b', \alpha d + \beta d')$

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Also we have,

$$2(\alpha a+\beta a')-3(\alpha b+\beta b')+5(\alpha c+\beta c')-(\alpha a+\beta d')=0$$

$\therefore$ $\alpha u+\beta v \in T$ and so T is a subspace of R^4

(3.) $W=\{(a,b,c) \mid a,b,c \in R$ and $a-2b+3c=5\}$ is not a subspace of R^3.

Sol: For $0 \in \mathbb{R}^3$, $0=(0,0,0) \in W$

$\because$ $0-2\cdot 0+3\cdot 0=0 \neq 5$ Hence T is non-empty

Suppose $u=(a,b,c)$ & $v=(a',b',c') \in W$

then $a-2b+3c=5$ & $a'-2b'+3c'=5$

For any $\alpha, \beta \in W$, then we have

$\alpha u+\beta v=(\alpha a+\beta a', \alpha b+\beta b', \alpha c+\beta c')$

Also we have, $(\alpha a+\beta a')-2(\alpha b+\beta b')+3(\alpha c+\beta c')=0$

$\therefore$ $\alpha u+\beta v \notin W$ and so W is not a subspace of R^3.

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Linear Combination :

Let V be vector space over the field F and let $v_1, v_2, v_3 \dots v_n \in V$ then any vector $v \in V$ is called a linear combination of $v_1, v_2, \dots v_n$ iff there exists scalar.

$\alpha_1, \alpha_2, \dots \alpha_n$ in F such that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

$$= \sum_{i=1}^{n} \alpha_i v_i$$

1. Write the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ as a linear combination of the matrices, $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ & $A_3 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

Sol: set A as a linear combination of A_1, A_2, A_3 using the unknowns α_1, α_2 & α_3 :

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$$A = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 \quad \ldots\ldots \quad (i)$$

i.e. $\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} \alpha_1 & \alpha_1 \\ 0 & -\alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & \alpha_2 \\ -\alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_3 & -\alpha_3 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 - \alpha_3 \\ -\alpha_2 & -\alpha_1 \end{bmatrix}$$

Equating the corresponding components

$$\alpha_1 + \alpha_2 + \alpha_3 = 3$$

$$\alpha_1 + \alpha_2 - \alpha_3 = -1$$

$$-\alpha_2 = 1$$

$$-\alpha_1 = -2$$

$\therefore \alpha_1 = 2, \ \alpha_2 = -1 \ \& \ \alpha_3 = 2$

From eq^n (i) $\Rightarrow$

$$A = 2A_1 - A_2 + 2A_3$$

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28/05/2017

Linear Independence & Dependence

Def^n: Let V be a vector space over the field F. The vectors $v_1, v_2, \ldots, v_m \in V$ are said to be linear dependence over F or simply dependent if there exist a non-trivial linear combination of them equal to the zero vector 0.

i.e. $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$

where $\alpha_i \neq 0$ for at least one i.

On the otherhand, the vectors $v_1, v_2, \ldots, v_m$ in V are said to be linearly independent over F or simply independent if the only linear combination of them equal to 0 is the trivial one, i.e.

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_m v_m = 0$

iff $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$

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Practice Sheet 3

Q1. Determine whether each of the following sets are linearly independent / dependent.

(a) $\{(2,1,2),(0,1,-1),(4,3,3)\}$

Sol: The vector equation,

$$k_1v_1 + k_2v_2 + k_3v_3 = 0$$

$$\Rightarrow k_1(2,1,2) + k_2(0,1,-1) + k_3(4,3,3) = (0,0,0)$$

$$\Rightarrow (2k_1 + 4k_3),\ (k_1 + k_2 + 3k_3),\ (2k_1 - k_2 + 3k_3) = (0,0,0)$$

Equating corresponding components we get,

$$2k_1 + 4k_3 = 0$$

$$k_1 + k_2 + 3k_3 = 0$$

$$2k_1 - k_2 + 3k_3 = 0$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 1 & 1 & 3 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 0 & 4 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right] \quad R_1 \rightleftarrows R_2$$

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$$= \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \quad \begin{array}{l} R_3' = R_3 - 2R_1 \\ R_4' = R_4 - 2R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = \left(\frac{-1}{2}\right) R_2 \\ R_3' = \left(\frac{-1}{3}\right) R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3' = R_3 - R_2$$

Since the row-echelon matrix has a zero row,

So the vectors are linearly dependent.

1. Show that the vector $(2,-1,4)$, $(3,6,2)$ & $(2,10,-4)$ are linearly independent

Sol: The vector equation

Method 1

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0$$

$$\Rightarrow \alpha_1 (2,-1,4) + \alpha_2 (3,6,2) + \alpha_3 (2,10,-4) = (0,0,0)$$

$$\Rightarrow (2\alpha_1 + 3\alpha_2 + 2\alpha_3), (-\alpha_1 + 6\alpha_2 + 10\alpha_3), (4\alpha_1 + 2\alpha_2 - 4\alpha_3) = (0,0,0)$$

Equating corresponding components we get,

$$2\alpha_1 + 3\alpha_2 + 2\alpha_3 = 0$$

$$-\alpha_1 + 6\alpha_2 + 10\alpha_3 = 0$$

$$4\alpha_1 + 2\alpha_2 - 4\alpha_3 = 0$$

The augmented matrix is,

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ -1 & 6 & 10 & 0 \\ 4 & 2 & -4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -6 & -10 & 0 \\ 2 & 3 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

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$$= \left[\begin{array}{ccc|c} 1 & -6 & -10 & 0 \\ 0 & 15 & 22 & 0 \\ 0 & 13 & -22 & 0 \end{array}\right] \qquad \begin{array}{l} r_2' = r_2 - 2r_1 \\ \\ r_3' = r_3 - 2r_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -6 & -10 & 0 \\ 0 & 1 & 22/15 & 0 \\ 0 & 0 & 264 & 0 \end{array}\right] \qquad \begin{array}{l} r_2' = \left(\frac{1}{15}\right) r_2 \\ \\ r_3' = r_3 - 13 r_2' \end{array}$$

$= -22 - 13 \times 22$

$= -22(1-13) = -22 \times 12$

$$= \left[\begin{array}{ccc|c} 1 & -6 & -10 & 0 \\ 0 & 1 & 22/15 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad r_3' = \left(\frac{1}{264}\right) r_3$$

The corresponding system of the linear equation is,

$\alpha_3 = 0$

$\alpha_2 + \frac{22}{15}\alpha_3 = 0 \quad \Rightarrow \alpha_2 = 0$

$\therefore \alpha_1 = 0$

Since the row-echelon matrix has no zero rows. So the vectors are linearly independent.

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(2.) Form the matrix, ~~where~~ whose rows are the given vectors (from previous question)

Method 2

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & -2 \\ 3 & 6 & 2 \\ 2 & -1 & 4 \end{bmatrix} \qquad r_1 \rightleftharpoons r_3$$

$$= \begin{bmatrix} 1 & 5 & -2 \\ 0 & -9 & 8 \\ 0 & -11 & 8 \end{bmatrix} \qquad \begin{aligned} r_2' &= r_2 - 3r_1 \\ r_3' &= r_3 - 2r_1 \end{aligned}$$

$$= \begin{bmatrix} 1 & 5 & -2 \\ 0 & 1 & 8/-9 \\ 0 & 0 & 96 \end{bmatrix} \qquad \begin{aligned} r_2' &= \left(\frac{-1}{9}\right) r_2 \\ r_3' &= r_3 + 11 r_2 \end{aligned}$$

$$= \begin{bmatrix} 1 & 5 & -2 \\ 0 & 1 & -8/9 \\ 0 & 0 & 1 \end{bmatrix} \qquad r_3' = \left(\frac{1}{96}\right) r_3$$

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$$\begin{bmatrix} 1 & -1/2 & 2 \\ 0 & 1 & -8/15 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the rowechelon matrix has non-zero row, so the vectors are linearly independent.

(3.) Show that the set of the vector $\{(3,0,1,-1), (2,-1,0,1), (1,1,1,-2)\}$ is linearly dependent.

Sol:

$$\begin{bmatrix} 3 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & -2 & 0 \end{bmatrix}$$

$R_1 \rightleftarrows R_3$

$$\begin{matrix} 1 & 1 & 1 & -2 \\ 2 & -1 & 0 & 1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$R_3' = R_3 - 3R_1$

$R_4' = R_4 - R_1$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \begin{matrix} R_3' = R_3 - 2R_2 \\ R_4' = R_4 + R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} R_3' = \left(\frac{-1}{4}\right) R_3 \\ R_4' = R_4 - 2R_3' \end{matrix}$$

Since the rowechelon matrix has one zero row, so the vectors are linearly dependent.

$$\begin{bmatrix} 1 & 1 & 1 & -2 \\ 0 & 1 & 2/3 & 5/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

dependent

Basis and Dimension

30/05/17

Defination: Let V be a vector space and $\{V_1, V_2, \ldots V_n\}$ finite set of vectors in V, we call $\{V_1, V_2, \ldots V_n\}$ a basis for V iff the following two conditions are satisfied.

(i) $\{V_1, V_2, \ldots V_n\}$ is linearly independent

(ii) $\{V_1, V_2, \ldots V_n\}$ spans V

Dimension: The dimension of a finite dimensional vector space is the number of vector in any basis of it.

① Prove that the vectors $(1,2,0), (0,5,7)$ and $(-1,1,3)$ from a basis for R^3 2(i)

Sol: Firstly, we have to show that the given vectors are span

Let us consider an arbitrary vector $b = (b_1, b_2, b_3)$

The linear combination: $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = b$ where, $\alpha_1, \alpha_2, \alpha_3$ are scalars.

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$\Rightarrow \alpha_1(1,2,0) + \alpha_2(0,5,7) + \alpha_3(-1,1,3) = (b_1, b_2, b_3)$

$\Rightarrow (\alpha_1 - \alpha_3,\ 2\alpha_1 + 5\alpha_2 + \alpha_3,\ 7\alpha_2 + 3\alpha_3) = (b_1, b_2, b_3)$

Equating corresponding components, we get,

$\alpha_1 - \alpha_3 = b_1$

$2\alpha_1 + 5\alpha_2 + \alpha_3 = b_2$

$7\alpha_2 + 3\alpha_3 = b_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 2 & 5 & 1 & b_2 \\ 0 & 7 & 3 & b_3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 5 & 3 & b_2 - 2b_1 \\ 0 & 7 & 3 & b_3 \end{array}\right] \qquad R_2' = R_2 - 2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 1 & 3/5 & \frac{1}{5}(b_2 - 2b_1) \\ 0 & 0 & -6/5 & b_3 - \frac{7}{5}b_2 + \frac{14}{5}b_1 \end{array}\right] \qquad R_2' = \left(\frac{1}{5}\right)R_2$$

$R_3' = R_3 - 7R_2'$

$3 - 7 \cdot \frac{3}{5} = 3 - \frac{21}{5} = \frac{15 - 21}{5}$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & b_1 \\ 0 & 1 & 3/5 & \frac{1}{5}(b_2-2b_1) \\ 0 & 0 & 1 & \frac{1}{6}(7b_2-14b_1-5b_3) \end{array}\right] \quad R_3' = \left(\frac{-5}{6}\right)R_3$$

The corresponding system of the linear equation is,

$$\alpha_1 - \alpha_3 = b_1$$

$$\alpha_2 + \frac{3}{5}\alpha_3 = \frac{1}{5}(-2b_1+b_2) \Rightarrow \alpha_2$$

$$\alpha_3 = \frac{1}{6}(7b_2-14b_1-5b_3)$$

This system is consistant. So there are many solutions.

Now we want to prove that the given vectors are linearly independent. Let us consider $b = (0,0,0)$

$\alpha_1 = 0$

$\alpha_2 = 0$

$\alpha_3 = 0$

So the given vectors are linearly independent.

Therefore the given vectors form a basis for R^3

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② Let U be the subspace of $\mathbb{R}^3$ spanned (generated) by the vectors. [4]

(1, 2, 1), (0, −1, 0) & (2, 0, 2) Find a basis and Dimension of U.

Sol: Form the matrix whose rows are given vectors and reduce the matrix to row-echelon form by the elementory row operation.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -4 & 0 \end{bmatrix} \qquad r_3' = r_3 - 2r_1$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} r_2' &= (-1)r_2 \\ r_3' &= \left(\frac{-1}{4}\right) r_3 \end{aligned}$$

This matrix is in row-echelon form and the non-zero rows in the matrix are (1, 2, 1) & (0, 1, 0)

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These non-zero rows form a basis of the row space

$\therefore$ Basis of $U = \{(1,2,1), (0,1,0)\}$

and Dimension $(U) = 2$

(3.) Let W be the subspace of R^5 spanned by the vectors

$(1,-2,0,0,3)$, $(2,-5,-3,-2,6)$, $(0,5,15,10,0)$ & $(2,6,18,8,6)$

Find the basis and dimension of W.

Sol: Form the matrix whose rows are given vectors and reduce the matrix to row-echelon form by the elementory row-operation.

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix} \quad \begin{matrix} r_2' = r_2 - 2r_1 \\ r_4' = r_4 - 2r_1 \\ r_3' = \left(\frac{1}{5}\right) r_3 \end{matrix}$$

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$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -12 & 0 \end{bmatrix}$$

$r_2' = (-1)\, r_2$

$r_3' = r_3 - r_2'$

$r_4' = r_4 - 10 r_2'$

$$= \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad r_4' = \left(\frac{-1}{12}\right) r_4$$

$$= \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad r_4 \rightleftharpoons r_3$$

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$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of $W = \{(1,-2,0,0,3), (0,1,3,2,0), (0,0,1,1,0)\}$

Dimension $(W) = 3$

(4.) Determine a basis and the dimension of the solution space of the homogeneous system:

$$\begin{aligned} 2x_1 + 2x_2 - x_3 \quad\quad + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 \quad\quad - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

Sol: The augmented matrix is, $A =$

$$\left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

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$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 1 & 1 & -2 & 3 & -1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$r_2' = r_2 - r_1$

$r_3' = r_3 - 2r_1$

$-1 + 4$

$$= \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basis $(A) = \{(1,1,-2,0,-1,0), (0,0,1,1,1,0), (0,0,0,1,0,0)\}$

Dimension $(A) = 3$

The corresponding system of linear equation

$x_1 + x_2 - 2x_3 - x_5 = 0$

$x_3 + x_4 + x_5 = 0$

$x_4 = 0$

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Let, $x_2 = s$

$x_5 = t$

$x_1 = -x_2 + 2x_3 + x_5 = -x_2 + 2x_5 + x_5 = -x_2 - 3x_5$

$x_3 = -x_4 - x_5 \Rightarrow x_3 = -x_5$

$x_4 = 0$

$\therefore x_1 = -s - 3t$

$x_2 = s$

$x_3 = -t$

$x_4 = 0$

$x_5 = t$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-3t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3t \\ 0 \\ -t \\ 0 \\ t \end{bmatrix}$$

Rank & Nullity

06/06/17

Rank

① The rank of matrix A is ~~a~~ the maximum number of linearly independent rows or columns in the matrix

② Let A be an $m \times n$ matrix and let A_R be the row echelon form of A. Then the rank of the matrix A is the number of non-zero rows of A_R.

The rank of a matrix A is denoted by rank(A) or $\rho(A)$

Row Rank : The maximum number of linearly independent rows of matrix A is called the row rank of A.

Column rank : The maximum number of linearly independent columns of matrix A is called the column rank of A.

Nullity : Let A be the matrix. The nullity of A is the dimension of the solution space of the linear system $Ax = 0$

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① $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}_{3\times 3}$, find rank (A)

Sol: $|A| = 0$. so the rank of the given matrix is less then 3

$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \neq 0$ $\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} \neq 0$

As the determinant value of the submatrix is not 0,

So the rank of the given matrix is 2

$\therefore$ rank $(A) = 2$

[* If the determinant value of the submatrix is 0

then the rank of the given matrix is is less then2. $\therefore$ rank = 1]

② $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ rank $(A) = 2$

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(3) Find the rank of matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

$\Rightarrow$ $|B| = 1(36-36) - 2(18-18) + 3(12-12)$

$= 1\times 0 - 2\times 0 + 3\times 0$

$= 0$

So $|B|$ rank is < 3

Submatrix $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4-4=0$

rank of $B = 1$

(4.) $C = \begin{bmatrix} 6 & 2 & 0 & 4 \\ -2 & -1 & 3 & 4 \\ -1 & -1 & 6 & 10 \end{bmatrix}_{3\times 4}$

$\Rightarrow$ $C_{11} = \begin{bmatrix} 6 & 2 & 0 \\ -2 & -1 & 3 \\ -1 & -1 & 6 \end{bmatrix}$ $C_{12} = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 3 & 4 \\ -1 & 6 & 10 \end{bmatrix}$

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$$C_{13} = \begin{bmatrix} 6 & 0 & 4 \\ -2 & 3 & 4 \\ 1 & 6 & 10 \end{bmatrix} \qquad C_{14} = \begin{bmatrix} 6 & 2 & 4 \\ -2 & -1 & 4 \\ -1 & -1 & 10 \end{bmatrix}$$

Sol: $C = \begin{bmatrix} 1 & 1 & -6 & -10 \\ -2 & -1 & 3 & 4 \\ 6 & 2 & 0 & 4 \end{bmatrix}$ $\quad R_1 \rightleftarrows r_3$

$$= \begin{bmatrix} 1 & 1 & -6 & -10 \\ 0 & 1 & -9 & -16 \\ 0 & -4 & 36 & -56 \end{bmatrix} \qquad \begin{aligned} r_2' &= r_2 + 2r_1 \\ r_3' &= r_3 - 6r_1 \end{aligned}$$

$$= \begin{bmatrix} 1 & 1 & -6 & -10 \\ 0 & 1 & -9 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad r_3' = r_3 + 4r_2$$

As there is 2 non-zero rows so the rank of the matrix is 2.

$\therefore$ Rank$(C) = 2$

Ans.

(5.) Find the rank and nullity of the matrix:

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}_{4\times 5}$$

Sol: $= \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix}$ $\quad r_2' = r_2 - 3r_1$

$r_3' = r_3 + r_1$

$r_4' = r_4 - 2r_1$

$$= \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 7 & 7 & 7 & 14 \\ 0 & 2 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As there is 2 non-zero rows, so the rank of the matrix is 2.

The corresponding system of the linear equation is

$$x_1 + 4x_2 + 5x_3 + 6x_4 + 9x_5 = 0$$

$$x_2 + x_3 + x_4 + 2x_5 = 0$$

Let, $x_3 = r$

$x_4 = s$

$x_5 = t$

$x_2 = -r - s - 2t$

$x_1 = -4(-r-s-2t) - 5r - 6s - 9t$

$= -r - 2s - t$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -r-2s-t \\ -r-s-2t \\ r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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$$v_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\{v_1, v_2, v_3\} = \text{or} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

So the nullity of the given matrix is 3.

* [rank + nullity = number of columns]

$2 + 3 = 5$

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⑥ Find the nullity of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & -1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$$

Sol: $= \begin{bmatrix} 1 & 0 & -5 & 6 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \\ 0 & -2 & 16 & -16 \end{bmatrix}$ $\quad R_2' = R_2 - 3R_1$, $\quad R_3' = R_3 - 5R_1$, $\quad R_4' = R_4 - 4R_1$

$= \begin{bmatrix} 1 & 0 & -5 & 6 \\ 0 & 1 & -8 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad R_2' = \left(\frac{-1}{2}\right) R_2'$, $\quad R_3' = R_3 + 2R_2'$, $\quad R_4' = R_4 + 2R_2'$

There are 2 non-zero rows in this matrix, the rank of it is 2

Ans.

The corresponding system of the linear equation is

$x_1 - 5x_3 + 6x_4 = 0$

$x_2 - 8x_3 + 8x_4 = 0$

Let, $x_3 = P$ & $x_4 = q$

$x_1 = 5P - 6q$

$x_2 = 8P - 8q$

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$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5p-6q \\ 8p-8q \\ p \\ q \end{bmatrix} = p\begin{bmatrix} 5 \\ 8 \\ 1 \\ 0 \end{bmatrix} + q\begin{bmatrix} -6 \\ -8 \\ 0 \\ 1 \end{bmatrix}$$

$$\{v_1, v_2\} \text{ or } \begin{bmatrix} 5 \\ 8 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -8 \\ 0 \\ 1 \end{bmatrix}$$

So the nullity of the matrix is 2 Ans.

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Linear Transformation

Defination: If $T: V \longrightarrow W$ is a function from a vector space V into a vector space W, T is called a linear transformation from V to W if for all vector u_1 & u_2 in V and every scalar C such that (i) $T(u_1+u_2) = T(u_1) + T(u_2)$

(ii) $T(Cu_1) = CT(u_1)$

① Which of the following define linear transformation ~~form~~ from R^3 into R^2: 18

(i) $T(x,y,z) = (x-y,\ x-z)$ (ii) $T(x,y,z) = (x+1,\ y+z)$

Sol: (i) Let, $u_1 = (x_1, y_1, z_1)$

& $u_2 = (x_2, y_2, z_2)$

$\therefore u_1 + u_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1+x_2,\ y_1+y_2,\ z_1+z_2)$

$T(u_1) = T(x_1, y_1, z_1) = (x_1-y_1,\ x_1-z_1)$

$T(u_2) = T(x_2, y_2, z_2) = (x_2-y_2,\ x_2-z_2)$

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$$T(u_1+u_2) = (x_1+x_2,\ y_1+y_2,\ z_1+z_2)$$

$$= (x_1+x_2-y_1-y_2,\ x_1+x_2-z_1-z_2)$$

$$= ((x_1-y_1,\ x_1-z_1) + (x_2-y_2,\ x_2-z_2))$$

$$= T(u_1) + T(u_2)$$

$$T(cu_1) = T(cx_1,\ cy_1,\ cz_1)$$

$$= (cx_1-cy_1,\ cx_1-cz_1)$$

$$= c(x_1-y_1,\ x_1-z_1)$$

$$= cT(u_1)$$

(ii) $T(u_1) = T(x_1, y_1, z_1) = (x_1+1,\ y_1+z_1)$

$T(u_2) = T(x_2, y_2, z_2) = (x_2+1,\ y_2+z_2)$

$T(u_1+u_2) = (x_1+x_2+1,\ y_1+y_2,\ z_1+z_2)$

$T(u_1)+T(u_2) = (x_1+1,\ y_1+z_1) + (x_2+1,\ y_2+z_2)$

$$= (x_1+x_2+2,\ y_1+y_2+z_1+z_2)$$

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Here, $x_1 + x_2 + 2 \neq x_1 + x_2 + 1$

$\therefore T(u_1 + u_2) \neq T(u_1) + T(u_2)$

[If there is numberical value like 1, 2, 3...etc, the transformation is never possible]

So the given T is not the linear transformation from R^3 into R^2.

※ Range: The range of T is the subset of W consisting of all $y \in W$ such that $T(x) = y$ for all $x \in V$

It is generally denoted by $R(T)$

※ Rank of a linear transformation: If $T: V \longrightarrow W$ is a linear transformation, the dimension of the range of T is called the rank of T and is denoted by rank (T)

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Kernel or Null space: Let T be a linear transformation of V ~~the dimension of the rang~~ into W. Then the Kernel of T is the subset of V consisting of all $x \in V$ for which $T(x) = 0$ when $0 \in W$. The Kernel of T is generally denoted by Ker(T).

Nullity of a linear transformation: If $T: V \longrightarrow W$ is a linear transformation then the dimension of the Kernel (Null space) is called the nullity of T and is denoted by Nullity(T).

(1.) Let $T: R^3 \longrightarrow R^3$ be the linear operator defined by $T(x,y,z) = (x+2y, y-z, x+2z)$. Find the rank and nullity of T.

Sol: Given that,

$$T(x,y,z) = (x+2y,\ y-z,\ x+2z)$$

The images of the generators of R^3 generate the Im T

(Image of T)

$T(1,0,0) = (1,0,1)$

$T(0,1,0) = (2,1,0)$

$T(0,0,1) = (0,-1,2)$

Form the matrix whose rows are the generators of Im T:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \quad r_2' = r_2 - 2r_1$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & +1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad r_3' = r_3 + \cancel{8}r_2$$

This matrix is in row-echolen form. there is ~~is~~ ~~one~~ two non-zero rows, form a basis So the rank of it 2. So rowspace $\{(1,0,1),(0,1,-2)\}$

Dimension = 2

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For nullity, we have to consider $T(x)=0$

$T(x, y, z) = (0,0,0)$

$\Rightarrow (x+2y, y-z, x+2z) = (0,0,0)$

The corresponding system of the linear equation is,

$x+2y=0$

$y-z=0$

$x+2z=0$

The augmented matrix is, $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right]$

$= \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \quad R_3' = R_3 - R_1$

$= \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3' = R_3 + 2R_2$

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$$\alpha = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ which is the nullspace.

As the dimension of this nullspace is 1, so the nullity is 1.

(2) Let $T : R^4 \to R^3$ be the linear transformation defined by

$T(x,y,z,t) = (x-y+z+t,\ x+2z-t,\ x+y+3z-3t)$ 19

Find the basis and the dimension of the (i) range of T &

(ii) Null space of T

Sol: The image of the generators of R^4 generate the Im T

$T(1,0,0,0) = (1,1,1)$

$T(0,1,0,0) = (-1,0,1)$

$T(0,0,1,0) = (1,2,3)$

$T(0,0,0,1) = (1,-1,-3)$

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Form the matrix whose rows are the generators of Im T

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -2 \end{bmatrix} \qquad \begin{aligned} r_2' &= r_2 + r_1 \\ r_3' &= r_3 - r_1 \\ r_4 &= r_4 - r_1 \end{aligned}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad \begin{aligned} r_3' &= r_3 - r_2 \\ r_4' &= r_4 + 2r_2 \end{aligned}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row-echolen form

$\{(1,1,1), (0,1,2)\}$ from a basis

Dimension $= 2$

For nullity we have to consider,

$T(x) = 0$

$T(x,y,z,t) = (0,0,0\cancel{0})$

$\Rightarrow (x-y+z+t,\ x+2z-t,\ x+y+3z-3t) = (0,0,0)$

The corresponding system of the linear equation is,

$x-y+z+t=0$

$x+2z-t=0$

$x+y+3z-3t=0$

The augmented matrix is,
$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & 2 & -1 & 0 \\ 1 & 1 & 3 & -3 & 0 \end{array}\right]$$

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$$= \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & -2 & 2 & -4 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 4 & -8 & 0 \end{array}\right] \quad R_3' = R_3 + 2R_2$$

$$= \left[\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x - y + z + t = 0 \\ y + z - 2t = 0 \end{array}$$

Let, $t = p$ & $z = q$

$\therefore y = 2p - q$

$\therefore x = \cancel{q - 2p - q - p = 2q - p} = 2p - q - q - p = p - 2q$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} p - 2q \\ 2p - q \\ q \\ p \end{bmatrix} = p \begin{bmatrix} +1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + q \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

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$\{(-2,-1,1,0), (1,2,0,1)\}$ is a basis of the nullspace of T

$\dim[N(T)] = 2$

So the nullity of the matrix is 2. Ans.

PS-3 Q-2

(iii) Prove that the following vector form a basis for R^4.

$\{\{1,1,1,1\}, \{0,1,1,1\}, \{0,0,1,1,\}, (0,0,0,1)\}$

Sol: $b = (b_1, b_2, b_3, b_4)$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = b$

$\Rightarrow \alpha_1(1,1,1,1) + \alpha_2(0,1,1,1) + \alpha_3(0,0,1,1) + \alpha_4(0,0,0,1) = (b_1, b_2, b_3, b_4)$

$\Rightarrow (\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3+\alpha_4) = (b_1, b_2, b_3, b_4)$

Equating corresponding components, we get,

$\alpha_1 = b_1$

$\alpha_1 + \alpha_2 = b_2 \quad \Rightarrow \alpha_2 = b_2 - \alpha_1 \quad \Rightarrow \alpha_2 = b_2 - b_1$

$\alpha_1 + \alpha_2 + \alpha_3 = b_3 \quad \Rightarrow \alpha_3 = b_3 - b_2 + b_1 - b_1 = b_3 - b_2$

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$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = b_4$

$\Rightarrow \alpha_4 = b_4 - \alpha_3 - \alpha_2 - \alpha_1$

$= b_4 - b_3 + \cancel{b_2} - \cancel{b_2} + b_1 - \cancel{b_1}$

$= b_4 - b_3$

→ Let, $b = (0,0,0,0)$

$\alpha_1 = 0$

$\alpha_2 = 0$

$\alpha_3 = 0$

$\alpha_4 = 0$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$\det(A) = 1 \cdot 1 \cdot 1 \cdot 1 = 1 \neq 0$

$\therefore$ A spans R^4

$\therefore$ The set form a basis for R^4.

~~Let~~ PS-3 Q-12

Que: Let $T: R^3 \to R^3$ be the linear transformation defined by

$T(x,y,z) = (x+2y-3z,\ 2x-y+4z,\ 4x+3y-2z)$

Find the basis & dimension of Kernel(T) or Ker(T) or nullspace(T).

Sol: $T(1,0,0) = (1,2,4)$

$T(0,1,0) = (2,-1,3)$

$T(0,0,1) = (-3,4,-2)$

From the matrix whose rows are the generator of Im T:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & -1 & 3 \\ -3 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -5 \\ 0 & 10 & 10 \end{bmatrix} \quad \begin{matrix} r_2' = -2r_1 + r_2 \\ r_3' = r_3 + 3r_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2' = \left(\frac{-1}{5}\right) R_2 \\ R_3' = R_3 - 10R_2' \end{matrix}$$

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$$= \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \{(1,2,4), (0,1,1)\}$ form a basis of Rang(T)

Dimension of Range$(T) = 2$

Rank of Range $(T) = 2$

For nullity we have to consider,

$T(x) = 0$

$\Rightarrow T(x,y,z) = (0,0,0)$

$\Rightarrow (x+2y-3z,\ 2x-y+4z,\ 4x+3y-2z) = (0,0,0)$

Equating corresponding components,

$x+2y-3z=0$

$2x-y+4z=0$

$4x+3y-2z=0$

The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & -1 & 4 & 0 \\ 4 & 3 & -2 & 0 \end{array}\right]$

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$$= \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & -5 & 10 & 0 \end{array}\right] \qquad \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 4R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} R_2' = \left(\frac{1}{-5}\right) R_2 \\ R_3' = R_3 + 5R_2' \end{array}$$

The corresponding system of the linear equation is

$x + 2y - 3z = 0$ . . . (i)

$y - 2z = 0$ . . . . . (ii)

Let, $z = t$

From (ii) $\rightarrow$ $y = 2t$ & (i) $\rightarrow$ $x = 3t - 2 \cdot 2t = 3t - 4t = -t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Basis of kernel $(T) = \{(-1, 2, 1)\}$

Dim of $Ker(T) = 1$

Nullity $(T) = 1$

Ans.

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Eigen Values & Eigen Vectors

02/07/17

Defination: Let A be an $m \times n$ matrix and x is a non-zero vector in $\mathbb{R}^n$ is called an eigen vector of A if Ax is a scalar multiple of ~~A if Ax is~~ x that is

$$Ax = \lambda x$$

$$\Rightarrow (\lambda I - A)x = 0$$

for some ~~selar~~ scalar λ

$\rightarrow \lambda x - Ax = 0$

$\rightarrow (\lambda - A)x = 0$

$\rightarrow (\lambda I - A)x = 0$

The scalar λ is an eigen value of A & x is said to be an eigen vector of A corresponding to λ.

(i) $(\lambda I - A) \longrightarrow$ is called the characteristic matrix

(ii) $\det(\lambda I - A) \rightarrow$ " " " " polynomial

(iii) $\det(\lambda I - A) = 0 \rightarrow$ " " " equation

1. Find the eigen values and corresponding eigen vectors of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$$

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Sol: The characteristic matrix is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{bmatrix} \quad \text{.....} \quad (i)$$

The characteristic equation is

$\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda-1 & -2 & 1 \\ 0 & \lambda+2 & 0 \\ 0 & 5 & \lambda-2 \end{vmatrix} = 0 \quad \text{.....}$$

$\Rightarrow \lambda-1\{(\lambda+2)(\lambda-2) - 5\times 0\} - 0\{(-2)(\lambda-2) - 5\times 1\} + 0(\lambda+2 - 2\times 0) = 0$

$\Rightarrow \lambda-1(\lambda^2-4) - 0 + 0 = 0$

$\Rightarrow \lambda^3 - 4\lambda - \lambda^2 + 4 = 0$

$\Rightarrow (\lambda-1)(\lambda+2)(\lambda-2) = 0$

$\therefore \lambda = 1, \ \lambda = 2, \ \lambda = -2 \rightarrow$ which are the eigen values

Ans.

Now let us consider, $(\lambda I - A)x = 0$

From equation (i) $\Rightarrow$

$$\left[\begin{array}{ccc|c} \lambda-1 & -2 & 1 & 0 \\ 0 & \lambda+2 & 0 & 0 \\ 0 & 5 & \lambda-2 & 0 \end{array}\right] \quad \text{-------- (ii)}$$

If $\lambda = 1$, from eq^n (ii) we get,

$$\left[\begin{array}{ccc|c} 0 & -2 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 5 & -1 & 0 \end{array}\right]$$

The corresponding system of the linear equation is,

$$-2x_2 + x_3 = 0$$

$$3x_2 = 0 \longrightarrow x_2 = 0$$

$$5x_2 - x_3 = 0 \longrightarrow x_3 = 0$$

Let, $x_1 = a$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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$P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$.

If $\lambda = 2$, then from equation (ii) we get,

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - 2x_2 + x_3 = 0 \ldots \text{(iii)} \rightarrow$ $\cancel{x_3 = -a}$

$x_2 = 0$

$\cancel{5x_2 = 0}$

$\therefore \cancel{x_2 = 0}$

$\therefore \cancel{x_3 = -a}$

Let, $z = s$

From eq^n (iii) $\rightarrow$

$x_1 = 2 \cdot 0 - s = -s$

~~Let, $x_1 = a$~~

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a \\ 0 \\ +a \end{bmatrix} = a \begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix}$$

$P_2 = \begin{bmatrix} -1 \\ 0 \\ +1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 2$

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If $\lambda = -2$, from eq^n (ii) $\longrightarrow$

$$\left[\begin{array}{ccc|c} -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & -4 & 0 \end{array}\right]$$

$$-3x_1 - 2x_2 + x_3 = 0$$

$$5x_2 - 4x_3 = 0 \quad \rightarrow x_2 = \frac{4b}{5}$$

Let, $x_3 = b$

$$\therefore 3x_1 = -2 \cdot \frac{4b}{5} + b$$

$$\therefore x_1 = \frac{-8}{15}b + \frac{b}{3} = \frac{-8b+5b}{15} = \frac{-3b}{15} = \frac{-b}{5}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -b/5 \\ 4b/5 \\ b \end{bmatrix} = b \begin{bmatrix} -1/5 \\ 4/5 \\ 1 \end{bmatrix} = \frac{b}{5} \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$$

$\therefore P_3 = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -2$

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04/07/17

(2) $A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ find eigen values & eigen vectors.

Sol: The characteristic matrix is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda+1 & 2 & 2 \\ -1 & \lambda-2 & -1 \\ +1 & 1 & \lambda \end{bmatrix}$$

The characteristic equation is,

$\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda+1 & 2 & 2 \\ -1 & \lambda-2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+1)\{\lambda(\lambda-2)+1\} - 2(\overset{-\lambda+1}{\cancel{-\lambda+1}}) + 2\{-1-\cancel{(\lambda+2)}\} = 0$$

$$\Rightarrow (\lambda+1)(\lambda^2 - 2\lambda + 1) - 2(-3\lambda) + 2(-\lambda-3) = 0$$

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$$\Rightarrow \lambda^3 - 2\lambda^2 + \lambda + \lambda^2 - 2\lambda + 1 + 6\lambda - 2\lambda - 6 = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 + 3\lambda - 5 = 0$$

$$\Rightarrow \lambda(\lambda^2 - \lambda + 3) = 5$$

$$(\lambda - 1)^2(\lambda + 1) = 0$$

$$\lambda = -1, \quad \lambda = 1, 1$$

Now,

$(\lambda I - A)x = 0$, where x is the eigen vector.

$$\left[\begin{array}{ccc|c} \lambda+1 & 2 & 2 & 0 \\ -1 & \lambda-2 & -1 & 0 \\ 1 & 1 & \lambda & 0 \end{array}\right] \quad \ldots\ldots (i)$$

If $\lambda = 1$, eq^n (i) $\rightarrow$

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of the linear equation is

$$x_1 + x_2 + x_3 = 0$$

Let, $x_2 = a$ & $x_3 = b$

$\therefore x_1 = -a-b$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a-b \\ a \\ b \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore P_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ & $P_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is the eigen vectors.

eigen space $= \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ corresponding to $\lambda = 1$

Again, if $\lambda = -1$, then equation (1) $\longrightarrow$

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ -1 & -3 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right]$$

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$$=\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_3' = R_3 - 2R_2$$

The corresponding system of the linear equation is,

$$x_1 + x_2 - x_3 = 0$$

$$x_2 + x_3 = 0$$

Let, $x_3 = a$

$\therefore x_2 = -a$

$\therefore x_1 - a - a = 0$

$\Rightarrow x_1 = 2a$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2a \\ -a \\ a \end{bmatrix} = a \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore P_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -1$

Ans.

(13) Find all eigen values and the corresponding eigen vectors of the following matrices:

(ii)
$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

$\Rightarrow$ The characteristic matrix is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda - 4 & -6 & -6 \\ -1 & \lambda - 3 & -2 \\ 1 & 4 & \lambda + 3 \end{bmatrix} \quad \text{------} \quad (i)$$

The characteristic equation is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda - 4 & -6 & -6 \\ -1 & \lambda - 3 & -2 \\ 1 & 4 & \lambda + 3 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 4)\left(\lambda^2 - 9 + 8\right) + 6\{\cancel{(3-\lambda)} + 2\} - 6\{(-4) + (3 - \lambda)\} = 0$$

$-1(\lambda+3)$

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$= \cancel{\lambda^3 - 9\lambda + 8\lambda - 4\lambda^2}$

$\Rightarrow (\lambda - 4)(\lambda^2 - 1) + 6(\cancel{5}- \lambda) + 6(\lambda + 1) = 0$

$\Rightarrow \lambda^3 - \lambda - 4\lambda^2 + 4 + 30 - 6\lambda + 6\lambda + 6 = 0$

$\Rightarrow \lambda^3 - 4\lambda^2 - \lambda + 40 = 0$

$\Rightarrow (\lambda-4)(\lambda^2-1) - 6(\lambda+1) + 6(\lambda+1) = 0$

$\Rightarrow (\lambda-4)(\lambda^2-1) = 0$

$\therefore \lambda = 4$ & $\therefore \lambda^2 = 1 \Rightarrow \lambda = 1$

Eigen values of the given matrix is 1 & 4 Ans.

Now let us consider, $(\lambda I - A)x = 0$

From equation (i) we get,

$$\left[\begin{array}{ccc|c} \lambda-4 & -6 & -6 & 0 \\ -1 & \lambda-3 & -2 & 0 \\ 1 & 4 & \lambda+3 & 0 \end{array}\right] \quad \ldots\ldots \text{(ii)}$$

If $\lambda = 1$, then from equation (ii) we get, $\left[\begin{array}{ccc|c} -3 & -6 & -6 & 0 \\ -1 & -2 & -2 & 0 \\ 1 & 4 & 4 & 0 \end{array}\right]$

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The corresponding system of the linear equation is,

$$x_1 + 4x_2 + 4x_3 = 0 \quad \ldots\ldots \text{(iii)}$$

$$x_1 + 2x_2 + 2x_3 = 0 \quad \ldots\ldots \text{(iv)}$$

$$3x_1 + 6x_2 + 6x_3 = 0 \quad \ldots\ldots \text{(v)}$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 4 & 0 \\ 1 & 2 & 2 & 0 \\ 3 & 6 & 6 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 4 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right] \qquad \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 - 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of the linear equation is,

$$x_1 + 4x_2 + 4x_3 = 0 \quad \ldots \text{(iii)}$$

$$x_2 + x_3 = 0 \quad \ldots\ldots \text{(iv)}$$

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Let, $x_3 = a$

~~Form~~ From eq^n (iv) $\rightarrow x_2 = -a$

From eq^n (iii) $\rightarrow$ $x_1 = -4x_2 - 4x_3$

$= -4(-a) - 4a \quad = 4a - 4a = 0$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -a \\ a \end{bmatrix} = a \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$P_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

If $\lambda = 4$, then from equation (ii) we get,

$$\left[\begin{array}{ccc|c} 0 & -6 & -6 & 0 \\ -1 & 1 & -2 & 0 \\ 1 & 4 & 7 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ -1 & 1 & -2 & 0 \\ 0 & 6 & 6 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + R_1 \\ R_3' = R_3 \times \left(\frac{1}{6}\right) \end{array}$$

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$$= \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} R_2' = \left(\frac{1}{5}\right) R_2 \\ R_3' = R_3 - R_2' \end{array}$$

The corresponding system of the linear equation is

$$x_1 + 4x_2 + 7x_3 = 0$$
$$x_2 + x_3 = 0$$

Let, $x_3 = b$

$\therefore\ x_2 = -b$

$\therefore\ x_1 = 4b - 7b = -3b$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3b \\ -b \\ b \end{bmatrix} = b \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

$P_2 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 4$.

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(iii) $A = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -8 & -2 \\ 1 & 2 & 2 \end{bmatrix}$

Sol: The characteristic matrix is

$$\lambda I_3 - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -2 & 1 \\ 2 & -8 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-2 & 2 & -1 \\ -2 & \lambda+8 & 2 \\ -1 & -2 & \lambda-2 \end{bmatrix} \quad \ldots\ldots \text{(i)}$$

The characteristic equation is,

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda-2 & 2 & -1 \\ -2 & \lambda+8 & 2 \\ -1 & -2 & \lambda-2 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-2)\{(\lambda+8)(\lambda-2) - (-2)2\} - 2\{(-2)(\lambda-2) - (-1)2\} - 1\{(-2)(-2) - (-1)(\lambda+8)\} = 0$$

$$\Rightarrow (\lambda-2)(\lambda^2 - 2\lambda + 8\lambda - 16 + 4) - 2\{(4-2\lambda)+2\} - 1(4+\lambda+8) = 0$$

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$\Rightarrow (\lambda-2)(\lambda^2+6\lambda-12)-2(6-2\lambda)-1(\lambda+12)=0$

$\Rightarrow (\lambda^3+6\lambda^2-12\lambda-2\lambda^2-12\lambda+24)-12+4\lambda-\lambda-12=0$

$\Rightarrow \lambda^3+4\lambda^2-21\lambda=0$

$\Rightarrow \lambda(\lambda^2+4\lambda-21)=0$

$\therefore \lambda=0$ & $\lambda^2+7\lambda-3\lambda-21=0$

$\Rightarrow \lambda(\lambda+7)-3(\lambda+7)=0$

$\Rightarrow (\lambda+7)(\lambda-3)=0$

$\therefore \lambda=-7, \quad \lambda=3$

The eigen values of the given matrix is 0, 3, −7 Ans.

Now let us consider $(\lambda I - A)x = 0$

From equation (i) $\Rightarrow$

$$\left[\begin{array}{ccc|c} \lambda-2 & 2 & -1 & 0 \\ -2 & \lambda+8 & 2 & 0 \\ -1 & -2 & \lambda-2 & 0 \end{array}\right] \quad \cdots\cdots \text{(ii)}$$

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If $\lambda = -7$, from eq^n (ii) we get,

$$\left[\begin{array}{ccc|c} -9 & 2 & -1 & 0 \\ -2 & 1 & 2 & 0 \\ -1 & -2 & -9 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 9 & 0 \\ -2 & 1 & 2 & 0 \\ 9 & -2 & 1 & 0 \end{array}\right] \quad \begin{array}{l} r_1 \rightleftharpoons r_3 \\ r_1' = (-1) r_1 \quad , \quad r_3' = (-1) r_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 9 & 0 \\ 0 & 5 & 20 & 0 \\ 0 & -20 & -80 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 - 9R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 9 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = \left(\frac{1}{5}\right) R_2 \\ R_3' = \left(\frac{-1}{20}\right) R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 9 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3' = R_3 - R_2$$

The corresponding system of the linear equation is

$$x_1 + 2x_2 + 9x_3 = 0$$
$$x_2 + 4x_3 = 0$$

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Let, $x_3 = a$

$\therefore x_2 = -4a$

$\therefore x_1 = -2x_2 - 9x_3 = -2(-4a) - 9a = 8a - 9a = -a$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a \\ -4a \\ a \end{bmatrix} = a \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$$

$P_1 = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -7$

If $\lambda = 0$, from equation (ii) we get,

$$\left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ -2 & 8 & 2 & 0 \\ -1 & -2 & -2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ -1 & 4 & 1 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \rightleftharpoons R_3 \\ R_1' = (-1)R_1 \\ R_3' = (-1)R_3 \\ R_2' = (\frac{1}{2})R_2 \end{array}$$

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$$= \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & -6 & -3 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + R_1 \\ R_3' = R_3 - 2R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3' = R_3 - R_2$$

The corresponding system of the linear equation is

$$x_1 + 2x_2 + 2x_3 = 0$$

$$x_2 + \frac{1}{2}x_3 = 0$$

Let, $x_3 = b$

$\therefore x_2 = \frac{-b}{2}$

$\therefore x_1 = -2x_2 - 2x_3 = -2\left(\frac{-b}{2}\right) - 2b = b - 2b = -b$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -b \\ -b/2 \\ b \end{bmatrix} = b \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

$P_2 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 0$.

If $\lambda = 3$, from eq^n (ii) we get,

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ -2 & 11 & 2 & 0 \\ -1 & -2 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 + R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_2' = \left(\frac{1}{15}\right) R_2$$

The corresponding system of the linear equation is

$$x_1 + 2x_2 - x_3 = 0$$

$$x_2 = 0$$

Let, $x_3 = c$

$\therefore \ x_1 = c$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ c \end{bmatrix} = c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore \ P_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$

Ans.

Q. 14. Find a matrix P that diagonalizes the following matrices, Also find $P^{-1}AP$.

(i) $$A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$$

Sol: The characteristic matrix is,

$$\lambda I_2 - A = \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda+14 & -12 \\ 20 & \lambda-17 \end{bmatrix} \quad \cdots\cdots (i)$$

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The characteristic equation is

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda+14 & -12 \\ 20 & \lambda-17 \end{vmatrix} = 0$$

$\Rightarrow (\lambda+14)(\lambda-17) - (-12)\times 20 = 0$

$\Rightarrow \lambda^2 - 17\lambda + 14\lambda - 238 + 240 = 0$

$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow \lambda^2 - 2\lambda - \lambda + 2 = 0$

$\Rightarrow \lambda(\lambda-2) - 1(\lambda-2) = 0$

$\Rightarrow (\lambda-2)(\lambda-1) = 0$

$\therefore \lambda = 2, \quad \lambda = 1 \rightarrow$ eigen values of the given matrix.

Now let us consider $(\lambda I - A)x = 0$

From equation (i) $\rightarrow$

$$\left[\begin{array}{cc|c} \lambda+14 & -12 & 0 \\ 20 & \lambda-17 & 0 \end{array}\right] \quad \cdots \text{(ii)}$$

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If $\lambda = 1$, from equation (ii) we obtain,

$$\left[\begin{array}{cc|c} 15 & -12 & 0 \\ 20 & -16 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & -12/15 & 0 \\ 20 & -16 & 0 \end{array}\right] \quad R_1' = \left(\frac{1}{15}\right) R_1$$

$$= \left[\begin{array}{cc|c} 1 & -4/5 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad R_2' = R_2 - 20R_1$$

The corresponding system of the linear equation is,

$$x_1 - \frac{4}{5} x_2 = 0$$

Let, $x_2 = R$

$\therefore x_1 = \frac{4R}{5}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{4R}{5} \\ R \end{bmatrix} = R \begin{bmatrix} 4/5 \\ 1 \end{bmatrix}$$

$\therefore P_1 = \begin{bmatrix} 4/5 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

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If $\lambda = 2$, from equation (ii) we obtain,

$$\left[\begin{array}{cc|c} 16 & -12 & 0 \\ 20 & -15 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 1 & -3/4 & 0 \\ 20 & -15 & 0 \end{array}\right] \qquad R_1' = \left(\frac{1}{16}\right) R_1$$

$$= \left[\begin{array}{cc|c} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad R_2' = R_2 - 20R_1$$

The corresponding system of the linear equation is,

$$x_1 - \frac{3}{4} x_2 = 0$$

Let, $x_2 = s$

$$\therefore x_1 = \frac{3}{4} s$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3/4\, s \\ s \end{bmatrix} = s \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$$

$P_2 = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 2$

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$$P = \begin{bmatrix} 4/5 & 3/4 \\ 1 & 1 \end{bmatrix} \qquad P_{co} = \begin{bmatrix} 1 & -3/4 \\ -1 & 4/5 \end{bmatrix}$$

$$\det(P) = \frac{4}{5} - \frac{3}{4} = \frac{1}{20}$$

$$\therefore P^{-1} = \frac{\operatorname{adj}(P)}{\det(P)} = 20 \begin{bmatrix} 1 & -3/4 \\ -1 & 4/5 \end{bmatrix} = \begin{bmatrix} 20 & -15 \\ -20 & 16 \end{bmatrix}$$

$\therefore P^{-1} AP$

$$= \begin{bmatrix} 20 & -15 \\ -20 & 16 \end{bmatrix} \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix} \begin{bmatrix} 4/5 & 3/4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -280+300 & 240-255 \\ 280-320 & -240+272 \end{bmatrix} \begin{bmatrix} 4/5 & 3/4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & -15 \\ -40 & +32 \end{bmatrix} \begin{bmatrix} 4/5 & 3/4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{20}\ 16-15 & 15-15 \\ -32+32 & -30+32 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \cancel{-64}0 & \cancel{-0}2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

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So the matrix A is diagonalizable.

The matrix P diagonalizes A.

(ii) $A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

The characteristic matrix is $= \lambda I - A$

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda+1 & 2 & 2 \\ -1 & \lambda-2 & -1 \\ 1 & 1 & \lambda \end{bmatrix} \quad \cdots\cdots \text{(i)}$$

The characteristic equation is $\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda+1 & 2 & 2 \\ -1 & \lambda-2 & -1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+1)\{(\lambda-2)\lambda - (-1)\cdot 1\} - 2\{(-1)\lambda - 1(-1)\} + 2\{(-1)\cdot 1 - 1\cdot(\lambda-2)\} = 0$$

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$\Rightarrow (\lambda+1)(\lambda^2-2\lambda+1) - 2(-\lambda+1) + 2(-1-\lambda+2) = 0$

$\Rightarrow (\lambda^3 - 2\lambda^2 + \lambda + \lambda^2 - 2\lambda + 1) - 2(1-\lambda) + 2(1-\lambda) = 0$

$\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0$

$\Rightarrow \lambda^2(\lambda-1) - 1(\lambda-1) = 0$

$\Rightarrow (\lambda-1)(\lambda^2-1) = 0$

$\therefore \lambda - 1 = 0$ & $\lambda^2 - 1 = 0$

$\therefore \lambda = 1$ $\therefore \lambda^2 = 1$

$\therefore \lambda = \pm 1$

The eigen values of the given matrix is $\lambda = -1, 1$

Now let us consider $(\lambda I - A)x = 0$

From equation (i) we get,

$$\left[\begin{array}{ccc|c} \lambda+1 & 2 & 2 & 0 \\ -1 & \lambda-2 & -1 & 0 \\ 1 & 1 & \lambda & 0 \end{array}\right] \text{.....(ii)}$$

If $\lambda = -1$, then from equation (ii) we obtain,

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$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ -1 & -3 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \rightleftharpoons R_3 \\ R_2' = (-1)R_2 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = (\frac{1}{2})R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R_2' = (\frac{1}{2})R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3' = R_3 - R_2$$

The corresponding system of the linear equation is

$$x + y - z = 0$$

$$y + z = 0$$

Let, $z = a$

$\therefore y = -a$

$\therefore\ x = z - y$

$= a - (-a) = a + a = 2a$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2a \\ -a \\ a \end{bmatrix} = a \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$P_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = -1$

Again, if $\lambda = 1$ then, from equation (ii) we obtain,

$$= \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_1' = \left(\frac{1}{2}\right) R_1 \\ R_2' = (-1) R_2 \\ \cancel{R_3' =} \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 - R_1 \end{array}$$

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The corresponding system of the linear equation is,

$$x + y + z = 0$$

Let, $z = b$ & $y = c$

$\therefore x = -b - c$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -b-c \\ c \\ b \end{bmatrix} = b \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$P_2 = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ eigen vector corresponding to $\lambda = 1$

$$P = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$(P|I) = \left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

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$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 2 & -1 & -1 & 1 & 0 & 0 \end{array}\right] \quad R_1' \rightleftharpoons R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & -3 & -1 & 1 & 0 & -2 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 + R_1 \\ R_3' = R_3 - 2R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 & 3 & 1 \end{array}\right] \quad R_3' = R_3 + 3R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1/2 & 3/2 & 1/2 \end{array}\right] \quad R_3' = \left(\frac{1}{2}\right) R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 3/2 & 1/2 \end{array}\right] \quad R_2' = R_2 - R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 3/2 & 1/2 \end{array}\right] \quad R_1' = R_1 - R_2$$

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$$P^{-1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 \\ 1/2 & 3/2 & 1/2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 \\ 1/2 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 + 1/2 - 1/2 & -1 + 1 - 1/2 & -1 + 1/2 + 0 \\ 1/2 - 1/2 - 1/2 & \cancel{-1+3-1/2} \quad 1 - 1 - 1/2 & \cancel{-1+3/2+0} \quad 1 - 1/2 + 0 \\ -1/2 + 3/2 - 1/2 & -1 + 3 - 1/2 & -1 + 3/2 + 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 \\ -1 & 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

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Diagonalize A

Defination: A square matrix A is called diagonaligable if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. the matrix P is said to be diagonalige of A.

Working rule

(i) At first find n linearly independent eigen vectors of A say, $P_1, P_2, P_3 \dots P_n$.

(ii) Secondly form the matrix P having $P_1, P_2, \dots P_n$ as its column vector.

(iii) Finally, the matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2, \dots \lambda_n$ as its successive entries, where λ_i is the eigen value corresponding to P_i for $i = 1, 2, \dots n$.

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Double Integrals

09/07/17

\# Iterated (or repeated) integration: A partial definate integral with respect to x is a function of y and hence can be integrated w.r.to y.

Similarly a partial definate integral w.r.to y can be integrated with respect to x.

ie. $$\int_c^d \int_a^b f(x,y)\, dx\, dy = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy$$

or, $$\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx$$

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14.1

~~Practice~~ Practice Sheet 4 Double Integrals

Ⅲ Evaluate the ~~inte~~ iterated integrals:

(1.) $\int_0^1 \int_0^2 (x+3)\, dy\, dx$

$= \int_0^1 [xy + 3y]_0^2\, dx$

$= \int_0^1 [(x\cdot 2 + 3\cdot 2) - (x\cdot 0 + 3\cdot 0)]\, dx$

$= \int_0^1 [2x+6]\, dx$

$= \left[\frac{2x^2}{2} + 6x\right]_0^1$

$= (1^2 + 6\cdot 1) - (0^2 + 6\cdot 0)$

$= 7$ Ans.

(2.) $\int_1^3 \int_{-1}^1 (2x-4y)\, dy\, dx$

$= \int_1^3 \left[2xy - \frac{4y^2}{2}\right]_{-1}^1 dx \quad = \int_1^3 [(2x-2) - (-2x-2)]\, dx$

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$$= \int_1^3 [2x - 2 + 2x + 2]\, dx$$

$$= \int_1^3 4x \, dx$$

$$= 4\left[\frac{x^2}{2}\right]_1^3$$

$$= 4\left(\frac{9^2}{2} - \frac{1}{2}\right)$$

$$= 16$$

Ans.

(3.) $$\int_{-1}^{0}\int_{2}^{5} dx\, dy$$

$$= \int_{-1}^{0} [x]_2^5 \, dy$$

$$= \int_{-1}^{0} [5-2]\, dy$$

$$= 3[y]_{-1}^{0}$$

$$= 3(0+1)$$

$$= 3$$

Ans.

(4.) $$\int_2^4 \int_0^1 x^2 y \, dx\, dy$$

$$= \int_2^4 \left[y\frac{x^3}{3}\right]_0^1 dy$$

$$= \int_2^4 y\left(\frac{1}{3} - 0\right) dy$$

$$= \frac{1}{3}\left[\frac{y^2}{2}\right]_2^4$$

$$= \frac{1}{3}\left(\frac{16}{2} - \frac{4}{2}\right)$$

$$= \frac{1}{3}(8-2)$$

$$= \frac{1}{3} \times 6$$

$$= 2$$

Ans.

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(5.)

$$\int_{-2}^{0}\int_{-1}^{2}(x^2+y^2)\,dx\,dy$$

$$=\int_{-2}^{0}\left[\frac{x^3}{3}+y^2x\right]_{-1}^{2}dy$$

$$=\int_{-2}^{0}\left[\left(\frac{8}{3}+2y^2\right)-\left(\frac{-1}{3}-y^2\right)\right]dy$$

$$=\int_{-2}^{0}\left(\frac{8}{3}+2y^2+\frac{1}{3}+y^2\right)dy$$

$$=\int_{-2}^{0}\left(3y^2+\frac{9}{3}\right)dy$$

$$=\left[3\frac{y^3}{3}+\frac{9}{3}y\right]_{-2}^{0}$$

$$=0-\left[(-2)^3+\frac{9}{3}(-2)\right]$$

$$=8+\frac{18}{3}$$

$$=14$$

Ans.

(6.)

$$\int_{0}^{\ln 3}\int_{0}^{\ln 2}e^{x+y}\,dy\,dx$$

$$=\int_{0}^{\ln 3}\int_{0}^{\ln 2}e^{x}.e^{y}\,dy\,dx$$

$$=\int_{0}^{\ln 3}e^{x}\left[e^{y}\right]_{0}^{\ln 2}dx$$

$$=\int_{0}^{\ln 3}e^{x}\left[e^{\ln 2}-e^{0}\right]dx$$

$$=\int_{0}^{\ln 3}e^{x}(2-1)\,dx$$

$$=\left[e^{x}\right]_{0}^{\ln 3}$$

$$=e^{\ln 3}-e^{0}$$

$$=3-1$$

$$=2$$

Ans.

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(7.) $\int_0^2 \int_0^1 y \sin x \, dy \, dx$

$= \int_0^2 \left[\sin x \frac{y^2}{2} \right]_0^1 dx$

$= \int_0^2 \left[\sin x \left(\frac{1}{2}\right) - \sin 0 \right] dx$

$= \int_0^2 \frac{\sin x}{2} \, dx$

$= \frac{-1}{2} \left[\cos x \right]_0^2$

$= \frac{-1}{2} (\cos 2 - \cos 0)$

$= \frac{-1}{2} (\cos 2 - 1)$ Ans.

(8.) $\int_4^6 \int_{-3}^7 dy \, dx$

$= \int_4^6 \left[y \right]_{-3}^7 dx$

$= \int_4^6 (7+3) \, dx$

$= \left[10x \right]_4^6$

$= 10 \, (6-4)$

$= 10 \times 2 = 20$

Ans.

(9) $\int_{\pi/2}^{\pi} \int_1^2 x \cos xy \, dy \, dx$

$\int_{\pi/2}^{\pi} x \left[\frac{\sin xy}{x} \right]_1^2 dx = \int_{\pi/2}^{\pi} \frac{x}{x} \left[\sin 2x - \sin x \right] dx$

$= \left[\frac{-\cos 2x}{2} + \cos x \right]_{\pi/2}^{\pi} = \left(\cos \pi - \frac{1}{2} \cos 2\pi\right) - \left(\cos \frac{\pi}{2} - \frac{1}{2} \cos \pi\right)$

$= (-1-1) - \left(0 + \frac{1}{2}\right) = -\frac{5}{2}$

Ans

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(10) $\int_0^1 \int_0^1 \frac{x}{(xy+1)^2} \, dy\,dx$

Let, $(xy+1) = z$

$\frac{d}{dy}(xy+1) = \frac{d}{dy}(z)$

$x = dz$

If $y=0$ then $z = 1$

If $y=1$ then $z = x+1$

$= \int_0^1 \int_1^{x+1} \frac{1}{z^2} \, dz\,dx$

$= -\int_0^1 \left[\frac{1}{z} \right]_1^{x+1} dx$

$= -\int_0^1 \left(\frac{1}{x+1} - 1 \right) dx$ $\quad = -\left[\ln(x+1) - x \right]_0^1$

$= -\int_0^1 \cancel{\left(\frac{1-x-1}{x+1} \right)} dx$ $\quad = -\left[\left\{ \ln(1+1) - 1 \right\} - \left\{ \ln(0+1) - 0 \right\} \right]$

$= -\left[\ln 2 - 1 - \ln 1 \right]$

$= 1 - \ln 2$

Ans.

(11) $$\int_0^{\ln 2}\int_0^1 xy\, e^{y^2x}\, dy\, dx$$

Let, $u = y^2x$

$$\frac{du}{dy} = 2yx$$

$$\frac{du}{2} = xy\, dy$$

if $y=0$ then, $u=0$

if $y=1$ then, $u=x$

$$= \int_0^{\ln 2}\left[\int_0^x \left(e^u \frac{du}{2}\right)\right]dx$$

$$= \int_0^{\ln 2} \frac{1}{2}\left[e^u\right]_0^x \, dx$$

$$= \frac{1}{2}\int_0^{\ln 2} (e^x - e^0)\, dx$$

$$= \frac{1}{2}\left[e^x - x\right]_0^{\ln 2}$$

$$= \frac{1}{2}\left\{(e^{\ln 2} - \ln 2) - (e^0 - 0)\right\}$$

$$= \frac{1}{2}(2 - \ln 2 - 1)$$

$$= \frac{1}{2}(1 - \ln 2)$$ Ans.

Evaluate the double integral over the rectangular region R.

(12) $$\iint_R 4xy^3 \, dA \qquad R = \{(x,y): -1 \le x \le 1, \; -2 \le y \le 2\}$$

$$= \int_{-1}^{1} \int_{-2}^{2} 4xy^3 \, dy \, dx$$

$$= \int_{-1}^{1} x \left[4x \frac{y^4}{4} \right]_{-2}^{2} dx$$

$$= \int_{-1}^{1} x \left(2^4 - (-2)^4\right) dx$$

$$= \int_{-1}^{1} x \times 0 \, dx$$

$$= 0$$

(13) $$\iint_R \frac{xy}{\sqrt{x^2+y^2+1}} \, dA \qquad R = \{(x,y): 0 \le x \le 1, \; 0 \le y \le 1\}$$

$$= \int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}} \, dx \, dy$$

Let, $z = x^2+y^2+1$

$$\frac{dz}{dx} = 2x$$

$$\therefore \frac{dz}{2} = x \, dx$$

If, $x = 0$, then $z = 1+y^2$

If, $x = 1$ then $z = 2+y^2$

$$= \frac{1}{2} \int_0^1 \int_{1+y^2}^{2+y^2} \frac{y}{\sqrt{z}} \, dz \, dy$$

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$$= \int_0^1 y \int_{1+y^2}^{2+y^2} \frac{1}{2\sqrt{z}}\, dz\, dy$$

$$= \int_0^1 y \left[\sqrt{z} \right]_{1+y^2}^{2+y^2} dy$$

$$= \int_0^1 y \left(\sqrt{2+y^2} - \sqrt{1+y^2} \right) dy$$

$$= \int_0^1 y\sqrt{2+y^2} - \int_0^1 y\sqrt{1+y^2}\, dy$$

$$= \frac{1}{2} \int_2^3 \sqrt{p}\, dp - \frac{1}{2} \int_1^2 \sqrt{q}\, dq$$

$$= \frac{1}{2} \times \frac{2}{3} \left[p^{3/2} \right]_2^3 - \frac{1}{2} \times \frac{2}{3} \left[q^{3/2} \right]_1^2$$

$$= \frac{1}{3} \left(3^{3/2} - 3^2 \right) - \frac{1}{3} \left(2^{3/2} - 1^{3/2} \right)$$

$$= \frac{1}{3} \left(3^{3/2} - 3^2 - 2^{3/2} + 1 \right) \text{ Ans.}$$

Let, $2 + y^2 = p$

$$\frac{dp}{dy} = 2y$$

$$\frac{dp}{2} = y\, dy$$

If, $y = 0$, then $p = 2$

if $y = 1$ then $p = 3$

Again, $1 + y^2 = q$

$$\frac{dq}{dy} = 2y$$

$$\frac{dq}{2} = y\, dy$$

If $y = 0$, then $q = 1$

if $y = 1$ then $q = 2$

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(14.) $\iint x\sqrt{1-x^2}\, dA$ $\quad R=\{(x,y): 0\le x\le 1;\ 2\le y\le 3\}$

$$= \int_2^3 \int_0^1 x\sqrt{1-x^2}\, dx\, dy$$

Let, $1-x^2 = z$

$$\frac{dz}{dx} = -2x$$

$$\frac{dz}{-2} = x\, dx$$

if, $x=0$ then $z=1$

if, $x=1$ then $z=0$

$$= -\frac{1}{2}\int_2^3 \int_1^0 \sqrt{z}\, dz\, dy$$

$$= -\frac{1}{2}\int_2^3 \left[\frac{z^{3/2}}{3/2}\right]_1^0 dy$$

$$= -\frac{1}{2}\times\frac{2}{3}\int_2^3 (0^{3/2} - 1^{3/2})\, dy$$

$$= \frac{+1}{3}\left[y\right]_2^3$$

$$= \frac{+1}{3}(3-2) = \frac{1}{3}$$

(15.) $\iint_R (x\sin y - y\sin x)\, dA$ $\quad R=\{(x,y): 0\le x\le \pi/2,\ 0\le y\le \frac{\pi}{3}\}$

$$= \int_0^{\pi/3}\int_0^{\pi/2} (x\sin y - y\sin x)\, dx\, dy$$

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$$= \int_0^{\pi/3} \left\{ \sin y \left[\frac{x^2}{2} \right]_0^{\pi/2} + y \left[\cos x \right]_0^{\pi/2} \right\} dy$$

$$= \int_0^{\pi/3} \left[\sin y \left\{ \frac{1}{2} \left(\frac{\pi}{2} \right)^2 \right\} + y \left(\cos \pi/2 - \cos 0 \right) \right] dy$$

$$= \int_0^{\pi/3} \left(\frac{\pi^2}{8} \sin y - y \right) dy$$

$$= -\frac{\pi^2}{8} \left[\cos y \right]_0^{\pi/3} - \left[\frac{y^2}{2} \right]_0^{\pi/3}$$

$$= \frac{-\pi^2}{8} \left(\cos \frac{\pi}{3} - \cos 0 \right) - \left\{ \frac{1}{2} \left(\frac{\pi}{3} \right)^2 - \left(\frac{0}{2} \right) \right\}$$

$$= \frac{-\pi^2}{8} \left(\frac{1}{2} - 1 \right) - \frac{\pi^2}{18}$$

$$= \frac{-\pi^2}{16} + \frac{\pi^2}{8} - \frac{\pi^2}{18}$$

$$= 0.07$$

Ans.

14.2

Double integrals over Nonrectangular Regions

① 16.

$$\int_0^1 \int_{x^2}^{x} xy^2 \, dy \, dx$$

$$= \int_0^1 x \left[\frac{y^3}{3} \right]_{x^2}^{x} dx$$

$$= \frac{1}{3} \int_0^1 x \left\{ x^3 - (x^2)^3 \right\} dx$$

$$= \frac{1}{3} \int_0^1 (x^4 - x^7) \, dx$$

$$= \frac{1}{3} \left[\frac{x^5}{5} - \frac{x^8}{8} \right]_0^1$$

$$= \frac{1}{3} \left(\frac{1}{5} - \frac{1}{8} \right)$$

$$= \frac{1}{40}$$ Ans.

② 17.

$$\int_1^{3/2} \int_y^{3-y} y \, dx \, dy$$

$$= \int_1^{3/2} y \left[x \right]_y^{3-y} dy$$

$$= \int_1^{3/2} y (3 - y - y) \, dy$$

$$= \int_1^{3/2} (3y - 2y^2) \, dy$$

$$= \left[3 \frac{y^2}{2} - 2 \frac{y^3}{3} \right]_1^{3/2}$$

$$= \left\{ \frac{3}{2} \cdot \left(\frac{3}{2}\right)^2 - \frac{2}{3} \left(\frac{3}{2}\right)^3 \right\} - \left\{ \frac{3}{2} \cdot 1 - \frac{2}{3} 1^2 \right\}$$

$$= \cancel{\left(\frac{27}{8} - \frac{3}{2} \right) - \frac{5}{6}}$$

$$= \frac{7}{24}$$ Ans.

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③ ~~18~~.

$$\int_0^3 \int_0^{\sqrt{9-y^2}} y \, dx \, dy$$

$$= \int_0^3 y [x]_0^{\sqrt{9-y^2}} dy$$

$$= \int_0^3 y \sqrt{9-y^2} \, dy$$

$$= \frac{1}{-2} \int_9^0 \sqrt{z} \, dz$$

$$= \frac{1}{-2} \times \frac{2}{3} [z^{3/2}]_9^0$$

$$= \frac{-1}{3} (-9^{3/2}) = 9 \quad \text{Ans.}$$

Let, $9 - y^2 = z$

$$\frac{dz}{dy} = -2y$$

$$\frac{dz}{-2} = y \, dy$$

if, $y = 0$ then $z = 9$

if, $y = 3$ then $z = 0$

④ ~~19~~.

$$\int_{1/4}^{1} \int_{x^2}^{x} \frac{\sqrt{x}}{\sqrt{y}} \, dy \, dx$$

$$= \int_{1/4}^{1} \sqrt{x} \int_{x^2}^{x} y^{-1/2} \, dy \, dx = \int_{1/4}^{1} \sqrt{x} \left[\frac{y^{1/2}}{1/2} \right]_{x^2}^{x} dx = \int_{1/4}^{1} \sqrt{x} \cdot 2(\sqrt{x} - x) \, dx$$

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$$= 2\left[\int_{1/4}^{1} x\,dx - \int_{1/4}^{1} x^{3/2}\,dx\right]$$

$$= 2\left[\frac{x^2}{2} - \frac{2}{5}x^{5/2}\right]_{1/4}^{1}$$

$$= 2\left[\left(\frac{1}{2} - \frac{2}{5}\right) - \left\{\frac{(1/4)^2}{2} - \frac{2}{5}\left(\frac{1}{4}\right)^{5/2}\right\}\right]$$

$$= 2\left\{\frac{1}{10} - \left(\frac{1}{32} - \frac{1}{80}\right)\right\}$$

$$= \frac{13}{80} \quad \text{Ans.}$$

(5) 20.

$$\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_{0}^{x^3} \sin\frac{y}{x}\,dy\,dx$$

$$= -\int_{\pi}^{\sqrt{2\pi}} x\left[\cos\frac{y}{x}\right]_0^{x^3} dx$$

$$= -\int_{\pi}^{\sqrt{2\pi}} x\left(\cos\frac{x^3}{x} - \cos\frac{0}{x}\right) dx \quad = -\int_{\pi}^{\sqrt{2\pi}} x(\cos x^2 - 1)\,dx$$

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$$= -\frac{1}{2}\int_{\sqrt{\pi}}^{\sqrt{2\pi}} 2x\cos x^2\,dx + \int_{\sqrt{\pi}}^{\sqrt{2\pi}} x\,dx$$

Let, $z = x^2$

$\therefore \frac{dz}{dx} = 2x$

$\therefore dz = 2x\,dx$

$$= -\frac{1}{2}\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \cos z\,dz + \frac{1}{2}\left[x^2\right]_{\sqrt{\pi}}^{\sqrt{2\pi}}$$

$$= -\frac{1}{2}\left[\sin z\right]_{\sqrt{\pi}}^{\sqrt{2\pi}} + \frac{1}{2}\left[(\sqrt{2\pi})^2 - (\sqrt{\pi})^2\right]$$

$$= -\frac{1}{2}\left(\sin\sqrt{2\pi} - \sin\sqrt{\pi}\right) + \frac{1}{2}(2\pi - \pi)$$

$$= -\frac{1}{2}(0-0) + \frac{1}{2}(\pi) = \frac{\pi}{2}$$ Ans.

(6) $$\int_{-1}^{1}\int_{-x^2}^{x^2} (x^2 - y)\,dy\,dx$$

$$= \int_{-1}^{1}\left[x^2 y - \frac{y^2}{2}\right]_{-x^2}^{x^2} dx$$

$$= \int_{-1}^{1}\left[\left(x^2 \cdot x^2 - \frac{x^4}{2}\right) - \left\{x^2(-x^2) - \frac{(-x^2)^2}{2}\right\}\right] dx$$

$$= \int_{-1}^{1}\left[x^4 - \frac{x^4}{2} + x^4 + \frac{x^4}{2}\right] dx = \int_{-1}^{1} 2x^4\,dx$$

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$$= 2\left[\frac{x^5}{5}\right]_{-1}^{1}$$

$$= 2\left(\frac{1}{5}+\frac{1}{5}\right)$$

$$= 2\cdot\frac{2}{5} = \frac{4}{5} \quad \text{Ans.}$$

(7.) $$\int_0^1\int_0^x y\sqrt{x^2-y^2}\,dy\,dx$$

Let, $x^2-y^2 = z$

$$\frac{dz}{dy} = -2y$$

$$\therefore dz = -2y\,dy$$

If $y=0$, then $z=x^2$

If $y=x$ then $z=0$

$$= -\frac{1}{2}\int_0^1\int_0^x -2y\sqrt{x^2-y^2}\,dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_{x^2}^{0}\sqrt{z}\,dz\,dx$$

$$= -\frac{1}{2}\int_0^1 \frac{2}{3}\left[z^{3/2}\right]_{x^2}^{0}\,dx$$

$$= \frac{-1}{3}\int_0^1 \cancel{x}-(x^2)^{3/2}\,dx$$

$$= \frac{1}{3}\int_0^1 x^3\,dx = \frac{1}{3}\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{3}\left(\frac{1}{4}-\frac{0}{4}\right) = \frac{1}{12} \quad \text{Ans.}$$

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(8) $$\int_1^2 \int_0^{y^2} e^{x/y^2}\, dx\, dy$$

$$= \int_1^2 \int_0^1 y^2 e^u\, du\, dy$$

$$= \int_1^2 y^2 \left[e^u \right]_0^1 dy$$

$$= \int_1^2 y^2 (e^1 - e^0)\, dy$$

$$= \int_1^2 y^2 (e-1)\, dy$$

$$= (e-1)\left[\frac{y^3}{3} \right]_1^2$$

$$= (e-1)\left[\frac{2^3}{3} - \frac{1^3}{3} \right]$$

$$= \frac{7}{3}(e-1)$$ Ans.

Let, $u = \frac{x}{y^2}$

$$\frac{du}{dx} = \frac{1}{y^2} \cdot 1$$

$$y^2\, du = dx$$

If $x = 0$ then $u = 0$

If $x = y^2$ then $u = 1$

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15. $\iint_R x^2 \, dA$; R is the region bounded by $y = \frac{16}{x}$, $y = x$, $x = 8$

Sol: $\because y = \frac{16}{x}$ & $y = x$

$\therefore x = \frac{16}{x}$

$\Rightarrow x^2 = 16$

$\therefore x = 4$

$$\int_4^8 \int_{16/x}^{x} x^2 \, dy \, dx$$

$$= \int_4^8 x^2 [\, y \,]_{16/x}^{x} \, dx$$

$$= \int_4^8 x^2 [\, x - 16/x \,] \, dx$$

$$= \int_4^8 x^2 \left[\frac{x^2 - 16}{x} \right] dx$$

$$= \int_4^8 x (x^2 - 16) \, dx = \int_4^8 (x^3 - 16x) dx = \left[\frac{x^4}{4} - \frac{16x^2}{2} \right]_4^8$$

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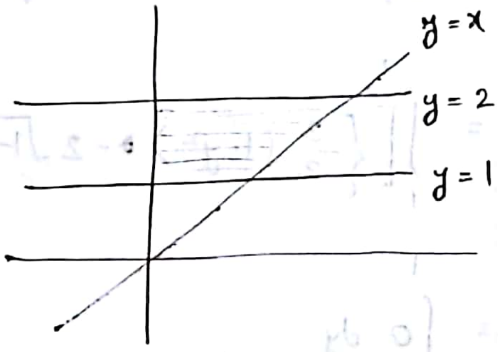
$$= \left(\frac{8^4}{4} - 8\cdot 8^2\right) - \left(\frac{4^4}{4} - 8\cdot 4^2\right)$$

$$= 512 + 64 = 576$$

Ans.

16. $\iint_R xy^2\, dA$ R is the region enclosed by $y=1$, $y=2$, $x=0$, $y=x$

Sol: $\int_1^2 \int_0^y xy^2\, dx\, dy$



$$= \int_1^2 y^2 \left[\frac{x^2}{2}\right]_0^y dy$$

$$= \int_1^2 y^2 \left(y^2/2 - 0\right) dy$$

$$= \frac{1}{2}\int_1^2 y^4\, dy$$

$$= \frac{1}{2}\left[\frac{y^5}{5}\right]_1^2$$

$$= \frac{1}{10}\left(2^5 - 1^5\right) = \frac{31}{10}$$ Ans.

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17. $\iint_R (3x-2y)\,dA$; R is region enclosed by the circle $x^2+y^2=1$

Sol:
$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (3x-2y)\,dx\,dy$$

$$= \int_{-1}^{1}\left[3\frac{x^2}{2} - 2xy\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy$$

$$= \int_{-1}^{1}\left[\left\{\cancel{\frac{3}{2}(1-y^2)} - \cancel{2\sqrt{1-y^2}\cdot y}\right\} - \left\{\cancel{\frac{3}{2}\cdot(1-y^2)} + \cancel{2y\sqrt{1-y^2}}\right\}\right] dy$$

$$= \int_{-1}^{1} 0\,dy$$

$$= 0$$ Ans.

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18. $\iint_R y\,dA$; R is the region in the first quardant enclosed between the circle $x^2+y^2=25$ and the line $x+y=5$.

Sol: $\int_0^5 \int_{5-y}^{\sqrt{25-y^2}} y\,dx\,dy$

$= \int_0^5 y[\,x\,]_{5-y}^{\sqrt{25-y^2}}\,dy$

$= \int_0^5 y(\sqrt{25-y^2} - 5 + y)\,dy$

$= \int_0^5 (y\sqrt{25-y^2} - 5y + y^2)\,dy$

$= \int_0^5 (y\sqrt{25-y^2})dy + \int_0^5 (y^2 - 5y)\,dy$

$= -\frac{1}{2}\int_{25}^{0} \sqrt{z}\,dz + \left[\frac{y^3}{3} - 5\cdot\frac{y^2}{2}\right]_0^5$

$= -\frac{1}{2}\cdot\frac{2}{3}\left[z^{3/2}\right]_{25}^{0} + \left(\frac{5^3}{3} - \frac{5}{2}\cdot 5^2\right)$

Let, $z = 25-y^2$

$\frac{dz}{dy} = -2y$

$dz = -2y\,dy$

If, $y=0$ then, $z = 25$

If $y=5$ then $z=0$

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$= \frac{-1}{3}\left(-25^{3/2}\right) + \frac{125}{3} - \frac{125}{2}$

$= \frac{125}{3} - \frac{125}{6}$

$= \frac{125}{6}$ Ans.

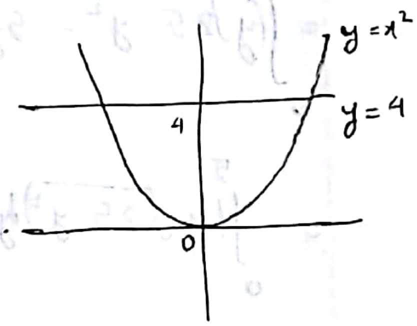
19. $\iint_R x(1+y^2)^{-1/2}$; R is the region in the first quardrant enclosed by $y=x^2$, $y=4$ and $x=0$.

Sol: $\int_0^4 \int_0^{\sqrt{y}} x(1+y^2)^{-1/2}\, dx\, dy$

$\because x=0 \therefore y = 0^2 = 0$

$\because y = x^2 \therefore x = \sqrt{y}$

$= \int_0^4 (1+y^2)^{-1/2} \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy$



$= \int_0^4 (1+y^2)^{-1/2} (y/2 - 0)\, dy$

$= \frac{1}{2}\int_0^4 y(1+y^2)^{-1/2}\, dy$

Let, $u = 1+y^2$

$\frac{du}{dy} = 2y$

$= \frac{1}{2}\int_0^4 \frac{y}{\sqrt{1+y^2}}\, dy$

$\therefore du = 2y\, dy$

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$$= \frac{1}{2} \cdot \frac{1}{2} \int_{1}^{17} \frac{du}{\sqrt{u}}$$

If $y = 0$ then $u = 1$

If $y = 4$ then $u = 17$

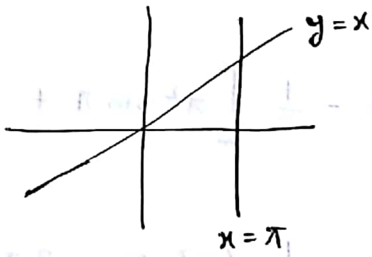
$$= \frac{1}{4} \left[2\sqrt{u} \right]_{1}^{17}$$

$$= \frac{1}{4} \left(2\sqrt{17} - 2\sqrt{1} \right)$$

$$= \frac{2}{4} \left(\sqrt{17} - 1 \right) = \frac{1}{2} \left(\sqrt{17} - 1 \right)$$ Ans.

20. $\iint_R x \cos y \, dA$; R is the triangular region bounded by lines $y = x$, $y = 0$ and $x = \pi$

Sol: $$\int_0^{\pi} \int_y^{\pi} x \cos y \, dx \, dy$$



$$= \int_0^{\pi} \cos y \left[\frac{x^2}{2} \right]_y^{\pi} dy$$

$$= \int_0^{\pi} \cos y \left(\frac{\pi^2}{2} - \frac{y^2}{2} \right) dy = \frac{\pi^2}{2} \int_0^{\pi} \cos y \, dy - \frac{1}{2} \int_0^{\pi} y^2 \cos y \, dy$$

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$$= \frac{\pi^2}{2}\left[\sin y\right]_0^{\pi} - \frac{1}{2}\left[y^2\int \cos y\,dy - \int\left\{\frac{d}{dy}(y^2)\int \cos y\,dy\right\}dy\right]_0^{\pi}$$

$$= \frac{\pi^2}{2}\left[\sin\pi - \sin 0\right] - \frac{1}{2}\left[y^2\sin y - 2\int y\sin y\,dy\right]_0^{\pi}$$

$$= \frac{\pi^2}{2}(0-0) - \frac{1}{2}\left[y^2\sin y - 2\left\{y\int \sin y\,dy - \int\left(\frac{d}{dy}(y)\int \sin y\,dy\right)dy\right\}\right]_0^{\pi}$$

$$= \frac{\pi^2}{2}\times 0 - \frac{1}{2}\left[y^2\sin y - 2\left\{-y\cos y + \int 1\cdot\cos y\,dy\right\}\right]_0^{\pi}$$

$$= 0 - \frac{1}{2}\left\{y^2\sin y - 2(-y\cos y - \sin y)\right\}_0^{\pi}$$

$$= -\frac{1}{2}\left[y^2\sin y + 2y\cos y + 2\sin y\right]_0^{\pi}$$

$$= -\frac{1}{2}\left[\pi^2\sin\pi + 2\pi\cos\pi + 2\sin\pi - 0\right]$$

$$= -\frac{1}{2}\left(\pi^2\cdot 0 + 2\pi(-1) + 2\cdot 0\right)$$

$$= -\frac{1}{2}(-2\pi)$$

$$= \pi$$

Ans.

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21. $\iint_R xy\, dA$; R is the region enclosed by $y=\sqrt{x}$, $y=6-x$, $y=0$

Sol: $\because$ $y=\sqrt{x}$ & $y=6-x$ Limit for $y = 0, 2$

$\therefore$ $x=y^2$ & $x=6-y$ & $x = y^2 , 6-y$

$\therefore$ $y^2 = 6-y$

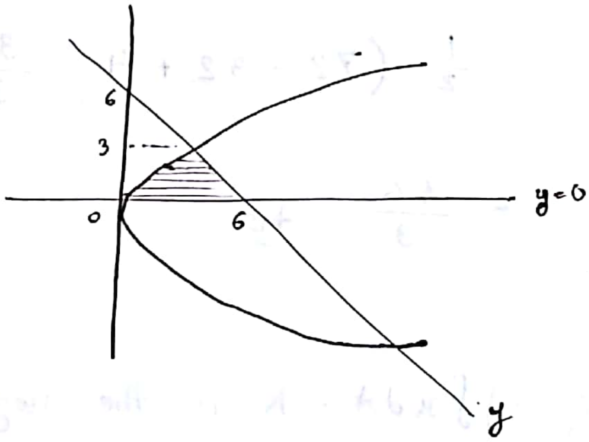
$\Rightarrow$ $y^2+y-6=0$

$\Rightarrow$ $y^2+3y-2y-6=0$

$\Rightarrow$ $y(y+3)-2(y+3)=0$

$\Rightarrow$ $(y+3)(y-2)=0$

$\therefore$ $y=-3$ & $y=2$



$\therefore \int_0^2 \int_{y^2}^{6-y} xy\, dx\, dy$

$= \int_0^2 y \left[\frac{x^2}{2}\right]_{y^2}^{6-y} dy$

$= \frac{1}{2}\int_0^2 y\{(6-y)^2-(y^2)^2\} dy = \frac{1}{2}\int_0^2 y(36-12y-y^2-y^4)\, dy$

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$$= \frac{1}{2} \int_0^2 (36y - 12y^2 + y^3 - y^5)\, dy$$

$$= \frac{1}{2} \left[\frac{36y^2}{2} - \frac{12y^3}{3} + \frac{y^4}{4} - \frac{y^6}{6} \right]_0^2$$

$$= \frac{1}{2} \left\{ \left(18 \cdot 2^2 - 4 \cdot 2^3 + \frac{2^4}{4} - \frac{2^6}{6}\right) - 0 \right\}$$

$$= \frac{1}{2} \left(72 - 32 + 4 - \frac{32}{3}\right)$$

$$= \frac{56}{3}$$ Ans.

22. $\iint x\, dA$; R is the region enclosed by $y = \sin^{-1} x$, $x = \frac{1}{\sqrt{2}}$, $y = 0$

Sol: $\int_0^{\pi/4} \int_{\sin y}^{1/\sqrt{2}} x\, dx\, dy$

Limit for $x = \frac{1}{\sqrt{2}}$, $\sin y$

& $y = 0$, $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45° = \frac{\pi}{4}$

$$= \frac{1}{2} \int_0^{\pi/4} \left[x^2\right]_{\sin y}^{1/\sqrt{2}} dy$$

$$= \frac{1}{2} \int_0^{\pi/4} \left(\frac{1}{2} - \sin^2 y\right) dy$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{2}\,dy - \frac{1}{2}\int_0^{\pi/4} \sin^2 y \; dy$$

$$= \frac{1}{4}\left[x\right]_0^{\pi/4} - \frac{1}{2}\times\frac{1}{2}\int_0^{\pi/4}(1-\cos 2y)\,dy \qquad \left[\because 2\sin^2\theta = 1-\cos 2\theta\right]$$

$$= \frac{1}{4}\times\frac{\pi}{4} - \frac{1}{4}\left[y - \frac{\sin 2y}{2}\right]_0^{\pi/4}$$

$$= \frac{\pi}{16} - \frac{1}{4}\left\{\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right\}$$

$$= \frac{\pi}{16} - \frac{\pi}{16} + \frac{1}{2}$$

$$= \frac{1}{2} \quad \text{Ans.}$$

23. $\iint_R (x-1)\,dA$; R is the region in the 1st quadrant enclosed between $y=x$ and $y=x^3$

Sol: $\because y=x$ & $y=x^3$

$\therefore x = x^3$

$\Rightarrow x^3 - x = 0 \qquad \Rightarrow x(x^2-1)=0$

$\therefore x = 0 \qquad$ & $\qquad x^2-1=0$

$\therefore x = \pm 1$

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$$\int_0^1 \int_{x^3}^{x} (x-1)\, dy\, dx$$

$$= \int_0^1 \cancel{(x-1)}\ \cancel{(x-x}[xy-y]_{x^3}^{x}\ dx$$

$$= \int_0^1 \{(x\cdot x - x) - (x\cdot x^3 - x^3)\}\ dx$$

$$= \int_0^1 (x^2 - x - x^4 + x^3)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - \frac{x^5}{5} + \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{2} - \frac{1}{5} + \frac{1}{4}$$

$$= \frac{-7}{60}$$ Ans.

24. $\iint x^2\, dA$; R is the region in the first quardrant enclosed by

$xy = 1$, $y = x$, $y = 2x$

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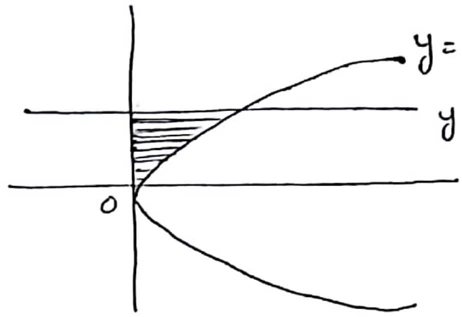
25. $\iint_R \sin(y^3)\, dA$; R is the region enclosed by $y=\sqrt{x}$, $y=2$, $x=0$

Sol: $\int_0^2 \int_0^{y^2} \sin y^3 \, dx\, dy$



$= \int_0^2 \left[x \sin y^3 \right]_0^{y^2} dy$

$= \int_0^2 (y^2 \sin y^3 - 0)\, dy$

Now,

$y^2 \int \sin y^3 \, dy - \int \left\{ \frac{d}{dy}(y^2) \int \sin y^3 \, dy \right\} dy$

$=$

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26. Evaluate $\iint_R x\,dA$; where R is the region bounded by $x = \ln y$, $x = 0$ and $y = e$

$\because x = \ln y$

$\therefore e^x = e^{\ln y}$

$\Rightarrow e^0 = y \quad [\because x = 0]$

$\therefore y = 1$

Sol:

$$\int_1^e \int_0^{\ln y} x\,dx\,dy$$

$$= \int_1^e \left[\frac{x^2}{2}\right]_0^{\ln y} dy$$

$$= \frac{1}{2}\int_1^e (\ln y^2 - 0)\,dy$$

$$= \frac{1}{2}\left\{\ln y^2 \int 1 \cdot dy - \int \frac{d}{dy}(\ln(y^2)) \int 1\,dy\right\}_1^e$$

$$= \frac{1}{2}\left\{y \ln y^2 - \int \left(\frac{1}{y^2} \cdot 2y \cdot y\right) dy\right\}_1^e$$

$$= \frac{1}{2}\left\{y \ln y^2 - 2\int dy\right\}_1^e$$

$$= \frac{1}{2}(y \ln y^2 - 2y) = \frac{1}{2}\left[(e \ln e^2 - 2e) - (1 \ln 1 - 2 \cdot 1)\right]$$

$$= \frac{1}{2}\left[(2e - 2e) - (0 - 2)\right] = \frac{1}{2}(0 + 2) = 1$$

Ans.

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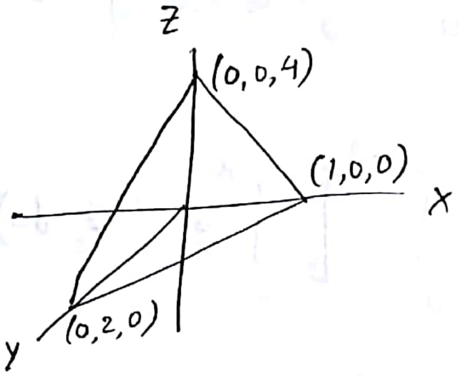
Double Integral

11/07/17

Find the volume of tera hedron bounded by co-ordinate planes & $z = 4 - 4x - 2y$ (Ans: $4/3$)



$z = 4 - 4x - 2y$

$x = 0,\ y = 0,\ z = 4$

$y = 0,\ z = 0,\ x = 1$

$x = 0,\ z = 0,\ y = 2$

If $z = 0$ then

$\Rightarrow 0 = 4 - 4x - 2y$

$\Rightarrow 2y = 4(1 - 2x)$

$\therefore y = 2 - 2x$

Sol: $V = \iint_R (4 - 4x - 2y)\, dA$

$$= \int_0^1 \int_{y=0}^{2-2x} (4 - 4x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[4y - 4xy - 2\frac{y^2}{2} \right]_0^{2-2x} dx$$

$$= \int_0^1 \left[4(2-2x) - 4x(2-2x) - (2-2x)^2 \right] dx$$

$$= \int_0^1 (8 - 8x - 8x + 8x^2 - 4 + 8x - 4x^2)\, dx$$

$$= \int_0^1 [4 - 8x + 4x^2]\, dx$$

$$= \left[4x - 8\cdot\frac{x^2}{2} + 4\cdot\frac{x^3}{3} \right]_0^1$$

$$= 4 - 4 + \frac{4}{3} = \frac{4}{3} \text{ Ans.}$$

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[1] Use a double integral to find the volume of the solid that is bounded by the plane $z = 4 - x - y$ and below the rectangle $R: \{(x,y): 0 \le x \le 1,\ 0 \le y \le 2\} \to$ (Ans: 5)

Sol: $V = \iint_R (4-x-y)\, dA$

$$= \int_{y=0}^{2} \int_{x=0}^{1} (4-x-y)\, dx\, dy$$

$$= \int_0^2 \left[4x - \frac{x^2}{2} - yx \right]_0^1 dy$$

$$= \int_0^2 \left[4 - \frac{1}{2} - y \right] dy$$

$$= \left[\frac{7}{2}y - \frac{y^2}{2} \right]_0^2$$

$$= \left[\frac{7}{2} \times 2 - \frac{2^2}{2} \right] = (7-2) = 5$$

Ans.

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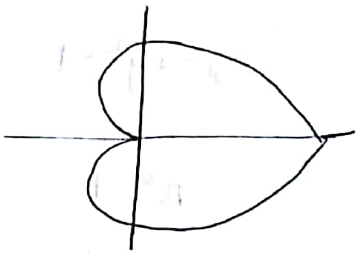
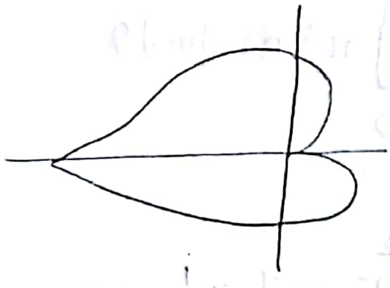
Practice Sheet 4

16/07/17

Cardioid:



$r = 1 + a\cos\theta$



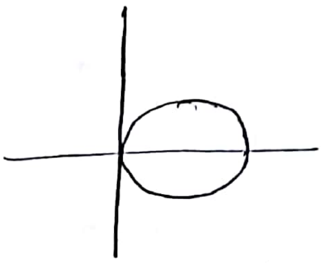
$r = 1 - a\cos\theta$

Rose Petals:

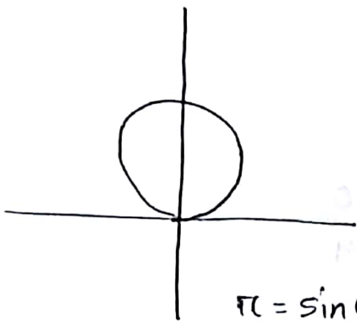
(i) $r = a\cos n\theta$

(ii) $r = a\sin n\theta$

$n \rightarrow$ no. of Petals



$r = \cos\theta$



$r = \sin\theta$

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14.3 Double Integrals in Polar Coordinates

Evaluate the iterated integral.

1. $$\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta \, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\cos\theta \, \frac{r^2}{2} \right]_0^{\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (\sin^2\theta \cos\theta - 0)\, d\theta$$

$$= \frac{1}{2}\int_0^{1} u^2\, du$$

$$= \frac{1}{2}\left[u^3/3 \right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3} - 0 \right)$$

$$= \frac{1}{6}$$ Ans.

Let, $u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$\therefore du = \cos\theta \, d\theta$

If, $\theta = 0$ then $u = 0$

if $\theta = \pi/2$ then, $u = 1$

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(2) $\int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta$

$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{1+\cos\theta} d\theta$

$= \frac{1}{2}\int_0^{\pi}\left\{(1+\cos\theta)^2 - 0\right\} d\theta$

$= \frac{1}{2}\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$

$= \frac{1}{2}\int_0^{\pi}\left\{1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right\} d\theta \qquad [\because 2\cos^2\theta = 1+\cos 2\theta]$

$= \frac{1}{2}\left[\theta - 2\sin\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi}$

$= \frac{1}{2}\left(\pi - 2\sin\pi + \frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right)$

$= \frac{1}{2}\left(\pi - 0 + \frac{\pi}{2} - 0\right)$

$= \frac{1}{2} \times \frac{3\pi}{2} = \frac{3\pi}{4}$ Ans.

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(3) $$\int_0^{\pi/2}\int_0^{a\sin\theta} r^2\,dr\,d\theta$$

$$=\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{a\sin\theta} d\theta$$

$$=\frac{1}{3}\int_0^{\pi/2}(a^3\sin^3\theta-0)\,d\theta$$

$$=\frac{a^3}{3}\int_0^{\pi/2}\sin\theta\,(1-\cos^2\theta)\,d\theta$$

$$=\frac{a^3}{3}\left[\int_0^{\pi/2}\sin\theta\,d\theta-\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta\right]$$

Let, $u=\cos\theta$

$\frac{du}{d\theta}=-\sin\theta$

$du=-\sin\theta\,d\theta$

If, $\theta=0$ then $u=1$

if $\theta=\frac{\pi}{2}$ then $u=0$

$$=\frac{a^3}{3}\left[-\cos\theta\right]_0^{\pi/2}+\frac{a^3}{3}\int_0^{\pi/2}u^2\,du$$

$$=\frac{a^3}{-3}\left[\cos\frac{\pi}{2}-\cos 0\right]+\frac{a^3}{3}\left[\frac{u^3}{3}\right]_1^0$$

$$=\frac{-a^3}{3}(0-1)+\frac{a^3}{3}\left(\frac{0}{3}-\frac{1}{3}\right)$$

$$=\frac{a^3}{3}-\frac{a^3}{9}\quad=\frac{2}{9}a^3$$

Ans.

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(4.) $\displaystyle\int_0^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta$

$$= \int_0^{\pi/6}\left[\frac{r^2}{2}\right]_0^{\cos 3\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6}\left(\cos^2 3\theta - 0\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6}\frac{1}{2}\left(1+\cos 6\theta\right)d\theta \qquad [\because 1+\cos 2\theta = 2\cos^2\theta]$$

$$= \frac{1}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi/6}$$

$$= \frac{1}{4}\left[\frac{\pi}{6} + \frac{1}{6}\sin\frac{6\pi}{6}\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{6}+0\right)$$

$$= \cancel{\frac{\pi}{6}} \quad \frac{\pi}{24}$$ Ans.

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(5.) $$\int_{0}^{\pi}\int_{0}^{1-\sin\theta} r^2\cos\theta \, dr\, d\theta$$

$$= \int_{0}^{\pi}\left[\frac{r^3}{3}\cos\theta\right]_0^{1-\sin\theta} d\theta$$

$$= \frac{1}{3}\int_{0}^{\pi}(1-\sin\theta)^3\cos\theta \, d\theta$$

$$= \frac{1}{3}\int_{0}^{\pi}\cos\theta\,(1-3\sin\theta+3\sin^2\theta-\sin^3\theta)\, d\theta$$

$$= \frac{1}{3}\int_{0}^{\pi}(\cos\theta-3\sin\theta\cos\theta+3\sin^2\theta\cos\theta-\sin^3\theta\cos\theta)\, d\theta$$

$$= \frac{1}{3}\left[\int_{0}^{\pi} du - 3\int_{0}^{\pi} u\, du + 3\int_{0}^{\pi} u^2\, du - \int_{0}^{\pi} u^3\, du\right]$$

Let, $u = \sin\theta$

$\therefore du = \cos\theta \, d\theta$

$$= \frac{1}{3}\left[u - 3\cdot\frac{u^2}{2} + 3\cdot\frac{u^3}{3} - \frac{u^4}{4}\right]_0^0$$

if, $\theta = 0$ then $\rightarrow u = 0$

if $\theta = \pi$ then $\rightarrow u = 0$

$$= 0$$

Ans.

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⑥ $$\int_0^{\pi/2}\int_0^{\cos\theta} r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{\cos\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}(\cos^4\theta - 0)\,d\theta = \frac{1}{4}\int_0^{\pi/2}\cos^2\theta\cdot\cos^2\theta\,d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\frac{1}{2}\times\frac{1}{2}(1+\cos 2\theta)(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{16}\int_0^{\pi/2}(1+\cos 2\theta)^2\,d\theta$$

$$= \frac{1}{16}\int_0^{\pi/2}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta$$

$$= \frac{1}{16}\int_0^{\pi/2}\left\{1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right\}d\theta$$

$$= \frac{1}{16}\int_0^{\pi/2}\left(1+2\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta\right)d\theta$$

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$$= \frac{1}{16}\left[\theta + \frac{2\sin 2\theta}{2} + \frac{\theta}{2} + \frac{\sin 4\theta}{2\cdot 4}\right]_0^{\pi/2}$$

$$= \frac{1}{16}\left[\frac{\pi}{2} + \sin\pi + \frac{\pi}{4} + \frac{1}{8}\sin 2\pi\right]$$

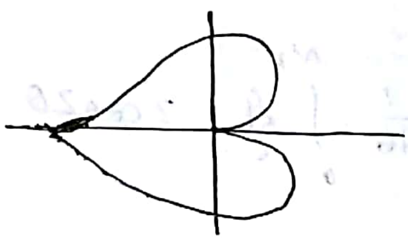
$$= \frac{1}{16}\left(\frac{\pi}{2} + 0 + \frac{\pi}{4} + 0\right)$$

$$= \frac{1}{16} \times \frac{3\pi}{4} = \frac{3\pi}{64}$$ Ans.

Use a double integral in polar coordinates to find the area of the region described.

(7) The region enclosed by the cardioid $r = 1 - \sin\theta$

Sol: $$\int_0^{2\pi}\int_0^{1-\cos\theta} r\,dr\,d\theta$$



$$= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{1-\cos\theta} d\theta = \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2\,d\theta$$

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$$= \frac{1}{2}\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta \qquad [\because 2\cos^2\theta = 1 + \cos 2\theta]$$

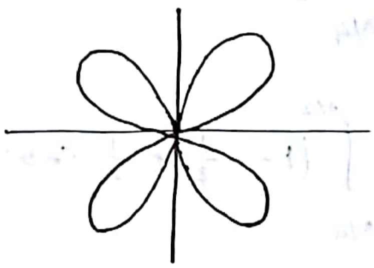
$$= \frac{1}{2}\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi - 0 + \pi + 0)$$

$$= \frac{\pi}{2} \quad \text{Ans.}$$

(8) The region enclosed by the rose $r = \sin 2\theta$

Sol: $$\int_0^{2\pi}\int_0^{\sin 2\theta} r\,dr\,d\theta$$



$$= \frac{1}{2}\int_0^{2\pi} \sin^2 2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1}{2}(1 - \cos 4\theta)\,d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi} = \frac{1}{4}\left(2\pi - \frac{1}{4}\sin 8\pi\right) = \frac{1}{4}(2\pi - 0) = \frac{\pi}{2}$$

Ans.

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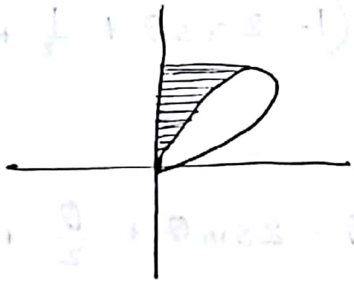
(9.) The region in the first quardrant bounded by $r = 1$ and $r = \sin 2\theta$ with $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

Sol: $\displaystyle\int_{\pi/4}^{\pi/2} \int_{\sin 2\theta}^{1} r\,dr\,d\theta$



$$= \int_{\pi/4}^{\pi/2} \left[\frac{r^2}{2}\right]_{\sin 2\theta}^{1} d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} (1 - \sin^2 2\theta)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} \left\{1 - \frac{1}{2}(1 - \cos 4\theta)\right\} d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2} \left(1 - \frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{\theta}{2} + \frac{\sin 4\theta}{8}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4} + \frac{1}{8}\sin 2\pi - \frac{\pi}{4} + \frac{\pi}{8} - \frac{1}{8}\sin \pi\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{2\pi}{4} + \frac{\pi}{8}\right) = \frac{\pi}{16} \quad \text{Ans.}$$

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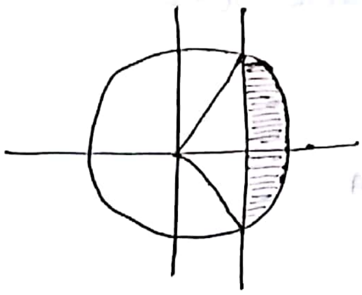
(10.) The region inside the circle $x^2+y^2=4$ and to the right of the line $x = 1$



Sol: $2\int_0^{\pi/3}\int_{\sec\theta}^{2} r\,dr\,d\theta$

$= 2\int_0^{\pi/3}\left[\frac{r^2}{2}\right]_{\sec\theta}^{2} d\theta$

$= \frac{2}{2}\int_0^{\pi/3}(2^2-\sec^2\theta)\,d\theta$

$= \left[4\theta - \tan\theta\right]_0^{\pi/3}$

$= \frac{4\pi}{3} - \tan\frac{\pi}{3} - 0$

$= \frac{4\pi - 3\sqrt{3}}{3}$

$\because x^2+y^2=4$

$\therefore r^2 = 2$

$\therefore r = \pm 2$

$\because x^2+y^2=4$

$\therefore y^2 = 4-x^2$

$= 4-1^2$

$= 3$

$\therefore y = \pm\sqrt{3}$

$x = 1$

$r\cos\theta = 1$

$r = \frac{1}{\cos\theta} = \sec\theta$

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

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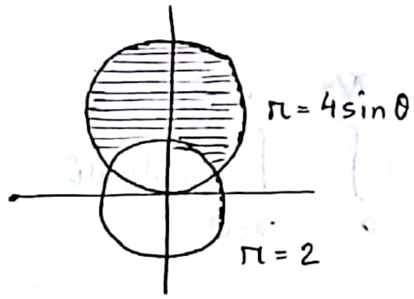
(11.) Find the area of the region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$



Sol:

Area $A = \iint_R dA$

$= \int_{\pi/6}^{5\pi/6} \int_{2}^{4\sin\theta} r\,dr\,d\theta$

$= \int_{\pi/6}^{5\pi/6} \left[\frac{r^2}{2}\right]_2^{4\sin\theta} d\theta$

$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (16\sin^2\theta - 4)\,d\theta$

$= \frac{4}{2}\int_{\pi/6}^{5\pi/6} (4\sin^2\theta - 1)\,d\theta$

$= 2\int_{\pi/6}^{5\pi/6} \{2(1-\cos 2\theta) - 1\}\,d\theta$

$= \int_{\pi/6}^{5\pi/6} (2 - 2\cos 2\theta - 1)\,d\theta$

$\because r = 4\sin\theta$ & $r = 2$

$\therefore 4\sin\theta = 2$

$\Rightarrow \sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

$\therefore \theta = \frac{\pi}{6}$

$\sin\theta = \sin\frac{\pi}{6}$

$= \sin(\pi - \frac{\pi}{6})$

$= \sin\frac{5\pi}{6}$

$\therefore \theta = \frac{5\pi}{6}$

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$$= 2\int_{\pi/6}^{5\pi/6} (1 - 2\cos 2\theta)\, d\theta$$

$$= 2\left[\theta - 2\cdot\frac{\sin 2\theta}{2}\right]_{\pi/6}^{5\pi/6}$$

$$= 2\left\{\left(\frac{5\pi}{6} - \sin\frac{10\pi}{6}\right) - \left(\frac{\pi}{6} - \sin\frac{\pi}{3}\right)\right\}$$

$$= 2\left(\frac{4\pi}{6} - \sin\frac{5\pi}{3} + \sin\frac{\pi}{3}\right)$$

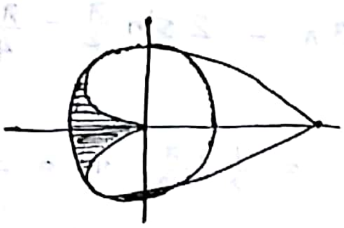
$$= 2\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)$$

$$= 2\left(\frac{2\pi}{3} + \frac{2\sqrt{3}}{2}\right)$$

$$= 4\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$$ Ans.

(12) The region inside the circle $r = 1$ and outside the cardioid $r = 1 + \cos\theta$

Sol:



$\because r = 1$ & $r = 1 + \cos\theta$

$\therefore 1 = 1 + \cos\theta$

$\therefore \cos\theta = 1 - 1 = 0$

$\therefore \theta = \cos^{-1}(0) = \frac{\pi}{2}$

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$$\therefore \int_{\pi/2}^{3\pi/2} \int_{1}^{1+\cos\theta} r\,dr\,d\theta$$

$$= \int_{\pi/2}^{3\pi/2} \left[\frac{r^2}{2} \right]_1^{1+\cos\theta} d\theta$$

$$= \frac{1}{2} \int_{\pi/2}^{3\pi/2} \{(1+\cos\theta)^2 - 1\} d\theta$$

$$= \frac{1}{2} \int_{\pi/2}^{3\pi/2} (1 + 2\cos\theta + \cos^2\theta - 1)\, d\theta$$

$$= \frac{1}{2} \int_{\pi/2}^{3\pi/2} \left\{2\cos\theta + \frac{1}{2}(1 + \cos 2\theta)\right\} d\theta$$

$$= \frac{1}{2} \int_{\pi/2}^{3\pi/2} \left(2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2} \left[2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{2}\left(2\sin\frac{3\pi}{2} + \frac{3\pi}{4} + \frac{1}{4}\sin 3\pi - 2\sin\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{4}\sin\pi\right)$$

$$= \frac{1}{2}\left(-2 + \frac{3\pi}{4} + 0 - 2 - \frac{\pi}{4} - 0\right) = \frac{1}{2}\left(\frac{\pi}{2} - 4\right) = \frac{1}{2}\left(\frac{\pi - 8}{2}\right) = \frac{\pi - 8}{4} \text{ Ans.}$$

$\because \cos\theta = \cos\frac{\pi}{2}$

$\therefore \cos\theta = \cos\left(2\pi - \frac{\pi}{2}\right)$

$= \cos\frac{3\pi}{2}$

$\therefore \theta = \frac{3\pi}{2}$

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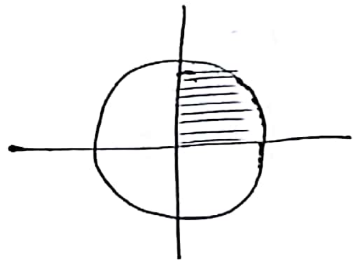
24. $\iint_R \sqrt{9-x^2-y^2}\, dA$ where R is the region in the first quardrant within the circle $x^2+y^2=9$

Sol:

$$\int_0^{\pi/2}\int_0^3 \sqrt{9-r^2}\, r\,dr\,d\theta$$



$\because x^2+y^2=9$

$\therefore r^2=9$

$\therefore r=3$

Let, $u=9-r^2$

$\frac{du}{dr}=-2r$

$\frac{du}{-2}=r\,dr$

$r=0 \longrightarrow u=9$

$r=3 \longrightarrow u=0$

$$= -\frac{1}{2}\int_0^{\pi/2}\int_0^3 \sqrt{u}\, du\, d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \frac{2}{3}\left[u^{3/2}\right]_9^0 d\theta$$

$$= -\frac{1}{2}\times\frac{2}{3}\int_0^{\pi/2} (0-9^{3/2})\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} 27\, d\theta$$

$$= \frac{1}{3}\left[27\theta\right]_0^{\pi/2}$$

$$= \frac{1}{3}\cdot 27\cdot\frac{\pi}{2} = 9\pi/2$$

Ans.

25. $\iint\limits_R \frac{1}{1+x^2+y^2}\, dA$; where R is the region in the first quardrant bounded by $y=0$, $y=x$ & $x^2+y^2=4$

Sol: $\int_0^{\pi/4}\int_0^2 \frac{1}{1+r^2}\, r\,dr\,d\theta$

Let, $u = 1+r^2$

$\frac{du}{dr} = 2r$

$\frac{du}{2} = r\,dr$

$r=0 \rightarrow u=1$

$r=2 \rightarrow u=5$

$\because x^2+y^2=4$

$\therefore r^2 = 2^2$

$\therefore r=2$

$\because x^2+y^2=4$

$\Rightarrow x^2+x^2=4$

$\Rightarrow 2x^2=4$

$\Rightarrow x^2=2$

$\Rightarrow x=\sqrt{2}$

$\Rightarrow r\cos\theta=\sqrt{2}$

$\Rightarrow \cos\theta = \frac{\sqrt{2}}{r} = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$\because x^2+y^2=4$

$\Rightarrow x^2+0=4$

$\Rightarrow x=2$

$\Rightarrow r\cos\theta = 2$

$\Rightarrow \cos\theta = \frac{2}{r} = 1$

$\therefore \theta = \cos^{-1}(1) = 0$

$= \int_0^{\pi/4}\int_0^2 \frac{1}{u}\,\frac{du}{2}\,d\theta$

$= \frac{1}{2}\int_0^{\pi/4} \left[\ln u\right]_1^5\, d\theta$

$= \frac{1}{2}\int_0^{\pi/4} (\ln 5 - \ln 1)\,d\theta$

$= \frac{1}{2}\ln 5 \int_0^{\pi/4} d\theta$

$= \frac{1}{2}\ln 5\,[\theta]_0^{\pi/4}$

$= \frac{1}{2}\ln 5 \cdot \frac{\pi}{4} = \frac{\pi}{8}\ln 5$

Ans.

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14.6

* Rectangular to Polar: $x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$ Polar $\longrightarrow$ rectangular

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $r \geq 0$, $0 \leq \theta \leq 2\pi$

* Cylindrical Cordinate

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

* rect $\longrightarrow$ cylindirical

$r = \sqrt{x^2 + y^2}$ $r \geq 0$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $0 \leq \theta \leq 2\pi$

$z \in (-\alpha, \alpha)$

* Cylindrical $\longrightarrow$ rectangular

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Practice sheet 4

4. (a) Find the volume of the solid that is bounded by the cylinder $y=x^2$ and by the planes $y+z=4$ and $z=0$

$\because y+z=4$

$\therefore z=4-y$

$\because y+z=4$

$\therefore y+0=4$

$\therefore y=4$

$\because y=x^2$

$\therefore x^2=4$

$\therefore x=\pm 2$

Sol:

$$\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{4-y} dz\,dy\,dx$$

$$=\int_{-2}^{2}\int_{x^2}^{4}[z]_0^{4-y}\,dy\,dx$$

$$=\int_{-2}^{2}\int_{x^2}^{4}(4-y-0)\,dy\,dx$$

$$=\int_{-2}^{2}\left[4y-\frac{y^2}{2}\right]_{x^2}^{4}dx$$

$$=\int_{-2}^{2}\left[\left(4\cdot4-\frac{4^2}{2}\right)-\left(4\cdot x^2-\frac{x^4}{2}\right)\right]dx$$

$$=\int_{-2}^{2}\left[(16-8)-\left(4x^2-\frac{x^4}{2}\right)\right]dx$$

$$=\left[8x-\frac{4}{3}x^3+\frac{x^5}{10}\right]_{-2}^{2} = \left(8\cdot2-\frac{4}{3}\cdot2^3+\frac{2^5}{10}\right)-\left(8(-2)-\frac{4}{3}(-2)^3+\frac{(-2)^5}{10}\right)$$

$$=\frac{128}{15}+\frac{128}{15}=\frac{256}{15}\approx 17 \text{ Ans.}$$

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4.(b) Use cylindrical/spherical coordinates to evaluate the integral

$$\int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2 \, dz \, dx \, dy.$$

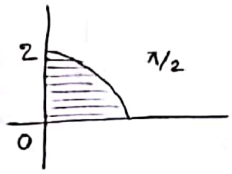
$\because x = \sqrt{4-y^2}$

$\therefore x^2+y^2=4$

$\therefore r^2 = 2^2$

$\therefore r = 2$

$r = \sqrt{x^2+y^2}$



$\because dz\,dx\,dy = r\,dz\,dr\,d\theta$

Sol:

$$\int_0^{\pi/2} \int_0^2 \int_r^{\sqrt{8-r^2}} z^2 r \, dz \, dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^2 r \left[\frac{z^3}{3} \right]_r^{\sqrt{8-r^2}} dr \, d\theta$$

$$= \int_0^{\pi/2} \int_0^2 \frac{r}{3} \left\{ (\sqrt{8-r^2})^3 - r^3 \right\} dr \, d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \int_0^2 \left\{ r(8-r^2)^{3/2} - r^4 \right\} dr \, d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \left[\int_0^2 r(8-r^2)^{3/2} \, dr - \int_0^2 r^4 \, dr \right] d\theta$$

Let, $u = 8 - r^2$

$\frac{du}{dr} = -2r$

$\frac{du}{-2} = r\,dr$

Now,

$$\int r(8-r^2)^{3/2} \, dr$$

$$= \int u^{3/2} \frac{du}{-2} = \frac{-1}{2} \cdot \frac{2}{5} u^{5/2} = \frac{-1}{5}(8-r^2)^{5/2} + c$$

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$$= \frac{1}{3}\int_0^{\pi/2}\left\{\left[-\frac{1}{5}(8-r^2)^{5/2}\right]_0^2 - \left[\frac{r^5}{5}\right]_0^2\right\} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}\left[\frac{-1}{5}\left\{(8-2^2)^{5/2} - (8-0)^{5/2}\right\} - \frac{1}{5}(2^5 - 0^5)\right] d\theta$$

$$= \frac{-1}{15}\int_0^{\pi/2}\left[(8-4)^{5/2} - 8^{5/2} + 2^5\right] d\theta$$

$$= \frac{-1}{15}\int_0^{\pi/2}\left[64 - (2\sqrt{2})^5\right] d\theta$$

$$= \frac{-1}{15}\left[64\theta - (2\sqrt{2})^5\theta\right]_0^{\pi/2}$$

$$= \frac{-1}{15}\left[64\frac{\pi}{2} - (2\sqrt{2})^5\frac{\pi}{2} - 0\right]$$

$$= \frac{-1}{15}\left[64\frac{\pi}{2} - 32(\sqrt{2})^5 \cdot \frac{\pi}{2}\right]$$

$$= \frac{-1}{15} \times 32\frac{\pi}{2}\left\{2 - (\sqrt{2})^5\right\}$$

$$= \frac{16\pi}{15}\left[(\sqrt{2})^5 - 2\right]$$

Ans.

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(5.) Evaluated the iterated integral by converting to polar coordinates:

(a) $$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\, dy\, dx$$

$\because y = \sqrt{1-x^2}$

$\therefore x^2+y^2 = 1$

$\therefore r^2 = 1$

$\therefore r = 1$

$\because x = 1$ | $x = 0$

$\therefore r\cos\theta = \cos 0$ | $r\cos\theta = \cos\frac{\pi}{2}$

$\therefore \theta = 0$ | $\therefore \theta = \frac{\pi}{2}$

Sol! $$= \int_0^{\pi/2} \int_0^1 r^2 \; r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_0^1 d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1}{4} - 0 \right) d\theta$$

$$= \frac{1}{4} \left[\theta \right]_0^{\pi/2}$$

$$= \frac{1}{4} \left(\frac{\pi}{2} - 0 \right)$$

$$= \frac{\pi}{8} \quad \text{Ans.}$$

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(b) $$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

Sol: $$= \int_0^{\pi/2} \int_0^{2\cos\theta} r \; r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \{(2\cos\theta)^3 - 0\} d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos^3\theta\, d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2} \cos\theta(1-\sin^2\theta)\, d\theta$$

$$= \frac{8}{3}\left[\int_0^{\pi/2} \cos\theta\, d\theta - \int_0^{\pi/2} \cos\theta \sin^2\theta\, d\theta\right]$$

$$= \frac{8}{3}\left[\left(\sin\frac{\pi}{2} - \sin 0\right) - \int_0^1 u^2\, du\right]$$

$$= \frac{8}{3}\left\{1 - \left[\frac{u^3}{3}\right]_0^1\right\} = \frac{8}{3}\left(1-\frac{1}{3}\right) = \frac{16}{9}$$

Ans:

$\because y = \sqrt{2x-x^2}$

$\therefore x^2+y^2 = 2x$

$\therefore r^2 = 2r\cos\theta$

$\Rightarrow r(r - 2\cos\theta) = 0$

$\therefore r = 0, \quad r = 2\cos\theta$

$\because x = 2$	$\because x = 0$
$\therefore r\cos\theta = 2$	$\therefore r\cos\theta = 0$
$\Rightarrow 2\cos^2\theta = 2$	$\Rightarrow 2\cos^2\theta = 0$
$\Rightarrow \cos\theta = 1$	$\Rightarrow \cos\theta = 0$
$\Rightarrow \cos\theta = \cos 0$	$\Rightarrow \cos\theta = \cos\frac{\pi}{2}$
$\therefore \theta = 0$	$\therefore \theta = \frac{\pi}{2}$

Let, $u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$du = \cos\theta\, d\theta$

if $\theta = 0$ then $u = 0$

if $\theta = \frac{\pi}{2}$ then $u = 1$

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(e.) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dy\,dx}{(1+x^2+y^2)^{3/2}}$, $a>0$

$\because y=\sqrt{a^2-x^2}$

$\therefore x^2+y^2=a^2$

$r^2=a^2$

$\therefore r=a$

Sol: $= \int_0^{\pi/2} \int_0^a \frac{r\,dr\,d\theta}{(1+r^2)^{3/2}}$

$\because x=a$

$r\cos\theta=a$

$a\cos\theta=a$

$\cos\theta=1$

$\cos\theta=\cos 0$

$\therefore \theta=0$

$\because x=0$

$\therefore r\cos\theta=0$

$\cos\theta=\cos\frac{\pi}{2}$

$\therefore \theta=\frac{\pi}{2}$

Now, $\int \frac{r\,dr}{(1+r^2)^{3/2}}$

$= \int \frac{\frac{du}{2}}{u^{3/2}}$

$= \frac{1}{2}\times\left(\frac{2}{-1}\right)u^{-1/2}+c$

$= \frac{-1}{\sqrt{1+r^2}}+c$

Let, $u=1+r^2$

$du=2r\,dr$

$\therefore \frac{du}{2}=r\,dr$

$\therefore \int_0^{\pi/2}\left[\frac{-1}{\sqrt{1+r^2}}\right]_0^a d\theta$

$= \int_0^{\pi/2}\left(\frac{-1}{\sqrt{1+a^2}}-\frac{-1}{\sqrt{1}}\right)d\theta = \int_0^{\pi/2}\left(\frac{-1}{\sqrt{1+a^2}}+1\right)d\theta = \left[\frac{-\theta}{\sqrt{1+a^2}}+\theta\right]_0^{\pi/2}$

$= \frac{-\pi/2}{\sqrt{1+a^2}}+\frac{\pi}{2} = \frac{\pi}{2}\left(1-\frac{1}{\sqrt{1+a^2}}\right)$

Ans.

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M-7

Jacobian in two variables

Defination: If T is the Transformation from the uv plane to the xy-plane defined by the equation $x = x(u,v)$, $y = y(u,v)$, then the Jacobian of T is denoted by $J(u,v)$ or $\frac{\partial(x,y)}{\partial(u,v)}$ and is defined by:

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Change of variable in the Double integrals:

If the transformation $x = x(u,v)$, $y = y(u,v)$, maps the region S in the uv-plane in the region R in the xy-plane and if the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is non-zero then

$$\iint_R f(x,y)\, dA_{xy} = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du.\, dv.$$

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14.7

Jacobian in two variables

$$\iint_R f(x,y)\, dA_{xy} = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(6) (a) Evaluate $\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\, dx\, dy$ by applying transformation T: where $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$ and integrating over an appropiate region uv-plane.

Sol: $\because v = \frac{y}{2}$

$\therefore y = 2v$

$\because u = \frac{2x-y}{2}$

$\therefore 2x - y = 2u$

$2x = 2u + y = 2u + 2v$

$\therefore x = u + v$

$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(2v) = 0$

$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(2v) = 2$

$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(u+v) = 1$

$\frac{\partial x}{\partial v} = \frac{\partial}{v}(u+v) = 1$

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For the limit of u,

$\because x = \frac{y}{2} = \frac{2v}{2} = v$

$\therefore x = u + v$

$\Rightarrow v = u + v$

$\therefore u = v - v = 0$

$\because x = \frac{y}{2} + 1 = v + 1$

$\therefore x = u + v$

$\Rightarrow v + 1 = u + v$

$\therefore u = v + 1 - v = 1$

& For the limit of v,

$\because y = 0$

$\therefore y = 2v$

$\Rightarrow 0 = 2v$

$\therefore v = 0$

$\because y = 4$

$\therefore y = 2v$

$\Rightarrow 4 = 2v$

$\therefore v = 2$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$$

$$\iint \frac{2x-y}{2}\, dx\, dy = \int_0^2 \int_0^1 u\, |J(u,v)|\, du\, dv = \int_0^2 \int_0^1 u\ 2\, du\, dv$$

$$= 2 \int_0^2 \left[\frac{u^2}{2} \right]_0^1 dv = 2 \int_0^2 \frac{1}{2}\, dv$$

$$= [v]_0^2 = 2$$

Ans.

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(6) (b) Evaluate $\iint\limits_R \frac{x-y}{x+y}\, dA$, where R is the region enclosed by the lines $x-y=0$, $x-y=1$, $x+y=1$ & $x+y=3$, using the transformation.

Sol: Let, $u = x-y$ & $v = x+y$

$\therefore u+v = 2x$ & $v-u = 2y$

$\therefore x = \frac{1}{2}(u+v)$ $\therefore y = \frac{1}{2}(v-u)$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

Now,

$$\iint \frac{x-y}{x+y}\, dx\, dy = \int_1^3 \int_0^1 \frac{u}{v} \left| J(u,v) \right| du\, dv$$

$$= \int_1^3 \int_0^1 \frac{u}{v} \left(\frac{1}{2}\right) du\, dv$$

$$= \frac{1}{2} \int_1^3 \frac{1}{v} \left[\frac{u^2}{2}\right]_0^1 dv = \frac{1}{2} \int_1^3 \frac{1}{v}\left(\frac{1}{2} - 0\right) dv$$

$$= \frac{1}{4}\left[\ln v\right]_1^3 = \frac{1}{4} \ln 3 \quad \text{Ans.}$$

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(3.) Evaluate $\iint_R e^{xy}\, dA$ where R is the region enclosed by the lines

$y = \frac{1}{2}x$, $y = x$, $y = \frac{1}{x}$ & $y = 2/x$

Sol: $\because y = \frac{1}{2}x$ $\therefore y/x = 1/2$

$\because y = x$ $\therefore y/x = 1$

$\because y = 1/x$ $\therefore xy = 1.$

$\because y = 2/x$ $\therefore xy = 2$

Let us consider, $u = y/x$ & $v = xy$

$\therefore u = \frac{1}{2}$ & $u = 1$

$\therefore v = 1$ & $v = 2$

$\because u = y/x$

$\therefore u = \frac{y}{v/y}$

$\therefore y^2 = uv$

$\therefore y = \sqrt{uv}$

$\because v = xy$

$\therefore v = x\sqrt{uv}$

$\therefore x = \frac{v}{\sqrt{uv}} = \frac{v}{\sqrt{v}} \cdot u^{-1/2}$

$\therefore x = \sqrt{\frac{v}{u}}$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(\sqrt{uv})$$

$$= \frac{\partial}{\partial P}(\sqrt{P}) \cdot \frac{\partial}{\partial u}(uv)$$

$$= \frac{1}{2\sqrt{P}} \cdot v = \frac{v}{2\sqrt{uv}} = \frac{1}{2}\sqrt{\frac{v}{u}}$$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\sqrt{\frac{v}{u}}$$

$$= \frac{\partial}{\partial P}(\sqrt{P}) \cdot \frac{\partial}{\partial u}\left(\frac{v}{u}\right)$$

$$= \frac{1}{2\sqrt{P}}(-1) \cdot \frac{v}{u^2} = \frac{-v}{2u^2\sqrt{\frac{v}{u}}}$$

$$= \frac{-1}{2u}\sqrt{\frac{v}{u}}$$

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$$\frac{\partial}{\partial v}(x) = \frac{\partial}{\partial v}\left(\sqrt{\frac{v}{u}}\right)$$

$$= \frac{\partial}{\partial p}(\sqrt{p}) \cdot \frac{\partial}{\partial v}\left(\frac{v}{u}\right)$$

$$= \frac{1}{2\sqrt{p}} \cdot \frac{1}{u}$$

$$= \frac{1}{2\sqrt{v/u}} \cdot \frac{1}{u}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{uv}}$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\sqrt{uv}$$

$$= \frac{1}{2}\sqrt{\frac{u}{v}}$$

$$\therefore J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2u}\sqrt{\frac{v}{u}} & \frac{1}{2}\cdot\frac{1}{\sqrt{uv}} \\ \frac{1}{2}\sqrt{\frac{u}{v}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix}$$

$$= \frac{-1}{4u} - \frac{1}{4u} = \frac{-1}{2u}$$

Now,

$$\iint_R e^{xy}\, dA = \int_1^2 \int_{1/2}^1 \left|\frac{-1}{2u}\right| e^v\, du\, dv = \frac{1}{2}\int_1^2 e^v\left[\int_{1/2}^1 \frac{1}{u}\, du\right] dv$$

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$$= \frac{1}{2}\int_{1}^{2} e^{v}\left[\ln u\right]_{1/2}^{1} dv$$

$$= \frac{1}{2}\left\{\ln(1) - \ln(1/2)\right\}\int_{1}^{2} e^{v}\, dv$$

$$= \frac{1}{2}\left(0 - \ln 1/2\right)\left[e^{v}\right]_{1}^{2}$$

$$= \frac{-1}{2}\left(\ln 1 - \ln 2\right)\left(e^{2} - e^{1}\right)$$

$$= \frac{1}{2}(\ln 2)(e^{2} - e)$$ Ans.

(4) Use the transformation $u = x - 2y$, $v = 2x + y$ to find $\iint_R \frac{x-2y}{2x+y}\, dA$ where R is the rectangular region enclosed by the lines $x - 2y = 1$, $x - 2y = 4$, $2x + y = 1$, $2x + y = 3$.

Sol: Let,

$u = 1$ & $u = 4$ $\qquad$ $v = 1$ & $v = 3$

$\because u = x - 2y$ $\qquad$ $\because y = \cancel{2x + y}$

$\therefore u = x - 2(v - 2x)$ $\qquad$ $\therefore y = \cancel{\frac{2(u+2v)}{5}}$

$u = x - 2v + 4x$

$5x = u + 2v$ $\qquad$ $\therefore x = \frac{u+2v}{5}$

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$$\because \; v = 2x + y$$

$$\therefore \; y = v - 2 \cdot \frac{u+2v}{5}$$

$$= \frac{5v - 2u - 4v}{5}$$

$$= \frac{v-2u}{5}$$

$$\therefore \; \cancel{\frac{x-2y}{2x+y}} = J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{vmatrix}$$

$$= \frac{1}{25} + \frac{4}{25} = \frac{1}{5}$$

Now,

$$\iint_R \frac{x-2y}{2x+y}\, dA = \int_1^3 \int_1^4 \left|1/5\right| \frac{u}{v}\, du\, dv$$

$$= \frac{1}{5} \int_1^3 \frac{1}{v} \left[\frac{u^2}{2}\right]_1^4 dv$$

$$= \frac{1}{5} \int_1^3 \frac{1}{v} \left(\frac{16}{2} - \frac{1}{2}\right) dv$$

$$= \frac{15}{2 \times 5} \int_1^3 \frac{1}{v}\, dv = \frac{3}{2} \left[\ln v\right]_1^3$$

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$= \frac{3}{2}\left[\ln 3 - \ln 1\right]$

$= \frac{3}{2}\ln 3$ Ans.

(5) Use the transformation $u = x+y$ and $v = x-y$ to find

$\iint_R (x-y)e^{x^2-y^2}\, dA$ over the rectangular region R enclosed by

the lines $x+y=0$, $x+y=1$, $x-y=1$, $x-y=4$

Sol: Here, $u=0, u=1$ & $v=1, v=4$

$\because u = x+y$

$\therefore u = x + (x-v)$

$\Rightarrow 2x = u+v$

$\therefore x = \frac{u+v}{2}$

$\because v = x-y$

$\therefore y = \frac{u+v}{2} - v$

$= \frac{u+v-2v}{2} = \frac{u-v}{2}$

$$\therefore J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \frac{-1}{4} - \frac{1}{4} = \frac{-1}{2}$$

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$\therefore\ x^2-y^2 = \left(\frac{u+v}{2}\right)^2 - \left(\frac{u-v}{2}\right)^2$

$= \frac{1}{4}\left(u^2+2uv+v^2-u^2+2uv-v^2\right)$

$= \frac{1}{4}(4uv)$

$= uv$

Now,

$$\iint_R (x-y)e^{x^2-y^2}\,dA = \int_1^4\int_0^1 v\,e^{uv}\left|-1/2\right|du\,dv$$

Let, $p = uv$

$\frac{dp}{du} = v$

$dp = v\,du$

$\int ve^{uv}du = \int e^p dp$

$= e^p$

$$= \frac{+1}{2}\int_1^4 \left[e^{uv}\right]_0^1 dv$$

$$= \frac{1}{2}\int_1^4 (e^v - e^0)\,dv$$

$$= \frac{1}{2}\left[e^v - v\right]_1^4$$

$$= \frac{1}{2}\left[(e^4-4)-(e^1-1)\right]$$

$$= \frac{1}{2}(e^4-e-3)$$

Ans.

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(6) Use the transformation $u = \frac{y}{x}$, $v = xy$ to find $\iint_R xy^3 \, dA$ over the region R in the first quardrant enclosed by $y=x$, $y=3x$, $xy=1$, $xy=4$

Sol: Here, $u = 1$, $u = 3$ & $v = 1$, $v = 4$

$\because u = \frac{y}{x}$

$\therefore u = \frac{v/x}{x}$ $[\because v = xy]$

$\therefore u = \frac{v}{x^2}$

$\therefore x = \sqrt{\frac{v}{u}}$

$\because y = \frac{v}{x}$

$\therefore y = \frac{v}{\sqrt{v/u}}$

$= v \times \frac{\sqrt{u}}{\sqrt{v}} = \sqrt{uv}$

$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u} \sqrt{\frac{v}{u}}$

Let, $p = \frac{v}{u}$

$dp = \frac{v}{u^2} du$

$= \frac{\partial}{\partial p}(\sqrt{p}) \cdot \frac{\partial}{\partial u}\left(\frac{v}{u}\right)$

$= \frac{-1}{2\sqrt{p}} \frac{v}{u^2}$

$= \frac{-v}{2\sqrt{\frac{v}{u}} \cdot u^2} = \frac{-1}{2u} \cdot \sqrt{\frac{v}{u}}$

$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u} \sqrt{uv}$

$= \frac{\partial}{\partial p} \sqrt{p} \quad \frac{\partial}{\partial u}(uv)$

$= \frac{1}{2\sqrt{p}} \cdot v$

$= \frac{v}{2\sqrt{uv}} = \frac{1}{2}\sqrt{\frac{v}{u}}$

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$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(\sqrt{\frac{v}{u}}\right)$$

$$= \frac{\partial}{\partial p}(\sqrt{p}) \cdot \frac{\partial}{\partial v}\left(\frac{v}{u}\right)$$

$$= \frac{1}{2\sqrt{p}} \cdot \frac{1}{u}$$

$$= \frac{1}{2\sqrt{v/u}} \cdot \frac{1}{u}$$

$$= \frac{1}{2} \frac{\sqrt{u}}{u} \cdot \frac{1}{\sqrt{v}}$$

$$= \frac{1}{2\sqrt{uv}}$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(\sqrt{uv})$$

$$= \frac{\partial}{\partial p}(\sqrt{p}) \cdot \frac{\partial}{\partial v}(uv)$$

$$= \frac{1}{2\sqrt{p}} \cdot u$$

$$= \frac{u}{2\sqrt{uv}}$$

$$= \frac{1}{2}\sqrt{\frac{u}{v}}$$

$$\therefore J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2u}\left(\sqrt{\frac{v}{u}}\right) & \frac{1}{2}\frac{1}{\sqrt{uv}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix}$$

$$= \frac{-1}{4u}\sqrt{\frac{v}{u} \cdot \frac{u}{v}} - \frac{1}{4}\sqrt{\frac{v}{u \cdot uv}}$$

$$= \frac{-1}{4u} - \frac{1}{4u}$$

$$= \frac{-1}{2u}$$

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$$\therefore xy^3 = \sqrt{\frac{v}{u}} \cdot \left(\sqrt{uv}\right)^3$$

$$= \sqrt{\frac{v}{u}} \cdot (uv)^{3/2}$$

$$= v^{1/2} \cdot u^{-1/2} \cdot u^{3/2} \cdot v^{3/2}$$

$$= v^{\frac{1}{2}+\frac{3}{2}} \cdot u^{\frac{3}{2}-\frac{1}{2}}$$

$$= v^2 \cdot u$$

Now,

$$\iint_R xy^3 \, dA = \int_1^4 \int_1^3 v^2 u \left| -\frac{1}{2u} \right| du\,dv$$

$$= \int_1^4 v^2 \int_1^3 \frac{1}{2} \, du\,dv$$

$$= \frac{1}{2} \int_1^4 v^2 \left[u \right]_1^3 dv$$

$$= \frac{1}{2} \int_1^4 v^2 (3-1) \, dv$$

$$= \frac{2}{2} \left[\frac{v^3}{3} \right]_1^4 = \frac{1}{3} (4^3 - 1^3) = 21$$

Ans.

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Line Integral

1. Evaluate $\int_c [(x^2-y)\,dx + (y^2+x)\,dy]$

(a) along the straight line c from (0, 1) to (1, 2)

(b) " " " " c " (0, 1) to (1, 1)

and then from (1,1) to (1, 2)

(c) along the parabola c: $x=t,\ y=t^2+1$ from (0, 1) to (1, 2)

Sol: (a) Here,

$(x_1, y_1) = (0, 1)$ & $(x_2, y_2) = (1, 2)$

We know,

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2}$$

$$\Rightarrow \frac{x-0}{0-1} = \frac{y-1}{1-2}$$

$$\Rightarrow \frac{x}{-1} = \frac{y-1}{-1}$$

$$\Rightarrow x = y-1$$

$$\therefore y = x+1$$

$$\therefore \int_c [(x^2-y)\,dx + (y^2+x)\,dy]$$

$$= \int_0^1 [\{x^2-(x+1)\}\,dx + \{(x+1)^2+x\}\,dx]$$

$$= \int_0^1 (x^2-x-1+x^2+2x+1+x)\,dx$$

$$= \int_0^1 (2x^2+2x)\,dx$$

$$= \left[\frac{2x^3}{3} + \frac{2x^2}{2}\right]_0^1$$

$$= \left(\frac{2}{3}\cdot 1^3 + 1^2\right) - 0 = \frac{5}{3} \text{ Ans.}$$

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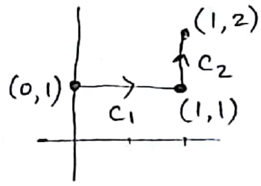
(b) For C_1: $y = 1$ & x: 0 to 1

$dy = 0$



$$\int_{C_1} [(x^2-y)dx + (y^2+x)dy]$$

$$= \int_0^1 (x^2-1)dx + 0 \qquad [\because dy = 0]$$

$$= \left[\frac{x^3}{3} - x\right]_0^1 = \left(\frac{1}{3} - 1\right) = \frac{-2}{3}$$

For C_2: $x = 1$ & y: 1 to 2

$\therefore dx = 0$

$$\int_{C_2} [(x^2-y)dx + (y^2+x)dy]$$

$$= \int_1^2 0 + (y^2+1)dy$$

$$= \left[\frac{y^3}{3} + 1\right]_1^2 = \left(\frac{2^3}{3} + 1\right) - \left(\frac{1}{3} + 1\right) = \frac{10}{3}$$

$$\therefore \int_C [(x^2-y)dx + (y^2+x)dy]$$

$$= \frac{-2}{3} + \frac{10}{3} = \frac{8}{3} \quad \text{Ans.}$$

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(e) Here,

$x = t$ & $y = t^2+1$

$\frac{dx}{dt} = 1$ $\frac{dy}{dt} = 2t + 0$

$\therefore dx = dt$ $\therefore dy = 2t\,dt$

$\because x = t$

$\therefore$ For $(0,1)$, $x = 0$, so $t = 0$

& $(1,2)$, $x = 1$ so $t = 1$

$$\int_c [(x^2-y)\,dx + (y^2+x)\,dy] = \int_0^1 [\{t^2 - (t^2+1)\}dt + \{(t^2+1)^2 + t\}\,2t\,dt]$$

$$= \int_0^1 \{t^2 - t^2 - 1 + (t^4 + 2t^2 + 1 + t)2t\}\,dt$$

$$= \int_0^1 (-1 + 2t^5 + 4t^3 + 2t + 2t^2)\,dt$$

$$= \left[-t + \frac{2t^6}{6} + \frac{4t^4}{4} + \frac{2t^2}{2} + \frac{2t^3}{3}\right]_0^1$$

$$= -1 + \frac{1^6}{3} + 1^4 + 1^2 + \frac{2}{3}\cdot 1^3 - 0$$

$$= 2$$

Ans.

② Evaluate $\int_C xy\,dx + x^2\,dy$ if

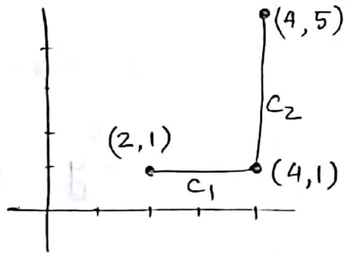
(a) C consists of line segments from (2, 1) to (4, 1) and from (4, 1) to (4, 5)

(b) C is the line segment from (2, 1) and (4, 5)

(c) Parametric equation for C are $x = 3t - 1$, $y = 3t^2 - 2t$; $1 \le t \le 5/3$

Sol: (a) For C_1: $y = 1$

$dy = 0$ $\qquad x : (2 \text{ to } 4)$



$$\int_{C_1} xy\,dx + x^2\,dy = \int_2^4 x\,dx + 0$$

$$= \left[\frac{x^2}{2}\right]_2^4 = \frac{16}{2} - \frac{4}{2} = 6$$

For C_2: $x = 4$

$dx = 0$ $\qquad y : (1 \text{ to } 5)$

$$\int_{C_2} xy\,dx + x^2\,dy = \int_1^5 0 + 4^2\,dy = [16y]_1^5 = (16 \cdot 5 - 16 \cdot 1) = 64$$

$$\therefore \int_{C_1} xy\,dx + x^2\,dy + \int_{C_2} xy\,dx + x^2\,dy = 6 + 64 = 70$$

Ans.

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(b) $(x_1, y_1) = (2, 1)$ & $(x_2, y_2) = (4, 5)$

We know,

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

$$\Rightarrow \frac{x-2}{2-4} = \frac{y-1}{1-5}$$

$$\Rightarrow -4(x-2) = -2(y-1)$$

$$\Rightarrow 2x - 4 = +y - \cancel{+}1$$

$$\Rightarrow y = 2x - 3$$

$$\therefore \int_C xy\,dx + x^2\,dy$$

$$= \int_2^4 (x(2x-3) + 2x^2)\,dx$$

$$= \left[\frac{2x^3}{3} - \frac{3x^2}{2} + \frac{2x^3}{3}\right]_2^4$$

$$= \left(\frac{4}{3}\cdot 4^3 - \tfrac{3}{2}4^2\right) - \left(\frac{4}{3}\cdot 2^3 - \tfrac{3}{2}2^2\right)$$

$$= \frac{220}{3} - \frac{14}{3} = \frac{170}{3} \quad \text{Ans.}$$

(c)

$x = 3t - 1$

$dx = 3\,dt$

& $y = 3t^2 - 2t$

$dy = (6t - 2)\,dt$

$$\int_C xy\,dx + x^2\,dy$$

$$= \int_C (3t-1)(3t^2-2t)3\,dt + (3t-1)^2(6t-2)\,dt$$

$$= \int_C \{(9t^3 - 6t^2 - 3t^2 + 2t)3 + (9t^2 - 6t + 1)(6t-2)\}\,dt$$

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$$= \int_C [(9t^3 - 9t^2 + 2t)3 + (54t^3 - 36t^2 + 6t - 18t^2 + 12t - 2)]\,dt$$

$$= \int_C (27t^3 - 27t^2 + 6t + 54t^3 - 54t^2 + 18t - 2)\,dt$$

$$= \int_C (81t^3 - 81t^2 + 24t - 2)\,dt$$

$$= \left[\frac{81t^4}{4} - \frac{81t^3}{3} + \frac{24t^2}{2} - 2t\right]_1^{5/3}$$

$$= \left\{\frac{81}{4}\left(\frac{5}{3}\right)^4 - \frac{81}{3}\left(\frac{5}{3}\right)^3 + 12\cdot\left(\frac{5}{3}\right)^2 - 2\cdot\frac{5}{3}\right\} - \left\{\frac{81}{4} - \frac{81}{3} + 12 - 2\right\}$$

$$= \frac{245}{4} - \frac{13}{4} = \frac{232}{14} = 58$$

Ans.

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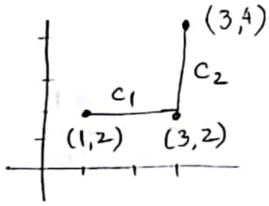
Independent of Path

Show that $\int_C [(6xy^2 - y^3)dx + (6x^2y - 3xy^2)dy]$ is independent of the path joining the points $(1,2)$ to $(3,2)$ and the $(3,2)$ to $(3,4)$ and evaluate the integral.

Sol: Let, $P = 6xy^2 - y^3$ & $Q = 6x^2y - 3xy^2$

$$\frac{\partial P}{\partial y} = 12xy - 3y^2 \qquad \frac{\partial Q}{\partial x} = 12xy - 3y^2$$

$\because \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\therefore$ The line is independent of path



For C_1: $y = 2$ & $x:(1 \text{ to } 3)$

$dy = 0$

$$\int_{C_1} (6x\cdot 2^2 - 2^3)dx + 0$$

$$= \left[24\cdot\frac{x^2}{2} - 8x\right]_1^3$$

$$= (12\cdot 3^2 - 8\cdot 3) - (12\cdot 1 - 8\cdot 1)$$

$$= 84 - 4$$

$$= 80$$

For C_2: $x = 3$ & $y:(2 \text{ to } 4)$

$dx = 0$

$$\int_{C_2} 0 + (6\cdot 3^2\cdot y - 3\cdot 3\cdot y^2)dy$$

$$= \left[\frac{54y^2}{2} - 9\cdot\frac{y^3}{3}\right]_2^4$$

$$= (\cancel{54}\,27\cdot 4^2 - 3\cdot 4^3) - (27\cdot 2^2 - 3\cdot 2^3)$$

$$= 240 - 84$$

$$= 156$$

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$$\therefore \int_C \left[(6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy \right]$$

$$= 80 + 156$$

$$= 236$$

Ans.

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Fourier Series

Let $f(x)$ be defined in the interval $(-1,1)$ and determind outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. then the Fourier series or Fourier exponsion corresponding to $f(x)$ is defined as:

$$f(x) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos\left(\frac{n\pi x}{L}\right)}_{\text{even part}} + \underbrace{\sum_{n=1}^{\alpha} b_n \sin\left(\frac{n\pi x}{L}\right)}_{\text{odd part}}$$

Co efficients:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3 \ldots\ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3 \ldots\ldots$$

Even & Odd Functions

(i) **Even:**

(1) $f(-x) = f(x) \longrightarrow$ even

(2) $f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x) \rightarrow$ "

(3) $f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x = f(x)$

even

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Fourier Analysis

Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Coefficients:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \qquad n = 1, 2, 3 \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \qquad n = 1, 2, 3 \ldots$$

even: $f(-x) = f(x)$ odd: $f(-x) = -f(x)$

1. Find the fourier coefficients to the function

$$f(x) = \begin{cases} 0, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases} \qquad \text{period} = 10$$

also write down the fourier series.

Sol: Fourier coefficients:

Here, $2L = 10$

$\therefore L = 5$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{5} \int_{-5}^{5} f(x) \cos\left(\frac{n\pi x}{5}\right) dx$$

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$$= \frac{1}{5}\left[\int_{-5}^{0} f(x)\cos\frac{n\pi x}{5}\,dx + \int_{0}^{5} f(x)\cos\frac{n\pi x}{5}\,dx\right]$$

$$= \frac{1}{5}\left[\int_{-5}^{0} 0\,dx + \int_{0}^{5} 3\cos\frac{n\pi x}{5}\,dx\right]$$

$$= \frac{3}{5}\int_{0}^{5}\cos\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left[\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right]_{0}^{5} \qquad \left[\because \int \cos mx\,dx = \frac{\sin mx}{m} + c\right]$$

$$= \frac{3}{n\pi}\left[\sin\frac{n\pi 5}{5} - \sin 0\right]$$

$$= \frac{3}{n\pi}(\sin n\pi), \qquad n = 1, 2, 3 \ldots$$

$$= \frac{3}{n\pi}\times 0$$

$$= 0$$

Ans.

If $n = 0$ then $a_0 = \frac{3}{5}\int_{0}^{5}\cos 0\,dx = \frac{3}{5}\int_{0}^{5} 1\,dx = \frac{3}{5}[x]_{0}^{5}$

$$= \frac{3}{5}(5-0)$$

$$= 3$$

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$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left[\int_{-5}^{0} f(x)\sin\frac{n\pi x}{5}\,dx + \int_{0}^{5} f(x)\sin\frac{n\pi x}{5}\,dx\right]$$

$$= \frac{1}{5}\left[\int_{-5}^{0} 0\,dx + \int_{0}^{5} 3\sin\frac{n\pi x}{5}\,dx\right]$$

$$= \frac{3}{5}\int_{0}^{5}\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left[\frac{5}{n\pi}\left(-\cos\frac{n\pi x}{5}\right)\right]_0^5$$

$$= \frac{-3}{n\pi}\left[\cos\frac{n\pi 5}{5} - \cos 0\right]$$

$$= \frac{3}{n\pi}(1-\cos n\pi) \quad , n = 1, 2, 3, \ldots$$

Ans.

The fourier series is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$= \frac{3}{2} + 0 + \sum_{n=1}^{\infty}\frac{3}{n\pi}(1-\cos n\pi)\sin\frac{n\pi x}{5} \quad , n = 1, 2, 3, \ldots$$

Ans.

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(2) Determine the Fourier series for

$$f(x) = \begin{cases} -x, & -4 \le x \le 0 \\ x, & 0 \le x \le 4 \end{cases} \quad \text{Period} = 8$$

Sol: Fourier coefficients:

Here, $2L = 8$

$\therefore L = 4$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx$$

$$= \frac{1}{4} \int_{-4}^{4} f(x) \cos \frac{n\pi x}{4} \, dx$$

$$= \frac{1}{4} \left[\int_{-4}^{0} f(x) \cos \frac{n\pi x}{4} \, dx + \int_{0}^{4} f(x) \cos \frac{n\pi x}{4} \, dx \right]$$

$$= \frac{1}{4} \left[\int_{-4}^{0} (-x) \cos \frac{n\pi x}{4} \, dx + \int_{0}^{4} x \cos \frac{n\pi x}{4} \, dx \right]$$

$$= \frac{1}{4} \left[\int_{0}^{4} x \cos \frac{n\pi x}{4} \, dx - \int_{-4}^{0} x \cos \frac{n\pi x}{4} \, dx \right] \quad \text{------} \quad (i)$$

$$\therefore \int x \cos \frac{n\pi x}{4} \, dx = x \int \cos \frac{n\pi x}{4} \, dx - \int \left\{ \frac{d}{dx}(x) \int \cos \frac{n\pi x}{4} \, dx \right\} dx$$

$$= x \cdot \frac{4}{n\pi} \sin \frac{n\pi x}{4} - \frac{4}{n\pi} \int \sin \frac{n\pi x}{4} \, dx$$

$$= \frac{4x}{n\pi} \sin \frac{n\pi x}{4} + \frac{16}{n^2\pi^2} \cos \frac{n\pi x}{4} + c \quad \text{------} \quad (ii)$$

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Substituting the value from eq^n (ii) in eq^n(i) we get,

$$a_n = \frac{1}{4}\left(\left[\frac{4x}{n\pi}\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_0^4 - \left[\frac{4x}{n\pi}\sin\frac{n\pi x}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right]_{-4}^0\right)$$

$$= \frac{1}{4}\left[\left(\frac{16}{n^2\pi^2}\cos n\pi - \frac{16}{n^2\pi^2}\right) - \left(\frac{16}{n^2\pi^2} - \frac{16}{n^2\pi^2}\cos(-n\pi)\right)\right]$$

$[\because \sin\pi = \sin 2\pi = \sin n\pi = 0]$

$$= \frac{1}{4}\cdot\frac{16}{n^2\pi^2}\left(\cos n\pi - 1 - 1 + \cos n\pi\right)$$

$$= \frac{4}{n^2\pi^2}\left(2\cos n\pi - 2\right)$$

$$= \frac{8}{n^2\pi^2}\left(\cos n\pi - 1\right)$$

$$a_0 = \frac{1}{4}\left[\int_0^4 x\cos\frac{n\pi x}{4}\,dx - \int_{-4}^0 x\cos\frac{n\pi x}{4}\,dx\right]$$

$$= \frac{1}{4}\left[\int_0^4 x\,dx - \int_{-4}^0 x\,dx\right]$$

$$= \frac{1}{4}\left(\left[\frac{x^2}{2}\right]_0^4 - \left[\frac{x^2}{2}\right]_{-4}^0\right) = \frac{1}{4}\left(\frac{4^2}{2} + \frac{(-4)^2}{2}\right) = \frac{1}{4}\left(\frac{16}{2} + \frac{16}{2}\right)$$

$$= 4$$

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$$\therefore b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

$$= \frac{1}{4} \int_{-4}^{4} f(x) \sin \frac{n\pi x}{4}\, dx$$

$$= \frac{1}{4} \left[\int_{-4}^{0} f(x) \sin \left(\frac{n\pi x}{4}\right) dx + \int_{0}^{4} f(x) \sin \left(\frac{n\pi x}{4}\right) dx \right]$$

$$= \frac{1}{4} \left[\int_{0}^{4} x \sin \frac{n\pi x}{4}\, dx - \int_{-4}^{0} x \sin \frac{n\pi x}{4}\, dx \right] \ldots\ldots \text{(iii)}$$

$$\therefore \int x \sin \frac{n\pi x}{4}\, dx = x \int \sin \frac{n\pi x}{4}\, dx - \int \left\{ \frac{d}{dx}(x) \int \sin \frac{n\pi x}{4}\, dx \right\} dx$$

$$= x \cdot \frac{-4}{n\pi} \cos \frac{n\pi x}{4} + \int \frac{4}{n\pi} \cos \frac{n\pi x}{4}\, dx$$

$$= \frac{-4x}{n\pi} \cos \frac{n\pi x}{4} + \frac{16}{n^2\pi^2} \sin \frac{n\pi x}{4} + C$$

$$\therefore b_n = \frac{1}{4} \left(\left[\frac{-4x}{n\pi} \cos \frac{n\pi x}{4} + \frac{16}{n^2\pi^2} \sin \frac{n\pi x}{4} \right]_0^4 - \left[\frac{-4x}{n\pi} \cos \frac{n\pi x}{4} + \frac{16}{n^2\pi^2} \sin \frac{n\pi x}{4} \right]_{-4}^{0} \right)$$

$$= \frac{1}{4} \left[\frac{-16}{n\pi} \cos(n\pi) + \frac{16}{n\pi} + \frac{16}{n\pi} - \frac{16}{n\pi} \cos(n\pi) \right]$$

$$= \frac{1}{4} \cdot \frac{16}{n\pi} (-\cos n\pi + 1 + 1 - \cos n\pi) = \frac{4}{n\pi}(2 - 2\cos n\pi) = \frac{8}{n\pi}(1 - \cos n\pi)$$

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Fourier Series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\alpha} b_n \sin \frac{n\pi x}{L}$$

$$= \frac{4}{2} + \sum_{n=1}^{\alpha} \frac{8}{n^2\pi^2} (\cos n\pi - 1) \cos \frac{n\pi x}{4} + \sum_{n=1}^{\alpha} \frac{8}{n\pi} (1 - \cos n\pi) \sin \frac{n\pi x}{L}$$

$$= 2 + \sum_{n=1}^{\alpha} \frac{8}{n^2\pi^2} (\cos n\pi - 1) \cos \frac{n\pi x}{4} + \sum_{n=1}^{\alpha} \frac{8}{n\pi} (1 - \cos n\pi) \sin \frac{n\pi x}{L}$$

Ans.

(3) Determine the fourier series of the function

$$f(x) = \begin{cases} 0, & -\pi \leq x \leq 0 \\ 1, & 0 \leq x \leq \pi \end{cases}$$

Sol: Fourier coefficients,

Here, $2L = 2\pi$

$\therefore L = \pi$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x) \cos nx \, dx + \int_{0}^{\pi} f(x) \cos nx \, dx \right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 1\cdot \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{1}{n\pi}\left[\sin n\pi - \sin 0\right]$$

$$= \frac{1}{n\pi}(0-0) \qquad n = 1, 2, 3 \dots$$

$$= 0$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \cos 0\,dx = \frac{1}{\pi}[x]_0^{\pi} = \frac{1}{\pi}(\pi - 0) = 1.$$

$$\therefore b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_0^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\quad 0 + \int_0^{\pi} 1\cdot \sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{-1}{n\pi}(\cos n\pi - \cos 0) = \frac{1}{n\pi}(1 - \cos n\pi)$$

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Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$= \frac{1}{2} + 0 + \sum_{n=1}^{\infty} \frac{1}{n\pi}(1-\cos n\pi)\sin\frac{n\pi x}{L}, \quad n=1,2,3\ldots$$

Ans.

Half-Range Fourier Sine or Cosines Series

It is a series where only sine or cosine terms are present.

$a_n = 0$, $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \longrightarrow$ odd function

$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \longrightarrow$ half-range sine series

$b_n = 0$, $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \longrightarrow$ even function

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \longrightarrow$ half-rang cosine serie

Que: Expand $f(x) = x$, $0 < x < 2$ in a half range

(a) sine series & (b) cosine series.

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Sol: (a) For sine series $a_n = 0$ Here, $L = 2$

$$\therefore b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{2}\int_0^2 x\cdot\sin\frac{n\pi x}{2}\,dx \;\ldots\ldots (i)$$

Now,

$$\int x\sin\frac{n\pi x}{2}\,dx = x\int\sin\frac{n\pi x}{2}\,dx - \int\left\{\frac{d}{dx}(x)\int\sin\frac{n\pi x}{2}\,dx\right\}dx$$

$$= x\cdot\frac{-2}{n\pi}\cos\frac{n\pi x}{2} - \int\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2} + C$$

$$\therefore b_n = \left[\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{-4}{n\pi}\cos n\pi + \frac{4}{n\pi}$$

$$= \frac{4}{n\pi}(1-\cos n\pi)$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{4}{n\pi}(1-\cos n\pi)\cdot\sin\frac{n\pi x}{2}$$ Ans.

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(b) For cosine series, $b_n = 0$ $\qquad L = 2$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx \quad \ldots\ldots \text{(ii)}$$

$$\therefore \int x\cos\frac{n\pi x}{2}\,dx = x\int\cos\frac{n\pi x}{2}\,dx - \int\left\{\frac{d}{dx}(x)\int\cos\frac{n\pi x}{2}\,dx\right\}dx$$

$$= \frac{2x}{n\pi}\sin\frac{n\pi x}{2} - \int\frac{2}{n\pi}\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2} + C$$

$$\therefore a_n = \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_0^2$$

$$= \frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

$$a_0 = \int_0^2 x\cos 0\,dx = \left[\frac{x^2}{2}\right]_0^2 = \left(\frac{2^2}{2} - 0\right) = 2$$

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$= \frac{2}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{L}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{L}$$

Ans.

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Differentiation	Integration
$\frac{d}{dx}(x) = 1$	$\int dx = x + c$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$
$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$	$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + c$
$\frac{d}{dx}(\log x) = \frac{1}{x}$	$\int \frac{dx}{x} = \log x + c = \ln x + c$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}(e^{mx}) = me^{mx}$	$\int e^{mx} dx = \frac{e^{mx}}{m} + c$
$\frac{d}{dx}(a^x) = a^x \cdot \log_e a$	$\int a^x dx = \frac{a^x}{\log_e a} + c$ $\int a\, dx = a\int dx = ax + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\, dx = \sin x + c$
$\frac{d}{dx}(\sin mx) = m\cos mx$	$\int \cos mx\, dx = \frac{1}{m}\sin mx + c$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\, dx = -\cos x + c$
$\frac{d}{dx}(\cos mx) = -m\sin mx$	$\int \sin mx\, dx = -\frac{1}{m}\cos mx + c$

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$$1+\cos 2\theta = 2\cos^2\theta$$

$$1-\cos 2\theta = 2\sin^2\theta$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\cancel{\frac{d}{dx}} \int \sec^2 x \, dx = \tan x + c$$

$$\int \sec x \cdot \tan x \, dx = \sec x + c$$

$$\int \operatorname{cosec}^2 x \, dx = -\cot x + c$$

$$\int \operatorname{cosec} x \cdot \cot x \, dx = -\operatorname{cosec} x + c$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$$

$$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1}x + c$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}x + c$$

$$\int \frac{dx}{1+x^2} = -\cot^{-1}x + c$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + c$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1}x + c$$

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