MAT120
Final Exam
KichuEkta
Set :05

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Question 1 (a)

Ans no 1

a

$$\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_0^{x^3} \sin(y/x)\, dy\, dx$$

Here, $\int_0^{x^3} \int \sin(y/x)\, dy$; let $u = y/x$.

$\frac{du}{dy} = \frac{1}{x}$.

$dy = x\, du$.

$= x \int \sin(u)\, du$

$= x \cdot -\cos(u) + c$

$= -x\cos(u) + c$

$= -x\cos(y/x) + c$

$\therefore \int_0^{x^3} \sin(y/x)\, dy$

$\Rightarrow -x\left[\cos y/x\right]_0^{x^3}$

$\Rightarrow -x\cos(x^2) + x.$

$\Rightarrow x - x\cos(x^2).$

Again, $\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \left[x - x\cos(x^2)\right] dx$

Here, $\int \left[x - x\cos(x^2)\right] dx$

$\Rightarrow \int x\, dx - \int x\cos x^2\, dx$

$\Rightarrow \frac{x^2}{2} + c_1 - \int x\cos x^2\, dx.$

$\Rightarrow \frac{x^2}{2} + c_1 - \frac{1}{2}\int \cos u\, du.$

let, $u = x^2$

$\frac{du}{dx} = 2x.$

$dx = \frac{1}{2x} du$

$\Rightarrow \frac{x^2}{2} + c_1 - \frac{1}{2}\sin(u) + c_2.$

$\Rightarrow \frac{x^2}{2} - \frac{1}{2}\sin(x^2) + c.$

$\therefore \int_{\sqrt{\pi}}^{\sqrt{2\pi}} \left[x - x\cos(x^2)\right]$

$\Rightarrow \left[\frac{x^2}{2} - \frac{1}{2}\sin(x^2)\right]_{\sqrt{\pi}}^{\sqrt{2\pi}}$

$$\Rightarrow \left[\frac{2\pi}{2} - \frac{1}{2}\sin(2\pi)\right] - \left[\frac{\pi}{2} - \frac{1}{2}\sin(\pi)\right]$$

$$\Rightarrow \pi - \pi/2.$$

$$\Rightarrow \pi/2.$$

$$\therefore \int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_0^{x^3} \sin(y/x)\, dy\, dx = \pi/2. \text{ (Ans.)}$$

Question 1 (b)

Ans no 1

b

$$\int_{-1}^{1}\int_{-x^2}^{x^2} (x^2-y)\,dy\,dx.$$

Here, $\int (x^2-y)\,dy.$

$\Rightarrow \int x^2\,dy - \int y\,dy.$

$\Rightarrow x^2 y - y^2/2 + c.$

$\therefore \int_{-x^2}^{x^2} (x^2-y)\,dy.$

$\Rightarrow \left[x^2 y - y^2/2\right]_{-x^2}^{x^2}$

$\Rightarrow \left[x^2 . x^2 - x^2/2\right] - \left[x^2\cdot(-x^2) - \frac{(-x^2)^2}{2}\right]$

$\Rightarrow x^4 - x^2/2 + x^4 + x^2/2.$

$\Rightarrow 2x^4.$

Again, $\int 2x^4 \, dx$

$\Rightarrow 2\left(\frac{x^5}{5}\right) + c.$

$\therefore \int_{-1}^{1} 2\left(\frac{x^5}{5}\right)$

$\Rightarrow 2\left(\frac{1^5}{5} - \frac{(-1)^5}{5}\right).$

$\Rightarrow 2\left(\frac{1}{5} + \frac{1}{5}\right).$

$\Rightarrow \frac{4}{5}.$

$\therefore \int_{-1}^{1} \int_{-x^2}^{x^2} (x^2 - y) \, dy \, dx = \frac{4}{5}.$ (Ans.)

Question 3 (a)

Ans to the Question no. 3

(a) $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-5+x^2+y^2}^{3-x^2-y^2} x \, dz\,dy\,dx$

integrating with respect to z we get

$= \int_0^2 \int_0^{\sqrt{4-x^2}} [xz]_{-5+x^2+y^2}^{3-x^2-y^2} \, dy\,dx$

$= \int_0^2 \int_0^{\sqrt{4-x^2}} \left(3x - x^3 - xy^2 + 5x - x^3 - xy^2\right) dy\,dx$

$= \int_0^2 \int_0^{\sqrt{4-x^2}} \left(8x - 2x^3 - 2xy^2\right) dy\,dx$

Interating with respef to y we get.

$= \int_0^2 \left[8xy - 2x^3y - 2x\frac{y^3}{3}\right]_0^{\sqrt{4-x^2}} dx$

$= \int_0^2 \left(8x\sqrt{4-x^2} - 2x^3\sqrt{4-x^2} - \frac{2x(4-x^2)^{3/2}}{3}\right) dx$

$\frac{4}{3}\int_0^2 x\left(4-x^2\right)^{3/2} dx$

$= -\frac{2}{3}\int_0^2 \left(4-x^2\right)^{3/2}(-2x)\,dx$

$= \frac{2}{3}\left[\frac{(4-x^2)^{5/2}}{5/2}\right]_0^2$

$= -\frac{4}{15}\left[0 - (4)^{5/2}\right] =$

$= \frac{128}{15}$

Ans.

Question 3 (b)

3(b) $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^1\int_0^{x^2} x\cos y \, dz\,dx\,dy$

Integrating with respect to z we get.

$$= \int_0^{\pi/4}\int_0^1 \left[xz\cos y\right]_0^{x^2} dx\,dy$$

$$= \int_0^{\pi/4}\int_0^1 \left(x^3\cos y\right) dx\,dy$$

Integrating with respect to x

$$= \int_0^{\frac{\pi}{4}} \left[\frac{x^4\cos y}{4}\right]_0^1 dy$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{\cos y}{4}\right) dy$$

integratee with respect to y we get.

$$= \frac{1}{4}\left[\sin y\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[\sin\frac{\pi}{4}\right]$$

$$= \frac{1}{4}\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2}}{8}$$

Ans:

Question 4(a)

Ans to the Q No 4

a) $\csc(y)\,dx + \sec^2(x)\,dy = 0$

Let, y be the dependent variable, devide by dx $\Rightarrow$

$\csc(y) + \sec^2(x)\frac{dy}{dx} = 0$

Substituting $\frac{dy}{dx}$ with y' $\Rightarrow$

$\csc(y) + \sec^2(x)y' = 0$

Substracting $\csc(y)$ from both sides $\Rightarrow$

$\csc(y) + \sec^2(x)y' - \csc(y) = 0 - \csc(y)$

$\Rightarrow \sec^2(x)\,y' = -\csc(y)$

Deviding both sides by $\sec^2(x)$ $\Rightarrow$

$$\frac{\sec^2(x)\,y'}{\sec^2(x)} = \frac{-\csc(y)}{\sec^2(x)}$$

$$\Rightarrow y' = \frac{-\csc(y)}{\sec^2(x)}$$

Deviding both sides by $\csc(y) \Rightarrow$

$$\frac{y'}{\csc(y)} = \frac{\frac{-\csc(y)}{\sec^2(x)}}{\csc(y)}$$

$$\Rightarrow \frac{y'}{\csc(y)} = -\frac{1}{\sec^2(x)} \quad \left[\because N(y) = \frac{1}{\csc(y)}\,, \; M(x) = \frac{-1}{\sec^2(x)}\right]$$

$$\Rightarrow \frac{1}{\csc(y)} y' = \frac{-1}{\sec^2(x)}$$

$$\Rightarrow \int \frac{1}{\csc(y)}\, dy = \int -\frac{1}{\sec^2(x)}\, dx$$

$\left[\because N(y) \cdot y' = M(x),\; y' = \frac{dy}{dx} \text{ then } \int N(y)\, dy = \int M(x)\, dx \text{ up to constant}\right]$

Integrating each side of the equation $\Rightarrow$

$$\int -\frac{1}{\sec^2(x)}\, dx = -\frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + c_1 \quad (1)$$

① $\Rightarrow$ L.H.S $\Rightarrow$

$$\Rightarrow \int \frac{1}{sec^2(x)}\,dx$$

$$\Rightarrow \frac{1}{sec^2 x}$$

$$\Rightarrow \frac{1}{\left(\frac{1}{\cos(x)}\right)^2}$$

$$\Rightarrow \cos^2(x)$$

$$\Rightarrow \int \frac{1+\cos 2(x)}{2}\,dx$$

$$= -\int \frac{1+\cos 2x}{2}\,dx$$

$$= -1/2 \int 1+\cos(2x)\,dx$$

$$= -1/2 \int 1\,dx + \int \cos(2x)\,dx$$

$$= -1/2 \cdot x + \int \cos(2x)\,dx \quad \text{— ②}$$

Here,

$u = 2x$

$\frac{du}{dx} = 2$

$\Rightarrow du = 2dx$

$\Rightarrow dx = 1/2\, du$

Again, from eq $\Rightarrow$

$1/2 \int \cos(u)\, du$

$\Rightarrow 1/2 \sin(u)$

$\Rightarrow 1/2 \sin 2(x) \Rightarrow$ (3)

$[\because u = 2x]$

We know,

$\int f(g(x)) \cdot g(x) dx = \int f(x)\, dx$

Now,

from (2) $\Rightarrow$

$-1/2\, x + \int \cos(2x)\, dx$

$= -1/2\, x + \int \cos(u)\, du$ [from (3)]

$= -1/2\, (x + 1/2 \sin 2x) + C1$

Ans

Question 4(b)

Ans to the Q No 4

b) $$x\frac{dy}{dx} + 4y = x^3 - x$$

substitute $\frac{dy}{dx}$ with $y' \Rightarrow$

$$xy' + 4y = x^3 - x$$

Deviding both sides by $x \Rightarrow$

$$\frac{xy'}{x} + \frac{4y}{x} = \frac{x^3}{x} - \frac{x}{x}$$

$$\Rightarrow y' + \frac{4y}{x} = x^2 - 1$$

$$\Rightarrow y' + \frac{4}{x}\, y = x^2 - 1$$

~~Let, [illegible]~~

Now, $\mu'(x) = \mu(x)P(x)$

Devide both sides by $\mu(x)$

$$\frac{\mu'(x)}{\mu(x)} = \frac{\mu(x)P(x)}{\mu(x)}$$

$$\Rightarrow \frac{\mu'(n)}{\mu(n)} = P(n)$$

$$\Rightarrow (\ln(\mu(n)))' = P(n) \quad \left[\because (\ln(\mu(n)))' = \frac{\mu'(n)}{\mu(n)}\right]$$

$$\Rightarrow (\ln(\mu(n)))' = 4/n \quad [\because P(n) = 4/n]$$

$$\Rightarrow \ln(\mu(n)) = \int 4/n \, dn$$

$$= 4 \cdot \int 1/n \, dn$$

$$= 4 \ln(n)$$

$$= 4 \ln(n) + c_1$$

$$\Rightarrow \mu(n) = e^{4\ln(n) + c_1} \quad \left[\because \ln(\mu(n)) = 4\ln(n) + c_1 = \mu(n)\right]$$

$$\left[\because \log_a(b) = c \text{ then } b = a^c\right]$$

$$\Rightarrow \mu(n) = e^{4\ln(n)} e^{c_1} \quad \left[\because a^{b+c} = a^b a^c \text{ exponent rule}\right]$$

$$\mu(x) = e^{4\ln(x)}$$

$$= \left(e^{\ln(x)}\right)^4 \quad \left[\because a^{bc} = (a^b)^c \text{ exponent rule}\right]$$

$$= x^4 \quad \left[\because e^{\ln(x)} = x\right]$$

$$\mu(x) = x^4 e^{c1}$$

$$= e^{c1} \cdot x^4$$

Now, Multiplying the integration factor $\mu(x)$ and the eqn $\Rightarrow$

$$y' + 4/x \, y = x^2 - 1$$

Multiplying both sides by integration factor $\Rightarrow$

$$y' x^4 + \frac{4}{x} y x^4 = x^2 x^4 - 1 \cdot x^4$$

$$\Rightarrow x^4 y' + 4x^3 y = x^6 - x^4$$

$$\Rightarrow (x^4 y)' = x^6 - x^4 \quad [\because \text{product rule}]$$

$$\Rightarrow x^4 y = \int x^6 - x^4 \, dx = \left(\frac{x^{6+1}}{6+1} - \frac{x^{4+1}}{4+1}\right.$$

$$\Rightarrow x^4 y = \frac{x^7}{7} - \frac{x^5}{5} + c_1$$

$$\Rightarrow \frac{x^4 y}{x^4} = \frac{\frac{x^7}{7}}{x^4} - \frac{\frac{x^5}{5}}{x^4} + \frac{c_1}{x^4}$$

[devide both side by x^4]

$$\Rightarrow y = \frac{x^3}{7} - \frac{x}{5} + \frac{c_1}{x^4}$$

$$\therefore y = \frac{x^3}{7} - x/5 + c_1/x^4$$ Ans.

Question 5(a)

Date: / /

☐Sat ☐Sun ☐Mon ☐Tue ☐wed ☐Thu ☐Fri

Theme:

5(a) $(4t^3y - 15t^2 - y)dt + (t^4 + 3y^2 - t)dy = 0$

This equation is in the form of $Mdt + Ndy = 0$

Let,

$M = (4t^3y - 15t^2 - y)$,

& $N = (t^4 + 3y^2 - t)$.

$\frac{\partial M}{\partial y} = (4t^3 - 0 - 1)$

$= 4t^3 - 1$

$\frac{\partial N}{\partial t} = 4t^3 + 0 - 1$

$= 4t^3 - 1$

So, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$ [exact]

Let $U = \int M dt$

$U \Rightarrow \int (4t^3y - 15t^2 - y) dt$

$\Rightarrow \left(\frac{4t^4}{4} y - \frac{15t^3}{3} - yt\right)$

$\Rightarrow$ unreachable.

Theme:

Date: / /

□Sat □Sun □Mon □Tue □wed □Thu □Fri

$$\frac{\partial u}{\partial y} = (t^4 - 0 - t)$$

$$= t^4 - t$$

$$\phi(y) = N - \frac{\partial u}{\partial y}$$

$$= (t^4 + 3y^2 - t) - (t^4 - t)$$

$$= (t^4 + 3y^2 - t - t^4 + t)$$

$$= 3y^2$$

$$\int \phi(y)\,dy = \int 3y^2\,dy$$

$$= \frac{3y^3}{3}$$

$$= y^3$$

General solution is $u + \int \phi(y)\,dy = C$.

$$(t^4 y - 5t^3 - yt) + y^3 = C$$

$$t^4 y - 5t^3 - yt + y^3 = C$$

$$(t^4 - t)y - 5t^3 + y^3 = C$$

[Ans]

Question 5(b)

Date: / /

□Sat □Sun □Mon □Tue □wed □Thu □Fri

Theme:

5 (b) $6xy\,dx + (4y + 9x^2)\,dy = 0$

Let,

$M = 6xy.$

$N = 4y + 9x^2$

$$\frac{\partial M}{\partial y} = 6x \;;\; \frac{\partial N}{\partial x} = 18x.$$

Now, $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{6xy}(18x - 6x)$

$$= \frac{12x}{6xy}$$

$$= \frac{2}{y}$$

$$= f(y)$$

$$e^{\int f(y)dy} = e^{\int \frac{2}{y} dy} = y^2$$

Now,

$\Rightarrow 6xy^3\,dx + (4y^3 + 9x^2y^2)\,dy = 0$ [Multiply by y^2 with the given equation]

So General ~~equation~~ solution:

$$\int_x 6xy^3\,dx + \int 4y^3\,dy = C$$

$$\Rightarrow 6\frac{x^2}{2}\cdot y^3 + 4\frac{y^4}{4} = C$$

$$\Rightarrow 3x^2y^3 + y^4 = C.$$

[Ans]

Question 6(a)

Answer to the question number-6

ⓐ We have the second order differential equation,

$$y''' - 5y'' + 3y' + 9y = 0 \quad \text{—— (i)}$$

For,

linear homogeneous differential equation,

$$a_n y^{(n)} + \cdots + a_1 y + a_0 y = 0$$

let,

assume a solution of the form $e^{\gamma t}$

Rewriting the equation with $y = e^{\gamma t}$

$$((e^{\gamma t}))''' - 5((e^{\gamma t}))'' + 3((e^{\gamma t}))' + 9e^{\gamma t} = 0 \quad \text{—— (ii)}$$

Here,

$$(e^{\gamma t})''' = (e^{\gamma t}\gamma)''$$
$$= (\gamma^2 e^{\gamma t})'$$
$$= \gamma^3 e^{\gamma t}$$

$$\therefore (e^{\gamma t})''' = \gamma^3 e^{\gamma t}$$
$$(e^{\gamma t})'' = \gamma^2 e^{\gamma t}$$
$$(e^{\gamma t})' = \gamma e^{\gamma t}$$

Putting all the values in (ii)

$$\gamma^3 e^{\gamma t} - 5\gamma^2 e^{\gamma t} + 3e^{\gamma t}\gamma + 9e^{\gamma t} = 0$$

$$e^{\gamma t}(\gamma^3 - 5\gamma^2 + 3\gamma + 9) = 0$$

Since $e^{\gamma t} \neq 0$, we have

$$\gamma^3 - 5\gamma^2 + 3\gamma + 9 = 0$$

$$(\gamma+1)\frac{\gamma^3 - 5y^2 + 3\gamma + 9}{\gamma+1} = 0$$

$$(\gamma+1)\left(\gamma^2 + \frac{-6\gamma^2 + 3y + 9}{\gamma+1}\right) = 0$$

$$(r+1)\left(r^2-6r+\frac{9r+9}{r+1}\right)=0$$

$$(r+1)(r^2-6r+9)=0$$

$$(r+1)(r-3)(r-3)=0$$

$$(r+1)(r-3)^2=0$$

Using the zero factor principle if $ab=0$ then $a=0$ or $b=0$

$r+1=0$ or, $r-3=0$

$\therefore r=-1$ or, $r=3$

For non repeated real root r, the general solution takes the form

$y = c_1 e^{rt}$

for, $r=-1$: $c_1 e^{-t}$

for, $r=3$: $c_2 e^{3t} + c_3 t e^{3t}$

~~From~~ Putting the value,

$$y = c_1 e^{-t} + c_2 e^{3t} + c_3 t e^{3t} \quad \text{(Ans)}$$

Question 6(b)

Answer to the question number 6

(b) We have the second order differential equation,

$$y'' - 10y + 25y = 0 \quad \text{———} (i)$$

$$y(0) = 1$$
$$y(1) = 0$$

Second order linear homogeneous differential equation has the form of,

$$ay'' + by' + cy = 0$$

lets,
assume a solution of the form $e^{\gamma t}$

Rewriting the equation with $y = e^{\gamma t}$

$$\therefore ((e^{\gamma t}))'' - 10((e^{\gamma t}))' + 25e^{\gamma t} = 0 \quad \text{———} (ii)$$

Here,

$$((e^{\gamma t}))'' = (e^{\gamma t}\gamma)'$$
$$= \gamma^2 e^{\gamma t}$$

$$(e^{\gamma t}\gamma)' = e^{\gamma t}\gamma$$

$$\therefore (e^{\gamma t})'' = \gamma^2 e^{\gamma t}$$

Putting the values ~~of e~~ in (ii)

$$\gamma^2 e^{\gamma t} - 10e^{\gamma t}\gamma + 25e^{\gamma t} = 0$$

$$e^{\gamma t}(\gamma^2 - 10\gamma + 25) = 0$$

Since $e^{\gamma t} \neq 0$, we have

$$\gamma^2 - 10\gamma + 25 = 0$$
$$\gamma^2 - 5\gamma - 5\gamma + 25 = 0$$
$$\gamma(\gamma - 5) - 5(\gamma - 5) = 0$$

$(\gamma-5)\cdot(\gamma-5)=0$

$(\gamma-5)^2=0$

$\gamma=5$

The solution to the quadratic equation is
$\gamma=5$ with multiplicity of 2

For one real root γ.

The general form, $y=c_1e^{\gamma t}+c_2te^{\gamma t}$

$\therefore\ y=c_1e^{5t}+c_2te^{5t}$

Now, $t=0$ as $y(0)=c_1e^{5\cdot0}+c_2 0e^{5\cdot0}$

and use initial condition $y(0)=1$

$1=c_1e^{5\cdot0}+c_2\cdot0\cdot e^{5\cdot0}$

or, $1=c_1+0$

$\therefore c_1=1$

for, $y=c_1e^{5t}+c_2te^{5t}$

$y=1\cdot e^{5t}+c_2te^{5t}$ $\quad[c_1=1]$

$y=e^{5t}+c_2e^{5t}t$

Now,

$t = 1$

$\therefore y(1) = e^{5 \cdot 1} + c_2 e^{5 \cdot 1} \cdot 1$

$0 = e^{5 \cdot 1} + c_2 e^{5 \cdot 1} \cdot 1$ [$y(1) = 0$]

$e^{5 \cdot 1} + c_2 e^{5 \cdot 1} \cdot 1 = 0$

$e^5 + e^5 c_2 = 0$

$e^5 + e^5 c_2 - e^5 = 0 - e^5$ [substracting e^5 from both side]

$e^5 c_2 = -e^5$

$\frac{e^5 c_2}{e^5} = \frac{-e^5}{e^5}$ [Divide e^5 from both sides]

$c_2 = -1$

for,

$y = e^{5t} + c_2 e^{5t} t$

$y = e^{5t} + (-1) e^{5t} t$

$y = e^{5t} - e^{5t} t$ (Ans.)