Acknowledgement

All thanks to almighty Allah for granting me this opportunity to help the ones in need. And, to my instructor Sanjeeda Nazneen (SAN) whose efforts have perfected me in knowledge and also my special without whom this would have been impossible.

Note: Use this unofficial document at your own risk. There might be some errors and wrong answers as well, since I am not that good at Mathematics so please do consider that. I wasn’t able to include few topics just because my final examination was knocking at the door.

Domain and Range

1| $f(x) = \dfrac{1}{x-3}$

$x - 3 \neq 0$

$\Rightarrow x \neq 3$

Domain : $\mathbb{R} - \{3\}$

Range : $\mathbb{R} - \{0\}$

$y = \dfrac{1}{x-3}$

$\Rightarrow xy - 3y = 1$

$\Rightarrow xy = 1 + 3y$

$\Rightarrow x = \dfrac{1+3y}{y}$

$\therefore y \neq 0$

11| $f(x) = \dfrac{2x}{x-4}$

$x - 4 \neq 0$

$\Rightarrow x \neq 4$

Domain : $\mathbb{R} - \{4\}$

Range : $\mathbb{R} - \{2\}$

$y = \dfrac{2x}{x-4}$

$\Rightarrow xy - 4y = 2x$

$\cancel{\Rightarrow x = \dfrac{2x + 4y}{y}}$

$\Rightarrow xy - 2x = 4y$

$\Rightarrow x(y-2) = 4y$

$\Rightarrow x = \dfrac{4y}{y-2}$

$\therefore y \neq 2$

12| $f(x) = \dfrac{1}{5x+7}$

$5x + 7 \neq 0$

$\Rightarrow 5x \neq -7$

$\Rightarrow x \neq -7/5$

Domain: $\mathbb{R} - \{-7/5\}$

Range: $\mathbb{R} - \{0\}$

$y = \dfrac{1}{5x+7}$

$\Rightarrow 5xy + 7y = 1$

$\Rightarrow 5xy = 1 - 7y$

$\Rightarrow x = \dfrac{1-7y}{5y}$

$\therefore y \neq 0$

2) $f(x) = \sqrt{x^2-9}$

$x^2 - 9 \geq 0$

$\Rightarrow x^2 - 3^2 \geq 0$

$\Rightarrow (x+3)(x-3) \geq 0$ —— ①

$(x+3)(x-3) = 0$

$\Rightarrow x = -3, 3$

$-\infty$ ... -3 ... 3 ... $+\infty$

let $x=-4$	let $x=2$	let $x=4$
from ①, $(-4+3)(-4-3)$	$(2+3)(2-3)$	$(4+3)(4-3)$
$= 7 > 0$	$= -5 < 0$	$= 7 > 0$
✓	×	✓

Domain: $(-\infty, -3] \cup [3, \infty)$

Range: $[0, \infty)$

To find Range:

x	$-\infty$	-3	3	∞
y	∞	0	0	∞

3) $f(x) = \sqrt{9-x^2}$

$9 - x^2 \geq 0$

$\Rightarrow 3^2 - x^2 \geq 0$

$\Rightarrow (3+x)(3-x) \geq 0$ —— ①

$(3+x)(3-x) = 0$

$\Rightarrow x = -3, 3$

$-\infty$ ... -3 ... 3 ... $+\infty$

let $x=-4$	let $x=2$	let $x=4$
from ①, $(3-4)(3+4)$	$(3+2)(3-2)$	$(3+4)(3-4)$
$= -7 \not> 0$	$= 5 > 0$	$= -7 \not> 0$
×	✓	×

Domain: $[-3, 3]$

Range: $[0, 3]$

to find Range:

x	~~∞~~	-3	3	0
y		0	0	3

4] $f(x) = \sqrt{x^2 - 5x + 6}$

$x^2 - 5x + 6 \geq 0$

$\Rightarrow x^2 - 3x - 2x + 6 \geq 0$

$\Rightarrow x(x-3) - 2(x-3) \geq 0$

$\Rightarrow (x-2)(x-3) \geq 0$ — ①

$(x-2)(x-3) = 0$

$\Rightarrow x = 2, 3$

$-\infty \quad 2 \quad 3 \quad \infty$

let $x = 1$	let $x = 2.5$	let $x = 4$
$(1-2)(1-3)$	$(2.5-2)(2.5-3)$	$(4-2)(4-3)$
$= 2 > 0$	$= -0.25 \not> 0$	$= 2 > 0$
✓	×	✓

<u>to find range:</u>

x	$-\infty$	2	3	∞
y	∞	0	0	∞

Domain: $(-\infty, 2] \cup [3, \infty)$

Range: $[0, \infty)$

10] $f(x) = -\sqrt{x^2 - 7x + 10}$

$x^2 - 7x + 10 \geq 0$

$\Rightarrow x^2 - 2x - 5x + 10 \geq 0$

$\Rightarrow x(x-2) - 5(x-2) \geq 0$

$\Rightarrow (x-2)(x-5) \geq 0$ — ①

$(x-2)(x-5) = 0$

$\Rightarrow x = 2, 5$

$-\infty \quad 2 \quad 5 \quad +\infty$

let $x = 1$	let $x = 3$	let $x = 6$
$(1-2)(1-5)$	$(3-2)(3-5)$	$(6-2)(6-5)$
$= 4 > 0$	$= -2 \not> 0$	$= 4 > 0$
✓	×	✓

<u>to find range:</u>

x	$-\infty$	2	5	∞
y	$-\infty$	0	0	$-\infty$

Domain: $(-\infty, 2] \cup [5, \infty)$

Range: $(-\infty, 0]$

19| $f(x) = \sqrt{2x+4}$

$2x+4 \geq 0$

$\Rightarrow 2(x+2) \geq 0$

$\Rightarrow (x+2) \geq 0$ —— ①

$x+2=0$

$\Rightarrow x=-2$

$-\infty \longleftarrow \quad -2 \quad \longrightarrow +\infty$

let $x=-3$	let $x=3$
$(-3+2)$	$(3+2)$
$=-1 \ngeq 0$	$=5>0$
×	✓

to find range:

x	-2	∞
y	0	∞

Domain: $[-2, \infty)$

Range: $[0, \infty)$

7| $f(x) = \begin{cases} x+2 & , x \leq -1 \\ x^3 & , |x| < 1 \\ -x+3 & , x \geq 1 \end{cases}$

$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$

$= \begin{cases} x+2 & , x \leq -1 \\ x^3 & , -1 < x < 1 \\ -x+3 & , x \geq 1 \end{cases}$

Domain: $(-\infty, -1] \cup (-1, 1) \cup [1, \infty)$

$= \mathbb{R}$

Range: $(-\infty, 1] \cup (-1, 1) \cup [-\infty, 2]$

$= (-\infty, 2]$

8| $f(x) = \begin{cases} \frac{x^2-1}{x-1} & , x \neq 1 \\ 2 & , x = 1 \end{cases}$

Domain: $\mathbb{R} - \{1\} \cup \{1\}$

$= \mathbb{R}$

Range: $\mathbb{R}$

14| $f(x) = \begin{cases} x^2 & , x<0 \\ x & , 0 \leq x \leq 1 \\ 1/x & , x>1 \end{cases}$

Domain: $(-\infty, 0) \cup [0,1] \cup (1, \infty)$

$= \mathbb{R}$

Range: $(0, \infty) \cup [0,1] \cup (1, 0)$ $\quad [\because 1/\infty = 0]$

$= [0, \infty)$

15| $f(x) = \begin{cases} 2x+6 & , -3 \leq x \leq 0 \\ 6 & , 0<x<2 \\ 2x-6 & , 2 \leq x \leq 5 \end{cases}$

Domain: $[-3,0] \cup (0,2) \cup [2,5]$

$= [-3, 5]$

Range: $[0,6] \cup \{6\} \cup [-2,4]$

$= [-2, 6]$

5] $f(x) = \frac{x}{|x|}$

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

$|x| \neq 0$

$\Rightarrow x \neq 0$

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & , x > 0 \\ -1 & , x < 0 \end{cases}$$

Domain: $\mathbb{R} - \{0\}$

Range: $\{-1, 1\}$

6] $f(x) = x^3 + 2$

Domain: $\mathbb{R}$

Range: $\mathbb{R}$

9] $f(x) = 3\sin x$

Domain: $\mathbb{R}$

Range: $[-3, 3]$

13] $f(x) = \ln(x^2+1)$

$x^2 + 1 > 0$

for all $x \in R$, $x^2+1 > 0$

x	$-\infty$	-1	0	1	∞
y	$\ln\infty = \infty$	$\ln 2 = +ve$	$\ln 1 = 0$	$\ln 2 = +ve$	$\ln\infty = \infty$

Domain: $\mathbb{R}$

Range: $[0, \infty)$

16| $f(x) = \sin^2 x$

$= (\sin x)^2$

Domain : $\mathbb{R}$

Range : $[0, 1]$

17| $f(x) = e^x$

Domain: $\mathbb{R}$

Range : $(0, \infty)$

18| $f(x) = \operatorname{Log} x$

$= \operatorname{Log}_e x$

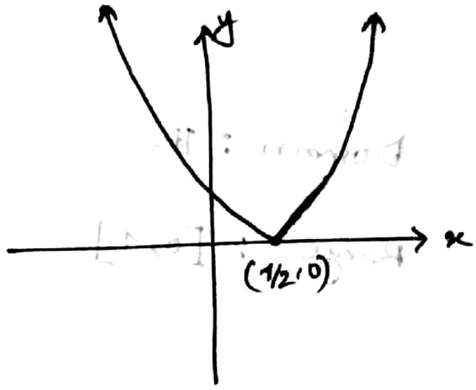
$= \ln x$

Domain: $(0, \infty)$

Range : $\mathbb{R}$

Sketch

1) $y = (2x-1)^2$



$y = 0$ if $(2x-1)^2 = 0$

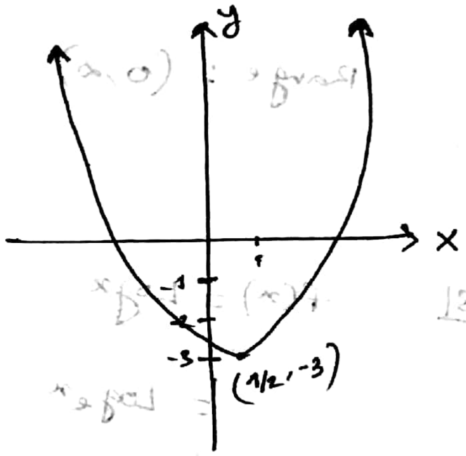
$\Rightarrow 2x - 1 = 0$

$\Rightarrow 2x = 1$

$\Rightarrow x = 1/2$

vertex: $(1/2, 0)$

2) $y = (2x-1)^2 - 3$



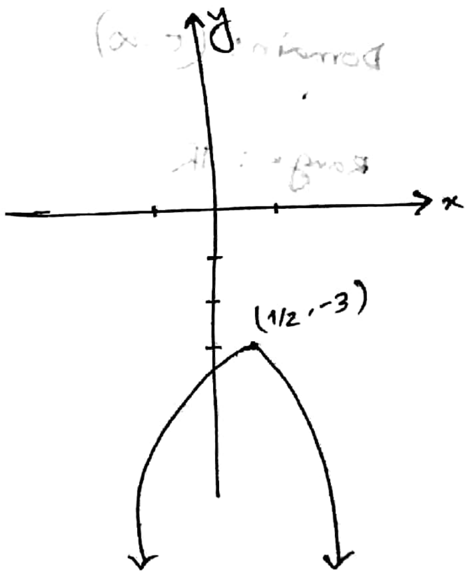
$\Rightarrow y + 3 = (2x-1)^2$

$y + 3 = 0$ if $(2x-1)^2 = 0$

$\Rightarrow y = -3$ $\Rightarrow x = 1/2$

vertex: $(1/2, -3)$

3) $y = -(2x-1)^2 - 3$



$\Rightarrow y + 3 = -(2x-1)^2$

$y + 3 = 0$ if $-(2x-1)^2 = 0$

$\Rightarrow y = -3$ $\Rightarrow x = 1/2$

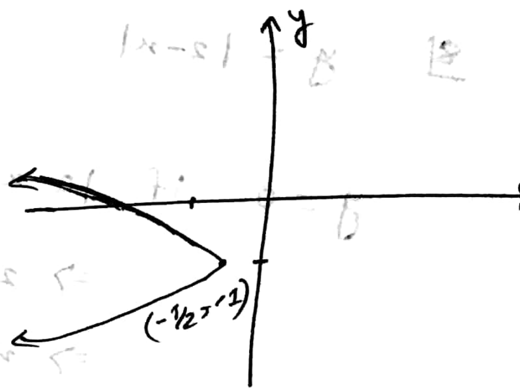
vertex $(1/2, -3)$

4| $(y+1)^2 = -x - 1/2$

$(y+1)^2 = 0$ if $-x - 1/2 = 0$

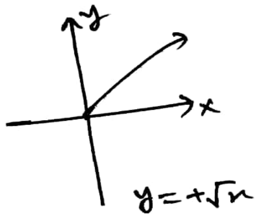
$\Rightarrow y = -1$ $\Rightarrow x = -1/2$

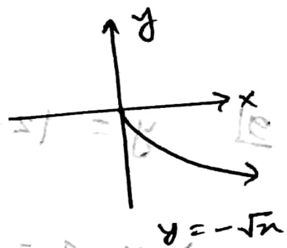
vertex: $(-1/2, -1)$



5| $y^2 = x$

$\Rightarrow y = \pm\sqrt{x}$





6| $y = \sqrt{1+2x} - 2$

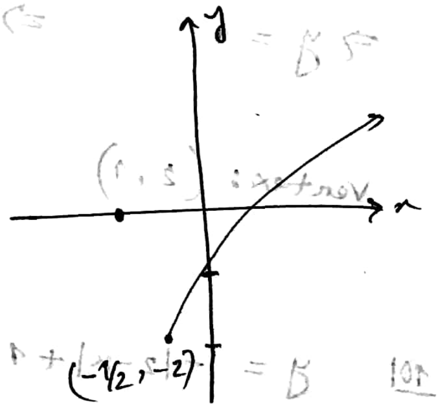
$\Rightarrow y + 2 = \sqrt{1+2x}$

$y + 2 = 0$ if $\sqrt{1+2x} = 0$

$\Rightarrow y = -2$ $\Rightarrow 1 + 2x = 0$

$\Rightarrow x = -1/2$

vertex: $(-1/2, -2)$



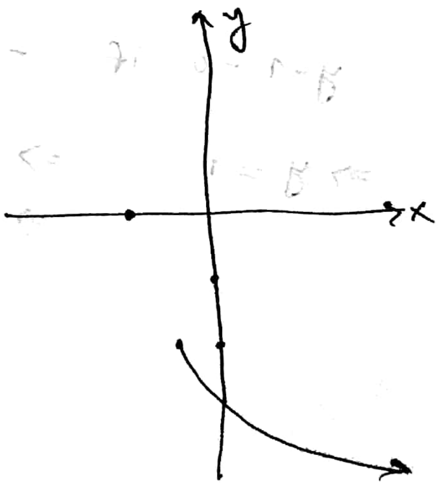
7| $y = -\sqrt{1+2x} - 2$

$\Rightarrow y + 2 = -\sqrt{1+2x}$

$y + 2 = 0$ if $-\sqrt{1+2x} = 0$

$\Rightarrow y = -2$ $\Rightarrow x = -1/2$

Vertex: $(-1/2, -2)$



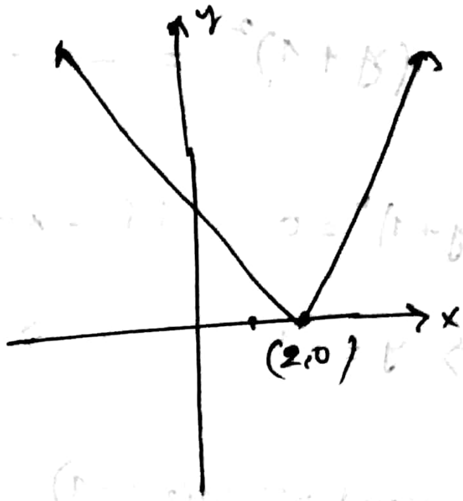
8] $y = |2-x|$

$y = 0$ if $|2-x| = 0$

$\Rightarrow 2-x = 0$

$\Rightarrow x = 2$

vertex: $(2,0)$



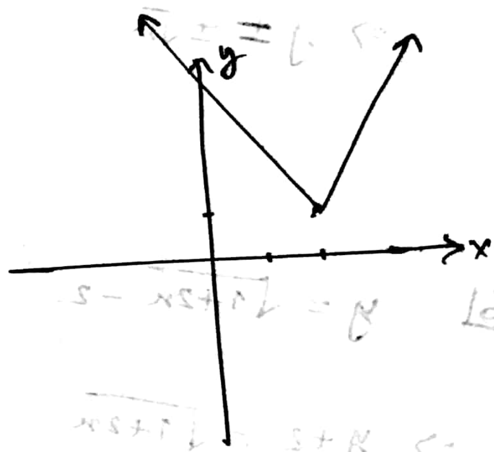
9] $y = |2-x| + 1$

$\Rightarrow y-1 = |2-x|$

$y-1 = 0$ if $|2-x| = 0$

$\Rightarrow y = 1$ $\Rightarrow x = 2$

vertex: $(2,1)$



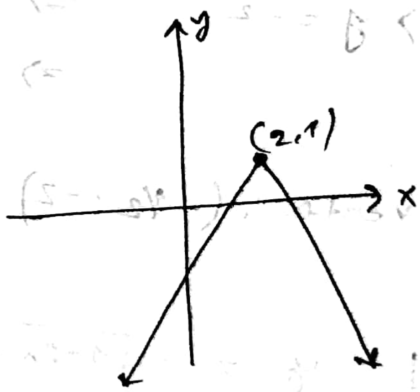
10] $y = -|2-x| + 1$

$y-1 = -|2-x|$

$y-1 = 0$ if $-|2-x| = 0$

$\Rightarrow y = 1$ $\Rightarrow 2-x = 0$

$\Rightarrow x = 2$



LIMIT

1) $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{x+1}-1}$

$= \lim\limits_{x\to 0} \dfrac{x(\sqrt{x+1}+1)}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}$

$= \lim\limits_{x\to 0} \dfrac{x(\sqrt{x+1}+1)}{(\sqrt{x+1})^2-1^2}$

$= \lim\limits_{x\to 0} \dfrac{x\sqrt{x+1}+1}{x+1-1}$

$= \lim\limits_{x\to 0} \sqrt{x+1}+1$

$= \sqrt{0+1}+1$

$= 2$

2) $\lim\limits_{x\to 2} \dfrac{2x^2-5x+2}{5x^2-7x-6}$

$= \lim\limits_{x\to 2} \cancel{\dfrac{x(2x-5)+2}{}} \quad \dfrac{2x^2-4x-x+2}{5x^2-10x+3x-6}$

$= \lim\limits_{x\to 2} \dfrac{2x(x-2)-1(x-2)}{5x(x-2)+3(x-2)}$

$= \lim\limits_{x\to 2} \dfrac{(x-2)(2x-1)}{(x-2)(5x+3)}$

$= \dfrac{2\times2-1}{5\times2+3}$

$= \dfrac{3}{13}$

3) $\lim_{x \to 0} \frac{x}{|x|}$

$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$ $\Rightarrow \frac{x}{|x|} = \begin{cases} 1 & , x > 0 \\ -1 & , x < 0 \end{cases}$

$L.H.L = \lim_{x \to 0^-} f(x)$

$= \lim_{x \to 0^-} (-1)$

$= -1$

$R.H.L = \lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0^+} (1)$

$= 1$

Doesn't exist.

4) $f(x) = \begin{cases} 2-x & , x < 1 \\ x^2+1 & , x > 1 \end{cases}$

$L.H.L = \lim_{x \to 1^-} f(x)$

$= \lim_{x \to 1^-} (2-x)$

$= 2-1$

$= 1$

$R.H.L = \lim_{x \to 1^+} f(x)$

$= \lim_{x \to 1^+} (x^2+1)$

$= 1^2+1$

$= 2$

Doesn't exist.

5) $f(x) = \begin{cases} 3x-1 & , x < 1 \\ 3-x & , x > 1 \end{cases}$

$L.H.L = \lim_{x \to 1^-} f(x)$

$= \lim_{x \to 1^-} (3x-1)$

$= 3 \cdot 1 - 1$

$= 2$

$R.H.L = \lim_{x \to 1^+} f(x)$

$= \lim_{x \to 1^+} (3-x)$

$= 3-1$

$= 2$

Limit exists.

6) $f(x) = \begin{cases} \frac{1}{x+2} & , x<-2 \\ x^2-5 & , -2<x<3 \\ \sqrt{x+13} & , x>3 \end{cases}$

$L.H.L = \lim_{x \to -2^-} f(x)$

$= \lim_{x \to -2^-} \left(\frac{1}{x+2}\right)$

$= \lim_{x \to -2^-} \frac{(x-2)}{(x+2)(x-2)}$

$= \frac{1}{-2+2}$

$= \frac{1}{0}$

$= \infty$

Doesn't exist.

$R.H.L = \lim_{x \to -2^+} f(x)$

$= \lim_{x \to -2^+} (x^2-5)$

$= (-2)^2-5$

$= -1$

$L.H.L = \lim_{x \to 3^-} f(x)$

$= \lim_{x \to 3^-} (x^2-5)$

$= (3)^2-5$

$= 4$

$R.H.L = \lim_{x \to 3^+} f(x)$

$= \lim_{x \to 3^+} (\sqrt{x+13})$

$= \sqrt{3+13}$

$= \sqrt{16}$

$= 4$

Limit exists.

7) $\lim_{x\to\infty} \frac{3x+5}{6x-8}$

$$= \lim_{x\to\infty} \frac{\frac{3x+5}{x}}{\frac{6x-8}{x}}$$

$$= \lim_{x\to\infty} \frac{3+\frac{5}{x}}{6-\frac{8}{x}}$$

$$= \frac{3+\frac{5}{\infty}}{6-\frac{8}{\infty}}$$

$$= \frac{3+0}{6-0}$$

$$= \frac{1}{2}$$

8) $\lim_{x\to\infty} 3\sqrt{\frac{3x+5}{6x-8}}$

$$= \lim_{x\to\infty} 3\sqrt{\frac{(3x+5)/x}{(6x-8)/x}}$$

$$= \lim_{x\to\infty} 3\sqrt{\frac{3+\frac{5}{x}}{6-\frac{8}{x}}}$$

$$= \cancel{\lim_{x\to\infty}}\, 3\sqrt{\frac{3+0}{6-0}}$$

$$= 3\sqrt{1/2}$$

9) $\lim_{x \to -\infty} \dfrac{4x^2 - x}{2x^3 - 5}$

$= \lim_{x \to -\infty} \dfrac{\frac{4x^2 - x}{x^3}}{\frac{2x^3 - 5}{x^3}}$

$= \lim_{x \to -\infty} \dfrac{\frac{4}{x} - \frac{1}{x^2}}{2 - \frac{5}{x^3}}$

$= \dfrac{\frac{4}{-\infty} - \frac{1}{(-\infty)^2}}{2 - \frac{5}{(-\infty)^3}}$

$= \dfrac{0 - 0}{2 - 0}$

$= 0$

10) $\lim_{x \to \infty} \left(\sqrt{x^6 + 5} - x^3\right)$

$= \lim_{x \to \infty} \dfrac{\left(\sqrt{x^6+5} - x^3\right)\left(\sqrt{x^6+5} + x^3\right)}{\sqrt{x^6+5} + x^3}$

$= \lim_{x \to \infty} \dfrac{x^6 + 5 - x^6}{\sqrt{x^6\left(1 + \frac{5}{x^6}\right)} + x^3}$

$= \lim_{x \to \infty} \dfrac{5}{x^3\sqrt{1 + \frac{5}{x^6}} + x^3}$

$= \lim_{x \to \infty} \dfrac{\frac{5}{x^3}}{\frac{x^3\sqrt{1 + \frac{5}{x^6}} + x^3}{x^3}}$

$$= \lim_{x \to \infty} \frac{\frac{5}{x^3}}{\sqrt{1+\frac{5}{x^6}}+1}$$

$$= \frac{\frac{5}{(\infty)^3}}{\sqrt{1+\frac{5}{(\infty)^6}}+1}$$

$$= \frac{0}{\sqrt{1+0}+1}$$

$$= 0$$

11| $f(x) = \begin{cases} x^2+1 & , \quad x > 0 \\ 1 & , \quad x = 0 \\ 1+x & , \quad x < 0 \end{cases}$

$$L.H.L = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} (1+x)$$

$$= 1+0$$

$$= 1$$

$$R.H.L = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} (x^2+1)$$

$$= 0^2+1$$

$$= 1$$

Limit exists.

12

$$\lim_{x\to\infty}\left(\sqrt{x^6+5x^3}-x^3\right)$$

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^6+5x^3}-x^3\right)\left(\sqrt{x^6+5x^3}+x^3\right)}{\left(\sqrt{x^6+5x^3}+x^3\right)}$$

$$=\lim_{x\to\infty}\frac{x^6+5x^3-x^6}{\sqrt{x^6\left(1+\frac{5x^3}{x^6}\right)}+x^3}$$

$$=\lim_{x\to\infty}\frac{5x^3}{x^3\sqrt{1+\frac{5}{x^3}}+x^3}$$

$$=\lim_{x\to\infty}\frac{\frac{5x^3}{x^3}}{\frac{x^3\sqrt{1+\frac{5}{x^3}}+x^3}{x^3}}$$

$$=\lim_{x\to\infty}\frac{5}{\sqrt{1+5/x^3}+1}$$

$$=\frac{5}{\sqrt{1+\frac{5}{(\infty)^3}}+1}$$

$$=\frac{5}{\sqrt{1+0}+1}$$

$$=\frac{5}{1+1}$$

$$=\frac{5}{2}$$

13)
$$f(x) = \begin{cases} e^{-\frac{|x|}{2}} & , -1 < x < 0 \\ x^2 & , 0 < x < 2 \end{cases}$$

$$L.H.L = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} \left(e^{-\frac{|x|}{2}}\right)$$

$$= \lim_{x \to 0^-} \left(e^{-\frac{(-x)}{2}}\right)$$

$$= \lim_{x \to 0^-} e^{x/2}$$

$$= e^{0/2}$$

$$= e^0$$

$$= 1$$

$$R.H.L = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} (x^2)$$

$$= 0^2$$

$$= 0$$

Doesn't exist.

14)
$$f(x) = \begin{cases} x^2 & , x < 1 \\ 2.4 & , x = 1 \\ x^2+1 & , x > 1 \end{cases}$$

$$L.H.L = \lim_{x \to 1^-} f(x)$$

$$= \lim_{x \to 1^-} (x^2)$$

$$= 1^2$$

$$= 1$$

$$R.H.L = \lim_{x \to 1^+} f(x)$$

$$= \lim_{x \to 1^+} (x^2+1)$$

$$= 1^2+1$$

$$= 2$$

Doesn't exist.

15| $f(x) = \begin{cases} 2x+1 & , x<1 \\ 3-x & , x>1 \end{cases}$

$$\text{L.H.L} = \lim_{x\to 1^-} f(x)$$
$$= \lim_{x\to 1^-} (2x+1)$$
$$= 2.1+1$$
$$= 3$$

$$\text{R.H.L} = \lim_{x\to 1^+} f(x)$$
$$= \lim_{x\to 1^+} (3-x)$$
$$= 3-1$$
$$= 2$$

Doesn't exist.

16| Let, $y = \left(1+\frac{1}{x}\right)^x$

$$\Rightarrow \ln y = \ln\left(1+\frac{1}{x}\right)^x$$
$$\Rightarrow \ln y = x\ln\left(1+\frac{1}{x}\right)$$

$$\left[\begin{array}{l} \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty \\ \ln\left(1+\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \dots \infty \end{array}\right.$$

$$\Rightarrow \ln y = x\left(1/x - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \dots \infty\right)$$
$$\Rightarrow \ln y = 1 - \frac{1}{2x} + \frac{1}{3x^2} - \frac{1}{4x^3} + \dots \infty$$
$$\Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty}\left(1 - \frac{1}{2x} + \frac{1}{3x^2} - \frac{1}{4x^3} + \dots \infty\right)$$
$$\Rightarrow \lim_{x\to\infty} \ln y = (1 - 0 + 0 - 0 + \dots \infty)$$
$$\Rightarrow \lim_{x\to\infty} \ln y = 1 \Rightarrow \lim_{x\to\infty} \operatorname{Log}_e y = 1$$

$$\left[\begin{array}{l} \because \operatorname{Log}_a y = x \\ \Rightarrow y = a^x \end{array}\right]$$

$$\Rightarrow \lim_{x\to\infty} y = e^1 = e$$

$$\therefore \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

(proved)

Continuity

1] $f(x) = \begin{cases} \cos x, & x \geq 0 \\ -\cos x, & x < 0 \end{cases}$ at $x = 0$

$L.H.L = \lim_{x \to 0^-} f(x)$

$= \lim_{x \to 0^-} (-\cos x)$

$= (-\cos 0^\circ)$

$= -1$

$R.H.L = \lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0^+} (\cos x)$

$= \cos 0^\circ$

$= 1$

$f(0) = \cos 0^\circ = 1$

$\therefore$ Discontinuous.

2] $f(x) = \begin{cases} x\cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

$L.H.L = \lim_{x \to 0^-} f(x)$

$= \lim_{x \to 0^-} (x\cos 1/x)$

$= \left(\lim_{x \to 0^-} x\right)\left(\lim_{x \to 0^-} \cos \frac{1}{x}\right)$

$= 0 \cdot$ [oscillates . . .]

$= 0$

$R.H.L = \lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0^+} \left(x\cos \frac{1}{x}\right)$

$= \left(\lim_{x \to 0^+} x\right)\left(\lim_{x \to 0^-} \cos \frac{1}{x}\right)$

$= 0 \cdot$ [oscillates . . .]

$= 0$

$f(0) = 0$

$\therefore$ Continuous.

3) $f(x) = \begin{cases} e^{1/x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$ at $x=0$

$$L.H.L = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} (e^{1/x})$$

$$= e^{1/0}$$

$$= e^{\infty}$$

$$= \infty$$

$$R.H.L = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} (e^{1/x})$$

$$= e^{1/0}$$

$$= e^{\infty}$$

$$= \infty$$

$$f(0) = 1$$

Discontinuous.

4) $f(x) = \begin{cases} \sqrt{|x|} & , x \geq 0 \\ -\sqrt{|x|} & , x < 0 \end{cases}$ at $x=0$

$$L.H.L = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} (-\sqrt{|x|})$$

$$= -\sqrt{0}$$

$$= 0$$

$$R.H.L = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} (\sqrt{|x|})$$

$$= \sqrt{0}$$

$$= 0$$

$$f(0) = \sqrt{0} = 0$$

continuous.

5| $f(x) = \begin{cases} e^{\frac{-|x|}{2}} & , -1 < x < 0 \\ x^2 & , 0 \le x < 2 \end{cases}$ at $x=0$

$$\text{L.H.L} = \lim_{x \to 0^-} f(x)$$
$$= \lim_{x\to 0^-} \left(e^{-\frac{|x|}{2}}\right)$$
$$= \lim_{x \to 0^-} e^{\frac{x}{2}}$$
$$= e^{0/2}$$
$$= e^0$$
$$= 1$$

$$\text{R.H.L} = \lim_{x \to 0+} f(x)$$
$$= \lim_{x\to 0+} x^2$$
$$= 0^2$$
$$= 0$$

$$f(0) = 0^2 = 0$$

Discontinuous.

6| $f(x) = \begin{cases} (x-a)\sin\frac{1}{x-a} & , x \ne a \\ 0 & , x = a \end{cases}$ at $x=a$

$$\text{L.H.L} = \lim_{x \to a^-} f(x)$$
$$= \lim_{x \to a^-} \left\{(x-a)\sin\frac{1}{x-a}\right\}$$
$$= \left(\lim_{x\to a^-} (x-a)\right)\left(\lim_{x \to a^-} \sin\frac{1}{x-a}\right)$$
$$= (a-a)\sin\frac{1}{a-a}$$
$$= 0 \cdot [\text{oscillates}\ldots]$$
$$= 0$$

$$\text{R.H.L} = \lim_{x \to a+} f(x)$$
$$= \lim_{x \to a+} \left\{(x-a)\sin\frac{1}{x-a}\right\}$$
$$= \left(\lim_{x\to a+} (x-a)\right)\left(\lim_{x \to a+} \sin\frac{1}{x-a}\right)$$
$$= (a-a)\sin\frac{1}{a-a}$$
$$= 0 \cdot [\text{oscillates}\ldots]$$
$$= 0$$

$$f(a) = 0$$

continuous.

7| $f(x) = \begin{cases} x\sin(1/x) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ at $x=0$

$L.H.L = \lim_{x \to 0^-} f(x)$

$= \lim_{x \to 0^-} \left(x \sin\frac{1}{x}\right)$

$= \left(\lim_{x \to 0^-} x\right)\left(\lim_{x \to 0^-} \sin\frac{1}{x}\right)$

$= 0 \cdot$ [Oscillates]

$= 0$

$R.H.L = \lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0^+} \left(x \sin\frac{1}{x}\right)$

$= \left(\lim_{x \to 0^+} x\right)\left(\lim_{x \to 0^+} \sin\frac{1}{x}\right)$

$= 0 \cdot$ [Oscillates]

$= 0$

$f(0) = 0$

continuous.

8| $f(x) = \begin{cases} 1 & , x < 0 \\ 1 + \sin x & , 0 \leq x < \pi/2 \\ 2 + (x - \pi/2)^2 & , x \geq \pi/2 \end{cases}$ at $x = 0$ and $x = \frac{\pi}{2}$

$L.H.L = \lim_{x \to 0^-} f(x)$

$= \lim_{x \to 0^-} (1)$

$= 1$

$R.H.L = \lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0^+} (1 + \sin x)$

$= 1 + \sin 0^\circ$

$= 1 + 0$

$= 1$

$f(0) = 1 + \sin 0^\circ$

$= 1$

$\therefore$ Continuous at $x = 0$

$$L.H.L = \lim_{x\to \pi/2^-} f(x)$$

$$= \lim_{x\to \pi/2^-} (1+\sin x)$$

$$= 1+\sin 90^\circ$$

$$= 1+1$$

$$= 2$$

$$R.H.L = \lim_{x\to \pi/2^+} f(x)$$

$$= \lim_{x\to\pi/2^+} \left\{2+\left(x-\frac{\pi}{2}\right)^2\right\}$$

$$= 2+\left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2$$

$$= 2+0$$

$$= 2$$

$$f\left(\frac{\pi}{2}\right) = 2+\left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2$$

$$= 2$$

continuous at $x=\frac{\pi}{2}$

9) $f(x) = |x|+|x-1|$ at $x=0$ and $x=1$

$$|x| = \begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$$

$$|x-1| = \begin{cases} x-1, & x\geq 1 \\ -(x-1), & x<1 \end{cases}$$

from real number line,

$-\infty \leftarrow$ ——— 0 ——— 1 ——— $\rightarrow +\infty$

$x<0$	$0<x<1$	$x\geq 1$
let $x=-1$	let $x=0.5$	let $x=2$
$\lvert x\rvert = -x$ $\lvert x-1\rvert = -x+1$	$\lvert x\rvert = x$ $\lvert x-1\rvert = -x+1$	$\lvert x\rvert = x$ $\lvert x-1\rvert = x-1$
$f(x) = -x-x+1$ $= -2x+1$	$f(x) = x-x+1$ $= 1$	$f(x) = x+x-1$ $= 2x-1$

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$$f(x) = \begin{cases} -2x+1, & x<0 \\ 1, & 0\le x<1 \\ 2x-1, & x\geq 1 \end{cases}$$

$$L.H.L = \lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-} (-2x+1)$$

$$= 1$$

$$R.H.L = \lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0^+} (1)$$

$$= 1$$

$$f(0) = 1$$

Continuous at $x=0$

$$L.H.L = \lim_{x\to 1^-} f(x)$$

$$= \lim_{x\to 1^-} (1)$$

$$= 1$$

$$R.H.L = \lim_{x\to 1^+} f(x)$$

$$= \lim_{x\to 1^+} (2x-1)$$

$$= 2\cdot 1-1$$

$$= 1$$

$$f(1) = 2\cdot 1-1$$

$$= 1$$

$\therefore$ Continuous at $x=1$

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10)
$$f(x) = \begin{cases} \dfrac{|x-3|}{x-3}, & x \neq 3 \\ 0, & x = 3 \end{cases} \quad \text{at } x=3$$

$$|x-3| = \begin{cases} x-3, & x \geq 3 \\ -(x-3), & x < 3 \end{cases}$$

$$f(x) = \begin{cases} 1, & x \geq 3 \\ -1, & x < 3 \end{cases}$$

$$\text{L.H.L} = \lim_{x \to 3^-} f(x)$$
$$= \lim_{x \to 3^-} (-1)$$
$$= -1$$

$$\text{R.H.L} = \lim_{x \to 3^+} f(x)$$
$$= \lim_{x \to 3^+} (1)$$
$$= 1$$

$$f(3) = 1$$

$\therefore$ Discontinuous.

11)
$$f(x) = \begin{cases} (1+x)^{1/x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \quad \text{at } x=0$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (1+x)^{1/x}$$

$$\Rightarrow \lim_{x \to 0} \ln f(x) = \lim_{x \to 0} \ln(1+x)^{1/x}$$

$$\Rightarrow \lim_{x \to 0} \ln(f(x)) = \lim_{x \to 0} \frac{1}{x} \ln(1+x)$$

$$\Rightarrow \lim_{x \to 0} \ln f(x) = \lim_{x \to 0} \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots \infty\right)$$

$$\Rightarrow \lim_{x \to 0} \ln f(x) = 1$$

$$\Rightarrow \lim_{x \to 0} f(x) = e^1 = e$$

$\therefore$ L.H.L = R.H.L = e

But $f(0) = 1$

Discontinuous.

12) $f(x) = \begin{cases} \dfrac{e^{1/x^2}}{e^{1/x^2}-1} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$ at $x = 0$

$$f(x) = \frac{e^{1/x^2}}{e^{1/x^2}-1}$$

$$= \frac{(e^{1/x^2}-1)+1}{(e^{1/x^2}-1)}$$

$$= 1 + \frac{1}{e^{1/x^2}-1}$$

$$\text{L.H.L} = \lim_{x \to 0^-} f(x)$$

$$= \lim_{x \to 0^-} 1 + \frac{1}{e^{1/x^2}-1}$$

$$= 1 + \frac{1}{e^{1/0}-1}$$

$$= 1 + \frac{1}{e^{\infty}-1}$$

$$= 1 + \frac{1}{\infty - 1}$$

$$= 1 + 0$$

$$= 1$$

$$\text{R.H.L} = \lim_{x \to 0^+} f(x)$$

$$= \lim_{x \to 0^+} 1 + \frac{1}{e^{1/x^2}-1}$$

$$= 1 + \frac{1}{e^{1/0}-1}$$

$$= 1 + \frac{1}{\infty - 1}$$

$$= 1$$

$f(0) = 1$

$\therefore$ continuous.

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Differentiability

23-Oct-2017

1] (i) $f(x) = \begin{cases} \cos x & , x \geq 0 \\ -\cos x & , x < 0 \end{cases}$ at $x = 0$

$$Lf'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0^-} \frac{-\cos x - 1}{x}$$

$$= \frac{-\cos 0^\circ - 1}{0}$$

$$= \frac{-1 - 1}{0}$$

$$= \frac{-2}{0}$$

$$= -\infty$$

$$Rf'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0^+} \frac{\cos x - 1}{x}$$

$$= \lim_{x \to 0^+} \frac{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \infty\right) - 1}{x}$$

$$= \lim_{x \to 0^+} \left(- \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots \infty\right)$$

$$= -\frac{0}{2!} + \frac{0}{4!} - \frac{0}{6!} + \cdots \infty$$

$$= 0$$

$\therefore$ $f(x)$ is not differentiable at $x = 0$

(ii) $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ at $x = 0$

$Lf'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$ $Rf'(0) =$ same

$= \lim_{x \to 0^-} \frac{x^2 \sin(1/x) - 0}{x}$

$= \lim_{x \to 0^-} x \sin(1/x)$

$= \left(\lim_{x \to 0^-} x\right)\left(\lim_{x \to 0^-} \sin 1/x\right)$

$= 0$ [oscillates]

$= 0$

Differentiable.

(iii) $f(x) = |x|$ at $x = 0$

$f(x) = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$

$Lf'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{x \to 0^-} \frac{-x - 0}{x}$

$= \lim_{x \to 0^-} (-1)$

$= -1$

$Rf'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{x \to 0^+} \frac{x - 0}{x}$

$= \lim_{x \to 0^+} (1)$

$= 1$

$\therefore$ not differentiable

(iv) $f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ at $x = 0$

$$Lf'(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0^-} \frac{x\cos\left(\frac{1}{x}\right) - 0}{x}$$

$$= \lim_{x\to 0^-} \left(\cos\frac{1}{x}\right)$$

$$= \lim_{x\to 0^-} \left(1 - \frac{1}{x^2 2!} + \frac{1}{x^4 4!} - \frac{1}{x^6 6!} + \cdots \infty\right)$$

$$= 1 - \infty + \infty - \infty + \cdots \infty$$

$$= \infty$$

$$Rf'(0) = \text{Same}$$

Not differentiable

2) $f(x) = \begin{cases} x^2 - 16x, & x < 9 \\ 12\sqrt{x}, & x \geq 9 \end{cases}$ at $x = 9$

$$Lf'(9) = \lim_{x \to 9^-} \frac{f(x) - f(9)}{x - 9}$$

$$= \lim_{x \to 9^-} \frac{x^2 - 16x - 36}{x - 9}$$

$$= \lim_{x \to 9^-} \frac{x^2 - 18x + 2x - 36}{x - 9}$$

$$= \lim_{x \to 9^-} \frac{(x+2)(x-18)}{x - 9}$$

$$= \frac{(9+2)(9-18)}{9-9}$$

$$= \infty$$

$$Rf'(9) = \lim_{x \to 9^+} \frac{f(x) - f(9)}{x - 9}$$

$$= \lim_{x \to 9^+} \frac{12\sqrt{x} - 36}{x - 9}$$

$$= \lim_{x \to 9^+} \frac{12(\sqrt{x} - 3)}{(\sqrt{x}+3)(\sqrt{x}-3)}$$

$$= \lim_{x \to 9^+} \frac{12}{\sqrt{x} + 3}$$

$$= \frac{12}{\sqrt{9} + 3}$$

$$= 2$$

$\therefore$ not differentiable.

$$L.H.L = \lim_{x\to 9^-} f(x)$$

$$= \lim_{x\to 9^-} (x^2-16x)$$

$$= (9^2-16\times 9)$$

$$= -63$$

$$R.H.L = \lim_{x\to 9^+} f(x)$$

$$= \lim_{x\to 9^+} (12\sqrt{x})$$

$$= 12\sqrt{9}$$

$$= 36$$

Discontinuous.

<u>31</u> $f(x) = \begin{cases} x^2, & x\le 1 \\ \sqrt{x}, & x>1 \end{cases}$ at $x=1$

$$Lf'(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to 1^-} \frac{x^2-1}{x-1}$$

$$= \lim_{x\to 1^-} \frac{(x+1)(x-1)}{(x-1)}$$

$$= \lim_{x\to 1^-} (x+1)$$

$$= 1+1$$

$$= 2$$

$$Rf'(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to 1^+} \frac{\sqrt{x}-1}{x-1}$$

$$= \lim_{x\to 1^+} \frac{(\sqrt{x}-1)}{(\sqrt{x}+1)(\sqrt{x}-1)}$$

$$= \lim_{x\to 1^+} \frac{1}{\sqrt{x}+1}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

not differentiable

$$L.H.L = \lim_{x \to 1^-} f(x)$$

$$= \lim_{x \to 1^-} (x^2)$$

$$= 1^2$$

$$= 1$$

$$R.H.L = \lim_{x \to 1^+} f(x)$$

$$= \lim_{x \to 1^+} (\sqrt{x})$$

$$= \sqrt{1}$$

$$= 1$$

$$f(1) = 1^2 = 1$$

continuous

4) $f(x) = \begin{cases} x^2+1 , & x \leq 1 \\ x , & x > 1 \end{cases}$ at $x = 1$

$$Lf'(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x \to 1^-} \frac{x^2+1-2}{x-1}$$

$$= \lim_{x \to 1^-} \frac{x^2-1}{x-1}$$

$$= \lim_{x \to 1^-} \frac{(x+1)(x-1)}{(x-1)}$$

$$= \lim_{x \to 1^-} (x+1)$$

$$= 1+1$$

$$= 2$$

$$Rf'(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x \to 1^+} \frac{x-2}{x-1}$$

$$= \lim_{x \to 1^+} \frac{(x-1)-1}{(x-1)}$$

$$= \lim_{x \to 1^+} 1 - \frac{1}{x-1}$$

$$= 1 - \frac{1}{1-1}$$

$$= 1 - \infty$$

$$= \infty$$

not Differentiable

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$$L.H.L = \lim_{x \to 1^-} f(x)$$

$$= \lim_{x \to 1^-} (x^2+1)$$

$$= 1^2+1$$

$$= 2$$

$$R.H.L = \lim_{x \to 1^+} f(x)$$

$$= \lim_{x \to 1^+} (x)$$

$$= 1$$

$$f(1) = 1^2+1 = 2$$

not continuous.

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Techniques of Differentiation

1 (i) $y = \sin x \sin 2x \sin 3x$

$\Rightarrow \frac{d}{dx} y = \sin x \sin 2x \frac{d}{dx}(\sin 3x) + \sin 3x \frac{d}{dx}(\sin x \sin 2x)$

$= \sin x \sin 2x \, 3\cos 3x + \sin 3x \left[\sin x \frac{d}{dx} \sin 2x + \sin 2x \frac{d}{dx} \sin x \right.$

$= 3 \sin x \sin 2x \cos 3x + 2 \sin x \sin 3x \cos 2x + \sin 3x \sin 2x \cos x$

(ii) $y = \operatorname{cosec}^3 x$

$\Rightarrow y = (\operatorname{cosec} x)^3$

$\Rightarrow \frac{dy}{dx} = 3 \operatorname{cosec}^2 x \frac{d}{dx}(\operatorname{cosec} x)$

$= 3 \operatorname{cosec}^2 x (-\operatorname{cosec} x \cot x)$

$= -3 \operatorname{cosec}^2 x \operatorname{cosec} x \cot x$

(iii) $y = \cos 2x \cos 3x$

$\Rightarrow \frac{dy}{dx} = \cos 2x \frac{d}{dx}(\cos 3x) + \cos 3x \frac{d}{dx}(\cos 2x)$

$= -3 \cos 2x \sin 3x - 2 \cos 3x \sin 2x$

(iv) $y = \sin^{-1}(x^2)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^4}} \cdot 2x$

$= \frac{2x}{\sqrt{1-x^4}}$

(v) $y = \tan(\sin^{-1}x)$

$$\Rightarrow \frac{dy}{dx} = \sec^2(\sin^{-1}x) \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{\sec^2(\sin^{-1}x)}{\sqrt{1-x^2}}$$

(vi) $y = \cot^{-1}\left(\frac{1+x}{1-x}\right)$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{1+\left(\frac{1+x}{1-x}\right)^2} \cdot \frac{(1-x)\frac{d}{dx}(1+x) - (1+x)\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$= -\frac{1}{1+\frac{(1+x)^2}{(1-x)^2}} \cdot \frac{(1-x)\cdot 1 - (1+x)\cdot(-1)}{(1-x)^2}$$

$$= -\frac{1}{\frac{(1-x)^2+(1+x)^2}{(1-x)^2}} \cdot \frac{1-x+1+x}{(1-x)^2}$$

$$= -\frac{(1-x)^2}{(1-x)^2+(1+x)^2} \cdot \frac{2}{(1-x)^2}$$

$$= -\frac{2}{(1-2x+x^2)+(1+2x+x^2)}$$

$$= -\frac{2}{1-2x+x^2+1+2x+x^2}$$

$$= -\frac{2}{2x^2+2}$$

$$= -\frac{1}{x^2+1}$$

(vii) $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

let, $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1}x$

$= \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\cos 2\theta)$

$= 2\theta$

$= 2\tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{2}{1+x^2}$

(viii) $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

let, $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1}x$

$= \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta$

$= 2\tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{2}{1+x^2}$

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(ix) $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$= \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$= \tan^{-1}(\tan 2\theta)$

$= 2\theta$

$= 2\tan^{-1}x$

$\therefore \frac{dy}{dx} = \frac{2}{1+x^2}$

(x) $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

let, $x = \sin\theta$

$\Rightarrow \theta = \sin^{-1}x$

$= \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$

$= \tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$

$= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$

$= \tan^{-1}(\tan\theta)$

$= \theta$

$= \sin^{-1}x$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

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2) (i) $y=(\sin x)^{\ln x}$

$\Rightarrow \ln y = \ln(\sin x)^{\ln x}$

$\Rightarrow \ln y = \ln x \ln(\sin x)$

$\Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx} \ln x \ln(\sin x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln x \frac{d}{dx}\ln(\sin x) + \ln(\sin x)\frac{d}{dx}\ln x$

$= \ln x \cdot \frac{1}{\sin x}\cdot \cos x + \ln(\sin x)\cdot\frac{1}{x}$

$= \ln x \cdot \cot x + \frac{\ln(\sin x)}{x}$

$\therefore \frac{dy}{dy} = (\sin x)^{\ln x}\left[\ln x \cot x + \frac{\ln(\sin x)}{x}\right]$

(ii) $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

let, $u = (\sin x)^{\cos x}$ —— ①

$v = (\cos x)^{\sin x}$ —— ⑪

from ①,

$\ln u = \ln(\sin x)^{\cos x}$

$\Rightarrow \ln u = \cos x \ln(\sin x)$

$\Rightarrow \frac{d}{dx}\ln u = \frac{d}{dx}\cos x \ln(\sin x)$

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$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \frac{d}{dx}\ln(\sin x) + \ln(\sin x)\frac{d}{dx}(\cos x)$$

$$= \cos x \cdot \frac{1}{\sin x}\cdot \cos x + \ln(\sin x)\cdot(-\sin x)$$

$$= \cot x\cos x - \sin x \ln(\sin x)$$

$$\therefore \frac{du}{dx} = (\sin x)^{\cos x}\left[\cot x\cos x - \sin x\ln(\sin x)\right]$$

from (ii)

$$\ln v = \ln(\cos x)^{\sin x}$$

$$\Rightarrow \ln v = \sin x \ln(\cos x)$$

$$\Rightarrow \frac{d}{dx}\ln v = \frac{d}{dx}\sin x\ln(\cos x)$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \sin x\frac{d}{dx}\ln(\cos x) + \ln(\cos x)\frac{d}{dx}(\sin x)$$

$$= \sin x\cdot\frac{1}{\cos x}\cdot(-\sin x) + \ln(\cos x)\cdot\cos x$$

$$= -\sin x\tan x + \cos x\ln(\cos x)$$

$$\therefore \frac{dv}{dx} = (\cos x)^{\sin x}\left[-\sin x\tan x + \cos x\ln(\cos x)\right]$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= (\sin x)^{\cos x}\left[\cot x\cos x - \sin x\ln(\sin x)\right] + (\cos x)^{\sin x}\left[-\sin x\tan x + \cos x\ln(\cos x)\right]$$

31 $y = \sqrt{\left(\frac{1+x}{1-x}\right)}$

$$\Rightarrow \ln y = \ln \left(\frac{1+x}{1-x}\right)^{1/2}$$

$$\Rightarrow \ln y = \frac{1}{2} \ln \left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow \ln y = \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{1+x}\cdot 1 - \frac{1}{1-x}\cdot(-1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{1-x}\right]$$

$$= \frac{1}{2}\left[\frac{(1-x)+(1+x)}{(1+x)(1-x)}\right]$$

$$= \frac{1}{2}\left[\frac{2}{1-x^2}\right]$$

$$= \frac{1}{1-x^2}$$

$$\therefore \frac{dy}{dx} = \sqrt{\frac{(1+x)}{(1-x)}} \cdot \frac{1}{1-x^2}$$

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4]

(i) $3x^4 - x^2y + 2y^3 = 0$

$\Rightarrow \frac{d}{dx}(3x^4 - x^2y + 2y^3) = \frac{d}{dx}(0)$

$\Rightarrow 12x^3 - \frac{d}{dx}x^2y + \frac{d}{dx}2y^3 = 0$

$\Rightarrow 12x^3 - [x^2\frac{d}{dx}y + y\frac{d}{dx}x^2] + 2[3y^2 \cdot \frac{d}{dx}y] = 0$

$\Rightarrow 12x^3 - x^2\frac{dy}{dx} - 2xy + 6y^2\frac{dy}{dx} = 0$

$\Rightarrow 6y^2\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy - 12x^3$

$\Rightarrow \frac{dy}{dx}(6y^2 - x^2) = 2xy - 12x^3$

$\therefore \frac{dy}{dx} = \frac{2xy - 12x^3}{6y^2 - x^2}$

(ii) $x^3 + y^3 + 4x^2y - 25 = 0$

$\Rightarrow \frac{d}{dx}(x^3 + y^3 + 4x^2y) = \frac{d}{dx}(25)$

$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} + 4\frac{d}{dx}x^2y = 0$

$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} + 4[x^2\frac{dy}{dx} + y\frac{d}{dx}x^2] = 0$

$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} + 4x^2\frac{dy}{dx} + 8xy = 0$

$\Rightarrow \frac{dy}{dx}(3y^2 + 4x^2) = -8xy - 3x^2$

$\therefore \frac{dy}{dx} = \frac{-3x^2 - 8xy}{3y^2 + 4x^2}$

(iii) $x^y = y^x$

$\Rightarrow \ln x^y = \ln y^x$

$\Rightarrow y \ln x = x \ln y$

$\Rightarrow \frac{d}{dx}(y \ln x) = \frac{d}{dx}(x \ln y)$

$\Rightarrow y \frac{d}{dx} \ln x + \ln x \frac{d}{dx} y = x \frac{d}{dx} \ln y + \ln y \frac{d}{dx} x$

$\Rightarrow y \cdot \frac{1}{x} + \ln x \frac{dy}{dx} = x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \ln y$

$\Rightarrow \ln x \frac{dy}{dx} - \frac{x}{y} \frac{dy}{dx} = \ln y - \frac{y}{x}$

$\Rightarrow \frac{dy}{dx}\left(\ln x - \frac{x}{y}\right) = \ln y - \frac{y}{x}$

$\therefore \frac{dy}{dx} = \dfrac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$

5| (i) $x = a\cos^3\theta$

$y = a\sin^3\theta$

$\frac{dx}{d\theta} = 3\cos^2\theta \frac{d}{d\theta}(\cos\theta)$

$= -3\cos^2\theta \sin\theta$

$\frac{dy}{d\theta} = 3\sin^2\theta \frac{d}{d\theta}(\sin\theta)$

$= 3\sin^2\theta \cos\theta$

$\therefore \frac{dy}{dx} = -\frac{3\sin^2\theta\cos\theta}{3\cos^2\theta \sin\theta}$

$= -\frac{\sin\theta}{\cos\theta} = -\tan\theta$

(ii) $x = \sin^2\theta$

$y = \tan\theta$

$$\frac{dx}{d\theta} = 2\sin\theta\cos\theta$$

$$\frac{dy}{d\theta} = \sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2\theta}{2\sin\theta\cos\theta}$$

(iii) $x = a\sec^2\theta$

$y = a\tan^2\theta$

$$\frac{dx}{d\theta} = 2\sec\theta \cdot \sec\theta\cancel{\tan}\theta$$

$$= 2\sec^2\theta\tan\theta$$

$$\frac{dy}{d\theta} = 2\tan\theta \cdot \sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{2\tan\theta\sec^2\theta}{2\sec^2\theta\tan\theta}$$

$$= 1$$

6) (i) let, $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$z = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$\cancel{\frac{dy}{dx}}$ $y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$ let, $x = \tan\theta$

$= \cos^{-1}(\cos 2\theta)$ $\theta = \tan^{-1}x$

$= 2\theta$

$= 2\tan^{-1}x$

$\frac{dy}{dx} = \frac{2}{1+x^2}$

$z = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$= \tan^{-1}(\tan 2\theta)$

$= 2\theta$

$= 2\tan^{-1}x$

$\frac{dz}{dx} = \frac{2}{1+x^2}$

$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = 1$

(ii) let, $y = x^{\sin^{-1}(x)}$

$z = \sin^{-1} x$

$\ln y = \ln x^{\sin^{-1} x}$

$\Rightarrow \ln y = \sin^{-1} x \ln x$

$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \sin^{-1} x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{\sqrt{1-x^2}}$

$= \frac{\sin^{-1} x}{x} + \frac{\ln x}{\sqrt{1-x^2}}$

$\therefore \frac{dy}{dx} = x^{\sin^{-1}(x)} \left[\frac{\sin^{-1} x}{x} + \frac{\ln x}{\sqrt{1-x^2}} \right]$

$\frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{dy}{dz} = \frac{x^{\sin^{-1} x} \left[\frac{\sin^{-1} x}{x} + \frac{\ln x}{\sqrt{1-x^2}} \right]}{\frac{1}{\sqrt{1-x^2}}}$

7] (i) $y = \ln\sqrt{\dfrac{1+\sin x}{1-\sin x}}$

$\Rightarrow y = \ln\left(\dfrac{1+\sin x}{1-\sin x}\right)^{1/2}$

$\Rightarrow y = \dfrac{1}{2}\ln\left(\dfrac{1+\sin x}{1-\sin x}\right)$

$\Rightarrow y = \dfrac{1}{2}\left[\ln(1+\sin x) - \ln(1-\sin x)\right]$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}\left[\dfrac{1}{1+\sin x}\cdot\cos x - \dfrac{1}{1-\sin x}\cdot(-\cos x)\right]$

$= \dfrac{1}{2}\left[\dfrac{\cos x}{1+\sin x} + \dfrac{\cos x}{1-\sin x}\right]$

$= \dfrac{1}{2}\left[\dfrac{\cos x - \sin x\cos x + \cos x + \sin x\cos x}{(1+\sin x)(1-\sin x)}\right]$

$= \dfrac{1}{2}\left[\dfrac{2\cos x}{1-\sin^2 x}\right]$

$= \dfrac{\cos x}{\cos^2 x}$

$= \dfrac{1}{\cos x}$

$= \sec x$

(ii) $y = \ln\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

$$\Rightarrow y = \ln\left(\frac{1-\cos x}{1+\cos x}\right)^{1/2}$$

$$\Rightarrow y = \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$\Rightarrow y = \frac{1}{2}\left[\ln(1-\cos x) - \ln(1+\cos x)\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{1-\cos x}\cdot \sin x + \frac{1}{1+\cos x}\cdot \sin x\right]$$

$$= \frac{1}{2}\left[\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x}\right]$$

$$= \frac{1}{2}\left[\frac{\sin x + \sin x\cos x + \sin x - \sin x\cos x}{(1+\cos x)(1-\cos x)}\right]$$

$$= \frac{1}{2}\left[\frac{2\sin x}{1-\cos^2 x}\right]$$

$$= \frac{\sin x}{\sin^2 x}$$

$$= \frac{1}{\sin x}$$

$$= \operatorname{cosec} x$$

(iii) $y = \tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

$= \tan^{-1}\sqrt{\dfrac{1-\cos 2(x/2)}{1+\cos 2(x/2)}}$

$= \tan^{-1}\sqrt{\dfrac{2\sin^2(\frac{x}{2})}{2\cos^2(\frac{x}{2})}}$

$= \tan^{-1}\left(\tan\frac{x}{2}\right)$

$= \frac{x}{2}$

$\frac{dy}{dx} = \frac{1}{2}$

(iv) $y = \tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$

$= \tan^{-1}\left[\dfrac{\frac{\cos x - \sin x}{\cos x}}{\frac{\cos x + \sin x}{\cos x}}\right]$

$= \tan^{-1}\left[\dfrac{1-\tan x}{1+\tan x}\right]$

$= \tan^{-1}\left[\dfrac{\tan 45^\circ - \tan x}{1+\tan 45^\circ(\tan x)}\right]$

$= \tan^{-1}\left[\tan(45^\circ - x)\right]$

$= 45^\circ - x$

$\therefore \frac{dy}{dx} = -1$

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(v) $y = \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$

$= \sin\left(2\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)$ let, $x = \cos\theta$

$\Rightarrow \theta = \cos^{-1}x$

$= \sin\left(2\tan^{-1}\sqrt{\frac{1-\cos 2(\theta/2)}{1+\cos 2(\theta/2)}}\right)$

$= \sin\left(2\tan^{-1}\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}\right)$

$= \sin\left(2\tan^{-1}\tan\theta/2\right)$

$= \sin\theta$

$= \sin\cdot\cos^{-1}x$

$\frac{dy}{dx} = \cos(\cos^{-1}x)\cdot\left(-\frac{1}{\sqrt{1-x^2}}\right)$

$= -\frac{x}{\sqrt{1-x^2}}$

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Maxima and Minima

1] (i) $f(x) = x^2 - 5x + 6$

$f'(x) = 2x - 5$

$f''(x) = 2$

Critical points:

$f'(x) = 0$ or $f'(x)$ is undefined

$\Rightarrow 2x - 5 = 0$ × [polynomial never gets undefined]

$\Rightarrow x = 5/2$

$-\infty \longleftarrow \quad 5/2 \quad \longrightarrow +\infty$

let $x = 1$	let $x = 3$
$f'(1) < 0$	$f'(3) > 0$
decreasing	increasing

f is increasing in the intervals $(5/2, \infty)$

f " decreasing in the intervals $(-\infty, 5/2)$

Inflection points:

$f''(x) = 0$ or $f''(x)$ is undefined

$\Rightarrow 2 = 0$ ×

$\because f''(x) = 2 > 0$

$\therefore$ function is concave up on $\mathbb{R}$.

(ii) $f(x) = 5 + 12x - x^3$

$f'(x) = 12 - 3x^2$

$f''(x) = -6x$

<u>critical points:</u>

$f'(x) = 0$ or $f'(x)$ is undefined

$\times$

$\Rightarrow 12 - 3x^2 = 0$

$\Rightarrow 3x^2 = 12$

$\Rightarrow x = \pm 2$

$-\infty \longleftarrow$ -2 $\quad$ 2 $\longrightarrow$

let $x=-3$	let $x=1$	let $x=3$
$f'(-3) < 0$	$f'(1) > 0$	$f'(3) < 0$
decreasing	increasing	decreasing

<u>Inflection points:</u>

$f''(x) = 0$ or $f''(x)$ is undefined

$\times$

$\Rightarrow -6x = 0$

$\Rightarrow x = 0$

$-\infty \longleftarrow$ 0 $\longrightarrow +\infty$

let $x=-1$	let $x=1$
$f''(-1) > 0$	$f''(1) < 0$
concave up	concave down

(iii) $f(x) = x^4 - 8x^2 + 16$

$f'(x) = 4x^3 - 16x$

$f''(x) = 12x^2 - 16$

__critical points :__

$f'(x) = 0$ or $f'(x)$ is undefined $\times$

$\Rightarrow 4x^3 - 16x = 0$

$\Rightarrow 4x(x^2-4) = 0$

$\Rightarrow x = 0, \pm 2$

$-\infty$ ... -2 ... 0 ... 2 ... $+\infty$

let $x=-3$	let $x=-1$	let $x=1$	let $x=3$
$f'(-3) < 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(3) > 0$
decreasing	increasing	decreasing	increasing

__Inflection points :__

$f''(x) = 0$ or $f''(x)$ is undefined $\times$

$\Rightarrow 12x^2 - 16 = 0$

$\Rightarrow 12x^2 = 16$

$\Rightarrow 3x^2 = 4$

$\Rightarrow x = \pm \frac{2}{\sqrt{3}}$

$-\infty$ ... $-2/\sqrt{3}$ ... $2/\sqrt{3}$ ... $+\infty$

let $x=-3$	let $x=1$	let $x=3$
$f''(-3) > 0$	$f''(1) < 0$	$f''(3) > 0$
concave up	concave down	concave up

(iv) $f(x) = \dfrac{x^2}{x^2+2}$

$$f'(x) = \frac{(x^2+2)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(x^2+2)}{(x^2+2)^2}$$

$$= \frac{(x^2+2)\cdot 2x - x^2 \cdot 2x}{(x^2+2)^2}$$

$$= \frac{(x^2+2)\cdot 2x - 2x^3}{(x^2+2)^2}$$

$$= \frac{2x^3+4x-2x^3}{(x^2+2)^2}$$

$$= \frac{4x}{(x^2+2)^2}$$

$$f''(x) = \frac{(x^2+2)^2\frac{d}{dx}4x - 4x\frac{d}{dx}(x^2+2)^2}{\{(x^2+2)^2\}^2}$$

$$= \frac{(x^2+2)^2\cdot 4 - 4x\cdot 2(x^2+2)\cdot 2x}{(x^2+2)^4}$$

$$= \frac{4(x^2+2)^2 - 16x^2(x^2+2)}{(x^2+2)^4}$$

$$= \frac{(x^2+2)\{4(x^2+2) - 16x^2\}}{(x^2+2)^4}$$

$$= \frac{4x^2+8-16x^2}{(x^2+2)^3}$$

$$= \frac{8-12x^2}{(x^2+2)^3}$$

Critical points:

$f'(x) = 0$

$\Rightarrow \frac{4x}{(x^2+2)^2} = 0$

$\Rightarrow 4x = 0$

$\Rightarrow x = 0$

or $f'(x)$ is undefined

denominator $= 0$

$\Rightarrow (x^2+2)^2 = 0$

$\Rightarrow x^2+2 = 0$

$\Rightarrow x = \pm\sqrt{-2}$

no real solution

$-\infty \leftarrow$ $x=-1$ | 0 | $x=1$ $\rightarrow \infty$

$f(-1) < 0$ | $f(1) > 0$

decreasing | increasing

Inflection points:

$f''(x) = 0$

$\Rightarrow \frac{8-12x^2}{(x^2+2)^3} = 0$

$\Rightarrow 8 - 12x^2 = 0$

$\Rightarrow 12x^2 = 8$

$\Rightarrow 3x^2 = 2$

$\Rightarrow x^2 = \frac{2}{3}$

$\therefore x = \pm\sqrt{2/3}$

or $f''(x)$ is undefined

$(x^2+2)^3 = 0$

$\Rightarrow x = \pm\sqrt{-2}$

No real solution

$-\infty \leftarrow$ $x=-2$ | $-\sqrt{2/3}$ | $x=0$ | $+\sqrt{2/3}$ | $x=2$ $\rightarrow +\infty$

$f''(-2) < 0$ | $f''(0) > 0$ | $f''(2) < 0$

concave down | concave up | concave down

(v) $f(x) = \sqrt[3]{x+2}$

$$= (x+2)^{1/3}$$

$$f'(x) = \frac{1}{3}(x+2)^{-2/3} \cdot 1$$

$$f''(x) = -\frac{2}{9}(x+2)^{-5/3} \cdot 1$$

Critical points:

$f'(x) = 0$ or $f(x)$ is undefined

$\Rightarrow \frac{1}{3}(x+2)^{-2/3} = 0$ ✗

$\Rightarrow x = -2$

$-\infty \leftarrow \quad -2 \quad \rightarrow +\infty$

let $x=-3$	let $x=3$
$f'(-3) < 0$	$f'(3) > 0$
decreasing	increasing

Inflection points:

$f''(x) = 0$ or $f''(x)$ is undefined

$\Rightarrow -\frac{2}{9}(x+2)^{-5/3} = 0$ ✗

$\Rightarrow x = -2$

$-\infty \leftarrow \quad -2 \quad \rightarrow +\infty$

let $x=-3$	let $x=3$
$f''(-3) > 0$	$f''(3) < 0$
up	down

2) (i) $f(x) = x^3 + 3x^2 - 9x + 1$

$$f'(x) = 3x^2 + 6x - 9$$

critical points: or $f'(x)$ is undefined

$f'(x) = 0$

$\Rightarrow 3x^2 + 6x - 9 = 0$

$\Rightarrow x^2 + 2x - 3 = 0$

$\Rightarrow x^2 - x + 3x - 3 = 0$

$\Rightarrow x(x-1) + 3(x+1) = 0$

$\therefore x = 1, -3$

stationary points:

$f'(x) = 0$

$\Rightarrow x = -3, 1$

(ii) $f(x) = 5 + 12x - x^3$

$f'(x) = 12 - 3x^2$

critical points:

(ii) $f(x) = x^4 - 6x^2 - 3$

$f'(x) = 4x^3 - 12x$

<u>Critical points:</u>

$f'(x) = 0$

$\Rightarrow 4x^3 - 12x = 0$

$\Rightarrow 4x(x^2 - 3) = 0$

$\Rightarrow x = 0, \pm\sqrt{3}$

<u>Stationary points:</u>

$f'(x) = 0$

$\Rightarrow x = 0, \pm\sqrt{3}$

(iii) $f(x) = \dfrac{x}{x^2+2}$

$$f'(x) = \frac{(x^2+2)\frac{d}{dx}x - x\frac{d}{dx}(x^2+2)}{(x^2+2)^2}$$

$$= \frac{x^2+2-2x^2}{(x^2+2)^2}$$

$$= \frac{2-x^2}{(x^2+2)^2}$$

Critical points:

$f'(x) = 0$

$\Rightarrow \frac{2-x^2}{(x^2+2)^2} = 0$

$\Rightarrow 2 - x^2 = 0$

$\Rightarrow x = \pm\sqrt{2}$

or $f'(x)$ is undefined

$(x^2+2)^2 = 0$

$\Rightarrow x^2 + 2 = 0$

$\Rightarrow x^2 = -2$

$\Rightarrow x = \pm\sqrt{-2}$

$\therefore$ no real solution

Stationary points:

$f'(x) = 0$

$\Rightarrow x = \pm\sqrt{2}$

(iv) $f(x) = x^{2/3}$

$f'(x) = \frac{2}{3} x^{-1/3}$

$= \frac{2}{3x^{1/3}}$

Critical points:

$f'(x) = 0$

$\Rightarrow \frac{2}{3x^{1/3}} = 0$

$\Rightarrow 2 = 0$

no solution

or $f'(x)$ is undefined

$3x^{1/3} = 0$

$\Rightarrow x = 0$

Stationary points:

$f'(x) = 0$

no stationary points.

(v) $f(x) = x^{1/3}(x+4)$

$\cancel{f'(x)} = x^{4/3} + 4x^{1/3}$

$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3}$

$= \frac{4}{3}x^{1/3} + \frac{4}{3}\frac{1}{x^{2/3}}$

$= \frac{4x^{1/3}.x^{2/3} + 4}{3x^{2/3}}$

$= \frac{4x+4}{3x^{2/3}}$

critical points:

$f'(x) = 0$

$\Rightarrow \frac{4x+4}{3x^{2/3}} = 0$

$\Rightarrow 4x+4 = 0$

$\Rightarrow x = -1$

or $f'(x)$ is undefined

$3x^{2/3} = 0$

$\Rightarrow x = 0$

Stationary points:

$f'(x) = 0$

$\Rightarrow x = -1$

(vi) $f(x) = \cos 3x$

$f'(x) = -3\sin 3x$

Critical points:

$f'(x) = 0$

$\Rightarrow -3\sin 3x = 0$

$\Rightarrow \sin 3x = 0$

$\Rightarrow \sin 3x = \sin n\pi$

$\Rightarrow 3x = n\pi$

$\Rightarrow x = \frac{n\pi}{3}$ $[n = 0, \pm 1, \pm 2, \dots]$

Stationary points:

$f'(x) = 0$

$\Rightarrow x = \frac{n\pi}{3}$

3 (i) $f(x) = 2x^3 - 9x^2 + 12x$

$f'(x) = 6x^2 - 18x + 12$

$f''(x) = 12x - 18$

Relative extrema using 1st derivative test:

<u>critical points:</u> or $f'(x)$ is undefined

×

$f'(x) = 0$

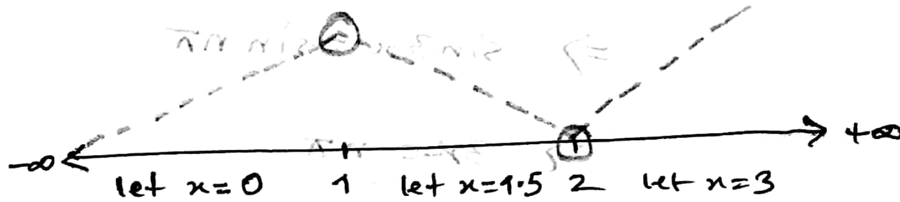
$\Rightarrow 6x^2 - 18x + 12 = 0$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow x^2 - x - 2x + 2 = 0$

$\Rightarrow (x-1)(x-2) = 0$

$\Rightarrow x = 1, 2$



$f'(0) > 0$	$f'(1.5) < 0$	$f'(3) > 0$
increasing	decreasing	increasing

$\therefore$ relative maximum at $x = 1$ and

" minimum at $x = 2$

Relative extrema using 2nd derivative test:

x	$f''(x)$	comment
1	$f''(1) < 0$	maximum
2	$f''(2) > 0$	minimum

(ii) $f(x) = \frac{x}{2} - \sin x$, $0 < x < 2\pi$

$f'(x) = \frac{1}{2} - \cos x$

$f''(x) = \sin x$

<u>Critical points:</u>

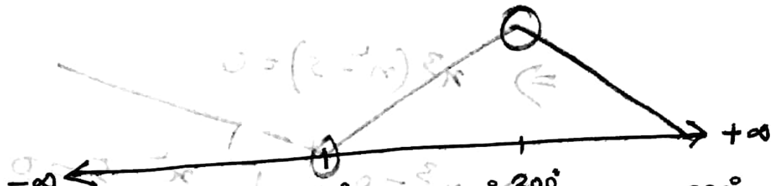
$f'(x) = 0$

$\Rightarrow \frac{1}{2} - \cos x = 0$

$\Rightarrow \cos x = \frac{1}{2}$

$\Rightarrow \cos x = \cos 60^\circ, \cos 300^\circ$

$\therefore x = 60^\circ, 300^\circ$



let $x = 40^\circ$	let $x = 80^\circ$	let $x = 320^\circ$
$f'(40^\circ) < 0$	$f'(80^\circ) > 0$	$f'(320^\circ) < 0$
decreasing	increasing	decreasing

$\therefore$ relative maximum at $x = 300^\circ$

" minimum at $x = 60^\circ$

Relative extrema using 2nd derivative test:

x	$f''(x)$	comment
60°	$f''(60^\circ) > 0$	minimum
300°	$f''(300^\circ) < 0$	maximum

4| (i) $f'(x) = x^3(x^2-5)$

$f''(x) = 5x^4 - 15x^2$

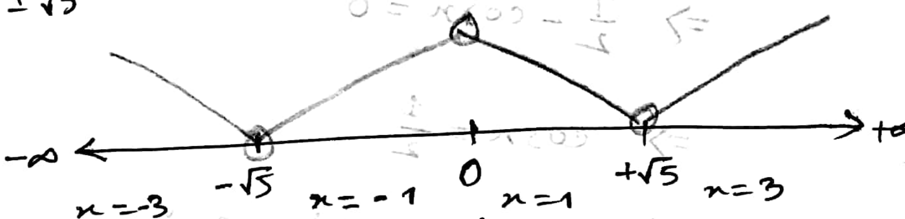
<u>critical points</u>: or $f'(x)$ is undefined ✗

$f'(x) = 0$

$\Rightarrow x^3(x^2-5) = 0$

$\Rightarrow x^3 = 0$	$x^2 - 5 = 0$
$\Rightarrow x = 0$	$\Rightarrow x = \pm\sqrt{5}$

$\therefore x = 0, \pm\sqrt{5}$



$f'(-3) \not> 0$	$f'(-1) > 0$	$f'(1) < 0$	$f'(3) > 0$
decreasing	increasing	decreasing	increasing

relative maximum at $x=0$

" minimum at $x = \pm\sqrt{5}$

<u>Using 2nd derivative</u>:

x	$f''(x)$	comment
0	$f''(0) = 0$	inconclusive
$+\sqrt{5}$	$f''(\sqrt{5}) > 0$	minimum
$-\sqrt{5}$	$f''(-\sqrt{5}) \not> 0$	minimum

Leibnitz's Theorem

08 - Nov - 20

12) $x = \tan(\ln y)$

$\Rightarrow \ln y = \tan^{-1} x$

$\Rightarrow y = e^{\tan^{-1} x}$ —— (i)

Differentiating (i) with respect to x,

$y_1 = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2}$

$\Rightarrow (1+x^2) y_1 = y$ —— (ii)

Differentiating (ii) with respect to x,

$(1+x^2) \frac{d}{dx}(y_1) + y_1 \frac{d}{dx}(1+x^2) = \frac{d}{dx}(y)$

$\Rightarrow (1+x^2) y_2 + 2x y_1 = y_1$

$\Rightarrow (1+x^2) y_2 + (2x-1) y_1 = 0$ —— (iii)

Applying Leibnitz's theorem in equation (iii)

$\frac{d^n}{dx^n}\left[(1+x^2) y_2\right] + \frac{d^n}{dx^n}\left[(2x-1) y_1\right] = \frac{d^n}{dx^n}[0]$

$\Rightarrow \left[y_{n+2}(1+x^2) + {}^nc_1\, y_{n+1} \cdot 2x + {}^nc_2\, y_n \cdot 2 + {}^nc_3\, y_{n-1} \cdot 0\right]$

$+ \left[y_{n+1}(2x-1) + {}^nc_1\, y_n \cdot 2 + {}^nc_2\, y_{n-1} \cdot 0\right] = 0$

$\Rightarrow (1+x^2) y_{n+2} + 2nx\, y_{n+1} + \frac{n(n-1)}{2} y_n \cdot 2 + (2x-1) y_{n+1} + 2n y_n = 0$

$\Rightarrow (1+x^2) y_{n+2} + (2nx + 2x - 1) y_{n+1} + n(n+1) y_n = 0$

Showed

9| $\log_e y = \tan^{-1}x$

$\Rightarrow \ln y = \tan^{-1}x$

$\Rightarrow y = e^{\tan^{-1}x}$

$\Rightarrow y_1 = e^{\tan^{-1}x} \cdot \frac{1}{1+x^2}$

$\Rightarrow y_1(1+x^2) = y$

$\Rightarrow (1+x^2)y_1 = y$

$\Rightarrow (1+x^2)\frac{d}{dx}(y_1) + y_1\frac{d}{dx}(1+x^2) = \frac{d}{dx}y$

$\Rightarrow (1+x^2)y_2 + 2xy_1 = y_1$

$\Rightarrow (1+x^2)y_2 + (2x-1)y_1 = 0$

$\Rightarrow [y_{n+2}(1+x^2) + {}^nc_1\, y_{n+1}\cdot 2x + {}^nc_2\, y_n \cdot 2 + {}^nc_3\, y_{n-1}\cdot 0]$

$+ [y_{n+1}(2x-1) + {}^nc_1\, y_n \cdot 2 + {}^nc_2\, y_{n-1}\cdot 0] = 0$

$\Rightarrow (1+x^2)y_{n+2} + y_{n+1}\cdot 2nx + \frac{n(n-1)}{2}y_n \cdot 2 + (2x-1)y_{n+1}$

$+ 2n y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + (n^2-n+2n)y_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + (2nx+2x-1)y_{n+1} + n(n+1)y_n = 0$

showed

4| same as 9.

1] $y = \tan^{-1} x$

$$\Rightarrow y_1 = \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2) y_1 = 1$$

$$\Rightarrow (1+x^2)\frac{d}{dx}(y_1) + y_1 \frac{d}{dx}(1+x^2) = \frac{d}{dx}(1)$$

$$\Rightarrow (1+x^2) y_2 + 2x \cdot y_1 = 0$$

$$\Rightarrow \left[y_{n+2}(1+x^2) + {}^nc_1\, y_{n+1} \cdot 2x + {}^nc_2\, y_n \cdot 2 + {}^nc_3 \cdot y_{n-1} \cdot 0\right]$$

$$+ \left[y_{n+1}\, 2x + {}^nc_1\, y_n \cdot 2 + {}^nc_2\, y_{n-1} \cdot 0\right] = 0$$

$$\Rightarrow (1+x^2) y_{n+2} + y_{n+1} \cdot 2nx + 2 \cdot \frac{n(n-1)}{2} y_n + 2x \cdot y_{n+1} + 2n y_n = 0$$

$$\Rightarrow (1+x^2) y_{n+2} + (2nx + 2x) y_{n+1} + (n^2 - n + 2n) y_n = 0$$

$$\Rightarrow (1+x^2) y_{n+2} + 2(n+1)x\, y_{n+1} + n(n+1) y_n = 0$$

Showed

3) $y = \cot^{-1} x$

$$\Rightarrow y_1 = -\frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2) y_1 = -1$$

$$\Rightarrow (1+x^2)\frac{d}{dx}(y_1) + y_1 \frac{d}{dx}(1+x^2) = \frac{d}{dx}(-1)$$

$$\Rightarrow (1+x^2) y_2 + y_1 \cdot 2x = 0$$

$$\Rightarrow [y_{n+2}(1+x^2) + {}^nc_1\, y_{n+1} \cdot 2x + {}^nc_2\, y_n \cdot 2 + {}^nc_3\, y_{n-1} \cdot 0]$$

$$+ [y_{n+1} 2x + {}^nc_1 y_n \cdot 2 + {}^nc_2 \cdot y_{n-1} \cdot 0] = 0$$

$$\Rightarrow (1+x^2) y_{n+2} + 2nx\, y_{n+1} + 2 \cdot \frac{n(n-1)}{2} + 2x\, y_{n+1} + 2n\, y_n = 0$$

$$\Rightarrow (1+x^2) y_{n+2} + (2nx + 2x) y_{n+1} + (n^2 - n - 2n) y_n = 0$$

$$\Rightarrow (1+x^2) y_{n+2} + 2(n+1)x\, y_{n+1} + n(n+1)\, y_n = 0$$

Showed

3] $y\sqrt{1-x^2} = \sin^{-1}x$

$$\Rightarrow y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1 = \frac{\sqrt{1-x^2}\,\frac{d}{dx}(\sin^{-1}x) - \sin^{-1}x\,\frac{d}{dx}(\sqrt{1-x^2})}{(\sqrt{1-x^2})^2}$$

$$\Rightarrow y_1 = \frac{\sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}} - \sin^{-1}x\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)}{1-x^2}$$

$$\Rightarrow y_1 = \frac{1 + \frac{x\sin^{-1}x}{\sqrt{1-x^2}}}{1-x^2}$$

$$\Rightarrow (1-x^2)y_1 = 1+xy$$

$$\cancel{\Rightarrow (1-x^2)\frac{d}{dx}(y_1) + y_1\frac{d}{dx}(1-x^2) = \frac{d}{dx}(1+xy)}$$

$$\Rightarrow (1-x^2)y_1 - xy = 1$$

$$\Rightarrow \left[y_{n+1}(1-x^2) + {}^nc_1\,y_n(-2x) + {}^nc_2\,y_{n-1}(-2) + {}^nc_3\,y_{n-2}\cdot 0\right]$$
$$-\left[y_n\cdot x + {}^nc_1\,y_{n-1}\cdot 1 + {}^nc_2\,y_{n-2}\cdot 0\right] = 0$$

$$\Rightarrow (1-x^2)y_{n+1} - 2nx\,y_n - n(n-1)\,y_{n-1} - xy_n - ny_{n-1} = 0$$

$$\Rightarrow (1-x^2)y_{n+1} - (2n+1)xy_n - n^2y_{n-1} = 0$$

showed

5 $y = e^{m\sin^{-1}x}$

$$\Rightarrow y_1 = m e^{m\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1 = \frac{my}{\sqrt{1-x^2}}$$

$$\Rightarrow \left(\sqrt{1-x^2}\right) y_1 = my$$

$$\Rightarrow (1-x^2) y_1^2 = m^2 y^2$$

$$\Rightarrow (1-x^2)\, 2y_1 \cdot y_2 + y_1^2(-2x) = m^2 2y y_1$$

$$\Rightarrow (1-x^2)\, 2y_1 y_2 - 2x y_1^2 - 2m^2 y y_1 = 0$$

$$\Rightarrow 2y_1\left[(1-x^2) y_2 - x y_1 - m^2 y\right] = 0$$

$$\Rightarrow (1-x^2) y_2 - x y_1 - m^2 y = 0$$

$$\Rightarrow \left[y_{n+2}(1-x^2) + {}^nc_1\, y_{n+1}(-2x) + {}^nc_2\, y_n(-2) + {}^nc_3\, y_{n-1} \cdot 0\right]$$

$$- \left[y_{n+1}\, x + {}^nc_1\, y_n\, 1 + {}^nc_2\, y_{n-1}\, 0\right] - m^2 y_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} + (-2x) n\, y_{n+1} - 2 \cdot \frac{n(n-1)}{2} y_n - (x y_{n+1} + n y_n) - m^2 y_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - (2nx + x) y_{n+1} - (n^2 - n) y_n - n y_n - m^2 y_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 - n + n + m^2) y_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 + m^2) y_n = 0$$

showed

6) $y = (\sin^{-1}x)^2$

$\Rightarrow y_1 = 2\sin^{-1}x \cdot \dfrac{1}{\sqrt{1-x^2}}$

$\Rightarrow (\sqrt{1-x^2})\, y_1 = 2\sqrt{y}$

$\Rightarrow (1-x^2) y_1^2 = 4y$

$\Rightarrow (1-x^2)\, 2y_1 y_2 + y_1^2(-2x) = 4y_1$

$\Rightarrow (1-x^2) 2y_1 y_2 - 2x y_1^2 - 4y_1 = 0$

$\Rightarrow 2y_1 [(1-x^2) y_2 - x y_1 - 2] = 0$

$\Rightarrow (1-x^2) y_2 - x y_1 - 2 = 0$

$\Rightarrow (1-x^2) y_2 - x y_1 = 2$

$\Rightarrow [y_{n+2}(1-x^2) + {}^nc_1\, y_{n+1}(-2x) + {}^nc_2\, y_n(-2) + {}^nc_3\, y_{n-1} \cdot 0]$

$- [y_{n+1} x + {}^nc_1 y_n \cdot 1 + y_{n-1} \cdot 0 \cdot {}^nc_2] = 0$

$\Rightarrow (1-x^2) y_{n+2} - 2nx\, y_{n+1} - 2 \cdot \dfrac{n(n-1)}{2} y_n - x y_{n+1}$

$- n y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2-n) y_n - n y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0$

showed

7) $\log_e y = a\sin^{-1}x$

$\Rightarrow \ln y = a\sin^{-1}x$

$\Rightarrow y = e^{a\sin^{-1}x}$

$\Rightarrow y_1 = a e^{a\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \left(\sqrt{1-x^2}\right) y_1 = ay$

$\Rightarrow (1-x^2)\, y_1^2 = a^2 y^2$

$\Rightarrow (1-x^2)\, 2y_1 y_2 + y_1^2(-2x) = a^2 2y y_1$

$\Rightarrow (1-x^2)\, 2y_1 y_2 - 2x\, y_1^2 - a^2 2y y_1 = 0$

$\Rightarrow 2y_1\left[(1-x^2)y_2 - xy_1 - a^2 y\right] = 0$

$\Rightarrow (1-x^2)\, y_2 - xy_1 - a^2 y = 0$

$\Rightarrow \left[y_{n+2}(1-x^2) + {}^nc_1\, y_{n+1}(-2x) + {}^nc_2\, y_n(-2) + {}^nc_3\, y_{n-1}\cdot 0\right]$

$- \left[y_{n+1}x + {}^nc_1 y_n\, 1 + {}^nc_2 y_{n-1}\cdot 0\right] - a^2 y_n = 0$

$\Rightarrow (1-x^2)\, y_{n+2} - 2nx y_{n+1} - 2\cdot\frac{n(n-1)}{2} y_n - xy_{n+1} - y_n n - a^2 y_n = 0$

$\Rightarrow (1-x^2)\, y_{n+2} - (2n+1)x y_{n+1} - (n^2-n)y_n - y_n n - a^2 y_n = 0$

$\Rightarrow (1-x^2)\, y_{n+2} - (2n+1)x y_{n+1} - (n^2+a^2)\, y_n = 0$

showed

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8] $y = e^{m\cos^{-1}x}$

$\Rightarrow y_1 = m e^{m\cos^{-1}x} \cdot \left(-\frac{1}{\sqrt{1-x^2}}\right)$

$\Rightarrow \left(-\sqrt{1-x^2}\right) y_1 = my$

$\Rightarrow (1-x^2) y_1^2 = m^2 y^2$

$\Rightarrow (1-x^2) 2y_1 y_2 + y_1^2 (-2x) = m^2 2y y_1$

$\Rightarrow (1-x^2) 2y_1 y_2 - 2x y_1^2 - m^2 2y y_1 = 0$

$\Rightarrow 2y_1 \left[(1-x^2) y_2 - x y_1 - m^2 y\right] = 0$

$\Rightarrow (1-x^2) y_2 - x y_1 - m^2 y = 0$

$\Rightarrow \left[y_{n+2} (1-x^2) + n_{C_1} y_{n+1} (-2x) + n_{C_2} y_n (-2) + n_{C_3} y_{n-1} \cdot 0\right]$

$- \left[y_{n+1} x + n_{C_1} y_n \cdot 1 + n_{C_2} y_{n-1} \cdot 0\right] - m^2 y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - 2nx y_{n+1} - 2 \frac{n(n-1)}{n} y_n - x y_{n+1} - n y_n - m^2 y_n =$

$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 + m^2) y_n = 0$

showed

9| $\log_e y = \tan^{-1} x$

$\Rightarrow \ln y = \tan^{-1} x$

$\Rightarrow y = e^{\tan^{-1} x}$

$\Rightarrow$ then same as 4

10| $y = (\cos^{-1} x)^2$

$\Rightarrow y_1 = 2\cos^{-1} x \left(-\frac{1}{\sqrt{1-x^2}}\right)$

$\Rightarrow \left(-\sqrt{1-x^2}\right) y_1 = 2\cos^{-1} x$

$\Rightarrow \left(-\sqrt{1-x^2}\right) y_1 = 2\sqrt{y}$

$\Rightarrow (1-x^2) y_1^2 = 4y$

$\Rightarrow$ then same as 6

11| $\ln y = m\cos^{-1} x$

$\Rightarrow y = e^{m\cos^{-1} x}$

$\Rightarrow$ then same as 8

Rolle's and Mean Value Theorem

a.

1) $f(x) = x^2 - 6x + 8$; $[2,4]$

Here $f(x)$ is differentiable on $(2,4)$ and continuous on $[2,4]$, if $f(2) = 2^2 - 6\cdot2 + 8 = 0$

$f(4) = 4^2 - 6\cdot4 + 8 = 0$

Since $f(2) = f(4) = 0$ there is at least one number $c \in (2,4)$ such that, $f'(c) = 0$

$f(x) = x^2 - 6x + 8$

$f'(x) = 2x - 6$

$f'(c) = 2c - 6$

Now, $2c - 6 = 0$

$\Rightarrow 2c = 6$

$\Rightarrow c = 3$

$\therefore c = 3 \in (2,4)$ (verified)

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2] $f(x) = \cos x \;;\; \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Here $f(x)$ is differentiable on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and continuous on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. If

$$f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$$

$$f\left(\frac{3\pi}{2}\right) = \cos\frac{3\pi}{2} = 0$$

since $f\left(\frac{\pi}{2}\right) = f\left(\frac{3\pi}{2}\right) = 0$ there is at least one number $c \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ such that $f'(c) = 0$

$$f'(x) = -\sin x$$

$$f'(c) = -\sin c$$

Now, $-\sin c = 0$

$\Rightarrow \sin c = 0$

$\Rightarrow \sin c = \sin n\pi$

$\Rightarrow c = n\pi$

$n = 0 \;;\; c = 0 \notin \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$n = 1 \;;\; c = \pi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$n = 2 \;;\; c = 2\pi \notin \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\therefore c = \pi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

(verified)

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3) $f(x) = \frac{x}{2} - \sqrt{x}$; $[0,4]$

Here $f(x)$ is differential and continuous on $(0,4)$. if,

$f(0) = \frac{0}{2} - \sqrt{0} = 0$

$f(4) = \frac{4}{2} - \sqrt{4} = 2-2 = 0$

since $f(0) = f(4) = 0$, by Rolle's theorem there is at least one number $c \in (0,4)$ such that $f'(c) = 0$

$f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{x}}$

$= \frac{1}{2}\left(1 - \frac{1}{\sqrt{x}}\right)$

$f'(c) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{c}}\right)$

NOW, $\frac{1}{2}\left(1 - \frac{1}{\sqrt{c}}\right) = 0$

$\Rightarrow \frac{1}{2} - \frac{1}{2\sqrt{c}} = 0$

$\Rightarrow \frac{\sqrt{c}-1}{2\sqrt{c}} = 0$

$\Rightarrow \sqrt{c} - 1 = 0$

$\Rightarrow \sqrt{c} = 1$

$\Rightarrow c = 1$

$\therefore c = 1 \in (0,4)$ (verified)

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b.

1] $f(x) = x^3 + x - 4$; $[-1, 2]$

Since polynomials are everywhere ($\mathbb{R}$) continuous and differentiable so $f(x)$ is differentiable and continuous on $(-1, 2)$. Then by Mean value Theorem there exists at least one number $c \in (-1, 2)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{——} (1)$$

$f(x) = x^3 + x - 4$

$f'(x) = 3x^2 + 1$

$f'(c) = 3c^2 + 1$

$f(-1) = (-1)^3 + (-1) - 4 = -6$

$f(2) = 2^3 + 2 - 4 = 6$

from (1),

$$3c^2 + 1 = \frac{6 - (-6)}{2 - (-1)}$$

$$\Rightarrow 3c^2 + 1 = \frac{12}{3}$$

$$\Rightarrow 3c^2 = 3$$

$$\Rightarrow c^2 = 1$$

$$\Rightarrow c = \pm 1$$

$\therefore c = 1 \in (-1, 2)$ (verified).

3) $f(x) = \sqrt{x+1}$; $[0,3]$

since $f(x)$ is differentiable and continuous on $[0,3]$, there is at least one number $c \in (0,3)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{——— (1)}$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} \cdot 1$$

$$f'(c) = \frac{1}{2\sqrt{c+1}}$$

$f(0) = 1$

$f(3) = 2$

from (1),

$$\frac{1}{2\sqrt{c+1}} = \frac{2-1}{3-0}$$

$$\Rightarrow 2\sqrt{c+1} = 3$$

$$\Rightarrow \sqrt{c+1} = \frac{3}{2}$$

$$\Rightarrow c+1 = \frac{9}{4}$$

$$\Rightarrow c = \frac{9}{4} - 1$$

$$\Rightarrow c = \frac{5}{4}$$

$\therefore c = \frac{5}{4} \in (0,3)$ (verified).

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3| $f(x) = \sqrt{25-x^2}$; $[0,5]$

Since $f(x)$ is differentiable and continuous on $[0,5]$ so by Mean value theorem there is at least one number $c \in (0,5)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{—— (1)}$$

$$f'(x) = \frac{1}{2\sqrt{25-x^2}} \cdot (-2x)$$

$$= -\frac{x}{\sqrt{25-x^2}}$$

$f(0) = 5$

$f(5) = 0$

$$f'(c) = -\frac{c}{\sqrt{25-c^2}}$$

from (1),

$$-\frac{c}{\sqrt{25-c^2}} = \frac{0-5}{5-0}$$

$$\Rightarrow 5c = 5\sqrt{25-c^2}$$

$$\Rightarrow c = \sqrt{25-c^2}$$

$$\Rightarrow c^2 = 25-c^2$$

$$\Rightarrow 2c^2 = 25$$

$$\Rightarrow c = \pm\frac{5}{\sqrt{2}}$$

$\therefore c = \frac{5}{\sqrt{2}} \in (0,5)$ (verified).

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Partial Derivative

09-NOV-2017

1] $f(x) = 3x^3y^2$

$f_x = 9x^2y^2$

$\therefore f_x(x,1) = 9x^2$

$f_x(1,2) = 9 \cdot 4$

$= 36$

$f_y = 6x^3y$

$\therefore f_y(1,y) = 6y$

$f_y(1,2) = 6 \cdot 2$

$= 12$

2] (i) $f(x,y) = xe^{-y} + 5y$

$f_x = e^{-y}$

$\therefore f_x(3,0) = e^{-0}$

$= 1$

$f_y = -x e^{-y} + 5$

$\therefore f_y(4,2) = -4 \cdot e^{-2} + 5$

$= -4e^{-2} + 5$

(ii) $f(x,y) = \sqrt{3x+2y}$

$$f_x = \frac{1}{2\sqrt{3x+2y}} \cdot (3+0)$$

$$= \frac{3}{2\sqrt{3x+2y}}$$

$$\therefore f_x(3,0) = \frac{3}{2\sqrt{3 \cdot 3+2 \cdot 0}}$$

$$= \frac{3}{6}$$

$$= \frac{1}{2}$$

$$f_y = \frac{1}{2\sqrt{3x+2y}} \cdot (0+2)$$

$$= \frac{1}{\sqrt{3x+2y}}$$

$$\therefore f_y(4,2) = \frac{1}{\sqrt{3 \cdot 4+2 \cdot 2}}$$

$$= \frac{1}{\sqrt{16}}$$

$$= \frac{1}{4}$$

3) (i) $f(x,y) = 4x^2 - 2y + 7x^4y^5$

$$f_x = \frac{\partial}{\partial x}\left(4x^2 - 2y + 7x^4y^5\right)$$

$$= 8x + 28x^3y^5$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x)$$

$$= 8 + 84x^2y^5$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x)$$

$$= 140y^4x^3$$

$$f_y = \frac{\partial}{\partial y}\left(4x^2 - 2y + 7x^4y^5\right)$$

$$= -2 + 35x^4y^4$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y)$$

$$= 140x^4y^3$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y)$$

$$= 140x^3y^4$$

(ii) $f(x,y) = e^{x^2+xy+y^2}$

$$f_x = e^{x^2+xy+y^2} \cdot (2x+y)$$

$$f_{xx} = \frac{\partial}{\partial x}(f_x)$$

$$= e^{x^2+xy+y^2} \frac{\partial}{\partial x}(2x+y) + (2x+y) \cdot \frac{\partial}{\partial x}\left(e^{x^2+xy+y^2}\right)$$

$$= e^{x^2+xy+y^2}(2+0) + (2x+y) \cdot e^{x^2+xy+y^2} \cdot (2x+y)$$

$$= 2e^{x^2+xy+y^2} + (2x+y)^2 \cdot e^{x^2+xy+y^2}$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x)$$

$$= e^{x^2+xy+y^2} \frac{\partial}{\partial y}(2x+y) + (2x+y) \cdot \frac{\partial}{\partial y}\left(e^{x^2+xy+y^2}\right)$$

$$= e^{x^2+xy+y^2} \cdot 1 + (2x+y) \cdot e^{x^2+xy+y^2} \cdot (x+2y)$$

$$= e^{x^2+xy+y^2} + (2x+y)(x+2y)\, e^{x^2+xy+y^2}$$

$$f_y = e^{x^2+xy+y^2} \cdot (x+2y)$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y)$$

$$= e^{x^2+xy+y^2} \frac{\partial}{\partial y}(x+2y) + (x+2y) \frac{\partial}{\partial y}\left(e^{x^2+xy+y^2}\right)$$

$$= e^{x^2+xy+y^2} \cdot 2 + (x+2y) \cdot e^{x^2+xy+y^2} \cdot (x+2y)$$

$$= 2e^{x^2+xy+y^2} + (x+2y)^2 \cdot e^{x^2+xy+y^2}$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y)$$

$$= e^{x^2+xy+y^2} \cdot 1 + (x+2y)\, e^{x^2+xy+y^2}(2x+y)$$

$$= e^{x^2+xy+y^2} + (x+2y)(2x+y) \cdot e^{x^2+xy+y^2}$$

4) (i) $z = \sin(y^2-4x)$

$f_x = \frac{\partial}{\partial x}(z)$

$= \cos(y^2-4x) \cdot (-4)$

$= -4\cos(y^2-4x)$

$f_y = \frac{\partial}{\partial y}(z)$

$= \cos(y^2-4x) \cdot 2y$

$= 2y\cos(y^2-4x)$

$\therefore f_x(2,1) = -4$

$\therefore f_y(-2,4) = 2\times 4 \cos(16+8)$

$= 7.30$

(ii) $z = (x+y)^{-1}$

$f_x = \frac{\partial}{\partial x}(z)$

$= -1 \cdot (x+y)^{-2} \cdot 1$

$= -\frac{1}{(x+y)^2}$

$f_y = \frac{\partial}{\partial y}(z)$

$= -1 \cdot (x+y)^{-2} \cdot 1$

$= -\frac{1}{(x+y)^2}$

$\therefore f_x(2,1) = -\frac{1}{(2+1)^2}$

$= -\frac{1}{9}$

$\therefore f_y(-2,4) = -\frac{1}{(-2+4)^2}$

$= -\frac{1}{4}$

51 $f(x,y,z) = x^3y^5z^7 + xy^2 + y^3z$

$f_x = 3x^2y^5z^7 + y^2 + 0$

$f_y = 5x^3y^4z^7 + 2xy + 3y^2z$

$f_z = 7x^3y^5z^6 + 0 + y^3$

$f_{xy} = \frac{\partial}{\partial y}(3x^2y^5z^7 + y^2)$

$= 15x^2y^4z^7 + 2y$

$f_{yz} = \frac{\partial}{\partial z}(5x^3y^4z^7 + 2xy + 3y^2z)$

$= 35x^3y^4z^6 + 3y^2$

$f_{xz} = \frac{\partial}{\partial z}(3x^2y^5z^7 + y^2)$

$= 21x^2y^5z^6$

$f_{zz} = \frac{\partial}{\partial z}(7x^3y^5z^6 + y^3)$

$= 42x^3y^5z^5$ 0

$f_{zyy} = ((f_z)_y)_y$

$\therefore f_{zy} = \frac{\partial}{\partial y}(7x^3y^5z^6 + y^3)$

$= 35x^3y^4z^6 + 3y^2$

$$\therefore f_{zyy} = \frac{\partial}{\partial y}\left(35x^3y^4z^6 + 3y^2\right)$$

$$= 140\, x^3y^3z^6 + 6y$$

$$f_{zxy} = \left((f_z)_x\right)_y$$

$$\therefore \quad f_{zx} = \frac{\partial}{\partial x}\left(7x^3y^5z^6 + y^3\right)$$

$$= 21x^2y^5z^6$$

$$\therefore \quad f_{zxy} = \frac{\partial}{\partial y}\left(21x^2y^5z^6\right)$$

$$= 105x^2y^4z^6$$

$$f_{zyx} = \left((f_z)_y\right)_x$$

$$f_{zy} = \frac{\partial}{\partial y}\left(7x^3y^5z^6 + y^3\right)$$

$$= 35x^3y^4z^6 + 3y^2$$

$$\therefore \quad f_{zyx} = \frac{\partial}{\partial x}\left(35x^3y^4z^6 + 3y^2\right)$$

$$= 105x^2y^4z^6 + 0$$

$$= 105x^2y^4z6$$

$$f_{xxyz} = \left(\left((f_x)_x\right)_y\right)_z$$

$$f_{xx} = \frac{\partial}{\partial x}\left(3x^2y^5z^7 + y^2\right)$$

$$= 6xy^5z^7$$

$$f_{xxy} = \frac{\partial}{\partial y}\left(6xy^5z^7\right)$$

$$= 30xy^4z^7$$

$$f_{xxyz} = \frac{\partial}{\partial z}\left(30xy^4z^7\right)$$

$$= 210\,xy^4z^6$$

6) $$f(x,y,z) = \sqrt{xy} + \ln(x^2z^3) - x\tan z$$

$$= x^{1/2}y^{1/2} + \ln(x^2z^3) - x\tan z$$

$$f_x = y^{1/2}\cdot\frac{1}{2}x^{1/2-1} + \frac{1}{x^2z^3}\cdot 2xz^3 - \tan z$$

$$= y^{1/2}\cdot\frac{1}{2}x^{-1/2} + \frac{2}{x} - \tan z$$

$$= y^{1/2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x}} + \frac{2}{x} - \tan z$$

$$f_{xy} = \frac{1}{2}\frac{1}{\sqrt{x}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{y}} + 0 - 0$$

$$= \frac{1}{4\sqrt{xy}}$$

$$f_z = 0 + \frac{1}{x^2 z^3} \cdot x^2 3z^2 - x\sec^2 z$$

$$= \frac{3}{z} - x\sec^2 z$$

$$f_{xyz} = 0$$

7) $f(x,y,z) = (x^2-y^2)\cos x + x^5 z^2 - 30z$

$$= x^2\cos x - y^2\cos x + x^5 z^2 - 30z$$

$$f_x = 2x\cos x - \sin x \cdot x^2 + y^2\sin x + 5x^4 x^2$$

$$f_{xy} = 0 - 0 + 2y\sin x + 0$$

$$= 2y\sin x$$

$$f_{xx} = -2x\sin x + 2\cos x - 2x\sin x - x^2\cos x + y^2\cos x + 20x^3 z^2$$

$$f_{xxz} = -0 + 0 - 0 - 0 + 0 + 40x^3 z$$

$$= 40x^3 z$$

$$f_z = 0 + 2x^5 z - 30$$

$$f_{zy} = 0$$

$$f_{zyz} = 0$$

8| $f(x,y) = x^3\ln(x^2y) + x^4y - e^{3x}x^2$

$$f_x = x^3 \cdot \frac{1}{x^2y} \cdot 2xy + \ln(x^2y) \cdot 3x^2 + 4x^3y - [e^{3x} \cdot 2x + x^2e^{3x} \cdot 3]$$

$$= 2x^2 + 3x^2\ln(x^2y) + 4x^3y - 2xe^{3x} - 3x^2e^{3x}$$

$$f_{xx} = 4x + 3\left[x^2 \frac{1}{x^2y} \cdot 2xy + \ln(x^2y) \cdot 2x\right] + 12x^2y - 2[xe^{3x} \cdot 3 + e^{3x} \cdot 1]$$
$$- 3[x^2e^{3x} \cdot 3 + e^{3x} \cdot 2x]$$

$$= 4x + 6x + 6x\ln(x^2y) + 12x^2y - 6xe^{3x} - 2e^{3x} - 9x^2e^{3x} - 6xe^{3x}$$

$$= 10x + 6x\ln(x^2y) + 12x^2y - 12xe^{3x} - 9x^2e^{3x} - 2e^{3x}$$

$$f_{xy} = 0 + 3x^2 \frac{1}{x^2y} \cdot x^2 + 4x^3 - 0 - 0$$

$$= \frac{3x^2}{y} + 4x^3$$

$$f_y = x^3 \frac{1}{x^2y} \cdot x^2 + x^4 - 0$$

$$= \frac{x^3}{y} + x^4$$

$$= x^3 \cdot y^{-1} + x^4$$

$$f_{xy} = x^3\left[(-1)y^{-1-1}\right] + 4x^3$$

$$= -x^3 \frac{1}{y^2} + 4x^3$$

$$f_{yy} = -x^3 \frac{1}{y^2} + 0$$

$$= -x^3 \frac{1}{y^2}$$

Successive Differentiation

17-Nov-20

a.

1) $y = x^n$

Differentiating y with respect to x successively.

$$y_1 = n x^{n-1}$$

$$y_2 = n(n-1) x^{n-2}$$

$$y_3 = n(n-1)(n-2) x^{n-3}$$

$$\therefore y_n = n! \, x^{n-n}$$

$$= n!$$

2) $y = (ax+b)^n$

Differentiating y with respect to x successively.

$$y_1 = n(ax+b)^{n-1} \cdot a$$ [chain rule]

$$y_2 = n(n-1)(ax+b)^{n-2} \cdot a^2$$

$$y_3 = n(n-1)(n-2)(ax+b)^{n-3} \cdot a^3$$

$$\therefore y_n = n! \, (ax+b)^{n-n} \cdot a^n$$

$$= a^n n!$$

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3]

$$y = \ln(ax+b)$$

$$y_1 = \frac{1}{ax+b} \cdot a$$

$$= a(ax+b)^{-1}$$

$$y_2 = (-1)(ax+b)^{-2} \cdot a^2$$

$$y_3 = (-1)(-2)(ax+b)^{-3} \cdot a^3$$

$$\therefore y_n = (-1)(-2) \ldots [-(n-1)] \cdot (ax+b)^{-n} \cdot a^n$$

$$= (-1)^{n-1} [1 \cdot 2 \ldots (n-1)] (ax+b)^{-n} \cdot a^n$$

$$= (-1)^{n-1} (n-1)! \cdot (ax+b)^{-n} \cdot a^n$$

4]

$$y = \frac{1}{x+a}$$

$$= (x+a)^{-1}$$

$$y_1 = (-1)(x+a)^{-2} \cdot 1$$

$$y_2 = (-1)(-2)(x+a)^{-3} \cdot 1$$

$$y_3 = (-1)(-2)(-3)(x+a)^{-4} \cdot 1$$

$$\therefore y_n = (-1)^n \cdot (1 \cdot 2 \cdot 3 \ldots n) \cdot (x+a)^{-(n+1)}$$

$$= (-1)^n \cdot n! \, (x+a)^{-(n+1)}$$

5|

$$y = e^{ax}$$

$$y_1 = a e^{ax}$$

$$y_2 = a^2 e^{ax}$$

$$y_3 = a^3 e^{ax}$$

$$\therefore y_n = a^n e^{ax}$$

6|

$$y = \sin(ax+b)$$

$$y_1 = \cos(ax+b) \cdot a$$

$$= a \sin\left(\frac{\pi}{2} + ax + b\right)$$

let, $\theta = ax+b$.

$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

$$y_2 = a^2 \cos\left(\frac{\pi}{2} + ax + b\right)$$

$$= a^2 \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + ax + b\right)$$

$$= a^2 \sin\left(2 \cdot \frac{\pi}{2} + ax + b\right)$$

$$\therefore y_n = a^n \sin\left(n \cdot \frac{\pi}{2} + ax + b\right)$$

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7| $y = \cos(ax+b)$

$$y_1 = -a\sin(ax+b)$$

$$= a\cos\left(\frac{\pi}{2} + ax + b\right) \qquad \left[\because \cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta\right]$$

$$y_2 = -a^2\sin\left(\frac{\pi}{2} + ax + b\right)$$

$$= a^2\cos\left(\pi/2 + \frac{\pi}{2} + ax + b\right)$$

$$= a^2\cos\left(2\cdot\frac{\pi}{2} + ax + b\right)$$

$$\therefore y_n = a^n\cos\left(n\cdot\frac{\pi}{2} + ax + b\right)$$

b| $y = e^{ax}\sin bx$

$$y_1 = e^{ax}\cdot b\cos bx + \sin bx \cdot a \cdot e^{ax}$$

$$= be^{ax}\cos bx + ae^{ax}\sin bx$$

$$= be^{ax}\cos bx + ay$$

$$y_2 = b\left[e^{ax}\cdot(-b\sin bx) + \cos bx\,(ae^{ax})\right] + ay_1$$

$$= -b^2e^{ax}\sin bx + abe^{ax}\cos bx + ay_1$$

$$= -b^2y + a(y_1 - ay) + ay_1$$

$$= -b^2y + ay_1 - a^2y + ay_1$$

$$= -(a^2+b^2)y + 2ay_1$$

showed.

c) $y = e^x \sin x$

$y_1 = e^x \cos x + \sin x\, e^x$

$y_2 = -e^x \sin x + \cos x\, e^x + e^x \cos x + \sin x \cdot e^x$

$= 2e^x \cos x$

$y_3 = 2[-e^x \sin x + \cos x\, e^x]$

$= 2[e^x \cos x - e^x \sin x]$

$y_4 = 2[\cos x\, e^x - e^x \sin x - e^x \cos x - e^x \sin x]$

$= 2[-2e^x \sin x]$

$= -4e^x \sin x$

$\therefore y_4 + 4y = -4e^x \sin x + 4e^x \sin x$

$= 0$

Maclaurin and Taylor Series

(i) Expand $y = \sin x$ in the power of $(x - \frac{\pi}{2})$ or

find the Taylor series for $y = \sin x$ at $x_0 = \frac{\pi}{2}$

$f(x) = \sin x$	$f(\pi/2) = 1$
$f'(x) = \cos x$	$f'(\pi/2) = 0$
$f''(x) = -\sin x$	$f''(\pi/2) = -1$
$f'''(x) = -\cos x$	$f'''(\pi/2) = 0$
$f^4(x) = \sin x$	$f^4(\pi/2) = 1$
$f^5(x) = \cos x$	$f^5(\pi/2) = 0$

$$\therefore \sin x = 1 + \left(x - \frac{\pi}{2}\right)\cdot 0 + \frac{\left(x - \frac{\pi}{2}\right)^2}{2!}\cdot(-1) + \frac{\left(x - \frac{\pi}{2}\right)^3}{3!}\cdot 0$$

$$+ \frac{\left(x - \frac{\pi}{2}\right)^4}{4!}\cdot 1 + \frac{\left(x - \frac{\pi}{2}\right)^5}{5!}\cdot 0 + \dots \infty$$

$$= 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} + \dots \infty$$

(ii) $y = \ln x$ at $x_0 = 2$ or $x - 2 = 0$

$y_1 = \frac{1}{x} = x^{-1}$	$f'(2) = 2^{-1} = \frac{1}{2}$
$y_2 = -x^{-2}$	$f''(2) = -2^{-2} = -\frac{1}{4}$
$y_3 = 2x^{-3}$	$f'''(2) = 2 \cdot 2^{-3} = \frac{2}{8} = \frac{1}{4}$
$y_4 = -6x^{-4}$	$f^4(2) = -6 \cdot 2^{-4} = -\frac{6}{16} = -\frac{3}{8}$
$y_5 = 24x^{-5}$	$f^5(2) = 24 \cdot 2^{-5} = \frac{24}{2^5} = \frac{3}{4}$

$$\therefore \ln x = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{4} \cdot \frac{(x-2)^2}{2!} + \frac{1}{4} \cdot \frac{(x-2)^3}{3!} - \frac{3}{8} \cdot \frac{(x-2)^4}{4!}$$
$$+ \frac{3}{4} \frac{(x-2)^5}{5!} + \ldots\ldots \infty$$

2) $y = e^{ax}$ in the power of $x - 1$.

$f(x) = e^{ax}$	$f(1) = e^a$
$f'(x) = ae^{ax}$	$f'(1) = ae^a$
$f''(x) = a^2 e^{ax}$	$f''(1) = a^2 e^a$
$f'''(x) = a^3 e^{ax}$	$f'''(1) = a^3 e^a$
$f^4(x) = a^4 e^{ax}$	$f^{iv}(1) = a^4 e^a$
$f^5(x) = a^5 e^{ax}$	$f^5(1) = a^5 e^a$

$$\therefore e^{ax} = e^a + (x-1) \cdot ae^a + \frac{(x-1)^2}{2!} a^2 e^a + \frac{(x-1)^3}{3!} a^3 e^a + \frac{(x-1)^4}{4!} a^4 \cdot e^a$$
$$+ \frac{(x-1)^5}{5!} a^5 \cdot e^a + \ldots\ldots \infty$$

31 $y = e^{ax}$ in the power of x.

$f(x) = e^{ax}$	$f(0) = 1$
$f'(x) = ae^{ax}$	$f'(0) = a$
$f''(x) = a^2e^{ax}$	$f''(0) = a^2$
$f'''(x) = a^3e^{ax}$	$f'''(0) = a^3$
$f^4(x) = a^4e^{ax}$	$f^4(0) = a^4$
$f^5(x) = a^5e^{ax}$	$f^5(0) = a^5$

$$\therefore e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \frac{a^4x^4}{4!} + \frac{a^5x^5}{5!} + \cdots\cdots \infty$$

田 $y = \cos x$ in the power of x.

$f(x) = \cos x$	$f(0) = 1$
$f'(x) = -\sin x$	$f'(0) = 0$
$f''(x) = -\cos x$	$f''(0) = -1$
$f'''(x) = \sin x$	$f'''(0) = 0$
$f^4(x) = \cos x$	$f^4(0) = 1$
$f^5(x) = -\sin x$	$f^5(0) = 0$

$$\therefore \cos x = 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 \cdots\cdots \infty$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\cdots \infty$$

41 $y = e^{x}\cos x$

$f(x) = e^{x}\cos x$

$f'(x) = e^{x}(-\sin x) + \cos x\, e^{x}$

$= -e^{x}\sin x + \cos x\, e^{x}$

$f''(x) = -e^{x}\cos x - \sin x\, e^{x} + \cos x\, e^{x} - e^{x}\sin x$

$= -2e^{x}\sin x$

$f'''(x) = -2\left[e^{x}\cos x + \sin x\, e^{x}\right]$

$= -2e^{x}\cos x - 2e^{x}\sin x$

$= -2e^{x}(\cos x + \sin x)$

$f^{4}(x) = -2\left[e^{x}(-\sin x + \cos x) + (\cos x + \sin x)\cdot e^{x}\right]$

$= -2\left[-e^{x}\sin x + e^{x}\cos x + e^{x}\sin x + e^{x}\cos x\right]$

$= -2\left[2e^{x}\cos x\right]$

$= -4e^{x}\cos x$

$f(0) = e^0 \cdot \cos 0 = 1$

$f'(0) = -0 + 1 = 1$

$f''(0) = -2e^0 \sin 0 = 0$

$f'''(0) = -2e^0(\sin 0 + \cos 0) = -2$

$f^4(0) = -4e^0 \cdot \cos 0 = 4$

$\therefore P_n(x) = 1 + 1.x + 0 + (-2) \cdot \frac{x^3}{3!} + (-4)\frac{x^4}{4!}$

$= 1 + x + 0 - \frac{2x^3}{3!} - \frac{4x^4}{4!}$

$\therefore P_0 = 1$

$P_1 = 1 + x$

$P_2 = 1 + x + 0 = 1 + x$

$P_3 = 1 + x + 0 - \frac{2x^3}{3!}$

$P_4 = 1 + x - \frac{2x^3}{3!} - \frac{4x^4}{4!}$

5] $y = \ln(x+1)$

$f(x) = \ln(x+1)$ | $f(0) = \ln 1 = 0$

$f'(x) = \frac{1}{x+1} = (x+1)^{-1}$ | $f'(0) = 1$

$f''(x) = -(x+1)^{-2}$ | $f''(0) = -1$

$f'''(x) = -2(x+1)^{-3}$ | $f'''(0) = 2$

$f^4(x) = -6(x+1)^{-4}$ | $f^4(0) = -6$

$f^5(x) = 24(x+1)^{-5}$ | $f^5(0) = 24$

$$\therefore \ln(x+1) = 0 + x - \frac{x^2}{2!} + 2\frac{x^3}{3!} - 6\frac{x^4}{4!} + 24\frac{x^5}{5!} + \dots \infty$$

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Indeterminate Forms

18-Nov-2017

1] $$\lim_{x \to 1} \frac{\ln x}{x-1} \qquad \left[\frac{0}{1-1} = \frac{0}{0} \text{ form}\right]$$

By L' Hospital's rule :

$$\lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)}$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{1}$$

$$= 1$$

2] $$\lim_{x \to 3} \frac{x-3}{3x^2-13x+12}$$

By L' Hospital's rule :

$$\lim_{x \to 3} \frac{\frac{d}{dx}(x-3)}{\frac{d}{dx}(3x^2-13x+12)}$$

$$= \lim_{x \to 3} \frac{1}{6x-13}$$

$$= \frac{1}{6 \cdot 3 - 13}$$

$$= \frac{1}{5}$$

4| $\lim_{x \to \pi} \frac{\sin x}{x-\pi}$

By L'Hospital's rule:

$$\lim_{x \to \pi} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x-\pi)}$$

$$= \lim_{x \to \pi} \frac{\cos x}{1}$$

$$= \frac{-1}{1}$$

$$= -1$$

5| $\lim_{x \to 0} \frac{x-\tan^{-1}x}{x^3}$

By L'Hospital's rule:

$$\lim_{x \to 0} \frac{\frac{d}{dx}(x-\tan^{-1}x)}{\frac{d}{dx}(x^3)}$$

$$= \lim_{x \to 0} \frac{1-\frac{1}{1+x^2}}{3x^2}$$

$$= \lim_{x \to 0} \frac{\frac{1+x^2-1}{1+x^2}}{3x^2}$$

$$= \lim_{x \to 0} \frac{x^2}{1+x^2} \times \frac{1}{3x^2}$$

$$= \frac{1}{3+0}$$

$$= 1/3$$

6] $$\lim_{x \to +\infty} \frac{e^{3x}}{x^2}$$

By L' Hospital's rule:

$$\lim_{x \to +\infty} \frac{3e^{3x}}{2x} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to +\infty} \frac{9e^{3x}}{2}$$

$$= \frac{9e^{3\cdot\infty}}{2}$$

$$= \frac{\infty}{2}$$

7] $$\lim_{x \to 0} \frac{a^x - 1 - x\log a}{x^2}$$

By L' Hospital's rule:

$$\lim_{x \to 0} \frac{a^x \ln a - 0 - \ln a . 1}{2x}$$

$$= \lim_{x \to 0} \frac{a^x \ln a - \ln a}{2x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\ln a . a^x . \ln a - 0}{2}$$

$$= \frac{\ln a . a^0 . \ln a}{2}$$

$$= \frac{(\ln a)^2}{2}$$

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11| $\lim_{x \to 0} \frac{\ln(\sin x)}{\ln(\tan x)}$

By L'Hospital's rule:

$$\lim_{x \to 0} \frac{\frac{1}{\sin x} \cdot \cos x}{\frac{1}{\tan x} \cdot \sec^2 x}$$

$$= \lim_{x \to 0} \frac{\cot x}{\cot x \sec^2 x}$$

$$= \lim_{x \to 0} \frac{1}{\sec^2 x}$$

$$= \frac{1}{\sec^2 0^\circ}$$

$$= 1$$

13| $\lim_{x \to 0} \frac{\sin 2x}{x}$

By L'Hospital's rule:

$$\lim_{x \to 0} \frac{2\cos 2x}{1}$$

$$= \lim_{x \to 0} 2\cos 2x$$

$$= 2 \times 1$$

$$= 2$$

14| $\lim_{x \to 0} \frac{\sin x}{x^2}$

By L' Hospital's rule :

$\lim_{x \to 0} \frac{\cos x}{2x}$

$= \lim_{x \to 0} \frac{-\sin x}{2}$

$= \frac{0}{2}$

$= 0$

15| $\lim_{x \to \infty} \frac{x}{e^x}$

By L' Hospital's rule :

$\lim_{x \to \infty} \frac{1}{e^x}$

$= \frac{1}{e^\infty}$

$= \frac{1}{\infty}$

$= 0$

17]

$$\lim_{x \to 0} \frac{\sin 2x}{\sin 5x}$$

By L'Hospital's rule :

$$\lim_{x \to 0} \frac{2\cos 2x}{5\cos 5x}$$

$$= \frac{2 \times 1}{5 \times 1}$$

$$= \frac{2}{5}$$

Transformation of Co-ordinates

26th Nov 2017

Formulas:

Relation between cartesian and Polar co-ordinates:

1. $r = \sqrt{x^2+y^2}$

Polar $\rightarrow (r, \theta)$

2. $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

rectangular co-ordinates/Cartesian $\rightarrow (x, y)$

3. $x = r\cos\theta$

4. $y = r\sin\theta$

Shifting Origin Formula:

5. $x = x' + h$

6. $y = y' + k$

Rotation of axes:

7. $x = x'\cos\theta - y'\sin\theta$

8. $y = x'\sin\theta + y'\cos\theta$

General equation of 2nd degree,

9. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Transformed equation,

10. $ax^2 + 2hxy + by^2 + c = 0$

1]

(i) Find the polar co-ordinates of the point $(2\sqrt{3}, -2)$

(r, θ) (x, y)

Here Given,

$$x = 2\sqrt{3}$$

$$y = -2$$

We know,

$$r = \sqrt{x^2+y^2}$$

$$= \sqrt{(2\sqrt{3})^2 + (-2)^2}$$

$$= \sqrt{12+4}$$

$$= 4$$

And,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

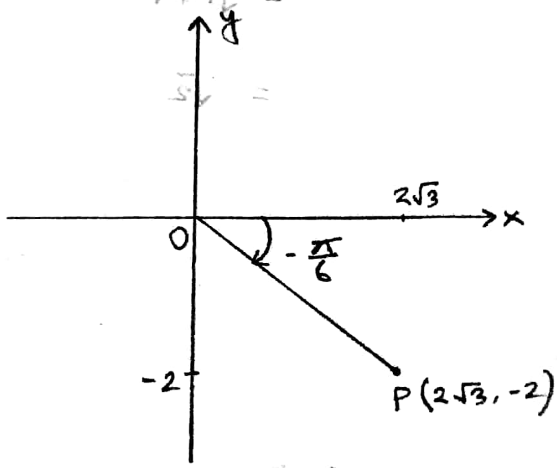
$$= \tan^{-1}\left(\frac{-2}{2\sqrt{3}}\right)$$

$$= \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$= \tan^{-1}\tan\left(-\frac{\pi}{6}\right)$$

$$= 2\pi - \frac{\pi}{6}$$

$$= \frac{11\pi}{6}$$



$\therefore$ Polar co-ordinates: $(r, \theta) = \left(4, \frac{11\pi}{6}\right)$

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(ii) $(0,-2)$

$$r = \sqrt{x^2+y^2}$$
$$= \sqrt{0^2+(-2)^2}$$
$$= 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$= \tan^{-1}\left(\frac{-2}{0}\right)$$
$$= \tan^{-1}(\infty)$$
$$= \tan^{-1}\tan\left(\frac{\pi}{2}\right)$$
$$= \frac{\pi}{2}$$

(iii) $(1,1)$

$$r = \sqrt{x^2+y^2}$$
$$= \sqrt{1^2+1^2}$$
$$= \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$= \tan^{-1}\left(\frac{1}{1}\right)$$
$$= \tan^{-1}(1)$$
$$= \tan^{-1}\tan\left(\frac{\pi}{4}\right)$$
$$= \frac{\pi}{4}$$

$\therefore$ Polar co-ordinates : $(r,\theta) = \left(\sqrt{2}, \frac{\pi}{4}\right)$

2)

(i) Find the rectangulare co-ordinates (x, y) of the point $(7, \frac{2\pi}{3})$ (r, θ)

We know,

$x = r\cos\theta$

$= 7\cos\frac{2\pi}{3}$

$y = r\sin\theta$

$= 7\sin\frac{2\pi}{3}$

(ii) $(8, \frac{9\pi}{4})$

$x = r\cos\theta$

$= 8\cos\frac{9\pi}{4}$

$y = r\sin\theta$

$= 8\sin\frac{9\pi}{4}$

(iii) $(0, \pi)$

$x = r\cos\theta$

$= 0\cos\pi$

$= 0$

$y = r\sin\theta$

$= 0\sin\pi$

$= 0$

$\therefore$ Rectangulare co-ordinates ; $(x, y) = (0, 0)$

3| (i) $r = a\sin\theta$

$\Rightarrow \sqrt{x^2+y^2} = a\sin\theta$

$\Rightarrow x^2+y^2 = a^2\sin^2\theta$

$\Rightarrow x^2+y^2 = a \cdot a\sin^2\theta$

$\Rightarrow x^2+y^2 = ay$

Ans.

We know,

$y = r\sin\theta$

$= a\sin\theta \cdot \sin\theta$ $[\because r = a\sin\theta]$

$= a\sin^2\theta$

(ii) $\sqrt{r} = \sqrt{a}\cos\left(\frac{\theta}{2}\right)$

$\Rightarrow r = a\cos^2\left(\frac{\theta}{2}\right)$

$\Rightarrow r = a\left[\frac{1+\cos 2\left(\frac{\theta}{2}\right)}{2}\right]$

$\left[\because 2\cos^2\theta = 1+\cos 2\theta \quad \therefore \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$

$\Rightarrow r = \frac{a}{2}(1+\cos\theta)$

Now, $\cos\theta = \frac{x}{r}$ $[\because x = r\cos\theta]$

$\therefore r = \frac{a}{2}\left(1+\frac{x}{r}\right)$

$\Rightarrow r = \frac{a}{2}\left(\frac{r+x}{r}\right)$

$\Rightarrow 2r^2 = a(r+x)$

$\Rightarrow 2(x^2+y^2) = a\left(\sqrt{x^2+y^2}+x\right)$

Ans.

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41 (i) $9x^2 + 4y^2 = 36$ —— ①

we know,

$x = r\cos\theta$

$y = r\sin\theta$

putting the values of x and y in eqn ① we get,

$9(r\cos\theta)^2 + 4(r\sin\theta)^2 = 36$

$\Rightarrow 9r^2\cos^2\theta + 4r^2\sin^2\theta = 36$

$\Rightarrow r^2(9\cos^2\theta + 4\sin^2\theta) = 36$

Ans.

(ii) $x^3 = y^2(2a - x)$

$\Rightarrow (r\cos\theta)^3 = (r\sin\theta)^2(2a - r\cos\theta)$

$\Rightarrow r^3\cos^3\theta = r^2\sin^2\theta(2a - r\cos\theta)$

$\Rightarrow r\cos^3\theta = 2a\sin^2\theta - r\cos\theta \cdot \sin^2\theta$

$\Rightarrow r\cos^3\theta + r\cos\theta \cdot \sin^2\theta = 2a\sin^2\theta$

$\Rightarrow r\cos\theta(\cos^2\theta + \sin^2\theta) = 2a\sin^2\theta$

$\Rightarrow r\cos\theta = 2a\sin^2\theta$

Ans.

5] $2x^2+y^2-4x+4y=0$ ——— ①

Given,

New origin, $O'(h,k) = (1,-2)$

We know,

$x = x'+h \Rightarrow x = x'+1$

$y = y'+k \Rightarrow y = y'-2$

substituting the values of x and y in eqn ①,

$2(x'+1)^2 + (y'-2)^2 - 4(x'+1) + 4(y'-2) = 0$

$\Rightarrow 2(x'^2+2x'+1) + (y'^2-4y'+4) - 4x'-4+4y'-8 = 0$

$\Rightarrow 2x'^2+4x'+2+y'^2-4y'+4-4x'-4+4y'-8 = 0$

$\Rightarrow 2x'^2+y'^2-6 = 0$

Removing suffixes,

$2x^2+y^2-6 = 0$

Ans.

6) $x^2+y^2-8x+14y+5=0$ ——— (1)

Given, New origin, $O'(h,k) = (4,-7)$

we know,

$x = x'+h \Rightarrow x = x'+4$

$y = y'+k \Rightarrow y = y'-7$

substituting the values of x and y in eqn (1)

$(x'+4)^2+(y'-7)^2-8(x'+4)+14(y'-7)+5=0$

$\Rightarrow x'^2+8x'+16+y'^2-14y'+49-8x'-32+14y'-98+5=0$

$\Rightarrow x'^2+y'^2-60=0$

Removing suffixes,

$x^2+y^2-60=0$

Ans.

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11| $2x^2 + 4xy + 5y^2 - 4x - 22y + 7 = 0$ —— ①

Given, New Origin $O'(h, k) = (-2, 3)$

we know,

$x = x' + h \Rightarrow x = x' - 2$

$y = y' + k \Rightarrow y = y' + 3$

substituting the values of x and y in eqn ①,

$$2(x'-2)^2 + 4(x'-2)(y'+3) + 5(y'+3)^2 - 4(x'-2) - 22(y'+3) + 7 = 0$$

$$\Rightarrow 2(x'^2 - 4x' + 4) + 4(x'y' + 3x' - 2y' - 6) + 5(y'^2 + 6y' + 9)$$
$$- (4x' - 8) - 22(y' + 3) + 7 = 0$$

$$\Rightarrow 2x'^2 - 8x' + 8 + 4x'y' + 12x' - 8y' - 24 + 5y'^2 + 30y'$$
$$+ 45 - 4x' + 8 - 22y' - 66 + 7 = 0$$

$$\Rightarrow 2x'^2 + 5y'^2 + 4x'y' + 4x' - 22 = 0$$

Removing suffixes,

$$2x^2 + 5y^2 + 4xy + 4x - 22 = 0$$

Ans.

7] $7x^2 - 2xy + y^2 + 1 = 0$ ——— ①

Given, $\theta = \tan^{-1}(1/2)$

$= 26.56^\circ$

We know,

$x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$

$\Rightarrow x = 0.9x' - 0.4y'$ $\Rightarrow y = 0.4x' + 0.9y'$

$\left[\because \cos 26.56^\circ = 0. \quad \sin 26.56^\circ = 0. \right.$

substituting the values of x and y in the eqn ①,

$$7(0.9x' - 0.4y')^2 - 2(0.9x' - 0.4y')(0.4x' + 0.9y') + (0.4x' + 0.9y')^2 + 1 = 0$$

$$\Rightarrow 7(0.81x'^2 - 0.72x'y' + 0.16y'^2) - 2(0.36x'^2 - 0.16x'y' + 0.81x'y' - 0.36y'^2)$$
$$+ (0.16x'^2 + 0.72x'y' + 0.81y'^2) + 1 = 0$$

$$\Rightarrow 5.67x'^2 - 5.04x'y' + 1.12y'^2 - 0.72x'^2 - 0.32x'y' + 1.62x'y' - 0.72y'^2$$
$$+ 0.16x'^2 + 0.72x'y' + 0.81y'^2 + 1 = 0$$

$$\Rightarrow 5.11x'^2 - 3.02x'y' + 1.21y'^2 + 1 = 0$$

Removing suffixes,

$$5.11x^2 - 3.02xy + 1.21y^2 + 1 = 0$$

Ans.

8] $11x^2 + 24xy + 4y^2 - 20x - 40y - 5 = 0$ —— (1)

Given,

New origin $O'(h, k) = (2, -1)$

We know,

$x = x' + h$ and $y = y' + k$

$\Rightarrow x = x' + 2$ $\Rightarrow y = y' - 1$

substituting the values of x and y in the eqn (1)

$$11(x'+2)^2 + 24(x'+2)(y'-1) + 4(y'-1)^2 - 20(x'+2) - 40(y'-1) - 5 = 0$$

$$\Rightarrow 11(x'^2 + 4x' + 4) + 24(x'y' + 2y' - x' - 2) + 4(y'^2 - 2y' + 1)$$
$$- 20x' - 40 - 40y' + 40 - 5 = 0$$

$$\Rightarrow 11x'^2 + 44x' + 44 + 24x'y' + 48y' - 24x' - 48 + 4y'^2 - 8y' + 4$$
$$- 20x' - 40 - 40y' + 40 - 5 = 0$$

$$\Rightarrow 11x'^2 + 4y'^2 + 24x'y' - 5 = 0$$

Removing suffixes,

$11x^2 + 4y^2 + 24xy - 5 = 0$ —— (11)

Also Given,

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$= 53.1^\circ$$

We know,

$x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$

$\left[\because \cos 53.1^\circ = 0.6,\ \sin 53.1^\circ = 0.8\right]$

$= 0.6x' - 0.8y'$ $\qquad$ $= 0.8x' + 0.6y'$

$= \frac{6x'}{10} - \frac{8y'}{10}$ $\qquad$ $= \frac{8x'}{10} + \frac{6y'}{10}$

$= \frac{3x'-4y'}{5}$ $\qquad$ $= \frac{4x'+3y'}{5}$

substituting the values of x and y in the eqn (ii)

$$11\left(\frac{3x'-4y'}{5}\right)^2 + 4\left(\frac{4x'+3y'}{5}\right)^2 + 24\left(\frac{3x'-4y'}{5}\right)\left(\frac{4x'+3y'}{5}\right) - 5 = 0$$

$$\Rightarrow 11(9x'^2 - 24x'y' + 16y'^2) + 4(16x'^2 + 24x'y' + 9y'^2)$$
$$+ 24(12x'^2 - 16x'y' + 9x'y' - 12y'^2) - 125 = 0$$

$$\Rightarrow 99x'^2 - 264x'y' + 176y'^2 + 64x'^2 + 96x'y' + 36y'^2 + 288x'^2 - 384x'y'$$
$$+ 216x'y' - 288y'^2 - 125 = 0$$

$$\Rightarrow 451x'^2 - 76y'^2 - 336x'y' - 125 = 0$$

Removing suffixes,

$$451x^2 - 76y^2 - 336xy - 125 = 0$$

Ans.

9) $9x^2 + 15xy + y^2 + 12x - 11y - 5 = 0$ —— (i)

comparing eqn (i) with the general eqn of 2nd degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 9$	$2h = 15$ $\Rightarrow h = \frac{15}{2}$	$b = 1$	$2g = 12$ $\Rightarrow g = 6$	$2f = -11$ $\Rightarrow f = -\frac{11}{2}$	$c = -5$

Let $F(x, y) = 9x^2 + 15xy + y^2 + 12x - 11y - 5 = 0$

$\therefore \frac{\partial F}{\partial x} = 18x + 15y + 12 = 0$

$\Rightarrow 6x + 5y + 4 = 0$ —— (ii)

$\therefore \frac{\partial F}{\partial y} = 15x + 2y - 11 = 0$

$\Rightarrow 15x + 2y - 11 = 0$ —— (iii)

Solving equations (ii) and (iii)

(ii) $\times 2$ − (iii) $\times 5$,

$$\begin{array}{r} 12x + \cancel{10y} + 8 = 0 \\ 75x + \cancel{10y} - 55 = 0 \\ \scriptstyle(-) \quad\quad (-) \quad\quad (+) \\ \hline -63x + 63 = 0 \end{array}$$

$\Rightarrow -63(x-1) = 0$

$\Rightarrow x - 1 = 0$

$\therefore x = 1$

$x = 1 \Rightarrow$ (ii)

$$6 \cdot 1 + 5y + 4 = 0$$

$$\Rightarrow 5y = -10$$

$$\therefore y = -2$$

$\therefore$ New origin $O'(\alpha, \beta) = (1, -2)$

We know,

Transformed eq^n, $ax^2 + 2hxy + by^2 + c = 0$ — (IV)

where,

$$c = g\alpha + f\beta + c$$

$$= (6 \times 1) + \left(-\frac{11}{2}\right)(-2) + (-5)$$

$$= 6 + 11 - 5$$

$$= 12$$

$\therefore$ from eq^n (IV),

$$9x^2 + 15xy + y^2 + 12 = 0$$

Ans.

141 $9x^2+24xy+2y^2-6x+20y+41=0$ ——— (1)

comparing eqn (1) with the general eqn of 2nd degree,

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

Here,

$a=9$	$2h=24$	$b=2$	$2g=-6$	$2f=20$	$c=41$
	$\Rightarrow h=12$		$\Rightarrow g=-3$	$\Rightarrow f=10$	

Let $F(x,y)=9x^2+24xy+2y^2-6x+20y+41=0$

$\therefore \frac{\partial F}{\partial x}=18x+24y-6=0$

$\Rightarrow 3x+4y-1=0$ ——— (II)

$\therefore \frac{\partial F}{\partial y}\Rightarrow 24x+4y+20=0$ ——— (III)

solving equations (II) and (III)

(III) − (II),

$$\begin{array}{r} 24x+4y+20=0 \\ 3x+4y-1=0 \\ (-)\quad(-)\quad(+) \\ \hline 21x+21=0 \end{array}$$

$\Rightarrow 21(x+1)=0$

$\Rightarrow x+1=0$

$\therefore x=-1$

$x = -1 \Rightarrow$ (ii)

$$3\cdot(-1) + 4y - 1 = 0$$

$$\Rightarrow 4y - 4 = 0$$

$$\Rightarrow 4y = 4$$

$$\therefore y = 1$$

$\therefore$ New origin $O'(\alpha, \beta) = (-1, 1)$

we know,

Transformed eqn, $ax^2 + 2hxy + by^2 + c = 0$ —— (iv)

where, $c = g\alpha + f\beta + c$

$$= (-3)(-1) + 10\cdot 1 + 41$$

$$= 54$$

$\therefore$ from eqn (iv),

$$9x^2 + 24xy + 2y^2 + 54 = 0$$

Ans.

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12| $9x^2 + 24xy + 2y^2 + 54 = 0$ —— (1)

comparing eqn (1) with the following eqn

$$ax^2 + 2hxy + by^2 + c = 0$$

Here,

$a = 9$ | $2h = 24$ $\Rightarrow h = 12$ | $b = 2$ | $c = 54$

Now, to remove xy term from eqn (1) both axes are rotated through an angle, $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$

$$= \frac{1}{2}\tan^{-1}\left(\frac{24}{9-2}\right)$$

$$= 36.9^\circ$$

we know,

$$x = x'\cos\theta - y'\sin\theta$$

$$= \frac{4x'}{5} - \frac{3y'}{5}$$

$$= \frac{4x' - 3y'}{5}$$

$\left[\because \cos 36.9^\circ = \frac{4}{5} \right.$

$\left. \sin 36.9^\circ = \frac{3}{5} \right.$

$$y = x'\sin\theta + y'\cos\theta$$

$$= \frac{3x' + 4y'}{5}$$

substituting the values of x and y in the eqn ①

$$9\left(\frac{4x'-3y'}{5}\right)^2+24\left(\frac{4x'-3y'}{5}\right)\left(\frac{3x'+4y'}{5}\right)+2\left(\frac{3x'+4y'}{5}\right)^2+54=0$$

$$\Rightarrow 9\left(16x'^2-24x'y'+9y'^2\right)+24\left(12x'^2-9x'y'+16x'y'-12y'^2\right)$$
$$+2\left(9x'^2+24x'y'+16y'^2\right)+1350=0$$

$$\Rightarrow 144x'^2-216x'y'+81y'^2+288x'^2-216x'y'+384x'y'-288y'^2$$
$$+18x'^2+48x'y'+32y'^2+1350=0$$

$$\Rightarrow 450x'^2-175y'^2+1350=0$$

$$\Rightarrow 18x'^2-7y'^2+54=0$$

Removing suffixes,

$$18x^2-7y^2+54=0$$

Ans.

10) $11x^2 + 3xy + 7y^2 + 19 = 0$ ——— (1)

comparing eqn (1) with the following eqn

$$ax^2 + 2hxy + by^2 + c = 0$$

Here,

$a = 11$ | $2h = 3 \Rightarrow h = \frac{3}{2}$ | $b = 7$ | $c = 19$

Now, to remove xy term from eqn (1) both axes are rotated through an angle, $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$

$$= \frac{1}{2}\tan^{-1}\left(\frac{3}{11-7}\right)$$

$$= 18.4^\circ$$

we know,

$x = x'\cos\theta - y'\sin\theta$

$= 0.95x' - 0.32y'$

$= \frac{95x'}{100} - \frac{32y'}{100}$

$= \frac{95x' - 32y'}{100}$

$\left[\because \cos 18.4^\circ = 0.95 \quad \sin 18.4^\circ = 0.32 \right.$

$y = x'\sin\theta + y'\cos\theta$

$= 0.32x' + 0.95y'$

$= \frac{32x' + 95y'}{100}$

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substituting the values of x and y in the eqn ①

$$11(0.95x' - 0.32y')^2 + 3(0.95x' - 0.32y')(0.32x' + 0.95y') + 7(0.32x' + 0.95y')^2 + 19 = 0$$

$$\Rightarrow 11(0.9x'^2 - 0.6x'y' + 0.1y'^2) + 3(0.3x'^2 + 0.9x'y' - 0.1x'y' - 0.3y'^2) + 7(0.1x'^2 + 0.6x'y' + 0.9y'^2) + 19 = 0$$

$$\Rightarrow 9.9x'^2 - 6.6x'y' + 1.1y'^2 + 0.9x'^2 + 2.7x'y' - 0.3x'y' - 0.9y'^2 + 0.7x'^2 + 4.2x'y' + 6.3y'^2 + 19 = 0$$

$$\Rightarrow 11.5x'^2 + 6.5y'^2 + 19 = 0$$

Removing suffixes,

$$11.5x^2 + 6.34y^2 + 19 = 0$$

Ans.

131 $\quad x^2 + 2\sqrt{3}\,xy - y^2 = 2a^2$ ——— ①

We know,

$x = x'\cos\theta - y'\sin\theta$ $\qquad$ $y = x'\sin\theta + y'\cos\theta$

$= x'\cos 30^\circ - y'\sin 30^\circ$ $\qquad$ $= x'\sin 30^\circ + y'\cos 30^\circ$

$\left[\because \cos 30^\circ = \frac{\sqrt{3}}{2},\ \sin 30^\circ = \frac{1}{2}\right]$

$= x'\frac{\sqrt{3}}{2} - y'\frac{1}{2}$ $\qquad$ $= x'\frac{1}{2} + y'\frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}x' - y'}{2}$ $\qquad$ $= \frac{x' + \sqrt{3}y'}{2}$

substituting the values of x and y in the eqn ①,

$$\left(\frac{\sqrt{3}x' - y'}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) - \left(\frac{x' + \sqrt{3}y'}{2}\right)^2 = 2a^2$$

$$\Rightarrow (3x'^2 - 2\sqrt{3}\,x'y' + y'^2) + 2\sqrt{3}(\sqrt{3}x'^2 + 3x'y' - x'y' - \sqrt{3}y'^2) - (x'^2 + 2\sqrt{3}\,x'y' + 3y'^2) = 8a^2$$

$$\Rightarrow 3x'^2 - 2\sqrt{3}x'y' + y'^2 + 6x'^2 + 6\sqrt{3}x'y' - 2\sqrt{3}x'y' - 6y'^2 - x'^2 - 2\sqrt{3}x'y' - 3y'^2 = 8a^2$$

$$\Rightarrow 8x'^2 - 6\sqrt{3}x'y' + 6\sqrt{3}x'y' - 8y'^2 = 8a^2$$

$$\Rightarrow 8x'^2 - 8y'^2 = 8a^2$$

$$\Rightarrow x'^2 - y'^2 = a^2$$

Removing suffixes,

$$x^2 - y^2 = a^2$$

Ans.

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Pair of Straight Lines

29th Nov 2017

1) (i) $2y^2 - xy - x^2 + y + 2x - 1 = 0$ —— ①

comparing eqn ① with the general eqn of 2nd degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = -1$	$2h = -1$ $\Rightarrow h = -\frac{1}{2}$	$b = 2$	$2g = 2$ $\Rightarrow g = 1$	$2f = 1$ $\Rightarrow f = \frac{1}{2}$	$c = -1$

eqn ① will represent a pair of straight lines if,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$= (-1)\cdot 2\cdot(-1) + 2\cdot\frac{1}{2}\cdot 1\cdot\left(-\frac{1}{2}\right) + 1\cdot\left(\frac{1}{2}\right)^2 - 2\cdot 1^2 + 1\cdot\left(-\frac{1}{2}\right)^2$$

$$= 2 - \frac{1}{2} + \frac{1}{4} - 2 + \frac{1}{4}$$

$$= 0$$

Hence eqn ① represents a pair of straight lines.

Now,

Let $F(x,y) = 2y^2 - xy - x^2 + y + 2x - 1 = 0$

$\therefore \frac{\partial F}{\partial x} = -y - 2x + 2 = 0$

$\Rightarrow 2x + y - 2 = 0$ —— (ii)

$\therefore \frac{\partial F}{\partial y} = 4y - x + 1 = 0$

$\Rightarrow x - 4y - 1 = 0$ —— (iii)

solving equations (ii) and (iii),

(ii) − (iii) × 2,

$$2x + y - 2 = 0$$
$$2x - 8y - 2 = 0$$
$$\underline{(-) \quad (+) \quad (+)}$$
$$9y = 0$$
$$\Rightarrow y = 0$$

$y = 0 \Rightarrow$ (ii),

$$2x - 2 = 0$$
$$\Rightarrow 2x = 2$$
$$\Rightarrow x = 1$$

So the point of intersection $(x, y) = (\alpha, \beta) = (1, 0)$

Then,

we know,

$$\text{Angle}, \theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a+b}\right)$$
$$= \tan^{-1}\left(\frac{2\sqrt{\frac{1}{4} + 2}}{1}\right)$$
$$= \tan^{-1}(3)$$
$$= 71.6^\circ$$

Ans.

(ii) $2x^2 - 2xy + x + 2y - 3 = 0$ ——— ①

comparing eqn ① with the general eqn of 2nd degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 2$	$2h = -2$ $\Rightarrow h = -1$	$b = 0$	$2g = 1$ $\Rightarrow g = \frac{1}{2}$	$2f = 2$ $\Rightarrow f = 1$	$c = -3$

eqn ① will represent a pair of straight lines if,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$= 0 + 2\cdot 1\cdot \frac{1}{2}\cdot(-1) - 2\cdot 1^2 + 0 - (-3)\cdot(-1)^2$$

$$= 0 - 1 - 2 + 3$$

$$= 0$$

Hence eqn ① represents a pair of straight lines

Now,

Let $F(x,y) = 2x^2 - 2xy + x + 2y - 3 = 0$

$$\therefore \frac{\partial F}{\partial x} = 4x - 2y + 1 = 0 \quad \text{——— (ii)}$$

$$\frac{\partial F}{\partial y} = -2x + 2 = 0 \quad \text{——— (iii)}$$

P. T. O

solving equations (ii) and (iii)

(ii) + (iii) × 2,

$$\begin{array}{r} 4x - 2y + 1 = 0 \\ -4x \quad + 4 = 0 \\ \hline -2y + 5 = 0 \end{array}$$

$$\Rightarrow y = \frac{5}{2}$$

$y = \frac{5}{2} \Rightarrow$ (ii),

$$4x - 2 \cdot \frac{5}{2} + 1 = 0$$

$$\Rightarrow 4x - 5 + 1 = 0$$

$$\Rightarrow x = 1$$

so the point of intersection $(x, y) = (\alpha, \beta) = (1, \frac{5}{2})$

We know,

$$\text{Angle}, \theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a+b}\right)$$

$$= \tan^{-1}\left(\frac{2\sqrt{(-1)^2 - 0}}{2+0}\right)$$

$$= \tan^{-1}\left(\frac{2}{2}\right)$$

$$= \tan^{-1}(1)$$

$$= 45°$$

Ans.

(iii) $x^2 + 3xy + 2y^2 + \frac{1}{8}x - \frac{1}{32} = 0$ —— (i)

comparing eqn (i) with the general eqn of 2nd degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 1$	$2h = 3$ $\Rightarrow h = \frac{3}{2}$	$b = 2$	$2g = \frac{1}{8}$ $\Rightarrow g = \frac{1}{16}$	$f = 0$	$c = -\frac{1}{32}$

eqn (i) will represent a pair of straight lines if,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$= 1 \cdot 2 \cdot \left(-\frac{1}{32}\right) + 0 - 0 - 2 \cdot \left(\frac{1}{16}\right)^2 - \left(-\frac{1}{32}\right)\left(\frac{3}{2}\right)^2 = 0$$

$$= -\frac{1}{16} - \frac{1}{128} + \frac{9}{128}$$

$$= \frac{-8 - 1 + 9}{128}$$

$$= \frac{0}{128}$$

$$= 0 \qquad \left[\because \frac{0}{\text{something}} = 0\right]$$

Hence eqn (i) represents a pair of straight lines.

Let $F(x,y) = x^2 + 3xy + 2y^2 + \frac{1}{8}x - \frac{1}{32} = 0$

$\therefore \frac{\partial F}{\partial x} = 2x + 3y + \frac{1}{8} = 0$ —— (ii)

$\frac{\partial F}{\partial y} = 3x + 4y = 0$ —— (iii)

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solving equations (ii) and (iii)

(ii) × 3 − (iii) × 2,

$$6x + 9y + \frac{3}{8} - 6x - 8y = 0$$

$$\Rightarrow \quad y = -\frac{3}{8}$$

$y = -\frac{3}{8} \Rightarrow$ (iii),

$$3x - 4 \cdot \frac{3}{8} = 0$$

$$\Rightarrow 3x = \frac{3}{2}$$

$$\Rightarrow x = \frac{1}{2}$$

so the point of intersection $(x, y) = (\alpha, \beta) = \left(\frac{1}{2}, -\frac{3}{8}\right)$

we know,

$$\text{Angle}, \theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a+b}\right)$$

$$= \tan^{-1}\left(\frac{2\sqrt{(3/2)^2 - 1.2}}{1+2}\right)$$

$$= \tan^{-1}\left(\frac{2\sqrt{9/4 - 2}}{3}\right)$$

$$= \tan^{-1}\left(\frac{2\sqrt{\frac{9-8}{4}}}{3}\right)$$

$$= \tan^{-1}\left(\frac{1}{3}\right)$$

$$= 18.4^\circ$$

Ans.

(iv) $21x^2 + 40xy - 21y^2 + 44x + 122y - 17 = 0$ —— ①

comparing eqn ① with the general eqn of 2nd degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 21$	$2h = 40$ $\Rightarrow h = 20$	$b = -21$	$2g = 44$ $\Rightarrow g = 22$	$2f = 122$ $\Rightarrow f = 61$	$c = -17$

eqn ① will represents a pair of straight lines if -

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= 21\cdot(-21)\cdot(-17) + 2\cdot61\cdot22\cdot20 - 21\cdot(61)^2 + 21\cdot(22)^2 + 17(20)^2$$

$$= 7497 + 53680 - 78141 + 10164 + 6800$$

$$= 0$$

Hence eqn ① represents a pair of straight lines.

Let $F(x,y) = 21x^2 + 40xy - 21y^2 + 44x + 122y - 17 = 0$

$\therefore \frac{\partial F}{\partial x} = 42x + 40y + 44 = 0$ —— (ii)

$\frac{\partial F}{\partial y} = 40x - 42y + 122 = 0$ —— (iii)

from ①, $42x + 40y + 44 = 0$

$\Rightarrow 42x = -44 - 40y$

$\Rightarrow x = \frac{-22 - 20y}{21}$ —— (iv)

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putting value of x in eqn (III)

$$y = 1$$

$y = 1 \Rightarrow$ (IV)

$$x = -2$$

so the point of intersection, $(x, y) = (\alpha, \beta) = (-2, 1)$

We know,

$$\text{Angle}, \theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a+b}\right)$$

$$= \tan^{-1} \frac{2\sqrt{(20)^2 - (21)\cdot(-21)}}{21 - 21}$$

$$= \tan^{-1} \frac{2\sqrt{841}}{0}$$

$$=$$

3| (i) $2\lambda xy - y^2 + 4x + 2y + 8 = 0$ ——— ①

comparing eqn ① with the general eqn of 2nd degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 0$	$2h = 2\lambda$ $\Rightarrow h = \lambda$	$b = -1$	$2g = 4$ $\Rightarrow g = 2$	$2f = 2$ $\Rightarrow f = 1$	$c = 8$

eqn ① will represent a pair of straight lines if,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow 0 + 4\lambda - 0 + 4 - 8\lambda^2 = 0$$

$$\Rightarrow 8\lambda^2 - 4\lambda - 4 = 0$$

$$\Rightarrow 2\lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow 2\lambda^2 - 2\lambda + \lambda - 1 = 0$$

$$\Rightarrow 2\lambda(\lambda - 1) + 1(\lambda - 1) = 0$$

$$\Rightarrow (\lambda - 1)(2\lambda + 1) = 0$$

$$\Rightarrow \lambda = 1, -\frac{1}{2}$$

Ans.

(ii) $2x^2 + xy - y^2 - 2x - 5y + k = 0$ ——— ①

comparing eqn ① with the general eqn of 2nd degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 2$ | $2h = 1$ $\Rightarrow h = \frac{1}{2}$ | $b = -1$ | $2g = -2$ $\Rightarrow g = -1$ | $2f = -5$ $\Rightarrow f = -5/2$ | $c = k$

eqn ① will represent a pair of straight lines if,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow 2(-1)k + 2(-5/2)(-1)\frac{1}{2} - 2(-5/2)^2 + 1\cdot(-1)^2 - k\left(\frac{1}{2}\right)^2 = 0$$

$$\Rightarrow -2k + \frac{5}{2} - \frac{25}{2} + 1 - \frac{k}{4} = 0$$

$$\Rightarrow \frac{-8k + 10 - 50 + 4 - k}{4} = 0$$

$$\Rightarrow -9k - 36 = 0$$

$$\Rightarrow -9k = 36$$

$$\Rightarrow k = -4$$

Ans.

(iii) $x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$ —— ①

comparing eqn ① with the general equation of 2nd degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 1$	$2h = -\lambda$ $\Rightarrow h = -\frac{\lambda}{2}$	$b = 2$	$2g = 3$ $\Rightarrow g = \frac{3}{2}$	$2f = -5$ $\Rightarrow f = -\frac{5}{2}$	$c = 2$

eqn ① will represent a pair of straight lines if,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow 1\cdot 2\cdot 2 + 2\cdot(-5/2)\cdot\frac{3}{2}\cdot\left(-\frac{\lambda}{2}\right) - 1\cdot\left(-\frac{5}{2}\right)^2 - 2\cdot\left(\frac{3}{2}\right)^2 - 2\cdot\left(-\frac{\lambda}{2}\right)^2 = 0$$

$$\Rightarrow 4 + \frac{15\lambda}{4} - \frac{25}{4} - \frac{9}{2} - \frac{\lambda^2}{2} = 0$$

$$\Rightarrow \frac{16 + 15\lambda - 25 - 18 - 2\lambda^2}{4} = 0$$

$$\Rightarrow -2\lambda^2 + 15\lambda - 27 = 0$$

$$\Rightarrow 2\lambda^2 - 15\lambda + 27 = 0$$

$$\Rightarrow 2\lambda^2 - 6\lambda - 9\lambda + 27 = 0$$

$$\Rightarrow 2\lambda(\lambda - 3) - 9(\lambda - 3) = 0$$

$$\Rightarrow (\lambda - 3)(2\lambda - 9) = 0$$

$$\Rightarrow \lambda = 3, \frac{9}{2}$$

Ans.

(iv) $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ ——— (1)

comparing eqn (1) with the general eqn of 2nd degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 12$	$2h = -10$ $\Rightarrow h = -5$	$b = 2$	$2g = 11$ $\Rightarrow g = \frac{11}{2}$	$2f = -5$ $\Rightarrow f = -\frac{5}{2}$	$c = \lambda$

eqn (1) will represent a pair of straight lines if,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow 12 \cdot 2 \cdot \lambda + 2 \cdot \left(-\frac{5}{2}\right) \cdot \frac{11}{2} \cdot (-5) - 12\left(-\frac{5}{2}\right)^2 - 2\left(\frac{11}{2}\right)^2 - \lambda \cdot (-5)^2 = 0$$

$$\Rightarrow 24\lambda + \frac{275}{2} - 75 - \frac{121}{2} - 25\lambda = 0$$

$$\Rightarrow \frac{48\lambda + 275 - 150 - 121 - 50\lambda}{2} = 0$$

$$\Rightarrow 4 - 2\lambda = 0$$

$$\Rightarrow -2\lambda = -4$$

$$\Rightarrow \lambda = 2$$

Ans.

3) (i) $x^2+xy-6y^2-x-8y-2=0$ ——— ①

comparing eqn ① with the general eqn of 2nd degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

Here,

$a=1$ | $2h=1 \Rightarrow h=\frac{1}{2}$ | $b=-6$ | $2g=-1 \Rightarrow g=-\frac{1}{2}$ | $2f=-8 \Rightarrow f=-4$ | $c=-2$

Let $F(x,y) \equiv x^2+xy-6y^2-x-8y-2=0$

$$\therefore \frac{\partial F}{\partial x} = 2x+y-1=0 \quad \text{——— (ii)}$$

$$\frac{\partial F}{\partial y} = x-12y-8=0 \quad \text{——— (iii)}$$

solving equations (ii) and (iii),

(ii) − (iii) × 2,

$$\begin{array}{r} 2x+y-1=0 \\ 2x-24y-16=0 \\ (-)\quad(+)\quad(+) \\ \hline 25y+15=0 \end{array}$$

$$\Rightarrow y=-\frac{3}{5}$$

$y=-\frac{3}{5} \Rightarrow$ (ii),

$$2x-\frac{3}{5}-1=0$$

$$\Rightarrow 2x=\frac{3}{5}+1$$

$$\Rightarrow x=\frac{4}{5}$$

$$\therefore \; (x, y)^{①} = (\alpha, \beta) = \left(\frac{4}{5}, -\frac{3}{5}\right)$$

We know,

$$\frac{(x-\alpha)^2 - (y-\beta)^2}{a-b} = \frac{(x-\alpha)(y-\beta)}{h}$$

$$\Rightarrow \frac{\left(x-\frac{4}{5}\right)^2 - \left(y+\frac{3}{5}\right)^2}{1-(-6)} = \frac{\left(x-\frac{4}{5}\right)\left(y+\frac{3}{5}\right)}{\frac{1}{2}}$$

$$\Rightarrow \frac{\left(\frac{5x-4}{5}\right)^2 - \left(\frac{5y+3}{5}\right)^2}{7} = 2\left(xy + \frac{3x}{5} - \frac{4y}{5} - \frac{12}{25}\right)$$

$$\Rightarrow \frac{25x^2-40x+16}{25} - \frac{25y^2+30y+9}{25} = 14\left(\frac{25xy+15x-20y-12}{25}\right)$$

$$\Rightarrow \frac{25x^2-40x+16-25y^2+30y+9}{25} = 14\left(\frac{25xy+15x-20y-12}{25}\right)$$

$$\Rightarrow 25x^2-40x+16-25y^2+30y+9 = 350xy+210x-280y-168$$

$$\Rightarrow 25x^2-350xy-25y^2-250x+310y+193 = 0$$

Ans.

(ii) $8x^2 - 14xy + 6y^2 + 2x - y - 1 = 0$ —— ①

comparing eqn ① with the general eqn of 2nd degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 8$	$2h = -14$ $\Rightarrow h = -7$	$b = 6$

Let $F(x,y) \equiv 8x^2 - 14xy + 6y^2 + 2x - y - 1 = 0$

$\therefore \frac{\partial F}{\partial x} = 16x - 14y + 2 = 0$

$\Rightarrow 8x - 7y + 1 = 0$ —— (ii)

$\frac{\partial F}{\partial y} = -14x + 12y - 1 = 0$ —— (iii)

solving equations (ii) and (iii),

from ①, $8x - 7y + 1 = 0$

$\Rightarrow 8x = 7y - 1$

$\Rightarrow x = \frac{7y-1}{8}$ —— (iv)

$x = \frac{7y-1}{8} \Rightarrow$ (iii)

$-14\left(\frac{7y-1}{8}\right) + 12y - 1 = 0$

$\Rightarrow \frac{-98y+14}{8} + 12y = 1$

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$$\Rightarrow \frac{14 - 98y + 96y}{8} = 1$$

$$\Rightarrow 14 - 2y = 8$$

$$\Rightarrow y = 3$$

$y = 3 \Rightarrow$ (IV)

$$x = \frac{7 \cdot 3 - 1}{8}$$

$$= \frac{5}{2}$$

$$\therefore (x, y) = (\alpha, \beta) = \left(\frac{5}{2}, 3\right)$$

We know,

$$\frac{(x-\alpha)^2 - (y-\beta)^2}{a-b} = \frac{(x-\alpha)(y-\beta)}{h}$$

$$\Rightarrow \frac{(x-\frac{5}{2})^2 - (y-3)^2}{8-6} = \frac{(x-\frac{5}{2})(y-3)}{-7}$$

$$\Rightarrow \frac{x^2 - 5x + \frac{25}{4} - y^2 + 6y - 9}{2} = \frac{xy - 3x - \frac{5y}{2} + \frac{15}{2}}{-7}$$

$$\Rightarrow -7x^2 + 35x - \frac{175}{4} + 7y^2 - 42y + 63 = 2xy - 6x - 5y + 15$$

$$\Rightarrow 28x^2 - 140x + 175 - 28y^2 + 168y - 252 + 8xy - 24x - 20y + 60 = 0$$

$$\Rightarrow 28x^2 + 8xy - 28y^2 - 164x + 148y - 17 = 0$$

Ans.

(iii) $2x^2 + xy - y^2 - 3x + 6y - 9 = 0$ —— ①

comparing eqn ① with the general eqn of 2nd degree.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 2$	$2h = 1$ $\Rightarrow h = \frac{1}{2}$	$b = -1$

Let $F(x,y) \equiv 2x^2 + xy - y^2 - 3x + 6y - 9 = 0$

$$\therefore \frac{\partial F}{\partial x} = 4x + y - 3 = 0 \quad \text{—— (ii)}$$

$$\frac{\partial F}{\partial y} = x - 2y + 6 = 0 \quad \text{—— (iii)}$$

solving equations (ii) and (iii),

(ii) − (iii) ×4,

$$4x + y - 3 - 4x + 8y - 24 = 0$$

$\Rightarrow 9y = 27$

$\Rightarrow y = 3$

$y = 3 \Rightarrow$ (ii),

$x = 0$

$$\therefore (x, y) = (\alpha, \beta) = (0, 3)$$

We know,
$$\frac{(x-\alpha)^2 - (y-\beta)^2}{a-b} = \frac{(x-\alpha)(y-\beta)}{h}$$

$$\Rightarrow \frac{(x-0)^2 - (y-3)^2}{2+1} = \frac{(x-0)(y-3)}{\frac{1}{2}}$$

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$$\Rightarrow \frac{x^2+0+0-y^2+6y-9}{3} = \frac{xy-3x-0-0}{1/2}$$

$$\Rightarrow \frac{x^2-y^2+6y-9}{3} = 2xy-6x$$

$$\Rightarrow x^2-6xy-y^2+18x+6y-9=0$$

Ans.

(iv) $2x^2+7xy+6y^2+13x+22y+20=0$ —— ①

comparing eqn ① with the general eqn of 2nd degree,

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

Here,

$a=2$ | $2h=7 \Rightarrow h=\frac{7}{2}$ | $b=6$

Let $F(x,y) \equiv 2x^2+7xy+6y^2+13x+22y+20=0$

$$\therefore \frac{\partial F}{\partial x} = 4x+7y+13=0 \quad \text{—— (ii)}$$

$$\frac{\partial F}{\partial y} = 7x+12y+22=0 \quad \text{—— (iii)}$$

solving equations (ii) and (iii),

(ii) × 7 − (iii) × 4,

$$28x+49y+91-28x-48y-88=0$$

$$\Rightarrow y=-3$$

$y = -3 \Rightarrow$ (ii),

$$4x - 21 + 13 = 0$$

$$\Rightarrow 4x = 8$$

$$\Rightarrow x = 2$$

$\therefore (x, y) = (\alpha, \beta) = (2, -3)$

we know,

$$\frac{(x-\alpha)^2 - (y-\beta)^2}{a-b} = \frac{(x-\alpha)(y-\beta)}{h}$$

$$\Rightarrow \frac{(x-2)^2 - (y+3)^2}{2-6} = \frac{(x-2)(y+3)}{7/2}$$

$$\Rightarrow \frac{x^2 - 4x + 4 - y^2 - 6y - 9}{-4} = \frac{2(xy + 3x - 2y - 6)}{7}$$

$$\Rightarrow 7x^2 - 28x + 28 - 7y^2 - 42y - 63 = -8xy - 24x + 16y + 48$$

$$\Rightarrow 7x^2 + 8xy - 7y^2 - 4x - 58y - 83 = 0$$

Ans.

Circle

11 - Dec - 2017

1) (i) Here,

centre $(-2,-1)$

radius 4

the equation of the circle,

$$(x-h)^2+(y-k)^2=r^2$$

$$\Rightarrow (x+2)^2+(y+1)^2=4^2$$

$$\Rightarrow x^2+4x+4+y^2+2y+1=16$$

$$\Rightarrow x^2+y^2+4x+2y-11=0$$

Ans.

(ii) Here,

centre $(9,0)$

radius 1

the equation of the circle,

$$(x-h)^2+(y-k)^2=r^2$$

$$\Rightarrow (x-9)^2+(y-0)^2=1^2$$

$$\Rightarrow x^2-18x+81+y^2=1$$

$$\Rightarrow x^2+y^2-18x+80=0$$

Ans.

(iii) Here,

centre $(0, 0)$

radius 5

the equation of the circle,

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\Rightarrow (x-0)^2 + (y-0)^2 = 5^2$$

$$\Rightarrow x^2 + y^2 = 25$$

$$\Rightarrow x^2 + y^2 - 25 = 0$$

Ans.

2| (i) $5x^2 + 5y^2 - 11x - 9y - 12 = 0$

$$\Rightarrow x^2 + y^2 - \frac{11}{5}x - \frac{9}{5}y - \frac{12}{5} = 0 \quad \text{——} \ (1)$$

comparing eqn (1) with the general eqn of a circle,

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Here,

$2g = -\frac{11}{5}$ | $2f = -\frac{9}{5}$ | $c = -\frac{12}{5}$

$\Rightarrow g = -\frac{11}{10}$ | $\Rightarrow f = -\frac{9}{10}$

$\therefore$ centre $(-g, -f) = \left(\frac{11}{10}, \frac{9}{10}\right)$.

radius, $r = \sqrt{g^2 + f^2 - c}$

$$= \sqrt{\left(-\frac{11}{10}\right)^2 + \left(-\frac{9}{10}\right)^2 + \frac{12}{5}}$$

$$= \sqrt{\frac{121 + 81 + 240}{100}}$$

$$= \frac{\sqrt{442}}{10}$$

Ans.

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(ii) $x^2+y^2+2x+2y+1=0$ —— ①

comparing eqn ① with the general eqn of a circle,

$$x^2+y^2+2gx+2fy+c=0$$

Here,

$2g=2$	$2f=2$	$c=1$
$\Rightarrow g=1$	$\Rightarrow f=1$	

$\therefore$ centre $(-g,-f)=(-1,-1)$

radius, $r=\sqrt{g^2+f^2-c}$

$=\sqrt{1^2+1^2-1}$

$=1$

Ans.

(iii) $x^2+y^2+2x-4y-8=0$ ——— ①

comparing eqn ① with the general eqn of a circle

$$x^2+y^2+2gx+2fy+c=0$$

Here,

$2g=2$	$2f=-4$	$c=-8$
$\Rightarrow g=1$	$\Rightarrow f=-2$	

$\therefore$ centre $(-g,-f)=(-1,2)$

radius, $r=\sqrt{g^2+f^2-c}$

$=\sqrt{(-1)^2+2^2+8}$

$=\sqrt{13}$ Ans.

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31 (i) $(1,3), (2,-1), (-1,1)$

Let the equation of the circle be,

$$x^2+y^2+2gx+2fy+c=0 \quad \text{——} \ (\mathrm{I})$$

circle (I) passing through the points :

$(1,3) \rightarrow$ (I), $\quad 10+2g+6f+c=0$ —— (II)

$(2,-1) \rightarrow$ (I), $\quad 5+4g-2f+c=0$ —— (III)

$(-1,1) \rightarrow$ (I), $\quad 2-2g+2f+c=0$ —— (IV)

Now,

(II) + (IV) $\Rightarrow$

$$12+8f+2c=0 \quad \text{——} \ (\mathrm{V})$$

(III) + (IV)$\times 2 \Rightarrow$

$$9+2f+3c=0 \quad \text{——} \ (\mathrm{VI})$$

(V) − (VI) $\times 4$,

$$-24-10c=0$$

$$\Rightarrow c=-\frac{24}{10}$$

$$\Rightarrow c=-\frac{12}{5}$$

$c = -\frac{12}{5} \rightarrow$ (vi),

$$9 + 2f + 3 \times \left(-\frac{12}{5}\right) = 0$$

$$\Rightarrow 2f = \frac{36}{5} - 9$$

$$\Rightarrow 2f = \frac{-9}{5}$$

$$\Rightarrow f = -\frac{9}{10}$$

$f = -\frac{9}{10} \rightarrow$ (iv),

$$2 - 2g + 2f + c = 0$$

$$\Rightarrow 2g = 2 + 2f + c$$

$$\Rightarrow 2g = 2 - 2 \cdot \frac{9}{10} - \frac{12}{5}$$

$$\Rightarrow 2g = \frac{10 - 21}{5}$$

$$\Rightarrow 2g = -\frac{11}{5}$$

$$\Rightarrow g = -\frac{11}{10}$$

from (i),

$$x^2 + y^2 - \frac{11}{5}x - \frac{9}{5}y - \frac{12}{5} = 0$$

$$\Rightarrow 5x^2 + 5y^2 - 11x - 9y - 12 = 0 \quad \text{Ans.}$$

(iii) $(3,1), (4,-3), (1,-1)$

Let the equation of the circle be,

$$x^2+y^2+2gx+2fy+c=0 \quad \text{—— (i)}$$

circle (i) passing through the points :

$(3,1) \rightarrow$ (i),

$$3^2+1^2+2g\cdot 3+2f\cdot 1+c=0$$

$$\Rightarrow 6g+2f+c=-10 \quad \text{—— (ii)}$$

$(4,-3) \rightarrow$ (i),

$$4^2+(-3)^2+2g\cdot 4+2f\cdot(-3)+c=0$$

$$\Rightarrow 8g-6f+c=-25 \quad \text{—— (iii)}$$

$(1,-1) \rightarrow$ (i),

$$1^2+(-1)^2+2g\cdot 1+2f\cdot(-1)+c=0$$

$$\Rightarrow 2g-2f+c=-2 \quad \text{—— (iv)}$$

Now,

(ii) $-$ (iv) $\times 3 \Rightarrow$

$$6g+2f+c+10-6g+6f-3c-6=0$$

$$\Rightarrow 8f-2c+4=0 \quad \text{—— (v)}$$

(iii) $-$ (iv) $\times 4 \Rightarrow$

$$8g-6f+c+25-8g+8f-4c-8=0$$

$$\Rightarrow 2f-3c+17=0 \quad \text{—— (vi)}$$

V − VI × 4 $\Rightarrow$

$$8f - 2c + 4 - 8f + 12c - 68 = 0$$

$$\Rightarrow \quad 10c - 64 = 0$$

$$\Rightarrow \quad c = \frac{64}{10}$$

$$\Rightarrow \quad c = \frac{32}{5}$$

$c = \frac{32}{5} \rightarrow$ VI,

$$2f - 3 \cdot \frac{32}{5} + 17 = 0$$

$$\Rightarrow \quad 2f = \frac{96}{5} - 17$$

$$\Rightarrow \quad f = \frac{11}{10}$$

from IV,

$$2g - 2 \cdot \frac{11}{10} + \frac{32}{5} = -2$$

$$\Rightarrow \quad 2g = \frac{22}{10} - \frac{32}{5} - 2$$

$$\Rightarrow \quad g = -\frac{31}{10}$$

from I,

$$x^2 + y^2 - 2 \cdot \frac{31}{10} x + 2 \cdot \frac{11}{10} y + \frac{32}{5} = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{31}{5} x + \frac{11}{5} y + \frac{32}{5} = 0$$

$$\Rightarrow \quad 5x^2 + 5y^2 - 31x + 11y + 32 = 0$$

Ans.

Tangent and Normal of Circle

12th Dec 2017

1) $x^2+y^2-2x+4y+3=0$ —— (i)

$x^2+y^2-8x-2y+9=0$ —— (ii)

comparing circles (i) and (ii) with the general eqn of a circle,

$$x^2+y^2+2gx+2fy+c=0$$

Here,

(i) $\Rightarrow$

$2g=-2$; $2f=4$; $c=3$

$\Rightarrow g=-1$ $\quad \Rightarrow f=2$

$\therefore$ centre, $c_1(-g,-f)=(1,-2)$

radius, $r_1=\sqrt{g^2+f^2-c}$

$=\sqrt{2}$

(ii) $\Rightarrow$

$2g=-8$; $2f=-2$; $c=9$

$\Rightarrow g=-4$ $\quad \Rightarrow f=-1$

$\therefore$ centre, $c_2=(4,1)$

radius, $r_2=\sqrt{8}$

$=2\sqrt{2}$

$c_1(1,-2)$ ———————— $c_2(4,1)$

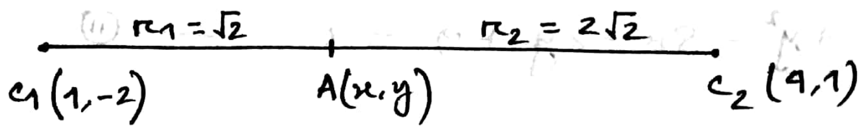
$$\therefore c_1c_2=\sqrt{(1-4)^2+(-2-1)^2} \quad \left[\because PQ=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\right]$$

$=\sqrt{18}$

$=3\sqrt{2}$

$=r_1+r_2$

Now,



$$\therefore \quad x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}$$

$$= \frac{\sqrt{2}\cdot 4 + 2\sqrt{2}\cdot 1}{\sqrt{2} + 2\sqrt{2}} \qquad [\text{Here, } r = m]$$

$$= \frac{4\sqrt{2} + 2\sqrt{2}}{3\sqrt{2}}$$

$$= 2$$

$$y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

$$= \frac{\sqrt{2}\cdot 1 + 2\sqrt{2}\cdot(-2)}{\sqrt{2} + 2\sqrt{2}}$$

$$= \frac{\sqrt{2} - 4\sqrt{2}}{3\sqrt{2}}$$

$$= -1$$

$$\therefore \quad A(x,y) = (2,-1)$$

Hence the circles touch one another at $(2,-1)$

Ans.

2|

$$x^2+y^2-9x+14y-7=0 \quad \text{——} \ (\text{i})$$

$$x^2+y^2+15x+14=0 \quad \text{——} \ (\text{ii})$$

Let the equation of the circle be,

$$x^2+y^2-9x+14y-7+k(x^2+y^2+15x+14)=0 \quad \text{——} \ (\text{iii})$$

equation (iii) passing through the point (2, 5).

$$(2^2+5^2-9\cdot 2+14\cdot 5-7)+k(2^2+5^2+15\cdot 2+14)=0$$

$$\Rightarrow (4+25-18+70-7)+k(4+25+30+14)=0$$

$$\Rightarrow k=-\frac{74}{73}$$

from (iii),

$$x^2+y^2-9x+14y-7+\left(-\frac{74}{73}\right)(x^2+y^2+15x+14)=0$$

$$\Rightarrow 73x^2+73y^2-657x+1022y-511-74x^2-74y^2-1110x \mp 1036=0$$

$$\Rightarrow -x^2-y^2-1767x+1022y \mp \overset{1547}{525}=0$$

$$\Rightarrow x^2+y^2+1767x-1022y \overset{+1547}{-525}=0$$

Ans.

31 $x^2+y^2-1=0 \quad \text{—— (i)}$

$x^2+y^2+2x+4y+1=0 \quad \text{—— (ii)}$

$x+2y+5=0 \quad \text{—— (iii)}$

Let the equation of the circle be,

$$(x^2+y^2-1)+k(x^2+y^2+2x+4y+1)=0$$

$$\Rightarrow x^2+y^2-1+x^2k+y^2k+2xk+4yk+k=0$$

$$\Rightarrow x^2+x^2k+y^2+y^2k+2xk+4yk+k-1=0$$

$$\Rightarrow (1+k)x^2+(1+k)y^2+2xk+4yk+(k-1)=0$$

$$\Rightarrow x^2+y^2+\left(\frac{2k}{1+k}\right)x+\left(\frac{4k}{1+k}\right)y+\left(\frac{k-1}{1+k}\right)=0 \quad \text{—— (iv)}$$

comparing eqn (iv) with the general eqn of a circle,

$$x^2+y^2+2gx+2fy+c=0$$

Here,

$2g=\frac{2k}{1+k}$	$2f=\frac{4k}{1+k}$	$c=\frac{k-1}{1+k}$
$\Rightarrow g=\frac{k}{1+k}$	$\Rightarrow f=\frac{2k}{1+k}$	

$\therefore$ centre $(-g,-f)=\left(-\frac{k}{1+k},-\frac{2k}{1+k}\right)$

radius, $r = \sqrt{g^2+f^2-c}$

$$= \sqrt{\left(\frac{k}{1+k}\right)^2 + \left(\frac{2k}{1+k}\right)^2 - \left(\frac{k-1}{1+k}\right)}$$

$$= \sqrt{\frac{k^2+4k^2-(k-1)(1+k)}{(1+k)^2}}$$

$$= \sqrt{\frac{5k^2-k^2+1}{(1+k)^2}}$$

$$= \frac{\sqrt{4k^2+1}}{\sqrt{(1+k)^2}}$$

$$= \frac{\sqrt{4k^2+1}}{1+k}$$

we know,

$OP = r$

$$\Rightarrow \left|\frac{\left(\frac{-k}{1+k}\right) + 2\left(\frac{-2k}{1+k}\right) + 5}{\sqrt{1^2+2^2}}\right| = \frac{\sqrt{4k^2+1}}{1+k} \qquad \left[\because OP = \left|\frac{ah+bk+c}{\sqrt{a^2+b^2}}\right|\right]$$

$$\Rightarrow \left|\frac{\left(\frac{-5k+5(1+k)}{1+k}\right)}{\sqrt{5}}\right| = \frac{\sqrt{4k^2+1}}{1+k}$$

$$\Rightarrow \left|\frac{(\sqrt{5})^2}{\sqrt{5}\,(1+k)}\right| = \frac{\sqrt{4k^2+1}}{1+k}$$

$$\Rightarrow |\sqrt{5}| = \sqrt{4k^2+1}$$

$$\Rightarrow \left(|\sqrt{5}|\right)^2 = \left(\sqrt{4k^2+1}\right)^2$$

$$\Rightarrow 5 = 4k^2+1$$

$$\Rightarrow 4k^2 = 4$$

$$\Rightarrow k^2 = 1$$

$$\Rightarrow k = \pm 1$$

$$\therefore k = 1 \qquad [\because k \neq -1]$$

from (iv),

$$x^2+y^2+\left(\frac{2.1}{1+1}\right)x + \left(\frac{4.1}{1+1}\right)y + \left(\frac{1-1}{1+1}\right) = 0$$

$$\Rightarrow x^2+y^2+x+2y = 0$$

Ans.

4)
$$x^2+y^2-4x+6y-3=0 \quad \text{--- (i)}$$

comparing eqn (i) with the general eqn of a circle,

$$x^2+y^2+2gx+2fy+c=0$$

Here,

$2g=-4$ | $2f=6$ | $c=-3$

$\Rightarrow g=-2$ | $\Rightarrow f=3$ |

$\therefore$ centre $(-g,-f) = (2,-3)$

radius, $r = \sqrt{g^2+f^2-c} = 4$

Let the equation of the tangent line be,

$$3x-4y+k=0 \quad \text{--- (ii)}$$

we know,

$$OP = r$$

$$\Rightarrow \left|\frac{3\times2-4\times(-3)+k}{\sqrt{3^2+(-4)^2}}\right| = 4$$

$$\Rightarrow \left|\frac{18+k}{5}\right| = 4$$

$$\Rightarrow \frac{18+k}{5} = \pm 4$$

$$\Rightarrow 18+k = \pm 20$$

from (ii),

$(+)\ k=2 \Rightarrow 3x-4y+2=0$

$(-)\ k=-38 \Rightarrow 3x-4y-38=0$

Ans.

5. $x^2+y^2+2gx+2fy=0$ —— (i)

$x^2+y^2+2g_1x+2f_1y=0$ —— (ii)

comparing circles (i) and (ii) with the general eqn of a circle

$$x^2+y^2+2gx+2fy+c=0$$

Here,

(i) $\Rightarrow$

$g=g$

$f=f$

$c=0$

$\therefore$ centre, $c_1=(-g,-f)$

radius, $r_1=\sqrt{g^2+f^2}$

(ii) $\Rightarrow$

$g=g_1$

$f=f_1$

$c=0$

$\therefore$ centre, $c_2=(-g_1,-f_1)$

radius, $r_2=\sqrt{g_1^2+f_1^2}$

We know,

$$c_1c_2=|r_1\pm r_2|$$

$$\Rightarrow \left(\sqrt{(-g+g_1)^2+(-f+f_1)^2}\right)^2=\left(\left|\sqrt{g^2+f^2}\pm\sqrt{g_1^2+f_1^2}\right|\right)^2$$

$$\Rightarrow \cancel{g^2}-2gg_1+\cancel{g_1^2}+\cancel{f^2}-2ff_1+\cancel{f_1^2}=\cancel{g^2}+\cancel{f^2}\pm2\sqrt{g^2+f^2}\sqrt{g_1^2+f_1^2}+\cancel{g_1^2}+\cancel{f_1^2}$$

$$\Rightarrow -2(gg_1+ff_1)=\pm2\sqrt{(g^2+f^2)(g_1^2+f_1^2)}\qquad [\because \sqrt{ab}=\sqrt{a}\cdot\sqrt{b}]$$

$$\Rightarrow \left[-2(gg_1 + ff_1)\right]^2 = \left[\pm 2\sqrt{(g^2+f^2)(g_1^2+f_1^2)}\right]^2$$

$$\Rightarrow \cancel{g^2g_1^2} + 2gg_1ff_1 + f^2f_1^2 = \cancel{g^2g_1^2} + g_1^2f^2 + g^2f_1^2 + \cancel{f^2f_1^2}$$

$$\Rightarrow g_1^2f^2 - 2gg_1ff_1 + g^2f_1^2 = 0$$

$$\Rightarrow (g_1f)^2 - 2(g_1f)(gf_1) + (gf_1)^2 = 0$$

$$\Rightarrow (g_1f - gf_1)^2 = 0$$

$$\Rightarrow g_1f - gf_1 = 0$$

$$\Rightarrow f_1g = fg_1$$

showed.

6) $x^2+y^2-6x+4y-12=0$ — ①

comparing eqn ① with the general eqn of a circle.

Here,

$2g=-6$ | $2f=4$ | $c=-12$

$\Rightarrow g=-3$ | $\Rightarrow f=2$

$\therefore$ centre $(-g,-f)=(3,-2)$

radius, $r=\sqrt{(-3)^2+2^2+12}$

$=5$

we know,

$OP=r$

$$\Rightarrow \left|\frac{12-2k+7}{\sqrt{4^2+k^2}}\right|=5$$

$$\Rightarrow \left(\left|\frac{19-2k}{\sqrt{4^2+k^2}}\right|\right)^2=5^2$$

$$\Rightarrow (19-2k)^2=25(16+k^2)$$

$$\Rightarrow 361-76k+4k^2=400+25k^2$$

$$\Rightarrow -21k^2-76k-39=0$$

$$\Rightarrow 21k^2+63k+13k+39=0$$

$$\Rightarrow 21k(k+3)+13(k+3)=0$$

$$\Rightarrow k=-3,\ -\frac{13}{21}$$

Ans.

BRAC UNIVERSITY

Inspiring Excellence

Assignment

Semester: Fall 2017

Course ID: MAT 110 (Section: 02)

Course Title: Differential Calculus and Coordinate Geometry

Answer all the questions:

Q1. Determine the equation $x^2 + 2\sqrt{3}xy - y^2 = 2a^2$ after rotating of axes through 30^0. [5]

Q2. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y,) = xy$ subject to the constraint $4x^2 + 8y^2 = 16$. Also find the points at which these extreme values occur. [5]

Q3. Locate all relative maxima, relative minima and saddle points, if any of [5]

$$f(x, y) = y^2 + xy + 3y + 2x + 3.$$

Q4. Find the value of k so that the following equation may represent a pair of straight lines: [5]

$$2x^2 + xy - y^2 - 2x - 5y + k = 0.$$

Q5. Find the equation of the circle passing through the points $(3, 1), (4, -3), (1, -1)$. [5]

Q1. $x^2 + 2\sqrt{3}\,xy - y^2 = 2a^2$ ——— ①

We know,

$x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$

$= x'\cos 30^\circ - y'\sin 30^\circ$ $\qquad$ $= x'\sin 30^\circ + y'\cos 30^\circ$

$= x'\frac{\sqrt{3}}{2} - y'\frac{1}{2}$ $\qquad$ $= x'\frac{1}{2} + y'\frac{\sqrt{3}}{2}$ $\qquad$ $\left[\because \cos 30^\circ = \frac{\sqrt{3}}{2}, \ \sin 30^\circ = \frac{1}{2}\right.$

$= \frac{\sqrt{3}x' - y'}{2}$ $\qquad$ $= \frac{x' + \sqrt{3}y'}{2}$

substituting the values of x and y in the equation ①,

$$\left(\frac{\sqrt{3}x'-y'}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}x'-y'}{2}\right)\left(\frac{x'+\sqrt{3}y'}{2}\right) - \left(\frac{x'+\sqrt{3}y'}{2}\right)^2 = 2a^2$$

$$\Rightarrow (3x'^2 - 2\sqrt{3}x'y' + y'^2) + 2\sqrt{3}(\sqrt{3}x'^2 + 3x'y' - x'y' - \sqrt{3}y'^2) - (x'^2 + 2\sqrt{3}x'y' + 3y'^2) = 8a^2$$

$$\Rightarrow 3x'^2 - 2\sqrt{3}x'y' + y'^2 + 6x'^2 + 6\sqrt{3}x'y' - 2\sqrt{3}x'y' - 6y'^2 - x'^2 - 2\sqrt{3}x'y' - 3y'^2 = 8a^2$$

$$\Rightarrow 8x'^2 - 6\sqrt{3}x'y' + 6\sqrt{3}x'y' - 8y'^2 = 8a^2$$

$$\Rightarrow 8x'^2 - 8y'^2 = 8a^2$$

$$\Rightarrow x'^2 - y'^2 = a^2$$

Removing suffixes,

$$x^2 - y^2 = a^2$$

Answer.

Q2. $f(x,y) = xy$

$g(x,y) = 0$

$\Rightarrow 4x^2 + 8y^2 - 16 = 0$ ——— (I)

We know,

$\bar{\nabla} f = \lambda \bar{\nabla} g$

$\Rightarrow \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \lambda\left(\frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j}\right)$

$\Rightarrow y\hat{i} + x\hat{j} = \lambda(8x\hat{i} + 16y\hat{j})$

$\because$ $8\lambda x = y$ and $16\lambda y = x$

$\Rightarrow \lambda = \frac{y}{8x}$ — (II) $\Rightarrow \lambda = \frac{x}{16y}$ ——— (III)

from (II) and (III)

$\frac{y}{8x} = \frac{x}{16y}$

$\Rightarrow 16y^2 = 8x^2$

$\Rightarrow 8y^2 = 4x^2$ ——— (IV)

from (I),

$4x^2 + 8y^2 - 16 = 0$

$\Rightarrow 4x^2 + 4x^2 = 16$

$\Rightarrow 8x^2 = 16$

$\Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$

$x = \pm\sqrt{2} \Rightarrow$ (IV),

$y = \pm 1$

$x = -\sqrt{2} \Rightarrow$ (IV),

$y = \pm 1$

(x,y)	$(\sqrt{2},1)$	$(\sqrt{2},-1)$	$(-\sqrt{2},1)$	$(-\sqrt{2},-1)$
$f(x,y) = xy$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$

$\therefore$ Maximum value $= \sqrt{2}$

Minimum value $= -\sqrt{2}$

Thus, the function $f(x,y)$ has an absolute maximum of $\sqrt{2}$ at $(\sqrt{2},1)$ and $(-\sqrt{2},-1)$, also an absolute minimum of $-\sqrt{2}$ at $(\sqrt{2},-1)$ and $(-\sqrt{2},1)$.

Answer.

Q3. $f(x,y) = y^2 + xy + 3y + 2x + 3$

to find critical points:

$f_x = 0$	$f_y = 0$	f_x is undefined	f_y is undefined
$\Rightarrow y + 2 = 0$	$\Rightarrow 2y + x + 3 = 0$	$f_x = y + 2$	$f_y = 2y + x + 3$
$\Rightarrow y = -2$	$\Rightarrow -4 + x + 3 = 0$	$\times$	$\times$
	$\Rightarrow x = 1$		

critical point $(x,y) = (1,-2)$

second partial test:

$$f_{xx} = \frac{\partial}{\partial x}(f_x)$$
$$= 0$$

$$f_{yy} = \frac{\partial}{\partial y}(f_y)$$
$$= 2$$

$$f_{xy} = \frac{\partial}{\partial y}(f_x)$$
$$= 1$$

we know,

$$D = f_{xx} \cdot f_{yy} - f_{xy}^2$$
$$= -1$$

(x_0, y_0)	$f_{xx} = 0$	$f_{yy} = 2$	$f_{xy} = 1$	D	comment
$(1, -2)$	$0 = 0$	$2 > 0$	$1 > 0$	$-1 < 0$	$\because D < 0$ so it has a saddle point

Answer.

Q4. $2x^2 + xy - y^2 - 2x - 5y + k = 0$ —— (1)

comparing equation (1) with the general equation of 2nd degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 2$	$2h = 1$	$b = -1$	$2g = -2$	$2f = -5$	$c = k$
	$\Rightarrow h = \frac{1}{2}$		$\Rightarrow g = -1$	$\Rightarrow f = -1$	

equation (1) will represent a pair of straight lines if,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow 2\cdot(-1)\cdot k + 2\cdot\left(-\frac{5}{2}\right)\cdot(-1)\cdot\frac{1}{2} - 2\left(-\frac{5}{2}\right)^2 + 1\cdot(-1)^2 - k\cdot\left(\frac{1}{2}\right)^2 = 0$$

$$\Rightarrow -2k + \frac{5}{2} - \frac{25}{2} + 1 - \frac{k}{4} = 0$$

$$\Rightarrow \frac{-8k + 10 - 50 + 4 - k}{4} = 0$$

$$\Rightarrow -9k - 36 = 0$$

$$\Rightarrow -9k = 36$$

$$\Rightarrow k = -4$$

Answer.

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Q5. Let the equation of the circle be,

$$x^2+y^2+2gx+2fy+c=0 \quad \text{——} (I)$$

circle (I) passing through the points $(3,1)$, $(4,-3)$ and $(1,-1)$

$(3,1) \rightarrow$ (I),

$$3^2+1^2+2g\cdot 3+2f\cdot 1+c=0$$

$$\Rightarrow \quad 6g+2f+c=-10 \quad \text{——} (II)$$

$(4,-3) \rightarrow$ (I),

$$4^2+(-3)^2+2g\cdot 4+2f\cdot(-3)+c=0$$

$$\Rightarrow \quad 8g-6f+c=-25 \quad \text{——} (III)$$

$(1,-1) \rightarrow$ (I),

$$1^2+(-1)^2+2g\cdot 1+2f\cdot(-1)+c=0$$

$$\Rightarrow \quad 2g-2f+c=-2 \quad \text{——} (IV)$$

(II) $-$ (IV) $\times 3 \rightarrow$

$$6g+2f+c+10-6g+6f-3c-6=0$$

$$\Rightarrow \quad 8f-2c+4=0 \quad \text{——} (V)$$

(III) $-$ (IV) $\times 4 \rightarrow$

$$8g-6f+c+25-8g+8f-4c-8=0$$

$$\Rightarrow \quad 2f-3c+17=0 \quad \text{——} (VI)$$

$\text{V} - \text{VI} \times 4 \longrightarrow$

$$8f - 2c + 4 - 8f + 12c - 68 = 0$$

$$\Rightarrow \quad 10c - 64 = 0$$

$$\Rightarrow \quad c = \frac{64}{10}$$

$$\Rightarrow \quad c = \frac{32}{5}$$

$c = \frac{32}{5} \rightarrow \text{VI}$,

$$2f - 3 \cdot \frac{32}{5} + 17 = 0$$

$$\Rightarrow \quad 2f = \frac{96}{5} - 17$$

$$\Rightarrow \quad f = \frac{11}{10}$$

from IV,

$$2g - 2 \cdot \frac{11}{10} + \frac{32}{5} = -2$$

$$\Rightarrow \quad 2g = \frac{22}{10} - \frac{32}{5} - 2$$

$$\Rightarrow \quad g = -\frac{31}{10}$$

from I,

$$x^2 + y^2 - 2 \cdot \frac{31}{10} x + 2 \cdot \frac{11}{10} y + \frac{32}{5} = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{31}{5} x + \frac{11}{5} y + \frac{32}{5} = 0$$

$$\Rightarrow \quad 5x^2 + 5y^2 - 31x + 11y + 32 = 0$$

Answer.

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Semester: Fall 2017
Course ID: MAT 110
Course Title: Mathematics I
Section: 02
Quiz: 01

Name: [redacted]
ID: [redacted]
Date: Oct 3, 2017
Time: 20 Minutes
Marks obtained: 25

Answer the following questions: (Total Marks: 25)

Q1. Find domain and range of the following functions: 6+6=12

(i) $f(x) = \frac{1}{x-3}$

(ii) $g(x) = \sqrt{2x+4}$

Q2. Sketch the graph of the functions below: 3+3=6

(i) $y = 1 - |2x + 4|$

(ii) $y = 2 + (x-3)^2$

Q3. Find $\lim_{x \to 1} f(x)$ for $f(x) = \begin{cases} 2x+1, & x < 1 \\ 3-x, & x > 1. \end{cases}$ 7

Q1 (i) $f(x) = \frac{1}{x-3}$

$x - 3 \neq 0$

$\Rightarrow x \neq 3$

Domain: $\mathbb{R} - \{3\}$

Range: $\mathbb{R} - \{0\}$

6

let $y = \frac{1}{x-3}$

$\Rightarrow xy - 3y = 1$

$\Rightarrow xy = 1 + 3y$

$\Rightarrow x = \frac{1+3y}{y}$

$\therefore y \neq 0$

(ii) $g(x) = \sqrt{2x+4}$

$2x + 4 \geq 0$ — ①

$\Rightarrow 2x + 4 = 0$

$\Rightarrow 2x = -4$

$\Rightarrow x = -2$

Domain: $[-2, \infty)$

Range: $[0, \infty)$

6

$-\infty \longleftarrow \; -2 \; \longrightarrow \infty$

$x \in (-\infty, -2)$	$x \in (-2, \infty)$
let $x = -4$	let $x = 2$
from ①	from ①
$2(-4) + 4$	$2(2) + 4$
$= -4 \not> 0$	$\Rightarrow 4 + 4$
	$\Rightarrow 8 > 0$

To find Range

x	-2	∞
y	0	∞

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Q2

(i) $y = 1 - |2x+4|$

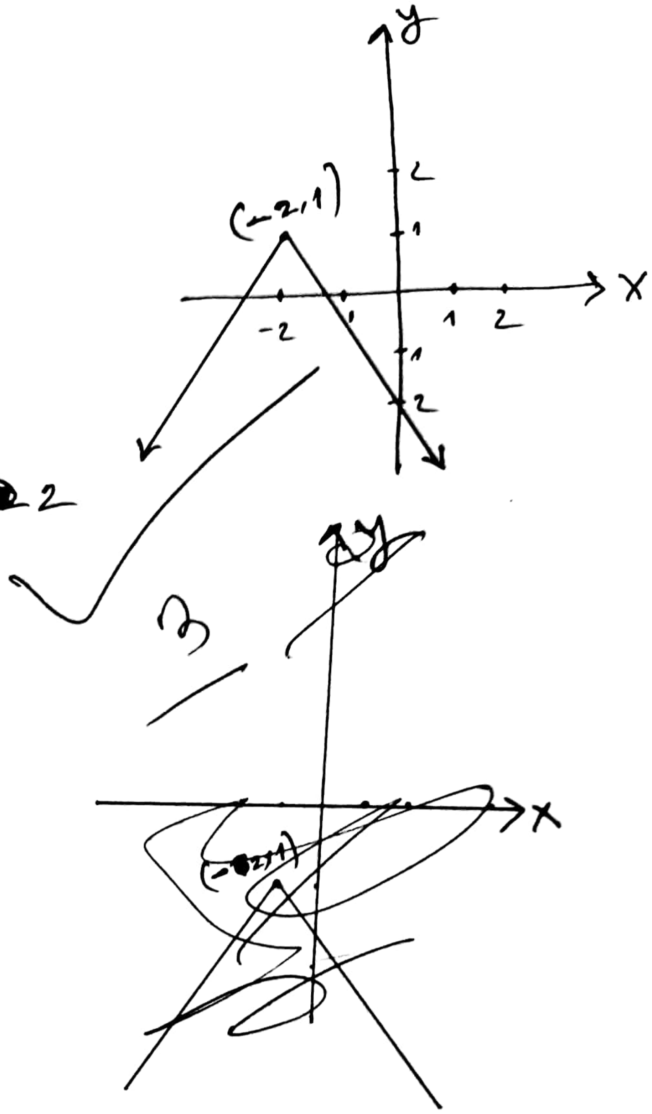
$\Rightarrow y - 1 = -|2x+4|$

$y - 1 = 0$ if $|2x+4| = 0$

$\Rightarrow y = 1$ | $\Rightarrow 2x = -4$

$\Rightarrow x = -2$

Vertex: ~~(-1/2, 1)~~ $(-2, 1)$



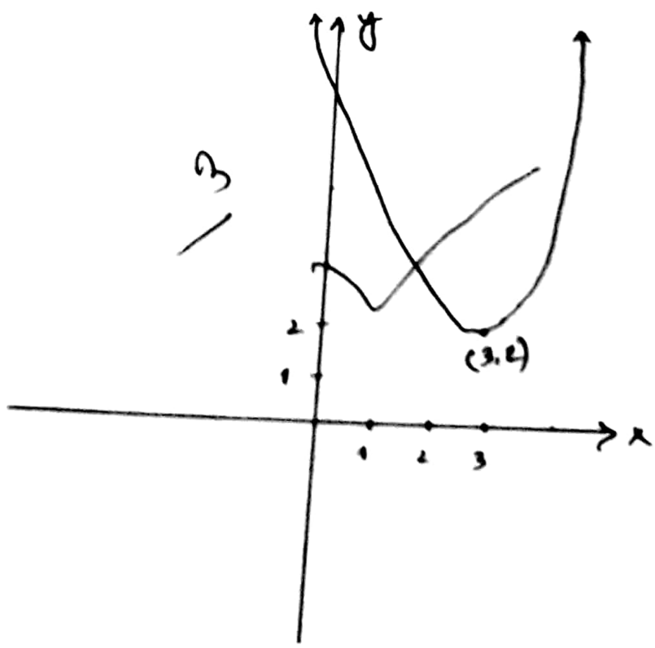
(ii) $y = 2 + (x-3)^2$

$\Rightarrow y - 2 = (x-3)^2$

$y - 2 = 0$ if $(x-3)^2 = 0$

$\Rightarrow y = 2$ | $x = 3$

Vertex: $(3, 2)$



Q3 $f(x) = \begin{cases} 2x+1 \ ; & x<1 \\ 3-x \ ; & x>1 \end{cases}$ $\lim_{x \to 1} f(x)$

$L.H.L = \lim_{x \to 1^-} f(x)$

$= \lim_{x \to 1^-} (2x+1)$

$= 2 \cdot 1 + 1$

$= 3$

$R.H.L = \lim_{x \to 1^+} f(x)$

$= \lim_{x \to 1^+} (3-x)$

$= 3-1$

$= 2$

7

$\because L.H.L \neq R.H.L$

$\therefore$ Limit does not exist.

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Semester: Fall 2017
Course ID: MAT 110
Course Title: Mathematics I
Section: 02
Quiz: 02

Name: [redacted]
ID: [redacted]
Date: Oct 15, 2017
Time: 20 Minutes
Marks obtained: 25

Answer the following questions: (Total Marks: 25)

Q1. Test the continuity and differentiability of the following function at $x = 0$: 12

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Q2. Find the differential coefficients of the following functions: 13

(i) $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

(ii) $y = (\sin x)^{\tan x}$.

Q1. $f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x=0$

$$Lf'(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0^-} \frac{x^2 \sin(1/x) - 0}{x - 0}$$

$$= \lim_{x \to 0^-} x \sin(1/x)$$

$$= \left(\lim_{x \to 0^-} x\right)\left(\lim_{x \to 0^-} \sin \frac{1}{x}\right)$$

$$= (0) \cdot \left(\sin \frac{1}{0}\right)$$

$= 0 \cdot \sin \cancel{\infty}$

$= 0 \cdot$ [it oscillates between -1 and 1 without approaching a limit]

$= 0$

R.T.O

$$Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0+} \frac{x^2 \sin(1/x) - 0}{x - 0}$$

12

$$= \lim_{x \to 0+} x \sin 1/x$$

$$= \left(\lim_{x \to 0+} x\right)\left(\lim_{x \to 0+} \sin \frac{1}{x}\right)$$

$= 0$. [Oscillates between -1 and 1, without approaching a limit]

$\because Lf'(0) = Rf'(0)$

$\therefore$ Differentiation at $x = 0$

And a Differentiate function is always continuous. So the the function is both continuous and differentiate at $x = 0$. Ans

Q21 (i) $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$= \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$ let, $x = \sin\theta$

$\Rightarrow \theta = \sin^{-1}x$

$= \tan^{-1}\left(\frac{\sin\theta}{\sqrt{(\cos\theta)^2}}\right)$

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$= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$

$= \tan^{-1}(\tan\theta)$

$= \theta$

$= \sin^{-1}x$

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$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$ Ans.

(ii) $y = (\sin x)^{\tan x}$

$\Rightarrow \ln y = \ln (\sin x)^{\tan x}$

$\Rightarrow \ln y = \tan x \ln(\sin x)$

$\Rightarrow \frac{d}{dx} \ln y = \frac{d}{dx}(\tan x \ \ln \sin x)$ [applying uv method]

$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \tan x \frac{d}{dx} \ln \sin x + \ln \sin x \frac{d}{dx} \tan x$

$= \tan x \frac{1}{\sin x} \cdot \cos x + \ln \sin x \cdot \sec^2 x$

$= \tan x \frac{\cos x}{\sin x} + \sec^2 x \ln \sin x$

$= \tan x \cdot \cot x + \sec^2 x \ln \sin x$

$= \tan x \cdot \frac{1}{\tan x} + \sec^2 x \ln \sin x$

$= 1 + \sec^2 x \ln \sin x$

$\Rightarrow \frac{dy}{dx} = y(1 + \sec^2 x \ln \sin x)$

$= (\sin x)^{\tan x} [1 + \sec^2 x \ln \sin x]$

6.5

Ans.

BRAC UNIVERSITY

Inspiring Excellence

Semester: Fall 2017
Course ID: MAT 110
Course Title: Mathematics I
Section: 02
Quiz: 03

Name: [redacted]
ID: [redacted]
Date: Nov 12, 2017
Time: 20 Minutes
Marks obtained: 25

Answer the following questions: (Total Marks: 25)

Q1. If $y = e^{m\sin^{-1}x}$, then show that **10**

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0.$$

Q2. Verify the statement of Mean Value Theorem for the function **7**

$$f(x) = \sqrt{x+1}; \quad [0,3].$$

Q3. Find f_{xy} for the function $f(x,y) = e^{x^2+xy+y^2}$. **8**

Q1) $y = e^{m\sin^{-1}x}$ —— (i)

Differentiating (i) with respect to x,

$y_1 = me^{m\sin^{-1}x} \cdot \dfrac{1}{\sqrt{1-x^2}}$

$\Rightarrow (\sqrt{1-x^2})\, y_1 = me^{m\sin^{-1}x}$

$\Rightarrow (\sqrt{1-x^2})\, y_1 = my$

$\Rightarrow (1-x^2)y_1^2 = m^2y^2$ —— (ii)

10

Differentiating (ii) with respect to x,

$(1-x^2)\, 2y_1 \cdot y_2 + y_1^2(-2x) = m^2 2yy_1$

$\Rightarrow 2y_1y_2(1-x^2) - 2xy_1^2 = m^2 2yy_1$

$\Rightarrow 2y_1y_2(1-x^2) - 2xy_1^2 - m^2 2yy_1 = 0$

$\Rightarrow 2y_1\left[(1-x^2)y_2 - xy_1 - m^2y\right] = 0$

$\Rightarrow (1-x^2)y_2 - xy_1 - m^2y = 0$ —— (iii)

Applying Leibnitz theorem in equation (iii)

$$\left[y_{n+2}(1-x^2) + {}^nc_1\, y_{n+1}(-2x) + {}^nc_2\, y_n(-2) + {}^nc_3\, y_{n-1}\cdot(0)\right]$$

$$-\left[y_{n+1}x + {}^nc_1\, y_n\cdot 1 + {}^nc_2\, y_{n-1}\cdot 0\right] - m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nx y_{n+1} - 2\cdot\frac{n(n-1)}{2}\cdot y_n - xy_{n+1} - ny_n - m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - xy_{n+1} - (n^2-n)y_n - ny_n - m^2 y_n = 0$$

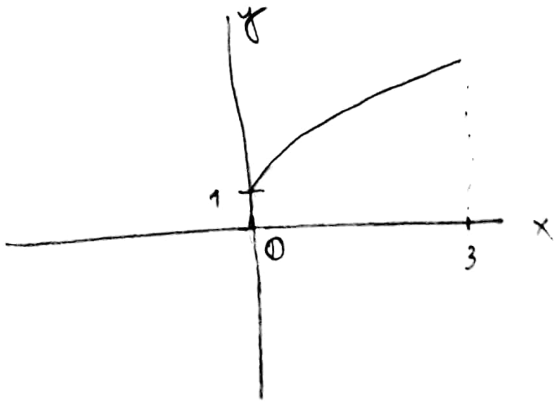
$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-n+n+m^2)y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

(showed)

Q21 $f(x) = \sqrt{x+1}$; $[0,3]$

Here, $y = \sqrt{x+1}$



Here, the function is differentiable on $(0,3)$ and continuous on $[0,3]$. So by Mean value theorem there is at least one number $c \in (0,3)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a} \quad \text{—} ①$$

$$f(0) = \sqrt{0+1} = 1$$

$$f(3) = \sqrt{3+1} = 2$$

$$f'(x) = \frac{1}{2\sqrt{x+1}} \cdot 1$$

$$= \frac{1}{2\sqrt{x+1}}$$

$$\therefore f'(c) = \frac{1}{2\sqrt{c+1}}$$

from ①,

$$\frac{1}{2\sqrt{c+1}} = \frac{2-1}{3-0}$$

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$$\Rightarrow 2\sqrt{c+1} = 3$$

$$\Rightarrow \sqrt{c+1} = \frac{3}{2}$$

$$\Rightarrow c+1 = \frac{9}{4}$$

$$\Rightarrow c = \frac{9}{4} - 1$$

$$= \frac{9-4}{4}$$

$$= \frac{5}{4}$$

$$\therefore c = \frac{5}{4} \in (0,3)$$ verified

93| $f(x,y) = e^{x^2+xy+y^2}$

$$f_x = \frac{\partial}{\partial x}\left(e^{x^2+xy+y^2}\right)$$

$$= e^{x^2+xy+y^2} \cdot (2x+y+0)$$

$$= (2x+y)\, e^{x^2+xy+y^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left[(2x+y)\, e^{x^2+xy+y^2}\right]$$

$$= (2x+y) \cdot e^{x^2+xy+y^2} \cdot (0+x+2y) + e^{x^2+xy+y^2} \cdot (0+1)$$

$$= (x+2y)(2x+y)\, e^{x^2+xy+y^2} + e^{x^2+xy+y^2}$$

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Ans.

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Semester: Fall 2017
Course ID: MAT 110
Course Title: Mathematics I
Section: 02
Quiz: 04

Name: [redacted]
ID: [redacted]
Date: Dec 03, 2017
Time: 40 Minutes
Marks obtained: 25

Answer the following questions: (Total Marks: 25)

Q1. Determine whether the following equation represents a pair of straight lines. Find their equations, point of intersection and the angle between them. 12

$$x^2 + 3xy + 2y^2 + \left(\frac{1}{8}\right)x - \left(\frac{1}{32}\right) = 0.$$

Q2. Remove the terms in x, y and xy from the equation $9x^2 + 24xy + 2y^2 - 6x + 20y + 41 = 0$ and write the transformed equation. 13

Q1) $x^2 + 3xy + 2y^2 + \left(\frac{1}{8}\right)x - \frac{1}{32} = 0$ ——— ①

comparing eqn ① with the general eqn of 2nd degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 1$	$2h = 3$ $\Rightarrow h = \frac{3}{2}$	$b = 2$	$2g = \frac{1}{8}$ $\Rightarrow g = \frac{1}{16}$	$2f = 0$ $\Rightarrow f = 0$	$c = -\frac{1}{32}$

eqn ① will represent a pair of straight lines if,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$= 1 \cdot 2 \cdot \left(-\frac{1}{32}\right) + 0 - 0 - 2\cdot\left(\frac{1}{16}\right)^2 - \left(-\frac{1}{32}\right)\cdot\left(\frac{3}{2}\right)^2$$

$$= -\frac{1}{16} - \frac{2}{256} + \frac{9}{128}$$

$$= \frac{-16 - 2 + 18}{256}$$

$$= \frac{0}{256} \quad \left[\because \frac{0}{\text{something}} = 0\right]$$

$$= 0$$

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$\therefore$ eqn ① represents a pair of straight lines.

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For the point of intersection,

Let $F(x,y) \equiv x^2 + 3xy + 2y^2 + \frac{1}{8}x - \frac{1}{32} = 0$

$\therefore \frac{\partial F}{\partial x} = 2x + 3y + \frac{1}{8} = 0$ —— (ii)

$\frac{\partial F}{\partial y} = 3x + 4y = 0$ —— (iii)

(ii) $\times 3$ − (iii) $\times 2 \Rightarrow$

$\cancel{6x} + 9y + \frac{3}{8} - \cancel{6x} - 8y = 0$

$\Rightarrow y = -\frac{3}{8}$

from (iii),

$3x + 4(-\frac{3}{8}) = 0$

$\Rightarrow 3x = \frac{12}{8}$

$\Rightarrow 3x = \frac{3}{2}$

$\Rightarrow x = \frac{1}{2}$

$\therefore (x,y) = (\alpha,\beta) = (\frac{1}{2}, -\frac{3}{8})$

To Find Angle:

$\theta = \tan^{-1}\left(\frac{2\sqrt{h^2-ab}}{a+b}\right)$

$= \tan^{-1}\left(\frac{2\sqrt{(\frac{3}{2})^2-2}}{3}\right)$

$= \tan^{-1}\left(\frac{2\sqrt{\frac{9}{4}-2}}{3}\right)$

$= \tan^{-1} \frac{\cancel{2}\cdot\frac{9}{2}-2}{3}$

$= \tan^{-1}(1/3)$

$= 18.43°$

Ans.

92) $9x^2 + 24xy + 2y^2 - 6x + 20y + 41 = 0$ ——— (i)

comparing eqn (i) with the general eqn of 2nd degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Here,

$a = 9$	$2h = 24$ $\Rightarrow h = 12$	$b = 2$	$2g = -6$ $\Rightarrow g = -3$	$2f = 20$ $\Rightarrow f = 10$	$c = 41$

Remove in the terms x, y:

Let $F(x,y) \equiv 9x^2 + 24xy + 2y^2 - 6x + 20y + 41 = 0$

$$\frac{\partial F}{\partial x} = 18x + 24y - 6 = 0$$

$\Rightarrow 3x + 4y - 1 = 0$ ——— (ii)

$$\frac{\partial F}{\partial y} = 24x + 4y + 20 = 0$$

~~$\Rightarrow 12x + 2y + 10y$~~

$\Rightarrow 6x + y + 5 = 0$ ——— (iii)

(ii) $\times 2$ − (iii)

$6\cancel{x} + 8y - 2 - \cancel{6x} - y - 5 = 0$

$\Rightarrow 7y - 7 = 0$

$\Rightarrow 7y = 7$

$\Rightarrow y = 1$

from (ii), $3x + 4.1 - 1 = 0$

$\Rightarrow 3x = -3$

$\Rightarrow x = -1$

$\therefore (x, y) = (\alpha, \beta) = (-1, 1)$

we know

Transformed equation,

$$ax^2 + 2hxy + by^2 + c = 0 \quad \text{—— (iv)}$$

where,

$$c = g\alpha + f\beta + c$$

$$= (-3)(-1) + 10 \cdot 1 + 41$$

$$= 3 + 10 + 41$$

$$= 54$$

from (iv),

$$9x^2 + 24xy + 2y^2 + 54 = 0 \quad \text{—— (v)}$$

<u>To remove xy term,</u>

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2\times 12}{9-2}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{24}{7}\right)$$

$$= 36.87^\circ$$

$\cos 36.87^\circ = 0.8$

$\sin 36.87^\circ = 0.6$

we know,

$$x = x'\cos\theta - y'\sin\theta \qquad\qquad y = x'\sin\theta + y'\cos\theta$$

$$= 0.8x' - 0.6y' \qquad\qquad = 0.6x' + 0.8y'$$

$$= \frac{8x' - 6y'}{10} \qquad\qquad = \frac{6x' + 8y'}{10}$$

substituting the values of x and y in the eqn (1)

$$9\left(\frac{8x'-6y'}{10}\right)^2 + 24\left(\frac{8x'-6y'}{10}\right).\left(\frac{6x'+8y'}{10}\right) + 2\left(\frac{6x'+8y'}{10}\right)^2 + 54 = 0$$

$$\Rightarrow \frac{9((8x')^2 - 2\cdot 8x'\cdot 6y' + (6y')^2)}{100} + \frac{24(48x'^2 + 64x'y' - 36x'y' - 48y'^2)}{100} + \frac{2((6x')^2 + 2\cdot 6x'\cdot 8y' + (8y')^2)}{100} + 54 = 0$$

$$\Rightarrow \frac{9(64x'^2 - 96x'y' + 36y'^2)}{100} + \frac{1152x'^2 + 1536x'y' - 864x'y' - 1152y'^2}{100}$$

$$+ \frac{2(36x'^2 + 96x'y' + 64y')}{100} + 54 = 0$$

$$\Rightarrow \frac{576x'^2 - 864x'y' + 324y'^2}{100} + \frac{1152x'^2 + 672x'y' - 1152y'^2}{100} + \frac{72x'^2 + 192x'y' + 128y'^2}{100} + 54 = 0$$

$$\Rightarrow 576x'^2 - 864x'y' + 324y'^2 + 1152x'^2 + 672x'y' - 1152y'^2 + 72x'^2 + 192x'y' + 128y'^2 + 5400 = 0$$

$$\Rightarrow 1800x'^2 - 700y'^2 + 5400 = 0$$

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$$\Rightarrow 18x'^2 - 7y'^2 + 54 = 0$$

removing suffixes,

$$18x^2 - 7y^2 + 54 = 0$$

Ans.