MAT120
Monthly Assignment

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Question 1:

Answer to the question number 1

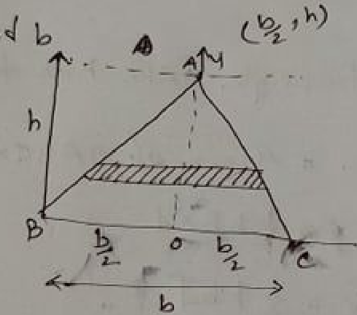
1. Given that

base b and height h

area $\frac{1}{2}hb$

So,

Ground b



Now,

equation of line AC 'straight line $(0,0)$ to $(\frac{b}{2}, h)$

$$\frac{y-0}{0-h} = \frac{x-0}{0-\frac{b}{2}}$$

$$\Rightarrow \frac{y}{-h} = \frac{x}{-\frac{b}{2}}$$

$$\Rightarrow \frac{y}{h} = \frac{2x}{b}$$

$$\Rightarrow y = \frac{2xh}{b}$$

So, area of $\Delta ABC = \Delta OAC \times 2$

$$= 2 \cdot \int_0^{b/2} \frac{2xh}{b}\, dx$$

$$= 2 \cdot \frac{2h}{b} \cdot \int_0^{b/2} x\, dx$$

$$= \frac{4h}{b} \left[\frac{x^2}{2}\right]_0^{b/2}$$

$$= \frac{4h}{b} \cdot \frac{1}{2}\left(\frac{b^2}{4} - 0\right)$$

$$= \frac{\cancel{4}h}{b} \cdot \frac{b^2}{\cancel{8}\,2}$$

$$= \frac{hb}{2} \text{ (Ans.)}$$

2. Let,
the triangle ΔABC has depth d

So, volume = Area of $\Delta ABC \times d$

$$= \frac{1}{2} hb \times d$$

$$\therefore v = \frac{1}{2} hbd$$

We know,

$$P = \frac{m}{v}$$

$$m = Pv$$

$$m = P \cdot \frac{1}{2} hbd$$

$$\therefore m = \frac{1}{2} Phbd \text{ (Ans)}$$

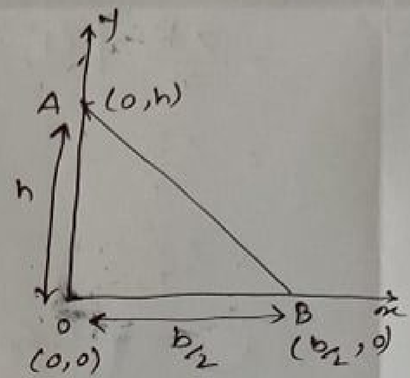
here,
P = density
m = mass
V = volume

[Putting value of v]

3. Let,

height = h

ground = b/2



So, the equation of the straight line AB is

(0, h) to (b/2, 0)

$$\frac{y-h}{h-0} = \frac{x-0}{0-b/2}$$

$$\frac{y-h}{h} = \frac{x}{-b/2}$$

$$x = -\frac{b(y-h)}{2h}$$

$$x = \frac{b(h-y)}{2h}$$

Now, Volume around y-axis

[applying disk washer method]

$$\int_{y_1}^{y_2} \pi \cdot [f(y)]^2 dy$$

$$= \int_0^h \pi \cdot \left[\frac{b(h-y)}{2h}\right]^2 dy$$

$$= \int_0^h \pi \frac{b^2(h-y)^2}{4h^2} dy$$

$$= \frac{\pi b^2}{4h^2} \int_0^h (h^2+y^2-2hy)dy$$

$$= \frac{\pi b^2}{4h} \left[h^2y + \frac{y^3}{3} - 2h \cdot \frac{y^2}{2}\right]_0^h$$

$$= \frac{\pi b^2}{4h^2} \left[h^2y + \frac{y^3}{3} - hy^2\right]_0^h$$

$$= \frac{\pi b^2}{4h^2}\left[\left(h^2\cdot h + \frac{h^3}{3} - h\cdot h^2\right) - 0\right]$$

$$= \frac{\pi b^2}{4h^2}\cdot\frac{h^3}{3}$$

$$= \frac{\pi b^2 h}{12}$$

$$\therefore v = \frac{\pi b^2 h}{12} \text{ (Ans.)}$$

4. Let,

Pyramid is a collection of squares and triangles,

So,

$$\frac{l}{y} = \frac{B}{H}$$

$$l = \frac{B}{H}\cdot y$$

$$\therefore V_{slice} = \left(\frac{B}{H}\times y\right)^2 dy$$

$$\therefore Force_{slice} = (P \times V_{slice})$$

$$= \frac{1}{H}\times\left(\frac{B}{H}\times y\right)^2 dy$$

$$\therefore W_{slice} = F_{slice} \times \text{distance}$$

$$= \left[\frac{1}{H}\times\left(\frac{B}{H}\cdot y\right)^2 dy\right]\times[H-y]$$

$$= \left[\frac{B^2 y^2}{H^3}dy\right]\times[H-y]$$

$\therefore$ Total work done, $W = \int_{y=0}^{y=H} \left[\frac{B^2}{H^3} y^2 dy\right] [H - y]$

$$= \int_0^H \left(\frac{B \times y^2 dy}{H^3}\right) \times H - \int_0^H \frac{B^2 y^3}{H^3} dy$$

$$= \frac{B^2}{H^2} \times \int_0^H y^2 dy - \frac{B^2}{H^3} \int_0^H y^3 dy$$

$$= \frac{B^2}{H^2} \cdot \frac{1}{3} \cdot \left[y^3\right]_0^H - \frac{B^2}{H^3} \cdot \frac{1}{4} \left[y^4\right]_0^H$$

$$= \frac{B^2}{H^2} \cdot \frac{1}{3} \cdot H^3 - \frac{B^2}{H^3} \cdot \frac{1}{4} \cdot H^4$$

$$= \frac{B^2 H}{3} - \frac{B^2 H}{4}$$

$$= \frac{4 B^2 H - 3 B^2 H}{12} = \frac{B^2 H}{12}$$

$\therefore W = \frac{B^2 H}{12}$ (Ans)

Question 2:

Problem no. 2

(1) Given.
$$S = \int dt\left(\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - \frac{1}{2}m\omega^2x^2\right)$$
$$= \int\left(\frac{1}{2}m\left(\frac{dx}{dt}\right)^2\right)dt - \int\left(\frac{1}{2}m\omega^2x^2\right)dt.$$

finding Integral we get,
$$\int\left(\frac{1}{2}m\left(\frac{dx}{dt}\right)^2\right)dt$$
$$= \frac{1}{2}m\int\left(\frac{dx}{dt}\cdot\frac{dx}{dt}\right)dt$$

Let,
$$u = \frac{dx}{dt}$$
$$\cancel{\frac{du}{dt}} =$$
$$\Rightarrow \frac{du}{dx} = \frac{d^2x}{dt^2}$$

$$\frac{dv}{dt} = \frac{dx}{dt}$$
$$\Rightarrow \int\frac{dv}{dt}\,dt = \int\frac{dx}{dt}\,dt$$
$$\therefore V = x$$

We know that, $\int u\frac{du}{dt} = uv - \int v\frac{du}{dt}\,dt$

putting the values we get
$$\frac{1}{2}m\int_{t_2}^{t_1}\left(\frac{dx}{dt}\,\frac{dx}{dt}\right)dt = \frac{1}{2}m\left[\frac{dx}{dt}x - \int x\frac{d^2x}{dt^2}dt\right]_{t_2}^{t_1}$$
$$\Rightarrow \frac{1}{2}m\int\left(\frac{dx}{dt}\right)^2dt = -\frac{1}{2}m\int x\frac{d^2x}{dt^2}\,dt$$

Substituting back we get
$$S = -\frac{1}{2}m\int x\,\frac{d^2x}{dt^2}\,dt - \int\left(\frac{1}{2}m\omega^2x^2\right)dt$$
$$= \int dt\left(-\frac{1}{2}mx\frac{d^2x}{dt^2} - \frac{1}{2}m\omega^2x^2\right)$$

(Showed)

② Given
$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2}\frac{dx}{dt} + \omega^2 x \frac{dx}{dt} = 0$$ [multiplying both side with $\frac{dx}{dt}$]

$$\Rightarrow \frac{d}{dt}\left(\frac{dx}{dt}\right)\frac{dx}{dt} - \omega^2 x \frac{dx}{dt} = 0$$

$$\Rightarrow d\left(\frac{dx}{dt}\right)\frac{dx}{dt} + \omega^2 x\,dx = 0$$

Integrating both sides we get.

$$\int\left(\frac{dx}{dt}\right)d\left(\frac{dx}{dt}\right) + \int \omega^2 x\,dx = 0$$

$$\Rightarrow \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + C_1 + \frac{1}{2}\omega^2 x^2 + C_2 = 0$$

$$\Rightarrow \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}Kx^2 = -m(C_1 + C_2)$$

$$\Rightarrow \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}Kx^2 = E$$ $[E = -m(C_1 + C_2)]$

Showed.

(3) Given $\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}Kx^2 = E$

$$\Rightarrow \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}Kx^2 = E \quad \text{[algebric Manipulation]}$$

$$\Rightarrow m\left(\frac{dx}{dt}\right)^2 + Kx^2 = 2E$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 + \frac{K}{m}x^2 = \frac{2E}{m}$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2E}{m} - \frac{K}{m}$$

$$\Rightarrow \frac{dx}{dt} = \pm\sqrt{\frac{2E}{m} - \frac{K}{m}}$$

$$\Rightarrow \frac{dx}{\sqrt{\frac{2E}{m} - \frac{K}{m}x^2}} = \pm dt \quad \text{[Showed]}$$

(4) given,

$$\frac{dx}{\sqrt{\frac{2E}{m} - \frac{K}{m}x^2}} = \pm dt$$

Integrating both sides

$$\int \left(\frac{1}{\sqrt{\frac{2E}{m} - \frac{K}{m}x^2}} \right) dx = \pm \int dt$$

$$\Rightarrow \int \left(\frac{1}{\sqrt{\frac{K}{m}\left(\frac{2E}{K} - x^2\right)}} \right) dx = \pm t$$

$$\Rightarrow \sqrt{\frac{m}{K}} \int \left(\frac{1}{\sqrt{\left(\frac{2E}{K} - x^2\right)}} \right) = \pm t + C_1$$

Let,

$$x = \sqrt{\frac{2E}{K}} \cos\theta$$

$$\theta = \cancel{\text{arc}} \ \text{arc}\cos\left(x\sqrt{\frac{K}{2E}} \right)$$

$$dx = -\sqrt{\frac{2E}{K}} \sin\theta \, d\theta$$

So, $$\sqrt{\frac{m}{K}} \int \frac{-\sqrt{\frac{2E}{K}} \sin\theta \, d\theta}{\sqrt{\left(\frac{2E}{K} - \frac{2E}{K}\cos^2\theta\right)}} = \pm t + C_1$$

$$\Rightarrow \sqrt{\frac{m}{K}} \int \frac{\sqrt{\frac{2E}{K}} \sin\theta \, d\theta}{\sqrt{\frac{2E}{K}}\left(1 - \cos^2\theta\right)} = \pm t + c_1$$

$$= \int d\theta = \pm \sqrt{\frac{K}{m}} t + C_2 \quad \left[C_2 = C_1\sqrt{\frac{K}{m}}\right]$$

$$\Rightarrow \theta + C_3 = \pm \sqrt{\frac{K}{m}} t + C_2$$

$$\Rightarrow \text{accross}\left(x\sqrt{\frac{K}{2E}} \right) = \pm \sqrt{\frac{K}{m}} t + K \quad [K = C_2 - C_3]$$

$$\Rightarrow x(t) = \pm \sqrt{\frac{2E}{K}} \cos\left(\sqrt{\frac{K}{m}} t + K \right)$$

(5) Let, $f(x) = x$

$g(x) = x^2$

Given $A = \frac{d}{dx}$

Now, $\int g(x)\,(Af(x))$

$= \int x^2 \left(\frac{d}{dx} x\right) dx$

$= \int x^2 dx$

$= \frac{x^3}{3} + C$

Again, $\int (Ag(x))\, f(x)\, dx$

$= \int \left(\frac{d}{dx} x^2\right) \cancel{dx}\, x\, dx$

$= \int 2x \cdot x\, dx$

$= 2\int x^2 dx$

$= 2\frac{x^3}{3} + C$

$\int g(x)\,(Af(x))\,dx \neq \int (Ag(x))\, f(x)\, dx$

$\therefore$ $A = \frac{d}{dx}$ is not a hermitian operator.

(6) $$\int f(x)\,(X\, f(x))\,dx = \int (X\, f(x))\, f(x)\,dx$$

$Xf(x) = xf(x)$ falls under Same relation as above

therefore X is a hermitian Operator.

Now, $$\langle x\rangle = \int_v^v f(x)\,(X\,f(x))$$

Here, $f(x) = \psi(x) = a\sin(kx)$. And $X\psi(x) = x\psi(x)$.

Plugging $$\langle x\rangle = \int_v^v a\sin(kx)\, x a \sin(kx)\,dx$$
$$= a^2 \int x \sin^2(kx)\,dx$$

Now.
$$\int x \sin^2(kx)\,dx$$
$$= \int x\left(\frac{1}{2}\left(2\sin^2(kx)\right)\right)dx$$
$$= \int x\left(\frac{1}{2}\left(1-\cos(2kx)\right)\right)dx$$
$$= \int\left(\frac{x}{2} - \frac{x\cos(2kx)}{2}\right)dx$$
$$= \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos(2kx)\,dx.$$

Now Solving,
$$\int x\cos(2kx)\,dx$$

Let $$u = x$$
$$\frac{du}{dx} = 1$$

Then,
$$\frac{dv}{dx} = \cos(2kx)$$
$$\int \frac{dv}{dx}\,dx = \int \cos(2kx)\,dx$$
$$v = \int \frac{1}{2k}\sin(2kx)\,dx$$

Applying interegating by parts we parts.

$$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$$

Substituting $\int x \sin(2kx) = x\left[\frac{1}{2k}\sin(2kx)\right] - \int\left[\frac{1}{2k}\sin(2kx)\right] \cdot dx$

Solving, $\int \frac{1}{2k}\sin(2kx)\,dx$

$u \to 2kx$, $\frac{du}{dx} \to 2k \to dx = \frac{du}{2k}$

$$\frac{1}{4k^2}\int \sin(u)$$

$$= -\frac{1}{4k^2}\cos(u)$$

$$= -\frac{1}{4k^2}\cos(2kx)$$

Putting in integrals.

$$\frac{x\sin(2kx)}{2k} - \int \frac{1}{2k}\sin(2kx)$$

$$= \frac{x\sin(2kx)}{2k} + \frac{\cos(2kx)}{4k^2}$$

So, $\frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos(2kx)\,dx$

$$= \frac{x^2}{4} - \frac{x\sin(2kx)}{4k} - \frac{\cos(2kx)}{8k^2}$$

Finally,

$$a^2\int x\sin^2(kx)\,dx = \frac{a^2x^2}{4} - \frac{a^2x\sin(2kx)}{4k} - \frac{a^2\cos(2kx)}{8k}$$

bound of the integral are domain of x:

$$\langle x\rangle = a^2\int_0^L x\sin^2(kx)\,dx = \frac{a^2L^2}{4} - \frac{a^2L\sin(2kL)}{4k} - \frac{a^2\cos(2kL)}{8k^2}$$

(7) Given Condition. $\int_{v}^{v} (\psi(x))^2 dx = 1$

Here $\psi(x) = a \sin(kx)$

Now Solving, $\int a(\sin(kx))^2 dx$

$$= \int a^2 \sin^2(kx) dx$$

$$= a^2 \int \sin^2(kx) dx$$

by Substituting $u \to kx$, $\frac{du}{dx} \to k$, $dx \to \frac{du}{k}$

$$a^2 \int \frac{1}{k} \sin^2(u) du$$

$$\frac{a^2}{k} \int \sin^2(u)$$

by applying redulction formula.

$$\sin^n(u) = \frac{n-1}{n} \int \sin^{n-2}(u) du - \frac{\cos(u) \sin^{n-1}(u)}{n}$$

$n = 2$ therefore $\frac{a^2}{k} \left[\frac{1}{2} \int 1 du - \frac{\cos(u) \sin(u)}{2} \right]$

$$= \frac{a^2}{k} \left[\frac{u}{2} - \frac{\cos(u) \sin(u)}{2} \right]$$

Undoing Substitution, $a^2 \left[\frac{x}{2} - \frac{\cos(kx) \sin(kx)}{2k} \right]$

Domain x starts from 0 to the length of the box

$$a^2 \int_0^L \sin^2(kx)\,dx = 1$$

$$\Rightarrow a^2 \left[\frac{x}{2} - \frac{\cos(kx)\sin(kx)}{2k}\right]_0^L = 1$$

$$\Rightarrow a^2 \left[\frac{xk - \cos(kx)\sin(kx)}{2k}\right]_0^L = 1$$

$$\Rightarrow a^2 \left[\frac{Lk - \cos(kL)\sin(kL)}{2k}\right] = 1$$

$$\Rightarrow a = \sqrt{\frac{2k}{Lk - \cos(kL)\sin(kL)}}$$ Ans.

Question 3:

Q-3, P-1

problem 3

1

given, $\rho_v = \frac{r}{r^2+4}\{\theta^2+2\theta\}$.

Mass, $M_v = \iiint \rho_v(r,\theta)\, r\, dr\, d\theta\, d\phi$

$$M_v = \int_0^{\pi}\int_0^{2\pi}\int_0^{R_1} \frac{r\,\theta^2}{r^2+4}\, r\, dr\, d\theta\, d\phi$$

$$+ \int_0^{\pi}\int_0^{2\pi}\int_0^{R_1} 2\theta\, r\, dr\, d\theta\, d\phi.$$

$$= \int_0^{\pi}\int_0^{2\pi}\left[\frac{r^3}{3(r^2+4)}\right]_0^{R_1} \theta^2 dr\, d\theta\, d\phi +$$

$$2\int_0^{\pi}\int_0^{2\pi} \theta\left[\frac{r^2}{2}\right]_0^{R_1} d\theta\, d\phi.$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{R_1^3\,\theta^2}{3(R_1^2+4)}\, d\theta\, d\phi + 2\int_0^{\pi}\int_0^{2\pi} \theta\frac{R_1^2}{2}\, d\theta\, d\phi.$$

Q3 - P2

$$= \frac{R_1^3}{3(R_1^2+4)} \int_0^{\pi} \left[\frac{\theta^3}{3}\right]_0^{2\pi} d\varphi$$

$$+ R_1^2 \int_0^{\pi} \left[\frac{\theta^2}{2}\right]_0^{2\pi} d\varphi$$

$$= \frac{R_1^3}{9(R_1^2+4)} \int_0^{\pi} \frac{(2\pi)^3}{3} d\varphi$$

$$+ R_1^2 \int_0^{\pi} \frac{(2\pi)^2}{2} d\varphi.$$

$$= \frac{R_1^3}{27(R_1^2+4)} \cdot 8\pi^3 \left[\varphi\right]_0^{\pi}$$

$$+ R_1^2/2 \; 4\pi^2 \left[\varphi\right]_0^{\pi}$$

$$= \frac{8R_1^3\pi^4}{27(R_1^2+4)} + 2R_1^2\pi^3.$$

(Ans.)

Q3-P3

(2)

$$M_{mars} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R_2} \rho_{mars}\, dv.$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R_2} \rho_{mars}\, r^2 \sin\theta\, dr\, d\theta\, d\varphi.$$

$$= \rho_{mars} \int_0^{R_2} r^2\, dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi.$$

$$= 4/3\, \pi R_2^3\, \rho_{mars}.$$

3

$F = GMm/r^2$.

potential energy at a distance r is.

$$\Delta u = \int_r^\infty F dr.$$

$$= \int_r^\infty \frac{GMm}{r^2} dr.$$

$$= GMm \int_r^\infty dr/r^2.$$

$$= GMm \left[-1/r\right]_r^\infty$$

$$= \frac{GMm}{r}.$$

$$U_\infty - U_r = \frac{GMm}{r}.$$

$$\Rightarrow U_r = -\frac{GMm}{r}. \quad [U_\infty = 0]$$

Total energy $= E_p + E_k$.

$$= -\frac{GMm}{r} + \frac{1}{2} Mmv^2.$$

$$= -\frac{GMm}{r} + \frac{1}{2} M \left(\sqrt{\frac{Gm}{r}}\right)^2.$$

$$= -\frac{1}{2} \frac{GMm}{r}.$$

4

let assume, at any instant 't'
distance between them is 'x'.

$$\left(\frac{dx}{dt}\right)^2 = 2GM_v\left(\frac{1}{x} - \frac{1}{d}\right).$$

$$\text{speed, } v = \sqrt{2GM_v\left(\frac{1}{x} - \frac{1}{d}\right)}$$

$$v = \frac{dx}{dt},$$

$$\therefore dt = dx/v$$

$$T = \int dt = \int_d^{R_1+R_2} \frac{-dx}{v}.$$

$$= \int_d^{R_1+R_2} \frac{-dx}{\sqrt{2GM_v\left(\frac{1}{x} - \frac{1}{d}\right)}}.$$

$$T = \sqrt{\frac{d}{2GM_v}} \int \frac{-dx\sqrt{x}}{\sqrt{d-x}} \qquad \left| \begin{array}{l} x = d\sin^2\theta \\ dx = d\sin 2\theta \, d\theta \end{array} \right.$$

$$= \sqrt{\frac{d}{2GM_v}} \int \frac{-d\sin 2\theta \sqrt{d\sin^2\theta} \, d\theta}{\sqrt{d} \quad \cos\theta}$$

$$= -\sqrt{\frac{d^3}{2GM_v}} \int 2\sin^2\theta \, d\theta$$

$$= -\sqrt{\frac{d^3}{2GM_v}} \int (1-\cos 2\theta) \, d\theta$$

$$= -\sqrt{\frac{d^3}{2GM_v}} \left(\theta - \frac{\sin 2\theta}{2}\right).$$

$$T = -\sqrt{\frac{d^3}{2GM_v}} \left(\sin^{-1}\sqrt{x/d} - \sqrt{x/d}\sqrt{\frac{1-x}{d}}\right)_d^{R_1+R_2}.$$

$$= -\sqrt{\frac{d^3}{2GM_v}} \left[\sin^{-1}\sqrt{\frac{R_1+R_2}{d}} - \frac{\sqrt{R_1+R_2}\sqrt{d-R_1-R_2}}{d} - \sin^{-1} 1 - 0\right].$$

$$\therefore T = \sqrt{\frac{d^3}{2GM_v}} \left[-\sin^{-1}\sqrt{\frac{R_1+R_2}{d}} + \frac{\sqrt{R_1+R_2}\sqrt{d-R_1-R_2}}{d} + \frac{\pi}{2}\right]$$

Q3-P6

if $(R_1+R_2 \to 0)$

$$T = \sqrt{\frac{d^3}{2GM_v}}\left[-0+0+\pi/2\right]$$

$\therefore$ time to collide with vulcan is

$$T = \pi/2 \sqrt{\frac{d^3}{2GM_v}}.$$

Q3-P7

Question 4:

04/

(i) The given equation.

$$\int d\theta\theta = \int d\theta(\theta+\theta_0)$$

Where θ_0 is another grassman number.

The gennal Grassman Variable is:

$$f(\theta) = a + b\theta$$

Since, $\theta^2 = 0$, Thus the prove is satisfied if,

$\int d\theta b\theta = 0$. which implies, $\int d\theta = 0$ [Showed]

(ii) We know that Berezin integrational property is

$$f(\theta)\, d\theta = \int (f_0 + f_1 \theta) = f_1$$

Let $f(\theta) = a + b\theta$

Now, $\int d\theta(a+b\theta) = \int (a+b\theta)\, d\theta$

Applying the property we get.

$$\int (a+b\theta)\, d\theta = b \quad \text{— (i)}$$

we also know that,

$$\int d\theta f(\theta) = \frac{\partial}{\partial\theta} f(\theta)$$

Again let,

$$f(\theta) = a + b\theta)$$

Applying the formula we get

$$\int d\theta(a+b\theta) = \frac{\partial}{\partial\theta}(a+b\theta) \quad \text{— (ii)}$$

So, from (i) & (ii) we get, $\int d\theta(a+b\theta)\, b = \frac{\partial}{\partial\theta}$ [Show

Question 5:

Answer to the bonus Problem 2

1. Given that,

$$I = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+m^2}$$

Here, $d^3k = dk_1 dk_2 dk_3$ and $k^2 = k_1^2 + k_2^2 + k_3^2$

Consider the integral,

$$I_d = \int_{-\infty}^{\infty} e^{-x_1^2 - x_2^2 - \cdots x_d^2} dV_d$$

$$= \int_0^{\infty} e^{-r^2} dV_d(r) \quad \text{——— (1)}$$

Where dV_d is the volume element in cartesian co-ordinate

$$dV_d = dx_1 dx_2 \cdots dx_d$$

and

$$dV_d(r) = S(d)\, r^{d-1}\, dr$$

is the volume element is spherical co-ordinate where in eq (1) the integral is a product of identical gaussian of one vasiable,

$$I_d = \left[\int_{-\infty}^{\infty} e^{-x^2} dx\right]^d = (\sqrt{\pi})^d = \pi^{d/2} \quad \text{——— (ii)}$$

The second integral is eq (i)

$$I_d = \int_0^{\infty} e^{-r^2} S(d)\, r^{d-1}\, dr$$

$$= S(d) \int_0^{\infty} e^{-r^2} r^{d-1}\, dr$$

$$= \frac{S(d)}{2} \int_0^{\alpha} e^{-r^2} (r^2)^{\frac{d}{2}-1} \, dr^2$$

$$= \frac{S(d)}{2} \int_0^{\alpha} e^{-t} \, t^{\frac{d}{2}-1} dt$$

$$= \frac{S(d)}{2} \Gamma\left(\frac{d}{2}\right) \quad \text{———— (iii)}$$

On comparing (ii) and (iii)

$$S(d) = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}$$

2. Show that,

$$\left(\frac{\pi}{\lambda}\right)^{d/2} = \int d^d k \, e^{-\lambda x^2}$$

$\because \; d^d k = dk_1 \, dk_2 \cdots\cdots dk_d \; , \; k^2 = k_1^2 + k_2^2 + \cdots + k_d^2$

So,
$$I_d = \int_{-\alpha}^{\alpha} e^{-\lambda k_1^2 - \lambda k_2^2 - \cdots\cdots - \lambda k_d^2} \, d^d k$$

$$= \left[\int_{-\alpha}^{\alpha} e^{-\lambda k^2} dk \right]^n$$

Product of the individual gaussians

$$I_d = \left[\frac{\Gamma\left(\frac{1}{2}\right)}{(\lambda)^{\frac{1}{2}}} \right]^d = \left(\frac{\pi}{\lambda}\right)^{d/2}$$

By using gaussian integral $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

3. As we develop in (1)

$$\int d^d k\, e^{-\lambda k^2} \longleftrightarrow \int_0^{\alpha} S_d\, dk\, k^{d-1} e^{-\lambda k^2}$$

$$S_d \int_0^{\alpha} dk\, k^{d-1} e^{-\lambda k^2} = \frac{S_d}{2} \int_0^{\alpha} e^{-\lambda k^2} (k^2)^{\frac{d}{2}-1} dk^2$$

$$= \frac{S_d}{2} \int_0^{\alpha} du\, u^{d/2-1} e^{-\lambda u}$$

$\{$put $u = k^2\}$

4. By using gamma function

$$I_d = \frac{S_d}{2} \int_0^{\alpha} u^{d/2-1} e^{-\lambda u} du$$

$$= \frac{S_d}{2} \cdot \frac{\Gamma(d/2)}{(\lambda)^{d/2}}$$

$$= \frac{S_d\, \Gamma(\frac{d}{2})}{2\lambda^{d/2}}$$

⑥ From the previous equation,

$$I = -\frac{1}{2} \frac{\lambda (m^2)^{d/2 - 1}}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right)$$

$$= -\frac{1}{2} \frac{\lambda (m^2)^{d/2 - 1}}{(4\pi)^{d/2}} \left(-\frac{d}{2}\right)!$$

Here,

The function should be started from (-1)

So, the starting point has to be,

$$(-1)! = \left(-\frac{2}{2}\right)! \quad \left[\left(-\frac{d}{2}\right)!\right]$$

So, the lower limit will be 2.

And while solving we use a series,

therefore the number of d will be increased

So here, $d \geq 2$ [showed]

7. Here,

$$I = -\frac{1}{2}\frac{\lambda}{(4\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right)(m^2)^{d/2-1}$$

now, $d = 4$

So, $I = -\frac{1}{2}\frac{\lambda}{(4\pi)^{4/2}}\Gamma\left(1-\frac{4}{2}\right)(m^2)^{4/2-1}$

$$\Rightarrow I = -\frac{m^2}{2} \times \frac{\lambda}{4^2\pi^2}\Gamma(-1)$$

When,

$m \to 0$

$$I = \lim_{m\to 0} -\frac{m^2}{2} \times \frac{\lambda}{4^2\pi^2}\Gamma(-1)$$

$$= -\frac{0}{2} \times \frac{\lambda}{4^2\pi^2}\Gamma(-1)$$

$$= 0$$

$\therefore$ The function is Converges.