The Indefinite integral → 5.2

Antiderivatives :- Let, $F(x)$ is called an antiderivative of $f(x)$,

if. $F'(x) = f(x)$

$$F(x) \xrightarrow{\text{derivative}} f(x) \quad \text{antiderivatives.}$$

Example : $F(x) = \frac{1}{3}x^3$, $f(x) = x^2$. antiderivative. derivetive

$$F'(x) = \frac{1}{3}x^3 \Rightarrow F'(x) = \frac{1}{3} \cdot 3x^2 = x^2 = f(x).$$

Hence. $\frac{1}{3}x^3$ is an antiderivative of x^2

$$G(x) = \frac{1}{3}x^3 + c$$
$$G'(x) = x^2.$$

" If two function F and G have same derivative then $F = G$" Provide example to justify or neglate the statement.

$$F(x) = \frac{1}{3}x^3 + 5 \quad , \quad F'(x) = x^2$$
$$G(x) = \frac{1}{3}x^3 - 9 \quad \quad G'(x) = x^2.$$

Hence, this statement is not true.

$$F \neq G$$

The process of finding antiderivative is called antidifferention or. Integration.

$$\frac{d}{dx}[F(x)] = f(x) \rightarrow \int f(x)\,dx = F(x) + c.$$

Examples from book →

2.(b) ⇒ $\int (x+x^2)\,dx$

$= \int x\,dx + \int x^2\,dx$

$= \frac{x^2}{2} + \frac{x^3}{3} + c$

formula.

$\left[\int \csc x \cdot \cot x\,dx = -\csc x + c\right]$

4. (a) ⇒ $\int \frac{\cos x}{\sin^2 x}\,dx = \int \cot x \cdot \csc x$

$= -\csc x + c$

(b). $\int \frac{t^2 - 2t^4}{t^4}\,dt = \int \left(\frac{1}{t^2} - 2\right)dt$

$= \int (t^{-2} - 2)\,dt$

$= \frac{t^{-1}}{-1} - 2t + c$

$= -\frac{1}{t} - 2t + c.$

(c) $\int \frac{x^2}{x^2+1}\,dx = \int \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right)dx$

$= \int \left(1 - \frac{1}{x^2+1}\right)dx$

$= x - \tan^{-1} x + c.$

Formula → $\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$

$* \ \frac{x^2}{2}+2x+\frac{1}{3}x^{3}$.
$\Downarrow$
W.S.S

Exercise set 5.2 $\Rightarrow$

15. $\int x(1+x^3)\,dx = \int (x+x^4)\,dx$
$= \frac{x^2}{2}+\frac{x^5}{5}+c.$

16. $\int (2+y^2)^2\,dy = \int (4+4y^2+y^4)\,dy$
$= 4y+\frac{4}{3}y^3+\frac{1}{5}y^5+c.$

17. $\int x^{1/3}(2-x)^2\,dx = \int x^{1/3}(4-4x+x^2)\,dx$
$= \int (4x^{1/3}-4x^{4/3}+x^{7/3})\,dx$
$= 4\cdot\frac{x^{4/3}}{4/3} - 4\cdot\frac{x^{7/3}}{7/3} + \frac{x^{10/3}}{10/3}+c.$

18. $\int (1+x^2)(2-x)\,dx = \int (2-x+2x^2-x^3)\,dx$
$= 2x-\frac{1}{2}x^2+\frac{2}{3}x^3-\frac{1}{4}x^4+c.$

19. $\int \frac{x^5+2x^2-1}{x^4}\,dx = \int (x+2x^{-2}-x^{-4})\,dx$
= Why so serious! $= \frac{x^2}{2}-2\frac{1}{x}+\frac{1}{3}x^{-3}+c.$

20. $\int \frac{1-2t^3}{t^3}\,dt = \int (t^{-3}-2)\,dt = -\frac{1}{2}\cdot t^{-2}-2t+c.$

21. $\int \left(\frac{2}{x}+3e^x\right)dx = \int (2x^{-1}+3e^x)\,dx$
$= 2\cdot\ln|x|+3e^x+c.$

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22. $\int \left(\frac{1}{2t} - \sqrt{2}\, e^{t} \right) dt = \frac{1}{2} \ln|t| - \sqrt{2}\, e^{t} + c$

23. $\int (3 \sin x - 2 \sec^2 x)\, dx = -3 \cos x - 2 \tan x + c$

24. $\int (\csc^2 t - \sec t \cdot \tan t)\, dt = -\cot t - \sec t + c$

25. $\int (\sec x (\sec x + \tan x))\, dx = \int (\sec^2 x + \sec x \cdot \tan x)\, dx$

$= \tan x + \sec x + c$

26. $\int \csc x (\sin x + \cot x)\, dx = \int (\operatorname{cosec} x \cdot \sin x + \cot x \cdot \csc x)\, dx$

$= \int (1 + \cot x \cdot \csc x)\, dx$

$= x - \csc x + c$

27. $\int \frac{\sec\theta}{\cos\theta}\, d\theta = \int \sec\theta \cdot \frac{1}{\cos\theta}\, d\theta = \tan\theta + c$

28. $\int \frac{dy}{\operatorname{cosec} y} = \int \sin y\, dy = -\cos y + c$

29. $\int \frac{\sin x}{\cos^2 x}\, dx = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\, dx = \int (\tan x \cdot \sec x)\, dx$

$= \sec x + c$

30. $\int \varphi + \frac{2}{\sin^2 \varphi}\, d\varphi = \int \varphi\, d\varphi + 2 \int \operatorname{cosec}^2 \varphi\, d\varphi$

$= \frac{\varphi^2}{2} + 2(-\cot\varphi) + c$

$= \frac{\varphi^2}{2} - 2\cot\varphi + c$

31. $\int (1+\sin^2\theta \cdot \operatorname{cosec}\theta)\, d\theta = \int \left(1+\sin\theta \cdot \frac{1}{\sin\theta} \cdot \sin\theta\right) d\theta$

$= \int (1+\sin\theta)\, d\theta$

$= \theta + (-\cos\theta) + c$

$= \theta - \cos\theta + c$

32. $\int \left(\frac{\sec x + \cos x}{2\cos x}\right) \cdot dx = \frac{1}{2}\int \sec^2 x\, dx + \frac{1}{2}\int 1 \cdot dx$

$= \frac{1}{2}\tan x + \frac{1}{2}x + c$

33. $\int \left(\frac{1}{2\sqrt{1-x^2}} - \frac{3}{1+x^2}\right) dx = \frac{1}{2}\int \frac{dx}{\sqrt{1-x^2}} - 3\int \frac{dx}{1+x^2}$

$= \frac{1}{2}\sin^{-1}x - 3\tan^{-1}x + c$

34. $\int \left(\frac{4}{x\sqrt{x^2-1}} + \frac{1+x+x^3}{1+x^2}\right) dx = 4\sec^{-1}x + \int \left(\frac{x(1+x^2)}{1+x^2} + \frac{1}{1+x^2}\right) dx$

$= 4\sec^{-1}x + \int x\, dx + \int \frac{1}{1+x^2}\, dx$

$= 4\sec^{-1}x + \frac{x^2}{2} + \tan^{-1}x + c$

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Integration By Substitution → 5.3

Examples from book →

3. $\int \cos 5x \, dx$

Let; $u = 5x$

$5dx = du$

$dx = \frac{du}{5}$

$= \int \cos u \cdot \frac{du}{5}$

$= \frac{1}{5} \sin u + c$

$= \frac{1}{5} \sin 5x + c.$

4. $\int \frac{dx}{(\frac{1}{3}x - 8)^5}$

Let, $u = \frac{1}{3}x - 8$

$du = \frac{1}{3} dx$

$dx = \frac{du\,3}{1}$

$= \int \frac{3du}{u^5}$

$= 3\int u^{-5} du = -\frac{3}{4} u^{-4} + c$

$= -\frac{3}{4}(\frac{1}{3}x - 8)^{-4} + c.$

5. $\int \frac{dx}{1+3x^2} = \int \frac{dx}{1+(\sqrt{3}x)^2}$

Let, $u = \sqrt{3}x$

$du = \sqrt{3} dx$

$dx = \frac{du}{\sqrt{3}}$

$= \int \frac{du/\sqrt{3}}{1+u^2}$

$= \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}x) + c.$

6. $\int (\frac{1}{x} + \sec^2 \pi x) dx$

Let, $u = \pi x$

$dx = \frac{du}{\pi}$

$= \int \frac{dx}{x} + \int \sec^2 \pi x \, dx$

$= \ln|x| + \int \sec^2 u \cdot du/\pi$

$= \ln|x| + \frac{1}{\pi} \tan u + c$

$= \ln|x| + \frac{1}{\pi} \tan(\pi x) + c.$

7. $\int \sin^2 x \cdot \cos x \, dx$ Let, $u = \sin x$

$du = dx \cos x$

$= \int \sin x \cdot \sin x \cos x \, dx$

$= \int u^2 \, du$

$= \frac{u^3}{3} + c$

$= \frac{\sin^3 x}{3} + c.$

8. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$ Let, $u = \sqrt{x}$

or, $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

or, $dx = 2\sqrt{x} \cdot du$

$= \int \frac{e^u}{\cancel{u}} \cdot 2\cancel{u} \, du$

$= \int 2e^u \, du$

$= 2e^{\sqrt{x}} + c.$

9. $\int t^4 \sqrt[3]{3 - 5t^5} \, dt$ Let, $u = 3 - 5t^5$

$\frac{du}{dt} = -25t^4$

$\therefore \frac{du}{-25} = t^4 dt$

$= \int \sqrt[3]{u} \cdot \frac{du}{-25}$

$= -\frac{1}{25} \int \sqrt[3]{u} \cdot du$

$= -\frac{1}{25} \cdot \frac{u^{4/3}}{4/3} + c$

$= -\frac{3}{100} (3 - 5t^5)^{4/3} + c.$

10. $\int \frac{e^x}{\sqrt{1 - e^{2x}}} = \int \frac{e^x}{\sqrt{1 - (e^x)^2}} dx$ Let, $u = e^x$

$du = e^x dx$

$= \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u + c$

$= \sin^{-1}(e^x) + c.$

p.t.o

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11. $\int x^2\sqrt{x-1}\,dx$

Let, $u=x-1$, or, $u+1=x$
$du=dx$

$= \int (u+1)^2\sqrt{u}\,du$

$= \int (u^2+2u+1)\sqrt{u}\,du$

$= \int (u^{5/2}+2u^{3/2}+u^{1/2})\,du$

$= \frac{2}{7}u^{7/2}+2\cdot\frac{u^{5/2}}{5/2}+\frac{u^{3/2}}{3/2}+c$

$= \frac{2}{7}(x-1)^{7/2}+\frac{4}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+c.$

12. $\int \cos^3x\,dx = \int \cos^2x\cdot\cos x\,dx = \int (1-\sin^2x)\cos x\,dx$

Let, $u=\sin x$
$du=\cos x\,dx$

$= \int (1-u^2)\,du = u-\frac{u^3}{3}+c$

$= \sin x-\frac{\sin^3x}{3}+c$

13. $\int \frac{dx}{a^2+x^2} = \int \frac{a\left(\frac{dx}{a}\right)}{a^2\left(1+\frac{x^2}{a^2}\right)} = \frac{1}{a}\int \frac{\frac{dx}{a}}{1+\left(\frac{x}{a}\right)^2}$

Let, $u=\frac{x}{a}$
$\frac{du}{dx}=\frac{1}{a}$
$du=\frac{dx}{a}$

$= \frac{1}{a}\int \frac{du}{1+u^2} = \frac{1}{a}\tan^{-1}u+c$

$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$ # direct formula can also be used.

14. $\int \frac{dx}{\sqrt{2-x^2}}$

Let, $u=x$
$du=dx$

$= \int \frac{dx}{\sqrt{(\sqrt{2})^2-(x)^2}}$

$= \int \frac{du}{\sqrt{(\sqrt{2})^2-(u)^2}}$

$= \sin^{-1}\frac{x}{\sqrt{2}}+c.$

Direct formula can also be ~~done~~ used.

* $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}+c.$
* $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}+c.$
* $\cancel{\int \frac{du}{u\sqrt{a^2-u^2}}}$
* $\int \frac{du}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right|+c.$

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Exercise set 5·3 =>

23. $\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$

Let, $u = 2x$

$\frac{du}{2} = dx$

$= \frac{1}{2}\sin^{-1}u + c$

$= \frac{1}{2}\sin^{-1}(2x) + c.$

24. $\int \frac{dx}{1+16x^2} = \int \frac{dx}{1+(4x)^2}$

Let, $u = 4x$

$\frac{du}{4} = dx$

$= \frac{1}{4}\int \frac{du}{1+u^2} = \frac{1}{4}\tan^{-1}u + c$

$= \frac{1}{4}\tan^{-1}(4x) + c.$

25. $\int t\sqrt{7t^2+12}\, dt$

Let, $u = 7t^2+12$

$\frac{du}{dx} = 7\cdot 2t$

$\frac{du}{14} = dt\cdot t$

$= \int \sqrt{u}\cdot\frac{du}{14}$

$= \frac{1}{14}\cdot\frac{u^{3/2}}{3/2} + c$

$= \frac{1}{21}(7t^2+12)^{3/2} + c.$

26. $\int \frac{x}{\sqrt{4-5x^2}}\, dx$

Let, $u = 4-5x^2$

$\frac{du}{dx} = -10x$

$\frac{du}{-10} = x\cdot dx$

$= -\frac{1}{10}\int \frac{du}{\sqrt{u}}$

$= -\frac{1}{10}\cdot\frac{u^{1/2}}{1/2} + c$

$= -\frac{1}{5}\sqrt{4-5x^2} + c.$

27. $\int \frac{6}{(1-2x)^3} dx$

Let, $u = 1-2x$

$\frac{du}{dx} = -2$

$dx = -\frac{du}{2}$

$= \int \frac{6}{u^3} \cdot -\frac{du}{2}$

$= -3 \int \frac{1}{u^3} \cdot du$

$= -3 \cdot \frac{u^2}{2} + c$

$= -3/2 \, (1-2x)^2 + c$.

28. $\int \frac{x^2+1}{\sqrt{x^3+3x}} dx$

Let, $u = x^3+3x$

$\frac{du}{dx} = 3x^2+3$

$= 3(x^2+1)$

$\frac{du}{3} = dx(x^2+1)$

$= \frac{1}{3} \int \frac{du}{\sqrt{u}} = \frac{1}{3} \cdot \frac{u^{1/2}}{1/2} + c$

$= \frac{2}{3} (x^3+3x)^{1/2} + c$.

29. $\int \frac{x^3}{(5x^4+2)^3} dx$

Let, $u = 5x^4+2$

$\frac{du}{dx} = 20x^3$

$\frac{du}{20} = x^3 dx$

$= \frac{1}{20} \int \frac{du}{u^3} = \frac{1}{20} \cdot u^2/2 + c$

$= \frac{1}{40} (5x^4+2)^2 + c$.

30. $\int \frac{\sin(1/x)}{3x^2} dx$

Rough

~~Let, $u = 3x^2$~~

~~$\frac{du}{dx} = 6x$~~

~~$du = 6x \cdot dx$~~

$= \int \frac{\sin u}{3} (-du)$

$= -\frac{1}{3} \int \sin u \cdot du$

$= -\frac{1}{3} \cdot -\cos u + c$

$= \frac{1}{3} \cos(\frac{1}{x}) + c$.

Let, $u = \frac{1}{x}$

$\Rightarrow \frac{du}{dx} = -\frac{1}{x^2}$

$\Rightarrow du \, x^2 = -dx$

$\therefore \frac{dx}{x^2} = -du$

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31. $\int e^{\sin x} \cdot \cos x \, dx$

Let, $u = \sin x$

$du = \cos x \cdot dx$

$= \int e^{u} \, du$

$= e^{\sin x} + c$.

32. $\int x^3 e^{x^4} dx$

Let, $u = x^4$

$\Rightarrow du = 4x^3 dx$

$\Rightarrow \frac{du}{4} = x^3 dx$

$= \frac{1}{4} \int e^{u} \, du$

$= \frac{1}{4} e^{x^4} + c$.

33. $\int x^2 e^{-2x^3} dx$

Let, $u = -2x^3$

$\frac{du}{dx} = -6x^2$

$\frac{du}{-6} = x^2 dx$

$= -\frac{1}{6} \int e^{u} \, du$

$= -\frac{1}{6} e^{-2x^3} + c$

34. $\int \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}} dx$

Let, $u = e^{x} - e^{-x}$

$\frac{du}{dx} = e^{x} + e^{-x}$

$du = e^{x} + e^{-x} \, dx$

Note, $\frac{d}{dx} e^{-x} = -e^{-x}$.

$= \int \frac{1}{u} du$

$= \ln|u| + c$

$= \ln|e^{x} - e^{-x}| + c$.

35. $\int \frac{e^{x}}{1 + e^{2x}} dx$

Let, $u = e^{x}$

$du = dx e^{x}$

$= \int \frac{e^{x}}{1^2 + (e^{x})^2} dx$

$= \int \frac{du}{1 + u^2}$

$= \tan^{-1} u + c$

$= \tan^{-1}(e^{x}) + c$

36. $\int \frac{t}{t^4+1}\,dt$

Let, $t^2 = u$

$\frac{du}{2} = dt.t$

$= \int \frac{t}{1^2+(t^2)^2}\,dt$

$= \frac{1}{2}\cdot \tan^{-1}(t^2)+c$

37. $\int \frac{\sin(5/x)}{x^2}\,dx$

Let, $u = 5/x$

$\frac{du}{dx} = -5/x^2$

$\frac{dx}{-x^2} = \frac{du}{5}$

$= -\frac{1}{5}\int \sin u\cdot du$

$= +\frac{1}{5}\cos\left(\frac{5}{x}\right)+c$

38. $\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx$

Let, $u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$\frac{du}{1} = \frac{dx}{2\sqrt{x}}$

$du\cdot 2 = \frac{dx}{\sqrt{x}}$

$= 2\int \sec^2 u\cdot du$

$= 2\tan\sqrt{x}+c$

39. $\int \cos^4 3t\cdot \sin 3t\cdot dt$

Let, $u = \cos 3t$

$du = -3\cdot\sin 3t\cdot dt$

$\therefore \frac{du}{-3} = \sin 3t\cdot dt$

$= \int \cos^3 3t\cdot\cos 3t\cdot\sin 3t\cdot dt$

$= -\frac{1}{3}\int u^4\cdot du$

$= -\frac{1}{3}\; u^5/5 + c$

$= -\frac{1}{15}(\cos 3t)^5+c$

40. $\int \cos 2t\cdot \sin^5 2t\; dt$

Let, $u = \sin 2t$

$\frac{du}{2} = \cos 2t\cdot dt$

$= \int \cos 2t\cdot \sin 2t\cdot \sin^4 2t\cdot dt$

$= \frac{1}{2}\int \sin 2t\cdot \sin^4 2t\cdot \cos 2t\cdot dt$

$= \frac{1}{2}\int u^5\, du$

$= \frac{1}{2}\cdot\frac{u^6}{6}+c$

$= \frac{1}{12}(\sin 2t)^6+c$

41. $\int x \sec^2(x^2)\, dx$

Let, $u = x^2$

or, $\frac{du}{dx} = 2x$

or, $\frac{du}{2} = x \cdot dx$

$= \frac{1}{2}\int \sec^2(u)\, du$

$= \frac{1}{2}\tan(x^2) + c$.

42. $\int \frac{\cos 4\theta}{(1+2\sin 4\theta)^4}\, d\theta$.

Let, $u = 1+2\sin 4\theta$

or, $\frac{du}{d\theta} = 8\cos\theta 4$

or, $\frac{du}{8} = \cos 4\theta \cdot d\theta$

$= \frac{1}{8}\int \frac{du}{u^4}$

$= \frac{1}{8}\; u^{-3}/_{-3} + c$

$= -\frac{1}{24}(1+2\sin 4\theta)^{-3} + c$.

43. $\int \cos 4\theta \sqrt{2-\sin 4\theta} \cdot d\theta$

Let, $u = 2-\sin 4\theta$

$\frac{du}{d\theta} = -\cos 4\theta \cdot 4$

$-\frac{du}{4} = \cos 4\theta \cdot d\theta$

$= -\frac{1}{4}\int \sqrt{u} \cdot du$

$= -\frac{1}{4}\; u^{3/2}/_{3/2} + c$

$= -\frac{1}{6}(2-\sin 4\theta)^{3/2} + c$.

44. $\int \tan^3(5x) \cdot \sec^2(5x)\, dx$

Let, $u = \tan(5x)$

or, $\frac{du}{dx} = \sec^2 5x \cdot 5$

or, $\frac{du}{5} = \sec^2 5x \cdot dx$

$= \int \tan^2(5x) \cdot \tan(5x) \cdot \sec^2(5x)\, dx$

$= \frac{1}{5}\int u^3 \cdot du$

$= \frac{1}{5} \cdot u^4/_4 + c$

$= \frac{1}{20}\tan^4(5x) + c$.

45. $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$

Let, $\tan x = u$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x \cdot dx$.

$= \int \frac{du}{\sqrt{1-u^2}}$

$= \sin^{-1}(\tan x) + c$.

46. $\int \frac{\sin\theta}{\cos^2\theta + 1} \cdot d\theta$.

Let, $u = \cos\theta$.

$\frac{du}{d\theta} = -\sin\theta$.

$-\frac{du}{1} = \sin\theta \cdot d\theta$

$= -\int \frac{du}{u^2+1^2}$

$= -\tan^{-1}(\cos\theta) + c$.

47. $\int \sec^3(2x) \cdot \tan(2x)\,dx$

Let, $u = \sec(2x)$

$\frac{du}{dx} = \sec 2x \cdot \tan 2x \cdot 2$

$\frac{du}{2} = \sec(2x) \cdot \tan(2x) \cdot dx$

$= \int \sec^2(2x) \cdot \sec(2x) \cdot \tan(2x)\,dx$

$= \frac{1}{2}\int u^2 \cdot du$

$= \frac{1}{2}\, u^3/3 + c$

$= \frac{1}{6}\sec^3(2x) + c$.

Note: $\left[\frac{d}{dx}\sec x = \sec x \cdot \tan x\right]$

48. $\int [\sin(\sin\theta)\cos\theta]\,d\theta$.

Let, $u = \sin\theta$

$du = d\theta \cdot \cos\theta$

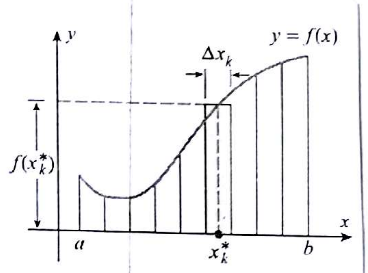
$= \int \sin u \cdot du$

$= -\cos u + c$

$= -\cos(\sin\theta) + c$.

The Definition of Area As a limit – 5.4

\# Sigma Notation: $\sum_{k=m}^{n} f(k)$ → index of summation.



▲ **Figure 6.1.1**

The wide of each subinterval, $\Delta x = \frac{b-a}{n}$.

$$\text{Area} \approx \sum_{k=1}^{n} f(x_k^*)\Delta x$$

1. Left endpoint approximation, $x_k^* = a + (k-1)\Delta x$

2. Right endpoint approximation, $x_k^* = a + k\Delta x$

3. Mid point approximation, $x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x$.

Formula →,

a. $\sum_{k=1}^{n} k = 1+2+3+\cdots+n$
$$= \frac{n(n+1)}{2}.$$

b. $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2$
$$= \frac{n(n+1)(2n+1)}{6}.$$

c. $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\cdots+n^3$
$$= \left[\frac{n(n+1)}{2}\right]^2.$$

① $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} 1 = 1$

⑪ $\lim_{n\to+\infty} \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{2}$.

⑪ $\lim_{n\to+\infty} \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{3}$.

⑫ $\lim_{n\to+\infty} \frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{1}{4}$.

Example–~~3~~ 4 ⇒

The length of each subinterval is:

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}.$$

$$x_k^* = a + k\Delta x = 0 + k\cdot\frac{1}{n} = \frac{k}{n}.$$

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x = \sum_{k=1}^{n} (x_k^*)^2\,\Delta x$$

$$= \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2\cdot\frac{1}{n} = \frac{1}{n^3}\sum_{k=1}^{n} k^2.$$

$$= \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\,\Delta x = \lim_{n\to+\infty}\left[\frac{(n+1)(2n+1)}{6n^2}\right]$$

$$= \lim_{n\to+\infty}\left[\frac{1}{6}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right]$$

$$= \lim_{n\to+\infty}\left[\frac{1}{6}\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= \frac{1}{6}(1)(2) = \frac{1}{3}$$

Ans: $\frac{1}{3}$

Example – 5 ⇒

The lenght of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}.$$

$$x_k^* = a + \left(k-\frac{1}{2}\right)\Delta x = \left(k-\frac{1}{2}\right)\left(\frac{3}{n}\right).$$

$$f(x_k^*)\,\Delta x = \left[9 - (x_k^*)^2\right]\Delta x$$

$$= \left[9 - \left(k-\frac{1}{2}\right)^2\left(\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$$= \left[9 - \left(k^2 - k + \frac{1}{4}\right)\left(\frac{9}{n^2}\right)\right]\left(\frac{3}{n}\right).$$

$$= \frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3}.$$

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \lim_{n\to+\infty} \sum_{k=1}^{n} \left(\frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3}\right)$$

$$= \lim_{n\to+\infty} 27\left[\frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 + \frac{1}{n}\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - \frac{1}{4n^2}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right].$$

$$= 27\left[1 - \frac{1}{3} + 0\cdot\frac{1}{2} - 0\cdot 1\right] = 18.$$ Ans: 18.

Example 17: The length of each subinterval.

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}.$$

$$x_k^* = a + (k-1)\Delta x$$

$$= (k-1)\left(\frac{2}{n}\right).$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left[\left((k-1)\left(\frac{2}{n}\right) - 1\right)\right]\left(\frac{2}{n}\right)$$

$$= \sum_{k=1}^{n}\left[\left(\frac{4}{n^2}\right)k - \frac{4}{n^2} - \frac{2}{n}\right]$$

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \lim_{n\to+\infty}\left[4\cdot\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - \frac{4}{n}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) - 2\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right]$$

$$= 4\cdot\frac{1}{2} - 0\cdot 1 - 2\cdot 1$$

$$= 0.$$

Ans: 0.

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Exercise set 5.4 $\Rightarrow$

35. The length of each subinterval is.

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}.$$

$$x_k^* = a + k\Delta x = 1 + k\cdot\frac{3}{n} = 1 + \frac{3k}{n}.$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left(\frac{1+\frac{3k}{n}}{2}\right)\cdot\frac{3}{n}$$

$$= \sum_{k=1}^{n}\left(\frac{n+3k}{2n}\right)\cdot\left(\frac{3}{n}\right)$$

$$= \sum_{k=1}^{n}\frac{3n+9k}{2n^2}.$$

$$= \sum_{k=1}^{n}\frac{3n}{2n^2} + \sum_{k=1}^{n}\frac{9k}{2n^2}.$$

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n}(f x_k^*)\Delta x = \lim_{n\to+\infty}\left[\frac{3}{2}\left(\frac{1}{n}\sum_{k=1}^{n}1\right) + \frac{9}{2}\left(\frac{1}{n^2}\sum_{k=1}^{n}k\right)\right]$$

$$= \frac{3}{2}\cdot 1 + \frac{9}{2}\cdot\frac{1}{2}$$

$$= \frac{15}{4}. \quad \text{Ans: } \frac{15}{4}.$$

Exercise $\rightarrow$ 36. The length of each subinterval is.

$$\Delta x = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n}.$$

$$x_k^* = a + k\Delta x = 0 + k\cdot\frac{5}{n} = \frac{5k}{n}.$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left(5 - \frac{5k}{n}\right)\cdot\frac{5}{n}$$

$$= \sum_{k=1}^{n}\left(\frac{25}{n} - \frac{25k}{n^2}\right)$$

$$= \sum_{k=1}^{n}\frac{25}{n} - \sum_{k=1}^{n}\frac{25k}{n^2}.$$

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty} \left[25\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) - 25\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) \right]$$

$$= 25\cdot 1 - 25\cdot \frac{1}{2}$$

$$= \frac{50-25}{2} = \frac{25}{2}.$$ Ans: $\frac{25}{2}$.

37. The length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}.$$

$$x_k^* = a + k\Delta x = 0 + k\cdot\frac{3}{n} = \frac{3k}{n}.$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} \left(9 - \left(\frac{3k}{n}\right)^2\right)\frac{3}{n}.$$

$$= \sum_{k=1}^{n} \left(\frac{27}{n} - \frac{27k^2}{n^3}\right)$$

$$= \sum_{k=1}^{n} \frac{27}{n} - \sum_{k=1}^{n} \frac{27k^2}{n^3}.$$

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty} \left[27\cdot\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) - 27\left(\frac{1}{n^3}\sum_{k=1}^{n} k^2\right) \right]$$

$$= 27\cdot 1 - 27\cdot \frac{1}{3}$$

$$= 27 - 9 = 18.$$ Ans: 18.

37+1=38. The length of each subinterval is,

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}.$$

$$x_k^* = a + k\Delta x = 0 + k\cdot\frac{3}{n}$$

$$= \frac{3k}{n}.$$

p.t.o

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left[4-\frac{1}{4}\left(\frac{3k}{n}\right)^2\right]\frac{3}{n}$$

$$= \sum_{k=1}^{n}\left(4-\frac{9k^2}{4n^2}\right)\frac{3}{n}$$

$$= \sum_{k=1}^{n}\left(\frac{12}{n}-\frac{27k^2}{4n^3}\right)$$

$$= \sum_{k=1}^{n}\frac{12}{n}-\sum_{k=1}^{n}\frac{27k^2}{4n^3}$$

$$A=\lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty}\left[12\left(\frac{1}{n}\sum_{k=1}^{n}1\right)-\frac{27}{4}\left(\frac{1}{n^3}\sum_{k=1}^{n}k^2\right)\right]$$

$$= 12\cdot 1-\frac{27}{4}\cdot\frac{1}{3}$$

$$= 12-\frac{27}{12}$$

$$= \frac{117}{12} \quad \text{Ans: } \frac{117}{2}.$$

39. The length of each subinterval is.

$$\Delta x = \frac{b-a}{n} = \frac{6-2}{n} = \frac{4}{n}.$$

$$x_k^* = a+k\Delta x = 2+k\cdot\frac{4}{n} = 2+\frac{4k}{n}.$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left(2+\frac{4k}{n}\right)^3\cdot\frac{4}{n}$$

$$= \sum_{k=1}^{n}\left(8+\frac{48k}{n}+\frac{96k^2}{n^2}+\frac{64k^3}{n^3}\right)\cdot\frac{4}{n}$$

$$= \sum_{k=1}^{n}\left(\frac{32}{n}+\frac{192k}{n^2}+\frac{384k^2}{n^3}+\frac{256k^3}{n^4}\right)$$

$$= \sum_{k=1}^{n}\frac{32}{n}+\sum_{k=1}^{n}\frac{192k}{n^2}+\sum_{k=1}^{n}\frac{384k^2}{n^3}+\sum_{k=1}^{n}\frac{256k^3}{n^4}.$$

p.t.o

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty} 32\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) + 192\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) + 384\left(\frac{1}{n^3}\sum_{k=1}^{n} k^2\right)$$

$$+ 256\left(\frac{1}{n^4}\sum_{k=1}^{n} k^3\right).$$

$$= 32\cdot 1 + 192\cdot\frac{1}{2} + 384\cdot\frac{1}{3} + 256\cdot\frac{1}{4}$$

$$= 32 + 96 + 128 + 64$$

$$= 320. \quad \text{Ans: } 320.$$

** 40. The length of each subinterval is,

$$\Delta x = \frac{b-a}{n} = \frac{-1+3}{n} = \frac{2}{n}$$

$$x_k^* = a + k\Delta x = -3 + \frac{2k}{n}.$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left[1 - \left(-3 + \frac{2k}{n}\right)^3\right]\frac{2}{n}$$

$$= \sum_{k=1}^{n}\left[1 + \left(3 - \frac{2k}{n}\right)^3\right]\frac{2}{n}.$$

$$= \sum_{k=1}^{n}\left[\frac{2}{n} + \left(27 - \frac{54k}{n} + \frac{36k^2}{n^2} - \frac{8k^3}{n^3}\right)\frac{2}{n}\right]$$

$$= \sum_{k=1}^{n}\left(\frac{2}{n} + \frac{54}{n} - \frac{108k}{n^2} + \frac{72k^2}{n^3} - \frac{2\times 8k^3}{n^4}\right)$$

$$A = \lim_{n\to+\infty} f(x_k^*)\Delta x = \cancel{\sum_{k=1}^{n}} \lim_{n\to+\infty}\left[2\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) + 54\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right.$$

$$\left. - 108\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) + 72\left(\frac{1}{n^3}\sum_{k=1}^{n} k^2\right) - 16\left(\frac{1}{n^4}\sum k^3\right)\right]$$

$$= 2\cdot 1 + 54\cdot 1 - 108\cdot\frac{1}{2} + 72\cdot\frac{1}{3} - 16\cdot\frac{1}{4}$$

$$= 2 + 54 - 54 + 24 - 4$$

$$= 22. \quad \text{Ans: } 22.$$

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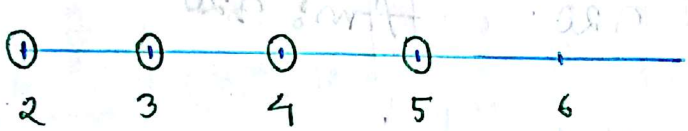
27. Here, it is given in the Que, $n=4$.

$$f(x) = 3x+1.$$

$$[a, b] = [2, 6]$$

$$\therefore \Delta x = \frac{b-a}{n} = \frac{6-2}{4} = 1.$$

a). Left-end point approximation :-



Note, → [n এর value দেওয়া থাকলে $k=1$ ধরে নেওয়া যায়]

$$x_k^* = a+(k-1)\Delta x = 2+0 \cdot 1 = 2.$$

$$A = \sum_{k=1}^{n} f(x_k^*) \Delta x$$

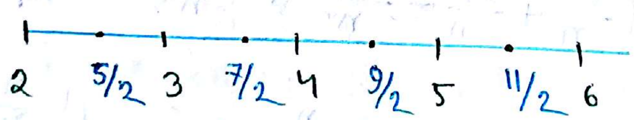
$$= [f(2)+f(3)+f(4)+f(5)] 1$$

$$= (7+10+13+16) \cdot 1$$

$$= 46.$$

Ans: 46

b) Mid- ~~end~~ point approximation :-



$$x_k^* = a+\left(k-\frac{1}{2}\right)\Delta x = 2+\left(1-\frac{1}{2}\right) \cdot 1$$

$$= 2+\frac{1}{2} = \frac{5}{2}.$$

$$A = \sum_{k=1}^{n} f(x_k^*) \Delta x$$

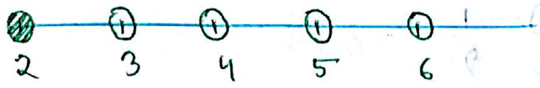
$$= [f(5/2)+f(7/2)+f(9/2)+f(11/2)] \cdot 1$$

$$= \frac{17}{2}+\frac{23}{2}+\frac{29}{2}+\frac{35}{2}$$

$$= \frac{104}{2} = 52.$$

Ans: 52.

c) Right - end point approximation :-



$$x_k^* = a + k\Delta x = 2 + 1 \cdot 1 = 2.$$

$$\Delta = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= [f(3) + f(4) + f(5) + f(6)] \cdot 1$$

$$= (10 + 13 + 16 + 19) \cdot 1$$

$$= 58. \qquad \text{Ans: } 58.$$

27+1
=28.

Here, it is given in the question, $n = 4$

$$f(x) = \frac{1}{x}.$$

$$[a, b] = [1, 9]$$

$$\Delta x = \frac{b-a}{n} = \frac{9-1}{4} = 2.$$

* a) Left - end point approximation :-

$$x_k^* = a + (k-1)\Delta x = 1 + 0 \cdot 2 = 1.$$

$$\Delta = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= [f(1) + f(3) + f(5) + f(7)] 2$$

$$= \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}\right) 2$$

$$= \left[\frac{105 + 35 + 21 + 15}{105}\right] 2$$

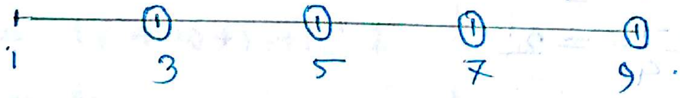
$$= \frac{352}{105}.$$

b). Mid-point approximation :-

$$x_k^* = a + \left(k - \tfrac{1}{2}\right)\Delta x$$
$$= a + \tfrac{1}{2}\cdot 2$$
$$= 1 + 1 = 2.$$

$$A = \sum_{k=1}^{n} f(x_k^*)\Delta x = [f(2) + f(4) + f(6) + f(8)]\,2$$
$$= \left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{8}\right)2$$
$$= \frac{25}{12}. \quad \text{Ans: } \frac{25}{12}.$$

c) Right end point approximation :-



$$x_k^* = a + k\Delta x = 1 + 1\cdot 2 = 3$$

$$A = \sum_{k=1}^{n} f(x_k^*)\Delta x$$
$$= [f(3) + f(5) + f(7) + f(9)]\,3$$
$$= \left[\tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{9}\right]3$$
$$= \frac{744}{315}$$
$$= \frac{248}{105}. \quad \text{Ans: } \frac{248}{105}.$$

30. Here it is given in the question, $n=4$.

$$f(x) = 2x - x^2;$$

$$[a, b] = [-1, 3]$$

$$\Delta x = \frac{b-a}{n} = \frac{3+1}{4} = 1$$

a) Left end-point approximation:-

$$x_k^* = a + (k-1)\Delta x = -1$$

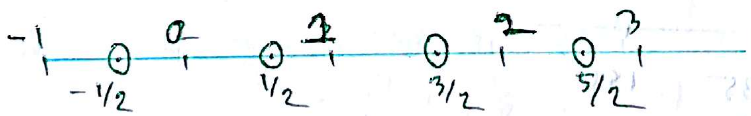
$$A = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= [f(-1) + f(0) + f(1) + f(2)]\, 1$$

$$= (-3+0+1+0)\, 1$$

$$= -2. \quad \text{Ans: } -2.$$

b) Mid point approximation:-



$$x_k^* = a + (k - \tfrac{1}{2})\Delta x = -1 + \tfrac{1}{2} \times 1 = -\tfrac{1}{2}$$

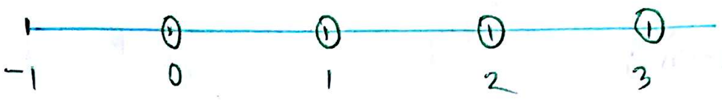
$$A = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= [f(-1/2) + f(1/2) + f(3/2) + f(5/2)]\, 1$$

$$= -5/4 + 3/4 + 3/4 + (-5/4)$$

$$= -1. \quad \text{Ans: } -1.$$

c) Right end point approximation :-



$x_k^* = a + k\Delta x = -1 + 1 \cdot = 0$.

$\Delta = \sum_{k=1}^{n} f(x_k^*)\Delta x$

$= [f(0) + f(1) + f(2) + f(3)] \cdot 1$

$= 0 + 1 + 0 - 3 = -2$; Ans: -2.

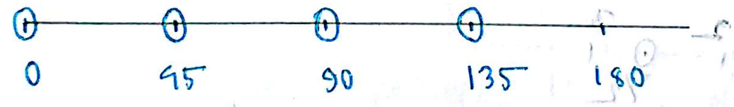
29. Here, it is given in the question, $n = 4$.

$f(x) = \cos x \; [0, \pi]$

$[a, b] = [0, \pi]$, $n = 4$.

$\Delta x = \frac{b-a}{n} = \frac{\pi}{4} = 45^\circ$.

a) Left end point approximation :-



$x_k^* = a + (k-1)\Delta x = 0 + (k-1) \cdot \Delta x = 0$.

$\Delta = \sum_{k=1}^{n} f(x_k^*)\Delta x$

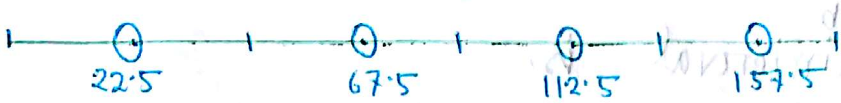
$= [f(0) + f(45) + f(90) + f(135)] \cdot 45^\circ$

$= (\cos 0 + \cos 45 + \cos 90 + \cos 135) \cdot 45^\circ$

$= [1 + 0.707 + 0 + (-0.707)]\, 45^\circ$

$= 45^\circ$. Ans: 45°.

Mid point approximation :-



$$x_k^* = a + (k - \tfrac{1}{2})\Delta x = 0 + \tfrac{1}{2} \cdot 45 = 22{\cdot}5$$

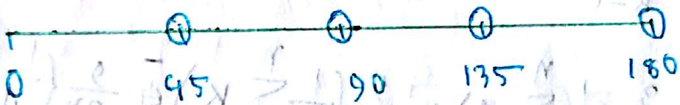
$$A = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \cancel{\sum_{k=1}^{n}} \left[f(22{\cdot}5) + f(67{\cdot}5) + f(112{\cdot}5) + f(157{\cdot}5) \right] 45^\circ$$

$$= \left[0{\cdot}923 + 0{\cdot}382 + (-0{\cdot}382) + (-0{\cdot}923) \right] 45^\circ$$

$$= 0 \cdot \quad \text{Ans: } 0 \cdot$$

Right end point approximation:-



$$x_k^* = a + k\Delta x = 0 + 1 \cdot 45 = 45$$

$$A = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \left[f(45) + f(90) + f(135) + f(180) \right] 45$$

$$= \left[\cos 45 + \cos 90 + \cos 135 + \cos 180 \right] 45$$

$$= \left[0{\cdot}707 + 0 - 0{\cdot}707 - 1 \right] 45$$

$$= -45^\circ \cdot \quad \text{Ans: } -45^\circ \cdot$$

p.t.o

41. $f(x) = x/2 . [1, 4]$

The length of each subinterval is,

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}.$$

$$x_k^* = a+(k-1)\Delta x = 1+(k-1)\frac{3}{n}$$

$$= \frac{n+3k-3}{n}$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} \left(\frac{n+3k-3}{2n}\right)\cdot\frac{3}{n}$$

$$= \sum_{k=1}^{n} \frac{3n+9k-9}{2n^2}.$$

$$= \sum_{k=1}^{n} \frac{3n}{2n^2} + \sum_{k=1}^{n} \frac{9k}{2n^2} - \sum_{k=1}^{n} \frac{9}{2n^2}.$$

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty}\left[\frac{3}{2}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) + \frac{9}{2}\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - \frac{9}{2n}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right]$$

$$= \frac{3}{2}\cdot 1 + \frac{9}{2}\cdot\frac{1}{2} - 0\cdot 1$$

$$= \frac{3}{2} + \frac{9}{4}$$

$$= \frac{15}{4}. \quad \text{Ans: } \frac{15}{4}.$$

**

42. The length of each subinterval is,

$$\Delta x = \frac{b-a}{n}. = \frac{5}{n}.$$

$$x_k^* = a+(k-1)\Delta x = (k-1)\frac{5}{n} = \frac{5k-5}{n}.$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left(5 - \frac{5k-5}{n}\right)\cdot\frac{5}{n}$$

$$= \sum_{k=1}^{n}\left(\frac{5n-5k-5}{n}\right)\frac{5}{n}$$

$$= \sum_{k=1}^{n} \frac{25n-25k-25}{n^2}.$$

$$= \sum_{k=1}^{n} \frac{25n}{n^2} - \sum_{k=1}^{n} \frac{25k}{n^2} - \frac{25}{n^2}\sum_{k=1}^{n}$$

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty} \left(\frac{25}{1}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) - \frac{1}{n^2} 25 \sum_{k=1}^{n} k - 25/n^2 \sum_{k=1}^{n} 1 \right)$$

$$= 25\cdot 1 - 25\cdot\frac{1}{2} - 0$$

$$= 25 - \frac{25}{2} = \frac{50-25}{2} = \frac{25}{2}. \quad \text{Ans: } \frac{25}{2}.$$

43. The length of each subinterval is,

$$\Delta x = \frac{a-b}{n} = \frac{3}{n}.$$

$$x_k^* = a + (k-1)\Delta x = \frac{3k-3}{n}.$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} \left(9 - \left(\frac{3k-3}{n}\right)^2\right)\frac{3}{n}$$

$$= \sum_{k=1}^{n} \left(9 - \frac{9k^2-18k+9}{n^2}\right)\frac{3}{n}$$

$$= \sum_{k=1}^{n} \left(\frac{27}{n} - \frac{27k^2-54k+27}{n^3}\right)$$

$$= \sum_{k=1}^{n} \frac{27}{n} - \sum_{k=1}^{n} \frac{27k^2}{n^3} + \sum_{k=1}^{n} \frac{54k}{n^3} - \sum_{k=1}^{n} \frac{27}{n^3}$$

$$A = \lim_{n\to+\infty} f(x_k^*)\Delta x = \lim_{n\to+\infty} 27\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) - 27\left(\frac{1}{n^3}\sum_{k=1}^{n} k^2\right) + 54/n\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - 27/n^2\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\Bigg]$$

$$= 27\cdot 1 - 27\cdot\frac{1}{3} + 54/n \cdot 1/2 - 0\cdot 1$$

$$= 27 - 27\cdot\frac{1}{3} + 0 - 0$$

$$= \cancel{\frac{27}{2}} = 18. \quad \text{Ans: } 18.$$

The length of each subinterval is,

$$\Delta x = \frac{b-a}{n} = \frac{3}{n}.$$

$$x_k^* = a+(k-1)\Delta x = \frac{3k-3}{n}.$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} \left[4 - \frac{1}{4}\cdot(3k-3/n)^2\right]\frac{3}{n}.$$

$$= \sum_{k=1}^{n} \left[4 - \frac{9k^2-18k+9}{4n^2}\right]\frac{3}{n}.$$

$$= \sum_{k=1}^{n} \left[\frac{12}{n} - \frac{27k^2-54k+27}{4n^3}\right]$$

$$= \sum_{k=1}^{n} \frac{12}{n} - \sum_{k=1}^{n} \frac{27k^2}{4n^3} + \sum_{k=1}^{n} \frac{54k}{4n^3} - \sum_{k=1}^{n} \frac{27}{4n^3}.$$

$$A = \lim_{n\to+\infty} f(x_k^*)\Delta x = \lim_{n\to+\infty}\left[12\left(\frac{1}{n}\sum_{k=1}^{n} 1\right) - \frac{27}{4}\left(\frac{1}{n^3}\sum_{k=1}^{n} k^2\right) + \frac{54}{4n}\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right)\right.$$

$$\left. - \frac{27}{4n^2}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right]$$

$$= 12\cdot 1 - \frac{27}{4}\cdot\frac{1}{3} + 0 - 0$$

$$= 12 - \frac{27}{12}$$

$$= \frac{117}{12}.$$

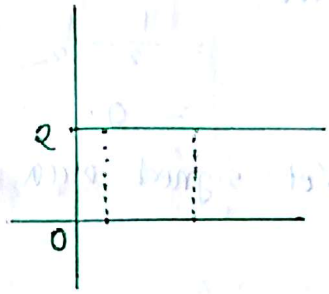
Ans: $\frac{117}{12}$.

The Definite Integral — 5.5

$$\lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*) \Delta x = \int_a^b f(x)\, dx \rightarrow \text{Riemann Sum.}$$

Example from book →

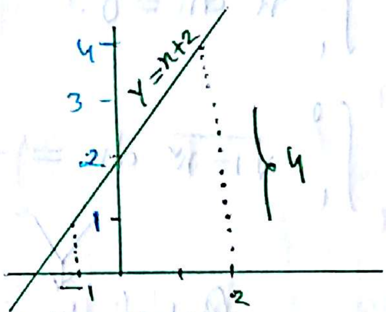
1. a) $\int_1^4 2\, dx$



Area of rectangle = base × hight

$= 3 \times 2 = 6.$

2. b) $\int_{-1}^{2} (x+2)\, dx$



Area of Trapezoid = $\frac{1}{2}$ × সমান্তরাল বাহুদ্বয়ের দূরত্ব × (সমান্তরাল বাহুদ্বয়ের দৈর্ঘ্যের যোগফল).

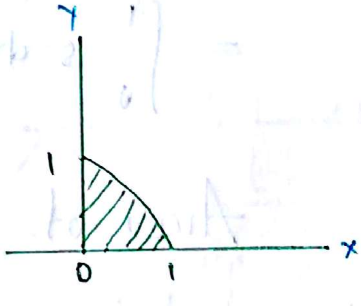
$= \frac{1}{2} \times 3 \times (1+4)$

$= \frac{15}{2}.$

3. c) $\int_0^1 \sqrt{1-x^2}\, dx$, Here, $y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1.$

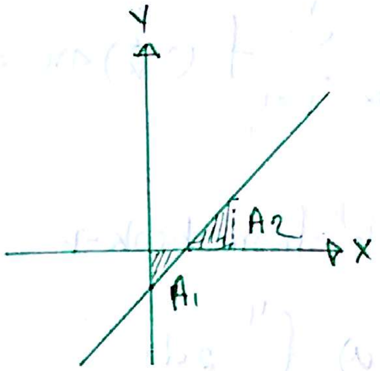


Area of circle $= \frac{1}{4} \pi r^2$

$= \frac{1}{4} \pi (1)^2$

$= \frac{\pi}{4}.$

2. a) $\int_0^2 (x-1)\,dx.$



Total area $A = A_2 - A_1$

$= \frac{1}{2} - \frac{1}{2}$

$= 0.$

Hence Net signed area 0.

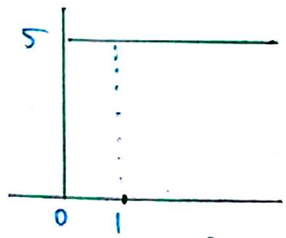
3. a) $\int_1^1 x^2\,dx = 0.$ [Direct formula]

b) $\int_1^0 \sqrt{1-x^2}\,dx = -\int_0^1 \sqrt{1-x^2}\,dx.$

Ans: $-\frac{\pi}{4}.$

Rest of the part similar to 1(c).

4. $\int_0^1 \left(5 - 3\sqrt{1-x^2}\right) dx$

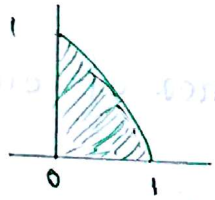


P_1

$= \int_0^1 5\,dx - 3\int_0^1 \sqrt{1-x^2}\,dx$

Area of rectangle = base × height

$= 1\times 5 = 5.$



P_2

Area of circle $= \frac{1}{4}\pi r^2$

$= \frac{\pi}{4}$

$\therefore A = 5 - 3\cdot\frac{\pi}{4}$

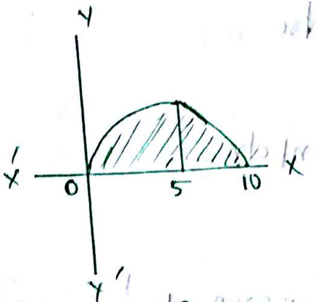
$= 5 - \frac{3\pi}{4}.$

Ans: $5 - \frac{3\pi}{4}.$

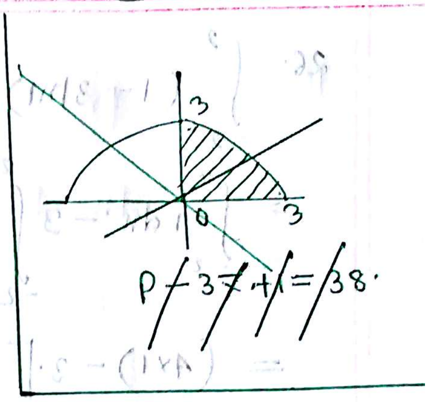
(Rough).

Exercise set 5.5 →

37. $\int_0^{10} \sqrt{10x-x^2}\, dx$



P-37.



P-37+1=38.

Here, $y=\sqrt{10x-x^2}$

$y^2 = 10x-x^2$

$(x-5)^2+y^2 = 5^2$

Area $=\frac{1}{2}\pi r^2 = \frac{1}{2}\pi\times 5^2$

$= \frac{25}{2}\pi$.

Ans: $\frac{25}{2}\pi$.

38. $\int_0^3 \sqrt{6x-x^2}\, dx$.

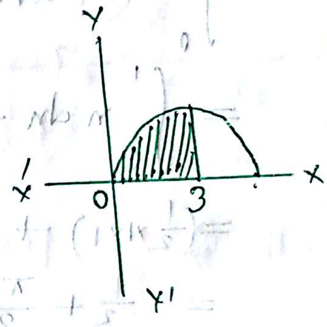
Here, $y=\sqrt{6x-x^2}$

$y^2 = 6x-x^2$

$y^2+x^2-6x=0$

$x^2-6x+9+y^2=9$

$(x-3)^2+y^2=3^2$

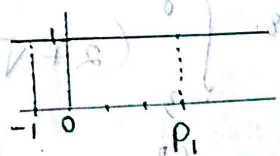


Area $= \frac{1}{4}\pi r^2$

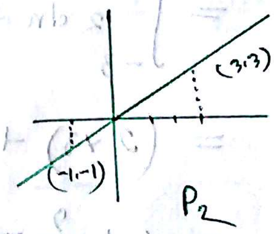
$= \frac{1}{4}\times\pi\times 3^2$

$= \frac{9\pi}{4}$. Ans: $\frac{9\pi}{4}$.

25. $\int_{-1}^{3} (4-5x)\,dx$.



$= 4\int_{-1}^{3} 1\cdot dx - 5\int_{-1}^{3} x\, dx$



$= 4(4\times1) - 5\left[\left(\frac{1}{2}\times3\times3\right) - \left(\frac{1}{2}\times1\times1\right)\right]$

$= 16 - 5\left(\frac{9}{2}-\frac{1}{2}\right)$

$= 16 - \frac{45}{2} + \frac{5}{2}$

$= \frac{32-45+5}{2}$

$= -4$. Ans: -4.

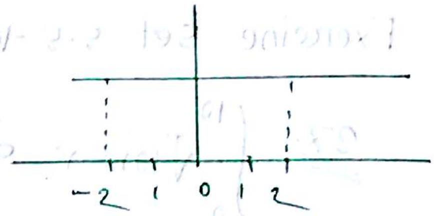
26. $\int_{-2}^{2} (1 - 3|x|)\, dx$

$= \int_{-2}^{2} 1\, dx - 3\int_{-2}^{2} |x|\, dx$

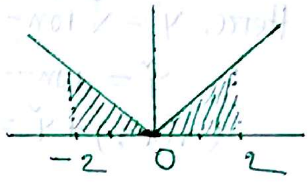
$= (4 \times 1) - 3 \cdot \left[\frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 2 \times 2\right]$

$= 4 - (3 \times 4)$

$= 4 - 12 = 8 \cdot$ Ans: 8 .



P_1



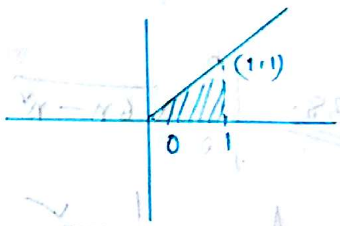
P_2

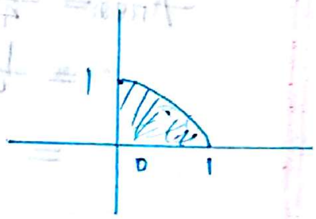
27. $\int_{0}^{1} \left(x + 2\sqrt{1 - x^2}\right) dx$

$= \int_{0}^{1} x\, dx + 2\int_{0}^{1} \sqrt{1 - x^2}\, dx$.

$= \left(\frac{1}{2} \times 1 \times 1\right) + 2\left(\frac{\pi}{4}\right)$

$= \frac{1}{2} + \frac{\pi}{2} = (1 + \pi)\frac{1}{2}$ · Ans . $(1 + \pi)\frac{1}{2}$.





28. $\int_{-3}^{0} \left(2 + \sqrt{9 - x^2}\right) dx$

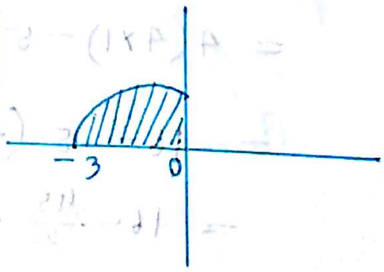
$= \int_{-3}^{0} 2\, dx + \int_{-3}^{0} \left(\sqrt{9 - x^2}\right) dx$

$= (2 \times 3) + \frac{1}{4}\pi (3)^2$

$= 6 + \frac{9}{4}\pi$

$= \frac{24 + 9\pi}{4}$.

Ans $\frac{24 + 9\pi}{4}$.



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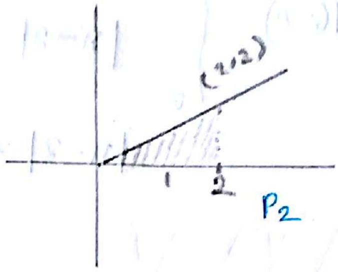
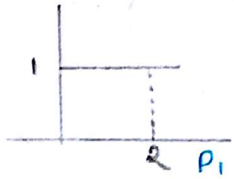
14. a. $\int_0^2 \left(1-\frac{1}{2}x\right)dx$

$= \int_0^2 1\,dx - \frac{1}{2}\int_0^2 x\,dx.$

$= (2\times1) - \frac{1}{2}\left(\frac{1}{2}\times2\times2\right)$

$= 2-0 = 2.$ Ans: 2.



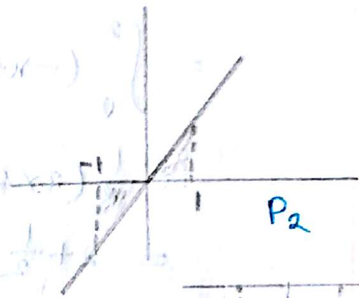


b. $\int_{-1}^{1} \left(1-\frac{1}{2}x\right)dx$

$= \int_{-1}^{1}(1)dx - \frac{1}{2}\int_{-1}^{1} x\,dx$

$= (2\times1) - \frac{1}{2}\times 0$ [From, P_2 it is easily visible that the positive and negative area are same]

$= 2.$

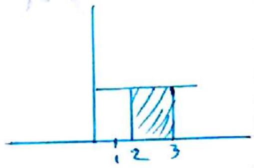


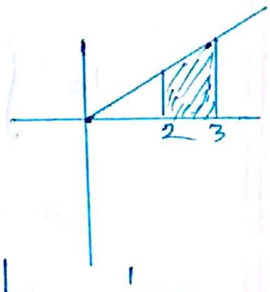
c. $\int_2^3 \left(1-\frac{1}{2}x\right)dx.$

$= \int_2^3 (1)\,dx - \frac{1}{2}\int_2^3 x\,dx$

$= (1\times1) - \frac{1}{2}\left(\frac{1}{2}\times1(2+3)\right)$ [Trapezoid].

$= 1-\frac{5}{4} = \frac{4-5}{4} = \frac{-1}{4}.$ Ans.





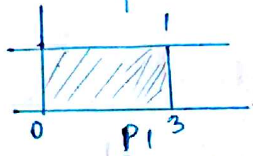
d. $\int_0^3 \left(1-\frac{1}{2}x\right)dx$

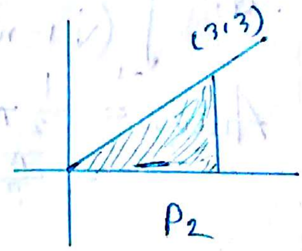
$= \int_0^3 (1)\,dx - \frac{1}{2}\int_0^3 x\,dx$

$= (3\times1) - \frac{1}{2}\left(\frac{1}{2}\times3\times3\right)$

$= 3-\frac{9}{4} = \frac{3}{4}.$

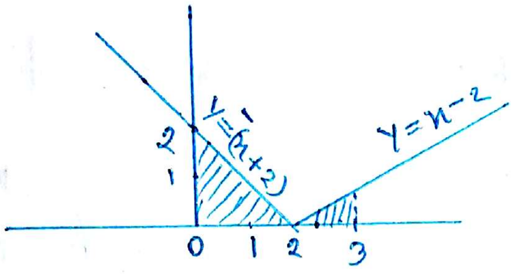
Ans: $\frac{3}{4}$.





16. c) $\int_0^3 |x-2|\, dx$

$$|x-2| = \begin{cases} x-2, & x-2 \geq 0 \Rightarrow x \geq 2 \\ -(x+2) & x-2 \leq 0 \Rightarrow x \leq 2 \end{cases}$$

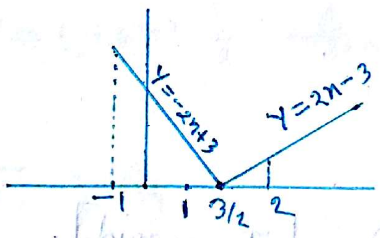


$$= \int_0^2 (-x+2)\,dx + \int_2^3 (x-2)\,dx$$

$$= \frac{1}{2}(2\times 2) + \frac{1}{2}(1\times 1)$$

$$= 2 + \frac{1}{2} = \frac{5}{2}. \quad \text{Ans: } \frac{5}{2}.$$

15. c) $\int_{-1}^{2} |2x-3|\, dx$, $|2x-3| = \begin{cases} 2x-3, & 2x-3 \geq 0, x \geq \frac{3}{2}, \\ -2x+3 & +(2x-3) \leq 0, x \leq \frac{3}{2} \end{cases}$



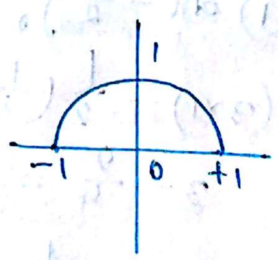
$$= \int_{-1}^{3/2} (-2x+3)\,dx + \int_{3/2}^{2} (2x-3)\,dx$$

$$= \frac{1}{2}(5\times 2.5) + \frac{1}{2}(0.5\times 1)$$

$$= 6.25 + 0.25$$

$$= 6.50. \quad \text{Ans: } 6.50.$$

d. $\int_{-1}^{1} \left(\sqrt{1-x^2}\right) dx$



Area $= \frac{1}{2}\pi r^2$

$= \frac{\pi}{2}$. Ans. $\frac{\pi}{2}$.

The Fundamental Theory of Calculus → 5.6

If 'f' is continuous on $[a,b]$ and 'F' is any derivative of 'f' on $[a,b]$, then,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Example From book ⇒.

2. $\int_0^3 (9-x^2)\,dx = \left[9x - \frac{x^3}{3}\right]_0^3$

$= (27-9) - (0-0)$

$= 18.$ Ans: 18.

3. $\int_0^\pi \cos x\,dx = [\sin x]_0^\pi = \sin\pi - \sin 0 = 0.$ Ans: 0.

4. $\int_1^9 \sqrt{x}\,dx = \int_1^9 x^{1/2}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_1^9 = \frac{2}{3}(27-1) = \frac{52}{3}.$ Ans: $\frac{52}{3}$.

5. b) $\int_0^{\pi/2} \frac{\sin x}{5}\,dx = \frac{1}{5}\int_0^{\pi/2} \sin x = -\frac{1}{5}[\cos x]_0^{\pi/2}$

$= -\frac{1}{5}\left(\cos\frac{\pi}{2} - \cos 0\right)$

$= -\frac{1}{5}(0-1) = \frac{1}{5}.$ Ans: $\frac{1}{5}$.

e) $\int_{-e}^{-1} \frac{1}{x}\,dx = \ln\left[|x|\right]_{-e}^{-1} = \ln|-1| - \ln|-e|$ $\boxed{\ln e = 1.}$

$= 0 - 1$

$= -1.$ Ans: -1.

f) $\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1} x\right]_{-1/2}^{1/2}$

$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$

$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)$

$= \frac{\pi}{3}.$ Ans: $\pi/3$.

7. $\int_0^3 f(x)\,dx$. $\quad f(x) = \begin{cases} x^2 & , x<2 \\ 3x-2 & , x \geq 2 \end{cases}$

$$= \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3$$

$$= \frac{8}{3} + \left(\frac{15}{2} - 2\right)$$

$$= \frac{49}{6}. \quad \text{Ans: } \frac{49}{6}.$$

8. $\int_0^2 |1-x^2|\,dx$ $\quad f(x) = \begin{cases} 1-x^2, & 1-x^2>0,\ x^2<1 \\ -1+x^2, & 1-x^2 \leq 0,\ x^2 \leq 1 \end{cases}$

$$= \int_0^1 (1-x^2)\,dx + \int_1^2 -(1-x^2)\,dx$$

$$= \left[x - \frac{x^3}{3}\right]_0^1 + (-)\left[x - \frac{x^3}{3}\right]_1^2$$

$$= \left[x - \frac{x^3}{3}\right]_0^1 - \left[x - \frac{x^3}{3}\right]_1^2$$

$$= \frac{2}{3} - \left(-\frac{4}{3}\right)$$

$$= 2. \quad \text{Ans: } 2.$$

10. $\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$

first step, $\int_1^x t^3\,dt = \left[\frac{t^4}{4}\right]_1^x = \frac{x^4}{4} - \frac{1}{4}$.

$$\frac{d}{dx}\left[\frac{x^4}{4} - \frac{1}{4}\right] = x^3. \quad \text{Ans: } x^3.$$

Exercise Set 5.6 ⇒

23. $\int_{\ln 2}^{3} 5e^{x}dx$

$= 5\cdot\left[e^{x}\right]_{\ln 2}^{3}$

$= 5(e^{3}-e^{\ln 2})$

$= 5(20-2)$

$= 90$. Ans: 90.

24. $\int_{1/2}^{1} \frac{1}{2x}dx$

$= \frac{1}{2}\ln\left[|x|\right]_{1/2}^{1}$

$= \frac{1}{2}\left[\ln(1)-\ln(1/2)\right]$

$= \frac{1}{2}(0-0.7)$

≈ -0.34. Ans: -0.34.

25. $\int_{0}^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}$

$= \left[\sin^{-1}x\right]_{0}^{1/\sqrt{2}}$

$= \sin^{-1}(1/\sqrt{2})-\sin^{-1}(0)$

$= \frac{\pi}{4}-\frac{\pi}{2} = \frac{\pi-2\pi}{4}$.

Ans: $\frac{\pi-2\pi}{4}$.

26. $\int_{-1}^{1}\frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_{-1}^{1}$

$= \tan^{-1}(1)-\tan^{-1}(-1)$

$= \frac{\pi}{4}+\frac{\pi}{4}$

$= \frac{\pi}{2}$. Ans: $\frac{\pi}{2}$.

27. $\int_{\sqrt{2}}^{2}\frac{dx}{x\sqrt{x^2-1}}$.

$= \left[\sec^{-1}|x|\right]_{\sqrt{2}}^{2}$

$= \sec^{-1}(2)-\sec^{-1}(\sqrt{2})$

$= \frac{\pi}{3}-\frac{\pi}{4}$.

Ans: $\frac{\pi}{3}-\frac{\pi}{4}$.

28. $\int_{-\sqrt{2}}^{-2/\sqrt{3}}\frac{dx}{x\sqrt{x^2-1}}$

$= \left[\sec^{-1}|x|\right]_{-\sqrt{2}}^{-2/\sqrt{3}}$

$= \sec^{-1}|-2/\sqrt{3}|-\sec^{-1}|-\sqrt{2}|$

$= \frac{\pi}{6}-\frac{\pi}{4}$

$= \frac{2\pi-3\pi}{12}$.

Ans: $\frac{2\pi-3\pi}{12}$.

29. $\int_1^4 \left(\frac{1}{\sqrt{t}} - 3\sqrt{t} \right) dt$

$= \int_1^4 (t^{-1/2})\, dt - 3\int_1^4 (t^{1/2})\, dt$

$= \left[\frac{t^{1/2}}{1/2} \right]_1^4 - 3\left[\frac{t^{3/2}}{3/2} \right]_1^4$

$= (4-2) - 2(8-1)$

$= 2 - 16 + 2 = -16$. Ans: -16.

30. $\int_{\pi/6}^{\pi/2} \left(x + \frac{2}{\sin^2 x} \right) dx$

$= \int_{\pi/6}^{\pi/2} (x)\, dx + 2\int_{\pi/6}^{\pi/2} \frac{1}{\sin^2 x} \cdot dx$

$= \left[\frac{x^2}{2} \right]_{\pi/6}^{\pi/2} + 2\int_{\pi/6}^{\pi/2} \operatorname{cosec}^2 x\, dx$

$= \left[\frac{x^2}{2} - 2\cot x \right]_{\pi/6}^{\pi/2}$

$= \left[\frac{(\pi/2)^2}{2} - 2\cot(\pi/2) \right] - \left[\frac{\pi^2/36}{2} - 2\cot(\pi/6) \right]$

$= \left(\frac{\pi^2}{8} - 0 \right) - \left(\frac{\pi^2}{72} - 2\sqrt{3} \right)$

$= \frac{\pi^2}{9} + 2\sqrt{3}$.

Ans: $\frac{\pi^2}{9} + 2\sqrt{3}$

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31. $\int_{-1}^{1} |2x-1|\,dx.$ $\quad |2x-1| = \begin{cases} 2x-1, & 2x-1 \geq 0 \Rightarrow x > \frac{1}{2} \\ -2x+1 & 2x-1 \leq 0 \Rightarrow x < \frac{1}{2}. \end{cases}$

$$= \int_{-1}^{1/2} (-2x+1)\,dx + \int_{1/2}^{1} (2x-1)\,dx$$

$$= \left[-2\cdot\frac{x^2}{2} + x\right]_{-1}^{1/2} + \left[\frac{2x^2}{2} - x\right]_{1/2}^{1}$$

$$= \left[-\frac{1}{4} + \frac{1}{2} + 1 + 1\right] + \left[1 - 1 - \frac{1}{4} + \frac{1}{2}\right]$$

$$= \frac{5}{2}. \quad \text{Ans: } \frac{5}{2}$$

* $\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2-1}}.$ $\quad$ formula $\rightarrow \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x|.$

$$= \left[\sec^{-1}|x|\right]_{\sqrt{2}}^{2}$$

$$= \sec^{-1}|2| - \sec^{-1}|\sqrt{2}|$$

$$= \frac{\pi}{3} - \frac{\pi}{4}.$$

Ans: $\frac{\pi}{3} - \frac{\pi}{3}.$

Evaluating definite integrals by Substitution – 5.9

Examples from book ⟹

1. $\int_0^2 x(x^2+1)^3 dx$

Let, $u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{du}{2} = x\cdot dx$

if, $x = 2$, $u = 5$

$x = 0$, $u = 1$.

$= \frac{1}{2}\int_1^5 u^3\, du$

$= \left[\frac{u^4}{8}\right]_2^5$

$= \frac{625}{8} - \frac{1}{8} = 78.$

Ans: 78.

3. a). $\int_0^{3/4} \frac{dx}{1-x}$.

Let, $u = 1-x$

$\frac{du}{dx} = -1$

$du = -dx$.

if, $x = 3/4$. $u = \frac{1}{4}$

$x = 0$. $u = 1$.

$= -\int_1^{1/4} \frac{du}{u}$.

$= -\left[\ln|u|\right]_1^{\frac{1}{4}} = -\left[\ln\frac{1}{4} - \ln 1\right]$

$= \ln 4$. Ans: $\ln 4$.

b) $\int_0^{\ln 3} e^x (1+e^x)^{1/2} dx$.

Let,

$u = 1+e^x$

$du = e^x\cdot dx$

if, $x = \ln 3$. $u = 4$.

$x = 0$. $u = 2$.

$= \int_2^4 u^{1/2}.du$

$= \left[\frac{u^{3/2}}{3/2}\right]_2^4$

$= \frac{2}{3}\left[4^{3/2} - 2^{3/2}\right]$

$= \frac{16-4\sqrt{2}}{3}$.

Ans: $\frac{16-4\sqrt{2}}{3}$.

2. a) $\displaystyle\int_0^{\pi/8} \sin^5 2x \cos 2x \, dx$

$= \dfrac{1}{2}\displaystyle\int_0^{1/\sqrt{2}} u^5 \, du$

$= \dfrac{1}{2} \left[\dfrac{u^6}{6}\right]_0^{1/\sqrt{2}}$

$= \dfrac{1}{2} \cdot \left[\dfrac{1}{(\sqrt{2})^6 \cdot 6}\right] = \dfrac{1}{96}.$

Ans: $\dfrac{1}{96}$.

Let,

$u = \sin 2x$

$\dfrac{du}{dx} = 2 \cdot \cos 2x$

$du = 2\cos 2x \cdot dx$

if, $x = 0$, $u = 0$

$x = \pi/8$, $u = 1/\sqrt{2}$.

b) $\displaystyle\int_2^5 (2x-5)(x-3)^9 \, dx.$

$= \displaystyle\int_{-1}^{2} (2u+1)u^9 \, du$

$= \displaystyle\int_{-1}^{2} (2u^{10}+u^9) \, du.$

$= \left[\dfrac{2u^{11}}{11} + \dfrac{u^{10}}{10}\right]_{-1}^{2}$

$= \left(\dfrac{2^{12}}{11} + \dfrac{2^{10}}{10}\right) - \left(-\dfrac{2}{11} + \dfrac{1}{10}\right)$

$= \dfrac{52233}{110}$

$\approx 474.28.$

Ans: ≈ 474.28.

Let, $u = x-3$

$du = dx.$

However, $x = u+3.$

Hence, $2x-5 = 2(u+3)-5$

$= 2u+6-5$

$= 2u+1.$

if, $x = 2$, $u = 2-3 = -1.$

$x = 5$, $u = 5-3 = 2.$

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Exercise 5.9 $\Rightarrow$

19. $\displaystyle\int_{-5/3}^{5/3} \sqrt{25-9x^2}\, dx,$

Let, $u = 3x$

or, $\dfrac{du}{dx} = 3$

or, $\dfrac{du}{3} = dx,$

if. $x = 5/3$, $u = 5$

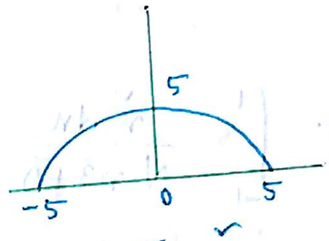
$x = -5/3$, $u = -5$,

$= \displaystyle\int_{-5/3}^{5/3} \sqrt{5^2-(3x)^2}\, dx$

$= \dfrac{1}{3}\displaystyle\int_{-5}^{5} \sqrt{25-u^2}\, du$

$= \dfrac{1}{3} \times \dfrac{1}{2}\pi 5^2$

$= \dfrac{25\pi}{6}$. Ans: $\dfrac{25\pi}{6}$.



$y^2+u^2=5^2$.

20. $\displaystyle\int_{0}^{2} x\sqrt{16-x^4}\, dx$

Let, $u = x^2$

or, $\dfrac{du}{dx} = 2x$

or, $\dfrac{du}{2} = x \cdot dx$

if, $x = 2$, $u = 4$.

$x = 0$, $u = 0$.

$= \displaystyle\int_{0}^{2} x\sqrt{16-(x^2)^2}\, dx.$

$= \dfrac{1}{2}\displaystyle\int_{0}^{4} \sqrt{16-u^2}\cdot du$

$= \dfrac{1}{2} \times \dfrac{1}{4}\pi 4^2 = \dfrac{4\pi}{2} = 2\pi.$

Ans: 2π.

21. $\displaystyle\int_{\pi/3}^{\pi/2} \sin\theta\sqrt{1-4\cos^2\theta}\, d\theta.$

Let, $u = 2\cos\theta$.

or, $\dfrac{du}{d\theta} = -2\sin\theta$.

or, $\dfrac{du}{-2} = \sin\theta \cdot d\theta$

if, $x = \pi/2$, $u = 0$

$x = \pi/3$, $u = 1$.

$= \displaystyle\int_{\pi/3}^{\pi/2} \sin\theta\sqrt{1-(2\cos\theta)^2}\, d\theta$

$= -\dfrac{1}{2}\displaystyle\int_{1}^{0} \sqrt{1-u^2}\, du$

$= \dfrac{1}{2}\displaystyle\int_{0}^{1} \sqrt{1-u^2}\, du$

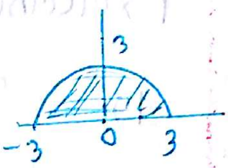
$= \dfrac{1}{2}\pi \times (1)^2 \times \dfrac{1}{4} = \dfrac{\pi}{8}$. Ans: $\dfrac{\pi}{8}$.

22. $\displaystyle\int_{e^{-3}}^{e^{3}} \frac{\sqrt{9-(\ln n)^2}}{n}\, dn$

Let, $u = \ln n$

$\frac{du}{dn} = \frac{1}{n}$

$du = \frac{dn}{n}$.



if, $n = e^3$. $u = 3$

$n = e^{-3}$, $u = -3$

$= \displaystyle\int_{-3}^{3} \sqrt{9-u^2}\,.du$

$= \frac{1}{2} \times \pi \times 3^2$

$= \frac{9\pi}{2}$. Ans: $\frac{9\pi}{2}$.

33. $\displaystyle\int_{-1}^{1} \frac{n^2\, dn}{\sqrt{n^3+9}}$

Let, $u = n^3 + 9$

or, $\frac{du}{dn} = 3n^2$

or, $du/_3 = n^2 dn$

if, $n = 1$, $u = 10$

$n = -1$, $u = 8$.

$= \displaystyle\int_{8}^{10} \frac{1}{\sqrt{u}} \cdot \frac{du}{3}$

$= \frac{1}{3} \displaystyle\int_{8}^{10} \frac{1}{\sqrt{u}} \cdot du$

$= \frac{1}{3} \left[\frac{u^{1/2}}{1/2} \right]_{8}^{10}$

$= 0.22$, ≈ 0.22. Ans: 0.22

34. $\displaystyle\int_{\pi/2}^{\pi} 6 \sin n (\cos n + 1)^5\, dn$

Let, $u = \cos n + 1$

$\frac{du}{dn} = -\sin n$

$du = -\sin n \cdot dn$

if, $n = \pi$, $u = 0$

$n = \pi/2$ $u = 1$.

$= -6 \displaystyle\int_{1}^{0} u \cdot du$

$= 6 \displaystyle\int_{0}^{1} u\, du$

$= 6 \cdot \left[\frac{u^2}{2} \right]_{0}^{1}$

$= 6 \left(\frac{1}{2} - 0 \right)$

$= 3$. Ans: 3.

35. $\int_1^3 \frac{x+2}{\sqrt{x^2+4x+7}} \cdot dx$

Let, $u = x^2+4x+7$

$\frac{du}{dx} = 2x+4$

$\frac{du}{dx} = 2(x+2)$

$\frac{du}{2} = (x+2) \cdot dx$

if, $x=3$, $u=28$

$x=1$, $u=12$.

$= \frac{1}{2}\int_{12}^{28} \frac{1}{\sqrt{u}} \cdot du$

$= \frac{1}{2}\left[\frac{u^{1/2}}{1/2}\right]_{12}^{28}$

$= \frac{1}{2}(10.58 - 6.92)$

≈ 1.83. Ans: 1.83.

36. $\int_1^2 \frac{dx}{x^2-6x+9}$

Let, $u = x-3$

$\frac{du}{dx} = 1$

$du = dx$

if, $x=2$, $u=-1$

$x=1$, $u=-2$

$= \int_1^2 \frac{dx}{(x-3)^2}$

$= \int_{-2}^{-1} \frac{du}{u^2} = \left[\frac{u^{-2+1}}{-2+1}\right]_{-2}^{-1}$

$= \left[\frac{u^{-1}}{-1}\right]_{-2}^{-1} = 1 - \frac{1}{2} = \frac{1}{2}$

Ans: $\frac{1}{2}$.

37. $\int_0^{\pi/4} 4\sin x\cos x\, dx$.

Let, $u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \cdot dx$

if, $x = \pi/4$, $u = \frac{1}{\sqrt{2}}$

$x=0$, $u=0$.

$= 4\int_0^{1/\sqrt{2}} u \cdot du$

$= 4 \cdot \left[\frac{u^2}{2}\right]_0^{1/\sqrt{2}}$

$= 4 \cdot \left[\left(\frac{1}{\sqrt{2}}\right)^2 / 2 - 0\right]$

$= 4 \cdot \frac{1}{4} = 1$. Ans: 1.

38. $\int_0^{\pi/4} \sqrt{\tan x} \cdot \sec^2 x \, dx$

Let, $u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x \cdot dx$

if, $x = \pi/4$, $u = 1$

$x = 0$. $u = 0$.

$= \int_0^1 \sqrt{u} \cdot du$

$= \left[\frac{u^{(3/2)}}{3/2} \right]_0^1$

$= \frac{2}{3} \cdot 1 = \frac{2}{3}$. Ans: $\frac{2}{3}$.

39. $\int_0^{\sqrt{\pi}} 5x \cos(x^2) \, dx$.

Let, $u = x^2$

$\frac{du}{dx} = 2x$

$\frac{du}{2} = x \cdot dx$.

if, $x = \sqrt{\pi}$, $u = \pi$

$x = 0$, $u = 0$.

$= \int_0^{\sqrt{\pi}} \cos u \cdot \frac{5 du}{2}$.

$= \frac{5}{2} \int_0^{\pi} \cos u \cdot du$.

$= \frac{5}{2} \left[\sin u \right]_0^{\pi}$

$= 5/2(0-0) = 0$. Ans: 0.

40. $\int_{\pi^2}^{4\pi^2} \frac{1}{\sqrt{x}} \sin\sqrt{x} \, dx$.

Let, $u = \sqrt{x}$

or, $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

or. $2du = \frac{dx}{\sqrt{x}}$.

if, $x = 4\pi^2$. $u = 2\pi$

$x = \pi^2$. $u = \pi$.

$= 2 \int_{\pi}^{2\pi} \sin u \cdot du$

$= 2\left[-\cos u \right]_{\pi}^{2\pi}$

$= -2 - 2$

$= -4$.

Ans: -4.

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45. $\displaystyle\int_{-1}^{4} \frac{x}{\sqrt{5+x}}\,dx.$

Let, $u = 5+x$ or, $u-5 = x$.

$\dfrac{du}{dx} = 1$

$du = dx$.

if, $x = 4$, $u = 9$

$x = -1$, $u = 4$.

$= \displaystyle\int_{-1}^{4} \frac{u-5}{\sqrt{u}}\,du.$

$= \displaystyle\int_{4}^{9} \left(\frac{u}{\sqrt{u}} - \frac{5}{\sqrt{u}}\right) du$

$= \displaystyle\int_{4}^{9} \left(\sqrt{u} - 5\cdot u^{-1/2}\right) du$

$= \left[\dfrac{u^{3/2}}{3/2} - 5\cdot\dfrac{u^{1/2}}{1/2}\right]_4^9$

$= \dfrac{8}{3}$ or ≈ 2.67. Ans: 2.67.

15. $\displaystyle\int_{-\ln 3}^{\ln 3} \frac{e^x}{e^x+4}\,dx.$

Let, $u = e^x+4$

$\dfrac{du}{dx} = e^x$

$du = e^x\,dx$

if, $x = \ln 3$, $u = 7$.

$x = -\ln 3$, $u = \dfrac{13}{3}$.

$= \displaystyle\int_{13/3}^{7} \frac{1}{u}\,du$

$= \left[\ln|u|\right]_{13/3}^{7}$

$= \ln|7| - \ln|13/3|$

$= 0.47$. Ans: ≈ 0.47.

49. $\displaystyle\int_{0}^{1/\sqrt{3}} \frac{1}{1+9x^2}\,dx$

Let, $u = 3x$

$du = 3\,dx$

$\dfrac{du}{3} = dx$.

$x = 0$, $u = 0$

$x = \dfrac{1}{\sqrt{3}}$, $u = \sqrt{3}$.

$= \dfrac{1}{3}\displaystyle\int_{0}^{\sqrt{3}} \frac{1}{1+u^2}\,du$

$= \dfrac{1}{3}\left[\tan^{-1}u\right]_0^{\sqrt{3}}$

$= \dfrac{1}{3}\left(\dfrac{\pi}{3}\right) = \dfrac{\pi}{9}$.

Ans: $\dfrac{\pi}{9}$.

* $\int_0^4 3x\sqrt{25-x^2}\,dx$

Let, $u = 25 - x^2$

$du = -2x\,dx$

$\frac{du}{-2} = x\,dx$

if, $x=0$, $u=25$

$x=4$, $u=9$.

$= 3\int_{25}^{9} \sqrt{u}\cdot\frac{du}{-2}$.

$= \frac{-3}{2}\int_{25}^{9} \sqrt{u}\cdot\frac{du}{1}$.

$= -\frac{3}{2}\left[\frac{u^{3/2}}{3/2}\right]_{25}^{9}$

$= -\left(9^{3/2} - 25^{3/2}\right)$

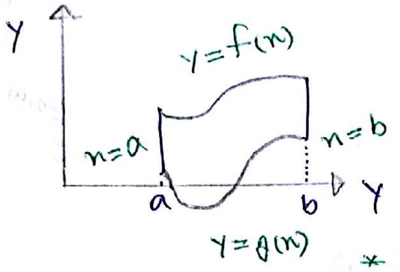
$= -(27-125)$

$= 98$. Ans: 98.

Applications of the Definite integral in GEOMETRY, SCIENCE and Engineering:

Area between two curves — 6.1

Area between $y=f(x)$ and $y=g(x)$.



* step-1

Suppose, f and g are continuous on interval $[a,b]$ and

$f(x) \geq g(x)$ for, $a \leq x \leq b$.

Area bounded by $f(x)$ and $g(x)$ with $x=a$ and $x=b$ line

$$A = \int_a^b [f(x) - g(x)]\, dx$$

It is not necessary to make an extremely accurate sketch in step-1, the only purpose of the sketch is to determine which curve is upper boundary and which is the lower boundary.

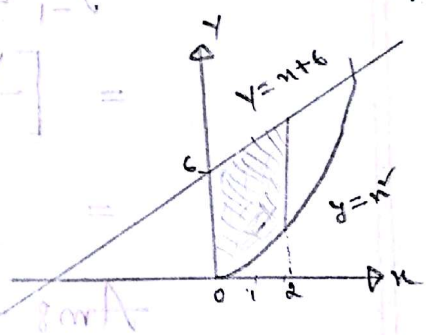
Example 1: $A = \int_0^2 [(x+6) - x^2]\, dx$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{34}{3} - 0$$

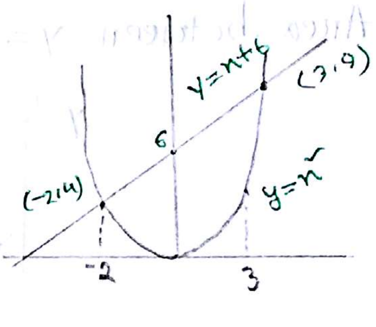
$$= \frac{34}{3}.$$

Ans: $\frac{34}{3}$.



P_1

Example-2: Here, $y = x^2$, $y = x+6$



P_2

Now, $x^2 = x+6$

or, $x^2 - x - 6 = 0$

or, $(x+2)(x-3) = 0$.

or, $x = -2, 3$

$$\text{Area} = \int_{-2}^{3} (x+6-x^2)\,dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= \frac{27}{2} - \left(-\frac{22}{3}\right) = \frac{125}{6}.$$

Ans: $\frac{125}{6}$.

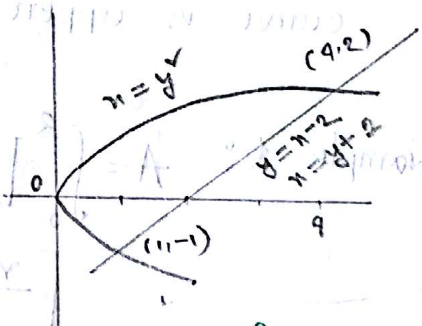
Example-4: Given that, $x = y^2$ and $x = y+2$.

$$\text{Area} = \int_{-1}^{2} [(y+2) - y^2]\,dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$$

$$= \frac{9}{2}.$$

Ans: $\frac{9}{2}$.



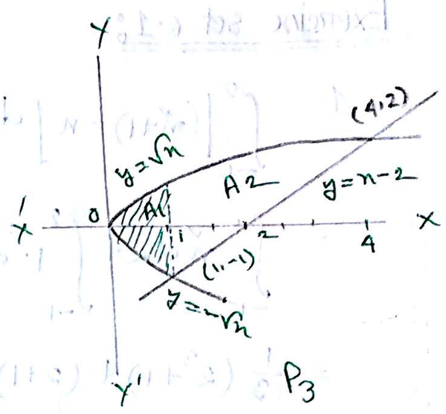
P_3

This math can also be done in another way, with respect to "dx".

Note: Outer−inner হলে Y অক্ষের limit টা নিতে হবে।

Alternative :-

Here, $x = y^2$ and $y = x - 2$ or, $y + 2 = x$,



P3

Now, $y^2 = y + 2$

or, $y^2 - y - 2 = 0$,

or, $(y-2)(y+1) = 0$

or, $y = 2, -1$

Hence, $x = 4, 1$

$y = \pm\sqrt{x}$, but, $y = -\sqrt{x}$ for $0 \le x \le 1$

and, $y = x - 2$ for $1 \le x \le 4$.

$$\text{Area}_1 = \int_0^1 \left[\sqrt{x} - (-\sqrt{x})\right] dx$$

$$= \int_0^1 2\sqrt{x}\, dx$$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^1 = \frac{4}{3},$$

$$\text{Area}_2 = \int_1^4 \left[\sqrt{x} - (x-2)\right] dx$$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x\right]_1^4$$

$$= \left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right)$$

$$= \frac{19}{6}$$

$$A = A_1 + A_2$$

$$= \frac{4}{3} + \frac{19}{6} = \frac{9}{2}. \quad \text{Ans: } \frac{9}{2}$$

Now, its upto you which one u will do.

Hardcore or Softcore.

Exercise set 6·1:

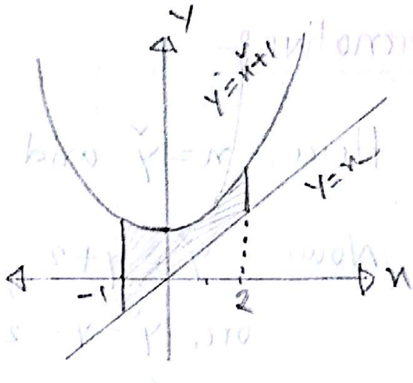
1. $$\int_{-1}^{2}\left[(x^2+1)-x\right]dx$$

$$= \int_{-1}^{2} x^2\,dx+\int_{-1}^{2} 1\cdot dx-\int_{-1}^{2} x\,dx$$

$$= \frac{1}{3}(2^3+1)+(2+1)-\frac{1}{2}(2^2-1)$$

$$= \frac{1}{3}\times 9+3-\frac{3}{2}$$

$$= \frac{9}{2}. \quad \text{Ans: } \frac{9}{2}.$$



P4

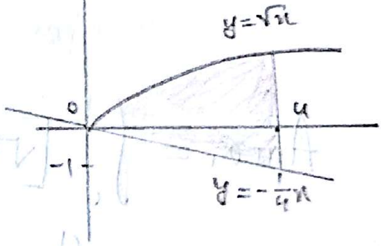
2. $$\int_{0}^{4}\left[\sqrt{x}+\frac{1}{4}x\right]dx$$

$$= \int_{0}^{4} x^{1/2}\,dx+\frac{1}{4}\int_{0}^{4} x\,dx$$

$$= \frac{2}{3}\left[x^{3/2}\right]_0^4+\frac{1}{4}\cdot\frac{1}{2}\left[x^2\right]_0^4$$

$$= \frac{2}{3}\times 8+\frac{16}{8}$$

$$= \frac{22}{3}. \quad \text{Ans: } \frac{22}{3}.$$



P5

3. $$\text{Area}=\int_{1}^{2}\left(y-\frac{1}{y^2}\right)dy$$

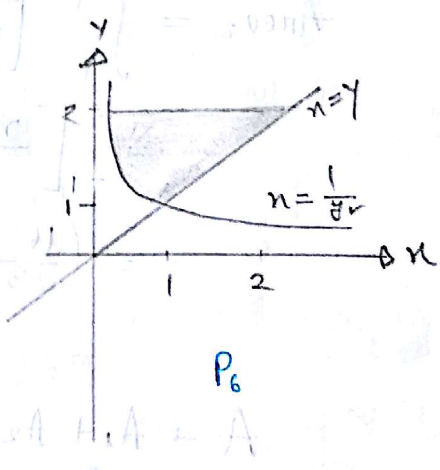
$$= \int_{1}^{2} y\,dy-\int_{1}^{2} y^{-2}\,dy$$

$$= \frac{1}{2}\left[y^2\right]_1^2-\left[\frac{y^{-1}}{-1}\right]_1^2$$

$$= \frac{4-1}{2}+\left(\frac{1}{2}-1\right)$$

$$= \frac{3+1-2}{2}$$

$$= 1. \quad \text{Ans: } 1.$$



P6

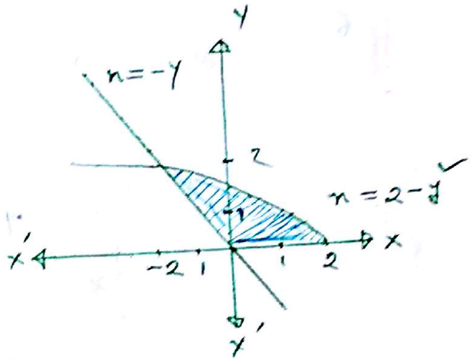
4. $\text{Area} = \int_0^2 \{(2-y^2) - (-y)\}\,dy$

$= 2\int_0^2 1\cdot dy - \int_0^2 y^2 dy + \int_0^2 y\,dy$

$= 2[y]_0^2 - \frac{1}{3}[y^3]_0^2 + \frac{1}{2}[y^2]_0^2$

$= 4 - \frac{8}{3} + 2$

$= \frac{10}{3}$. Ans: $\frac{10}{3}$.



P7

5. $\text{Area} = \int_0^2 (2x - x^2)\,dx$

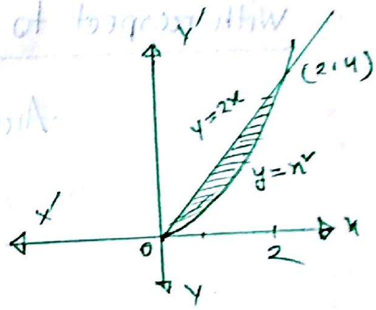
$= 2\int_0^2 x\,dx - \int_0^2 x^2\,dx$

$= 2\cdot\left[\frac{x^2}{2}\right]_0^2 - \frac{1}{3}[x^3]_0^2$

$= (4-0) - \frac{8}{3}$

$= \frac{12-8}{3} = \frac{4}{3}$.

Ans: $\frac{4}{3}$.



P8

with respect to Y : $\text{Area} = \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right)dy$

$= \int_0^4 (y^{1/2})\,dy - \frac{1}{2}\int_0^4 y\cdot dy$

$= \frac{2}{3}\cdot[y^{3/2}]_0^4 - \frac{1}{2}\cdot\frac{1}{2}[y^2]_0^4$

$= \frac{2}{3}\cdot 8 - \frac{1}{4}\cdot 16$

$= \frac{4}{3}$.

Ans: $\frac{4}{3}$.

in graph. $x = \sqrt{y}$

$2x = y$

or. $x = \frac{y}{2}$.

P8.

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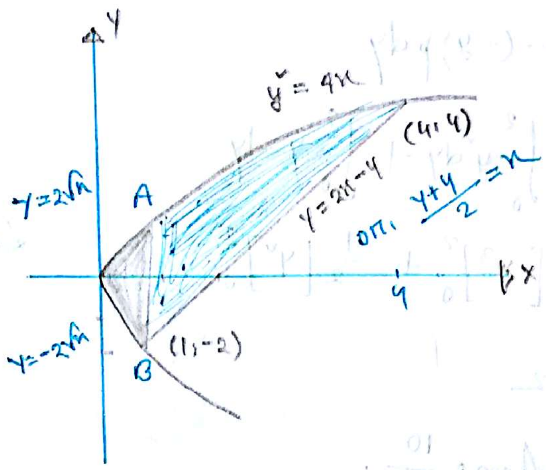
6.



with respect to y :-

$$\text{Area} = \int_{-2}^{4}\left(\frac{y+4}{2}\right)dy - \int_{-2}^{4}\left(\frac{y^2}{4}\right)dy$$

$$= \frac{1}{2}\int_{-2}^{4} y \cdot dy + 2\int_{-2}^{4} 1 \cdot dy - \frac{1}{4}\int_{-2}^{4} y^2 dy$$

$$= \frac{1}{4} \cdot \left[y^2\right]_{-2}^{4} + 2\left[y\right]_{-2}^{4} - \frac{1}{4} \cdot \frac{1}{3}\left[\frac{y^3}{1}\right]_{-2}^{4}$$

$$= \frac{12}{4} + 12 - \frac{72}{12}$$

$$= 3 + 12 - 6 = 15 - 6 = 9. \quad \text{Ans: } 9.$$

with respect to x :-

$$\text{Area} = \int_{0}^{1}\left[2\sqrt{x} - (-2\sqrt{x})\right]dx + \int_{1}^{4}\left[2\sqrt{x} - (2x-4)\right]dx$$

$$= \left[2 \cdot \frac{2}{3}\left[x^{3/2}\right]_0^1 + 2 \cdot \frac{2}{3}\left[x^{3/2}\right]_0^1\right] + 2 \times \frac{\left[x^{3/2}\right]_1^4}{3/2} - 2 \cdot \frac{\left[x^2\right]_1^4}{2} + 4 \cdot \left[x\right]_1^4$$

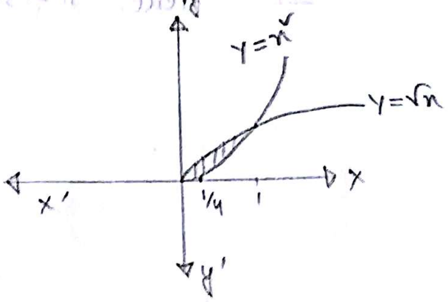
$$= \frac{4}{3}\left[(1-0) + (1-0)\right] + \left(\frac{4}{3} \times 7 - 15 + 12\right)$$

$$= \frac{8}{3} + \frac{19}{3}$$

$$= \frac{27}{3} = 9.$$

$$\text{Ans: } 9.$$

7. $y = x^2, \quad y = \sqrt{x}, \quad x = \frac{1}{4}, \quad x = 1.$



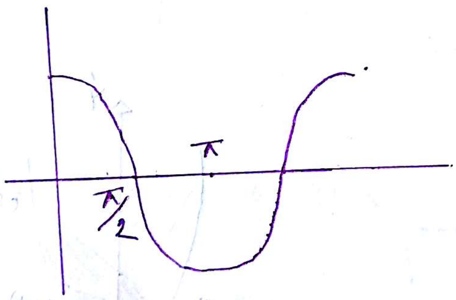
$$\text{Area} = \int_{1/4}^{1} (\sqrt{x} - x^2)\, dx$$

$$= \int_{1/4}^{1} x^{1/2} - \int_{1/4}^{1} x^2\, dx$$

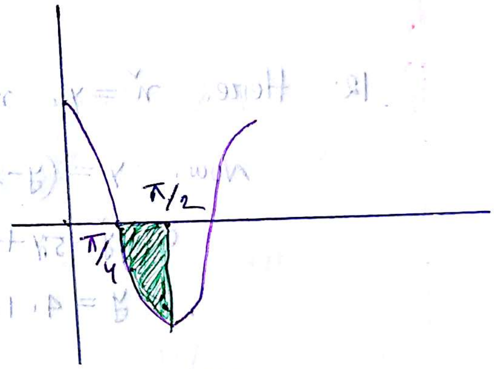
$$= \left[\frac{x^{3/2}}{3/2}\right]_{1/4}^{1} - \left[\frac{x^3}{3}\right]_{1/4}^{1}$$

$$= \frac{49}{192}. \quad \text{Ans: } \frac{49}{192}.$$

9. $y = \cos 2x, \quad y = 0, \quad x = \frac{\pi}{4}, \quad x = \frac{\pi}{2}.$



$P_1 = \cos x$



$P_2 = \cos 2x$

P_1 no need to draw, Just for better understanding.

$$\text{Area} = \int_{\pi/4}^{\pi/2} (0 - \cos 2x)\, dx$$

$$= -\int_{\pi/4}^{\pi/2} \cos 2x\, dx$$

$$= -\frac{1}{2}\left[\sin 2x\right]_{\pi/4}^{\pi/2}$$

$$= -\left[\frac{\sin 2x}{2}\right]_{\pi/4}^{\pi/2}$$

$$= -\frac{1}{2}\left[\sin \pi - \sin \pi/2\right]$$

$$= -\frac{1}{2}\left[0 - 1\right]$$

$$= \frac{1}{2}. \quad \text{Ans: } \frac{1}{2}.$$

$$= \frac{1}{2}. \quad \text{Ans: } \frac{1}{2}.$$

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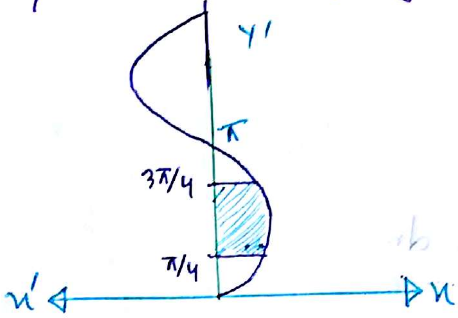
11. Here, $x = \sin y$, $x = 0$, $y = \frac{\pi}{4}$, $y = \frac{3\pi}{4}$.



$$\text{Area} = \int_{\pi/4}^{3\pi/4} (\sin y - 0)\, dy$$

$$= -\left[\cos y\right]_{\pi/4}^{3\pi/4}$$

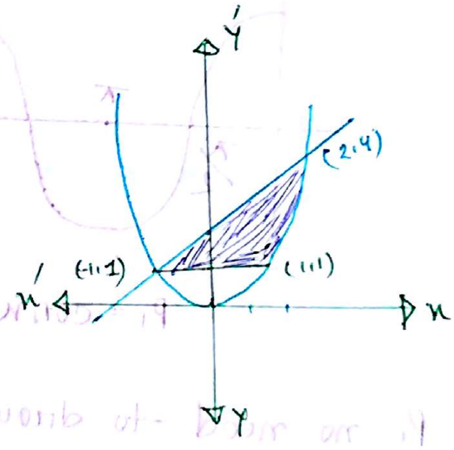
$$= \sqrt{2}. \quad \text{Ans: } \sqrt{2}.$$

12. Here, $x^2 = y$, $x = y - 2$.

Now, $y = (y-2)^2$

or, $y^2 - 5y + 4 = 0$

$\therefore y = 4, 1$.



$$\text{Area} = \int_1^4 (y^{1/2} - y + 2)\, dy$$

$$= \int_1^4 y^{1/2} dy - \int_1^4 y \cdot dy + 2\int_1^4 1 \cdot dy$$

$$= \frac{3}{2}\left[y^{3/2}\right]_1^4 - \left[\frac{y^2}{2}\right]_1^4 + 2\left[y\right]_1^4$$

$$= \frac{14}{3} - \frac{15}{2} + 6$$

$$= \frac{28 - 45 + 36}{6}$$

$$= \frac{19}{6}. \quad \text{Ans: } \frac{19}{6}.$$

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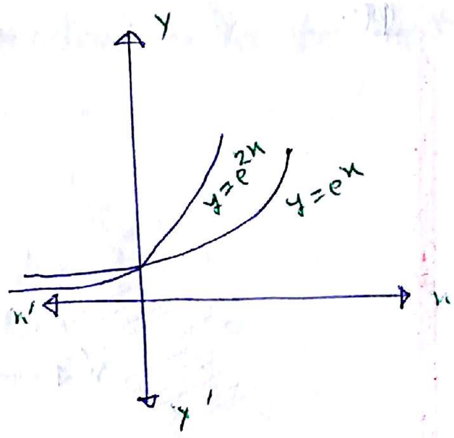
13. Here, $y=e^x$, $y=e^{2x}$ & $x=0$, $x=\ln 2$.



$$\text{Area} = \int_0^{\ln 2} e^{2x}\,dx - \int_0^{\ln 2} e^x\,dx$$

$$= \frac{1}{2}\left[e^{2x}\right]_0^{\ln 2} - \left[e^x\right]_0^{\ln 2}$$

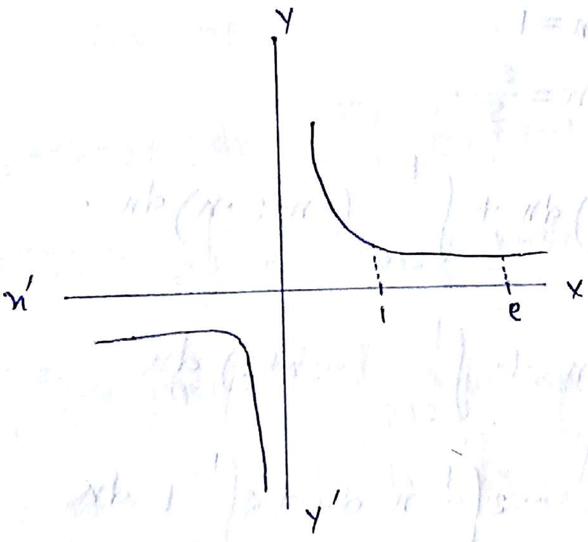
$$= \frac{1}{2}\left[e^{2\ln 2} - e^0\right] - (2-1)$$

$$= \frac{1}{2}\left[e^{\ln 4} - e^0\right] - 1$$

$$= \frac{1}{2}(4-1) - 1$$

$$= \frac{3}{2} - 1 = \frac{1}{2}.$$ Ans: $\frac{1}{2}$.

14. Here, $x=\frac{1}{y}$, $x=0$, $y=1$, $y=e$.



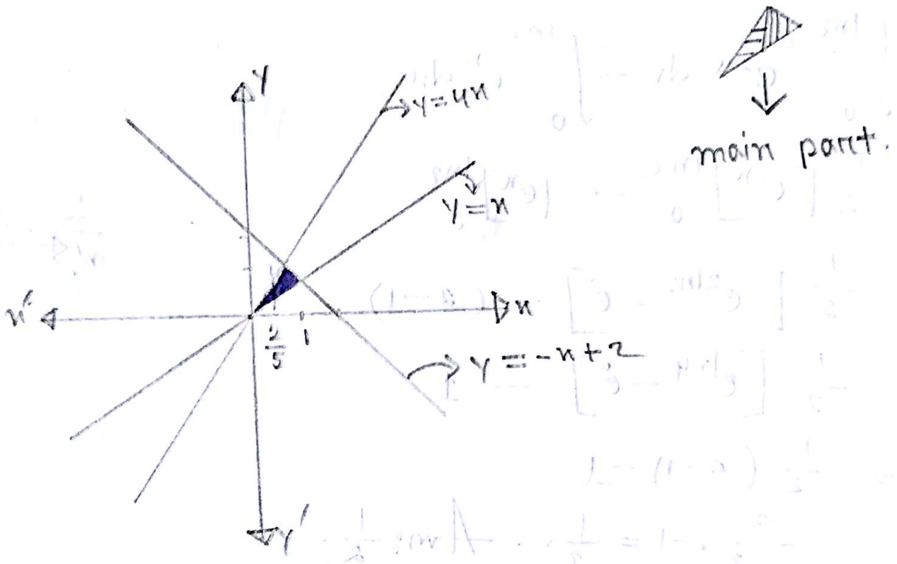
$$\text{Area} = \int_1^e \left(\frac{1}{y} - 0\right) dy$$

$$= \left[\ln y\right]_1^e$$

$$= \ln e - \ln 1$$

$$= 0.$$ Ans: 0.

** 18.



Here, $y=x$ —— (i), $y=4x$ —— (ii), $y=-x+2$ —— (iii)

From (i) and (ii) $\Rightarrow x=0$,

(i) and (iii) $\Rightarrow x=1$

(ii) and (iii) $\Rightarrow x=\frac{2}{5}$.

$$Area = \int_0^{2/5} (4x-x)\,dx + \int_{2/5}^{1} (-x+2-x)\,dx.$$

$$= \int_0^{2/5} 3x\cdot dx + \int_{2/5}^{1} (-2x+2)\,dx$$

$$= 3\int_0^{2/5} x\cdot dx - 2\int_{2/5}^{1} x\,dx + 2\int_{2/5}^{1} 1\,dx$$

$$= 3\left[x^2\right]_0^{2/5} - \left[x^2\right]_{2/5}^{1} + 2\left[x\right]_{2/5}^{1}$$

$$= \frac{3}{5}.$$

Ans: $\frac{3}{5}$.

V.V.I

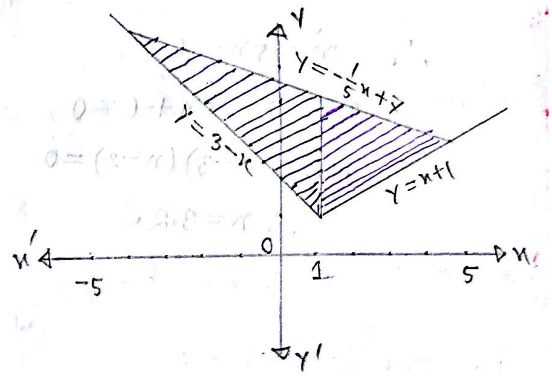
17. $y = 2+|x-1|$, $y = -\frac{1}{5}x+7$. Find out the limit or intersecting point.

Solution → Here, $y = -\frac{1}{5}x+7$ —— (i)

$y = 2+|x-1| = \begin{cases} 1+x, & x>1 \\ 3-x, & x<1 \end{cases}$

$y = 1+x$ —— (ii)

$y = 3-x$ —— (iii)



Now, from (i) and (ii) ⇒

$x+1 = -\frac{1}{5}x+7$

$\therefore x = 5$

from (i) and (iii) ⇒

$3-x = -\frac{1}{5}x+7$

$\therefore x = -5$

Suppose, for graph,

$x=5,\ y=6$

$x=0,\ y=7$.

$$A = \int_{-5}^{1}\left(-\frac{1}{5}x+7-3+x\right)dx + \int_{1}^{5}\left(-\frac{1}{5}x+7-1-x\right)dx$$

$$= \int_{-5}^{1}\left(\frac{-x+5x}{5}+4\right)dx + \int_{1}^{5}\left(\frac{-x-5x}{5}+6\right)dx$$

$$= \int_{-5}^{1}\left(\frac{4x}{5}+4\right)dx + \int_{1}^{5}\left(\frac{-6x}{5}+6\right)dx$$

$$= \int_{-5}^{1}\frac{4x}{5}\,dx + \int_{-5}^{1}4\,dx + \int_{1}^{5}-\frac{6x}{5}\,dx + \int_{1}^{5}6\,dx$$

$$= \frac{4}{5}\int_{-5}^{1}(x)\,dx + 4\int_{-5}^{1}(1)\,dx - \frac{6}{5}\int_{1}^{5}(x)\,dx + 6\int_{1}^{5}1\,dx$$

$$= \frac{4}{5}\cdot\left(\frac{-24}{2}\right) + 24 + 36 - \frac{6}{5}\left(\frac{24}{2}\right)$$

$$= \frac{-96}{10} + 24 + 36 - \frac{144}{10}$$

$$= 36$$

Ans: 36.

$y = x^2$, $x = \frac{y+6}{5}$.

or, $5x = y+6$

or, $5x - 6 = y$

$* \; 5x - y = 6$

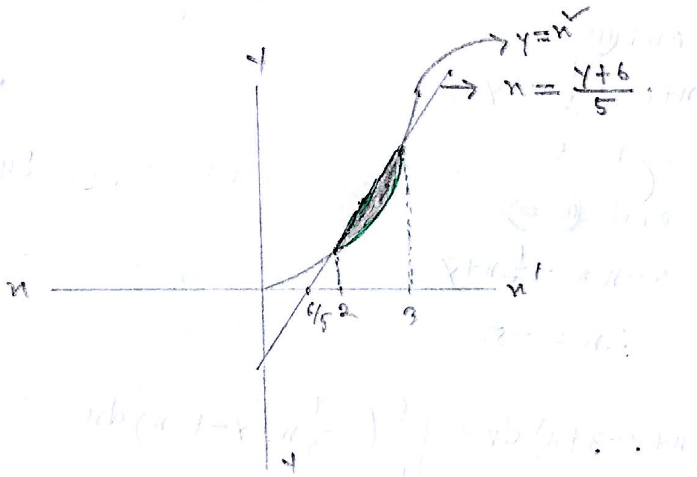
$\frac{x}{6/5} - \frac{y}{6} = 1$.

$\therefore \; x^2 = 5x - 6$

or, $x^2 - 5x + 6 = 0$

or, $(x-3)(x-2) = 0$

$\therefore x = 3, 2$.



$$A = \int_2^3 (5x - 6 - x^2)\,dx$$

$$= 5\left[\frac{x^2}{2}\right]_2^3 - 6\left[x\right]_2^3 - \left[\frac{x^3}{3}\right]_2^3$$

$$= 5 \cdot \frac{5}{2} - 6 - \frac{19}{3}$$

$$= \frac{75 - 36 - 35}{6}$$

$$= \frac{1}{6}. \quad \text{Ans: } \frac{1}{6}.$$

Volumes By Slicing; Disks and Washers — 6.2

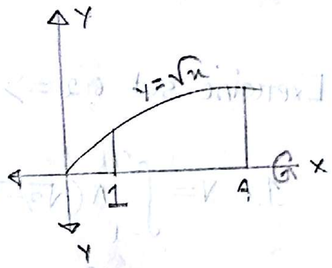
Formula → Volume of each disk $= \pi[f(x)]^2 dx$

Total $V = \int_a^b \pi [f(x)]^2 dx$

Example from book →

1. Find the volume of the solid obtained by $y=\sqrt{x}$ over $[1,4]$ is revolved about x-axis.

Solution: $V = \int_0^b \pi [f(x)]^2 dx$



$= \int_1^4 \pi [\sqrt{x}]^2 dx$

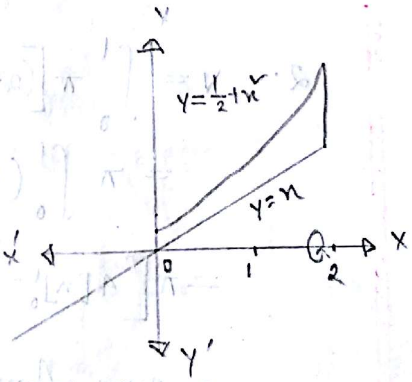
$= \int_1^4 \pi x \, dx$

$= \left[\frac{\pi x^2}{2}\right]_1^4$

$= 8\pi - \frac{\pi}{2} = \frac{15\pi}{2}$. Ans: $\frac{15\pi}{2}$.

Formula → $V = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right) dx$

4. $V = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right) dx$



$= \int_0^2 \pi\left[\left(\frac{1}{2}+x^2\right)^2 - x^2\right] dx$

$= \int_0^2 \pi\left(\frac{1}{4}+x^4\right) dx$

$= \pi \left[\frac{x}{4} + \frac{x^5}{5}\right]_0^2$

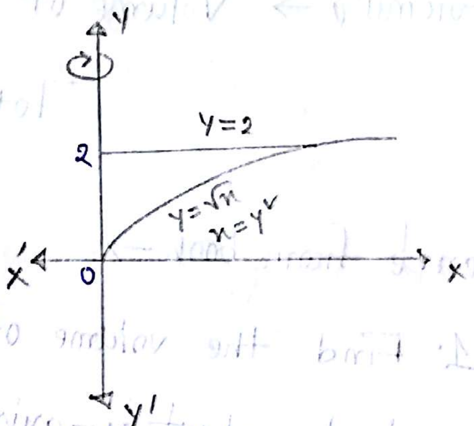
$= \pi \left(\frac{1}{2} + \frac{32}{5}\right)$

$= \frac{69\pi}{10}$. Ans: $\frac{69\pi}{10}$.

Example 5:

$$V = \int_a^b \pi [f(y)]^2 \, dy$$

$$V = \int_0^2 \pi [(y^2)^2 - (0)^2] \, dy$$

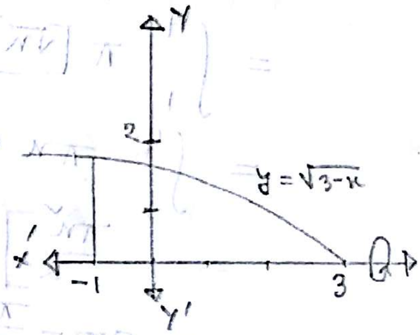


$$= \pi \left[\frac{y^5}{5}\right]_0^2$$

$$= \frac{32\pi}{5}. \quad \text{Ans: } \frac{32\pi}{5}.$$

Exercise set 6.2 =>.

1. $V = \int_{-1}^{3} \pi (\sqrt{3-x})^2 \, dx$



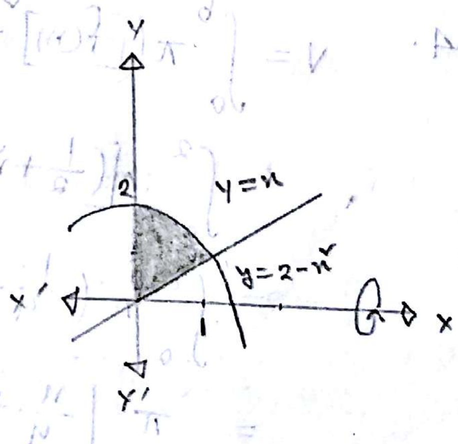
$$= \pi \int_{-1}^{3} (3-x) \, dx$$

$$= \pi \int_{-1}^{3} 3 \, dx - \pi \int_{-1}^{3} x \, dx$$

$$= 3\pi [x]_{-1}^{3} - \pi \left[\frac{x^2}{2}\right]_{-1}^{3}$$

$$= \frac{16\pi}{2} = 8\pi. \quad \text{Ans: } 8\pi.$$

2. $V = \int_0^1 \pi [(2-x^2)^2 - x^2] \, dx$



$$= \pi \int_0^1 (4 - 4x^2 + x^4 - x^2) \, dx$$

$$= \pi \left[4[x]_0^1 - \frac{4}{3}[x^3]_0^1 + \frac{1}{5}[x^5]_0^1 - \frac{1}{3}[x]_0^1 \right]$$

$$= 4\pi \quad \text{Ans: } 4\pi.$$

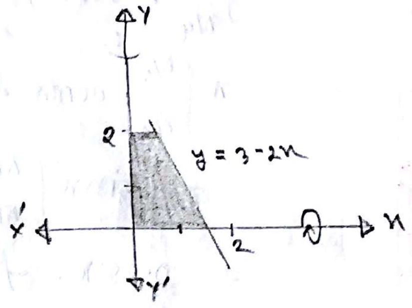
3. $V = \int_0^2 \pi (3-2x)^2 dx$

$= \pi \int_0^2 (9 - 12x + 4x^2) dx$

$= \pi \left[9[x]_0^2 - 6[x^2]_0^2 + \frac{4}{3}[x^3]_0^2 \right]$

$= \pi \left[18 - 24 + \frac{32}{3} \right]$

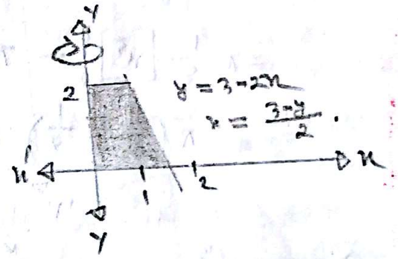
$= \frac{14\pi}{3}$. Ans: $\frac{14\pi}{3}$.



This solve is wrong. in terms of x axis.

This solve is only for revolving through x-axis.

for y-axis ⇒



$\therefore V = \int_0^2 \pi \left(\frac{3-y}{2}\right)^2 dy$

$= \frac{\pi}{4} \int_0^2 (9 - 6y + y^2) dy$

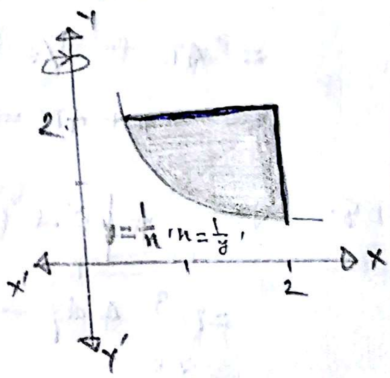
$= \frac{\pi}{4} \left[\int_0^2 9\, dy - \int_0^2 6y\, dy + \int_0^2 y^2\, dy \right]$

$= \frac{\pi}{4} \left[9[y]_0^2 - 6 \cdot \left[\frac{y^2}{2}\right]_0^2 + \left[\frac{y^3}{3}\right]_0^2 \right]$

$= \frac{13\pi}{6}$. Ans: $\frac{13\pi}{6}$.

This solve is correct. in terms of y axis.

4. $V = \int_0^2 \pi \left(2^2 - \left(\frac{1}{y}\right)^2\right) dy$



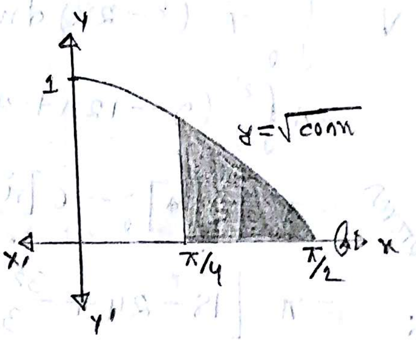
$= \pi \int_0^2 4\, dy - \pi \int_0^2 y^{-2}\, dy$

$= \pi [4 \times (2-0)] + \pi \left[\frac{y^{-1}}{1}\right]_0^2$

$= \pi \left(8 + \frac{1}{2}\right)$

$= \frac{(16+1)\pi}{2}$ Ans: $\frac{(16+1)\pi}{2}$.

5. $V = \int_{\pi/4}^{\pi/2} \pi(\sqrt{\cos x})^2\, dx$

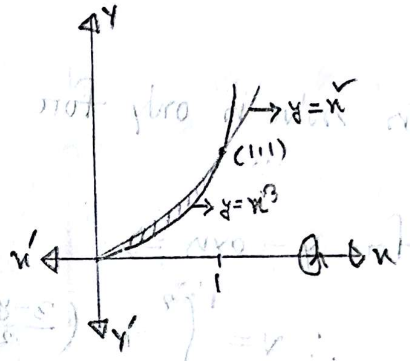


$= \pi \int_{\pi/4}^{\pi/2} \cos x\, dx$

$= \pi \left[\sin x\right]_{\pi/4}^{\pi/2}$

$= 0.29\pi$ Ans: 0.29π

6. $V = \int_0^1 \pi\left[(x^2)^2 - (x^3)^2\right] dx$

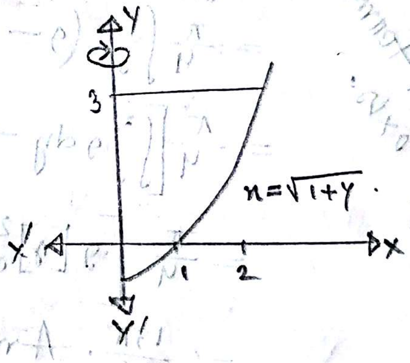


$= \pi \int_0^1 x^4\, dx - \pi \int_0^1 x^6\, dx$

$= \pi \cdot \frac{1}{5}\left[x^5\right]_0^1 - \pi \cdot \frac{1}{7}\left[x^7\right]_0^1$

$= \pi\left(\frac{1}{5} - \frac{1}{7}\right)$ Ans: $\pi\left(\frac{1}{5} - \frac{1}{7}\right)$

7. $V = \int_{-1}^{3} \pi \left(\sqrt{1+y}\right)^2 dy$



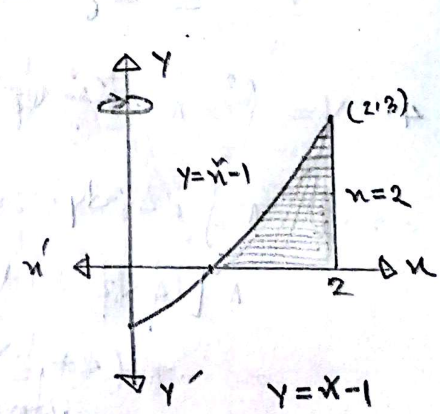
$= \pi \int_{-1}^{3} 1 \cdot dy + \pi \int_{-1}^{3} y\, dy$

$= \pi\left[y\right]_{-1}^{3} + \pi\left[\frac{y^2}{2}\right]_{-1}^{3}$

$= 4\pi + \frac{\pi}{2}\left(3^2 - (-1)^2\right)$

$= 4\pi + 4\pi = 8\pi$ Ans: 8π

8. $V = \int_0^3 \pi\left[2^2 - \left(\sqrt{y+1}\right)^2\right] dy$



$Y = X - 1$

$x^2 = y + 1$

$\therefore x = \sqrt{y+1}$

$= \pi \int_0^3 4\, dy - \pi \int_0^3 (y+1)\, dy$

$= \pi\left(12 - \frac{9}{2} - 3\right)$

$= \frac{9\pi}{2}$ Ans: $\frac{9\pi}{2}$

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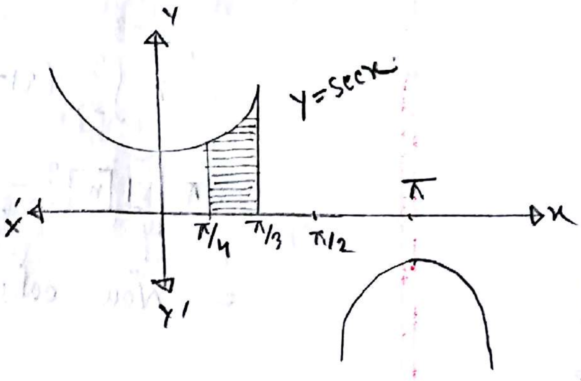
10. $y = \sec x$ and x axis $x = \frac{\pi}{4}, x = \frac{\pi}{3}$.

$$V = \int_{\pi/4}^{\pi/3} \pi (\sec x)^2 \, dx$$

$$= \pi \int_{\pi/4}^{\pi/3} \sec^2 x \, dx$$

$$= \pi \left[\tan x\right]_{\pi/4}^{\pi/3}$$

$$= 0.73\pi. \quad \text{Ans: } 0.73\pi.$$



11. Important one (***).

$\therefore y = \sqrt{25-x^2}$

or $y^2 = 25 - x^2$

$\therefore x^2 + y^2 = 25$.

$$V = \int_{-4}^{4} \pi\left[\left(\sqrt{25-x^2}\right)^2 - (3)^2\right] dx$$

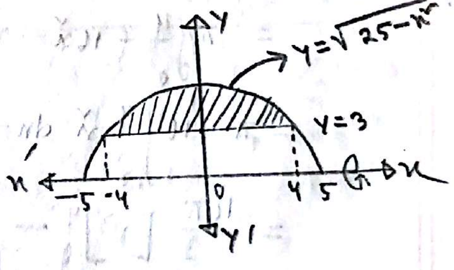
$$= \pi \int_{-4}^{4} (25 - x^2 - 9)\, dx$$

$$= \pi \int_{-4}^{4} (16 - x^2)\, dx$$

$$= \pi \int_{-4}^{4} 16\, dx - \pi \int_{-4}^{4} x^2\, dx$$

$$= \pi \cdot 16(4+4) - \frac{\pi}{3}\left[x^3\right]_{-4}^{4}$$

$$= \frac{256}{3}\pi. \quad \text{Ans: } \frac{256\pi}{3}.$$



$\therefore 3 = \sqrt{25-x^2}$

or, $9 = 25 - x^2$

or, $x^2 = 16$

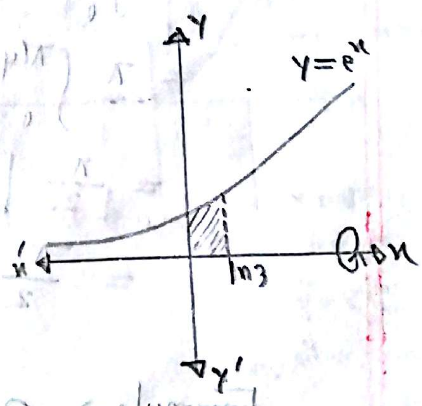
$\therefore x = \pm 4$.

15. $$V = \pi \int_0^{\ln 3} f(x)^2 - g(x)^2 \, dx$$

$$= \pi \int_0^{\ln 3} \left((e^x)^2 - 0^2\right) dx$$

$$= \pi \int_0^{\ln 3} e^{2x} \, dx$$

$$= \pi \cdot \frac{1}{2}\left[e^{2x}\right]_0^{\ln 3} = \frac{\pi}{2}(9-0) = \frac{9\pi}{2}. \quad \text{Ans: } \frac{9\pi}{2}.$$

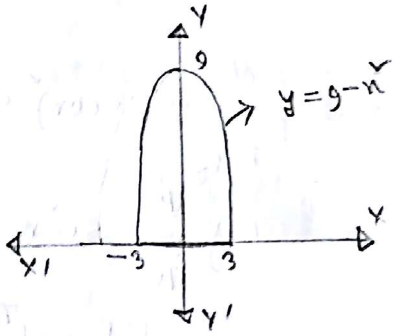


12. $V = \pi\int_{-3}^{3}(9-x^2)^2\,dx$

$= \pi\int_{-3}^{3}(81-18x^2+x^4)\,dx$

$=\pi\left[81[x]_{-3}^{3} - \frac{18}{3}[x^3]_{-3}^{3} + \frac{1}{5}[x^5]_{-3}^{3}\right]$

= Now calculate yourself. ☺



13. $V=\pi\int_{a}^{b} f(x)^2 - g(x)^2\,dx$

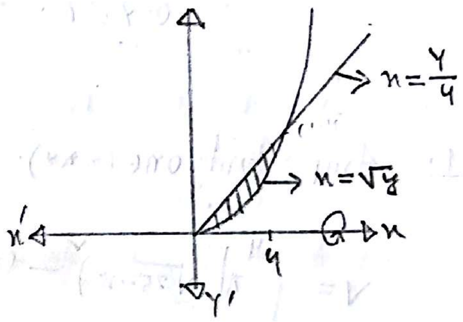
$=\pi\int_0^4\left[(4x)^2-(x^2)^2\right]dx$

$=\pi\int_0^4(16x^2-x^4)\,dx$

$=\pi\,16\int_0^4 x^2\,dx-\pi\int_0^4 x^4\,dx$

$=\frac{16\pi}{3}[x^3]_0^4-\frac{\pi}{5}[x^5]_0^4$

$=136.53\pi$. Ans: 136.53π.



Intersecting point —

$x^2-4x=0$, $x(x-4)=0$.

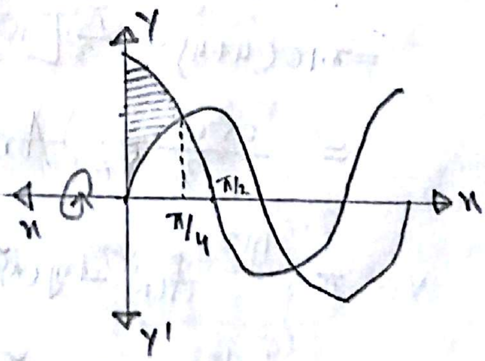
$\therefore x=0, 4$.

14. $V=\pi\int_0^{\pi/4}(\cos^2x-\sin^2x)\,dx$

$=\pi\int_0^{\pi/4}\cos 2x\,dx$

$=\frac{\pi}{2}\left[\sin 2x\right]_0^{\pi/4}$

$=\frac{\pi}{2}$ Ans: $\frac{\pi}{2}$.



Formula → $\cos 2x=\cos^2x-\sin^2x$.

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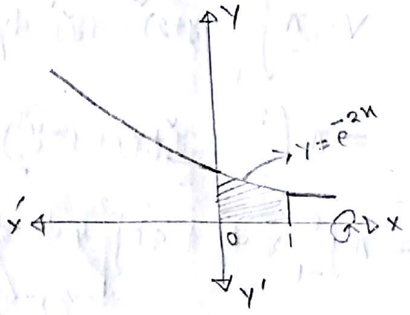
16. $y = e^{-2x}, \; y = 0 \cdot x = 0, \; x = 1.$

$$V = \pi \int_0^1 (e^{-2x})^2 \, dx$$

$$= \pi \int_0^1 e^{-4x} \, dx$$

$$= \pi \cdot \left(-\frac{1}{4}\right) \left[e^{-4x} \right]_0^1$$

$$= \frac{-\pi}{4} (e^{-4} - 1). \quad \text{Ans: } \frac{-\pi}{4}(e^{-4} - 1).$$



19.

$$V = \int_0^1 \pi \, (1)^2 dy - \int_0^1 (y^{2/3})^2 dy$$

$$= \pi \int_0^1 1 \, dy - \int_0^1 \pi \, y^{4/3} \, dy$$

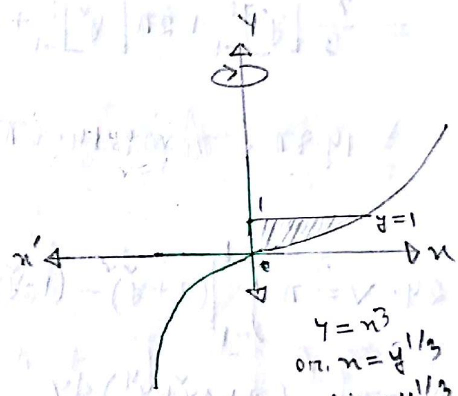
$$= \pi \left[y \right]_0^1 - \pi \left[\frac{y^{4/3+1}}{4/3+1} \right]_0^1$$

$$= \pi \left\{ 1 - \frac{(1-0)}{7/3} \right\}$$

$$= \pi - \frac{3\pi}{7} = \frac{7\pi - 3\pi}{7} = \frac{4\pi}{7}.$$

$$\text{Ans: } \frac{4\pi}{7}$$

~~Wrong, have to do it again.~~



$y = x^3$

or, $x = y^{1/3}$

$\therefore x = y^{1/3}.$

22.

$$V = \int_0^1 \pi \left[f(y)^2 - g(y)^2 \right] dy$$

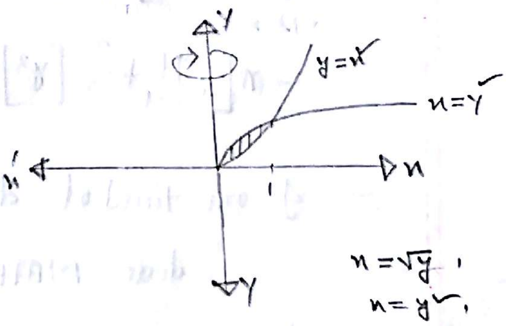
$$= \pi \int_0^1 \left[(\sqrt{y})^2 - (y^2)^2 \right] dy$$

$$= \pi \int_0^1 y \, dy - \pi \int_0^1 y^4 \, dy$$

$$= \pi \left[\frac{y^2}{2} \right]_0^1 - \pi/5 \left[y^5 \right]_0^1$$

$$= \frac{\pi}{2} - \frac{\pi}{5}.$$

$$= \frac{3\pi}{10}. \quad \text{Ans: } \frac{3\pi}{10}.$$



$x = \sqrt{y}$,

$x = y^2$,

$\therefore y^2 = \sqrt{y}$

or, $y^4 = y$

or, $y(y^3 - 1) = 0$

$\therefore y = 0, 1$

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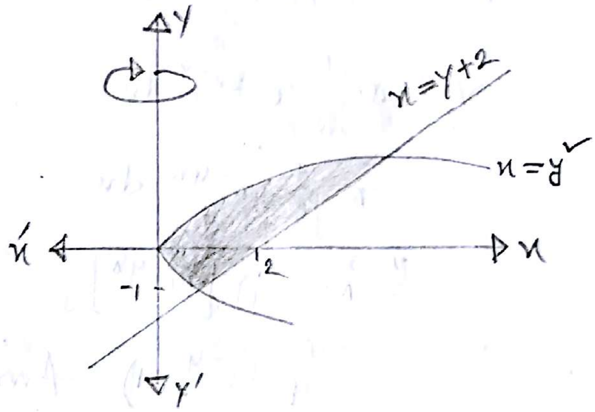
23.

$$V=\pi\int_{-1}^{2}(y+2)^2-(y^2)^2\,dy$$

$$=\pi\int_{-1}^{2}(y^2+4y+4-y^4)\,dy$$

$$=\pi\int_{-1}^{2}y^2\,dy+4\pi\int_{-1}^{2}y\,dy+4\pi\int_{-1}^{2}1\cdot dy-\pi\int_{-1}^{2}y^4\cdot dy$$



$$=\frac{\pi}{3}\left[y^3\right]_{-1}^{2}+2\pi\left[y^2\right]_{-1}^{2}+4\pi\left[y\right]_{-1}^{2}-\frac{\pi}{5}\left[y^5\right]_{-1}^{2}$$

$=14.8\pi$. Ans: 14.8π.

$y^2-y-2=0$.

or, $y^2-2y+y-2=0$.

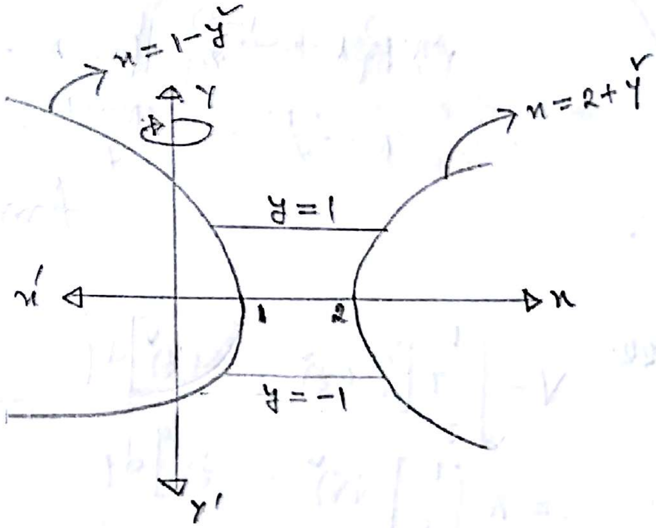
or, $y(y-2)+1(y-2)=0$

or, $(y-2)(y+1)=0$

$\therefore y=2,-1$.

24. $$V=\pi\int_{-1}^{1}\left[(2+y^2)^2-(1-y^2)^2\right]dy$$

$$=\pi\int_{-1}^{1}(4+4y^2+y^4)\,dy-\pi\int_{-1}^{1}(1-2y^2+y^4)\,dy$$



$$=4\pi\left[y\right]_{-1}^{1}+\frac{4\pi}{3}\left[y^3\right]_{-1}^{1}+\frac{1}{5}\left[y^5\right]_{-1}^{1}$$

$$-\pi\left[y\right]_{-1}^{1}+\frac{2}{3}\left[y^3\right]_{-1}^{1}-\frac{1}{5}\left[y^5\right]_{-1}^{1}$$

= I am tired of doing the same thing

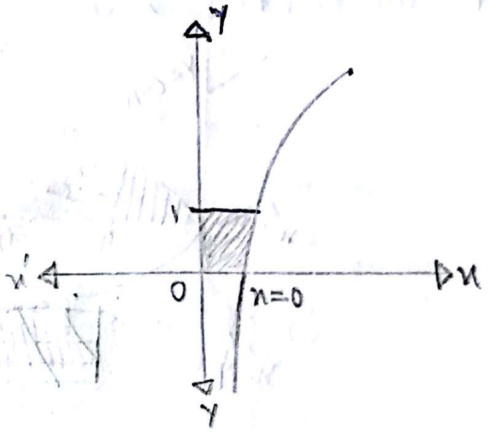
dear MATH plz grow up and solve ur own problem.

25. $y = \ln x$, $x = 0$, $y = 0$, $y = 1$,

or, $e^y = e^{\ln x}$

or, $e^y = x$

$\therefore x = e^y$



$$V = \pi \int_0^1 (e^y)^2 - (0)^2 \, dy$$

$$= \pi \int_0^1 (e^{2y}) \, dy$$

$$= \frac{\pi}{2} \cdot \left[e^{2y}\right]_0^1$$

$$= \frac{\pi e^2}{2}$$ Ans: $\frac{\pi e^2}{2}$.

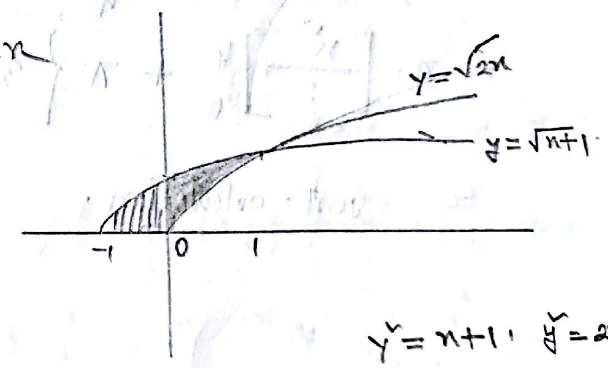
33. $$V = \pi \int_{-1}^{0} (\sqrt{x+1})^2 + \pi \int_0^1 \{(\sqrt{x+1})^2 - (\sqrt{2x})^2\} \, dx$$

$$= \pi \int_{-1}^{0} (x+1) \, dx + \pi \int_0^1 (x+1) \, dx - \pi \int_0^1 2x \, dx$$

$$= \pi \left\{ \frac{[x^2]_{-1}^{0}}{2} + [x]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1 + [x]_0^1 - 2 \cdot \left[\frac{x^2}{2}\right]_0^1 \right\}$$

$$= \frac{\pi}{2} + \pi + \frac{\pi}{2} + \pi - \pi$$

$$= \frac{4\pi}{2} = 2\pi$$ Ans: 2π.



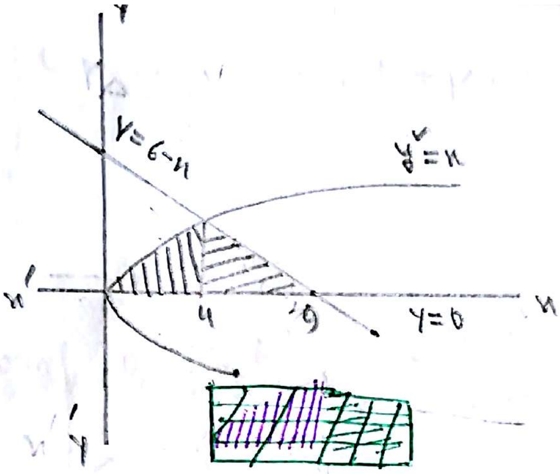
$y^2 = x+1$, $y^2 = 2x$.

$\therefore x+1 = 2x$

or, $2x - x = 1$

$\therefore x = 1$.

34.



$y^2=x,\ y=6-x$

$\therefore 6-x=\sqrt{x}$

or, $36-12x+x^2=x$

or, $x^2-13x+36=0$

$\therefore x=9,4$.

$$\therefore V=\pi\int_0^4\{(\sqrt{x})^2-(0)^2\}\,dx+\pi\int_4^9\{(6-x)^2-0^2\}\,dx$$

$$=\pi\int_0^4 x\,dx+\pi\int_4^9(36-12x+x^2)\,dx$$

$$=\pi\cdot\left[\frac{x^2}{2}\right]_0^4+\pi\left\{36[x]_4^9-6[x^2]_4^9+\frac{1}{3}[x^3]_4^9\right\}$$

= Just calculation.

Volumes by Cylindrical Shells — 6.3

এই chapter এ x অক্ষের সাপেক্ষে Revolve করলে y এর limit নিতে হবে and (outer − inner).

y অক্ষের সাপেক্ষে Revolve করলে x এর limit নিতে হবে and (upper − lower).

Tanvir boss ☺

Some Formula → $V = \int_a^b 2\pi x f(x)\, dx$.

$$V = \int_a^b 2\pi x [f(x) - g(x)]\, dx.$$

Example from book:

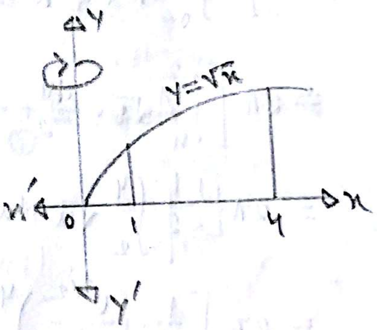
1. $V = \int_1^4 2\pi x \sqrt{x}\, dx$

$= 2\pi \int_1^4 x^{3/2}\, dx$

$= 2\pi \cdot \frac{2}{5} \left[x^{\frac{5}{2}}\right]_1^4$

$= \frac{4\pi}{5}(32-1) = \frac{124\pi}{5}$.

Ans: $\frac{124\pi}{5}$.



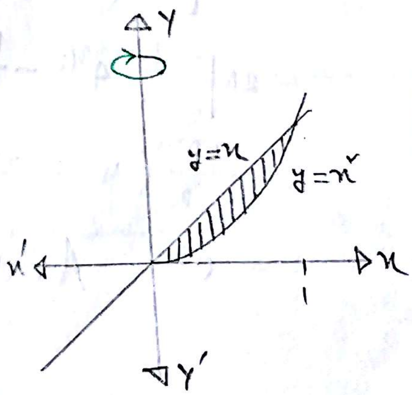
2. $V = \int_0^1 2\pi x (x - x^2)\, dx$

$= 2\pi \int_0^1 (x^2 - x^3)\, dx$

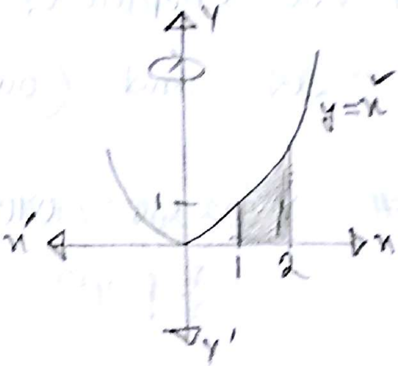
$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$

$= 2\pi \left(\frac{1}{3} - \frac{1}{4}\right)$

$= \frac{\pi}{6}$. Ans: $\frac{\pi}{6}$.



Exercise set 6.3 ⇒



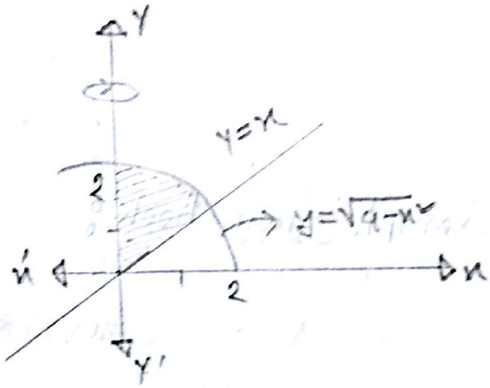
1. $V = \int_1^2 2\pi x \cdot f(x)\,dx$

$= 2\pi \int_1^2 x \cdot x^2\,dx$

$= 2\pi \int_1^2 x^3\,dx$

$= \frac{\pi}{2}\left[x^4\right]_1^2 = \frac{15\pi}{2}$

Ans: $\frac{15\pi}{2}$



2. $V = \int_0^{\sqrt{2}} 2\pi x\left(\sqrt{4-x^2} - x\right)dx$

$= 2\pi\left[\int_0^{\sqrt{2}} x\sqrt{4-x^2}\,dx - \int_0^{\sqrt{2}} x^2\,dx\right]$

$= 2\pi\left[\int_4^2 \sqrt{u}\,\frac{du}{-2} - \int_0^{\sqrt{2}} x^2\,dx\right]$

$= 2\pi\left[\frac{1}{2}\int_2^4 \sqrt{u}\,du - \frac{1}{3}\left[x^3\right]_0^{\sqrt{2}}\right]$

$= 2\pi\left[\frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right)_2^4 - \frac{1}{3}(\sqrt{2})^3\right]$

$= 2\pi\left[\frac{1}{3}\left(4^{3/2} - 2^{3/2}\right) - \frac{1}{3}(\sqrt{2})^3\right]$

$= 2\pi\left[\frac{1}{3}\,4^{3/2} - \frac{1}{3}\cdot 2^{3/2} - \frac{1}{3}\cdot 2^{3/2}\right]$

$= \frac{2\pi}{3}\left(4-2-2\right)^{3/2}$

$= 0$ Ans: 0

$x^2 = 4 - x^2$

or, $2x^2 - 4 = 0$

$\therefore x = \pm\sqrt{2}$.

Let, $u = 4 - x^2$

$du = -2x\,dx$

$\therefore \frac{du}{-2} = x\,dx$.

if, $x = \sqrt{2}$, $u = 2$

$x = 0$, $u = 4$.

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3.

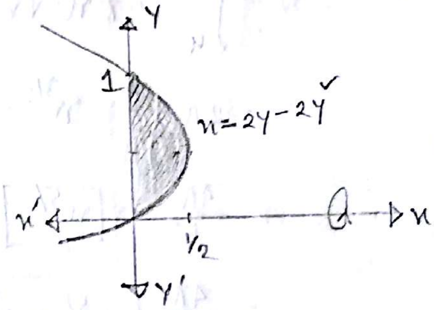
$$V = \int_0^1 2\pi y\,(2y - 2y^2)\,dy$$

$$= 2\pi \int_0^1 2y^2 \cdot dy - 2\pi \int_0^1 2y^3\,dy$$

$$= \frac{4\pi}{3}\left[y^3\right]_0^1 - \frac{4\pi}{4}\left[y^4\right]_0^1$$

$$= \frac{4\pi}{3} - \frac{4\pi}{4} = \frac{1}{3}\pi.$$

Ans: $\frac{1}{3}\pi$.



4. $$V = \int_0^2 2\pi y\,(y - y^2 + 2)\,dy$$

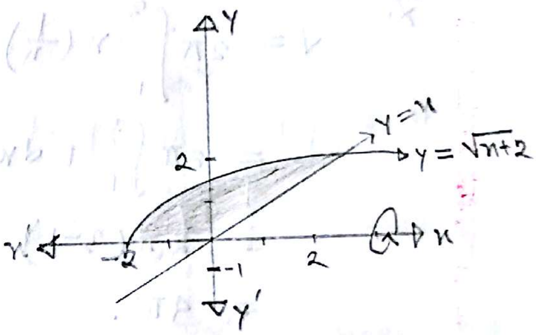
$$= 2\pi \int_0^2 (y^2 - y^3 + 2y)\,dy$$

$$= 2\pi \left\{ \frac{\left[y^3\right]_0^2}{3} - \frac{\left[y^4\right]_0^2}{4} + 2 \cdot \frac{\left[y^2\right]_0^2}{2} \right\}$$

$$= 2\pi \left(\frac{8}{3} - \frac{15}{4} + 4 \right)$$

$$= \frac{35\pi}{6}.$$

Ans: $\frac{35\pi}{6}$.



Note: $0^0 = 1$. ☺

$y^2 = x+2,\ y = x,$

$y^2 - 2 = y$

or, $y^2 - 2y + y - 2 = 0$

or, $(y-2)(y+1) = 0$

$\therefore y = 2, -1.$

5.

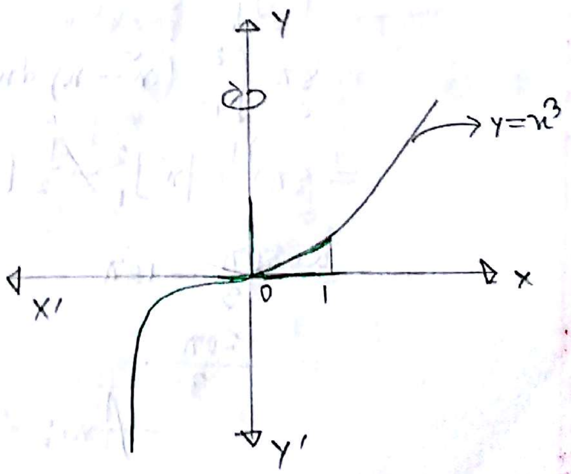
$$V = \int_0^1 2\pi x \cdot x^3\,dx$$

$$= 2\pi \int_0^1 x^4\,dx$$

$$= 2\pi \cdot \frac{1}{5}\left[x^5\right]_0^1$$

$$= \frac{2\pi}{5}.$$

Ans: $\frac{2\pi}{5}$.



6.

$$V = \int_4^9 2\pi x \cdot \sqrt{x}\, dx$$

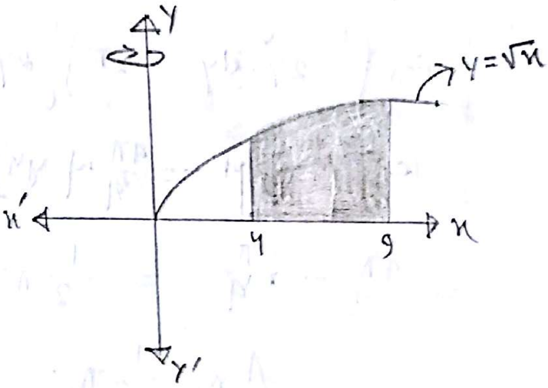
$$= 2\pi \int_4^9 x^{3/2}\, dx$$

$$= \frac{4\pi}{5} \left[x^{5/2}\right]_4^9$$

$$= \frac{4\pi}{5}\left(9^{5/2} - 4^{5/2}\right)$$

$$= \frac{844\pi}{5}$$

Ans: $\frac{844\pi}{4}$.



7.

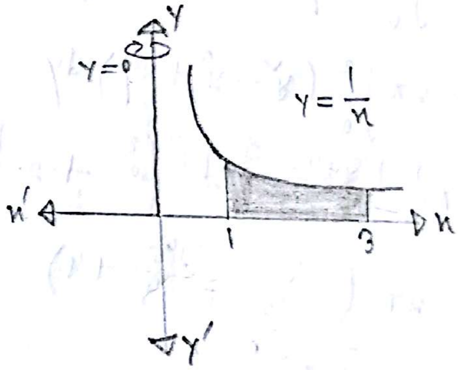
$$V = 2\pi \int_1^3 x\left(\frac{1}{x}\right) dx$$

$$= 2\pi \int_1^3 1\, dx$$

$$= 2\pi(3-1)$$

$$= 4\pi.$$

Ans: 4π.



9.

$$V = \int_1^2 2\pi x\,(2x-1+2x-3)\, dx$$

$$= 2\pi \int_1^2 (4x^2 - 4x)\, dx$$

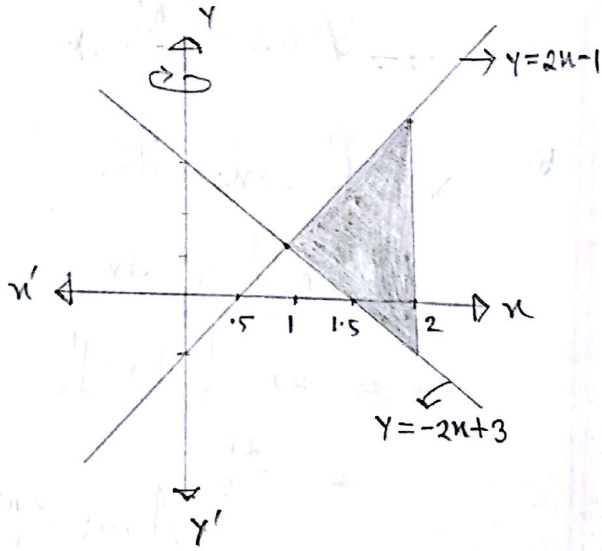
$$= 8\pi \int_1^2 (x^2 - x)\, dx$$

$$= 8\pi \left\{\frac{1}{3}\left[x^3\right]_1^2 - \frac{1}{2}\left[x^2\right]_1^2\right\}$$

$$= \frac{56\pi}{3} - 12\pi$$

$$= \frac{20\pi}{3}.$$

Ans: $\frac{20\pi}{3}$.



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10. $V = \int_0^2 2\pi x(2x - x^2)\,dx$

$= 2\pi \int_0^2 (2x^2 - x^3)\,dx$

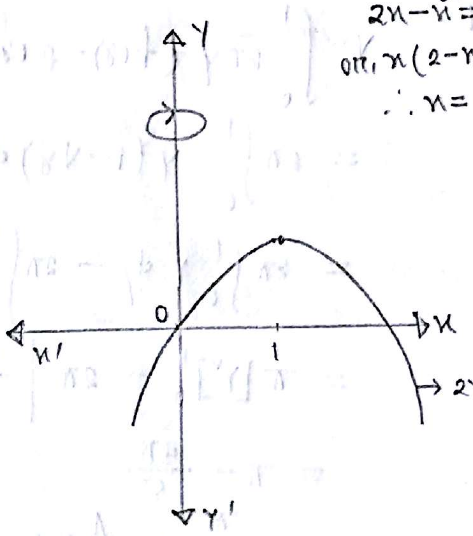
$= 4\pi \int_0^2 x^2\,dx - 2\pi \int_0^2 x^3\,dx$

$= 4\pi \cdot \frac{1}{3}\left[x^3\right]_0^2 - 2\pi \cdot \frac{1}{4}\left[x^4\right]_0^2$

$= \frac{32\pi}{3} - 8\pi$

$= \frac{8\pi}{3}$. Ans: $\frac{8\pi}{3}$.

$2x - x^2 = 0$
or, $x(2-x) = 0$
$\therefore x = 0, 2$

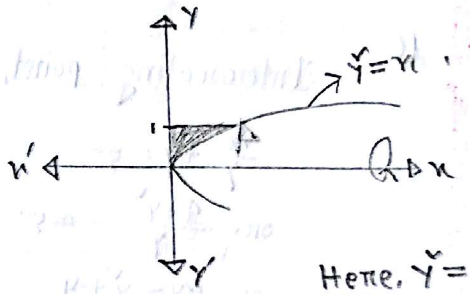


13. $V = \int_0^1 2\pi y \cdot y^2\,dy$

$= 2\pi \int_0^1 y^3\,dy$

$= 2\pi \cdot \frac{1}{4}\left[y^4\right]_0^1$

$= \frac{1}{2}\pi$. Ans: $\frac{\pi}{2}$.



Here, $y^2 = 0$
$\therefore y = 0$.

14. $V = \int_2^3 2\pi y \cdot 2y\,dy$

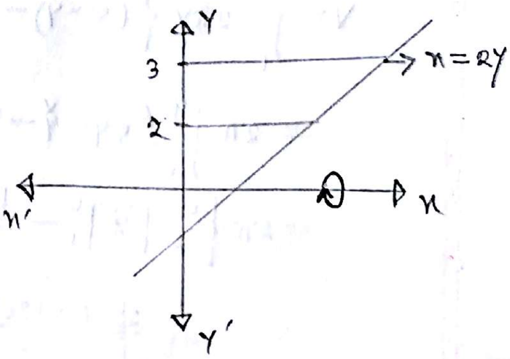
$= 4\pi \int_2^3 y^2\,dy$

$= \frac{4\pi}{3}\left[y^3\right]_2^3$

$= \frac{4\pi}{3}(27 - 9)$

$= \frac{72\pi}{3}$

Ans: $\frac{72\pi}{3}$.



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Q.

15. $V = \int_0^1 2\pi y\,(f(y) - g(y))\,dy$

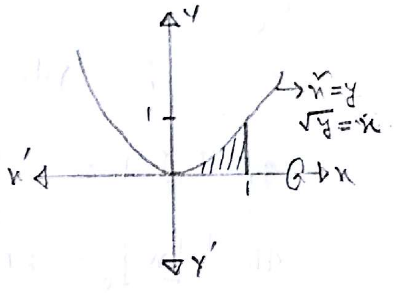
$= 2\pi \int_0^1 y(1-\sqrt{y})\,dy$

$= 2\pi \int_0^1 y\,dy - 2\pi \int_0^1 y^{3/2}\,dy$

$= \pi[y^2]_0^1 - 2\pi \left[\frac{y^{5/2}}{5/2}\right]_0^1$

$= \pi - \frac{4\pi}{5}$

$= \frac{\pi}{5}$. Ans: $\frac{\pi}{5}$.



16. Intersecting point;

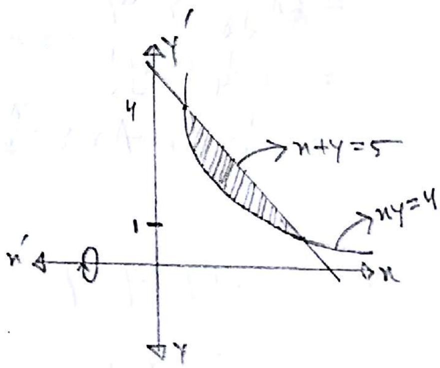
$\frac{4}{y} + y = 5$

or, $\frac{4+y^2}{y} = 5$

or, $5y = y^2 + 4$

or, $y^2 - 4y - y + 4 = 0$

$\therefore y = 4, 1$



$V = \int_1^4 2\pi y \left\{(5-y) - \frac{4}{y}\right\} dy$

$= 2\pi \int_1^4 (5y - y^2 - 4)\,dy$

$= 2\pi \left[\frac{5}{2}[y^2]_1^4 - \frac{1}{3}[y^3]_1^4 - 4[y]_1^4\right]$

$= \frac{225\pi - 126\pi - 24\pi}{3}$

$= 9\pi$

Ans: 9π.

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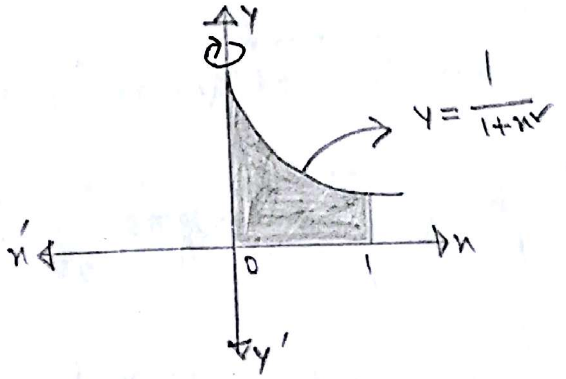
$y = \frac{1}{1+x^2}$, $x=0$, $x=1$, $y=0$. revolve about y-axis.



Solution →

$$V = 2\pi \int_0^1 x \cdot \left(\frac{1}{1+x^2}\right) dx$$

$$= 2\pi \int_0^1 \frac{x}{1+x^2}\, dx$$

$$= 2\pi \int_1^2 \frac{1}{2} \cdot \frac{du}{u}$$

$$= \pi \cdot \left[\ln u\right]_1^2$$

$$= \pi \ln 2 \qquad \text{Ans: } \pi \ln 2$$

Let, $u = 1+x^2$

$\frac{du}{2} = x\,dx$

if, $x=0$, $u=1$

$x=1$, $u=2$

Enough.

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Area of a Surface of Revolution — 6.5

Let, $y=f(x)$ on $[a,b]$ bounded by $x=a$ and $x=b$ and revolving by x-axis.

The surface area, $S=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

$$S=\int_c^d 2\pi g(x)\sqrt{1+[g'(y)]^2}\,dy.$$

Example from book: Through x-axis.

1. $S=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

Here, $y=x^3=f(x)$

$f'(x)=3x^2$

$$=\int_0^1 2\pi x^3\sqrt{1+(3x^2)^2}\,dx$$

$$=2\pi\int_0^1 x^3(1+9x^4)^{1/2}\,dx$$

Let, $u=1+9x^4$

or, $\frac{du}{dx}=36x^3.$

or, $du=36x^3\,dx$

$$=2\pi/36\int_1^{10} u^{1/2}\,du$$

Now, $x=1,\ u=10$

$x=0,\ u=1$

$$=\frac{2\pi}{36}\cdot\frac{2}{3}\left[u^{3/2}\right]_1^{10}$$

$$=\frac{\pi}{27}(10^{3/2}-1)$$

$= 3.56$ Approximately Ans: ≈ 3.56.

2. Through x-axis.

Here, $y=x^2 \Rightarrow x=\sqrt{y}=g(y)$

$g'(y)=\frac{1}{2}y^{-1/2}=\frac{1}{2\sqrt{y}}.$

$$S=\int_1^4 2\pi g(x)\sqrt{1+[g'(y)]^2}\,dy$$

$$=\int_1^4 2\pi\cdot\sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy$$

Let, $u=4y+1.$

$du=4dy.$

$$=\int_1^4 2\pi\cdot\sqrt{y}\sqrt{1+\frac{1}{4y}}\,dy$$

Now, $y=4,\ u=17$

$y=1.\ u=5.$

$$=\int_1^4 2\pi\cdot\sqrt{y}\,\frac{\sqrt{4y+1}}{2\sqrt{y}}\,dy$$

$$=\pi\int_1^4\sqrt{4y+1}\,dy$$

p.t.o

$$= \frac{\pi}{4}\int_5^{17} u^{1/2}\,du$$

$$= \frac{\pi}{4}\cdot\frac{2}{3}\left[u^{3/2}\right]_5^{17}$$

$$= \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right). \approx 30.85$$

Ans: 30.85

Exercise Set 6.5:- (1–4) Find the area of the surface generated by revolving the given curve about the x-axis.

1. $y = 7x, \quad 0 \le x \le 1.$

Here, $f(x) = 7x$

$f'(x) = 7$

$\therefore \sqrt{1+7^2} = \sqrt{50}$

$= 5\sqrt{2}.$

$$S = \int_0^1 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$= \int_0^1 2\pi\, 7x \cdot 5\sqrt{2}\cdot dx$$

$$= 70\sqrt{2}\pi \int_0^1 x\,dx$$

$$= 70\sqrt{2}\pi\,\frac{[x^2]_0^1}{2}$$

$$= 70\pi/\sqrt{2} = 35\sqrt{2}\pi.$$

Ans: $35\sqrt{2}\pi$.

2. $y=\sqrt{x}$, $f(x)=\sqrt{x}$, $f'(x)=\frac{1}{2\sqrt{x}}$.

Here, $\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}$

$= \sqrt{\frac{4x+1}{4x}}$

$= \frac{\sqrt{4x+1}}{2\sqrt{x}}.$

$$S = \int_1^4 2\pi\sqrt{x}\cdot\frac{\sqrt{4x+1}}{2\sqrt{x}}\,dx$$

$$= \pi\int_1^4 \sqrt{4x+1}\,dx$$

Let, $4x+1=u$.

$du = 4dx$

$x=4,\ u=17.$

$x=1,\ u=5$

$$= \frac{\pi}{4}\int_5^{17}\sqrt{u}\,du$$

p.t.o

$$= \frac{\pi}{4} \int_{5}^{17} u^{1/2} \cdot du$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} \left[u^{3/2} \right]_{5}^{17}$$

$$= \frac{\pi}{6} \times 58.9$$

Ans: $\frac{\pi}{6} \times 58.9$.

3. $y = \sqrt{4 - x^2}, \quad -1 \leq x \leq 1$.

$f(x) = \sqrt{4 - x^2}$,

$f'(x) = \frac{-2x}{2\sqrt{4-x^2}} = \frac{-x}{\sqrt{4-x^2}}$.

Here,

$$\sqrt{1 + \frac{x^2}{4-x^2}} = \sqrt{\frac{4 - x^2 + x^2}{4-x^2}}$$

$$= \sqrt{\frac{4}{4-x^2}}$$

$$= \frac{2}{\sqrt{4-x^2}}$$

$$\therefore \quad S = \int_{-1}^{1} 2\pi \cdot \sqrt{4-x^2} \cdot \frac{2}{\sqrt{4-x^2}} \, dx.$$

$$= 4\pi \int_{-1}^{1} 1 \cdot dx$$

$$= 4\pi \left[x \right]_{-1}^{1}$$

$$= 4\pi \times 2 = 8\pi.$$ Ans: 8π.

4. $x = \sqrt[3]{y} = y^{1/3} \quad \therefore y = x^3. \quad 1 \leq y \leq 8.$

$f'(x) = 3x^2$.

Here,

$\sqrt{1 + (3x^2)^2} = \sqrt{1 + 9x^4}$.

Let, $1 + 9x^4 = u$

or, $du = 36x^3 dx$.

if, $x = 8 \cdot u = 36865$

$x = 1 \cdot u = 10$.

$$S = 2\pi \int_{1}^{8} x^3 \sqrt{1 + 9x^4} \cdot dx$$

$$= \frac{2\pi}{36} \int_{1}^{8} u^{1/2} \cdot du$$

$$= \frac{2\pi}{36} \times \frac{2}{3} \left[u^{3/2} \right]_{10}^{36865}$$

$$\therefore S = 1365\pi$$

Ans: 1365π.

5–8, Find the area of the surface generated by revolving the given curve about the y-axis.

5. $x = 9y + 1, \quad 0 \le y \le 2.$

$g(y) = 9x + 1, \quad g'(y) = 9.$

$\sqrt{1 + g'(y)^2} = \sqrt{1 + 9^2} = \sqrt{82}.$

$$\therefore \; S = \int_0^2 2\pi\,(9y+1)\cdot\sqrt{82}\; dy$$

$$= 2\sqrt{82}\cdot\pi\left[9\int_0^2 y\cdot dy + \int_0^2 1\, dy\right]$$

$$= 2\sqrt{82}\cdot\pi\cdot\frac{9}{2}\left[y^2\right]_0^2 + 2\sqrt{82}\,\pi\left[y\right]_0^2$$

$$= 36\sqrt{82}\,\pi + 4\sqrt{82}\,\pi.$$

Ans: $36\sqrt{82}\,\pi + 4\sqrt{82}\,\pi$

6. $x = y^3, \quad 0 \le y \le 1.$

$$S = 2\pi\int_0^2 y^3\sqrt{1+9y^4}\; dy$$

$$= \frac{2\pi}{36}\int_1^{10} u^{1/2}\cdot du$$

$$= \frac{\pi}{18}\times\frac{2}{3}\left[u^{3/2}\right]_1^{10}$$

$$= \frac{\pi}{27}\left(10^{3/2} - 1^{3/2}\right).$$

Ans: $\frac{\pi}{27}\left(10^{3/2} - 1^{3/2}\right)$

Here,

$g(y) = y^3$

$g'(y) = 3y^2$

$\sqrt{1 + (3y^2)^2} = \sqrt{1 + 9y^4}.$

Let, $1 + 9y^4 = u,$

$36y^3\, dy = du.$

if, $y = 1, \; u = 10$

$y = 0, \; u = 1.$

p.t.o

7. $x = \sqrt{9-y^2}, \quad -2 \le y \le 2$

$$S = \int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}\,dy$$

$$= 2\pi \int_{-2}^{2} \sqrt{9-y^2} \cdot \frac{3}{\sqrt{9-y^2}}\,dy$$

$$= 6\pi \int_{-2}^{2} 1 \cdot dy$$

$$= 6\pi\,[y]_{-2}^{2} = 6\pi \times 4 = 24\pi.$$

Ans: 24π.

Here,

$$g(y) = \sqrt{9-y^2}$$

$$g'(y) = \frac{(-2)y}{2\sqrt{9-y^2}}.$$

Now, $\sqrt{1+\frac{y^2}{9-y^2}}$

$$= \sqrt{\frac{9-y^2+y^2}{9-y^2}}$$

$$= \frac{3}{\sqrt{9-y^2}}$$

8. $x = 2\sqrt{1-y}. \quad -1 \le y \le 0.$

$$\sqrt{1+g'(y)^2} = \sqrt{1+\frac{1}{1-y}}$$

$$= \sqrt{\frac{1-y+1}{1-y}}$$

$$= \sqrt{\frac{2-y}{1-y}}.$$

$$S = \int_{-1}^{0} 2\pi \cdot 2\sqrt{1-y} \cdot \frac{\sqrt{2-y}}{\sqrt{1-y}} \cdot dy$$

$$= 4\pi \int_{-1}^{0} \sqrt{2-y} \cdot dy$$

$$= -4\pi \int_{-1}^{0} u^{1/2} \cdot du$$

$$= -4\pi \cdot \frac{2}{3}\left[u^{3/2}\right]_3^2$$

$$\therefore S = -\frac{8\pi}{3} \times (-2.36)$$

Ans: $\frac{8\pi}{3} \times 2.36$

Here,

$$g(y) = 2\sqrt{1-y}.$$

$$g'(y) = 2 \cdot \frac{(-1)}{2\sqrt{1-y}}.$$

$$= \frac{-1}{\sqrt{1-y}}.$$

Let, $u = 2-y$

$du = -dy.$

if, $y=0$, $u=2$

$y=-1$, $u=3$.

Length of a plane curve – 6.4.

General Formula →.

$$L = \int_a^b \sqrt{1+f'(x)^2}\,dx \quad \text{along } x\text{-axis}.$$

$$L = \int_c^d \sqrt{1+f'(y)^2}\,dy \quad \text{along } y\text{-axis}.$$

For perpendicular curve :-

$$x = f(t).$$
$$y = g(t).$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,.\,dt. \qquad a \leq t \leq b. \quad \text{——} ①$$

Examples from book →.

1. a) $f(x) = x^{3/2}$
$f'(x) = 3/2\, x^{1/2}$

$$L = \int_1^2 \sqrt{1+(3/2\, x^{1/2})^2}\, dx$$

Let, $u = 1 + \frac{9}{4}x$
$du = \frac{9}{4}\, dx$

if, $x=1$, $u = \frac{13}{4}$.
$x=2$, $u = \frac{22}{4}$.

$$= \int_1^2 \sqrt{1+\frac{9}{4}x}\, dx$$

$$= \frac{4}{9}\int_{13/4}^{22/4} u^{1/2}.du$$

$$= \frac{4}{9}.\frac{2}{3}\left[u^{3/2}\right]_{13/4}^{22/4}$$

$$= \frac{8}{27}.\left[\left(\frac{22}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

$= 2.09$ .Approximately.

Ans: ≈ 2.09.

b). $y = x^{3/2} \Rightarrow x = y^{2/3}$

$f(y) = y^{2/3}$

$f'(y) = \frac{2}{3} y^{-1/3}$

$$L = \int_1^{2\sqrt{2}} \sqrt{1+\frac{4}{9}y^{-2/3}} \cdot dy$$

$$= \int_1^{2\sqrt{2}} \sqrt{\frac{9y^{2/3}+4}{9y^{2/3}}} \, dy$$

$$= \frac{1}{3}\int_1^{2\sqrt{2}} \cdot y^{-1/3} \sqrt{9y^{2/3}+4} \, dy$$

$$= \frac{1}{18}\int_{13}^{22} u^{1/2} \cdot du$$

Let, $u = 9y^{2/3}+4$

$du = 6y^{-1/3} \cdot dy$

if, $y=1$, $u=13$

$y = 2\sqrt{2}$, $u = 22$.

$$= \frac{1}{27}\left[u^{3/2}\right]_{13}^{22}$$

$$= \frac{1}{27}\left[(22)^{3/2} - (13)^{3/2}\right]$$

$$= \frac{22\sqrt{22} - 13\sqrt{13}}{27}.$$

Ans; $\frac{22\sqrt{22}-13\sqrt{13}}{27}$.

2. $f(x) = \sin x$

$f'(x) = \cos x$

$$L = \int_0^{\pi} \sqrt{1+(\cos x)^2} \, dx$$

$= 3.8202$ · Approximately.

Ans; ≈ 3.8202.

Exercise Set 6·4 => Find the exact arc length of the curve over the interval

3. $Y = 3x^{3/2} - 1, \quad x=0, x=1$

$f(x) = 3x^{3/2} - 1$

$f'(x) = 3 \cdot \frac{3}{2} x^{1/2} = \frac{9}{2}\sqrt{x}$

$\{f'(x)\}^2 = \frac{81}{4} x$

$\therefore L = \int_0^1 \sqrt{1+\frac{81x}{4}}\, dx = \int_0^1 \sqrt{\frac{4+81x}{4}}\, dx$

Let, $u = 4+81x$

$du = 81 dx$

if. $x=0, \; u=4$

$x=1, \; u=85$.

$= \frac{1}{2}\int_0^1 \sqrt{4+81x}\, dx$

$= \frac{1}{2} \times \frac{1}{81} \int_0^1 u^{1/2}\, du$

$= \frac{1}{162} \times \frac{2}{3} \left[u^{3/2}\right]_4^{85}$

$= 3.19$. Ans: 3.19.

4. $x = \frac{1}{3}(y^2+2)^{3/2}$ from $y=0$ to $y=1$

$f(y) = \frac{1}{3}(y^2+2)^{3/2}$

$f'(y) = \frac{1}{3} \cdot \frac{3}{2}(y^2+2)^{1/2} \cdot 2y$

$= y\sqrt{(y^2+2)}$.

$\therefore L = \int_0^1 \sqrt{1+f'(y)^2}\, dy$

$= \int_0^1 \sqrt{1+y^2(y^2+2)}\, dy$

$= \int_0^1 \sqrt{(1+y^4+2y^2)}\, dy$

$= \int_0^1 \sqrt{(y^2+1)^2}\, dy$

$= \int_0^1 (y^2+1)\, dy$

$= \frac{1}{3}[y^3]_0^1 + [y]_0^1$

$= \frac{1}{3}+1 = \frac{4}{3}$. Ans: $\frac{4}{3}$.

5. $y = x^{2/3}$ from $x = 1$ to $x = 8$.

$\therefore f(x) = x^{2/3}$

$f'(x) = \frac{2}{3} x^{-1/3}$.

Let $u = 9x^{2/3} + 4$

$du = 6x^{-1/3} \cdot dx$

if $x = 8$, $u = 32$

$x = 1$, $u = 13$

$\therefore L = \int_1^8 \sqrt{1 + \frac{4}{9} \cdot \frac{1}{x^{2/3}}}\, dx$

$= \int_1^8 \sqrt{\frac{9x^{2/3} + 4}{3x^{1/3}}}\, dx$

$= \frac{1}{3} \int_1^8 x^{-1/3} \sqrt{9x^{2/3} + 4}\, dx$

$= \frac{1}{18} \int_{13}^{32} u^{1/2} \cdot du$

$= \frac{1}{27} \left[u^{3/2} \right]_{13}^{32}$

$= \frac{1}{27} \left[(32)^{3/2} - (13)^{3/2} \right]$. Ans: $\frac{1}{27} \left[(32)^{3/2} - (13)^{3/2} \right]$

6.

$y = (x^6 + 8)/16x^2$, from $x = 2$ to $x = 3$.

$\therefore f(x) = \frac{x^6 + 8}{16x^2} = \frac{1}{16} x^4 + \frac{1}{2} x^{-2}$

$f'(x) = \frac{d}{dx} \left(\frac{1}{16} x^4 + \frac{1}{2} x^{-2} \right)$

$= \frac{1}{16} \times 4x^3 + (-2)/2 \cdot x^{-3}$

$= \frac{1}{4} x^3 - x^{-3}$

$= \frac{1}{4} x^3 - \frac{1}{x^3}$

$= \frac{x^6 - 4}{4x^3}$.

$\therefore L = \int_2^3 \sqrt{1 + \left(\frac{x^6 - 4}{4x^3} \right)^2}\, dx$

$= \int_2^3 \sqrt{\frac{16x^6 + x^{12} - 8x^6 + 16}{16x^6}} \cdot dx$

p.t.o

$$= \int_2^3 \frac{\sqrt{x^{12}+8x^6+16}}{4x^3}\, dx$$

$$= \int_2^3 \frac{\sqrt{(x^6+4)^2}}{4x^3}\, dx$$

$$= \int_2^3 \frac{x^6+4}{4x^3}\, dx = \frac{1}{4}\int_2^3 x^3\, dx + \int_2^3 x^{-3}\, dx.$$

$$= \frac{1}{16}\left[x^4\right]_2^3 + \frac{1}{(-2)}\cdot\left[x^{-2}\right]_2^3$$

$$= \frac{1}{16}\times 64 + \frac{1}{-2}\times(-0.138)$$

$$= 4.0694. \quad \text{Ans: } 4.0694.$$

**.
7. $24xy = y^4+48$ or, $x = \frac{y^4+48}{24y} = f(y)$

from, $y=2$ to $y=4$, or, $f(y) = \frac{y^3}{24}+\frac{2}{y}$

$$f'(y) = \frac{3y^2}{24} - \frac{2}{y^2}$$

$$= \frac{y^2}{8} - \frac{2}{y^2}.$$

$$f'(y)^2 = \left(\frac{y^2}{8} - \frac{2}{y^2}\right)^2$$

$$= \frac{y^4}{64} - 2\cdot\frac{y^2}{8}\cdot\frac{2}{y^2} + \frac{4}{y^4}.$$

$$= \frac{y^4}{64} - \frac{1}{2} + \frac{4}{y^4}.$$

$$\therefore L = \int_2^4 \sqrt{1+\frac{y^4}{64}-\frac{1}{2}+\frac{4}{y^4}}\, dy$$

$$= \int_2^4 \sqrt{\frac{y^4}{64}+\frac{4}{y^4}+\frac{1}{2}}\, dy$$

$$= \int_2^4 \sqrt{\frac{(y^4)^2+4\cdot 64+32y^4}{64\cdot y^4}}\, dy$$

$$= \int_2^4 \sqrt{\frac{(y^4)^2+2\cdot y^4\cdot 16+(16)^2}{64\, y^4}}\, dy$$

$$= \int_2^4 \frac{\sqrt{(y^4+16)^2}}{8y^2}\, dy$$

$$= \int_2^4 \frac{y^4+16}{8y^2}\, dy$$

$$= \int_2^4 \left(\frac{y^2}{8}+\frac{2}{y^2}\right) dy$$

$$= \left[\frac{y^3}{24}\right]_2^4 + 2\left[\frac{-1}{y}\right]_2^4$$

$$= \frac{17}{6}.$$

Ans: $\frac{17}{6}$.

8. $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$ from $y=1$ to $y=4$

$f(y) = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$

$f'(y) = \frac{1}{8}\cdot 4y^3 - \frac{2}{4}y^{-3}$

$= \frac{1}{2}y^3 - \frac{1}{2}y^{-3}$

$= \frac{1}{2}(y^3 - \frac{1}{y^3})$.

$\therefore f'(y)^2 = \frac{1}{4}(y^6 - 2\cdot y^3\cdot\frac{1}{y^3} + \frac{1}{y^6})$

$= \frac{1}{4}(y^6 + \frac{1}{y^6} - 2)$.

$\therefore 1 + f'(y)^2 = 1 + \frac{y^6}{4} + \frac{1}{4y^6} - \frac{2}{4}$

$= \frac{4y^6 + y^{12} + 1 - 2y^6}{4y^6}$

$= \frac{2y^6 + y^{12} + 1}{4y^6}$

$= \frac{(y^6)^2 + 2\cdot y^6\cdot 1 + (1)^2}{4y^6}$

$= \frac{(y^6+1)^2}{4y^6}$.

$\therefore L = \int_1^4 \sqrt{\frac{(y^6+1)^2}{4y^6}}\, dy$

$= \frac{1}{2}\int_1^4 \frac{y^6+1}{y^3}\, dy$

$= \frac{1}{2}\int_1^4 (y^3 + y^{-3})\, dy$.

$= \frac{1}{2}\times\frac{1}{4}[y^4]_1^4 + \frac{1}{-2}[y^{-2}]_1^4$

$= \frac{15}{8} + \frac{15}{32}$

$= \frac{160+15}{32}$

$= \frac{75}{32}$

Ans: $\frac{75}{32}$.

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27–32: Use the arc length formula from — ① to find the arc length of the curve.

27. $x = \frac{1}{3}t^3,\ y = \frac{1}{2}t^2.\quad (0 \le t \le 1).$

$\frac{dx}{dt} = \frac{1}{3} \times 3t^2 = t^2.$

$\frac{dy}{dt} = \frac{1}{2} \times 2t = t.$

$\therefore L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^1 \sqrt{(t^4 + t^2)}\, dt$

$= \int_0^1 \sqrt{t^2(t^2+1)}\, dt$

$= \int_0^1 t\sqrt{t^2+1}\, .dt$

$= \frac{1}{2}\int_1^2 z^{1/2}\, dz$

$= \frac{1}{2}\left[\frac{z^{3/2}}{3/2}\right]_1^2$

$= \frac{1}{3}(2^{3/2} - 1).$ Ans: $\frac{1}{3}(2^{3/2} - 1)$.

Let, $t^2 + 1 = z.$

or, $2t\,dt = dz.$

or, $t.dt = \frac{dz}{2}.$

$\therefore$ if $t = 0,\ z = 1.$

$t = 1,\ z = 2.$

29. ~~28.~~ $x = \cos 2t,\ y = \sin 2t,\quad 0 \le t \le \frac{\pi}{2}$

$\frac{dx}{dt} = -2\sin 2t,\quad \frac{dy}{dt} = 2\cos 2t.$

$\therefore L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

$= \int_0^{\pi/2} \sqrt{4(\sin^2 2t + \cos^2 2t)}\, dt$

$= 2\int_0^{\pi/2} 1 \cdot dt$

$= 2[t]_0^{\pi/2} = 2 \times \frac{\pi}{2} = \pi.$ Ans: π.

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30. $x = \cos t + t\sin t \qquad \therefore 0 \leq t \leq \pi.$

$$\frac{dx}{dt} = -\sin t + \sin t + t\cos t$$
$$= t\cos t.$$

$$y = \sin t - t\cos t$$
$$= \cos t - \cos t + t\sin t$$
$$= t\sin t.$$

$$L = \int_0^{\pi} \sqrt{t^2\cos^2 t + t^2\sin^2 t}\, dt$$
$$= \int_0^{\pi} t\sqrt{\cos^2 t + \sin^2 t}\, dt$$
$$= \int_0^{\pi} t\, dt$$
$$= \left[\frac{t^2}{2}\right]_0^{\pi} = \frac{\pi^2}{2}. \quad \text{Ans: } \frac{\pi^2}{2}.$$

31. $x = e^t\cos t \qquad \frac{dy}{dt} = e^t\sin t + e^t\cos t = e^t(\sin t + \cos t).$

$y = e^t\sin t \qquad \frac{dx}{dt} = e^t\cos t - e^t\sin t = e^t(\cos t - \sin t).$

$0 \leq t \leq \pi/2.$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t)^2\{(\cos t - \sin t)^2 + (\cos t + \sin t)^2\}$$
$$= (e^t)^2\left[\cos^2 t + \sin^2 t - 2\sin t\cdot\cos t + \sin^2 t + \cos^2 t + 2\sin t\cdot\cos t\right]$$
$$= (e^t)^2\left[1+1+0\right]$$
$$= 2(e^t)^2$$

$$\therefore \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2(e^t)^2} = \sqrt{2}\, e^t$$

$$\therefore L = \sqrt{2}\int_0^{\pi/2} e^t\, dt$$
$$= \sqrt{2}\left(e^{\pi/2} - e^0\right)$$
$$= \sqrt{2}\left(e^{\pi/2} - 1\right). \quad \text{Ans:}$$

32. $n = e^t(\sin t + \cos t)$

$y = e^t(\cos t - \sin t)$.

$[1 \leq t \leq 4]$

$\frac{dn}{dt} = e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t$

$= 2e^t \cos t$.

$\frac{dy}{dt} = e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t$.

$= -2e^t \sin t$.

$$L = \int_1^4 \sqrt{(2e^t \cos t)^2 + (-2e^t \sin t)^2}\, dt$$

$$= \int_1^4 \sqrt{4(e^t)^2(\cos^2 t + \sin^2 t)}\, dt.$$

$$= \int_1^4 2e^t\, dt$$

$$= 2 \cdot [e^t]_1^4 = 2(e^4 - e^1) = 103.75.$$ Ans.

28.

Principles of integral Evaluation.

Intrigation by parts — 7.2

Formula → $\int f(x)\, g(x)\, dx = f(x)\, G(x) - \int f'(x)\, G(x)\, dx$.

Here, $G(x)$ will be the anti derivative of $g(x)$.

$$G(x) = \int g(x)\, dx.$$

L I A T E → Exponential.

L ↓ logarithmic

I ↓ Inverse trigonometric

A ↓ Algebric

T ↓ Trigonometric

Examples from book →

1. $\int x \cos x\, dx$.

Here, $f(x) = x$, $g(x) = \cos x$

$f'(x) = 1$. $G(x) = \int \cos x\, dx = \sin x$

$= x \cdot \sin x - \int 1 \cdot \sin x\, dx$

$= x \cdot \sin x + \cos x + c$.

Ans

2. $\int x e^x dx$,

Here, $f(x) = x$, $g(x) = e^x$

$f'(x) = 1$. $G(x) = \int e^x dx = e^x$

$= x e^x - \int 1 \cdot e^x dx$

$= x e^x - e^x + c$ Ans

3. $\int \ln x\, dx$.

Here, $f(x) = \ln x$, $g(x) = 1$

$f'(x) = \frac{1}{x}$. $G(x) = \int 1\, dx = x$.

$= \ln x \cdot x - \int \frac{1}{x} \cdot x\, dx$

$= x \ln x - \int 1\, dx$

$= x \ln x - x + c$.

Ans

Repeated Intrigation By Parts :-

4. $\int x^2 e^{-x} dx$.

Here, $f(x) = x^2$ $\quad g(x) = e^{-x}$

$f'(x) = 2x$ $\quad G(x) = \int e^{-x} dx$

$= -e^{-x}$

$= x^2 (-e^{-x}) + \int 2x e^{-x} dx$ → Applying the same theory as before.

$= -x^2 e^{-x} + 2\left[x(-e^{-x}) + \int e^{-x} dx\right]$

$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$

$= -e^{-x}(x^2 + 2x + 2) + c$ Ans

5. $\int e^x \cos x\, dx = \cos x \cdot e^x + \int \sin x \cdot e^x dx$

$= e^x \cos x + e^x \sin x - \int e^x \cos x\, dx$

Here

$f(x) = \cos x$. $f'(x) = -\sin x$

$g(x) = e^x$ $G(x) = \int e^x dx$

$= e^x$.

or, $2\int e^x \cos x\, dx = e^x (\cos x + \sin x)$.

or, $\int e^x \cos x\, dx = \dfrac{e^x}{2}(\cos x + \sin x) + c$

Tabular Integration by parts :-

* $\int (x^2 - x) \cos x\, dx$.

Repeated Differentiation		Repeated Integration.
$x^2 - x$	+	$\cos x$
$2x - 1$	−	$\sin x$
2	+	$-\cos x$
0		$-\sin x$

$= (x^2 - x)\sin x - (2x-1)(-\cos x) + 2(-\sin x) + c$

$= (x^2 - x)\sin x + (2x-1)\cos x - 2\sin x + c$

$= (x^2 - x - 2)\sin x + (2x-1)\cos x + c$

Ans

6. $\int x^2 \sqrt{x-1}\, dx$

$= x^2 \cdot \frac{2}{3}(x-1)^{3/2} - 2x \cdot \frac{4}{15}(x-1)^{5/2} + \frac{16}{105}(x-1)^{7/2} + c$

$= \frac{2}{3} x^2 (x-1)^{3/2} - \frac{8}{15} x (x-1)^{5/2} + \frac{16}{105}(x-1)^{7/2} + c$

Repeated Differention		Repeated Integration
x^2	$+$	$(x-1)^{1/2}$
$2x$	$-$	$\frac{2}{3}(x-1)^{3/2}$
2	$+$	$\frac{4}{15}(x-1)^{5/2}$
0	$-$	$\frac{8}{105}(x-1)^{7/2}$

Ans

Reduction formula →

$\int x^n \cdot e^{mx}\, dx$

Here, $f(x) = x^n$, $f'(x) = n x^{n-1}$

$g(x) = e^{mx}$, $G(x) = \int e^{mx}\, dx = \frac{e^{mx}}{m}$

$= \frac{1}{m} e^{mx} x^n - \int n x^{n-1}\, e^{mx}/m \, dx$

$= \frac{1}{m} x^n e^{mx} - \frac{n}{m} \int x^{n-1} e^{mx}\, dx$

7. Example → $\int x^3 e^{2x}\, dx$

Here, $f(x) = x^3$, $f'(x) = 3x^2$

$g(x) = e^{2x}$, $G(x) = \frac{e^{2x}}{2}$

$= \frac{1}{2} e^{2x} x^3 - \int 3x^2 \cdot \frac{e^{2x}}{2}\, dx$

$= \frac{1}{2} x^3 e^{2x} - \frac{3}{2} \int x^2 e^{2x}\, dx$

$= \frac{1}{2} x^3 e^{2x} - \frac{3}{2}\left[x^2 \cdot \frac{e^{2x}}{2} - \int \cancel{2}x \cdot \frac{e^{2x}}{\cancel{2}}\, dx\right]$

$= \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{2} \int x e^{2x}\, dx$

$= \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{2}\left[x \cdot e^{2x}/2 - \int 1 \cdot e^{2x}/2\, dx\right]$

$= \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{4} x e^{2x} - \frac{3}{4} \cdot \frac{1}{2} e^{2x} + c$

$= \frac{1}{2} x^3 e^{2x} - \frac{3}{4} x^2 e^{2x} + \frac{3}{4} x e^{2x} - \frac{3}{8} e^{2x} + c$

Ans

$\int x^3 e^{2x}\, dx$.

Tabulare Integration by parts:

Repeated Differention		Repeated Integration.
		e^{2x}
x^3	+	$\frac{1}{2} e^{2x}$
$3x^2$	−	$\frac{1}{4} e^{2x}$
$6x$	+	$\frac{1}{8} e^{2x}$
6	−	$\frac{1}{16} e^{2x}$
0	+	

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4} x e^{2x} - \frac{3}{8} e^{2x} + c.$$

$\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1}x \cos x + \frac{n-1}{n}\int \sin^{n-2}x\, dx$

Reduction $\rightarrow \int \sin^n x\, dx$

Here, $f(x) = \sin^{n-1}x$

$f'(x) = (n-1)\sin^{n-2}x \cdot \cos x$

$g(x) = \sin x$

$G(x) = -\cos x$

$= \int \sin^{n-1}x \cdot \sin x\, dx$

$= \sin^{n-1}x(-\cos x) - \int (n-1)\sin^{n-2}x \cdot \cos x \cdot (-\cos x)\, dx$

$= -\sin^{n-1}x \cdot \cos x + (n-1)\int \sin^{n-2}x \cdot \cos^2 x\, dx$

$= -\sin^{n-1}x \cdot \cos x + (n-1)\int \sin^{n-2}x \cdot (1-\sin^2 x)\, dx$

$= -\sin^{n-1}x \cdot \cos x + (n-1)\int (\sin^{n-2}x - \sin^n x)\, dx$

$\Rightarrow \int \sin^n x\, dx = -\sin^{n-1}x \cdot \cos x + (n-1)\int \sin^{n-2}x\, dx - (n-1)\int \sin^n x\, dx$

$\Rightarrow \int \sin^n x\, dx + (n-1)\int \sin^n x\, dx = -\sin^{n-1}x \cdot \cos x + (n-1)\int \sin^{n-2}x\, dx$

$\therefore \int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1}x \cdot \cos x + \frac{n-1}{n}\int \sin^{n-2}x\, dx$

$\int \sin^6 x\, dx$

$= -\frac{1}{6}\sin^{6-1}x \cos x + \frac{6-1}{6}\int \sin^{6-2}x\, dx$

$= -\frac{1}{6}\sin^5 x \cos x + \frac{5}{6}\left[-\frac{1}{4}\sin^3 x \cdot \cos x + \frac{3}{4}\int \sin^2 x\, dx\right]$

$= -\frac{1}{6}\sin^5 x \cos x + \frac{5}{6}\left[-\frac{1}{4}\sin^3 x \cdot \cos x + \frac{3}{4}\left\{-\frac{1}{2}\sin x \cos x + \frac{1}{2}\int 1\, dx\right]\right.$

$+c$

Ans

$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

Reduction $\rightarrow$ $\int \cos^n x\,dx$

Here, $f(x) = \cos^{n-1}x$

$f'(x) = (n-1)\cos^{n-2}x(-\sin x)$

$g(x) = \cos x$

$G(x) = \int \cos x\,dx$

$= \sin x$

$= \int \cos^{n-1}x \cdot \cos x\,dx$

$= \cos^{n-1}x \cdot \sin x + (n-1)\int \cos^{n-2}x \cdot \sin x \cdot \sin x\,dx$

$= \cos^{n-1}x \cdot \sin x + (n-1)\int \cos^{n-2}x \cdot \sin^2 x\,dx$

$= \cos^{n-1}x \cdot \sin x + (n-1)\int \cos^{n-2}x \cdot (1-\cos^2 x)\,dx$

$= \cos^{n-1}x \cdot \sin x + (n-1)\int (\cos^{n-2}x - \cos^n x)\,dx$

$= \cos^{n-1}x \cdot \sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x \cdot dx$

$\Rightarrow \int \cos^n x\,dx + (n-1)\int \cos^n x\,dx = \cos^{n-1}x \cdot \sin x + (n-1)\int \cos^{n-2}x\,dx$

$\therefore \int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x \cdot \sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$.

Example_8 $\Rightarrow$ $\int \cos^4 x\,dx$

$= \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\,dx$

$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}\cos x \cdot \sin x + \frac{1}{2}\int 1 \cdot dx\right)$

$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + c$. Ans.

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Exercise Set 7.2 ⇒

1. $\int x e^{-2x}\,dx$

Here, $f(x) = x$, $f'(x) = 1$.

$g(x) = e^{-2x}$

$G(x) = -\dfrac{e^{-2x}}{2}$.

$= x \cdot -\frac{1}{2}e^{-2x} - \int 1\cdot(-\frac{1}{2})e^{-2x}\,dx$

$= \dfrac{-x}{2}e^{-2x} - \dfrac{1}{4}e^{-2x} + c$.

2. $\int x e^{3x}\,dx$

Here, $f(x) = 1$, $f'(x) = 1$.

$g(x) = e^{3x}$

$G(x) = \dfrac{e^{3x}}{3}$.

$= x\cdot\frac{1}{3}e^{3x} - \int 1\cdot\frac{1}{3}e^{3x}\,dx$

$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$.

3. $\int x^2 e^{x}\,dx$.

Here, $f(x) = x^2$, $f'(x) = 2x$

$g(x) = e^{x}$.

$G(x) = e^{x}$.

$= x^2e^{x} - \int 2x\cdot e^{x}\,dx$

$= x^2e^{x} - 2\int x\cdot e^{x}\,dx$

$= x^2e^{x} - 2\left[xe^{x} - \int 1\,e^{x}\,dx\right]$

$= x^2e^{x} - 2xe^{x} + 2e^{x} + c$.

4. $\int x^2 e^{-2x}\,dx$

Here, $f(x) = x^2$, $f'(x) = 2x$

$g(x) = e^{-2x}$

$G(x) = -\dfrac{e^{-2x}}{2}$.

$= x^2\cdot\dfrac{e^{-2x}}{-2} - \int 2x\cdot -\dfrac{e^{-2x}}{2}\,dx$

$= -\frac{1}{2}x^2e^{-2x} + \int x\cdot e^{-2x}\,dx$

$= -\frac{1}{2}x^2e^{-2x} + x\cdot -\frac{1}{2}e^{-2x} - \int 1\cdot e^{-2x}\,dx$

$= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} + \frac{1}{2}e^{-2x} + c$

$= -\frac{1}{2}e^{-2x}(x^2 + x - 1) + c$.

5. $\int x\sin 3x\,dx$

$= x\cdot -\frac{1}{3}\cos 3x - \int 1\cdot -\frac{1}{3}\cos 3x$

$= -\frac{1}{3}x\cos 3x + \int \frac{1}{3}\cos 3x$

$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x$

Here, $f(x) = x$, $f'(x) = 1$

$g(x) = \sin 3x$

$G(x) = \int \sin 3x\,dx$

$= -\frac{1}{3}\cos 3x$

6. $\int x\cos 2x\,dx$

$= x\cdot\frac{1}{2}\sin 2x - \int 1\,\frac{1}{2}\sin 2x$

$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$

Here,

$f(x) = x$, $f'(x) = 1$

$g(x) = \cos 2x$

$G(x) = \frac{1}{2}\sin 2x$

7. $\int x^2\cos x\,dx$

Applying tobular method

$= x^2\sin x + 2x\cos x + 2\sin x + c$

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upto '0'	⋮

8. $\int x^2\sin x\,dx$

Applying tobular method

$= -x^2\cos x + 2x\sin x + \cos x + c$

9. $\int x\ln x\,dx$

$= \frac{x^2}{2}\ln x - \int \frac{1}{x}\cdot\frac{x^2}{2}\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c$

Here, $f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$g(x) = x$

$G(x) = \frac{x^2}{2}$

10. $\int \sqrt{x}\cdot\ln x\,dx$

$= \ln x\cdot\frac{2}{3}x^{3/2} - \int \frac{1}{x}\cdot\frac{2}{3}x^{3/2}\,dx$

$= \frac{2}{3}\ln x\, x^{3/2} - \frac{2}{3}\int \frac{1}{x}\cdot x\sqrt{x}\,dx$

$= \frac{2}{3}\ln x\cdot x^{3/2} - \frac{2}{3}\cdot\frac{2}{3}x^{3/2} + c$

Here, $g(x) = \sqrt{x} = x^{1/2}$

$G(x) = \frac{2}{3}x^{3/2}$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

11. $\int (\ln x)^2 dx = \int \ln x \cdot \ln x \, dx$

$= \ln x \cdot (x\ln x - x) - \int \frac{1}{x} \cdot (x\ln x - x) dx$

$=$

Here, $f(x) = \ln x$
$f'(x) = \frac{1}{x}$
$g(x) = \ln x$
$G(x) = \int \ln x \, dx$
$= x\ln x - x$

12. $\int \frac{\ln x}{\sqrt{x}} dx$

$= \ln x \cdot 2x^{1/2} - \int \frac{1}{x} \cdot 2x^{1/2} dx$

$= 2\ln x\sqrt{x} - 2\int \frac{x^{1/2}}{x} dx$

$= 2\ln x \cdot \sqrt{x} - 2 \cdot \int x^{-1/2} dx$

$= 2\ln x\sqrt{x} - 4\sqrt{x}$

Here, $f(x) = \ln x$, $f'(x) = \frac{1}{x}$
$g(x) = x^{-1/2}$
$G(x) = \frac{x^{1/2}}{1/2}$

13. $\int \ln(3x-2) \, dx$

$= x\ln(3x-2) - \int \frac{3}{3x-2} \cdot x \, dx$

$= x\ln(3x-2) - \int \frac{(u+2)}{3u} du$

$= x\ln(3x-2) - \frac{1}{3}\int \frac{u+2}{u} du$

$= x\ln(3x-2) - \frac{1}{3}\int \left(1 + \frac{2}{u}\right) du$

$= x\ln(3x-2) - \frac{1}{3}\left[u + 2\ln u\right]$

$= x\ln(3x-2) - \frac{1}{3}(3x-2) - \frac{2}{3}(3x-2) + c$

Here, $f(x) = \ln(3x-2)$
$f'(x) = \frac{3}{3x-2}$
$g(x) = 1$
$G(x) = x$

Let, $3x - 2 = u$
$\frac{du}{dx} = 3$
$du = 3dx$
$u + 2 = 3x$
$\therefore x = \frac{u+2}{3}$

* $\int \ln(1+x)\,dx$

$= x\ln(1+x) - \int \frac{1}{1+x} \cdot x\,dx$

$= x\ln(1+x) - \int \frac{u-1}{u}\,du$

$= x\ln(1+x) - \int \left(1-\frac{1}{u}\right)du$

$= x\ln(1+x) - u + \ln u + c$

$= x\ln(1+x) - (1+x) + \ln(1+x) + c$

Let, $1+x=u$.
$dx = du$,
$x = u-1$.
$f(x) = \ln(1+x)$
$f'(x) = \frac{1}{1+x}$
$g(x) = 1$.
$G(x) = x$.

14. $\int \ln(x^2+4)\,dx$

$= \ln(x^2+4)\cdot x - \int \frac{2x}{x^2+4}\cdot x\,dx$

$= x\ln(x^2+4) - \int \frac{2x^2}{x^2+4}\,dx$

$= x\ln(x^2+4) - 2\int \frac{x^2}{x^2+4}\,dx$

$= x\ln(x^2+4) - 2\int \frac{x^2+4-4}{x^2+4}\,dx$

$= x\ln(x^2+4) - 2\left[\int 1\,dx - 4\int \frac{1}{x^2+4}\,dx\right]$

$= x\ln(x^2+4) - 2x + \frac{8}{2}\tan^{-1}\frac{x}{2} + c$

$= x\ln(x^2+4) - 2x + 4\tan^{-1}\frac{x}{2} + c$.

Here,
$f(x) = \ln(x^2+4)$, $f'(x) = \frac{1}{x^2+4}\cdot 2x$
$g(x) = 1$. $G(x) = x$

Note, $\int \frac{1}{x^2+a^2}\,dx$
$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c$.

15. $\int \sin^{-1}x\,dx = x\sin^{-1}x - \int x\cdot\frac{1}{\sqrt{1-x^2}}\,dx$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$.

$= x\sin^{-1}x - \int -\frac{1}{2}\frac{du}{\sqrt{u}}$.

$= x\sin^{-1}x + \frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right) + c$

$= x\sin^{-1}x + \sqrt{1-x^2} + c$. Ans

Here.
$f(x) = \sin^{-1}x$
$f'(x) = \frac{1}{\sqrt{1-x^2}}$
$g(x) = 1$. $G(x) = x$.

Let, $u = 1-x^2$
$du = -2x\,dx$

16. $\int \cos^{-1}(2x)\,dx$

$= \cos^{-1}(2x)\,x - \int \frac{-2}{\sqrt{1-4x^2}}\cdot x\,dx$

$= x\cos^{-1}(2x) + 2\int \frac{x}{\sqrt{1-4x^2}}\,dx$

$= x\cos^{-1}(2x) - 2\cdot\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du$

$= x\cos^{-1}(2x) - \frac{1}{4}\left(\frac{u^{1/2}}{1/2}\right) + c$

$= x\cos^{-1}(2x) - \frac{1}{2}\sqrt{1-4x^2} + c$

Here,

$f(x) = \cos^{-1}2x$

$f'(x) = \frac{-2}{\sqrt{1-4x^2}}$

$g(x) = 1,\ G(x) = x$

Let, $u = 1-4x^2$

$\frac{du}{dx} = -8x$

$\frac{du}{-8} = x\,dx$

17. $\int \tan^{-1}(3x)\,dx$

$= x\tan^{-1}(3x) - \int \frac{3}{1+9x^2}\cdot x\,dx$

$= x\tan^{-1}(3x) - 3\cdot\frac{1}{18}\int \frac{1}{u}\cdot du$

$= x\tan^{-1}(3x) - \frac{1}{6}\ln u$

$= x\tan^{-1}3x - \frac{1}{6}\ln(1+9x^2) + c$

Here,

$f(x) = \tan^{-1}(3x)$

$f'(x) = \frac{1}{1+9x^2}\cdot 3$

$g(x) = 1,\ G(x) = x$

Let, $u = 1+9x^2$

$\frac{du}{dx} = 18x$

18. $\int x\tan^{-1}x\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2}\cdot\frac{1}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1}\,dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + c$

Here.

$f(x) = \tan^{-1}x$

$f'(x) = \frac{1}{1+x^2}$

$g(x) = x$

$G(x) = \frac{x^2}{2}$

19. $\int e^x \sin x \, dx$

$= e^x \sin x - \int \cos x \cdot e^x dx$

$= e^x \sin x - e^x \cos x + \int \sin x \cdot e^x$

$\Rightarrow 2\int e^x \sin x = e^x \sin x - e^x \cos x$

$\Rightarrow 2\int e^x \sin x = e^x(\sin x - \cos x)$

$\therefore \int e^x \sin x = \frac{e^x}{2}(\sin x - \cos x)$

Here:
$f(x) = \sin x$
$f'(x) = \cos x$
$g(x) = e^x$. $G(x) = e^x$.

20. $\int e^{3x} \cos 2x \, dx$

$= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\int e^{3x} \sin 2x \, dx$

$= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{3}\left[\frac{1}{3}e^{3x}\sin 2x - \frac{2}{3}\int e^{3x}\cos 2x \, dx\right]$

$= \frac{1}{3}e^{3x}\cos 2x + \frac{2}{9}e^{3x}\sin 2x - \frac{4}{9}\int e^{3x}\cos 2x \, dx$

$\Rightarrow \frac{13}{9}\int e^{3x}\cos 2x \, dx = \frac{1}{3}e^{3x}\cdot\cos 2x + \frac{2}{9}e^{3x}\sin 2x$

$\therefore \int e^{3x}\cos 2x \, dx = \frac{3e^{3x}}{13}\cos 2x + \frac{2e^{3x}}{13}\sin 2x + c$

Here:
$f(x) = \cos 2x$
$f'(x) = -\sin 2x \cdot 2$
$g(x) = e^{3x}$
$G(x) = \frac{e^{3x}}{3}$.

22. $\int \cos(\ln x)dx = \int \cos u \cdot e^u du$

$= \int e^u \cos u \, du$

$= e^u \cdot \cos u + \int e^u \cdot \sin u \, du$

$= e^u \cos u + e^u \sin u - \int e^u \cos u \, du$

$\Rightarrow 2\int e^u \cos u \, du = e^u \cos u + e^u \sin u$

$\Rightarrow \int \cos u \cdot e^u du = \frac{e^u}{2}(\cos u + \sin u)$

$\therefore \int \cos(\ln x)dx = \frac{e^{\ln x}}{2}\left[\cos(\ln x) + \sin(\ln x)\right]$

Let, $u = \ln x$
or, $e^u = x$
or, $e^u du = dx$.

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21. $\int \sin(\ln x)dx$.

$= \int \sin u \cdot e^u du = \int e^u \cdot \sin u \cdot du$

$= \sin u \cdot e^u - \int \cos u \cdot e^u du$

$= \sin u \cdot e^u - \cos u \cdot e^u - \int \sin u \cdot e^u du$

$\Rightarrow 2\int \sin u \cdot e^u du = e^u(\sin u - \cos u)$

$\Rightarrow \int \sin u\, e^u du = \frac{e^u}{2}(\sin u - \cos u)$.

$\therefore \int \sin(\ln x)\, dx = \frac{e^{\ln x}}{2}\left[\sin(\ln x) - \cos(\ln x)\right] + c$.

Let,
$\ln x = u$.
or, $e^u = x$
or, $e^u du = dx$.
$f(x) = \sin u$
$f'(x) = \cos u$.
$g(x) = e^u$
$G(x) = e^u$.

23. $\int x \sec^2 x\, dx$

$= x\tan x - \int 1 \cdot \tan x\, dx$

$= x\tan x - \int \frac{\sin x}{\cos x} dx$

$= x\tan x + \int \frac{1}{z} dz$

$= x\tan x + \ln(\cos x) + c$.

Here,
$f(x) = x$
$f'(x) = 1$.
$g(x) = \sec^2 x$
$G(x) = \tan x$.

Let, $z = \cos x$
$dz = -\sin x\, dx$

24. $\int x \tan^2 x\, dx$.

$= x\tan x - x^2 - \int (\tan x - x)\, dx$

$= x\tan x - x^2 - \left[-\ln(\cos x) - \frac{x^2}{2}\right]$

$= x\tan x - x^2 + \ln(\cos x) + \frac{x^2}{2} + c$

$= x\tan x - \frac{x^2}{2} + \ln(\cos x) + c$.

Here.
$f(x) = x$, $f'(x) = 1$.
$g(x) = \tan^2 x$
$G(x) = \int \tan^2 x\, dx$
$= \int (\sec^2 x - 1)\, dx$
$= \tan x - x$.

Let, $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$
$= \int \frac{-du}{u}$
$= -\ln|\cos x|$

Some Formula→

* $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

* $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$.

* $\int \frac{1}{\sqrt{a^2+x^2}}dx = \ln\left|x+\sqrt{a^2+x^2}\right|$.

$\int \frac{1}{2x^2+5x+3}dx$.

$= \int \frac{1}{2(x^2+\frac{5}{2}x+\frac{3}{2})}dx$.

$= \frac{1}{2}\int \frac{1}{x^2+2x\cdot\frac{5}{4}+(\frac{5}{4})^2+\frac{3}{2}-\frac{25}{16}}dx$.

$= \frac{1}{2}\int \frac{1}{(x+\frac{5}{4})^2+(\frac{-1}{16})}dx$.

$= \frac{1}{2}\int \frac{1}{(x+5/4)^2-(1/4)^2}dx$.

$= \frac{1}{2}\cdot\frac{1}{2\cdot\frac{1}{4}}\ln\left|\frac{x+\frac{5}{4}-\frac{1}{4}}{x+\frac{5}{4}+\frac{1}{4}}\right|+c$.

Let

$u=\sin^{-1}x$.

$du=\frac{1}{\sqrt{1-x^2}}dx$.

$x=\sin u$.

$\int \frac{x\cdot\sin^{-1}x}{\sqrt{1-x^2}}dx$.

$= \int \sin u\cdot u\cdot du$

$= \int u\sin u\cdot du$

$= -\cos u\cdot u+\int\cos u\,du$.

$= -u\cos u+\sin u+c$

$\Rightarrow -\sin^{-1}x\cos(\sin^{-1}x)+\sin(\sin^{-1}x)+c$.

$\int \frac{x\tan^{-1}x}{\sqrt{1+x^2}}dx$.

$= \tan^{-1}x\cdot\sqrt{1+x^2}-\int\frac{1}{1+x^2}\cdot\sqrt{1+x^2}\,dx$

$= \tan^{-1}x\cdot\sqrt{1+x^2}-\int\frac{1}{\sqrt{1+x^2}}dx$

$= \tan^{-1}x\sqrt{1+x^2}-\ln\left|x+\sqrt{1+x^2}\right|+c$.

Here

$f(x)=\tan^{-1}x,\ f'(x)=\frac{1}{1+x^2}$.

$g(x)=\frac{x}{\sqrt{1+x^2}}$.

$G(x)=\int\frac{x}{\sqrt{1+x^2}}dx$.

Let, $u=1+x^2$, $\frac{du}{2}=x\,dx$

$=\frac{1}{2}\int\frac{du}{\sqrt{u}}dx$

$=\frac{1}{2}\cdot\frac{\sqrt{u}}{1/2}=\sqrt{u}$

$=\sqrt{1+x^2}$.

Integrating Rational Functions By Partial Fractions — 7.5

Fraction: (i) Proper fraction $\rightarrow \frac{2}{3}, \frac{5}{8}, \frac{-3}{7}$ $\left[\frac{a}{b}, a<b\right]$

(ii) Improper fraction $\rightarrow \frac{5}{2}, \frac{7}{3}$ $[a>b]$

(iii) Mixed fraction $\rightarrow 1\frac{1}{2}, 2\frac{1}{3}$.

$\frac{2x-8}{x^2+4x+4} \rightarrow$ proper fraction.

$\frac{3x^2-10}{x-3} \rightarrow$ Improper.

$$\begin{array}{r|l|l} x-3 & 3x^2-10 & 3x \\ & 3x^2-9x & \\ \hline & 9x-10 & \end{array}$$

$$= 3x + \frac{9x-10}{x-3}$$

$$= 3x + \frac{9x}{x-3} - \frac{10}{x-3}.$$

$\frac{3x^2-10}{x^2-4x+4}$.

$$\begin{array}{r|l|l} x^2-4x+4 & 3x^2-10 & 3 \\ & 3x^2-12x+2 & \\ \hline & 12x-22 & \end{array}$$

$$= 3 + \frac{12x-22}{x^2-4x+4}.$$

* $\frac{x+3}{x^3+3x^2+2} = \frac{Ax^2+Bx+C}{x^3+3x^2+2}.$

* $\frac{2x+3}{x^2+2} = \frac{Ax+B}{x^2+2}.$

* $\frac{1-x^2}{x^3(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1}.$

* $\frac{1-3x^2}{(x-2)(x^2+1)^2} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$

* $\frac{1}{(ax^2+bx+c)^2} = \frac{Ax+B}{(ax^2+bx+c)^2} + \frac{Cx+D}{ax^2+bx+c}.$

* $\frac{1}{(ax^2+bx+c)(x+d)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{x+d}.$

Example 1:

$$\int \frac{dx}{x^2+x-2}\,dx$$

Integrand: $\dfrac{1}{x^2+x-2} = \dfrac{1}{(x-1)(x+2)}$

Let, $\dfrac{1}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$

or, $1 = A(x+2) + B(x-1)$

Now,
$x=-2$ $\quad$ $x=1$
$B = -\frac{1}{3}$. $\quad$ $A = \frac{1}{3}$.

$$\int \left(\frac{1/3}{x-1} + \frac{-1/3}{x+2} \right) dx$$

$$= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{dx}{x+2}$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + c$$

Example 2:

$$\int \frac{2x+4}{x^3-2x^2}\,dx.$$

Integrand: $\dfrac{2x+4}{x^3-2x^2} = \dfrac{2x+4}{x^2(x-2)}$

Let, $\dfrac{2x+4}{x^2(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2}$.

$\Rightarrow 2x+4 = Ax(x-2) + B(x-2) + Cx^2$

$= Ax^2 - 2Ax + Bx - 2B + Cx^2$

$= (A+C)x^2 + (-2A+B)x - 2B$.

$A=-2$. $B=-2$. $C=2$.

$$\int \left(\frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2} \right) dx$$

$$= -2\int \frac{dx}{x} - 2\int \frac{dx}{x^2} + 2\int \frac{dx}{x-2}$$

$$= -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + c$$

Ans

Example-3:

$$\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx.$$

Integrand : $= \dfrac{x^2+x-2}{x^2(3x-1)+1(3x-1)}$

$$= \frac{x^2+x-2}{(3x-1)(x^2+1)}$$

Let, $\dfrac{x^2+x-2}{(3x-1)(x^2+1)} = \dfrac{A}{3x-1} + \dfrac{Bx+C}{x^2+1}$

or, $x^2+x-2 = A(x^2+1)+(Bx+C)(3x-1)$

$$= Ax^2+A+3Bx^2-Bx+3Cx-C$$

$$= (A+3B)x^2+(-B+3C)x+(A-C)$$

$$\begin{bmatrix} A+3B & & =1 \\ & -B+3C & =1 \\ A & & C=-2 \end{bmatrix}$$

Now, $A=-1.4$, $B=0.8$, $C=0.6$

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{-1.4}{3x-1} + \frac{0.8x+0.6}{x^2+1}.$$

$$\int \frac{x^2+x-2}{(3x-1)(x^2+1)}\,dx = -1.4\int \frac{dx}{3x-1} + 0.8\int \frac{x}{x^2+1}\,dx + 0.6\int \frac{1}{x^2+1}\,dx$$

$$= -1.4\ln|3x-1| - 0.4\ln|x^2+1| + 0.6\tan^{-1}x + c.$$

Example 4:

$$\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx.$$

$$= \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{dx+E}{(x^2+3)^2}$$

or, $3x^4+4x^3+16x^2+20x+9 = A(x^2+3)^2 + (Bx+C)(x+2)(x^2+3) + (dx+E)(x+2)$. — ①

$= A(x^4+6x^2+9) + (Bx+C)(x^3+3x+2x^2+6) + Dx^2+Ex+2Dx+2E$

$= (A+B)x^4 + (2B+C)x^3 + (6A+3B+2C+D)x^2$

$+ (6B+3C+2D+E)x + (9A+6C+2E)$ — ①

Here, $A+B=3$

$2B+C=4$

$6A+3B+2C+D=16$.

$6B+3C+2D+E=20$.

$9A+6C+2E=9$

Now, puttin the value of $x=-2$ in equation ①. we can find $A=1$.

Hence, $B=2$

$2B+C=4$

$3B+2C+D=10$

$6B+3C+2D+E=20$

$6C+2E=0$.

Now, $A=1$, $B=2$, $C=0$, $D=4$, $E=0$.

Now, $$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{1}{x+2} + \frac{2x}{x^3+3} + \frac{4x}{(x^2+3)^2}$$

P.t.o

So. $$\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx$$

$$= \int \frac{dx}{x+2} + \int \frac{2x}{x^2+3}\,dx + 4\int \frac{x}{(x^2+3)^2}\,dx$$

$$= \ln|x+2| + \ln|x^2+3| - \frac{2}{x^2+3} + c.$$

Example 5: $$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx.$$

$$\begin{array}{r|l|l}
x^2+x-2 & 3x^4+3x^3-5x^2+x-1 & 3x^2+1 \\
 & \underline{3x^4+3x^3-6x^2} & \\
 & x^2+x-1 & \\
 & \underline{x^2+x-2} & \\
 & 1 &
\end{array}$$

Integrand can be expressed,

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = (3x^2+1) + \frac{1}{x^2+x-2}.$$

Hence, $$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = \int (3x^2+1)\,dx + \int \frac{dx}{x^2+x-2}$$ → from example '1'

$$= x^3+x+\frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + c.$$

Note:

$$\int \frac{3x^2+2}{x^3+2x-8}\,dx = \ln|x^3+2x-8| + c$$

$$\int \frac{2x-1}{2x^2+1}\,dx = \int \frac{2x}{x^2+1}\,dx - \int \frac{dx}{x^2+1}$$

$$= \ln(x^2+1) - \tan^{-1}x + c.$$

It would be inefficient to use partial fractions to perform the integration. Since the integrand is already in partial fraction form.

Exercise set 7.5 ⇒

9. $\int \frac{dx}{x^2-3x-4} = \int \frac{dx}{x^2-4x+x-4}$

$= \int \frac{dx}{(x-4)(x+1)}$.

Now, $x=4$, $A=\frac{1}{5}$.

$x=-1$, $B=-\frac{1}{5}$.

Let, $\frac{1}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1}$

or, $1 = A(x+1)+B(x-4)$.

$\therefore \int \frac{1/5}{(x-4)}dx - \int \frac{1/5}{(x+1)}dx$

$= \frac{1}{5}\ln(x-4) - \frac{1}{5}\ln(x+1)+c$.

10. $\int \frac{dx}{x^2-6x-7} = \int \frac{dx}{x^2-7x+x-7}$

$= \int \frac{dx}{(x-7)(x+1)}$.

Let, $\frac{1}{(x-7)(x+1)} = \frac{A}{(x-7)} + \frac{B}{(x+1)}$

or, $1 = A(x+1)+B(x-7)$

Now, $x=-1$, $1=-8B$ $\therefore B=-\frac{1}{8}$

$x=7$, $1=8A$ $\therefore A=\frac{1}{8}$.

$\therefore \int \frac{1/8}{(x-7)}dx - \int \frac{1/8}{(x+1)}dx$

$= \frac{1}{8}\ln(x-7) - \frac{1}{8}\ln(x+1)+c$.

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11. $\int \frac{11x+17}{2x^2+7x-4}dx = \int \frac{11x+17}{(2x-1)(x+4)}dx$

$2x^2+7x-4=0$
or. $2x^2+8x-x-4=0$
or. $2x(x+4)-1(x+4)=0$
$(x+4)(2x-1)=0$.

Let. $\frac{11x+17}{(2x-1)(x+4)} = \frac{A}{2x-1} + \frac{B}{x+4}$

or. $11x+17 = A(x+4)+B(2x-1)$. —①.

now, $x=-4$ ① →
$11(-4)+17 = A\cdot 0 + B(-8-1)$
$\therefore B = 3$.

$x=\frac{1}{2}$ ① → $11\cdot\frac{1}{2}+17 = A(\frac{1}{2}+4)+B\cdot 0$

$\therefore A=5$

$\therefore \int \frac{5}{2x-1}dx + \int \frac{3}{x+4}dx$

* $= \frac{5}{2}\ln(2x-1) + 3\ln(x+4)+c$.

12. $\int \frac{5x-5}{3x^2-8x-3}dx = \int \frac{5x-5}{(3x+1)(x-3)}dx$

$3x^2-8x-3$
$=3x^2-9x+x-3$
$=(3x+1)(x-3)$.

Let. $\frac{5x-5}{(3x+1)(x-3)} = \frac{A}{3x+1} + \frac{B}{x-3}$.

or. $5x-5 = A(x-3)+B(3x+1)$.

now. $x=3$. $10=10B$ $\therefore B=1$.
$x=\frac{-1}{3}$. $-5/3-5 = A(-\frac{1}{3}-3)$
$A=2$.

$\therefore \int \frac{2}{3x+1}dx + \int \frac{1}{x-3}dx$

$= \frac{2}{3}\ln(3x+1) + \ln(x-3)+c$.

13. $\int \frac{2x^2-9x-9}{x^3-9x}dx = \int \frac{2x^2-9x-9}{x(x^2-9)}dx = \int \frac{2x^2-9x-9}{x(x+3)(x-3)}dx$.

Let, $\frac{2x^2-9x-9}{x(x+3)(x-3)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-3}$.

or. $2x^2-9x-9 = A(x+3)(x-3) + B(x-3)x + Cx(x+3)$.

$= A(x^2-3x+3x-9) + Bx^2-3Bx+Cx^2+3Cx$

$= Ax^2 - \cancel{3Ax} + \cancel{3Ax} - 9A + Bx^2 - 3Bx + Cx^2 + 3Cx$

$= x^2(A+B+C) + x(3C-3B) - 9A$

now, $A+B+C = 2$

$3C-3B = -9$

$-9A = -9$.

So, $A=1$, $B=2$, $C=-1$. $\boxed{\text{By calculator}}$

$\therefore \int \frac{1}{x}dx + \int \frac{2}{x+3}dx + \int \frac{-1}{x-3}$

$= \ln x + 2\ln(x+3) - \ln(x-3) + c$.

14. $\int \frac{dx}{x(x^2-1)}dx = \int \frac{dx}{x(x+1)(x-1)}$.

Let, $\frac{1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$

or, $1 = A(x+1)(x-1) + B(x-1)x + C(x+1)x$.

or, $1 = A(x^2-1) + Bx(x-1) + Cx(x+1)$.

now, $x=1$, $1=2C$ $\therefore C=\frac{1}{2}$

$x=-1$, $1=2B$ $\therefore B=\frac{1}{2}$

$x=0$, $1=-A$ $\therefore A=-1$.

p.t.o

$$\therefore \int \frac{-1}{x}\,dx + \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{1}{x-1}\,dx.$$

$$= -\ln x + \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) + c.$$

15. $\int \frac{x^2-8}{x+3}\,dx$

Here. $x+3=z$. $dx=dz$.

$(z-3)^2-8$

$= z^2-6z+9-8$

$= z^2-6z+1$.

$= \int \frac{z^2-6z+1}{z}\,dz$.

$= \int \left(z-6+\frac{1}{z}\right)dz$.

$= \frac{1}{2}z^2-6z+\ln z+c$

$= \frac{1}{2}(x+3)^2-6(x+3)+\ln(x+3)+c$.

17. $\int \frac{3x^2-10}{x^2-4x+4}\,dx$.

$$x^2-4x+4 \,\big)\, 3x^2-10 \,\big(\, 3$$
$$3x^2-12x+12$$
$$\overline{12x-22}$$

$= 3+\frac{12x-22}{x^2-4x+4}$

$= 3+\frac{12x-22}{(x-2)^2}$.

Let, $\frac{12x-22}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$

or, $12x-22 = A(x-2)+B$

now. $x=2$, $A=12$

$\therefore B=2$.

$$\therefore \int\left(3+\frac{12x-22}{(x-2)^2}\right)dx$$

$$= \int 3\,dx + \int \frac{12}{x-2}\,dx + \int \frac{2}{(x-2)^2}$$

$$= 3x+12\ln(x-2)-\frac{2}{x-2}+c.$$

18. $\int \frac{x^2}{x^2-3x+2} dx$.

$$x^2-3x+2 \,\Big)\, x^2 \,\Big(\, 1$$
$$\underline{x^2-3x+2}$$
$$3x-2$$

$$= 1 + \frac{3x-2}{x^2-3x+2}$$

$$= 1 + \frac{3x-2}{(x-2)(x-1)}.$$

Let, $\frac{3x-2}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$.

or, $3x-2 = A(x-1) + B(x-2)$.

now, $x=1$ $B=-1$

$x=2$ $A=4$

$\therefore \int \left(1 + \frac{3x-2}{(x-2)(x-1)}\right) dx$

$= \int 1\,dx + \int \frac{4/3}{x-2} dx + \int \frac{-1}{x-1} dx$

$= x + 4 \ln(x-2) - \ln(x-1) + c$.

19. $\int \frac{2x-3}{x^2-3x-10} dx$.

$$x^2-3x-10$$
$$x^2-5x+2x-10$$
$$(x-5)(x+2).$$

$= \int \frac{2x-3}{(x+2)(x-5)} dx$

Let, $\frac{2x-3}{(x+2)(x-5)} = \frac{A}{x+2} + \frac{B}{x-5}$

or, $2x-3 = A(x-5) + B(x+2)$.

now, $x=5$, $7=7B$. $\therefore B=1$.

$x=-2$, $-7=-7A$ $\therefore A=1$.

$\therefore \int \frac{1}{x+2} dx + \int \frac{1}{x-5} dx$

$= \ln(x+2) + \ln(x-5) + c$.

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20. $\int \frac{3x+1}{3x^2+2x-1}\,dx = \int \frac{3x+1}{(3x-1)(x+1)}\,dx.$

$3x^2+2x-1$
or, $3x^2+3x-x-1$
or, $3x(x+1)-1(x+1)$
$\therefore (x+1)(3x-1).$

Let, $\frac{3x+1}{(3x-1)(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1}.$

or, $3x+1 = A(x+1)+B(3x-1).$

now, $x=-1$, $B=\frac{1}{2}$

$x=\frac{1}{3}$, $A=\boxed{\frac{3}{4-2}}=\frac{3}{2}.$

$\therefore \int \frac{3/2}{3x-1}\,dx + \int \frac{1/2}{x+1}\,dx = \frac{3}{2}\ln(3x-1)+\frac{1}{2}\ln(x+1)+c.$

21. $\int \frac{x^5+x^2+2}{x^3-x}\,dx.$

$$\begin{array}{r|l|l} x^3-x & x^5+x^2+2 & x^2+1 \\ & \underline{x^5-x^3} & \\ & x^3+x^2+2 & \\ & \underline{x^3-x} & \\ & x^2+x+2 & \end{array}$$

$=(x^2+1)+\frac{x^2+x+2}{x^3-x}$

$=(x^2+1)+\frac{x^2+x+2}{x(x^2-1)}$

$=(x^2+1)+\frac{x^2+x+2}{x(x+1)(x-1)}.$

Let, $\frac{x^2+x+2}{x(x+1)(x-1)} = \frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}.$

or, $x^2+x+2 = A(x+1)(x-1)+B(x-1)x$
$+C(x+1)x.$

now, $x=1$, $A=-2$
$x=-1$, $B=1$
$x=0$, $C=2$.

$\therefore \int \left(x^2+1+\frac{x^2+x+2}{x^3-x}\right)dx$

$=\int (x^2+1)\,dx + \int\left(\frac{-2}{x}+\frac{1}{x+1}+\frac{2}{x-1}\right)dx$

$=\frac{x^3}{3}+x-2\ln(x)+\ln(x+1)+2\ln(x-1)+c.$

22. $\int \frac{x^5-4x^3+1}{x^3-4x}\,dx$

$$x^3-4x \,)\, x^5-4x^3+1 \,(\, x^2$$
$$\underline{x^5-4x^3}$$
$$1$$

$= x^2 + \frac{1}{x^3-4x}$

$= x^2 + \frac{1}{x(x^2-4)}$

$= x^2 + \frac{1}{x(x+2)(x-2)}$

Let, $\frac{1}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$

or, $1 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$

or, $1 = A(x^2-4) + Bx(x-2) + Cx(x+2)$.

now, $x=2$, $C = \frac{1}{8}$

$x=-2$, $B = \frac{1}{8}$

$x=0$, $A = -\frac{1}{4}$.

$\therefore \int \left(x^2 + \frac{1}{x(x+2)(x-2)}\right) dx$

$= \int x^2\,dx + \int \frac{-1/4}{x}\,dx + \int \frac{1/8}{x+2}\,dx + \int \frac{1/8}{x-2}\,dx$

$= \frac{x^3}{3} - \frac{1}{4}\ln(x) + \frac{1}{8}\ln(x+2) + \frac{1}{8}\ln(x-2) + c$.

23. $\int \frac{2x^2+3}{x(x-1)^2}\,dx$.

Let, $\frac{2x^2+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

or, $2x^2+3 = A(x-1)^2 + Bx(x-1) + Cx$.

24. $\int \frac{3x^2-x+1}{x^3-x^2}\,dx$

$= \int \frac{3x^2-x+1}{x^2(x-1)}\,dx$

Let, $\frac{3x^2-x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$.

or, $3x^2-x+1 = Ax(x-1) + B(x-1) + Cx^2$.

25. $$\int \frac{2x^2-10x+4}{(x+1)(x-3)^2}\,dx.$$

Let, $$\frac{2x^2-10x+4}{(x+1)(x-3)^2} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

26. $$\int \frac{2x^2-2x-1}{x^3-x^2}\,dx = \int \frac{2x^2-2x-1}{x^2(x-1)}\,dx.$$

Let, $$\frac{2x^2-2x-1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

27. $$\int \frac{x^2}{(x+1)^3}\,dx.$$

Let, $$\frac{x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

This can also be done in another way.

$$= \int \frac{(z-1)^2}{z^3}\,dz$$

$$= \int \frac{z^2-2z+1}{z^3}\,dz$$

$$= \int \frac{1}{z}\,dz - 2\int \frac{1}{z^2}\,dz + \int \frac{1}{z^3}\,dz.$$

$$= \ln(z) + 2\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} + c$$

$$= \ln(x+1) + \frac{2}{(x+1)} - \frac{1}{2(x+1)^2} + c.$$

29. $$\int \frac{2x^2-1}{(4x-1)(x^2+1)}\,dx$$

Let, $$\frac{2x^2-1}{(4x-1)(x^2+1)} = \frac{A}{4x-1} + \frac{Bx+C}{x^2+1}.$$

or, $2x^2-1 = A(x^2+1)+(Bx+C)(4x-1)$

30. $$\int \frac{dx}{x^3+2x} = \int \frac{dx}{x(x^2+2)}$$

Let, $$\frac{1}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}.$$

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28. $\displaystyle\int \frac{2x^2+3x+3}{(x+1)^3}\,dx.$

Let. $\displaystyle\frac{2x^2+3x+3}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$

31. $\displaystyle\int \frac{x^3+3x^2+x+9}{(x^2+1)(x^2+3)}\,dx.$

Let. $\displaystyle\frac{x^3+3x^2+x+9}{(x^2+1)(x^2+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3}.$

16. $\displaystyle\int \frac{x^2+1}{x-1}\,dx.$

Integrating Trigonometric Functions — 7.3

\# $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$

\# $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$

\# $\cos 2x = 1 - 2\sin^2 x.$

$\cos 2x = 2\cos^2 x - 1.$

\# $\int \sin^4 x\,dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c.$ \# $\int \sin^3 x\,dx = \frac{1}{3}\cos^3 x - \cos x + c.$

\# $\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c.$ \# $\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + c.$

$\int \sin^m x \cos^n x\,dx$	Procedure	Relevant Indentities.
n odd	1. Split off a factor of $\cos x$ 2. Apply relevant identity 3. Make the substitution $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$
m odd	1. Split offs a factor of $\sin x$ 2. Apply relevant identity 3. Make the substitution $u = \cos x$	$\sin^2 x = 1 - \cos^2 x$
m even n even	1. Use the relevant identities to reduce the powers on $\sin x$ and $\cos x$	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

2. a) $\int \sin^4 x \cos^5 x\,dx = \int \sin^4 x \cos^4 x \cdot \cos x\,dx$

Let, $u = \sin x$

$du = \cos x\,dx$

$$= \int \sin^4 x (1 - \sin^2 x)^2 \cos x\,dx$$

$$= \int u^4 (1 - u^2)^2\,du$$

$$= \int (u^4 - 2u^6 + u^8)\,du$$

$$= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + c$$

$$= \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + c.$$

b). $\int \sin^4 x \cos^4 x\, dx$

$= \int \left[\frac{1}{2}(1-\cos 2x)\right]^2 \cdot \left[\frac{1}{2}(1+\cos 2x)\right]^2 dx$

$= \frac{1}{16}\int (1-\cos 2x)^2 (1+\cos 2x)^2 dx$

$= \frac{1}{16}\int (1-\cos^2 2x)^2 dx$

$= \frac{1}{16}\int (\sin^2 2x)^2 dx$

$= \frac{1}{16}\int \sin^4 2x\, dx$

Let, $u = 2x$

$du = 2dx$

$\therefore \frac{du}{2} = dx$

$= \frac{1}{32}\int \sin^4 u\, du$

$= \frac{1}{32}\left[\frac{3}{8}u - \frac{1}{4}\sin 2u + \frac{1}{32}\sin 4u\right] + c$ formula $\int \sin^4 x\, dx$

$= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + c$

* $\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$
* $\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$
* $\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

Example 3: $\int \sin 7x \cos 3x\, dx = \frac{1}{2}\int [\sin(7x-3x) + \sin(7x+3x)]dx$

$= \frac{1}{2}\int (\sin 4x + \sin 10x)\, dx$

$= -\frac{1}{8}\cos 4x - \frac{1}{20}\cos 10x + c$

$\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

$\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

$\int \tan x\,dx = \ln|\sec x| + c$.   # $\int \sec x\,dx = \ln|\sec x + \tan x| + c$.

$\int \tan^2 x\,dx = \tan x - x + c$.   # $\int \sec^2 x\,dx = \tan x + c$.

* $1 + \tan^2 x = \sec^2 x$.

$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + c$.

$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + c$.

$\int \tan^m x \sec^n x\,dx$	Procedure	Relevant Identities.
n even	1. Split off a factor of $\sec^2 x$ 2. Apply the relevant identity 3. Make the substitution $u = \tan x$.	$\sec^2 x = 1 + \tan^2 x$.
m odd.	1. Split off a factor of $\sec x \tan x$ 2. Apply the relevant identity. 3. Make the substitution $u = \sec x$	$\tan^2 x = \sec^2 x - 1$.
m even n odd	1. Use the relevant identities to reduce the integrand to powers of $\sec x$ alone. 2. Then use the reduction formula for powers of $\sec x$.	$\tan^2 x = \sec^2 x - 1$.

Example 4: a. $\int \tan^2 x \sec^4 x\,dx$.

$= \int \tan^2 x \cdot \sec^2 x \cdot \sec^2 x\,dx$

$= \int \tan^2 x\,(1 + \tan^2 x)\sec^2 x\,dx$ Let, $u = \tan x$, $du = \sec^2 x\,dx$.

$= \int u^2(u^2+1)\,du$

$= \frac{1}{5}u^5 + \frac{1}{3}u^3 + c$

$= \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + c$.

b). $\int \tan^3 x \sec^3 x\, dx$

$= \int \tan^2 x \sec^2 x (\sec x \cdot \tan x)\, dx$

$= \int (\sec^2 x - 1)\sec^2 x (\sec x \cdot \tan x)\, dx$

Let, $u = \sec x$
$du = \sec x \tan x\, dx$

$= \int (u^2 - 1) u^2\, du$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + c$

$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + c.$

c). $\int \tan^2 x \sec x\, dx$

$= \int (\sec^2 x - 1)\sec x\, dx$

$= \int \sec^3 x\, dx - \int \sec x\, dx$

$= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| - \ln|\sec x + \tan x| + c$

$= \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + c.$

* $\int \sin^3 x\, dx = \int (\sin^2 x)\sin x\, dx$

$= \int (1 - \cos^2 x)\sin x\, dx$

$= -\int (1 - u^2)\, du$

Let, $u = \cos x$
$du = -\sin x\, dx.$

$= \frac{1}{3}u^3 - u + c$

$= \frac{1}{3}\cos^3 x - \cos x + c.$

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