Practice Sheet - 01

~~1.(i)~~

1.(i) $(i-2)\{2(1+i)-3(i-1)\}$

$= (i-2)(2+2i-3i+3)$

$= (i-2)(5-i)$

$= 5i-i^2-10+2i$

$= 7i+1-10$

$= 7i-9$

(ii) $\dfrac{(2+i)(3-2i)(1-i)}{(1-i)^2}$

$= \dfrac{(6-4i+3i-2i^2)(1-i)}{(1-i)^2}$

$= \dfrac{(6-i+2)(1-i)}{(1-i)^2}$

$= \dfrac{(8-i)(1-i)}{(1-i)^2}$

$= \dfrac{8-8i-i+i^2}{1-2i+i^2}$

$= \dfrac{7-9i}{-2i}$

$= -\dfrac{7}{2}i+\dfrac{9}{2}$

(iii) $\left\{\frac{4}{1-i}+\frac{2-i}{1+i}\right\}$

$= \left\{\frac{4+4i+(2-i)(1-i)}{(1-i)(1+i)}\right\}$

$= \frac{4+4i+2-2i-i-i^2}{1-i^2}$

$= \frac{4+4i+2-3i+1}{1+1}$

$= \frac{7-i}{2}$

$= \frac{7}{2}-\frac{1}{2}i$

(iv) $3\left(\frac{1+i}{1-i}\right)^2 - 2\left(\frac{1-i}{1+i}\right)^3$

$= 3\left(\frac{1+2i+i^2}{1-2i+i^2}\right) - 2\left(\frac{1-3i+3i^2-i^3}{1+3i+3i^2+i^3}\right)$

$= 3\cdot\frac{\cancel{2i}}{-\cancel{2i}} - 2\left(\frac{1-3i-3+i}{1+3i-3-i}\right)$

$= -3 - 2\left(\frac{-2i-2}{2i-2}\right)$

$$= -3 + 2\left(\frac{i+2}{i-2}\right)$$

$$= -3 + 2\left\{\frac{(i+2)(i+2)}{(i-2)(i+2)}\right\}$$

$$= -3 + 2\left(\frac{i^2+4i+4}{i^2-4}\right)$$

$$= -3 + 2\left(\frac{-1+4i+4}{-5}\right)$$

$$= -3 - 2\left(\frac{3+4i}{5}\right)$$

$$= -3 - \frac{6}{5} - \frac{8i}{5}$$

$$= \frac{-15-6}{5} - i\frac{8}{5}$$

$$= -\frac{21}{5} - \frac{8}{5}i$$

(V) $\dfrac{3i^{10} - i^{19}}{2i-1}$

$= \dfrac{3(i^2)^5 - (i^2)^9 \cdot i}{2i-1}$

$= \dfrac{3(-1)^5 - (-1)^9 \cdot i}{2i-1}$

$= \dfrac{-3+i}{2i-1}$

$= \dfrac{(i-3)(2i+1)}{(2i-1)(2i+1)}$

$= \dfrac{2i^2+i-6i-3}{4i^2-1}$

$= \dfrac{-2-5i-3}{-5}$

$= \dfrac{-5-5i}{-5}$

$= 1+i$

(VI) $\dfrac{i^4 - i^9 + i^{16}}{2 - i^5 + i^{10} - i^{15}}$

$= \dfrac{(i^2)^2 - (i^2)^4 . i + (i^2)^8}{2 - (i^2)^2 . i + (i^2)^5 - (i^2)^7 . i}$

$= \dfrac{(-1)^2 - (-1)^4 . i + (-1)^8}{2 - (-1)^2 . i + (-1)^5 - (-1)^7 . i}$

$= \dfrac{1 - \cancel{i} + 1}{2 - \cancel{i} - 1 + \cancel{i}}$

$= 2 - i$

2 (i) $(5+3i) + \{(-1+2i) + (7-5i)\}$

$= (5+3i) + (-1+2i+7-5i)$

$= (5+3i) + (6-3i)$

$= 11$

$\therefore$ The result illustrate the associative law of addition.

(ii) $\{(5+3i)+(-1+2i)\}+(7-5i)$

$=(5+3i-1+2i)+(7-5i)$

$=(4-5i)+(7-5i)$

$=11$

$\therefore$ The result illustrate the associative law of addition

3.(i) $|2z_2-3z_1|^2$

$=|2(-2+4i)-3(1-i)|^2$

$=|-4+8i-3+3i|^2$

$=|-7+11i|^2$

$=\left(\sqrt{(-7)^2+(11)^2}\right)^2$

$=\left(\sqrt{170}\right)^2$

$=170$

(ii) $\left|\dfrac{z_1+z_2+1}{z_1-z_2+i}\right|$

$= \left|\dfrac{1-i-2+4i+1}{1-i+2-4i+i}\right|$

$= \left|\dfrac{3i}{3-4i}\right|$

$= \dfrac{|3i|}{|3-4i|}$ $= \left|\dfrac{3i\times(3+4i)}{(3-4i)(3+4i)}\right|$

$= \dfrac{\sqrt{(3)^2}}{\sqrt{(3)^2+(-4)^2}}$ $= \left|\dfrac{9i-12}{9+16}\right|$

$= \dfrac{3}{\sqrt{25}}$ $= \left|\dfrac{-12+9i}{25}\right|$

$= \dfrac{3}{5}$ $= \left|\dfrac{-12}{25}+i\left(\dfrac{9}{25}\right)\right|$

$= \sqrt{\left(-\dfrac{12}{25}\right)^2+\left(\dfrac{9}{25}\right)^2}$

$= \sqrt{\dfrac{144+81}{625}}$

$= \sqrt{\dfrac{225}{625}}$ $= \dfrac{15}{25} = \dfrac{3}{5}$

(iii) $\overline{(z_1+z_3)(z_1-z_3)}$

$= \overline{(z_1+z_3)}\ \overline{(z_1-z_3)}$

$= (\overline{z}_1+\overline{z}_3)(\overline{z}_1-\overline{z}_3)$

(iii) $\overline{(z_1+z_3)(z_1-z_3)}$

$= \overline{z_1^2 - z_3^2}$

$= \overline{z_1^2} - \overline{z_3^2}$

$= \overline{(1-i)^2} - \overline{(\sqrt{3}-2i)^2} =$

$(\overline{z}_1)^2 - (\overline{z}_3)^2$

$= (1+i)^2 - (\sqrt{3}+2i)^2$

$= 1+2i+i^2 - (3+4\sqrt{3}i+4i^2)$

$= 2i - (-1+4\sqrt{3}i)$

$= 2i+1-4\sqrt{3}i$

=

(iii) $\overline{(z_1+z_3)(z_1-z_3)}$

$= \overline{(z_1+z_3)}\ \overline{(z_1-z_3)}$

$= (\overline{z}_1+\overline{z}_3)(\overline{z}_1-\overline{z}_3)$

$= (\overline{1-i} + \overline{\sqrt{3}-2i})(\overline{1-i} - \overline{\sqrt{3}-2i})$

$= (1+i+\sqrt{3}+2i)\{1+i-(\sqrt{3}+2i)\}$

$= (1+\sqrt{3}+3i)(1-\sqrt{3}-i)$

$= 1-\sqrt{3}-i+\sqrt{3}-3-\sqrt{3}i+3i-3\sqrt{3}i-3i^2$

$= 1-\cancel{\sqrt{3}}-i+\cancel{\sqrt{3}}-\cancel{3}-\sqrt{3}i+3i-3\sqrt{3}i+\cancel{3}$

$= 1+2i-4\sqrt{3}i$

$= 1+i(2-4\sqrt{3})$

(IV) $Re\{2z_1^3+3z_2^3-5z_3^2\}$

$z_1 = 1-i$

$z_2 = -2+4i$

$z_3 = \sqrt{3}-2i$

Now, $2z_1^3+3z_2^3-5z_3^2$

$= \{2(1-i)^3+3(-2+4i)^3-5(\sqrt{3}-2i)^2\}$

$= \{2(1-3i+3i^2-i^3)+3(-8+48i-96i^2+64i^3)-5(3-4\sqrt{3}i+4i^2)\}$

$= \{2(1-3i-3+i)+3(-8+48i+96-64i)-5(3-4\sqrt{3}i-4)\}$

$= 2(-2-2i)+3(88-16i)-5(-1-4\sqrt{3}i)$

$= -4-4i+274-48i+5+20\sqrt{3}i$

$= 275 - 52i + 20\sqrt{3}i$

$= 275 - i(52 - 20\sqrt{3})$

$\therefore Re\{2z_1^3 + 3z_2^3 - 5z_3^2\}$

$= 275$

(V) $Im\left\{\dfrac{z_1 z_2}{z_3}\right\}$

$z_1 = 1 - i$

$z_2 = -2 + 4i$

$z_3 = \sqrt{3} - 2i$

Now, $\dfrac{z_1 z_2}{z_3} = \dfrac{(1-i)(-2+4i)}{(\sqrt{3}-2i)}$

$= \dfrac{-2+4i+2i-4i^2}{\sqrt{3}-2i}$

$= \dfrac{-2+6i+4}{\sqrt{3}-2i}$

$= \dfrac{4+6i}{\sqrt{3}-2i}$

$$= \frac{(4+6i)(\sqrt{3}+2i)}{(\sqrt{3}-2i)(\sqrt{3}+2i)}$$

$$= \frac{4\sqrt{3}+8i+6\sqrt{3}i+12i^2}{(\sqrt{3})^2-(2i)^2}$$

$$= \frac{4\sqrt{3}+8i+6\sqrt{3}i-12}{3-4i^2}$$

$$= \frac{4\sqrt{3}-12+i(8+6\sqrt{3})}{7}$$

$$= \frac{4}{7}\sqrt{3}-\frac{12}{7}+i\left(\frac{8}{7}+\frac{6}{7}\sqrt{3}\right)$$

$$\therefore \mathrm{Im}\left\{\frac{z_1 z_2}{z_3}\right\} = \left(\frac{8}{7}+\frac{6}{7}\sqrt{3}\right)$$

(VI) $z_2^2 + 2z_1 - 3$

$= (1-i)^2 + 2(1-i) - 3$

$= 1 - 2i + i^2 + 2 - 2i - 3$

$= -4i - 1$

$= -1 - 4i$

$= -(1+4i)$

(VII) $|z_2\bar{z}_2 + z_2\bar{z}_1|$

$= |(1-i)\overline{(-2+4i)} + (-2+4i)\overline{(1-i)}|$

$= |(1-i)(-2-4i) + (-2+4i)(1+i)|$

$= |(1-i)(-2-4i) - 2 - 2i + 4i + 4i^2|$

$= |-2 - 4i + 2i + 4i^2 - 2 - 2i + 4i - 4|$

$= |-2 - 2i - 4 - 2 - 2i + 4i - 4|$

$= |-12|$

$= \sqrt{(-12)^2}$

$= \underline{12}$

(VIII) $\frac{1}{2}\left(\frac{z_3}{\overline{z_3}} + \frac{\overline{z_3}}{z_3}\right)$

$= \frac{1}{2}\left\{\frac{z_3^2 + (\overline{z_3})^2}{z_3 \cdot \overline{z_3}}\right\}$

$= \frac{1}{2}\left\{\frac{z_3^2 + (\overline{z_3})^2}{|z_3|^2}\right\}$

$= \frac{1}{2}\left\{\frac{(\sqrt{3}-2i)^2 + (\overline{\sqrt{3}-2i})^2}{\{|(\sqrt{3}-2i)|\}^2}\right\}$

$= \frac{1}{2}\left\{\frac{(\sqrt{3}-2i)^2 + (\sqrt{3}+2i)^2}{\left(\sqrt{(\sqrt{3})^2+(-2)^2}\right)^2}\right\}$

$= \frac{1}{2}\left\{\frac{3 - \cancel{4\sqrt{3}i} - 4 + 3 + \cancel{4\sqrt{3}i} - 4}{\cancel{\sqrt{7}}\ 7}\right\}$

$= \frac{1}{2} \times \cancel{\frac{-2}{\sqrt{7}}} - \frac{2}{7}$

$= -\cancel{\frac{1}{\sqrt{7}}} - \frac{1}{7}$

(IX) $\left(z_3 - \overline{z}_3\right)^5$

$= \left\{\sqrt{3} - 2i - \left(\overline{\sqrt{3} - 2i}\right)\right\}^5$

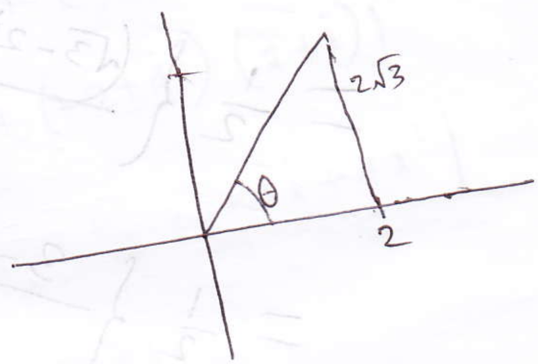
$= \left\{\sqrt{3} - 2i - \left(\sqrt{3} + 2i\right)\right\}^5$

$= \left(\cancel{\sqrt{3}} - 2i - \cancel{\sqrt{3}} - 2i\right)^5$

$= \left(-4i\right)^5$

$= 1024i$

4 (1) $z = 2 + 2\sqrt{3}\,i$



~~$\theta = \tan^{-1}$~~$\left(\frac{2\sqrt{3}}{2}\right)$

$r = |z|$

$= \sqrt{(2)^2 + (2\sqrt{3})^2}$

$= 4$

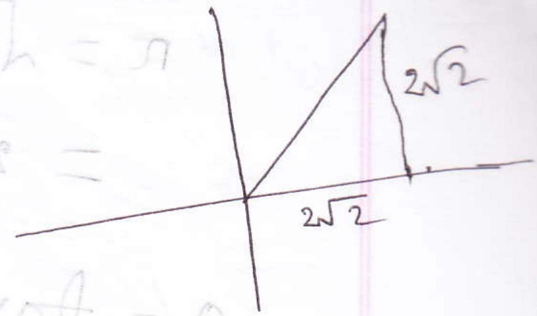
$\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$

$= \tan^{-1}\left(\sqrt{3}\right) = \tan^{-1}\left(\tan\frac{\pi}{3}\right) = \frac{\pi}{3}$

(ii) $\cancel{2\sqrt{2}}\ z = 2\sqrt{2} + 2\sqrt{2}\, i$

$$r = |z| = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2}$$

$$= \sqrt{16} = 4$$



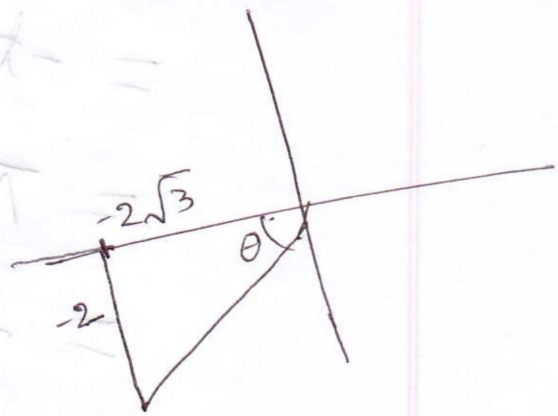
$$\theta = \tan^{-1}\left(\frac{2\sqrt{2}}{2\sqrt{2}}\right)$$

$$= \tan^{-1}(1)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4}$$

(iii) $z = -2\sqrt{3} - 2i$



$$r = \sqrt{(-2\sqrt{3})^2 + (-2)^2}$$

$$= 4$$

$$\therefore \theta = \tan^{-1}\left(\frac{-2}{-2\sqrt{3}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{6}\right) = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

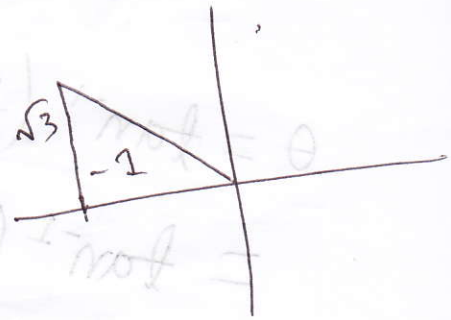
$$\cancel{= \frac{\pi}{6} + \pi = \frac{7\pi}{6}}$$

(IV) $z = -1 + \sqrt{3}i$

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2}$$

$$= 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$$



$$\cancel{= -\tan^{-1}(}$$

$$= -\tan^{-1}(\sqrt{3})$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{3}\right)$$

$$= \pi - \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

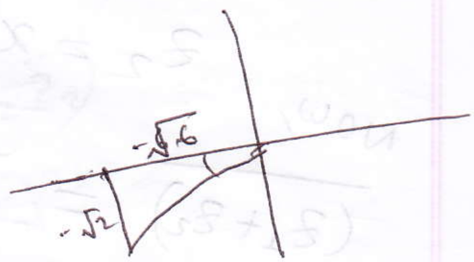
(v) $z = -\sqrt{6} - \sqrt{2}\,i$

$$r = \sqrt{(-\sqrt{6})^2 + (-\sqrt{2})^2}$$

$$= \sqrt{6+2}$$

$$= 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{2}}{-\sqrt{6}}\right)$$



$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{6}\right)$$

$$= \pi + \frac{\pi}{6}$$

$$= \frac{7\pi}{6}$$

5 (i)

proof:

Prove that, $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

Let, $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

Now, $\overline{(z_1 + z_2)} = \overline{(x + iy_1) + (x_2 + iy_2)}$

$= \overline{x_1 + x_2 + i(y_1 + y_2)}$

$= (x_1 + x_2) - i(y_1 + y_2)$

$= x_1 - iy_1 + x_2 - iy_2$

$= \overline{x_1 + iy_1} + \overline{x_2 + iy_2}$

$= \bar{z}_1 + \bar{z}_2$ (proved)

(ii) Proove that,
$$|z_1 z_2| = |z_1||z_2|$$

Now, $|z_1 \cdot z_2|^2 = (z_1 \cdot z_2) \overline{(z_1 \cdot z_2)}$ $\left[\because |z|^2 = z \cdot \overline{z}\right]$

$$= (z_1 \cdot z_2)(\overline{z_1} \cdot \overline{z_2})$$

$$= (z_1 \cdot \overline{z_1})(z_2 \cdot \overline{z_2})$$

$$\Rightarrow |z_1 \cdot z_2|^2 = |z_1|^2 \cdot |z_2|^2$$

$$\Rightarrow |z_1 \cdot z_2| = |z_1||z_2|$$ (Prooved)

(iii) Proove that, $|z_1 + z_2| \le |z_1| + |z_2|$ & $|z_1 - z_2| \le |z_1| + |z_2|$

$$|z_1 \pm z_2| \le |z_1| + |z_2|$$

Now, $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$

$$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$

$$\cancel{= z_1 \cdot \overline{z_1} + z_2 \overline{z_2} +}$$

$$= z_1 \cdot \overline{z_1} + z_1 \cdot \overline{z_2} + z_2 \cdot \overline{z_1} + z_2 \cdot \overline{z_2}$$

$$= z_1 \cdot \overline{z_1} + (z_1 \cdot \overline{z_2} + \overline{\overline{z_1} \cdot z_2}) + z_2 \cdot \overline{z_2}$$

$$= |z_1|^2 + (\overline{z_1} \cdot z_2 + \overline{\overline{z_1} \cdot z_2}) + |z_2|^2$$

$$= |z_1|^2 + 2\,\mathrm{Re}(\overline{z_1} \cdot z_2) + |z_2|^2$$

$$= |z_1|^2 +$$

$$\leq |z_1|^2 + 2|\bar{z}_1||z_2| + |z_2|^2 \quad \boxed{Re(z) \leq z}$$

$$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

$$= (|z_1| + |z_2|)^2$$

$$\Rightarrow |z_1+z_2|^2 \leq (|z_1|+|z_2|)^2$$

$$\Rightarrow |z_1+z_2| \leq |z_1|+|z_2| \quad \text{(proved)}$$

Now, Replacing z_2 by $-z_2$,

$$|z_1+(-z_2)| \leq |z_1| + |(-z_2)|$$

$$\Rightarrow |z_1-z_2| \leq |z_1| + |z_2| \quad \text{(proved)}$$

(iv) Prove that, $|z_1 - z_2| \geq |z_1| - |z_2|$

$|z_1 + z_2| \geq |z_1| - |z_2|$ &

Now, $\cancel{|z_1| = |z_1 + z_2 - z_2|}$

$\cancel{= |z_1 - z_2 + z_2|}$

$\cancel{\leq}$

Now, $|z_1| = |z_1 + z_2 - z_2|$

$\leq |z_1 + z_2| + |z_2|$ $[\because |z_1 - z_2| \leq |z_1| + |z_2|]$

$|z_1| \leq |z_1 + z_2| + |z_2|$

$|z_1| - |z_2| \leq |z_1 + z_2|$

$\Rightarrow |z_1 + z_2| \geq |z_1| - |z_2|$

$\Rightarrow |z_1 + z_2|$

Eulars formulla:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

proof:

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Substituting ~~$x = i\theta$~~, x by $i\theta$. we get,

$$e^{i\theta} = 1 + \frac{i\theta}{\cancel{\theta}} + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

⑥ state & prove the De Moivres theorem :-

Theorem:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Proof:

Now,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \quad \text{--- (1)}$$

For $n = 1$ (1) is true,

Let, (1) is true for $n = k$,

$$\therefore (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta \quad \text{--- (11)}$$

Let, for $n = k+1$ the eqn (1) become,

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)$$

$$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$= \cos k\theta \cdot \cos\theta + i\sin k\theta \cos\theta + i\sin\theta \cos k\theta - \sin\theta \sin k\theta$$

$$= (\cos k\theta \cos\theta - \sin k\theta \sin\theta) + i(\sin k\theta \cos\theta + \cos k\theta \sin\theta)$$

$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$

$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

$\therefore$ (1) is true for all value of n.

(6) state & proove the De Moivres theorem :-

Theorem:

$$(\cos\theta + i\sin n\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

proof:

Now,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + \cancel{i\sin(n\theta)}\, i\sin(n\theta) \quad \text{--- (1)}$$

For $n = 1$ (1) is true,

Let, (1) is true for $n = k$,

$$\therefore (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta \quad \text{--- (11)}$$

Let, for $n = k+1$ the eqn (1) become,

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)$$

$$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$= (\cos k\theta + i\sin k\theta\cos\theta + i\sin\theta\cos k\theta$$

$$= \cos k\theta \cdot \cos\theta + i\sin k\theta\cos\theta - \sin\theta\sin k\theta$$

$$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$$

$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$

$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

$\therefore$ (1) is true for all value of n

7 (1), $\dfrac{(8\,\text{Cis}\,40^\circ)^3}{(2\,\text{Cis}\,60^\circ)^4}$

$$= \frac{[8(\cos 40^\circ + i\sin 40^\circ)]^3}{[2(\cos 60^\circ + i\sin 60^\circ)]^{4}}$$

$$= \frac{512(\cos 120^\circ + i\sin 120^\circ)}{16(\cos 240^\circ + i\sin 240^\circ)}$$

$$= 32\,\frac{e^{i(120^\circ)}}{e^{i(240^\circ)}}$$

$$= 32\,e^{i(-120^\circ)}$$

$$= 32\left[\cos(-120^\circ) + i\sin(-120^\circ)\right]$$

$$= 32\left[\cos(120^\circ) - i\sin(120^\circ)\right]$$

$$= 32\left[\cos(120^\circ)\right.$$

$$= 32\left(-\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right) \quad \text{(Ans)}$$

(ii) $$\frac{\left(3e^{\pi i/6}\right)\left(2e^{-\frac{5\pi i}{4}}\right)\left(6e^{\frac{5\pi i}{3}}\right)}{\left(4e^{\frac{2\pi}{3}}\right)^{2}}$$

$$= \frac{9}{4}e^{\pi i/6-\frac{5\pi i}{4}+\frac{5\pi i}{3}-\frac{4\pi i}{3}}$$

$$= \frac{9}{4}e^{\frac{2\pi i-15\pi i+20\pi i-16\pi i}{12}}$$

$$= \frac{9}{4}e^{-\frac{9\pi i}{12}}$$

$$= \frac{9}{4}e^{-\frac{3\pi i}{4}}$$

$$= \frac{9}{4}e^{i\left(-\frac{3\pi}{4}\right)}$$

$$= \frac{9}{4}\left[\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right]$$

$$\cancel{= 9[\cos(-135}$$

$$= \frac{9}{4}\left[\cos\left(\frac{3\pi}{4}\right)-i\sin\left(\frac{3\pi}{4}\right)\right]$$

$$= \frac{9}{4}\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right) \text{ (Ans)}$$

$$\frac{1}{\sqrt{2}} \quad \frac{\sqrt{2}}{2}$$

$$= \frac{9\sqrt{2}}{8}\left(-1-i\right)$$

(iii) $(5\text{cis}20^\circ)(3\text{cis}40^\circ)$

$= \{5(\cos 20^\circ + i\sin 20^\circ)\}\{3(\cos 40^\circ + i\sin 40^\circ)$

$= 15\, e^{i(20^\circ)} \cdot e^{i(40^\circ)}$

$= 15\, e^{i(20^\circ) + i(40^\circ)}$

$= 15 e^{i(60^\circ)}$

$= 15[\cos(60^\circ) + i\sin(60^\circ)]$

$= 15\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$

$=$

(iv) $(2\text{cis}50^\circ)^6$

$= 2^6[\cos(300^\circ) + i\sin(300^\circ)]$

$= 64\, e^{i(300^\circ)}$

$= 64[\cos(300^\circ) + i\sin(300^\circ)]$

$= 64\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$

$= 32(1 - i\sqrt{3})$

(3) (i) $z = (-1+i)^{1/3}$

$= \left\{\sqrt{2}\, e^{i\left(\frac{3\pi}{4}+2n\pi\right)}\right\}^{1/3}$

$= (\sqrt{2})^{1/3}\, e^{i\left(\frac{3\pi}{4}+2n\pi\right)\frac{1}{3}}$

$= (\sqrt{2})^{1/3}\, e^{i\left(\frac{\pi}{4}+\frac{2n\pi}{3}\right)}$

Here,

$r = \sqrt{(-1)^2+(1)^2}$

$= \sqrt{2}$

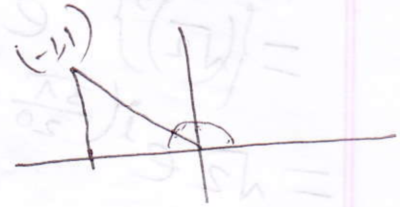
$\theta = \tan^{-1}\left(\frac{1}{-1}\right)$

$= -\tan^{-1}(1)$

$= -\tan^{-1}\left(\tan\frac{\pi}{4}\right)$

$= \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$



For, $n=0$,

$z_0 = (\sqrt{2})^{1/3}\, e^{i\left(\frac{\pi}{4}\right)}$

For $n=1$,

$z_1 = (\sqrt{2})^{1/3}\, e^{i\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)}$

$= (\sqrt{2})^{1/3}\, e^{i\left(\frac{11\pi}{12}\right)}$

For, $n=2$,

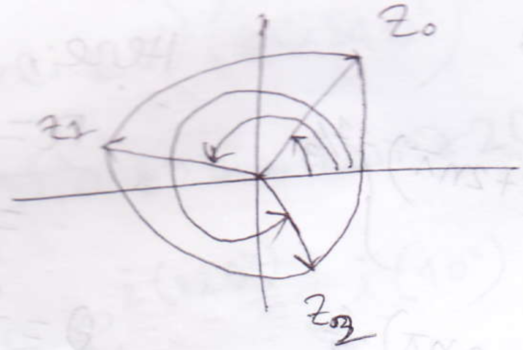
$z_2 = (\sqrt{2})^{1/3}\, e^{i\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)}$

$= (\sqrt{2})^{1/3}\, e^{i\left(\frac{19\pi}{12}\right)}$

~~For, $n=3$,~~

~~$z_3 = (\sqrt{2})^{1/3}\, e^{i\left(\frac{\pi}{4}+\frac{6\pi}{3}\right)}$~~

~~$= (\sqrt{2})^{1/3}\, e^{i\left(\frac{9\pi}{4}\right)}$~~



(11) $z = (-4+4i)^{1/5}$

$= \left\{4\sqrt{2}\, e^{i\left(\frac{3\pi}{4}+2n\pi\right)}\right\}^{1/5}$

$= (4\sqrt{2})^{1/5} e^{i\left(\frac{3\pi}{4}+2n\pi\right)^{1/5}}$

$= \left\{(\sqrt{2})^5\right\}^{1/5} e^{i\left(\frac{3\pi}{20}+\frac{2n\pi}{5}\right)}$

$= \sqrt{2}\, e^{i\left(\frac{3\pi}{20}+\frac{2n\pi}{5}\right)}$

For, $n=0$,

$z_0 = \sqrt{2}\, e^{i\left(\frac{3\pi}{20}\right)}$

For $n=1$,

$z_1 = \sqrt{2}\, e^{i\left(\frac{3\pi}{20}+\frac{2\pi}{5}\right)}$

$= \sqrt{2}\, e^{\left(\frac{11\pi}{20}\right)}$

For $n=2$,

$z_2 = \sqrt{2}\, e^{i\left(\frac{3\pi}{20}+\frac{4\pi}{5}\right)}$

$= \sqrt{2}\, e^{i\left(\frac{19\pi}{20}\right)}$

Here,

$r = \sqrt{(-4)^2+(4)^2}$

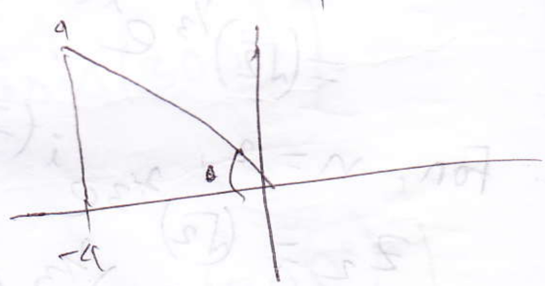
$= 4\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{4}{-4}\right)$

$= -\tan^{-1}\left(\tan\frac{\pi}{4}\right)$

$= \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$



For, $n=3$,

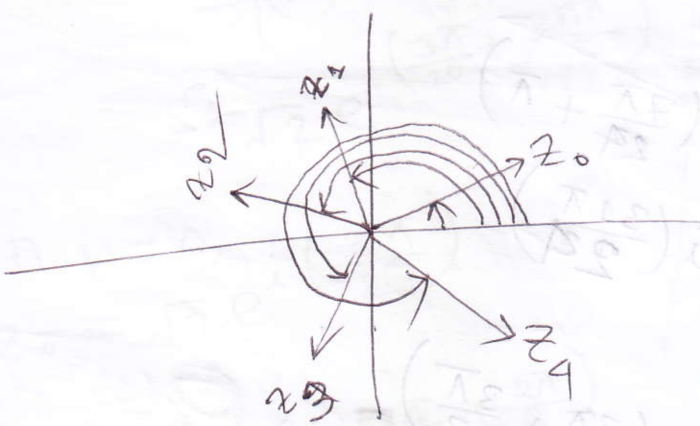
$$z_3 = \sqrt{2}\, e^{i\left(\frac{3\pi}{20} + \frac{6\pi}{5}\right)}$$

$$= \sqrt{2}\, e^{i\left(\frac{27\pi}{20}\right)}$$

For, $n=4$

$$z_4 = \sqrt{2}\, e^{i\left(\frac{3\pi}{20} + \frac{8\pi}{5}\right)}$$

$$= \sqrt{2}\, e^{i\left(\frac{35\pi}{20}\right)}$$



(iii) $z = \left(-2\sqrt{3} - 2i\right)^{1/4}$

$$= \left\{4\, e^{i\left(\frac{7\pi}{6} + 2n\pi\right)}\right\}^{1/4}$$

$$= (4)^{1/4}\, e^{i\left(\frac{7\pi}{6} + 2n\pi\right)\cdot\frac{1}{4}}$$

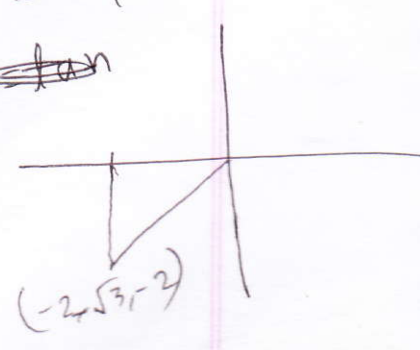
$$= \left\{(\sqrt{2})^4\right\}^{1/4} e^{i\left(\frac{7\pi}{6} + 2n\pi\right)\frac{1}{4}}$$

$$= \sqrt{2}\, e^{i\left(\frac{7\pi}{24} + \frac{n\pi}{2}\right)}$$

Here, $r = \sqrt{(-2\sqrt{3})^2 + (-2)^2}$

$= 4$

~~tan~~



For, $n=0$,

$z_0 = \sqrt{2}e^{i\left(\frac{7\pi}{24}\right)}$

For $n=1$,

$z_1 = \sqrt{2}e^{i\left(\frac{7\pi}{24}+\frac{\pi}{2}\right)}$

$= \sqrt{2}e^{i\left(\frac{7\pi+12\pi}{24}\right)}$

$= \sqrt{2}e^{i\left(\frac{19\pi}{24}\right)}$

For, $n=2$

$z_2 = \sqrt{2}e^{i\left(\frac{7\pi}{24}+\pi\right)}$

$= \sqrt{2}e^{i\left(\frac{31\pi}{24}\right)}$

For, $n=3$,

$z_3 = \sqrt{2}e^{i\left(\frac{7\pi}{24}+\frac{3\pi}{2}\right)}$

$= \sqrt{2}e^{i\left(\frac{7\pi+36\pi}{24}\right)}$

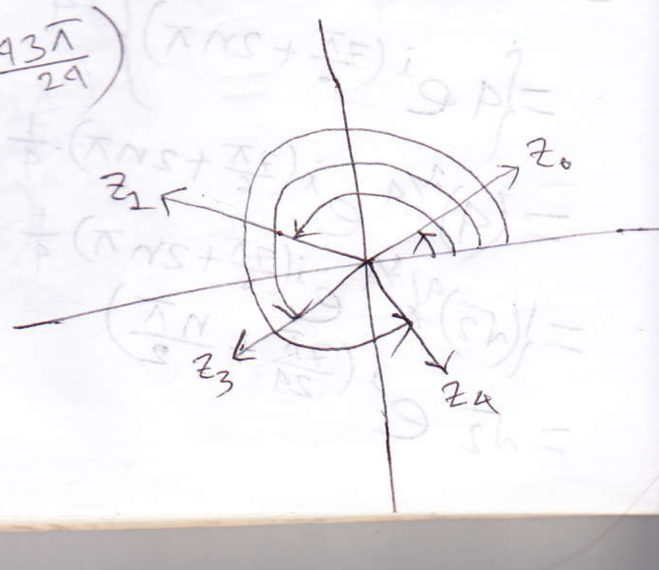
$= \sqrt{2}e^{i\left(\frac{43\pi}{24}\right)}$

$\theta = \tan^{-1}\left(\frac{-2}{-2\sqrt{3}}\right)$

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \pi + \frac{\pi}{6}$

$= \frac{7\pi}{6}$



~~8. (1) (1+i~~

9 (1) $z^5 = -4+4i$

$\Rightarrow z = (-4+4i)^{1/5}$

$\cancel{= (4\sqrt{2})\, e^{i\left(\frac{3\pi}{4}+2n\pi\right)}}$

$= \left\{4\sqrt{2}\, e^{i\left(\frac{3\pi}{4}+2n\pi\right)}\right\}^{1/5}$

$= \left\{(\sqrt{2})^5\right\}^{1/5} e^{i\left(\frac{3\pi}{4}+2n\pi\right)\cdot\frac{1}{5}}$

$= \sqrt{2}\, e^{i\left(\frac{3\pi}{20}+\frac{2n\pi}{5}\right)}$

For $n=0$,

$z_0 = \sqrt{2}\, e^{i\left(\frac{3\pi}{20}\right)}$

For $n=1$,

$z_1 = \sqrt{2}\, e^{i\left(\frac{3\pi}{20}+\frac{2\pi}{5}\right)}$

$= \sqrt{2}\, e^{i\left(\frac{11\pi}{20}\right)}$

For $n=2$,

$z_2 = \sqrt{2}\, e^{i\left(\frac{3\pi}{20}+\frac{4\pi}{5}\right)}$

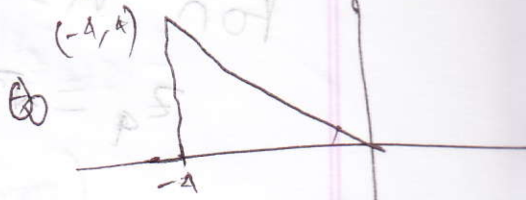
$= \sqrt{2}\, e^{i\left(\frac{19\pi}{20}\right)}$

$\cancel{= \sqrt{2},}$

$r = \sqrt{(-4)^2+(4)^2}$

$= \sqrt{32}$

$= 4\sqrt{2}$



$\theta = \tan^{-1}\left(\frac{4}{-4}\right)$

$= -\tan^{-1}\left(\tan\frac{\pi}{4}\right)$

$= \pi - \frac{\pi}{4}$

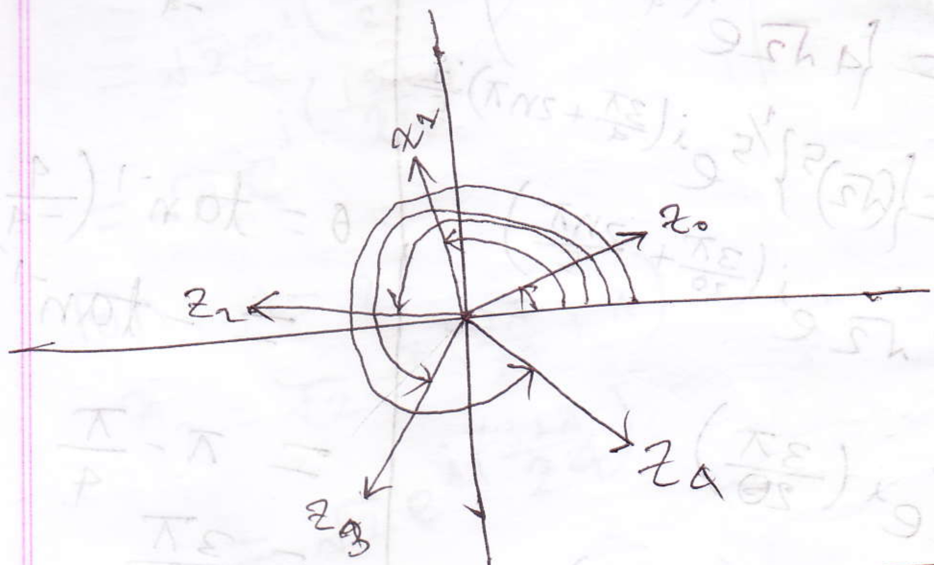
$= \frac{3\pi}{4}$

For $n=3$,

$$z_3 = \sqrt{2}\, e^{i\left(\frac{27\pi}{20}\right)}$$

For $n=4$,

$$z_4 = \sqrt{2}\, e^{i\left(\frac{35\pi}{20}\right)}$$



(11) $z^5 = 1$

$$\Rightarrow z = (1)^{1/5}$$

$$= \left\{1 \cdot e^{i(0+2n\pi)}\right\}^{1/5}$$

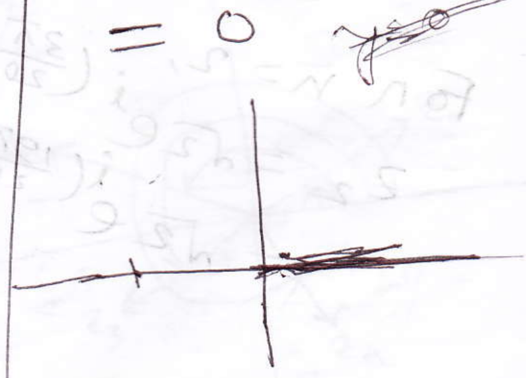
$$= e^{i\left(\frac{2n\pi}{5}\right)}$$

$r = \sqrt{(1)^2 + (0)^2}$

$= 1$

$\theta = \tan^{-1}\left(\frac{0}{1}\right)$

$= 0$



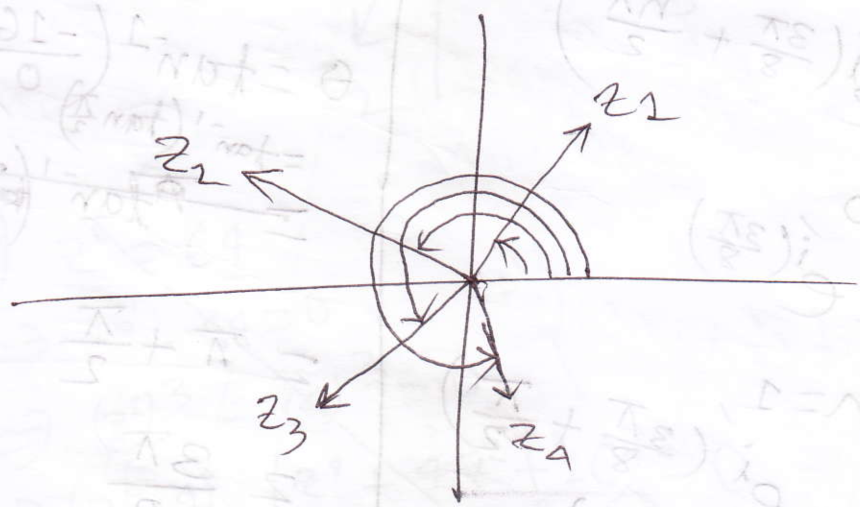
For $n=0$, $z_0 = e^{i.0} = 1$

For $n=1$, $z_1 = e^{i\left(\frac{2\pi}{5}\right)}$

For $n=2$, $z_2 = e^{i\left(\frac{4\pi}{5}\right)}$

For $n=3$, $z_3 = e^{i\left(\frac{6\pi}{5}\right)}$

For $n=4$, $z_4 = e^{i\left(\frac{8\pi}{5}\right)}$



(iii) $z^4 = -16i$

$$\Rightarrow z = (-16i)^{1/4}$$

$$= \left\{16 e^{i\left(\frac{3\pi}{2}+2n\pi\right)}\right\}^{\frac{1}{4}}$$

$$= (16)^{1/4} e^{i\left(\frac{3\pi}{2}+2n\pi\right)\frac{1}{4}}$$

$$= (2^4)^{1/4} e^{i\left(\frac{3\pi}{8}+\frac{n\pi}{2}\right)}$$

$$= 2e^{i\left(\frac{3\pi}{8}+\frac{n\pi}{2}\right)}$$

For $n=0$

$$z_0 = 2e^{i\left(\frac{3\pi}{8}\right)}$$

For $n=1$,

$$z_1 = 2e^{i\left(\frac{3\pi}{8}+\frac{\pi}{2}\right)}$$

$$= 2e^{i\left(\frac{7\pi}{8}\right)}$$

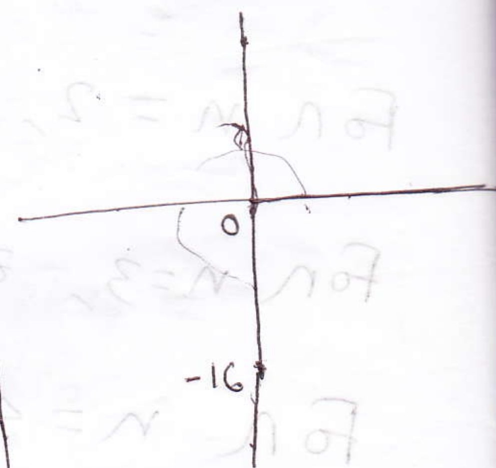
For $n=2$,

$$z_2 = 2e^{i\left(\frac{3\pi}{8}+\pi\right)}$$

$$= 2e^{i\left(\frac{11\pi}{8}\right)}$$

$$r = \sqrt{(0)^2+(-16)^2}$$

$$= 16$$



$$\theta = \tan^{-1}\left(\frac{-16}{0}\right)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{2}\right)$$

$$= \cancel{\tan^{-1}\left(\tan\frac{\pi}{2}\right)}$$

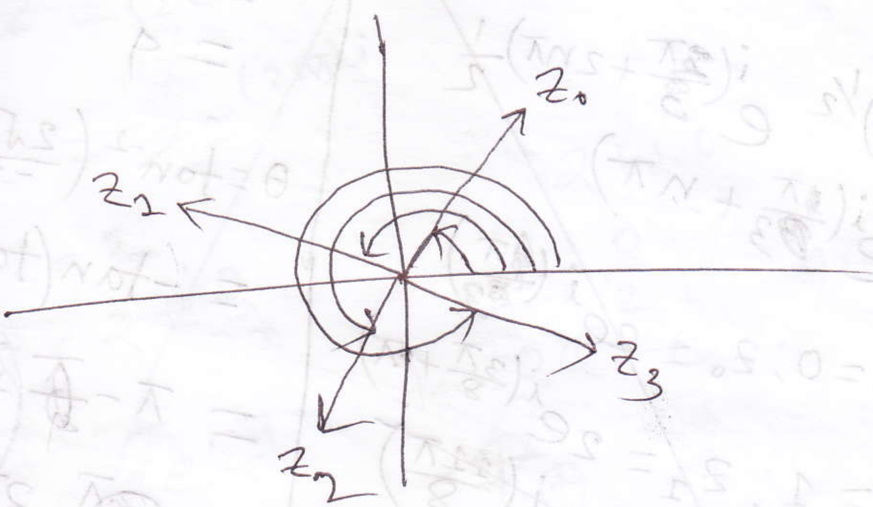
$$= \pi + \frac{\pi}{2}$$

$$= \frac{3\pi}{2}$$

For $n = 3$,

$$z_3 = 2e^{i\left(\frac{3\pi}{8}+\frac{3\pi}{2}\right)}$$

$$= 2e^{i\left(\frac{15\pi}{8}\right)}$$



(iv) $z^6 = 64$

$\Rightarrow z^6 - 64 = 0$

$\Rightarrow (z^2)^3 - (4)^3 = 0$

$\Rightarrow (z^2 - 4)(z^4 + 4z^2 + 16) = 0$

Now, $z^2 - 4 = 0$

$\Rightarrow z = 2, -2$

or, $z^4 + 4z^2 + 16 = 0$

$\Rightarrow \omega^2 + 4\omega + 16 = 0$ [let, $\omega = z^2$]

$\Rightarrow \omega = \frac{-4 \pm \sqrt{16-64}}{2}$

$= \frac{-4 \pm 4\sqrt{3}\,i}{2}$

$= -2 \pm 2\sqrt{3}\,i$

Again,

$z = \left\{2 \cdot e^{i(0+2n\pi)}\right\}^{\frac{1}{2}}$

$= 2e^{(2n\pi)i}$

$z = \left\{2e^{i(0+2n\pi)}\right\}^{1}$

$= 2e^{(2n\pi)i}$

$z = 2$

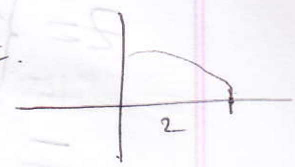
$z = 2e^{i0}$

$z = -2$

$= 2e^{i\pi}$

$r = \sqrt{(2)+(0)^2}$

$= 2$



$\theta = \tan^{-1}\left(\frac{0}{2}\right)$

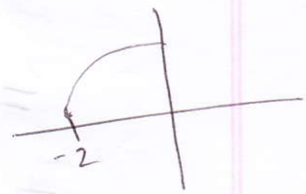
$= 0$

$2 + i0$

$\frac{0}{2}$

$\tan^{-1} 0 \; \theta = 0$

$r = \sqrt{(-2)^2+(0)^2} = 2$



(V) $z^4 + z^2 + 1 = 0$ $\quad [z = \omega^2]$

$\Rightarrow \omega^2 + \omega + 1 = 0$

$\Rightarrow \omega = \frac{-1 \pm \sqrt{1-4\cdot1\cdot1}}{2\cdot1}$

$\Rightarrow z^2 = \frac{-1 \pm \sqrt{3}i}{2}$

$\Rightarrow z = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{1/2}$ & $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{1/2}$

NOW,

$$z = \left\{1\cdot e^{i\left(\frac{2\pi}{3}+2n\pi\right)}\right\}^{1/2}$$

$$= e^{i\left(\frac{2\pi}{3}+2n\pi\right)\frac{1}{2}}$$

$$= e^{i\left(\frac{\pi}{3}+n\pi\right)}$$

Again,

$$z = \left\{1\cdot e^{i\left(\frac{4\pi}{3}+2n\pi\right)}\right\}^{1/2}$$

$$= e^{i\left(\frac{4\pi}{3}+2n\pi\right)\frac{1}{2}}$$

$$= e^{i\left(\frac{2\pi}{3}+n\pi\right)}$$

For,

$n=0$, $z_0 = e^{i\left(\frac{\pi}{3}\right)}$

For, $n=1$,

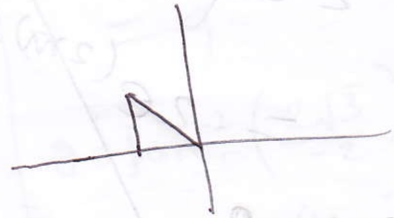
$$z_1 = e^{i\left(\frac{\pi}{3}+\pi\right)}$$

$$= e^{i\left(\frac{4\pi}{3}\right)}$$

$$R = \sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4}+\frac{3}{4}}$$

$$= 1$$



$$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{3}\right)$$

$$= \pi - \frac{\pi}{3}$$

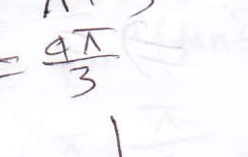
$$= \frac{2\pi}{3}$$

$$\theta = \tan^{-1}\left(\frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{3}\right)$$

$$= \pi+\frac{\pi}{3}$$

$$= \frac{4\pi}{3}$$



Again,

$$z = \left\{1 \cdot e^{i\left(\frac{4\pi}{3} + 2n\pi\right)}\right\}^{1/2}$$

$$= e^{i\left(\frac{4\pi}{3} + 2n\pi\right)\frac{1}{2}}$$

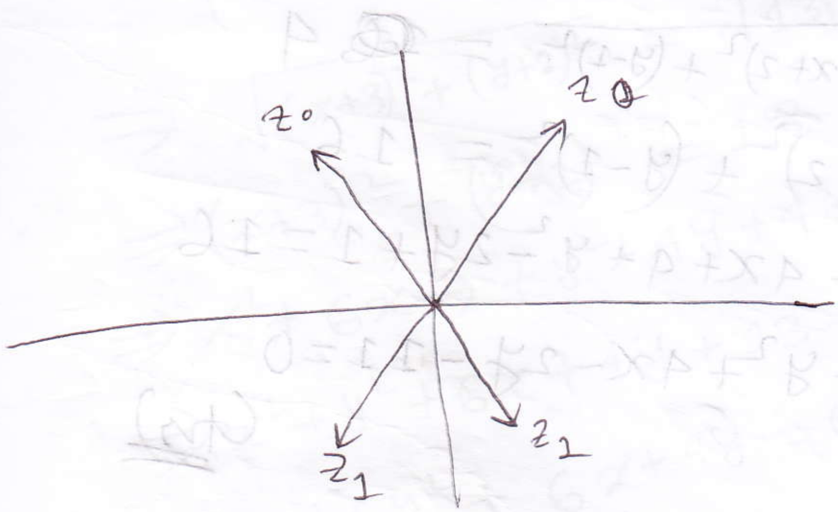
$$= e^{i\left(\frac{2\pi}{3} + n\pi\right)}$$

For, $n = 0$,

$$z_0 = e^{i\left(\frac{2\pi}{3}\right)}$$

For, $n = 1$, $z_1 = e^{i\left(\frac{2\pi}{3} + \pi\right)}$

$$= e^{i\left(\frac{5\pi}{3}\right)}$$



10 (1) Find an equation for a circle of radius 4 with $(-2, 1)$

$\Rightarrow$ Here,

radius, $R = 4$

centre, $z_0 = (-2, 1)$

Now, $|z - z_0| = R$

$\Rightarrow |x + iy - (-2 + i)| = 4$

$\Rightarrow |(x+2) + i(y-1)| = 4$

$\Rightarrow \sqrt{(x+2)^2 + (y-1)^2} = 4$

$\Rightarrow (x+2)^2 + (y-1)^2 = 16$

$\Rightarrow x^2 + 4x + 4 + y^2 - 2y + 1 = 16$

$\Rightarrow x^2 + y^2 + 4x - 2y - 11 = 0$

(Ans)

(11) Find the equation ~~of~~ for a circle of radius $\sqrt{13}$ with center at $(-3,-2)$

$\Rightarrow$ Here, Radius, $R = \sqrt{13}$

Centre, $z_0 = (-3,-2)$

Now, $|z - z_0| = R$

$\Rightarrow |x + iy - (-3-2i)| = \sqrt{13}$

$\Rightarrow |x + iy + 3 + 2i| = \sqrt{13}$

$\Rightarrow |(x+3) + i(y+2)| = \sqrt{13}$

$\Rightarrow \sqrt{(x+3)^2 + (y+2)^2} = \sqrt{13}$

$\Rightarrow (x+3)^2 + (y+2)^2 = 13$

$\Rightarrow x^2 + 6x + 9 + y^2 + 4y + 4 = 13$

$\Rightarrow x^2 + y^2 + 6x + 4y + 13 - 13 = 0$

$\Rightarrow x^2 + y^2 + 6x + 4y = 0$ (Ans)

$\Rightarrow x^2 + y^2 + 6x + 4y$

11 ① $|z-1|+|z+1|=4$ — ①

Let, $z=x+iy$

① $\Rightarrow |x+iy-1|+|x+iy+1|=4$

$\Rightarrow |(x-1)+iy|+|(x+1)+iy|=4$

$\Rightarrow \sqrt{(x-1)^2+y^2}+\sqrt{(x+1)^2+y^2}=4$

$\Rightarrow \sqrt{(x+1)^2+y^2}=4-\sqrt{(x-1)^2+y^2}$

$\Rightarrow (x+1)^2+y^2=16-8\sqrt{(x-1)^2+y^2}+(x-1)^2+y^2$

$\Rightarrow x^2+2x+1=16-8\sqrt{(x-1)^2+y^2}+x^2-2x+1$

$\cancel{\Rightarrow 4x \quad +16=}$

$\Rightarrow 4x-16=-8\sqrt{(x-1)^2+y^2}$

$\cancel{\Rightarrow x-4=}\Rightarrow x-4=-2\sqrt{(x-1)^2+y^2}$

$\Rightarrow x^2-8x+16=4\{(x-1)^2+y^2\}$

$\Rightarrow x^2-8x+16=4x^2-8x+4+4y^2$

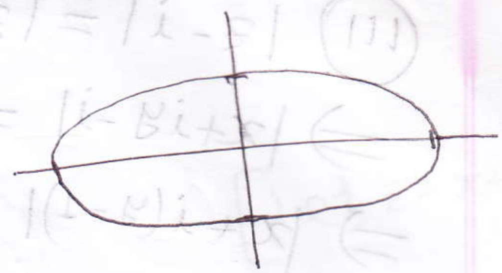
$\Rightarrow 3x^2+4y^2-12=0$

$\Rightarrow 3x^2+4y^2=12$

$\Rightarrow \frac{3x^2}{12}+\frac{4y^2}{12}=1$

$\Rightarrow \dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

$\Rightarrow \dfrac{x^2}{2^2} + \dfrac{y}{(\sqrt{3})^2} = 1$



(11) $|z-i| = 2$ [considering $z = x+iy$]

$\Rightarrow |x+iy-i| = 2$

$\Rightarrow |x+i(y-1)| = 2$

$\Rightarrow \sqrt{x^2+(y-1)^2} = 2$

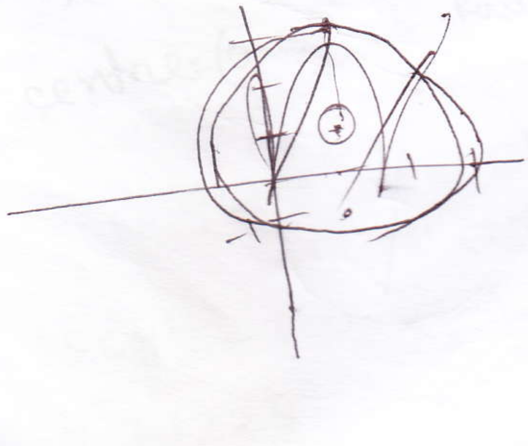
$\Rightarrow \cancel{x^2+y^2-2y+1=2}$

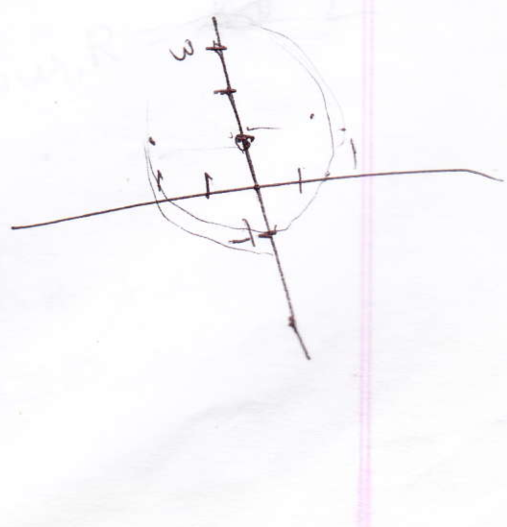
$\Rightarrow \cancel{x^2+y^2-2y-1=0}$

$\Rightarrow x^2+(y-1)^2 = 4$

$\Rightarrow (x-0)^2+(y-1)^2 = 4$

centre $(0,1)$ & radius $= 2$





(iii) $|z-i| = |z+i|$

$\Rightarrow |x+iy-i| = |x+iy+i|$

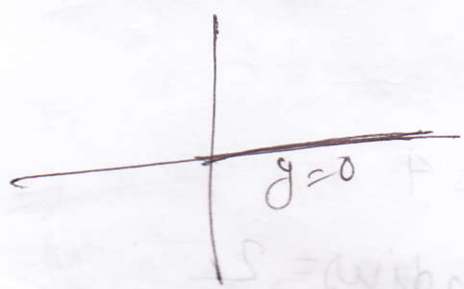
$\Rightarrow |x+i(y-1)| = |x+i(y+1)|$

$\Rightarrow \sqrt{x^2+(y-1)^2} = \sqrt{x^2+(y+1)^2}$

$\Rightarrow \cancel{x^2}+(y-1)^2 = \cancel{x^2}+(y+1)^2$

$\Rightarrow \cancel{y^2}-2y+\cancel{1}-\cancel{y^2}-2y-\cancel{1}=0$

$\Rightarrow y=0$



(iv) $1 < |z+i| \leq 2$

$\left[\text{Let, } z = x + iy\right]$

Now,

$|z+i| > 1$

$\Rightarrow |x+iy+i| > 1$

$\Rightarrow |x + i(y+1)| > 1$

$\Rightarrow \sqrt{x^2 + (y+1)^2} > 1$

$\Rightarrow x^2 + (y+1)^2 > (1)^2$

Here, centre $z_0 = (0,-1)$ & Radius, $R = 1$

Again, $|z+i| \leq 2$

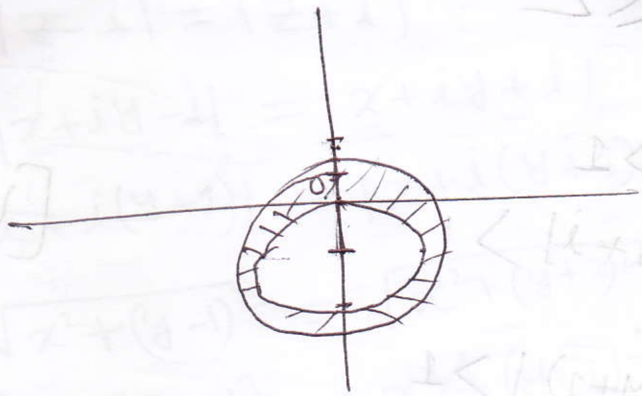
$\Rightarrow |x+iy+i| \leq 2$

$\Rightarrow |x+i(y+1)| \leq 2$

$\Rightarrow \sqrt{x^2+(y+1)^2} \leq 2$

$\Rightarrow x^2 + (y+1)^2 \leq (2)^2$

Here, centre $= (0,-1)$ & Radius, $R = 2$



(V) $|z+3i| > 4$

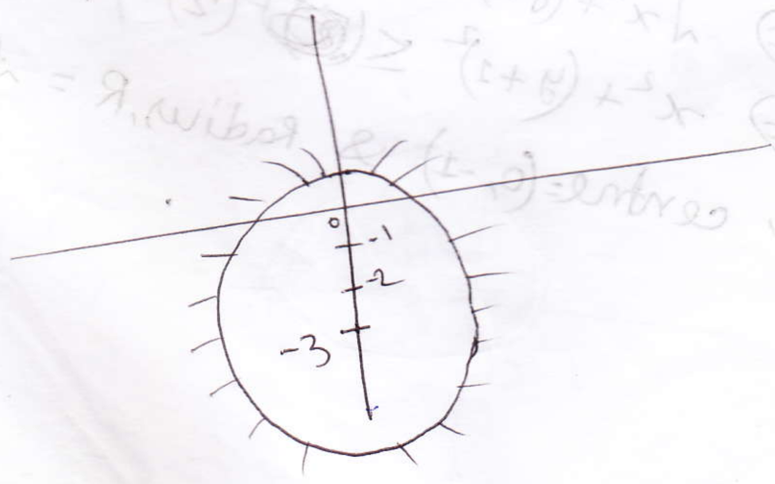
$\Rightarrow |x+iy+3i| > 4$

$\Rightarrow |x+i(y+3)| > 4$

$\Rightarrow \sqrt{x^2+(y+3)^2} > 4$

$\Rightarrow (x-0)^2+(y+3)^2 > (4)^2$

Here, centre $z_0 = (0,-3)$ & radius, $r = 4$



(vi) $|z+2-3i|+|z-2+3i|<10$

$$\Rightarrow |x+iy+2-3i|+|x+iy-2+3i|<10$$

$$\Rightarrow |x+2+i(y-3)|+|x-2+i(y+3)|<10$$

$$\Rightarrow \sqrt{(x+2)^2+(y-3)^2}+\sqrt{(x-2)^2+(y+3)^2}<10$$

$$\Rightarrow \sqrt{(x+2)^2+(y-3)^2}<10-\sqrt{(x-2)^2+(y+3)^2}$$

$$\Rightarrow (x+2)^2+(y-3)^2<100-20\sqrt{(x-2)^2+(y+3)^2}+(x-2)^2+(y+3)^2$$

$$\Rightarrow \cancel{x^2}+4x+\cancel{4}+\cancel{y^2}-6y+\cancel{9}<100-20\sqrt{(x-2)^2+(y+3)^2}+\cancel{x^2}-4x+\cancel{4}+\cancel{y^2}+6y+\cancel{9}$$

$$\Rightarrow 8x-12y<100-20\sqrt{(x-2)^2+(y+3)^2}$$

$$\Rightarrow 8x-12y-100<-20\sqrt{(x-2)^2+(y+3)^2}$$

$$\Rightarrow 64x^2+144y^2+10000-192xy+2400y-1600x<400(x^2-4x+4+y^2+6y+9)$$

$$\Rightarrow 64x^2+144y^2+10000-192xy+\cancel{2400y}-\cancel{1600x}<400x^2-\cancel{1600x}+1600+400y^2+\cancel{2400y}+3600$$

$$\Rightarrow 336x^2+256y^2+192xy\ \cancel{1600x}-4800>0$$

$$\Rightarrow 21x^2+16y^2+12xy-300>0$$

Practice sheet-3

(11)

① ① show that, $\mathrm{Exp}(2 \pm 3\pi i) = -e^2$

~~Now~~ $\cancel{\mathrm{Exp}(2 \pm 3\pi i) = e^{2 \pm 3\pi i}}$

$\cancel{= e^2 e^{\pm 3\pi i}}$

Now,

$L.S = \mathrm{Exp}(2 \pm 3\pi i)$

$= e^{2 \pm 3\pi i}$

$= e^2 \cdot e^{\pm 3\pi i}$

$= e^2 \{\cos(\pm 3\pi) + i\sin(\pm 3\pi)\}$

$= e^2(-1 + i \cdot 0)$

$= -e^2$

$= R.S$ (proved)

(ii) $L.S = Exp\left(\frac{2+\pi i}{4}\right)$

$= e^{1/2} \cdot e^{\frac{\pi i}{4}}$

$= e^{1/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= e^{1/2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$= \frac{\sqrt{e}}{\sqrt{2}}(1+i)$

$= \sqrt{\frac{e}{2}}(1+i) = R.S$ (proved)

(iii) $L.S = Exp(z+\pi i)$

$= e^{z} \cdot e^{\pi i}$

$= e^{z}(\cos\pi + i\sin\pi)$

$= e^{z}(-1+0)$

$= -e^{z}$

$= R.S$ (proved)

2 (i) $e^{z} = -2$

$\Rightarrow e^{x+iy} = -2$ —— ① [considering $z = x+iy$]

$\Rightarrow$ polar form of $-2 = 2e^{i(1+2n)\pi}$

① $\Rightarrow e^{x+iy} = 2e^{i(1+2n)\pi}$

$\Rightarrow e^{x} \cdot e^{iy} = 2e^{i(1+2n)\pi}$

$\therefore e^{x} = 2$ or $e^{iy} = e^{i(1+2n)\pi}$

$\Rightarrow \ln e^{x} = \ln 2$ $\quad \Rightarrow y = (1+2n)\pi$ for $n = 0, \pm 1, \pm 2, ---$

$\Rightarrow x \ln e = \ln 2$

$\Rightarrow x = \ln 2$

(ii) $e^{z} = 1 + \sqrt{3}\, i$ $\qquad r = \sqrt{(1)^2 + (\sqrt{3})^2}$

$= 2$

$\Rightarrow e^{x+iy} = 1 + \sqrt{3}\, i$ $\qquad \theta = \tan^{-1}(\sqrt{3})$

$\Rightarrow e^{x} \cdot e^{iy} = 2e^{i(\frac{\pi}{3} + 2n\pi)}$ $\qquad = \tan^{-1}\left(\tan\frac{\pi}{3}\right)$

& $e^{iy} = e^{i(\frac{\pi}{3} + 2n\pi)}$ $\qquad = \frac{\pi}{3}$

Now,

$e^{x} = 2$ $\qquad \Rightarrow y = \frac{\pi}{3} + 2n\pi$

$\Rightarrow \ln e^{x} = \ln 2$ $\qquad$ for $n = 0, \pm 1, \pm 2, \pm 3, ----$

$\Rightarrow x \ln e = \ln 2$

$\Rightarrow x = \ln 2$

(iii) $e^{2z-1} = 1$

$\Rightarrow e^{\{2(x+iy)-1\}} = 1$

$\Rightarrow e^{2x+2iy-1} = e^0 1$

$\Rightarrow e^{(2x-1)+2iy} = 1 e^{i(0+2n\pi)}$

$\Rightarrow e^{(2x-1)} \cdot e^{i(2y)} = 1 \cdot e^{i(2n\pi)}$

$r = \sqrt{(1)^2+(0)^2}$
$= 1$

$\theta = \tan^{-1}\left(\frac{1}{0}\right)$
$= 0$

Now, $e^{(2x-1)} = 1$

$\Rightarrow e^{(2x-1)} = e^0$

$\Rightarrow 2x-1 = 0$

$\Rightarrow x = \frac{1}{2}$

& $e^{i(2y)} = e^{i(2n\pi)}$

$\Rightarrow 2y = 2n\pi$

$\Rightarrow y = n\pi$

(iv) $e^z = -1$

$\Rightarrow e^{x+iy} = -1$

$\Rightarrow e^x \cdot e^{iy} = 1 \cdot e^{i(0+2n\pi)}$

$r = \sqrt{(-1)^2+(0)^2}$
$= 1$

$\theta = \tan^{-1}\left(\frac{0}{-1}\right)$
$= 0$

Now, $e^x = 1$

$\Rightarrow e^x = e^0$

$\Rightarrow x = 0$

& $e^{iy} = e^{i(2n\pi)}$

$\Rightarrow y = 2n\pi$

where $n = 0, \pm 1, \pm 2, \pm$ ---

Some formula

① $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

(ii) $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

(iii) $\sinh z = \frac{e^{z} - e^{-z}}{2}$

(iv) $\cosh z = \frac{e^{z} + e^{-z}}{2}$

~~(v) sin z =~~

(v) $\cos(iy) = \cosh y$

(vi) $\sin(iy) = i\sinh y$

3 ① Prove that, $\sin z = \sin x \cosh y + i \cos x \sinh y$

$L.S = \sin z$

$= \sin(x+iy)$

$= \sin x \cos(iy) + \cos x \sin(iy)$

$= \sin x \cosh y + \cos x (i\sinh y)$

$= \sin x \cosh y + i \cos x \sinh y$

$= R.S$ (proved)

(ii) ~~L.S = e~~

Prove that, $\cos z = \cos x \cosh y - i \sin x \sinh y$

$L.S = \cos z$

$= \cos(x+iy)$

$= \cos x \cos(iy) - \sin \cancel{x} \sin(iy)$

$= \cos x \cosh y - \sin \cancel{x}(i \sinh y)$

$= \cos x \cosh y - i \sin x \sinh y$

$= R.S$ (proved)

Prove that,

(iii) $\sin(z+2\pi) = \sin z$

$L.S = \sin(z+2\pi)$

$= \sin(x+iy+2\pi)$

$= \sin(x+2\pi+iy)$

$= \sin(x+2\pi)\cos(iy) + \cos(x+2\pi)\sin(iy)$

$= \sin(x+2\pi)\cos(iy) + \cos x \sin(iy)$

$= \sin x \cos(iy) + \cos x \sin(iy)$

$= \sin(x+iy)$

$= \sin z$

$= R.S$ (proved)

(IV) proved that,
$$\cos(z+2\pi) = \cos z$$

$$L.S = \cos(z+2\pi)$$
$$= \cos(x+iy+2\pi)$$
$$= \cos(x+2\pi+iy)$$
$$= \cos(x+2\pi)\cos(iy) - \sin(x+2\pi)\sin(iy)$$
$$= \cos x\cos(iy) - \sin x\sin(iy)$$
$$= \cos(x+iy)$$
$$= \cos z$$
$$= R.S \text{ (proved)}$$

4(i) ~~s~~ prove that,

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$

$L.S. = \sinh z$

$= -i\sin(iz)$

$= -i\sin[i(x+iy)]$

$= -i\sin(ix-y)$

$= -i[\sin(ix)\cos y - \cos(ix)\sin y]$

$= -i[i\sinh x \cos y - \cosh x \sin y]$

$= \sinh x \cos y + i\cosh x \sin y$

$= R.S.$ (proved)

(ii) Prove that,

$\cosh z = \cosh x \cos y + i \sin x \sin y$

$L.S = \cosh z$

$= \cos(iz)$

$= \cos[i(x+iy)]$

$= \cos(ix-y)$

$= \cos(ix)\cos y + \sin(ix)\sin y$

$= \cosh x \cos y + i \sinh x \sin y$

$= R.S$ (proved)

5 (i) Show that,

(a) $\sin^{-1} z = -i \ln\left[iz \pm (1-z^2)^{1/2}\right]$

Let, $\sin^{-1} z = \omega$

$\Rightarrow z = \sin \omega$

$\Rightarrow z = \dfrac{e^{i\omega} - e^{-i\omega}}{2i}$

$\Rightarrow e^{i\omega} - e^{-i\omega} = 2z$

$\Rightarrow e^{i\omega} - e^{-i\omega} = 2iz$

$\Rightarrow e^{i\omega} - \dfrac{1}{e^{i\omega}} = 2iz$

5 (1) Show that, $\sin^{-1}z = -i\ln\left[iz \pm (1-z^2)^{1/2}\right]$

Let, $\sin^{-1}z = \omega$

$\Rightarrow z = \sin\omega$

$\Rightarrow z = \dfrac{e^{i\omega} - e^{-i\omega}}{2i}$

$\Rightarrow e^{i\omega} - e^{-i\omega} = 2iz$

$\Rightarrow e^{i\omega} - \dfrac{1}{e^{i\omega}} = 2iz$

$\Rightarrow (e^{i\omega})^2 - 1 = 2iz\, e^{i\omega}$

$\Rightarrow (e^{i\omega})^2 - 2iz e^{i\omega} - 1 = 0$

$\Rightarrow e^{i\omega} = \dfrac{-(-2iz) \pm \sqrt{(-2iz)^2 - 4\cdot 1(-1)}}{2\cdot 1}$

$\Rightarrow e^{i\omega} = \dfrac{2iz \pm \sqrt{4i^2z^2+4}}{2}$

$\Rightarrow e^{i\omega} = \dfrac{2iz \pm 2\sqrt{1-z^2}}{2}$

$\Rightarrow e^{i\omega} = iz \pm (1-z^2)^{1/2}$

$\Rightarrow i\omega = \ln\left[iz \pm (1-z^2)^{1/2}\right]$

$$\Rightarrow \omega = \frac{1}{i} \ln \left[iz \pm (1-z^2)^{1/2} \right]$$

$$\Rightarrow \omega = \frac{i}{i \cdot i} \ln \left[iz \pm (1-z^2)^{1/2} \right]$$

$$\Rightarrow \omega = -i \ln \left[iz \pm (1-z^2)^{1/2} \right]$$ (proved)

(11) show that,

$$\cos^{-1} z = -i \ln \left[z \pm i(1-z^2)^{1/2} \right]$$

Let, $\cos^{-1} z = \omega$

$$\Rightarrow z = \cos \omega$$

$$\Rightarrow z = \frac{e^{i\omega} + e^{-i\omega}}{2}$$

$$\Rightarrow e^{i\omega} + e^{-i\omega} = 2z$$

$$\Rightarrow e^{i\omega} + \frac{1}{e^{i\omega}} = 2z$$

$$\Rightarrow (e^{i\omega})^2 + 1 = 2ze^{i\omega}$$

$$\Rightarrow (e^{i\omega})^2 - 2ze^{i\omega} + 1 = 0$$

$$\Rightarrow e^{i\omega} = \frac{-(-2z) \pm \sqrt{(-2z)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$\Rightarrow e^{i\omega} = \frac{2z \pm \sqrt{4z^2 - 4}}{2}$$

$$\Rightarrow e^{iw} = \frac{2z \pm 2\sqrt{z^2-1}}{2}$$

$$\Rightarrow e^{iw} = z \pm \sqrt{(z^2-1)}$$

$$\Rightarrow e^{iw} = z \pm \sqrt{-1(1-z^2)}$$

$$\Rightarrow e^{iw} = z \pm \sqrt{i^2(1-z^2)}$$

$$\Rightarrow e^{iw} = z \pm i(1-z^2)^{1/2}$$

$$\Rightarrow iw = \ln\left[z \pm i(1-z^2)^{1/2}\right]$$

$$\Rightarrow w = \frac{1}{i}\ln\left[z \pm i(1-z^2)^{1/2}\right]$$

$$\Rightarrow w = \frac{i}{i\cdot i}\ln\left[z \pm i(1-z^2)^{1/2}\right]$$

$$\therefore w = -\ln\left[z \pm i(1-z^2)^{1/2}\right] \quad \text{(Proved)}$$

6 (1) P~~...~~

$$\cosh z = \frac{1}{2}$$

$$\Rightarrow \frac{e^z + e^{-z}}{2} = \frac{1}{2}$$

$$\Rightarrow e^z + e^{-z} = 1$$

$$\cancel{\Rightarrow \frac{e^z + e^{-z}}{e^z} =}$$

$$\Rightarrow e^z\left(e^z + e^{-z}\right) = e^z$$

$$\Rightarrow e^{2z} + 1 = e^z$$

$$\Rightarrow \left(e^z\right)^2 - e^z + 1 = 0$$

$$\Rightarrow e^z = \frac{-(-1) \pm \sqrt{(1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$\Rightarrow e^z = \frac{1 \pm \sqrt{-3}}{2}$$

$$\Rightarrow e^z = \frac{1 \pm i\sqrt{3}}{2}$$

$$\Rightarrow z = \ln\left(\frac{1 \pm i\sqrt{3}}{2}\right)$$

(ii) $\sinh z = i$

$\Rightarrow \dfrac{e^z - e^{-z}}{2} = i$

$\Rightarrow e^z - e^{-z} = 2i$

$\Rightarrow e^z(e^z - e^{-z}) = 2ie^z$

$\Rightarrow e^{2z} - 1 = 2ie^z$

$\Rightarrow (e^z)^2 - 2ie^z - 1 = 0$

$\Rightarrow e^z = \dfrac{-(-2i) \pm \sqrt{(-2i)^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1}$

$\Rightarrow e^z = \dfrac{2i \pm \sqrt{0 \cancel{-4} \cancel{+4}}}{2}$

$\Rightarrow e^z = i$

$\Rightarrow z = \ln(i)$

7 (1) Show that: $\ln(-1+\sqrt{3}i) = \ln 2 + 2\left(n+\frac{1}{3}\right)\pi i$

$L.S = \ln(1+\sqrt{3}i)$

$= \ln\left\{2e^{i\left(\frac{2\pi}{3}+2n\pi\right)}\right\}$

$= \ln 2 + \ln\left\{e^{i\left(\frac{2\pi}{3}+2n\pi\right)}\right\}$

$= \ln 2 + i\left(\frac{2\pi}{3}+2n\pi\right)$

$= \ln 2 + 2\left(n+\frac{1}{3}\right)\pi i$

$= R.S$

(Proved)

Here,

$r = \sqrt{(1)^2+(\sqrt{3})^2}$

$= 2$

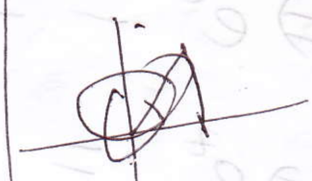
$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$

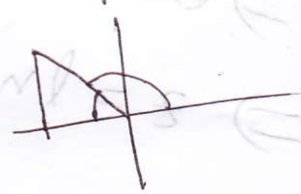
$= -\tan^{-1}(\sqrt{3})$

$= -\tan\left(\frac{\pi}{3}\right)$

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$





(ii) Show that, $\ln(1-i) = \frac{1}{2}\ln 2 + \left(2n + \frac{7}{4}\pi i\right)$

$$L.S = \ln(1-i)$$

$$= \ln\left\{\sqrt{2}\, e^{i\left(\frac{7\pi}{4} + 2n\pi\right)}\right\}$$

$$= \ln\sqrt{2} + \ln\left\{e^{i\left(\frac{7\pi}{4} + 2n\pi\right)}\right\}$$

$$= \ln(2)^{1/2} + i\left(\frac{7\pi}{4} + 2n\pi\right)$$

$$= \frac{1}{2}\ln 2 + \left(2n + \frac{7}{4}\right)\pi i$$

$$= R.S \text{ (Proved)}$$

$$r = \sqrt{(1)^2 + (-1)^2}$$

$$= \sqrt{2}$$

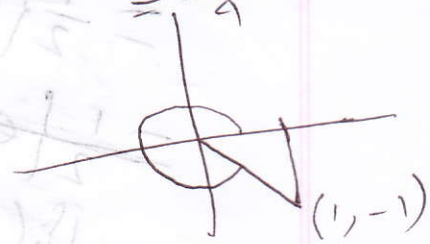
$$\theta = \tan^{-1}\left(\frac{-1}{1}\right)$$

$$= -\tan^{-1}\left(\frac{\pi}{4}\right)$$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4} + \pi$$

$$= \frac{7\pi}{4}$$



(iii) ~~Show~~

Practise sheet - 2

③ $f(z) = \begin{cases} \frac{z^2+4}{z-2i}, & z \neq 2i \\ & z = 2i \end{cases}$

$\lim\limits_{z \to 2i} f(z) = f(2i)$

$\lim\limits_{z \to 2i} \frac{z^2+4}{z-2i}$

$= \lim\limits_{z \to 2i} \frac{z^2 - 4i^2}{z-2i}$

$= \lim\limits_{z \to 2i} \frac{(z+2i)\cancel{(z-2i)}}{\cancel{(z-2i)}}$

$= \lim\limits_{z \to 2i} (z+2i)$

$= 2i + 2i$

$= 4i$

(4) $f(z) = \dfrac{2z-3}{z^2+2z+2}$

NOW, $z^2+2z+2 = 0$

$$\Rightarrow z = \frac{-2 \pm \sqrt{(2)^2-4\cdot1\cdot2}}{2\cdot1}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

5 (1), $f(z) = \dfrac{2z-i}{z+2i}$ at, $z = -i$

NOW, $f'(z_0) = \lim\limits_{\Delta z \to 0} \dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{\frac{2(z_0+\Delta z)-i}{(z_0+\Delta z)+2i} - \frac{2z_0-i}{z_0+2i}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\frac{(2z_0+2\Delta z-i)(z_0+2i)-(2z_0-i)(z_0+\Delta z-2i)}{\{(z_0+\Delta z)+2i\}(z_0+2i)}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(2z_0+2\Delta z-i)(z_0+2i)-(2z_0-i)(z_0+\Delta z-2i)}{(z_0+\Delta z+2i)(z_0+2i)}$$

$$= \frac{(2z_0+0-i)(z_0+2i)-(2z_0-i)(z_0+0-2i)}{(z_0+0+2i)(z_0+2i)}$$

$$= \frac{(2z_0-i)(z_0+2i)-(2z_0-i)(z_0-2i)}{(z_0+2i)^2}$$

$$= \frac{2z_0^2+4iz_0-iz_0+2-(2z_0^2-4iz_0-iz_0-2)}{(z_0+2i)^2}$$

$$= \frac{\cancel{2z_0^2}+3iz_0+2-\cancel{2z_0^2}+5iz_0+2}{(z_0+2i)^2}$$

$$= \frac{8iz_0+4}{(z_0+2i)^2}$$

$$= \frac{8i(-i)+4}{(-i+2i)^2}$$

$$= \frac{8+4}{i^2}$$

$$= \frac{12}{-1}$$

$$= -12$$

(1) $f(z) = 3z^{-2}$ at $z = 1+i$

Now, $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{\frac{3}{(z_0+\Delta z)^2} - \frac{3}{z_0^2}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3z_0^2 - 3(z_0+\Delta z)^2}{(z_0+\Delta z)^2 \cdot z_0^2 \Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3(\cancel{z_0^2} - \cancel{z_0^2} - 2z_0\Delta z - \Delta z^2)}{(z_0+\Delta z)^2 z_0^2 \Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{-3\cancel{\Delta z}(2z_0 + \Delta z)}{(z_0+\Delta z)^2 z_0^2 \cancel{\Delta z}}$$

$$= \lim_{\Delta z \to 0} \frac{-3(2z_0 + \Delta z)}{(z_0+\Delta z)^2 z_0^2}$$

$$= \frac{-3(2z_0 + 0)}{(z_0+0)^2 \cdot z_0^2}$$

$$= \frac{-6z_0}{z_0^4}$$

$$= \frac{-6}{z_0^3}$$

$$= \frac{-6}{(1+i)^3}$$

$$= \frac{-6}{1+3i+3i^2-i}$$

$$= \frac{-6}{1+3i-3-i}$$

$$= \frac{-6}{-2+2i}$$

$$= \frac{3}{(1-i)}$$

$$= \frac{3(1+i)}{(1+i)(1-i)}$$

$$= \frac{3(1+i)}{1-i^2}$$

$$= \frac{3(1+i)}{1+1}$$

$$= \frac{3}{2}(1+i)$$

6 (1) $$\lim_{z \to 2i} \frac{2z^2+4}{2z^2+(3-4i)z-6i}$$

$$= \lim_{z \to 2i} \frac{2z}{4z+(3-4i)}$$

$$= \frac{4i}{8i+3-4i}$$

$$= \frac{4i}{3+4i}$$

$$= \frac{4i(3-4i)}{(3+4i)(3-4i)}$$

$$= \frac{12i-16i^2}{9-16i^2}$$

$$= \frac{12i+16}{9+16}$$

$$= \frac{16+12i}{25}$$

$$= \frac{16}{25} + i\left(\frac{12}{25}\right)$$

$$= \cancel{\frac{16}{25} + i\,(}$$

(11) $\lim_{z \to 0} \frac{z - \sin z}{z^3}$

$= \lim_{z \to 0} \frac{1 - \cos z}{3z^2}$

$= \lim_{z \to 0} \frac{\sin z}{6z}$

$= \lim_{z \to 0} \frac{\cos z}{6}$

$= \frac{\cos 0^\circ}{6}$

$= \frac{1}{6}$

7 (ii) $u = xe^x \cos y - ye^x \sin y$

$U_x = e^x \cos y + xe^x \cos y - ye^x \sin y$

$U_{xx} = e^x \cos y + e^x \cos y + xe^x \cos y - ye^x \sin$

$= 2e^x \cos y + xe^x \cos y - ye^x \sin y$

Again,

$U_y = -xe^x \sin y - (e^x \sin y + ye^x \cos y)$

$U_{yy} = -xe^x \cos y - (e^x \cos y + e^x \cos y - ye^x \sin$

$= -xe^x \cos y - e^x \cos y - e^x \cos y + ye^x \sin y$

$= -xe^x \cos y - 2e^x \cos y$

$\therefore U_{xx} + U_{yy} = 0$

$\therefore$ The function is harmonic

Find conjuge harmonic function:

$U_x = \cos y\,(xe^x + e^x) - y\sin y\, e^x = V_y$

$\Rightarrow$ Now, $\frac{\partial v}{\partial y} = V_y = \cos y\,(xe^x + e^x) - y\sin y\, e^x$

$\Rightarrow \int \partial v = \int (xe^x \cos y + e^x \cos y - y\sin y\, e^x)\, \partial y$

$\Rightarrow v = xe^x \sin y + e^x \sin y - e^x \int y \sin y\, \partial y$

$\Rightarrow v = xe^x \sin y + e^x \sin y - e^x \left[y \int \sin y - \int \left\{ \frac{d}{dy}(y) \int \sin y\, dy \right\} dy \right]$

$\Rightarrow v = xe^x \sin y + e^x \sin y - e^x \left[-y\cos y - \int (-\cos y)\, dy \right]$

$\Rightarrow v = xe^x \sin y + e^x \sin y - e^x \left[-y\cos y + \sin y \right]$

$\Rightarrow v = xe^x \sin y + e^x \sin y \cancel{+ye^x \cos y} - e^x \left[-y\cos y + \sin y \right]$

$\Rightarrow v = xe^x \sin y + e^x \sin y + e^x y \cos y - e^x \cancel{\sin y} + \phi(x)$

$\Rightarrow v = xe^x \sin y + \cancel{e^x \sin y} + e^x y \cos y - \cancel{e^x \sin y} + \phi(x)$

$\Rightarrow v = xe^x \sin y + ye^x \cos y + \phi(x)$

$\phi'(x)$

$-v_x =$

$\phi'(x) = 0$

$\phi(x) = c$

$(u_y) = -v_x$

NOW,
$\cancel{U_y = -xe^x \sin y - e^x[y\cos y + \sin y] = -V_x}$

NOW,
$\cancel{U_y = -xe^x \sin y -}$

$$V_x = \sin y[xe^x + e^x] + ye^x \cos y + \phi'(x)$$

$$\Rightarrow V_x = xe^x \sin y + e^x \sin y + ye^x \cos y + \phi'(x)$$

$$\Rightarrow -V_x = -xe^x \sin y - e^x \sin y - ye^x \cos y - \phi'(x)$$

NOW, $U_y = -xe^x \sin y - e^x\left[y\cos y + \sin y\right]$

$$= -xe^x \sin y - ye^x \cos y - e^x \sin y$$

NOW,

$$U_y = -V_x$$

$$\Rightarrow \phi'(x) = 0$$

$$\Rightarrow \int \phi'(x)\, dx = \int 0 \cdot dx$$

$$\Rightarrow \phi(x) = C$$

$$\therefore V = xe^x \sin y + ye^x \cos y + C \text{ (Ans)}$$

(1) $U = 3x^2y + 2x^2 - y^3 - 2y^2$

$\Rightarrow U_x = 6xy + 4x$

$\Rightarrow U_{xx} = 6y + 4$

Again, $U_y = 3x^2 - 3y^2 - 4y$

$U_{yy} = -6y - 4$

$\therefore U_{xx} + U_{yy} = 0$

$\therefore$ The function is harmonic

Find conjugate harmonic function:

$U_x = 6xy + 4x = V_y$

$\Rightarrow \frac{\partial v}{\partial y} = V_y = 6xy + 4x$

$\Rightarrow \int \partial v = \int (6xy + 4x)\partial y$

$\Rightarrow \int \partial v = 6x\int y\partial y + 4x\int \partial y$

$\Rightarrow v = 3xy^2 + 4xy + \phi(x)$

$\Rightarrow v = 3y^2 + 4y + \phi'(x)$

$\cancel{\Rightarrow V_x =}$

Now, $V_x = -3y^2 - 4y - \phi'(x)$

$\Rightarrow -V_x = $

$$U_y = 3x^2 - 3y^2 - 4y$$

$$U_y = -V_x$$

$$\Rightarrow 3x^2 - \cancel{3y^2} - \cancel{4y} = -\cancel{3y^2} - \cancel{4y} - \phi'(x)$$

$$\Rightarrow 3x^2 = -\phi'(x)$$

$$\Rightarrow \int 3x^2 dx = -\int \phi'(x)\,dx$$

$$\Rightarrow x^3 = -\phi(x)$$

$$\Rightarrow \phi(x) = -x^3.$$

$$\therefore V = 3xy^2 + 4xy - x^3$$

(iii) $u = e^{-x}(x\sin y - y\cos y)$

$$U_x = \sin y[e^{-x} - xe^{-x}] + y\cos y e^{-x}$$

$$= e^{-x}\sin y - xe^{-x}\sin y + y\cos y e^{-x}$$

$$U_{xx} = -e^{-x}\sin y - \sin y[e^{-x} - xe^{-x}] - y\cos y e^{-x}$$

$$= -2e^{-x}\sin y + xe^{-x}\sin y - y\cos y e^{-x}$$

Again, $U_y = xe^{-x}\cos y - e^{-x}[y(-\sin y) + \cos$

$$= xe^{-x}\cos y + y\sin y e^{-x} - e^{-x}\cos$$

$$U_{yy} = -xe^{-x}\sin y + e^{-x}(y\cos y + \sin y) + e$$

$$= -xe^{-x}\sin y + 2e^{-x}\sin y + y\cos y e^{-x}$$

$\therefore U_{xx} + U_{yy} = 0$

so the given function is harmonic

Find conjugate harmonic function:

$U_x = e^{-x}\sin y - xe^{-x}\sin y + y\cos y\, e^{-x} = V_y$

$\Rightarrow \frac{\partial V}{\partial y} = V_y = e^{-x}\sin y - xe^{-x}\sin y + y\cos y e^{-x}$

$\Rightarrow \int \partial V = \int (e^{-x}\sin y - xe^{-x}\sin y + e^{-x} y\cos y)\, \partial y$

$\Rightarrow V = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}\int y\cos y\, \partial y$

$= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}\left[y\int \cos y\, \partial y - \int\left\{\frac{d}{dy}(y)\int \cos y\, dy\right\}dy\right]$

$= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}[y\sin y - \int \sin y\, dy]$

$= -e^{-x}\cos y + xe^{-x}\cos y + y\sin y e^{-x} + \cancel{e^{-x}\cos y} + \phi(x)$

$\Rightarrow V = \cancel{-e^{-x}\cos y} + xe^{-x}\cos y + y\sin y e^{-x} + \phi(x)$

now, $V_x = e^{-x}\cos y + (e^{-x} - xe^{-x})\cos y - y\sin y e^{-x} - e^{-x}\cos y + \phi'(x)$

$= e^{-x}\cos y + \cancel{e^{-x}\cos y} - xe^{-x}\cos y - y\sin y e^{-x} - \cancel{e^{-x}\cos y} + \phi'(x)$

$\Rightarrow -V_x = -e^{-x}\cos y + xe^{-x}\cos y + y\sin y e^{-x} - \phi'(x)$

$U_y = -V_x$

$\Rightarrow \phi'(x) = 0$

$\Rightarrow \int \phi'(x)\, dx = \int 0 \cdot dx$

$\Rightarrow \phi(x) = c$

$\therefore V = xe^{-x}\cos y + y\sin y e^{-x} + c$ (Ans)

Practice-sheet-4

A set of points $z=(x,y)$ in the complex is an arc if $x=x(t)$, $y=y(t)$ $(a\le t\le b)$ where $x(t)$ & $y(t)$ are continuous functions of the real parameter t.

<u>simple arc</u>: An arc is simple if it does not cross itself

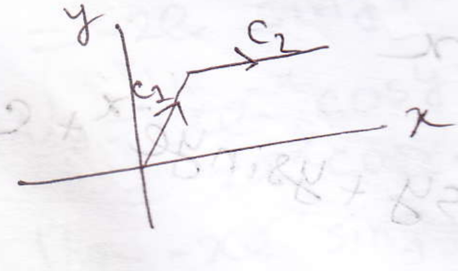
ie $t_1 \neq t_2$ then $z(t_1) \neq z(t_2)$

simple closed curve

An arc which is simple except

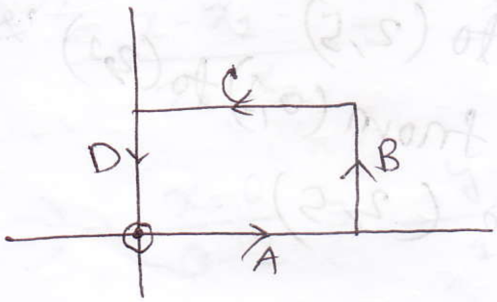
$z(b) = z(c)$

<u>contour</u>: A contour on apricewise smooth arc is an arc consisting of a finite number of smoots arcs joined end to end.



simple closed contour:

$z(a) = z(b)$ ie initial & final values of the arc $z(t)$ are same



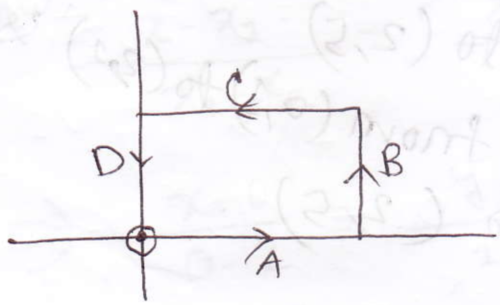
$\int_C f(z)\,dz$

$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

$\int_C f(z) = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

simple closed contour:

$z(a) = z(b)$. ie initial & final values of the arc $z(t)$ are same



$\int_C f(z)\,dz$

$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

$\int_C f(z) = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

① $\int_{(0,1)}^{(2,5)} (3x+y)\,d\cancel{x} + (2y-x)\,dy$ along

a) the curve $y = x^2+1$

b) the straight line joining $(0,1)$ & $(2,5)$

c) the straight lines from $(0,1)$ to $(0,5)$ & then from $(0,5)$ to $(2,5)$

d) the straight lines from $(0,1)$ to $(2,1)$ & then from $(2,1)$ to $(2,5)$

a)

$y = x^2+1$

$\Rightarrow \frac{dy}{dx} = 2x$

$\Rightarrow dy = \cancel{2x\,dy}\ 2x\,dx$

$\therefore \int_0^2 (3x+x^2+1)\,dx + (2x^2+2-x)\cdot 2x\,dx$

$= \left[\frac{3x^2}{2} + \frac{x^3}{3} + x\right]_0^2 + \left[\frac{4x^4}{4} + \frac{4x^2}{2} - \frac{2x^3}{3}\right]_0^2$

$= \left[x^4 + 2x^2 - \frac{2x^3}{3}\right]_0^2$

$= \left(\frac{3\cdot 4}{2} + \frac{8}{3} + 2\right) + \left(16+8-\frac{16}{3}\right)$

$= \left(8+\frac{8}{3}\right) + \left(24-\frac{16}{3}\right)$

$= \frac{32}{3} + \frac{56}{3}$

$= \frac{88}{3}$

(b)

$(0,1)$ —————— $(2,5)$

(x_1, y_1) $\qquad$ (x_2, y_2)

Now, $\dfrac{x - x_1}{x_1 - x_2} = \dfrac{y - y_1}{y_1 - y_2}$

$\Rightarrow \dfrac{x-0}{0-2} = \dfrac{y-1}{1-5}$

$\Rightarrow \dfrac{x}{2} = \dfrac{y-1}{4}$

$\Rightarrow y - 1 = 2x$

$\Rightarrow y = 2x + 1 \Rightarrow \dfrac{dy}{dx} = 2 ; \Rightarrow dy = 2dx$

Now, $\displaystyle\int_0^2 (3x + 2x + 1)\, dx + (4x + 4 - x) \cdot 2dx$

$= \displaystyle\int_0^2 (5x+1)\, dx + (6x+8)\, dx$

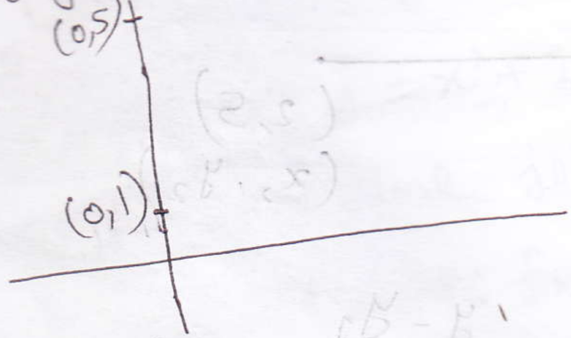
~~$= \int 11x$~~

$= \displaystyle\int_0^2 (5x+1)\, dx + \int_0^2 (6x+8)\, dx$

$= \left[\dfrac{5x^2}{2}\right]_0^2 + [x]_0^2 + 3\left[x^2\right]_0^2 + 8[x]_0^2$

$= 10 + 2 + 12 + 16 = 40$

(c) along the line (0,1) to (0,5)



$x = 0$

$\therefore dx = 0$

Now, $\int_1^5 (3.0+y).0 + (2y-0)\,dy$

$= \cancel{\int 2y\,y}\; 2\int_1^5 y\,dy$

$= \left[y^2\right]_1^5$

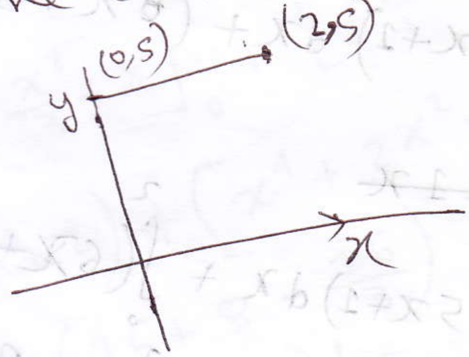
$= 25-1$

$= 24$

along the line (0,5) to (2,5)

$y = 5$

$\therefore dy = 0$



$$\int_0^2 (3x+5)\,dx + (10-x)\cdot 0$$

$$= \int_0^2 (3x+5)\,dx$$

$$= \frac{3}{2}[x^2]_0^2 + 5[x]_0^2$$

$$= 6+10$$

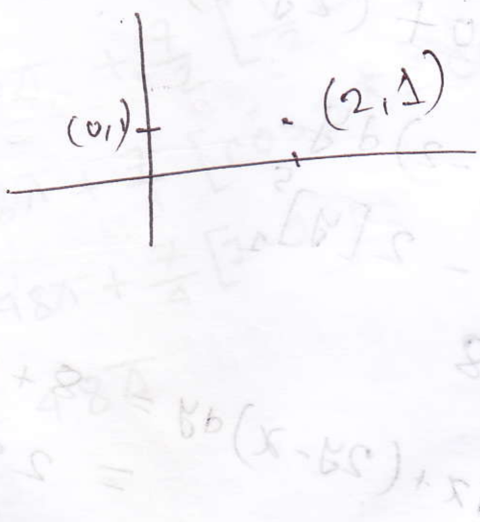
$$= 16$$

$$\therefore \int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$$

$$= 24+16$$

$$= 40$$

(d) along the line (0,1) to (2,1) ~~& then~~ fn



$y = 1$

$\Rightarrow dy = 0$

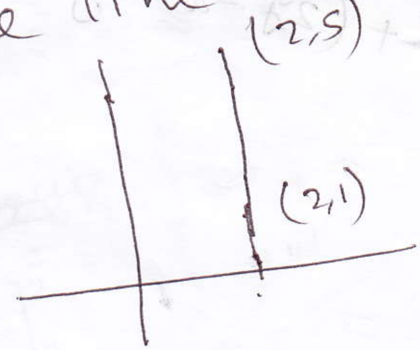
$$\int_0^2 (3x+1)\,dx + (2-x)\cdot 0$$

$$= \int_0^2 (3x+1)\,dx$$

$$= \frac{3}{2}[x^2]_0^2 + [x]_0^2$$

$$= 6+2 = 8$$

along the line (2,1) to (2,5)



$x = 2$

$\Rightarrow dx = 0$

$$\therefore \int_1^5 (6+y)\cdot 0 + (2y-2)\,dy$$

$$= \int_1^5 (2y-2)\,dy$$

$$= [y^2]_1^5 - 2[y]_1^5$$

$$= 24 - 8$$

$$= 16$$

$$\therefore \int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy = 8+16$$

$$= 24$$

(2) $\oint_C (x+2y)\,dx + (y-2x)\,dy$, C: eclipse:

$x = 4\cos\theta$

$y = 3\sin\theta$

$0 \le \theta \le 2\pi$

$dx = -4\sin\theta\,d\theta$

$dy = 3\cos\theta\,d\theta$

$$= \int_0^{2\pi} (4\cos\theta + 6\sin\theta)(-4\sin\theta\,d\theta) + (3\sin\theta - 8\cos\theta)(3\cos\theta\,d\theta)$$

$$= \int_0^{2\pi} (-16\sin\theta\cos\theta - 24\sin^2\theta)\,d\theta - 24\cos^2\theta)\,d\theta + (9\sin\theta\cos\theta$$

$$= \int_0^{2\pi} \{-24(\sin^2\theta + \cos^2\theta) - 7\sin\theta\cos\theta\}\,d\theta$$

$$= -24\int_0^{2\pi} d\theta - 7\int_0^{2\pi} \sin\theta\cos\theta\,d\theta$$

$$= -48\pi - \frac{7}{2}\int_0^{2\pi} 2\sin\theta\cos\theta$$

$$= -48\pi - \frac{7}{2}\int_0^{2\pi} \sin 2\theta$$

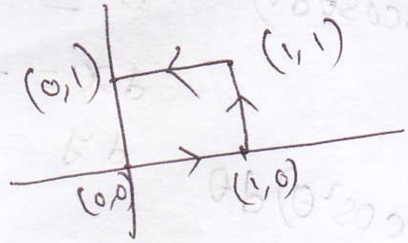
$$= -48\pi + \frac{7}{2}\left[\frac{1}{2}\cos 2\theta\right]_0^{2\pi}$$

$$= -48\pi + \frac{7}{4}\left[\cos 4\pi - \cos 0\right]$$

$$= -48\pi + \frac{7}{4}[1-1]$$

$$= -48\pi$$

④ $\oint_C |z|^2 dz$ around the square with verticles at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$

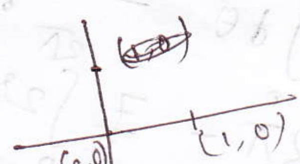


let, $z = x + iy$

$\Rightarrow dz = dx + i dy$

$\therefore \oint_C |z|^2 dz = \oint_C (x^2+y^2)(dx + i dy)$

① $(0,0)$ to $(1,0)$



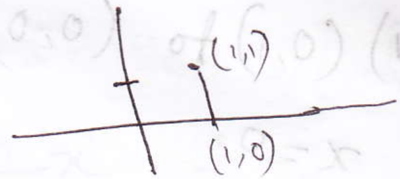
$y = 0$

$\Rightarrow dy = 0$

$\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1$

$= \frac{1}{3}$

(ii) $(1,0)$ to $(1,1)$



$x = 1$

$\Rightarrow dx = 0$

$\therefore \int_0^1 (1+y^2)\, i\, dy$

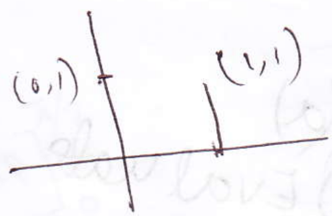
$= i \int_0^1 (1+y^2)\, dy$

$= i \left\{ [y]_0^1 + \left[\frac{y^3}{3}\right]_0^1 \right\}$

$= i\left(1+\frac{1}{3}\right)$

$= \frac{4}{3} i$

(iii) $(1,1)$ to $(0,1)$



$y = 1$

$\Rightarrow dy = 0$

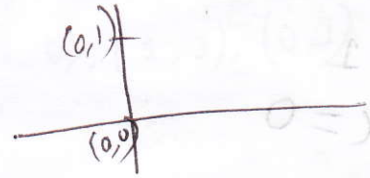
$\int_1^0 (x^2+1)\, dx$

$= \left[\frac{x^3}{3}\right]_1^0 + [x]_1^0$

$= -\frac{1}{3} - 1$

$= -\frac{4}{3}$

(iv) $(0,1)$ to $(0,0)$



$x = 0.$

$dx = 0$

$$\int_1^0 y^2 \cdot i\,dy$$

$$= i\left[\frac{y^3}{3}\right]_1^0$$

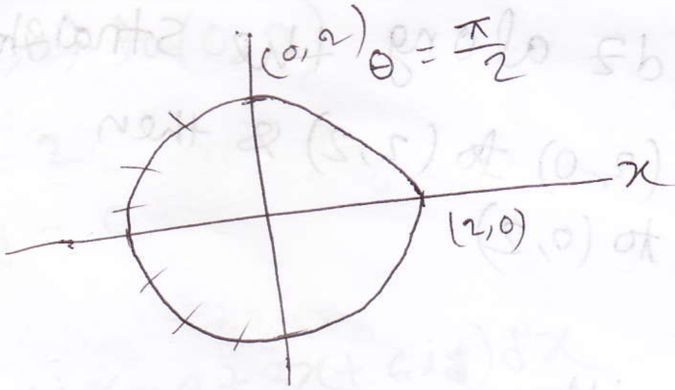
$$= -\frac{1}{3}i$$

$$\therefore \oint_c |z^2|\,dz = \frac{1}{3} + \frac{4}{3}i - \frac{4}{3} - \frac{1}{3}i$$

$$= \frac{1}{3} - \frac{4}{3} + i\left(\frac{4}{3} - \frac{1}{3}\right)$$

$$= -1 + i$$

(5) (a) Evaluate $\int_c (z^2 + 3z)\,dz$ ~~Now~~ along the circle $|z| = 2$ from $(2,0)$ to $(0,2)$ in a counterclockwise direction.



Now, $|z| = 2$

$\Rightarrow z = Re^{i\theta}$

$\Rightarrow z = 2e^{i\theta}$

$\Rightarrow dz = 2ie^{i\theta} d\theta$

$\int_C (z^2+3z)\, dz$

$= \int_0^{\pi/2} \left(4e^{i2\theta} + 6e^{i\theta}\right) \cdot 2ie^{i\theta} d\theta$

$= \int_0^{\pi/2} \left(8ie^{3i\theta} + 12ie^{2i\theta}\right) d\theta$

$= \frac{8i}{3i}\left[e^{3i\theta}\right]_0^{\pi/2} + \frac{12i}{2i}\left[e^{2i\theta}\right]_0^{\pi/2}$

$= \frac{8}{3}\left[e^{i\left(\frac{3\pi}{2}\right)} - 1\right] + 6\left[e^{i\left(2\frac{\pi}{2}\right)} - 1\right]$

$= \frac{8}{3}\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} - 1\right] + 6\left(\cos\pi + i\sin\pi - 1\right)$

$= \frac{8}{3}\left[0 - i - 1\right] + 6\left[-1 + 0 - 1\right]$

$= -\frac{8}{3}i - \frac{8}{3} - 12 = -\frac{44}{3} - i\frac{8}{3}$

(b) $\int_c (z^2+3z)\,dz$ along the straight line from (2,0) to (2,2) & then from (2,2) to (0,2)

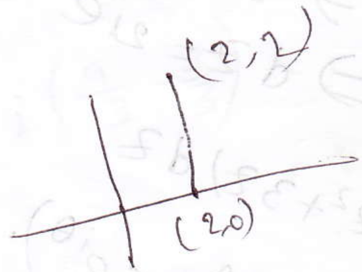
$\Rightarrow$ Now,

$z = x+iy$

$\Rightarrow dz = dx+i\,dy$

$\int_c (z^2+3z)\,dz = \int (x^2+2ixy-y^2+3x+3iy)(dx+$

(1) (2,0) to (2,2)



$x=2$

$\Rightarrow dx=0$

$\int_0^2 (4+4iy-y^2+6+3iy)\cdot i\,dy$

$= i\left[4\int_0^2 dy + 4i\int_0^2 y\,dy - \int_0^2 y^2\,dy + 6\int_0^2 dy + 3i\int_0^2 y\right.$

$= i\left[8+2i(4-0) - \frac{1}{3}(8-0) + 6(2-0) + \frac{3}{2}i(4-\right.$

$= i\left[8+8i-\frac{8}{3}+12+6i\right]$

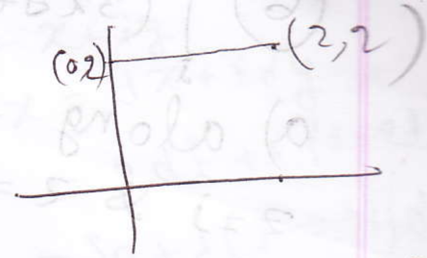
$= i\left[\frac{52}{3}+14i\right]$

$= -14 + i\left(\frac{52}{3}\right)$

(ii) $(2,2)$ to $(0,2)$

$y = 2$

$\Rightarrow dy = 0$



$\int_0^2 (x^2 + 4ix - 4 + 3x + 6i)\, dx$

$= \frac{1}{3}[x^3]_0^2 + \frac{4i}{2}[x^2]_0^2 - 4[x]_0^2 + \frac{3}{2}[x^2]_0^2 + 6i[x]_0^2$

$= \frac{8}{3} + 8i - 8 + 6 + 12i$

$= \frac{2}{3} + 20i$

$\therefore \int_C (z^2 + 3z)\, dz$

$= -14 + \frac{52}{3}i + \frac{2}{3} + 20i$

$= -\frac{40}{3} + \frac{112}{3}i$

~~$= -\frac{4}{3}(10 + 28i)$~~

$= -\frac{8}{3}(5 + 14i)$

(6) $\int_{i}^{2-i} (3xy + iy^2)\, dz$

a) along the straight line joining $z = i$ & $z = 2-i$

$\Rightarrow z = x + iy \qquad dz = dx + i dy$

$\Rightarrow z = 0 + i.1$

Now, $\int_0^2 (3xy + iy^2)(dx + i dy)$

$= \int_0^2 3xy\, dx + 3xy\, i\, dy + iy^2 dx - y^2 dy$

$= 3y \int_0^2 x\, dx + 3xi \int_0^2 y\, dy + iy^2 \int_0^2 dx - \int_0^2 y^2 dy$

$=$

(b) along the parabola $x = 2t - 2,\ y = 1 + t - t^2$

$x = 2t - 2$

$\Rightarrow dx = 2dt$

$y = 1 + t - t^2$

$\Rightarrow dy = (1-2t)dt$

$z = x + iy$

$\Rightarrow dz = dx + i\,dy$

$= 2dt + i(1-2t)dt$

$= \{2 + i(1-2t)\}dt$

if $x = 0$; $2t - 2 = 0$

$\Rightarrow t = 1$

$x = 2$, $2t - 2 = 2$

$\Rightarrow t = 2$

Now,

$$\int_1^2 \{(2t-2)(1+t-t^2) + i(1+t-t^2)^2\}\{2+i(1-2t)\}dt$$

$$= \int \{(2t-2)(1+t-t^2) + i(1+t^2+t^4+2t-2t^3-2t^2)\}(2+i-2it)\,dt$$

$$= \int \{(2t-2)(1+t-t^2) + i(1+2t-t^2-2t^3+t^4)\}\{2+i(1-2t)\}dt$$

$$= \int \{6(t-1)(1+t-t^2)$$

$=$

7 (i) $\oint_c (\bar{z})^2 \, dz$

arround the circles $|z| = 1$

$\Rightarrow |z| = 1$

$\Rightarrow z = 1 . e^{i\phi}$

$\Rightarrow dz = ie^{i\phi} d\phi$

Now, $\oint_c (\bar{z})^2 dz$

$= \int_0^{2\pi} (\overline{e^{i\phi}})^2 . ie^{i\phi} d\phi$

$= \int_0^{2\pi} (e^{-i\phi})^2 \, ie^{i\phi} d\phi$

$= i \int_0^{2\pi} e^{-i\phi} d\phi$

$= \frac{i}{-i} \left[e^{-i\phi} \right]_0^{2\pi}$

$= -1 \left[e^{i(-2\pi)} - 1 \right]$

$= -1 \left[\cos(-2\pi) + i \sin(-2\pi) - 1 \right]$

$= -1 \left[\cos(2\pi) - i \sin 2\pi - 1 \right]$

$= -1 [1 - 0 - 1] = 0$

(11) $|z-1| = 1$

$\Rightarrow z = 1 + 1 \cdot e^{i\phi}$

$\Rightarrow dz = i e^{i\phi} \, d\phi$

$$\int_0^{2\pi} \left(\overline{1+e^{i\phi}}\right)^2 \cdot i e^{i\phi} \, d\phi$$

$$= \int_0^{2\pi} (1+e^{-i\phi})^2 \cdot i \cdot e^{i\phi} \, d\phi$$

$$= \int_0^{2\pi} (1+2e^{-i\phi}+e^{-2i\phi}) \cdot i e^{i\phi} \, d\phi$$

$$= i \int_0^{2\pi} (e^{i\phi}+2+e^{-i\phi}) \, d\phi$$

$$= i \left[\int_0^{2\pi} e^{i\phi} \, d\phi + 2\int_0^{2\pi} d\phi + \int_0^{2\pi} e^{-i\phi} \, d\phi \right]$$

$$= i \left[\frac{1}{i} \left[e^{i\phi}\right]_0^{2\pi} + 2 \times 2\pi + \frac{1}{-i} \left[e^{-i\phi}\right]_0^{2\pi} \right]$$

$$= e^{i(2\pi)} - \cancel{1} + 4\pi i - e^{i(-2\pi)} + \cancel{1}$$

$$= \cos(2\pi) + i\sin(2\pi) + 4\pi i - [\cos(-2\pi) + i\sin(-2$$

$$= \cos(2\pi) + i\sin(2\pi) + 4\pi i - \cancel{\cos(2\pi)} + i\sin$$

$$= \cancel{\cos}(2\pi) + i\sin(2\pi) + 4\pi i$$

$$= 2i\sin(2\pi) + 4\pi i$$

$$= 0 + 4\pi i = 4\pi i$$

8. a. $\oint_C \frac{dz}{z-2}$ arround the circle $|z-2| = 4$

$|z-2| = 4$

$\Rightarrow z = 2 + 4e^{i\phi}$

$\Rightarrow dz = 4ie^{i\phi}$

$$\therefore \int_0^{2\pi} \frac{4ie^{i\phi}\, d\phi}{2+4e^{i\phi}-2}$$

$$= \int_0^{2\pi} \frac{4ie^{i\phi}\, d\phi}{4e^{i\phi}}$$

$$= i\int_0^{2\pi} d\phi$$

$$= 2\pi i$$

b) $|z-1| = 9$

$\Rightarrow z = 1 + 9e^{i\phi}$

$\Rightarrow dz = 9ie^{i\phi}\, d\phi$

$$\therefore \int_0^{2\pi} \frac{9ie^{i\phi}\, d\phi}{1+9e^{i\phi}-2}$$

$$= \int_0^{2\pi} \frac{9ie^{i\phi}\, d\phi}{9e^{i\phi}-1}$$

let, $u = 9\cancel{i}e^{i\phi} - 1$

$\Rightarrow du = 9ie^{i\phi}\, d\phi$

ϕ	2π	0
u	8	8

$$= \int_8^8 \frac{1}{u}\, du$$

$$= \left[\ln u\right]_8^8$$

$$= \ln 8 - \ln 8$$

$$= 0 \text{ (Ans)}$$

(9) Evaluate $\oint_C (5z^4 - z^3 + 2)\, dz$ around the circle $|z| = 1$

$\Rightarrow$ Now,

$$|z| = 1$$

$$\Rightarrow z = 1 \cdot e^{i\phi}$$

$$\Rightarrow dz = i e^{i\phi}\, d\phi$$

$$\int_0^{2\pi} \{5(e^{i\phi})^4 - (e^{i\phi})^3 + 2\} \cdot i\, e^{i\phi}\, d\phi$$

$$\int_0^{2\pi} (5e^{i\cdot 4\phi} - e^{i\cdot 3\phi} + 2) \cdot i\, e^{i\phi}\, d\phi$$

$$= \int_0^{2\pi} (5e^{i\cdot 4\phi} - e^{i4\phi} + 2i e^{i\phi})\, d\phi$$

$$= \int_0^{2\pi} (5i\, e^{i\cdot 5\phi} - i\, e^{i4\phi} + 2i \int_0^{2\pi} e^{i\phi}\, d\phi$$

$$= 5i \int_0^{2\pi} e^{i\cdot 5\phi}\, d\phi - i \int_0^{2\pi} e^{i4\phi}\, d\phi + \frac{2i}{i}\left[e^{i\phi}\right]_0^{2\pi}$$

$$= \frac{5i}{5i}\left[e^{i\cdot 5\phi}\right]_0^{2\pi} - \frac{i}{4i}\left[e^{i\cdot 4\phi}\right]_0^{2\pi}$$

$$= \left\{ e^{i \cdot (10\pi)} - 1 \right\} - \frac{1}{4} \left\{ e^{i(8\pi)} - 1 \right\} + 2 \left\{ e^{i(2\pi)} - 1 \right\}$$

$$= \left[\cos(10\pi) + i \sin(10\pi) - 1 \right] - \frac{1}{4} \left[\cos(8\pi) + i \sin(8\pi) - 1 \right]$$
$$+ 2 \left[\cos(2\pi) + i \sin(2\pi) - 1 \right]$$

$$= [1 + 0 - 1] - \frac{1}{4} [1 + 0 - 1] + 2 [1 + 0 - 1]$$

$$= 0 - 0 + 0 = 0$$

$$(a+b)^n = a^n + nc_1 a^{n-1} b + nc_2 a^{n-2} b^2 + \cdots + nc_n b^n$$

Proof:

$$\cos 5\phi + i\sin 5\phi = (\cos\phi + i\sin\phi)^5$$

Now,

$$R.S = (\cos\phi + i\sin\phi)^5$$

$$= \cos^5\phi + 5c_1 \cos^4\phi (i\sin\phi) + 5c_2 \cos$$
$$+ 5c_3 \cos^2\phi (i\sin\phi)^3 +$$
$$+ 5c_5 ($$

$$= \cos^5\phi + 5\cos^4\phi\, i\sin\phi - 10\cos^3$$
$$- 10\cos^2\phi\, i\sin^3\phi + 5\cos\phi \sin$$

$$= (\cos^5\phi - 10\cos^3\phi\sin^2\phi + 5\cos\phi\sin^4\phi$$
$$+ i(5\cos^4\phi\sin\phi - 10\cos^2\phi\sin^3\phi$$

For a

$$\cos 5\phi = \cos^5\phi - 10\cos^3\phi\sin^2\phi + 5\cos$$
$$= \cos^5\phi - 10\cos^3\phi(1-\cos^2\phi) + 5c$$
$$= \cos^5\phi - 10\cos^3\phi + 10\cos^5\phi + 5\cos$$
$$= \cos^5\phi - 10\cos^3\phi + 5\cos$$
$$= 16\cos^5\phi - 20\cos^3\phi + 5\cos$$

for b

$\sin 5\phi = 5\cos^4\phi \cdot \sin\phi - 10\cos^2\phi \sin^3\phi + \sin^5\phi$

$= \sin\phi(5\cos^4\phi - 10\cos^2\phi \sin^2\phi + \sin^4\phi)$

$= \sin\phi(5\cos^4\phi - 10\cos^2\phi(1-\cos^2\phi)$

Now, $\dfrac{\sin 5\phi}{\sin\phi} = 5\cos^4\phi - 10\cos^2\phi(1-\cos^2\phi) + (1-\cos^2\phi)^2$

$\lim\limits_{z\to 1} f(z) = \frac{i}{2}$ for $f(z) = i\frac{z}{2}$ in the open disk $|z|<1$ using defination of limit.

$|f(z) - \frac{1}{2}| < \epsilon$ for $0 < |z-1| < \delta$

$\Rightarrow \left|\frac{iz}{2} - \frac{i}{2}\right| < \epsilon$

$\Rightarrow \left|\frac{i(z-1)}{2}\right| < \epsilon$

$\Rightarrow |z-1| < 2\epsilon$

$0 < |z-1| < \delta$

$|z-1| < 2\epsilon$

if $\delta = 2\epsilon$

$|z-1| \epsilon < \epsilon$ when $0 < |z-1| < 2\epsilon$

$\lim\limits_{z\to z_0} f(z) = w_0$

$|f(z) - w_0| < \epsilon$

for

$0 < |z - z_0| < \delta$