P.S-4

❀❀❀ **Formulas:**

* ① $f(t) = u(t) + iv(t)\ ;\quad t \in [a,b]$

$$\boxed{\int_a^b f(t).dt = \int_a^b u(t).dt + i.\int_a^b v(t).dt}$$

* ② $g(t) = p(t) + i.q(t)$

$$\boxed{\int_a^b f(t).g(t).dt = \int_a^b (up - vq).dt + i.\int_a^b (vp + uq).dt}$$

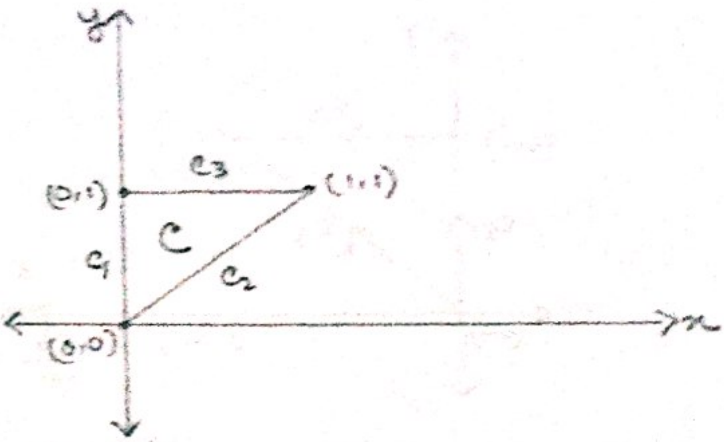
* ③ $$\boxed{Re\int_a^b f(t).dt = \int_a^b Re\big(f(t)\big).dt}$$

* ④ $$\boxed{\left|\int_a^b f(t).dt\right| \leq \int_a^b |f(t)|.dt}$$

———x———

Contour/ Path:

when c is constructed by joining finitely many smooth curves end to end, is called a contour.



Contour Integral:

$\boxed{\oint_c f(z).dz}$ → integration of $f(z)$ around the boundary of c.

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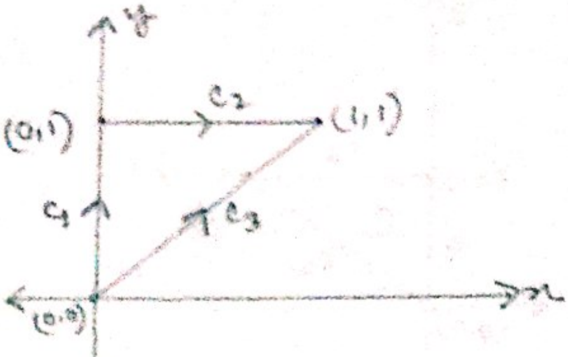
① Evaluate $\int_c f(z).dz$ over the closed triangular contour with vertices $(0,0)$, $(0,1)$, $(1,1)$ and where $f(z) = (y - x - 3ix^2)$.

$\Rightarrow$



Here,

$$\int_c f(z).dz = \int_{c_1} f(z)\, dz + \int_{c_2} f(z).dz - \int_{c_3} f(z).dz$$

Now, $f(z) = (y - x - 3ix^2)$.

C_1: $x = 0$,

$\therefore z = x + iy$

$\Rightarrow z = 0 + iy$

$\Rightarrow z = iy$

$\Rightarrow \frac{dz}{dy} = i$

$\therefore dz = i.dy$

$$\therefore \int_{c_1} f(z).dz = \int_0^1 (y - 0 - 3i.0^2).i.dy$$

$$= i\int_0^1 y \cdot dy$$

$$= i \cdot \left[\frac{y^2}{2}\right]_0^1$$

$$\therefore \int_{c_1} f(z) \cdot dz = (i/2)$$

c_2 : $y = 1$, $\therefore z = x + iy$

$$\Rightarrow z = (x + i)$$

$$\Rightarrow \frac{dz}{dx} = 1 + 0$$

$$\therefore dz = dx$$

$$\therefore \int_{c_2} f(z) \cdot dz = \int_0^1 (1 - x - 3ix^2) \cdot dx$$

$$= \int_0^1 dx - \int_0^1 x \cdot dx - 3i \cdot \int_0^1 x^2 \cdot dx$$

$$= [x]_0^1 - \left[\frac{x^2}{2}\right]_0^1 - 3i \cdot \left[\frac{x^3}{3}\right]_0^1$$

$$= 1 - \frac{1}{2} - 3i \cdot \frac{1}{3}$$

$$\therefore \int_{c_2} f(z) \cdot dz = \left(\frac{1}{2} - i\right)$$

C_3: $x=1,\ y=1 \quad \therefore x=y$

$\therefore z = x+iy$

$\Rightarrow z = x+ix$

$\Rightarrow \frac{dz}{dx} = 1+i$

$\therefore dz = (1+i).dx$

$$\therefore \int_{C_3} f(z).dz = \int_0^1 (x-x-3ix^2).(1+i).dx$$

$$= \int_0^1 (-3ix^2).(1+i).dx$$

$$= \int_0^1 (-3i.x^2 - 3i^2x^2).dx$$

$$= -3i.(1+i).\int_0^1 x^2.dx$$

$$= -3i.(1+i).\left[\frac{x^3}{3}\right]_0^1$$

$$= -3i.(1+i).\frac{1}{3}$$

$$= -i-i^2$$

$$\therefore \int_{C_3} f(z).dz = (1-i)$$

$$\therefore \int_c f(z).\,dz = i/2 + (1/2 - i) - (1-i)$$

$$= i/2 + 1/2 - i - 1 + i$$

$$= \frac{1}{2}.(i-1)$$

*****② Evaluate,** $\displaystyle\int_{(0,1)}^{(2,5)} \left[(3x+y).\,dx + (2y-x).\,dy\right]$

ⓐ along the curve, $y = (x^2+1)$

ⓑ along the line joining $(0,1)$ and $(2,5)$.

ⓒ along line from $(0,1)$ to $(0,5)$ and then from $(0,5)$ to $(2,5)$.

$\Rightarrow$ ⓐ along the curve, $y = (x^2+1)$.

Now, $y = (x^2+1)$ — ①

$$\Rightarrow \frac{dy}{dx} = 2x$$

$$\therefore dy = 2x.\,dx$$

$$\therefore \int_{(0,1)}^{(2,5)} \left[(3x+y)\cdot dx + (2y-x)\cdot dy\right]$$

$$= \int_0^2 \left[(3x+x^2+1)\cdot dx + (2x^2+2-x)\cdot 2x\, dx\right]$$

$$= \int_0^2 \left(3x+x^2+1+(2x^2+2-x)\cdot 2x\right)\cdot dx$$

$$= \int_0^2 \left(3x+x^2+1+4x^3+4x-2x^2\right)\cdot dx$$

$$= \int_0^2 (7x-x^2+4x^3+1)\cdot dx$$

$$= 7\cdot\int_0^2 x\, dx - \int_0^2 x^2\cdot dx + 4\cdot\int_0^2 x^3\cdot dx + \int_0^2 dx$$

$$= 7\cdot\left[\frac{x^2}{2}\right]_0^2 - \left[\frac{x^3}{3}\right]_0^2 + 4\cdot\left[\frac{x^4}{4}\right]_0^2 + \left[x\right]_0^2$$

$$= 7\cdot\left(\frac{2^2}{2}\right) - \left(\frac{2^3}{3}\right) + 4\left(\frac{2^4}{4}\right) + 2$$

$= 14 - \frac{8}{3} + 16 + 2$

$= \left(\frac{88}{3}\right)$ (ans)

(b) along the line joining (0,1) and (2,5).

$\Rightarrow$ the equation of line, $\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2}$

$\Rightarrow \frac{x-0}{0-2} = \frac{y-1}{1-5}$

$\Rightarrow \frac{x}{-2} = \frac{y-1}{-4}$

$\Rightarrow -4x = -2y+2$

$\Rightarrow -2x = -y+1$

$\Rightarrow y = (2x+1)$ —— (1)

$\Rightarrow \frac{dy}{dx} = 2.$

$\therefore dy = 2 \cdot dx$

so, $$\int_{(0,1)}^{(2,5)} \left[(3x+y).dx + (2y-x).dy\right]$$

$$= \int_0^2 \left[(3x+2x+1).dx + (4x+2-x).2.dx\right]$$

$$= \int_0^2 \left[3x+2x+1+8x+4-2x\right].dx$$

$$= \int_0^2 (11x+5).dx$$

$$= 11.\int_0^2 x.dx + 5.\int_0^2 .dx$$

$$= 11.\left[\frac{x^2}{2}\right]_0^2 + 5.\left[x\right]_0^2$$

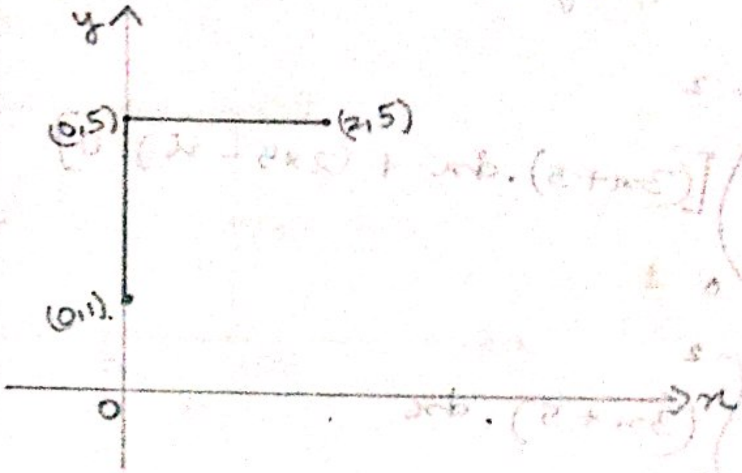
$$= 11.\left(\frac{2^2}{2}\right) + 5\times 2$$

$$= 11\times 2 + 10$$

$$= 32$$

(c) along line from (0,1) to (0,5) and then from (0,5) to (2,5).

⇒ Here,



Along line from (0,1) to (0,5),

we get, $x=0$,

$\therefore dx=0$

$$\therefore \int_1^5 [(0+y)\cdot 0 + (2y-0)\cdot dy]$$

$$= 2\cdot\int_1^5 y \cdot dy$$

$$= 2\cdot\left[\frac{y^2}{2}\right]_1^5$$

$$= 5^2 - 1^2$$

$$= 24$$

Again, along the line from $(0,5)$ to $(2,5)$,

we get, $y=5$

$\therefore dy = 0$

So,
$$\int_0^2 [(3x+5).dx + (2\times5 - x).0]$$

$$= \int_0^2 (3x+5).dx$$

$$= \left[3.\left(\frac{x^2}{2}\right) + 5x\right]_0^2$$

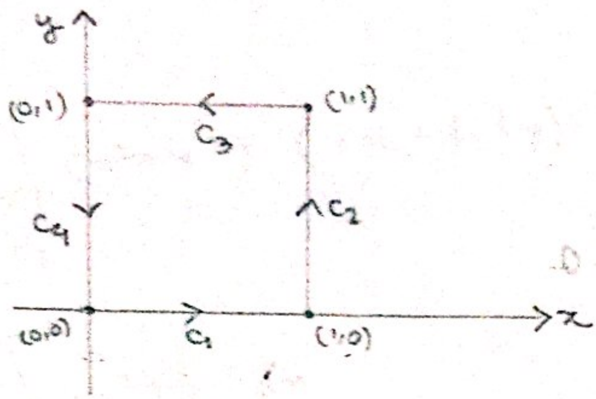
$$= 3\left(\frac{2^2}{2}\right) + 5\times2$$

$$= 16$$

$\therefore$ The required value $= (16+24) = 40$ (Ans)

***(4) Evaluate $\oint_C |z|^2 . dz$ around the square with (0,0), (0,1), (1,1), (1,0).

$\Rightarrow$



we know,

$$z = (x+iy) \quad \text{and} \quad |z|^2 = (x^2+y^2) \text{ --- (I)}$$

$$\Rightarrow \frac{dz}{dx} = 1 + i.\frac{dy}{dx}$$

$$\therefore dz = (dx + i.dy) \text{ --- (II)}$$

Now,

C_1: $y = 0$,

$$\Rightarrow \frac{dy}{dz} = 0$$

$$\therefore dy = 0$$

So, $$\int_{C_1} f(z).dz = \int_0^1 (x^2+y^2).(dx+i.dy) = \int_0^1 x^2.dx$$

$$= \left[\frac{x^3}{3}\right]_0^1$$

$$\therefore \int_{C_1} |z|^2 . dz = \left(\tfrac{1}{3}\right)$$

C_2: $\quad x = 1$

$$\Rightarrow \frac{dx}{dz} = 0$$

$$\therefore \quad dx = 0$$

$$\therefore \int_{C_2} |z|^2 . dz = \int_0^1 (x^2+y^2) . (dx + i . dy)$$

$$= \int_0^1 (1+y^2)(0 + i . dy)$$

$$= \int_0^1 (1+y^2) . i . dy$$

$$= i . \left[\int_0^1 dy + \int_0^1 y^2 . dy\right]$$

$$= i . \left[[y]_0^1 + \left[\frac{y^3}{3}\right]_0^1\right]$$

$$= i\left(1 + \tfrac{1}{3}\right)$$

$$\therefore \int_{C_2} |z|^2 . dz = \left(\frac{4i}{3}\right)$$

C_3: $y = 1$

$\Rightarrow \frac{dy}{dx} = 0$

$\therefore dy = 0$

$$\therefore \int_{C_3} |z|^2 \cdot dz = \int_1^0 (x^2+y^2) \cdot (dx + i \cdot dy)$$

$$= \int_1^0 (x^2+1) \cdot dx$$

$$= \int_1^0 x^2 \cdot dx + \int_1^0 dx$$

$$= \left[\frac{x^3}{3}\right]_1^0 + \left[x\right]_1^0$$

$$= -\frac{1}{3} - 1$$

$$\therefore \int_{C_3} |z|^2 \cdot dz = -\frac{4}{3}$$

C_4 : $x = 0$

$\Rightarrow \frac{dx}{dz} = 0$

$\therefore dx = 0$

so,

$$\int_{C_4} |z|^2 . dz = \int_1^0 (x^2+y^2) . (dx + i.dy)$$

$$= \int_1^0 y^2 . i . dy$$

$$= i . \left[\frac{y^3}{3}\right]_1^0$$

$$\therefore \int_{C_4} |z|^2 . dz = \left(\frac{-i}{3}\right)$$

$$\therefore \oint_C |z|^2 . dz = \int_{C_1} |z|^2 . dz + \int_{C_2} |z|^2 . dz + \int_{C_3} |z|^2 . dz + \int_{C_4} |z|^2 . dz$$

$$= 1/3 + \frac{4i}{3} - 4/3 - i/3$$

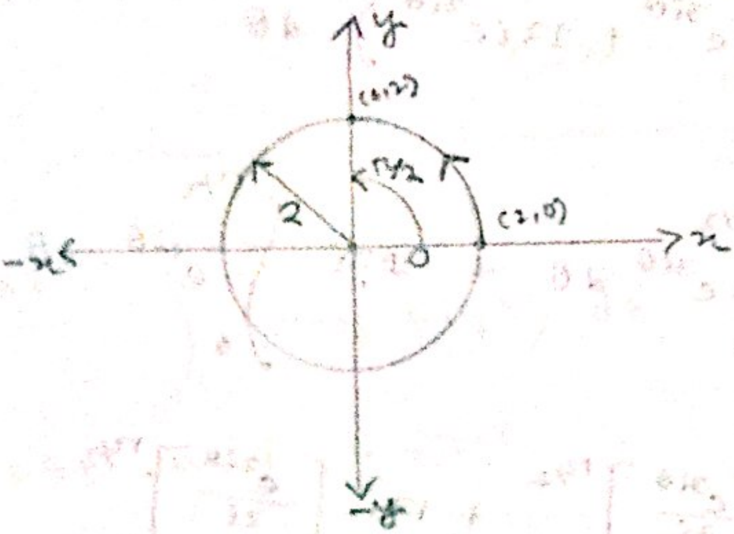
$$= (-1+i)$$

(Ans)

Evaluate $\oint_c (z^2+3z).dz$

(a) along the circle $|z|=2$ from $(2,0)$ to $(0,2)$ in a counter clockwise direction.

$\Rightarrow$



Now, $|z|=2$

$\Rightarrow z = 2.e^{i\theta}$ ——— (i)

$\Rightarrow \frac{dz}{d\theta} = 2.i.e^{i\theta}$

$\therefore dz = 2i.e^{i\theta}.d\theta$ ——— (ii)

$$\therefore \oint_c (z^2+3z).dz = \int_0^{\pi/2} \{(2e^{i\theta})^2 + 3.2e^{i\theta}\}.2i.e^{i\theta}.d\theta$$

$$= \int_0^{\pi/2} (4.e^{2i\theta} + 6.e^{i\theta}) . 2.i\, e^{i\theta} . d\theta$$

$$= \int_0^{\pi/2} (8ie^{3i\theta} + 12ie^{2i\theta}) . d\theta$$

$$= 8i. \int_0^{\pi/2} e^{3i\theta} . d\theta + 12i. \int_0^{\pi/2} e^{2i\theta} . d\theta$$

$$= 8i. \left[\frac{e^{3i\theta}}{3i}\right]_0^{\pi/2} + 12i. \left[\frac{e^{2i\theta}}{2i}\right]_0^{\pi/2}$$

$$= \frac{8i}{3i} . (e^{3\pi/2 . i} - e^0) + \frac{12i}{2i} . (e^{\pi i} - e^0)$$

$$= \frac{8}{3} . (e^{\frac{3\pi}{2}.i} - 1) + 6 . (e^{\pi i} - 1)$$

$$= \frac{8}{3} . \left[\cos\left(\frac{3\pi}{2}\right) + i . \sin\left(\frac{3\pi}{2}\right) - 1\right] + 6 . \left[\cos\pi + i . \sin\pi - 1\right]$$

$$= \frac{8}{3} . \left[0 + i(-1) - 1\right] + 6 . \left[-1 + i.0 - 1\right]$$

$$= \frac{8}{3} (-i - 1) - 12$$

$$= \frac{-8i-8-36}{3}$$

$$= \frac{-44-8i}{3}$$

$$= -\frac{1}{3}\cdot(44+8i)$$

(Ans)

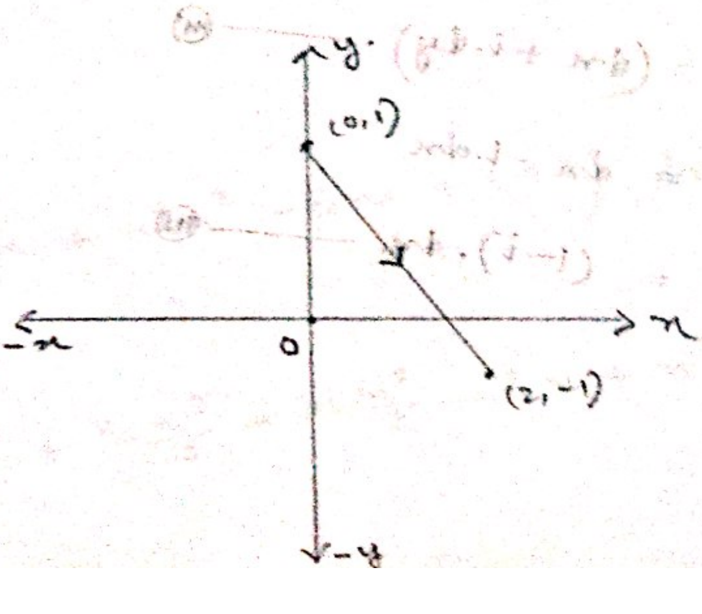
5. (a) Evaluate, $\int_{i}^{2-i} (3xy+iy^2).dz$ along the line joining $z=i$ and $z=2-i$.

$\Rightarrow$ Here, $z=i$ and $z=2-i$

$\therefore x=0$ $\quad\therefore x=2$

$\therefore y=1$ $\quad\therefore y=-1$

$\therefore z=(0,1)$ $\quad\therefore z=(2,-1)$



Equation of the line joining $(0,1)$ and $(2,-1)$,

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2}$$

$$\Rightarrow \frac{x-0}{0-2} = \frac{y-1}{1-(-1)}$$

$$\Rightarrow \frac{x}{-2} = \frac{y-1}{2}$$

$$\Rightarrow x = -y+1$$

$$\Rightarrow y = (-x+1) \quad \text{—— (I)}$$

$$\Rightarrow \frac{dy}{dx} = -1$$

$$\therefore dy = -dx \quad \text{—— (II)}$$

Now,

$$z = x+iy$$

$$\Rightarrow \frac{dz}{dx} = 1 + i.\frac{dy}{dx}$$

$$\Rightarrow dz = (dx + i.dy)$$

$$\Rightarrow dz = dx - i.dx$$

$$\therefore dz = (1-i).dx \quad \text{—— (III)}$$

So, $\int_0^2 \left[3x(-x+1) + i(-x+1)^2\right](1-i)\,dx$

$$= \int_0^2 \left[(-3x^2+3x) + i\cdot(x^2-2x+1)\right](1-i)\,dx$$

$$= \int_0^2 (-3x^2+3x+ix^2-2xi+i)\cdot(1-i)\,dx$$

$$= \int_0^2 (-3x^2+3x+ix^2-2xi+i+3ix^2-3xi-i^2\cdot x^2+2xi^2 - i^2)\cdot dx$$

$$= \int_0^2 (-3x^2+3x+i\cdot x^2-2xi+i+3ix^2-3xi+x^2-2x+1)\,dx$$

$$= \int_0^2 (-2x^2+x+4ix^2-5xi+i+1)\cdot dx$$

$$= \left[-2\cdot\frac{x^3}{3} + \frac{x^2}{2} + 4i\cdot\frac{x^3}{3} - 5i\cdot\frac{x^2}{2} + i\cdot x + x\right]_0^2$$

$$= -2\left(\frac{2^3}{3}\right) + \left(\frac{2^2}{2}\right) + 4i\left(\frac{2^3}{3}\right) - 5i\left(\frac{2^2}{2}\right) + 2i + 2$$

$$= -\frac{16}{3} + 2 + \frac{32i}{3} - 10i + 2i + 2$$

$$= \frac{-16 + 6 + 32i - 30i + 6i + 6}{3}$$

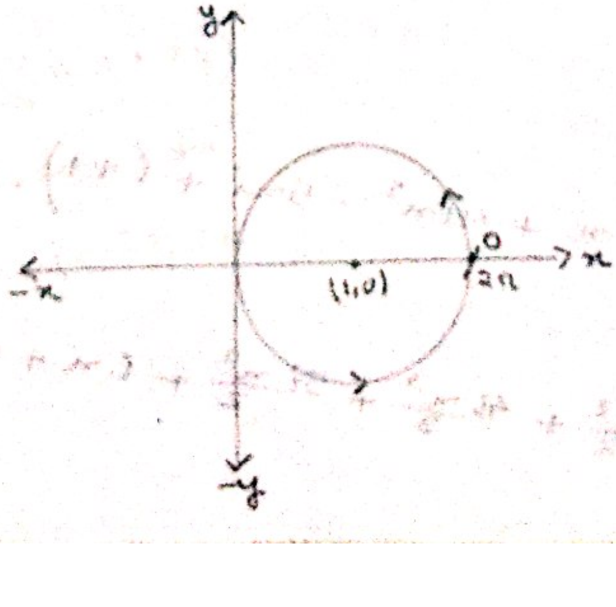
$$= \left(\frac{-4 + 8i}{3}\right)$$

$$= \frac{4}{3}(-1 + 2i)$$

(Ans)

(b) Evaluate $\oint_c (\bar{z})^2 \cdot dz$ around the circle $|z-1| = 1$

$\Rightarrow$



Now,

$$|z-1| = 1$$

$$\Rightarrow z-1 = e^{i\theta}$$

$$\Rightarrow z = (1+e^{i\theta}) \qquad \therefore \bar{z} = (1+e^{-i\theta})$$

$$\Rightarrow \frac{dz}{d\theta} = ie^{i\theta} \qquad \Rightarrow (\bar{z})^2 = (1+e^{-i\theta})^2$$

$$\therefore dz = i.e^{i\theta}.d\theta \quad \text{--- (i)} \qquad \therefore (\bar{z})^2 = (1 + 2.e^{-i\theta} + e^{-2i\theta}) \quad \text{--- (ii)}$$

So,

$$\oint_C (\bar{z})^2.dz = \int_0^{2\pi} (1+2.e^{-i\theta}+e^{-2i\theta}).ie^{i\theta}.d\theta$$

$$= \int_0^{2\pi} (i.e^{i\theta} + 2i + ie^{-i\theta}).d\theta$$

$$= i.\int_0^{2\pi} (e^{i\theta} + 2 + e^{-i\theta}).d\theta$$

$$= i.\left[\frac{e^{i\theta}}{i} + 2\theta + \frac{e^{-i\theta}}{-i}\right]_0^{2\pi}$$

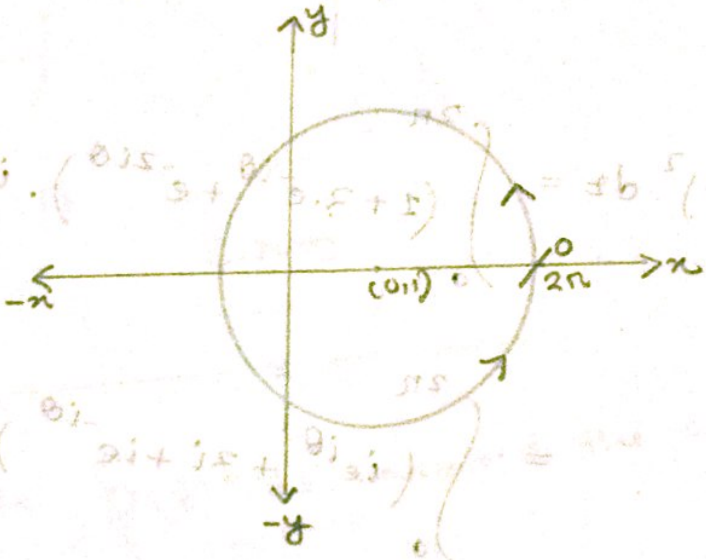
$$= i.\left[\left(\frac{e^{2\pi i}}{i} - \frac{1}{i}\right) + 4\pi - \left(\frac{e^{-2\pi i}}{i} - \frac{1}{+i}\right)\right]$$

$$= i\left[\frac{e^{2\pi i}}{i} - \frac{1}{i} + 4\pi - \frac{e^{-2\pi i}}{i} + \frac{1}{i}\right]$$

$$= 4\pi i \quad \text{(Ans)}$$

(ii) ***⑧ Evaluate, $\oint_c \frac{dz}{z-2}$ around the circle $|z-1| = 9$

$\Rightarrow$ (b)



Now,

$$|z-1| = 9$$

$$\Rightarrow z-1 = 9e^{i\theta}$$

$$\Rightarrow z = (1 + 9 \cdot e^{i\theta}) \text{ ——— ①}$$

$$\Rightarrow \frac{dz}{d\theta} = 9 \cdot i e^{i\theta}$$

$$\therefore dz = 9i \cdot e^{i\theta} \cdot d\theta \text{ ——— ⑪}$$

$$\therefore \oint_c \frac{dz}{z-2} = \int_0^{2\pi} \frac{9i.e^{i\theta}.d\theta}{1+9e^{i\theta}-2}$$

$$= \int_0^{2\pi} \frac{9ie^{i\theta}.d\theta}{9.e^{i\theta}-1}$$

Now, Let, $(9.e^{i\theta}-1)=u$

$$\Rightarrow 9.i.e^{i\theta} = \frac{du}{d\theta}$$

$$\therefore du = 9.i.e^{i\theta}.d\theta$$

when, $\theta = 2\pi$, $u = (9.e^{2\pi i}-1)$

$\therefore u = 8$ $[e^{2\pi i}=1]$

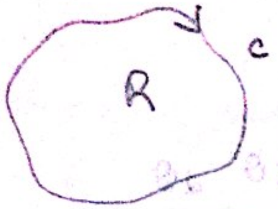
$\theta = 0$, $u = (9.e^{0}-1) = 8$

$$\therefore \oint_c \frac{dz}{z-2} = \int_8^8 \frac{du}{u}$$

$$= 0$$

(Ans)

⊛⊛⊛ Cauchy's Theorem:

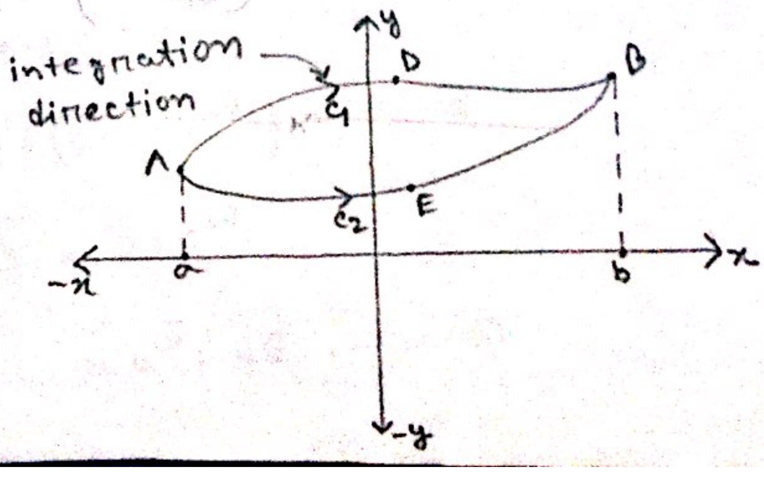


Let $f(z)$ be analytic in a Region R and on its boundary c.

Then, $\boxed{\oint_c f(z).dz = 0}$ cauchy's equation.

⊛⊛⊛ Theorem: Independent of Path:

If function $f(z)$ is analytic in a simply connected region R, then $\int_a^b f(z).dz$ is independent of the path in R joining any two points a and b in R.



Proof: By cauchy's Theorem,

$$\int_{ADBEA} f(z).dz = 0$$

$$\Rightarrow \int_{ADB} f(z).dz + \int_{BEA} f(z).dz = 0$$

$$\Rightarrow \int_{ADB} f(z).dz - \int_{AEB} f(z).dz = 0$$

$$\Rightarrow \int_{ADB} f(z).dz = \int_{AEB} f(z).dz$$

$$\Rightarrow \int_{C_1} f(z).dz = \int_{C_2} f(z).dz = \int_a^b f(z).dz$$

$\therefore$ The integration is independent of path.

[Proved]

① $f(z) = \begin{cases} 1, & y<0 \\ 4y, & y>0 \end{cases}$

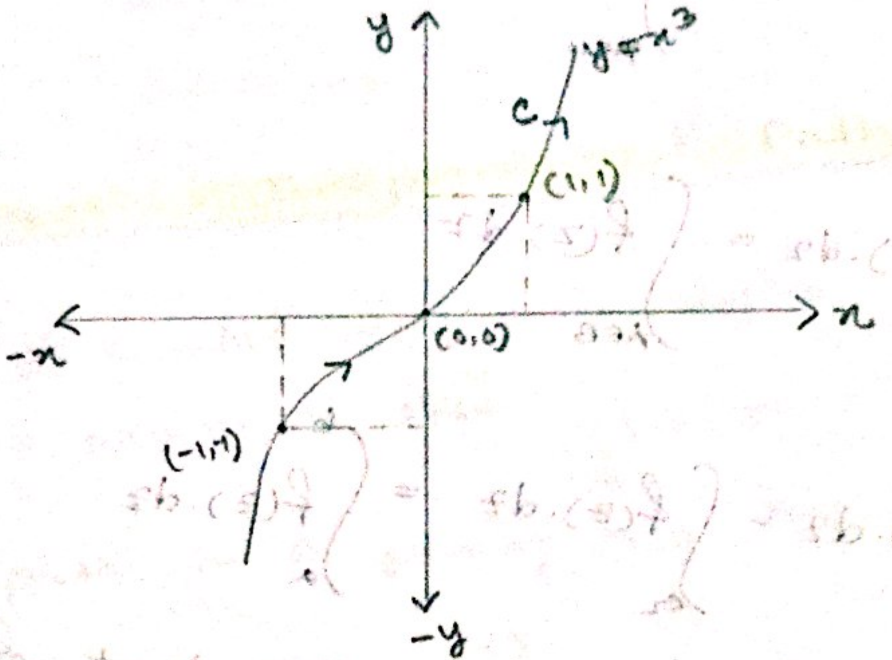
Evaluate $\int_c f(z).dz$; where c is the arc from $z=(-1-i)$ to $z=(1+i)$ along the curve $y=x^3$.

$\Rightarrow$ Here, $z=(-1-i)$ and $z=(1+i)$

$\therefore x=-1$ $\quad\quad$ $\therefore x=1$

$\therefore y=-1$ $\quad\quad$ $\therefore y=1$

$\therefore z=(-1,-1)$ $\quad\quad$ $\therefore z=(1,1)$



Now, $z = x+iy$ and $y=x^3$

$\Rightarrow z = x + i.x^3$

$\Rightarrow \frac{dz}{dx} = 1 + 3i.x^2$

$\therefore dz = (1+3i.x^2).dx$ —— ①

Now,

$$\int_c f(z).dz = \int_{(-1,-1)}^{(0,0)} f(z).dz + \int_{(0,0)}^{(1,1)} f(z).dz$$

Here,

$$\int_{(-1,-1)}^{(0,0)} f(z).dz = \int_{-1}^{0} 1.(1+3i.x^2).dx$$

$$= \int_{-1}^{0} dx + 3i.\int_{-1}^{0} x^2.dx$$

$$= [x]_{-1}^{0} + 3i.\left[\frac{x^3}{3}\right]_{-1}^{0}$$

$$= [0-(-1)] + 3i\left[0-\left(\frac{-1}{3}\right)\right]$$

$$= 1 + 3i.\frac{1}{3}$$

$$\therefore \int_{(-1,-1)}^{(0,0)} f(z).dz = (1+i)$$

Now,

$$\int_{(0,0)}^{(1,1)} f(z).dz = \int_{(0,0)}^{(1,1)} 4y\,(1+3ix^2).dx$$

$$= \int_0^1 4x^3\,(1+3i.x^2).dx$$

$$= \int_0^1 (4x^3 + 12ix^5).dx$$

$$= \left[4.\frac{x^4}{4} + 12i.\frac{x^6}{6}\right]_0^1$$

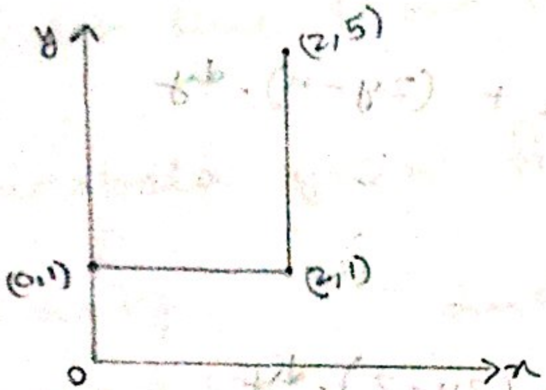
$$= \left[x^4 + 2i.x^6\right]_0^1$$

$$\therefore \int_{(0,0)}^{(1,1)} f(z).dz = (1+2i)$$

$$\therefore \int_c f(z).dz = (1+i) + (1+2i) = (2+3i)$$ (ans)

Evaluate $\int_{(0,1)}^{(2,5)} (3x+y).dx + (2y-x).dy$ along the straight lines from (0,1) to (2,1) and then from (2,1) to (2,5).

$\Rightarrow$



Now, along the path (0,1) to (2,1),

Here, $y = 1$

$\Rightarrow \frac{dy}{dx} = 0$

$\therefore dy = 0$

$\therefore \int_{(0,1)}^{(2,5)} (3x+y).dx + (2y-x).dy$

$= \int_0^2 (3x+1).dx + (2.1-x).0$

$= \int_0^2 (3x+1).dx$

$= \left[3.\frac{x^2}{2} + x\right]_0^2$

$= 3.\left(\frac{2^2}{2}\right) + 2$

$= 8$

Now, along the path, (2, 1) to (2, 5)

Here, $x = 2$

$$\Rightarrow \frac{dx}{dy} = 0$$

$$\therefore dx = 0$$

$$\therefore \int_{(,1)}^{(2,5)} (3x+y).dx + (2y-x).dy$$

$$= \int_1^5 (3.2+y).0 + (2y-2).dy$$

$$= \int_1^5 (2y-2).dy$$

$$= \left[2.\frac{y^2}{2} - 2y\right]_1^5$$

$$= \left[y^2 - 2y\right]_1^5$$

$$= (5^2 - 2.5) - (1-2)$$

$$= 25 - 10 + 1 = 16$$

$\therefore$ The required va

$= (16+8) = 24$

❀❀❀③ Evaluate $\oint_C (x^2 - iy^2).dz$ along,

ⓐ the parabola $y = 2x^2$ from $(1,1)$ to $(2,8)$.

ⓑ the straight lines from $(1,1)$ to $(1,8)$ and then from $(1,8)$ to $(2,8)$.

ⓒ the straight line from $(1,1)$ to $(2,8)$.

$\Rightarrow$ ⓐ The parabola $y = 2x^2$ from $(1,1)$ to $(2,8)$

Here, $z = x + iy$, and $y = 2x^2$

$\Rightarrow z = x + i.2x^2$

$\Rightarrow \frac{dz}{dx} = (1 + 4ix)$

$\therefore dz = (1 + 4ix).dx$

Now, $\displaystyle\int_{(1,1)}^{(2,8)} (x^2 - iy^2).dz$

$= \displaystyle\int_1^2 \{x^2 - i.(2x^2)^2\}.(1 + 4ix).dx$

$= \displaystyle\int_1^2 (x^2 - 4ix^4).(1 + 4ix).dx$

$$= \int_1^2 (x^2 - 4i.x^4 + 4i.x^3 + 16x^5).dx$$

$$= \left[\frac{x^3}{3} - \frac{4}{5}.i.x^5 + i.x^4 + \frac{16}{6}.x^6\right]_1^2$$

$$= \left(\frac{2^3}{3} - \frac{4}{5}.i.2^5 + i.2^4 + \frac{16}{6}.2^6\right) - \left(\frac{1}{3} - \frac{4}{5}i + i + \frac{16}{6}\right)$$

$$= \left(\frac{520}{3} - \frac{48i}{5}\right) - \left(3 + \frac{i}{5}\right)$$

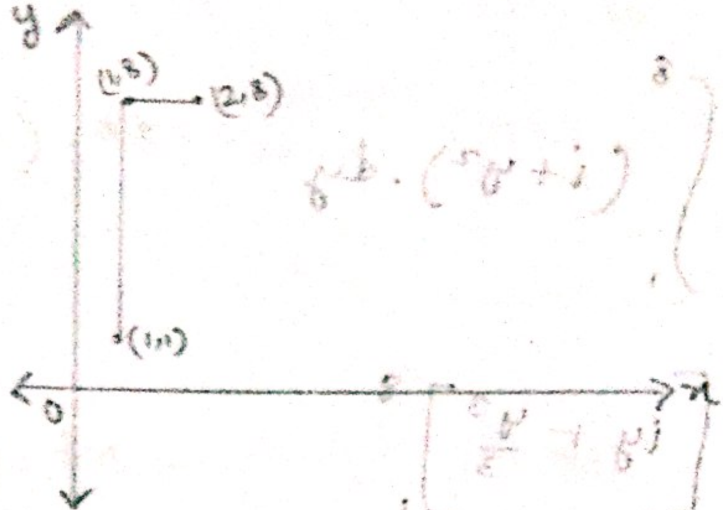
$$= \frac{520}{3} - 3 - \frac{48i}{5} - \frac{i}{5}$$

$$= \left(\frac{511}{3}\right) - \left(\frac{49}{5}\right)i$$

(ans).

(b) The straight lines from (1,1) to (1,8) a
then from (1,8) to (2,8)

$\Rightarrow$ Here,



Now, along the path (1,1) to (1,8)

Here, $x = 1$

$\therefore z = x + iy$

$\Rightarrow z = 1 + iy$

$\Rightarrow \frac{dz}{dy} = i$

$\therefore dz = i \cdot dy$

$$\therefore \oint_c (x^2 - iy^2) \cdot dz = \int_{(1,1)}^{(1,8)} (x^2 - i \cdot y^2) \cdot dz = \int_1^8 (1 - i \cdot$$

$$= \int_1^8 (i - i^2 \cdot y^2) \cdot dy$$

$$= \int_1^8 (i + y^2) \cdot dy$$

$$= \left[iy + \frac{y^3}{3} \right]_1^8$$

$$= \left(8i + \frac{8^3}{3}\right) - \left(i + \frac{1}{3}\right)$$

$$= \left(7i + \frac{511}{3}\right)$$

Again, along the path from (1,8) to (2,8)

Here, $y = 8$

$\therefore z = x + iy$

$\Rightarrow z = x + 8i$

$\Rightarrow \frac{dz}{dx} = 1$

$\therefore dz = dx$

$\therefore \oint_C (x^2 - iy^2).dz$

$$= \int_{(1,8)}^{(2,8)} (x^2 - iy^2).dz$$

$$= \int_1^2 (x^2 - 64.i).dx$$

$$= \left[\frac{x^3}{3} - 64xi\right]_1^2$$

$$= \left(\frac{2^3}{3} - 128i\right) - \left(\frac{1}{3} - 64i\right)$$

$$= \left(\frac{7}{3} - 64i\right)$$

$\therefore$ The required value is, $\left(\frac{511}{3} + 7i\right) + \left(\frac{7}{3} - 64i\right)$

$$= \left(\frac{518}{3} - 57i\right) \quad (\text{ans})$$

(e) The straight line from (1,1) to (2,8)

$\Rightarrow$ NOW,

The equation of straight line,

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2}$$

$$\Rightarrow \frac{x-1}{1-2} = \frac{y-1}{1-8}$$

$$\Rightarrow \frac{x-1}{-1} = \frac{y-1}{-7}$$

$$\Rightarrow 7x-7 = y-1$$

$$\therefore y = (7x-6)$$

Now, $z = x+iy$

$$\Rightarrow z = x + 7ix - 6i$$

$$\Rightarrow \frac{dz}{dx} = 1+7i$$

$$\therefore dz = (1+7i).dx$$

Here, $\oint_c (x^2 - iy^2).dz = \int_{(1,1)}^{(2,8)} (x^2 - iy^2).dz$

$$= \int_1^2 \left[x^2 - i\cdot(7x-6)^2\right](1+7i)\cdot dx$$

$$= \int_1^2 \left[x^2 - i\cdot\left(49x^2 - 84x + 36\right)\right]\cdot(1+7i)\cdot dx$$

$$= \int_1^2 \left(x^2 - 49i\cdot x^2 + 84\cdot i\cdot x - 36i\right)\cdot(1+7i)\cdot dx$$

$$= \int_1^2 \left(x^2 - 49i\cdot x^2 + 84i\cdot x - 36i + 7i\cdot x^2 + 343x^2 - 588x + 252\right)\cdot dx$$

$$= \left[\frac{x^3}{3} - \frac{49}{3}\cdot i\cdot x^3 + 42\cdot i\cdot x^2 - 36i\cdot x + \frac{7}{3}\cdot i\cdot x^3 + \frac{343}{3}\cdot x^3 - \frac{588}{2}x^2 + 252x\right]_1^2$$

$$= \left[\frac{x^3}{3} + \frac{343}{3}\cdot x^3 - \frac{588}{2}\cdot x^2 + 252\cdot x + i\cdot\left(-\frac{49}{3}\cdot x^3 + 42x^2 - 36x + \frac{7}{3}\cdot x^3\right)\right]_1^2$$

$$= \left[\frac{2^3}{3} + \frac{343}{3} \times 2^3 - \frac{588}{2} \times 2^2 + 252 \times 2 + i \left(-\frac{49}{3} \cdot 2^3 + 42 \cdot \right. \right.$$

$$\left. \left. -36 \times 2 + \frac{7}{3} \cdot 2^3 \right) \right] - \left[\frac{1}{3} + \frac{343}{3} - \frac{588}{2} + 252 + i \right.$$

$$\left. +42 - 36 + \frac{7}{3} \right) \Big]$$

$$= \left[\frac{736}{3} + i(-16) \right] - \left[\frac{218}{3} + i(-8) \right]$$

$$= \left(\frac{736}{3} - \frac{218}{3} \right) + \left(-16i + 8i \right)$$

$$= \left(\frac{518}{3} - 8i \right)$$

(ans)

***② Evaluate $\oint_c (x+2y)\,dx + (y-2x).\,dy$ around the ellipse c defined by, $x=4.\cos\theta$, $y=3.\sin\theta$, $0\le\theta\le 2\pi$ if c is described in a counterclockwise direction.

$\Rightarrow$ Here,

$x = 4.\cos\theta$ $\qquad$ $y = 3\sin\theta$

$\Rightarrow \dfrac{dx}{d\theta} = 4.(-\sin\theta)$ $\qquad$ $\Rightarrow \dfrac{dy}{d\theta} = 3.\cos\theta$

$\therefore dx = -4.\sin\theta.d\theta$ $\qquad$ $\therefore dy = 3.\cos\theta.d\theta$

Now,

$$\oint_c (x+2y).\,dx + (y-2x).\,dy \quad [0\le\theta\le 2\pi]$$

$$= \int_0^{2\pi} (4\cos\theta + 2.3.\sin\theta).(-4.\sin\theta).d\theta + (3.\sin\theta - 2.4.\cos\theta).3.\cos\theta.d\theta$$

$$= \int_0^{2\pi} (-16.\cos\theta.\sin\theta - 24.\sin^2\theta).d\theta + (9.\sin\theta.\cos\theta - 24.\cos^2\theta).d\theta$$

$$= \int_0^{2\pi} [-16.\sin\theta.\cos\theta + 9.\sin\theta.\cos\theta - 24.(\sin^2\theta + \cos^2\theta)].d\theta$$

$$= \int_0^{2\pi} (-7 \cdot \sin\theta \cdot \cos\theta) \cdot d\theta + \int_0^{2\pi} -24 \cdot d\theta$$

$$= -7 \int_0^{2\pi} \sin\theta \cdot \cos\theta \, d\theta - 24 \cdot \int_0^{2\pi} d\theta$$

Now, Let, $\sin\theta = u$ $\quad \therefore \theta = 2\pi, \; u = \sin 2\pi = 0$

$\Rightarrow \cos\theta = \frac{du}{d\theta}$ $\quad \therefore \theta = 0, \; u = \sin 0 = 0$

$\therefore du = \cos\theta \cdot d\theta$

$$\therefore -7 \cdot \int_0^0 u \cdot du = 0$$

$$\therefore \oint_C (x+2y) \cdot dx + (y-2x) \cdot dy = -24 \cdot \int_0^{2\pi} d\theta$$

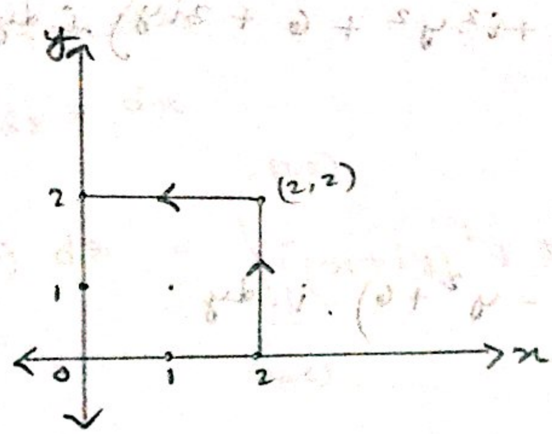
$$= -24 \cdot [\theta]_0^{2\pi}$$

$$= -24 \cdot 2\pi$$

$$= -48\pi \quad \text{(Ans)}$$

***⑤⑥ Evaluate $\oint_c (z^2+3z).dz$ the straight line from $(2,0)$ to $(2,2)$ and then from $(2,2)$ to $(0,2)$.

$\Rightarrow$ Here,



Now, along the path from $(2,0)$ to $(2,2)$,

Here, $x=2$

$\therefore z = x+iy$

$\Rightarrow z = 2+iy$

$\Rightarrow \frac{dz}{dy} = i$

$\therefore dz = i.dy$

$$\therefore \oint_c (z^2+3z).dz = \int_{(2,0)}^{(2,2)} \left[(x+iy)^2+3.(x+iy)\right].dz$$

$$= \int_0^2 \left[(2+iy)^2 + 3.(2+iy)\right] i.dy$$

$$= \int_0^2 (4+4iy+i^2.y^2+6+3iy).i.dy$$

$$= \int_0^2 (4+7iy-y^2+6).i.dy$$

$$= \int_0^2 (10i+7i^2y-iy^2).dy$$

$$= \int_0^2 (10i-7y-iy^2).dy$$

$$= \left[10i.y - \frac{7}{2}.y^2 - \frac{i}{3}.y^3\right]_0^2$$

$$= 20i - 14 - \frac{8}{3}i$$

$$= \left(-14 + \frac{52}{3}i\right)$$

Again along the path from $(2,2)$ to $(0,2)$.

Here, $y = 2$

$\therefore z = x + iy$

$\Rightarrow z = x + 2i$

$\Rightarrow dz = dx$

$$\therefore \oint_c (z^2+3z).dz = \int_{(2,2)}^{(0,2)} \left[(x+iy)^2 + 3.(x+iy)\right].dz$$

$$= \int_2^0 \left[(x+2i)^2 + 3.(x+2i)\right].dx$$

$$= \int_2^0 \left[x^2 + 4ix + 4i^2 + 3x + 6i\right].dx$$

$$= \int_2^0 (x^2 + 4ix - 4 + 3x + 6i).dx$$

$$= \left[\frac{x^3}{3} + 2i.x^2 - 4x + \frac{3}{2}.x^2 + 6ix\right]_2^0$$

$$= \left[\frac{x^3}{3} - 4x + \frac{3}{2}\cdot x^2 + i\cdot(2x^2+6x)\right]_2^0$$

$$= -\left[\frac{2^3}{3} - 4\cdot(+2) + \frac{3}{2}\cdot 2^2 + i\cdot(2\cdot 2^2 + 6\cdot 2)\right]$$

$$= -\left[\frac{2}{3} + 20i\right]$$

$$= \left(-\frac{2}{3} - 20i\right)$$

$\therefore$ The required value is, $-14 + \left(+\frac{52}{3}\right)i + \left(-\frac{2}{3}\right)$

$$= \left(-\frac{44}{3} - \frac{8}{3}i\right) \quad \text{(ans)}$$

*** ⓒ ⓑ Evaluate $\int_{i}^{2-i} (3xy + iy^2).\, dz$ along the parabola $x = (2t-2)$, $y = (1+t-t^2)$.

$\Rightarrow$ Here, $x = (2t-2)$

$y = (1+t-t^2)$

$\therefore\ z = x+iy$

$\Rightarrow\ z = (2t-2) + i.\,(1+t-t^2)$

$\Rightarrow\ z = 2t-2+i+it-it^2$

$\Rightarrow\ \frac{dz}{dt} = 2+i-2it$

$\therefore\ dz = (2+i-2it).\, dt$

Now, $z = (2-i)$ $\therefore\ x = 2$, $y = -1$ $\therefore\ (x,y) = (2,-1)$

$z = i$ $\therefore\ x = 0$, $y = 1$ $\therefore\ (x,y) = (0,1)$

$$\therefore \int_{i}^{2-i} (3xy + iy^2).\, dz = \int_{(0,1)}^{(2,-1)} \left[3.(2t-2).(1+t-t^2) + i.\,(1+t-t^2)^2\right].(2+i-2it).\, dt$$

Now, $x=2$ $\qquad$ $x=0$

$\therefore 2t-2=2$ $\qquad$ $\therefore 2t-2=0$

$\Rightarrow 2t=4$ $\qquad$ $\Rightarrow 2t=2$

$\therefore t=2$ $\qquad$ $\therefore t=1$

$$\therefore \int_1^2 \left[3\cdot(2t-2)\cdot(1+t-t^2)+i\,(1+t-t^2)^2\right]\cdot(2+i-2it)\cdot dt$$

*** 7(a) Evaluate $\oint_c (\bar{z})^2 \cdot dz$ around the circles $|z| = 1$.

$\Rightarrow$ Here,

$$|z| = 1$$

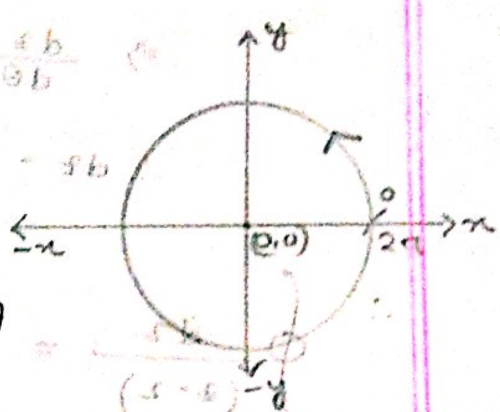
$$\Rightarrow z = 1.e^{i\theta}$$

$$\Rightarrow \frac{dz}{d\theta} = i e^{i\theta}$$

$$\therefore dz = i.e^{i\theta} \cdot d\theta$$

$$\therefore \bar{z} = e^{-i\theta}$$

$$\Rightarrow (\bar{z})^2 = e^{-2i\theta}$$



Now,

$$\oint_c (\bar{z})^2 \cdot dz = \int_0^{2\pi} (e^{-2i\theta}) \cdot i.e^{i\theta} \cdot d\theta$$

$$= i \int_0^{2\pi} e^{-2i\theta + i\theta} \cdot d\theta$$

$$= i \int_0^{2\pi} e^{-i\theta} \cdot d\theta$$

$$= i \left[\frac{e^{-i\theta}}{-i} \right]_0^{2\pi}$$

$$= -1 \cdot \left[e^{-2\pi i} - e^{0} \right] = -1 \left[1 - 1 \right] = 0$$

(Ans)

***⑧ⓐ Evaluate $\oint_c \frac{dz}{z-2}$ around the circle

$\Rightarrow$ Here,

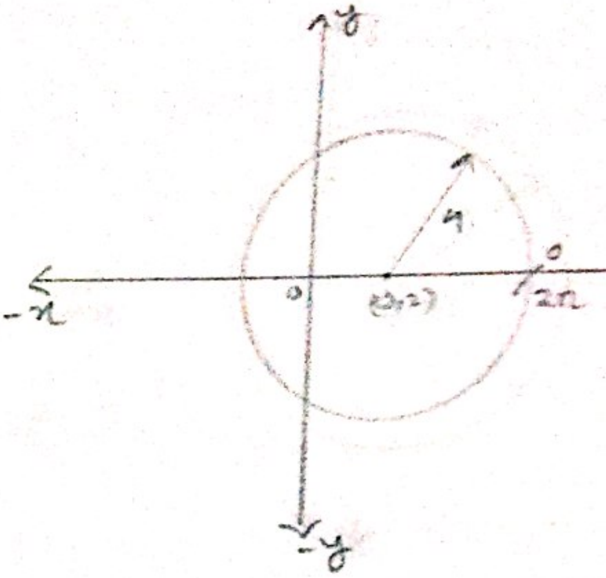
$$|z-2|=4$$

$$\Rightarrow z-2 = 4.e^{i\theta}$$

$$\Rightarrow z = (2+4.e^{i\theta})$$

$$\Rightarrow \frac{dz}{d\theta} = 4.i.e^{i\theta}$$

$$\therefore dz = 4.i.e^{i\theta}.d\theta$$



Now, $$\oint_c \frac{dz}{z-2} = \int_0^{2\pi} \frac{4.i.e^{i\theta}.d\theta}{(2+4.e^{i\theta})-2}$$

$$= \int_0^{2\pi} \frac{4.i.e^{i\theta}}{4.e^{i\theta}}.d\theta$$

$$= \int_0^{2\pi} i.d\theta$$

$$= i.[\theta]_0^{2\pi}$$

$$= 2\pi i \quad (\text{ans})$$

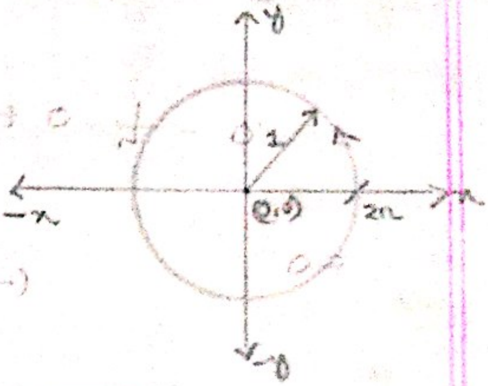
⑤ Evaluate $\oint_C (5z^4 - z^3 + 2).dz$ around the circle $|z|=1$

$\Rightarrow$ Here,

$$|z| = 1$$

$$\Rightarrow z = e^{i\theta}$$

$$\therefore dz = ie^{i\theta}.d\theta$$



Now,

$$\int_0^{2\pi} (5.e^{4i\theta} - e^{3i\theta} + 2).i.e^{i\theta}.d\theta$$

$$= \int_0^{2\pi} (5i.e^{5i\theta} - ie^{4i\theta} + 2.i.e^{i\theta}).d\theta$$

$$= 5i.\int_0^{2\pi} e^{5i\theta}.d\theta - i.\int_0^{2\pi} e^{4i\theta}.d\theta + 2i.\int_0^{2\pi} e^{i\theta}.d\theta$$

$$= 5i.\left[\frac{e^{5i\theta}}{5i}\right]_0^{2\pi} - i.\left[\frac{e^{4i\theta}}{4i}\right]_0^{2\pi} + 2i\left[\frac{e^{i\theta}}{i}\right]_0^{2\pi}$$

$$= \left[e^{10\pi i} - e^0\right] - \frac{1}{4}\left[e^{8\pi i} - e^0\right] + 2.\left[e^{2\pi i} - e^0\right]$$

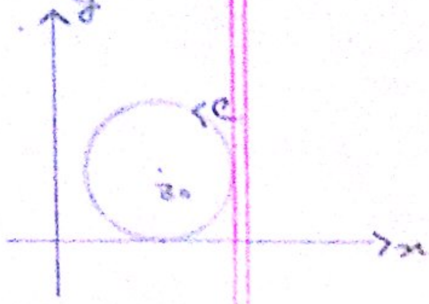
$$= \left[1^5 - 1\right] - \frac{1}{4}\cdot\left[1^4 - 1\right] + 2\cdot\left[1 - 1\right]$$

$$= 0 - \frac{1}{4}\cdot 0 + 2\cdot 0$$

$$= 0 \quad \text{(ans)}$$

[illegible] Cauchy's Integral formula:

Suppose $f(z)$ is analytic everywhere on and inside a closed contour C. If z_0 is any arbitrary point inside the contour C, then.



$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)} \cdot dz}$$

and the nth derivative of $f(z)$ at point z_0 is given by.

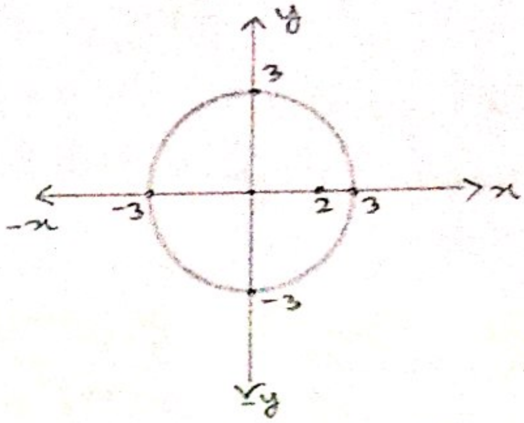
$$\boxed{f^n(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} \cdot dz} \quad ; \; n = 1, 2, 3 \ldots\ldots$$

———— x ————

***① Evaluate $\frac{1}{2\pi i} \oint \frac{e^z}{(z-2)} . dz$. if c is

(i) the circle $|z| = 3$

(ii) the circle $|z| = 1$



⇒ (i) $|z| = 3$

Here, $f(z) = e^z$

$z_0 = 2$

Now, $z_0 = 2$, lies inside the circle, $|z| = 3$.

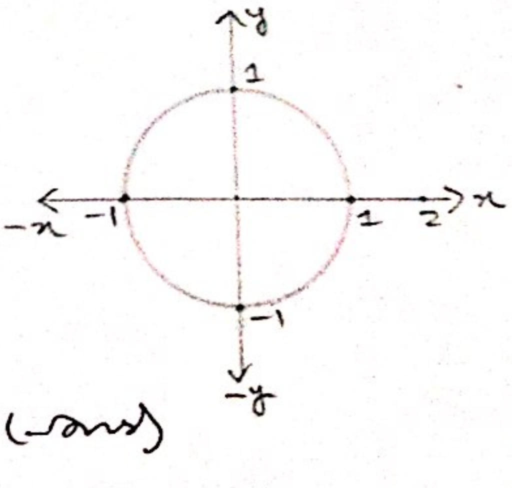
$$\therefore \frac{1}{2\pi i} \oint \frac{e^z}{(z-2)} . dz = f(z_0) = f(2) \quad \text{[By cauchy's integral formula]}$$

$$= e^2$$

(ii) $|z| = 1$

Here, $z_0 = 2$ lies outside the circle $|z| = 1$

$$\therefore \frac{1}{2\pi i} \oint \frac{e^z}{z-2} . dz = 0$$



(ans)

---x---

**③ Find $\oint_c \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} . dz$, where c is the circle $|z|=1$.

$\Rightarrow$ Here,

$z_0 = \left(\frac{\pi}{6}\right)$ and $(n+1) = 3$ $\therefore f(z) = \sin^6 z$

$\therefore n = 2$

Now, $f^n(z_0) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz$

$\Rightarrow \frac{2\pi i}{n!} . f^n(z_0) = \oint_c \frac{f(z)}{(z-z_0)^{n+1}} . dz$

$\Rightarrow \frac{2\pi i}{2!} . f^2(z_0) = \oint_c \frac{\sin^6 z}{(z-z_0)^3} . dz$

$\Rightarrow \frac{2\pi i}{2!} f^2 . \left(\frac{\pi}{6}\right) = \oint_c \frac{\sin^6 z}{(z-\pi/6)^3} . dz$

$\therefore \oint_c \frac{\sin^6 z}{(z-\pi/6)^3} . dz = \frac{2\pi i}{2!} . f^2(\pi/6)$ ——— ①

NOW,

$$f(z) = \sin^6 z$$

$$\therefore f'(z) = 6.\sin^5 z \cdot \cos z$$

$$\therefore f''(z) = 6.\left[5.\sin^4 z \cdot \cos^2 z + \sin^5 z \cdot (-\sin z)\right]$$

$$\therefore f^2(z) = 6\left[5.\sin^4 z \cdot \cos^2 z - \sin^6 z\right]$$

$$\therefore f^2(\pi/6) = 6.\left[5.\sin^4(\pi/6) \cdot \cos^2(\pi/6) - \sin^6(\pi/6)\right]$$

$$= 6.\left[5.\left(\frac{1}{2}\right)^4 \cdot \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^6\right]$$

$$= 6.\left[5 \cdot \frac{1}{16} \cdot \left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{64}\right]$$

$$= 6.\left[\frac{15}{64} - \frac{1}{64}\right]$$

$$= \frac{21}{16}$$

$$\therefore \oint_c \frac{\sin^6 z}{(z-\pi/6)^3}.dz = \frac{2\pi i}{2!} \cdot \frac{21}{16}$$

$$= \left(\frac{21}{16} \cdot \pi i\right)$$

(Ans)

*** (3) $\frac{1}{2\pi i} \cdot \oint \frac{e^{zt}}{(z^2+1)^2} \cdot dz$; if $t>0$ and c is the circle $|z|=3$.

$\Rightarrow$ NOW, $(n+1)=2 \therefore n=1$

$$\frac{1}{(z^2+1)} = \frac{1}{(z^2-i^2)} = \frac{1}{(z-i).(z+i)} = \frac{1}{2i}\left[\frac{1}{(z-i)} - \frac{1}{(z+i)}\right]$$

$$\therefore \frac{1}{2\pi i} \cdot \oint_c \frac{e^{zt}}{(z^2+1)} \cdot dz = \frac{1}{2\pi i} \cdot \oint_c \frac{1}{2i} \cdot \left[\frac{1}{(z-i)} - \frac{1}{(z+i)}\right] \cdot e^{zt} \cdot dz$$

$$= \frac{1}{4\pi i^2} \cdot \oint_c \left[\frac{1}{(z-i)} - \frac{1}{(z+i)}\right] e^{zt} \cdot dz$$

$$= \frac{1}{4\pi i^2} \cdot \left[\oint_c \frac{e^{zt}}{(z-i)} \cdot dz - \oint_c \frac{e^{zt}}{(z+i)} \cdot dz\right]$$

$$= \frac{1}{4\pi i^2} \cdot \left[2\pi i . f(i) - 2\pi i . f(-i)\right]$$

[By cauchy's integral formula]

$$\therefore \frac{1}{2\pi i} \cdot \oint_c \frac{e^{zt}}{(z^2+1)} \cdot dz = \frac{1}{2i} \cdot \left[f(i) - f(-i)\right]$$

Now,

$$\frac{1}{2\pi i} \oint \frac{e^{zt}}{(z^2+1)^{1+1}} \cdot dz = \frac{1}{2i} \cdot \left[f'(i) - f'(-i) \right]$$

$$\therefore \frac{1}{2\pi i} \oint \frac{e^{zt}}{(z^2+1)^2} \cdot dz = \frac{1}{2i} \cdot \left[f'(i) - f'(-i) \right] \text{——} ①$$

Here,

$$f(z) = e^{zt}$$

$$\therefore f'(z) = t.e^{zt}$$

$$\therefore f'(i) = t.e^{it} \quad \text{and} \quad f'(-i) = t.e^{-it}$$

$\therefore$ ① $\Rightarrow$

$$\frac{1}{2\pi i} \oint \frac{e^{zt}}{(z^2+1)^2} \cdot dz = \frac{1}{2i} \left[t.e^{it} - te^{-it} \right]$$

$$= \frac{t}{2i} \cdot \left[(\cos t + i.\sin t) - (\cos t - i.\sin t) \right]$$

$$= \frac{t}{2i} \left[\cos t + i.\sin t - \cos t + i.\sin t \right]$$

$$= \frac{t}{2i} \cdot 2.i.\sin t$$

$$= t.\sin t. \quad \text{(Ans)}$$

***④ Evaluate $\oint_c \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1).(z-2)}\,dz$ where c is the circle $|z|=3$.

$\Rightarrow$ Now,

$$\frac{1}{(z-1).(z-2)} = \left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

$$\therefore \oint_c \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1).(z-2)}\,dz = \oint_c \left[\frac{1}{z-2} - \frac{1}{z-1}\right]\left[\sin(\pi z^2) + \cos(\pi z^2)\right]dz$$

$$= \oint_c \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-2}\,dz - \oint_c \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-1}\,dz$$

$$\therefore \oint_c \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1).(z-2)}\,dz = 2\pi i.f(2) - 2\pi i.f(1) \quad \text{——— ①}$$

Now, $f(z) = \sin(\pi z^2) + \cos(\pi z^2)$

$\therefore f(2) = (\sin 4\pi + \cos 4\pi)$ $\qquad$ $\therefore f(1) = \sin\pi + \cos\pi$

$= 0+1$ $\qquad$ $= 0-1$

$\therefore f(2) = 1$ $\qquad$ $\therefore f(1) = -1$

$\therefore$ ① $\Rightarrow$

$$\oint_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-1).(z-2)}.dz = 2\pi i - 2\pi i.(-1)$$

$$= 4\pi i. \quad (\text{Ans})$$

****⑤** Evaluate $\oint_c \frac{e^{2z}}{(z+1)^4}.dz$; where c is circle $|z|=3$,

$\Rightarrow$ Here, $(n+1)=4$ and $f(z)=e^{2z}$ $\therefore z_0=-1$

$\therefore n=3$

We know, cauchy integral formula,

$$f^n(z_0) = \frac{n!}{2\pi i}.\oint_c \frac{f(z)}{(z-z_0)^{n+1}}.dz$$

$$\Rightarrow f^3(z_0) = \frac{3!}{2\pi i}.\oint_c \frac{e^{2z}}{(z+1)^4}.dz$$

$$\Rightarrow \oint_c \frac{e^{2z}}{(z+1)^4}.dz = \frac{2\pi i}{3!}.f^3(-1) \text{ ——— ①}$$

$\therefore \quad f(z) = e^{2z}$

$\therefore f'(z) = 2 \cdot e^{2z}$

$\therefore f''(z) = 4 \cdot e^{2z}$

$\therefore f'''(z) = 8 \cdot e^{2z}$

$\therefore f'''(-1) = 8 \cdot e^{-2}$

$\therefore$ ① $\Rightarrow$

$$\oint_c \frac{e^{2z}}{(z+1)^4} \cdot dz = \frac{2\pi i}{3!} \cdot 8 \cdot e^{-2}$$

$$= \frac{2\pi i}{1\times2\times3} \cdot 8 e^{-2}$$

$$= \left(\frac{8\pi i \cdot e^{-2}}{3}\right)$$

(Ans)

**6 Evaluate $\oint_c \frac{e^{3z}}{(z-\pi i)} \cdot dz$ where c is the circle $|z-1|=4$.

$\Rightarrow$ Now,

$$z_0 = \pi i$$

$$f(z) = e^{3z}$$

$$\therefore f(z_0) = e^{3\pi i}$$

$$= \cos(3\pi) + i.\sin(3\pi)$$

$$\therefore f(z_0) = -1$$

we know, Cauchy's Integral formula,

$$f(z_0) = \frac{1}{2\pi i} \cdot \oint_c \frac{e^{3z}}{(z-\pi i)} \cdot dz$$

$$\Rightarrow \oint_c \frac{e^{3z}}{(z-\pi i)} \cdot dz = 2\pi i \cdot f(z_0)$$

$$= 2\pi i \cdot (-1)$$

$$= -2\pi i.$$ (Ans)

P.S → 5

Theorem of Residue:

If $f(z)$ is analytic on and inside closed curve c except for a pole of order m at $z=a$ inside,

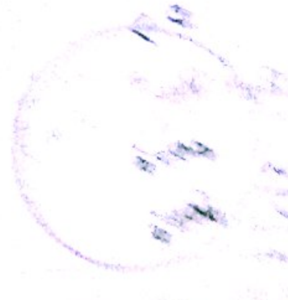
then:

$$\boxed{\frac{1}{2\pi i}\oint_c f(z)\,dz = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\}}$$

Pole:

$$\lim_{z\to a}(z-a)^m f(z) = A \neq 0$$

then $f(z)$ has pole of order m at $z=a$



If more than one pole inside c, suppose $z=a_1$ and $z=a_2$.

$z=a_1$ is a pole of order m_1.

$z=a_2$ is a pole of order m_2.

then.

$$\frac{1}{2\pi i}\oint_c f(z).dz = \lim_{z\to a_1} \frac{1}{(m_1-1)!}\cdot\frac{d^{(m_1-1)}}{dz^{(m_1-1)}}\cdot\left\{(z-a)^{m_1}\cdot f(z)\right\}$$

$$+\lim_{z\to a_2}\cdot\frac{1}{(m_2-1)!}\cdot\frac{d^{(m_2-1)}}{dz^{(m_2-1)}}\cdot\left\{(z-a)^{m_2}\cdot f(z)\right\}$$

$$\Rightarrow \frac{1}{2\pi i}\cdot\oint_c f(z).dz = R_1+R_2$$

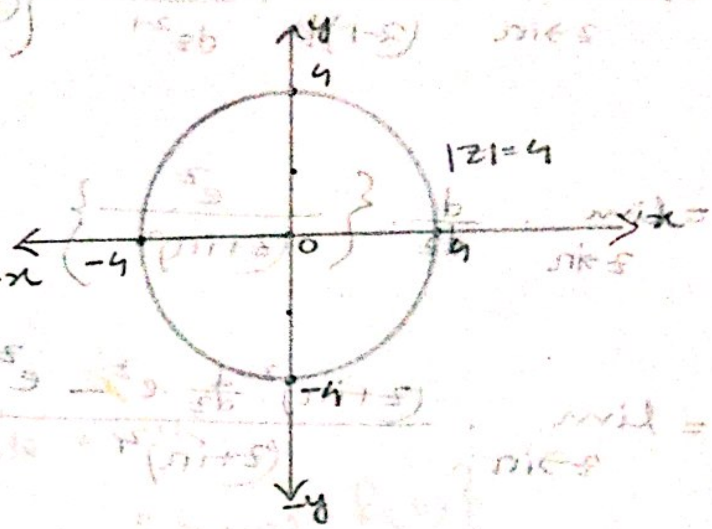
$$\therefore \oint_c f(z).dz = 2\pi i.(R_1+R_2)$$ [where R_1 an R_2 represents residue of $f(z)$]

$$= 2\pi i.(R_1+R_2+\cdots\cdots+R_n)$$

$$= 2\pi i.\sum_{i=1}^{n}\cdot R_i$$

*** ① Evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2} \cdot dz$ using the residue at the poles; where C is the circle $|z|=4$.

$\Rightarrow$ Now, $|z|=4$



Here, $\frac{1}{(z^2+\pi^2)} = \frac{1}{(z^2-\pi^2 i^2)} = \frac{1}{(z-i\pi)\cdot(z+i\pi)}$

$\therefore f(z) = \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z-i\pi)^2\cdot(z+i\pi)^2}$

Now, $(z-i\pi)^2\cdot(z+i\pi)^2=0$

$\Rightarrow (z-i\pi)\cdot(z+i\pi)=0$

$\therefore z=+i\pi, -i\pi$

$\therefore f(z)$ has a pole of order 2 at $z=-i\pi$.

$\therefore f(z)$ has a pole of order 2 at $z=i\pi$.

to $\therefore$ Residue at pole $z = i\pi$, here, $a = i\pi$, $m = 2$

$$R_1 = \lim_{z \to a} \cdot \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \cdot \left\{ (z-a)^m \cdot f(z) \right\}$$

$$= \lim_{z \to i\pi} \frac{1}{(2-1)!} \cdot \frac{d^{2-1}}{dz^{2-1}} \cdot \left\{ (z-i\pi)^2 \cdot \frac{e^z}{(z+i\pi)^2 \cdot (z-i\pi)^2} \right\}$$

$$= \lim_{z \to i\pi} \cdot \frac{d}{dz} \cdot \left\{ \frac{e^z}{(z+i\pi)^2} \right\}$$

$$= \lim_{z \to i\pi} \cdot \frac{(z+i\pi)^2 \cdot \frac{d}{dz} \cdot e^z - e^z \cdot \frac{d}{dz} \cdot (z+i\pi)^2}{(z+i\pi)^4}$$

$$= \lim_{z \to i\pi} \frac{(z+i\pi)^2 \cdot e^z - 2 \cdot e^z \cdot (z+i\pi)}{(z+i\pi)^4} = \lim_{z \to i\pi} \frac{(z+i\pi)\left\{(z+i\pi) \cdot e^z - 2e^z\right\}}{(z+i\pi)^4}$$

$$= \frac{(i\pi + i\pi) \cdot e^{i\pi} - 2 \cdot e^{i\pi}}{(i\pi + i\pi)^3}$$

$$= \frac{(2i\pi) \cdot e^{i\pi} - 2 \cdot e^{i\pi}}{(2i\pi)^3}$$

$$= \frac{-2i\pi + 2}{8i^3 \cdot \pi^3} \quad \left[e^{i\pi} = (\cos\pi + i \cdot \sin\pi) = -1 \right]$$

$$= \frac{-2(i\pi-1)}{-8i\cdot\pi^3}$$

$$= \frac{i\pi-1}{4i\pi^3}$$

$$= \frac{i\pi+i^2}{4i\pi^3}$$

$$= \frac{i(\pi+i)}{4i\pi^3}$$

$$\therefore R_1 = \left(\frac{\pi+i}{4\pi^3}\right)$$

$\therefore$ Residue at pole $z=-i\pi$,

$$\therefore R_2 = \lim_{z\to -i\pi} \frac{1}{1!}\cdot\frac{d}{dz}\cdot\left\{(z+i\pi)^2\cdot f(z)\right\}$$

$$\therefore R_2 = \left(\frac{\pi-i}{4\pi^3}\right)$$

$$\therefore \oint_c f(z)\cdot dz = 2\pi i\cdot(R_1+R_2)$$

$$= 2\pi i\cdot\left(\frac{\pi+i}{4\pi^3}+\frac{\pi-i}{4\pi^3}\right)$$

$$= 2\pi i\cdot\left(\frac{\pi+i+\pi-i}{4\pi^3}\right)$$

$$= 2\pi i\cdot\left(\frac{2\pi}{4\pi^3}\right) = \left(\frac{i}{\pi}\right) \text{ (Ans)}$$

*****②** Evaluate, $\frac{1}{2\pi i} \oint_c \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} \, dz$; where c is $|z|=4$.

$\Rightarrow$ Now,

$$z^4 + 10z^2 + 9 = 0$$

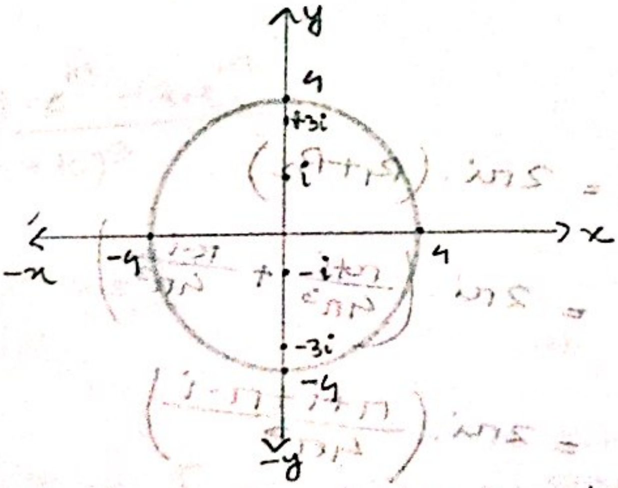
$$\Rightarrow z^4 + 9z^2 + z^2 + 9 = 0$$

$$\Rightarrow z^2 (z^2 + 9) + (z^2 + 9) = 0$$

$$\Rightarrow (z^2 + 9) \cdot (z^2 + 1) = 0$$

$\therefore (z^2 + 9) = 0$	$\therefore z^2 + 1 = 0$
$\Rightarrow z^2 = -9$	$\Rightarrow z^2 = -1$
$\Rightarrow z^2 = 9i^2$	$\Rightarrow z^2 = i^2$
$\therefore z = \pm 3i$	$\therefore z = \pm i$

$\therefore$ The simple poles are $\pm 3i$, $\pm i$.



$\therefore$ Both $\pm 3i$ and $\pm i$ poles are valid.

Now, Residue at $z=3i$

Here, $a=3i$

$m=1$

$$\therefore R_1 = \lim_{z\to a} \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \cdot \left\{(z-a)^m \cdot f(z)\right\}$$

$$= \lim_{z\to 3i} \frac{1}{0!} \cdot \frac{d^0}{dz^0} \cdot \left\{\cancel{(z-3i)} \cdot \frac{(z^2-z+2)}{(z+3i)\cdot \cancel{(z-3i)}(z+i)\cdot(z-i)}\right\}$$

$$= \lim_{z\to 3i} \frac{z^2-z+2}{(z+3i)\cdot(z+i)\cdot(z-i)}$$

$$= \frac{(3i)^2-3i+2}{(3i+3i)\cdot(3i+i)\cdot(3i-i)}$$

$$= \frac{-9-3i+2}{6i\cdot 4i\cdot 2i}$$

$$= \frac{-7-3i}{-48i}$$

$$= \frac{7+3i}{48i}$$

$$= \frac{7i-3}{-48}$$

$$\therefore R_1 = \left(\frac{3-7i}{48}\right)$$

$\therefore$ Residue at $z=-3i$

Here, $a=-3i$

$m=1$

$$\therefore R_2 = \left(\frac{3+7i}{48}\right)$$

Now, Residue at $z = i$

Here, $a = i$

$m = 1$

$$\therefore R_3 = \lim_{z \to a} \cdot \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \cdot \left\{ (z-a)^m \cdot f(z) \right\}$$

$$= \lim_{z \to i} \frac{1}{0!} \cdot \frac{d^0}{dz^0} \cdot \left\{ (z-i) \cdot \frac{z^2 - z + 2}{(z+3i) \cdot (z-3i) \cdot (z+i)\,(z-i)} \right\}$$

$$= \lim_{z \to i} \frac{z^2 - z + 2}{(z+3i) \cdot (z-3i) \cdot (z+i)}$$

$$= \frac{i^2 - i + 2}{4i \cdot (-2i) \cdot (2i)}$$

$$= \frac{-1 - i + 2}{+16i}$$

$$= \frac{1-i}{16i}$$

$$= \frac{i - i^2}{-16}$$

$$= \frac{1+i}{-16}$$

$$\therefore R_3 = \left(\frac{-1-i}{16} \right)$$

$\therefore$ Residue at $z = -i$

$$\therefore R_4 = \left(\frac{-1+i}{16} \right)$$

$$\therefore \frac{1}{2\pi i} \cdot \oint_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} \cdot dz$$

$$= R_1 + R_2 + R_3 + R_4$$

$$= \left(\frac{3-7i}{48} \right) + \left(\frac{3+7i}{48} \right) + \left(\frac{-1-i}{16} \right) + \left(\frac{-1+i}{16} \right)$$

$$= \frac{6}{48} - \frac{2}{16}$$

$$= \frac{1}{8} - \frac{1}{8}$$

$$= 0$$

$$\therefore \frac{1}{2\pi i}\oint_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}\, dz = 0$$

(Ans)

*** ③ Evaluate, $\oint_c \frac{z^2}{2z^2+5z+2} \cdot dz$, where c is $|z|=1$.

$\Rightarrow$ Now,

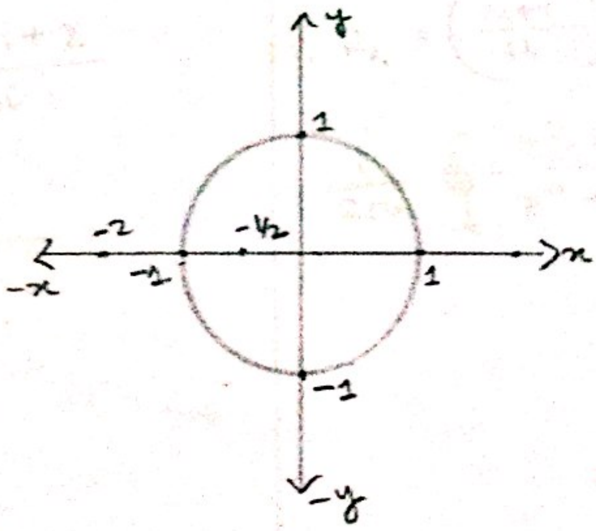
$$2z^2+5z+2=0$$

$$\Rightarrow 2z^2+4z+z+2=0$$

$$\Rightarrow 2z(z+2)+1(z+2)=0$$

$$\Rightarrow (z+2)\cdot(2z+1)=0$$

$\therefore (z+2)=0$ | $\therefore (2z+1)=0$
$\Rightarrow z=-2$ | $\Rightarrow z=-\frac{1}{2}$
$\therefore z=-2$ | $\therefore z=-\frac{1}{2}$



$\therefore z=-\frac{1}{2}$ is valid for $|z|=1$.

$\therefore$ Residue at $z = -1/2$

Here, $a = -1/2$

$m = 1$

$$\therefore R_1 = \lim_{z \to a} \cdot \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \cdot \left\{ (z-a)^m f(z) \right\}$$

$$= \lim_{z \to -1/2} \cdot \frac{1}{0!} \cdot 1 \cdot \cancel{(z+1/2)} \cdot \frac{z^2}{(z+2) \cdot \cancel{(z+1/2)}}$$

$$= \lim_{z \to -1/2} \cdot \frac{z^2}{(z+2)}$$

$$= \frac{(-1/2)^2}{-1/2 + 2}$$

$$= \frac{1/4}{\frac{-1+4}{2}}$$

$$= \frac{1}{4} \cdot \frac{2}{3}$$

$$\therefore R_1 = \frac{1}{6}$$

$$\therefore \oint_c \frac{z^2}{2z^2+5z+2} \cdot dz = 2\pi i \cdot R_1$$

$$= 2\pi i \cdot \frac{1}{6}$$

$$= \left(\frac{\pi i}{3}\right) \text{ (ans)}$$

***④ Evaluate, $\oint_c \frac{z^2+4}{z^3+2z^2+2z} \cdot dz$; where c is $|z|=3$.

$\Rightarrow$ Now,

$$z^3+2z^2+2z=0$$

$$\Rightarrow z(z^2+2z+2)=0$$

$$\Rightarrow z\cdot(z^2+2z+1+1)=0$$

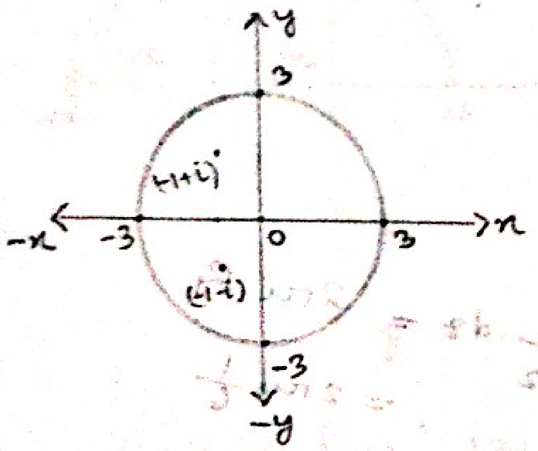
$$\Rightarrow z\cdot\{(z+1)^2+1\}=0$$

$$\Rightarrow z\cdot\{(z+1)^2-i^2\}=0$$

$$\Rightarrow z\cdot(z+1+i)\cdot(z+1-i)=0.$$

$$\therefore z=0, (-1-i), (-1+i)$$

Here,



$\therefore z=0, (-1-i), (-1+i)$ are valid at $|z|=3$.

$\therefore$ Residue at $z=0$

Here, $a=0$

$m=1$

$$\therefore R_1 = \lim_{z \to 0} \frac{1}{0!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m \cdot f(z) \cdot \right\}$$

$$= \lim_{z \to 0} \cdot 1 \cdot 1 \cdot (z-0) \frac{z^2+4}{z \cdot (z+1+i) \cdot (z+1-i)}$$

$$= \lim_{z \to 0} \cdot z \cdot \frac{z^2+4}{z \cdot (z+1+i) \cdot (z+1-i)}$$

$$= \lim_{z \to 0} \cdot \frac{z^2+4}{(z+1+i) \cdot (z+1-i)}$$

$$= \frac{0+4}{(1+i) \cdot (1-i)}$$

$$= \frac{4}{1-i^2}$$

$$= \frac{4}{2}$$

$\therefore R_1 = 2$

Now, Residue at $z = (-1-i)$

Here, $a = (-1-i)$

$m = 1$

$$\therefore R_2 = \lim_{z \to a} \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m \cdot f(z) \right\}$$

$$= \lim_{z \to (-1-i)} \frac{1}{0!} \cdot \frac{d^0}{dz^0} \cdot \cancel{(z+1+i)} \cdot \frac{z^2+4}{z \cdot \cancel{(z+1+i)}(z+1-i)}$$

$$= \lim_{z \to (-1-i)} 1 . 1 . \frac{z^2+4}{z \cdot (z+1-i)}$$

$$= \frac{(-1-i)^2+4}{(-1-i) \cdot (-1-i+1-i)}$$

$$= \frac{1 - 2i \cdot (-1) + i^2 + 4}{(-1-i) \cdot (-2i)}$$

$$= \frac{1+2i-1+4}{2i+2i^2}$$

$$= \frac{2i+4}{2i-2}$$

$$= \frac{(2i+4) \cdot (2i+2)}{(2i-2) \cdot (2i+2)}$$

$$= \frac{4i^2+4i+8i+8}{(2i)^2 - 2^2}$$

$$= \frac{-4+12i+8}{(2i)^2 - 4}$$

$$= \frac{12i+4}{4i^2-4}$$

$$= \frac{12i+4}{-4-4}$$

$$= \frac{4(1+3i)}{-8}$$

$$= \frac{1+3i}{-2}$$

$$\therefore R_2 = -\left(\frac{1}{2}\right) - \left(\frac{3}{2}\right) \cdot i$$

$\therefore$ Residue at $z = (-1+i)$

Here, $a = (-1+i)$

$m = 1$

$$\therefore R_3 = -\left(\frac{1}{2}\right) + \left(\frac{3}{2}\right) i$$

$$\therefore \oint_c \frac{z^2+4}{z^3+2z^2+2z} \cdot dz = 2\pi i \cdot (R_1 + R_2 + R_3)$$

$$= 2\pi i \cdot \left(2 + \frac{-1}{2} + \cancel{\frac{-3}{2}i} + \frac{-1}{2} + \cancel{\frac{3}{2}i}\right)$$

$$= 2\pi i \cdot (2-1)$$

$$= 2\pi i. \text{ (Ans)}$$

***(5) Evaluate, $\oint_c \dfrac{z \cdot e^{i\pi z}}{(z^2+2z+5)\cdot(z^2+1)^2} \cdot dz$ where

c is upper half circle of $|z|=2$.

$\Rightarrow$ Now,

$$(z^2+2z+5)\cdot(z^2+1)^2=0$$

$$\Rightarrow (z^2+2z+1+4)\cdot(z^2-i^2)^2=0$$

$$\Rightarrow \{(z+1)^2+4\}\cdot(z+i)^2\cdot(z-i)^2=0$$

$$\Rightarrow \{(z+1)^2-4i^2\}\cdot(z+i)^2\cdot(z-i)^2=0$$

$$\Rightarrow (z+1+2i)\cdot(z+1-2i)\cdot(z+i)^2\cdot(z-i)^2=0$$

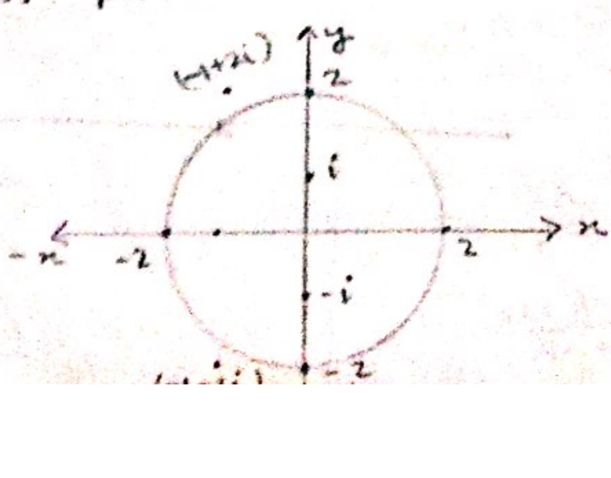
$$\therefore z=(-1-2i),\ (-1+2i),\ -i,\ +i.$$

Here,

$f(z)$ has simple poles at $z=(-1-2i),\ (-1+2i)$

$f(z)$ has pole of order 2 at $z=-i, i$.

Now, $|z|=2$



$\therefore$ Only $z = \pm i$ is valid for $|z|=2$

$\therefore z = +i$ is valid for upper half circle of $|z|=2$.

Now, Residue at $z=i$,

$$a = i$$

$$m = 2$$

$$\therefore R_1 = \lim_{z \to a} \cdot \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m \cdot f(z) \right\}$$

$$= \lim_{z \to i} \cdot \frac{1}{1!} \cdot \frac{d}{dz} \cdot \left\{ (z-i)^2 \cdot \frac{z \cdot e^{i\pi z}}{(z+1+2i) \cdot (z+1-2i) \cdot (z+i)^2 \cdot \cancel{(z-i)^2}} \right\}$$

$$= \lim_{z \to i} \cdot \frac{d}{dz} \cdot \left\{ \frac{z \cdot e^{i\pi z}}{(z^2+2z+5)(z+i)^2} \right\}$$

$$= \lim_{z \to i} \cdot \frac{(z^2+2z+5) \cdot (z+i)^2 \cdot (z \cdot e^{i\pi z} \cdot \pi i + e^{i\pi z}) - z \cdot e^{i\pi z} \cdot \left\{ (z^2+2z+5) \cdot 2(z+i) + (z+i)^2 \cdot (2z+2) \right\}}{(z^2+2z+5)^2 \cdot (z+i)^4}$$

$$= \frac{(i^2+2i+5) \cdot 4i^2 \cdot (i \cdot e^{i^2 \pi} \cdot \pi i + e^{i^2 \pi}) - i \cdot e^{i^2 \pi} \cdot \left\{ (i^2+2i+5) \cdot 4i + 4i^2 (2i+2) \right\}}{(i^2+2i+5)^2 \cdot 16 \cdot i^4}$$

$$= \frac{(-1+2i+5) \cdot (-4) (-\pi \cdot e^{-\pi} + e^{-\pi}) - i \cdot e^{-\pi} \cdot \left\{ (-1+2i+5) \cdot 4i - 4 \cdot (2i+2) \right\}}{(-1+2i+5)^2 \cdot 16}$$

$$= \frac{(-16-8i)\cdot(e^{-\pi}-\pi\cdot e^{-\pi}) - i\cdot e^{+\pi}\cdot\{(16i+8i^2)-8(i+1)\}}{(4+2i)^2\cdot 16}$$

$$= \frac{(-16-8i)\cdot(e^{-\pi}-\pi\cdot e^{-\pi}) - i\cdot e^{-\pi}\cdot\{16i-8-8i-8\}}{(16+16i+4i^2)\cdot 16}$$

$$= \frac{(-16-8i)\cdot(e^{-\pi}-\pi\cdot e^{-\pi}) - i\cdot e^{-\pi}\cdot(-16+8i)}{(16i+12)\cdot 16}$$

$$= \frac{-16\cdot e^{-\pi} + 16\pi\cdot e^{-\pi} - 8i\cdot e^{-\pi} + 8i\pi\cdot e^{-\pi} + 16i\cdot e^{-\pi} - 8i^2\cdot e^{-\pi}}{(16i+12)\cdot 16}$$

$$= \frac{-16\cdot e^{-\pi} + 16\pi\cdot e^{-\pi} + 8\cdot i\, e^{-\pi} + 8i\cdot\pi\cdot e^{-\pi} + 8\cdot e^{-\pi}}{16(16i+12)}$$

$$= \frac{-8\cdot e^{-\pi} + 16\cdot\pi\cdot e^{-\pi} + i\cdot(8e^{-\pi} + 8\pi\cdot e^{-\pi})}{16\cdot(16i+12)}$$

$$= \frac{-\frac{1}{2}\cdot e^{-\pi} + \pi\cdot e^{-\pi} + i\cdot(\frac{1}{2}\cdot e^{-\pi} + \frac{1}{2}\cdot\pi\cdot e^{-\pi})}{(12+16i)}$$

$$= \frac{e^{-\pi}\cdot\left(\pi-\frac{1}{2}\right)+\frac{i}{2}\cdot e^{-\pi}\cdot(1+\pi)}{(12+16i)}$$

$$= \frac{e^{-\pi}\cdot(2\pi-1)+i\cdot e^{-\pi}\cdot(\pi+1)}{2}\times\frac{1}{12+16i}$$

$$\therefore R_1 = \frac{e^{-\pi}\cdot(2\pi-1)+i\cdot e^{-\pi}\cdot(\pi+1)}{(24+32i)}$$

$$\therefore \oint_C \frac{z\cdot e^{i\pi z}}{(z^2+2z+5)(z^2+1)^2}\cdot dz = 2\pi i\cdot R_1$$

$$= 2\pi i\cdot\frac{e^{-\pi}\cdot(2\pi-1)+i\cdot e^{-\pi}\cdot(\pi+1)}{(24+32i)}$$

$$= \frac{ie^{-\pi}\cdot(4\pi^2-2\pi)-e^{-\pi}\cdot(2\pi^2+2\pi)}{(24+32i)}$$

$$= 2\pi\cdot\left[\frac{-e^{-\pi}\cdot(\pi+1)+i\cdot e^{-\pi}\cdot(2\pi-1)}{(24+32i)}\right]$$

$$= 2\pi\cdot e^{-\pi}\cdot\left[\frac{-(\pi+1)+i\cdot(2\pi-1)}{(24+32i)}\right]$$

Taylor's series/Theorem:

Let $f(z)$ be analytic on and inside a closed curve c. Let "a" and "(a+h)" be two points inside c, then,

$$f(a+h) = f(a) + f'(a).h + \frac{f''(a)}{2!}.h^2 + \cdots\cdots + \frac{f^n(a)}{n!}.h^n + \cdots\cdots$$

Let, $(a+h) = z$ $\therefore h = (z-a)$

$$\therefore f(z) = f(a) + f'(a).(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots\cdots + \frac{f^n(a)}{n!}.(z-a)^n + \cdots$$

This is called Taylor's Theorem or series for $f(z)$.

*** ① Expand $f(z) = (z^3 - z^2 + 4z + 2)$ at $z = 2$. ⑤ ***

$\Rightarrow$ Here,

$f'(z) = 3z^2 - 2z + 4$ | $\therefore f(2) = 2^3 - 2^2 + 4.2 + 2 = 14$

$\therefore f'(2) = 3.2^2 - 2.2 + 4 = 12$

$f''(z) = 6z - 2$ | $\therefore f''(2) = 6.2 - 2 = 10$

$f'''(z) = 6$ | $\therefore f'''(2) = 6$

$f^{iv}(z) = 0$ | $\therefore f^{iv}(2) = 0$

$$\therefore f(z) = f(2) + f'(2).(z-2) + \frac{f''(2)}{2!}.(z-2)^2 + \frac{f'''(2)}{3!}.(z-2)^3 + \frac{f^{iv}(2)}{4!}.(z-2)^4.$$

$$= 14 + 12.(z-2) + \frac{10}{2}.(z-2)^2 + \frac{6}{2\times3}.(z-2)^3 + 0$$

$$\therefore (z^3 - z^2 + 4z + 2) = 14 + 12.(z-2) + 5.(z-2)^2 + (z-2)^3.$$

(Ans)

—x—

***② Expand $f(z) = \cos z$ at $z = \left(\frac{\pi}{2}\right)$.

$\Rightarrow$ Here,

$f(z) = \cos z$	$\therefore f(\pi/2) = \cos(\pi/2) = 0$
$\therefore f'(z) = -\sin z$	$\therefore f'(\pi/2) = -\sin(\pi/2) = -1$
$\therefore f''(z) = -\cos z$	$\therefore f''(\pi/2) = -\cos(\pi/2) = 0$
$\therefore f'''(z) = \sin z$	$\therefore f'''(\pi/2) = \sin(\pi/2) = 1$
$\therefore f^{iv}(z) = \cos z$	$\therefore f^{iv}(\pi/2) = \cos(\pi/2) = 0$
$\therefore f^{v}(z) = -\sin z$	$\therefore f^{v}(\pi/2) = -\sin(\pi/2) = -1$

$$\therefore f(z) = f(\pi/2) + f'(\pi/2)\cdot(z-\pi/2) + \frac{f''(\pi/2)}{2!}\cdot(z-\pi/2)^2 + \frac{f'''(\pi/2)}{3!}(z-\pi/2)^3$$
$$+ \frac{f^{iv}(\pi/2)}{4!}(z-\pi/2)^4 + \frac{f^{v}(\pi/2)}{5!}\cdot(z-\pi/2)^5$$

$$= 0 + (-1)(z-\pi/2) + 0 + \frac{1}{3!}(z-\pi/2)^3 + 0 + \frac{(-1)}{5!}\cdot(z-\pi/2)^5$$

$$= (-1)\cdot\frac{1}{1!}\cdot(z-\pi/2)^1 + (-1)^2\cdot\frac{1}{3!}\cdot(z-\pi/2)^3 + (-1)^3\cdot\frac{1}{5!}\cdot(z-\pi/2)^5$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\cdot(-1)^{n+1}\cdot(z-\pi/2)^{2n+1}$$ (Ans)

***③ Expand, e^{-z} at $z=0$

$\Rightarrow$ NOW,

$f(z) = e^{-z}$ | $\therefore f(0) = e^{-0} = 1$

$\therefore f'(z) = -e^{-z}$ | $\therefore f'(0) = -e^{-0} = -1$

$\therefore f''(z) = e^{-z}$ | $\therefore f''(0) = e^{-0} = 1$

$\therefore f'''(z) = -e^{-z}$ | $\therefore f'''(0) = -e^{-0} = -1$

$\therefore f^{iv}(z) = e^{-z}$ | $\therefore f^{iv}(0) = e^{-0} = 1$

$$\therefore f(z) = f(0) + f'(0).(z-0) + \frac{f''(0)}{2!}.(z-0)^2 + \frac{f'''(0)}{3!}(z-0)^3 + \frac{f^{iv}(0)}{4!}.(z-0)^4$$

$$= 1 + (-1).z + \frac{1}{2!}.z^2 + \frac{(-1)}{3!}z^3 + \frac{1}{4!}z^4$$

$$= 1 + (-1).\frac{z}{1!} + \frac{z^2}{2!} + (-1).\frac{z^3}{3!} + \frac{z^4}{4!}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!}.(-1)^n \quad \text{(Ans)}$$

***④ Expand, ze^{2z} at $z=-1$

$\Rightarrow$ NOW,

$$f(z) = z.e^{2z}$$

$$\therefore f'(z) = (2.z.e^{2z} + e^{2z})$$

$$\therefore f''(z) = 4.z.e^{2z} + 2.e^{2z} + 2.e^{2z}$$

$$= (4.z.e^{2z} + 4.e^{2z})$$

$$\therefore f'''(z) = 8z.e^{2z} + 4.e^{2z} + 8e^{2z}$$

$$= (8z.e^{2z} + 12.e^{2z})$$

$$\therefore f(-1) = (-1).e^{-2} = (-)e^{-2}$$

$$\therefore f'(-1) = [2.(-1).e^{-2} + e^{-2}] = (-)e^{-2}$$

$$\therefore f''(-1) = 4.(-1).e^{-2} + 4.e^{-2}$$

$$= 0$$

$$\therefore f'''(-1) = 8(-1).e^{-2} + 12.e^{-2}$$

$$= 4.e^{-2}$$

$$\therefore f(z) = f(-1) + f'(-1).(z+1). + \frac{f''(-1)}{2!}.(z+1)^2 + \frac{f'''(-1)}{3!}.(z+1)^3$$

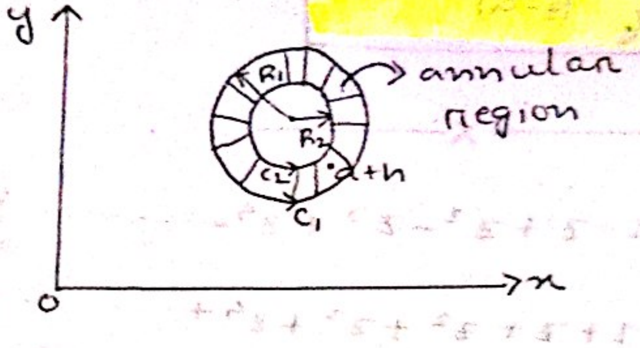
$$= -e^{-2} + (-e^{-2}).(z+1) + 0 + \frac{4.e^{-2}}{3!}.(z+1)^3$$

$$= (-)e^{-2} - e^{-2}.(z+1) + \frac{4}{3!}(z+1)^3.e^{-2}$$

$$= e^{-2}.\left[-1 - (z+1) + \frac{(z+1)^3}{3!}.4 + \cdots\cdots\right]$$

(ans)

*** Laurent Theorem/ Series:



Here,

$$f(a+h) = a_0 + a_1 h + a_2 \cdot h^2 + \dots\dots + \frac{a_{-1}}{h} + \frac{a_{-2}}{h^2} + \frac{a_{-3}}{h^3} + \dots\dots$$

where,

$$a_n = \frac{1}{2\pi i} \cdot \int_c \frac{f(z)}{(z-a)^{(n+1)}} \cdot dz \; ; \; n = 0, 1, 2, 3 \dots\dots$$

$$a_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^{(-n+1)}} \cdot dz \; ; \; n = -1, -2, -3 \dots\dots$$

Let, $(a+h) = z$

$\therefore h = (z-a)$.

$$\therefore f(z) = \underbrace{a_0 + a_1 \cdot (z-a) + a_2 \cdot (z-a)^2 + \dots\dots}_{\text{positive part} \to \text{analytic part}}$$

$$+ \underbrace{\frac{a_{-1}}{(z-a)} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-3}}{(z-a)^3} + \dots\dots}_{\text{negative part} \to \text{principal part}}$$

*** If principal part is zero than Laurent series becomes Taylor series.

where,

$$\boxed{a_n = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} \cdot dz} \; ; \; n = 0, \pm 1, \pm 2 \cdots\cdots$$

———x———

*** $(1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - \cdots\cdots \; ; \; |z| < 1$

*** $(1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + \cdots\cdots \; ; \; |z| < 1$

———x———

***① Expand $\frac{1}{z(z-2)}$ in a laurent series valid for

ⓐ $0 < |z| < 2$ ⓑ $|z| > 2$.

⇒ Now,

$$\frac{1}{z \cdot (z-2)} = \frac{A}{z} + \frac{B}{(z-2)}$$

$$\Rightarrow \; 1 = A \cdot (z-2) + B \cdot z$$

Now, $z = 0$

$\therefore 1 = A(-2)$

$\therefore A = -\left(\frac{1}{2}\right)$

$z = 2$,

$\therefore 1 = 2B$

$\therefore B = (1/2)$

$$\therefore \quad \frac{1}{z\cdot(z-2)} = \frac{-1}{2\cdot z} + \frac{1}{2\cdot(z-2)}$$

Here,

(a) $|z| < 2$

$$\therefore \quad \frac{|z|}{2} < 1$$

$$\therefore \quad \frac{-1}{2z} + \frac{1}{2\cdot(z-2)}$$

$$= \frac{-1}{2\cdot z} + \frac{1}{-4\cdot(1-z/2)}$$

$$= \frac{-1}{2\cdot z} - \frac{1}{4}\cdot(1-z/2)^{-1}$$

$$= \frac{-1}{2\cdot z} - \frac{1}{4}\cdot\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\cdots\right]$$

$$= \frac{-1}{2\cdot z} - \frac{1}{4}\cdot\sum_{n=0}^{\alpha}\left(\frac{z}{2}\right)^n.$$

(Ans)

ⓑ $|z| > 2$

$\Rightarrow \frac{|z|}{2} > 1$

$\therefore \frac{2}{|z|} < 1.$

$$\therefore \frac{1}{z.(z-2)} = \frac{-1}{2z} + \frac{1}{2.(z-2)}$$

$$= \frac{-1}{2.z} + \frac{1}{2.z.\left(1-\frac{2}{z}\right)}$$

$$= \frac{-1}{2.z} + \frac{1}{2z}.\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{-1}{2.z} + \frac{1}{2z}.\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right]$$

$$= \frac{-1}{2.z} + \frac{1}{2z}.\sum_{n=0}^{\alpha}\left(\frac{2}{z}\right)^n$$

$$= \frac{-1}{2.z} + \frac{1}{2}.\sum_{n=0}^{\alpha}\frac{2^n}{z^{n+1}}$$

(Ans)

***② Expand $f(z) = \dfrac{z}{(z-1)\cdot(2-z)}$ in a Laurent series valid for ⓐ $|z-1|>1$; ⓑ $0<|z-2|<1$.

$\Rightarrow$ Now,

ⓐ Let, $(z-1)=u$

$\Rightarrow z=(u+1)$.

$\therefore |z-1|>1$

$\Rightarrow |u|>1$.

$\therefore \dfrac{1}{|u|}<1$.

$$\frac{1}{u}+\frac{1}{1-u}$$
$$\frac{1-\cancel{u}+\cancel{u}}{u(1-u)}$$

$$\therefore f(z) = \frac{z}{(z-1)\cdot(2-z)} = \frac{u+1}{u\cdot(1-u)}$$

$$= \frac{1}{u}+\frac{2}{1-u}$$

$$= \frac{1}{u}-\frac{2}{u-1}$$

$$= \frac{1}{u}-\frac{2}{u(1-\frac{1}{u})}$$

$$= \frac{1}{u}-\frac{2}{u}\cdot\left(1-\frac{1}{u}\right)^{-1}$$

$$= \frac{1}{u}-\frac{2}{u}\cdot\left[1+\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots\right]$$

$$= \frac{1}{(z-1)}-\frac{2}{(z-1)}\cdot\left[1+\frac{1}{(z-1)}+\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+\cdots\right]$$

$$= \frac{1}{(z-1)} - 2.\left[\frac{1}{(z-1)} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} + \cdots\cdots\right]$$

$$= \frac{1}{(z-1)} - 2.\sum_{n=0}^{\infty} \frac{1}{(z-1)^{n+1}}$$

(Ans)

ⓑ Let, $(z-2) = u$

$\therefore z = (u+2)$

Now, $|z-2| < 1$

$\therefore |u| < 1$

$$\therefore f(z) = \frac{z}{(z-1).(2-z)} = \frac{1}{z-1} + \frac{2}{2-z}$$

$$= \frac{1}{u+2-1} + \frac{2}{2-(u+2)}$$

$$= \frac{1}{u+1} + \frac{2}{2-u-2}$$

$$= \frac{1}{u+1} + \frac{2}{-u}$$

$$= \frac{1}{u+1} - \frac{2}{u}$$

$$= -\frac{2}{u} + (u+1)^{-1}$$

$$= -\frac{2}{u} + \left[1 - u + u^2 - u^3 + \cdots\cdots\cdots\right]$$

$$= -\frac{2}{u} + \sum_{n=0}^{\alpha} \cdot (-1)^n \cdot u^n$$

$$= -\frac{2}{(z-2)} + \sum_{n=0}^{\alpha} \cdot (-1)^n \cdot (z-2)^n \quad \text{(Ans)}$$

***(3) Expand $f(z) = \frac{z}{(z-1)\cdot(z-2)}$ in a Laurent series valid for (a) $|z|<1$; (b) $1<|z|<2$; (c) $|z|>2$

$\Rightarrow$ (a) Here,

$|z|<1$

Now,

$$f(z) = \frac{z}{(z-1)(2-z)}$$

$$= \frac{1}{z-1} + \frac{2}{2-z}$$

$$= \frac{1}{-(1-z)} + \frac{2}{2(1-z/2)}$$

$$= -(1-z)^{-1} + (1-z/2)^{-1}$$

$$= -\left[1+z+z^2+z^3+\cdots\right] + \left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

$$= -\left[z+z^2+z^3+\cdots\right] + \left[\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

$$= \sum_{n=0}^{\alpha} \cdot(-1)\cdot z^n + \sum_{n=0}^{\alpha} \cdot \frac{z^n}{2^n}$$

$$= \sum_{n=0}^{\infty} \cdot \left(-1 + \frac{1}{2^n}\right) \cdot z^n$$

(Ans)

ⓑ $1 < |z| < 2$

Here, $f(z) = \dfrac{z}{(z-1)\cdot(2-z)}$

$$\Rightarrow f(z) = \frac{1}{(z-1)} + \frac{2}{(2-z)}$$

Now,

$1 < |z|$ $\Rightarrow |z| > 1$ $\therefore \frac{1}{|z|} < 1$

$|z| < 2$ $\therefore \frac{|z|}{2} < 1$

$$= \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{2}{2\left(1-z/2\right)}$$

$$= \frac{1}{z}\cdot\left(1-1/z\right)^{-1} + \left(1-z/2\right)^{-1}$$

$$= \frac{1}{z}\cdot\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\ldots\right] + \left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\ldots\right]$$

$$= \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

(Ans)

Ⓒ $|z|>2$

Now,

$|z|>2$

$\Rightarrow \frac{|z|}{2}>1$

$\therefore \frac{2}{|z|}<1$

Here, $f(z) = \dfrac{z}{(z-1)\cdot(2-z)}$

$= \dfrac{1}{(z-1)} + \dfrac{2}{(2-z)}$

$= \dfrac{1}{z(1-\frac{1}{z})} + \dfrac{2}{z\cdot(\frac{2}{z}-1)}$

$= \frac{1}{z}\cdot(1-1/z)^{-1} + \frac{-2}{z}\cdot(1-\frac{2}{z})^{-1}$

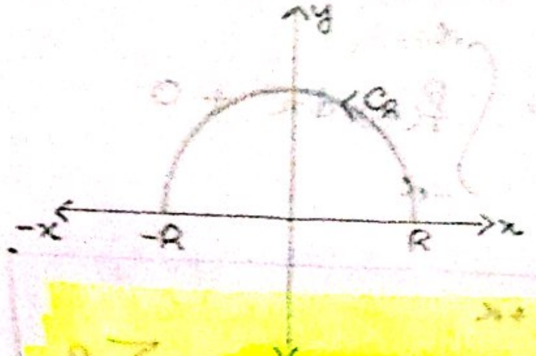
$= \frac{1}{z}\cdot\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right] - \frac{2}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3+\cdots\right]$

$= \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{2}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$ (Ans)

P.S→76:

Improper Integral:

Application of Residue:



Here,

$$\boxed{\int_c f(z).dz = \int_{-R}^{R} f(z).dz + \int_{C_R} f(z)\,dz} \quad \text{——— (i)}$$

Now, along x-axis, $y=0$ $\quad\Big|\ \therefore z=(x+i.y)=(x+i.0)=x$

$\therefore dy=0 \quad\Big|\ \therefore dz=dx$

$\therefore \text{(i)} \Rightarrow \quad \int_c f(z).dz = \int_{-R}^{R} f(x+iy).(dx+i.dy) + \int_{C_R} f(z).dz$

***

$$\therefore \int_c f(z).dz = \int_{-R}^{R} f(x).dx + \int_{C_R} f(z).dz \quad \text{——— (ii)}$$

Now, suppose;

$R \to \alpha$; then the integral over C_R approaches to zero.

$\therefore$ (11) $\Rightarrow$

$$\int_c f(z).dz = \int_{-\alpha}^{\alpha} f(x).dx + 0$$

$$\therefore \boxed{\int_c f(z).dz = \int_{-\alpha}^{+\alpha} f(x).dx = 2\pi i \sum R_n}$$

***① Show that $\int_0^{\infty} \frac{\ln(x^2+1)}{(x^2+1)} \cdot dx = \pi \cdot \ln 2.$

$\Rightarrow$ Now, consider, $\int_c \frac{\ln(z+i)}{(z^2+1)} \cdot dz$

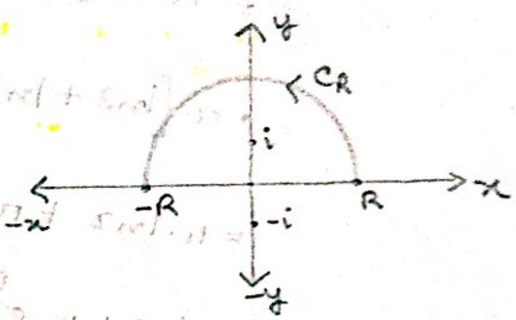
Let, $f(z) = \frac{\ln(z+i)}{(z^2+1)}$

$\therefore z^2+1=0$

$\Rightarrow z^2 = -1$

$\Rightarrow z^2 = i^2$

$\therefore z = \pm i$



$\therefore$ Only $z=i$ is valid for above region.

$\therefore$ Residue at $z=i$,

$a=i$,

$m=1$

$\therefore R = \lim_{z \to a} \cdot \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m \cdot f(z) \right\}$

$= \lim_{z \to i} \frac{1}{0!} \cdot 1 \cdot \cancel{(z-i)} \cdot \frac{\ln(z+i)}{(z+i) \cdot \cancel{(z-i)}}$

$$= \lim_{z \to i} \cdot \frac{\ln(z+i)}{(z+i)}$$

$$\therefore R = \frac{\ln(2i)}{2i}$$

$$\therefore \int_c f(z).dz = 2\pi i . R$$

$$= 2\pi i . \frac{\ln(2i)}{2i}$$

$$= \pi . \ln(2i)$$

$$= \pi . [\ln 2 + \ln i]$$

$$= \pi . \ln 2 + \pi . \ln i$$

$$= \pi . \ln 2 + \pi . e^{i(\pi/2)}$$

$$= \pi . \ln 2 + \pi . i . \pi/2$$

$$\therefore \int_c f(z).dz = \pi . \ln 2 + i . \left(\frac{\pi^2}{2}\right).$$

[Here, $x = 0,\ y = 1$

$\therefore r = 1$

$\therefore \theta = \tan^{-1}(1/0)$

$= \tan^{-1}(\infty)$

$= \tan^{-1}.(\tan \pi/2)$

$\therefore \theta = \pi/2$

$\therefore z = r.e^{i\theta}$

$\therefore z = e^{i.(\pi/2)}$.]

NOW,

$$\int_c f(z).dz = \int_{-R}^{R} f(z).dz + \int_{C_R} f(z).dz$$

$$\Rightarrow \pi \cdot \ln 2 + i\left(\frac{\pi^2}{2}\right) = \int_{-R}^{R} \frac{\ln(x+i)}{(x^2+1)} \cdot dx + \int_{C_R} f(z) \cdot dz$$

$\Big[$ along x-axis $\boxed{y=0}$ $\therefore z = (x+iy)$ $\therefore \boxed{z = x}$ $\therefore \boxed{z^2 = x^2}$

$$\therefore \pi \cdot \ln 2 + i\left(\frac{\pi^2}{2}\right) = \int_{-R}^{0} \frac{\ln(x+i)}{x^2+1} \cdot dx + \int_{0}^{R} \frac{\ln(x+i)}{x^2+1} \cdot dx + \int_{C_R} f(z)\, dz \quad \text{—— (1)}$$

Now, $\displaystyle\int_{-R}^{0} \frac{\ln(x+i)}{(x^2+1)} \cdot dx$

$\therefore$ Replacing x by $(-x)$,

$$\int_{R}^{0} \frac{\ln(-x+i)}{(x^2+1)} \cdot (-dx)$$

$$= \int_{0}^{R} \frac{\ln(i-x)}{(x^2+1)} \cdot dx$$

Now;

① ⇒

$$\pi \cdot \ln 2 + i \cdot \left(\frac{\pi^2}{2}\right) = \int_0^R \frac{\ln(x+i) \cdot dx}{x^2+1} + \int_0^R \frac{\ln(i-x)}{x^2+1} \cdot dx + \int_{C_R} f(z) \cdot dz$$

$$= \int_0^R \frac{1}{x^2+1} \cdot \ln\left[(x+i) \cdot (i-x)\right] \cdot dx + \int_{C_R} f(z) \cdot dz$$

$$= \int_0^R \frac{1}{x^2+1} \cdot \ln(-1-x^2) \cdot dx + \int_{C_R} f(z) \cdot dz$$

$$= \int_0^R \frac{1}{(x^2+1)} \cdot \ln\left[(-1) \cdot (1+x^2)\right] \cdot dx + \int_{C_R} f(z) \cdot dz$$

$$= \int_0^R \frac{1}{(x^2+1)} \cdot \left[\ln(-1) + \ln(1+x^2)\right] \cdot dx + \int_{C_R} f(z) \cdot dz$$

$$= \int_0^R \frac{1}{(x^2+1)} \cdot \left[\ln . e^{\pi i} + \ln(1+x^2)\right] . dx + \int_{C_A} f(z) . dz$$

Here,

$x = -1, y = 0$

$\therefore r = 1$

$\therefore \theta = \tan^{-1}\left(\frac{0}{-1}\right)$

$= \tan^{-1}(\tan\pi)$

$\therefore \theta = \pi$

$\therefore \boxed{z = e^{i\pi}}$

$$= \lim_{R\to\infty} \int_0^R \frac{1}{(x^2+1)} \cdot \left[\pi i + \ln(1+x^2).\right] . dx$$

$$+ \lim_{R\to\infty} \int_{C_A} f(z) . dz$$

$$= \int_0^\infty \frac{1}{(x^2+1)} \cdot \left[\pi i + \ln(1+x^2)\right] . dx + 0$$

$$\therefore \pi . \ln 2 + i\left(\frac{\pi^2}{2}\right) = \int_0^\infty \frac{1}{(x^2+1)} \cdot \left[\ln(1+x^2) + \pi i\right]$$

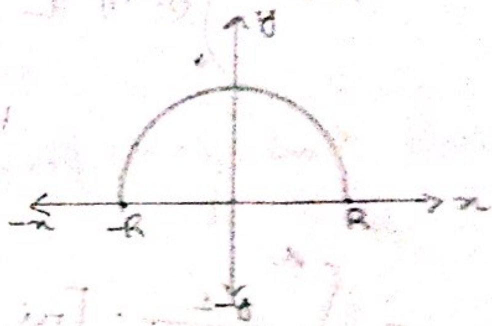
Now by equating real part,

$$\int_0^\infty \frac{1}{(x^2+1)} . \ln(1+x^2) dx = \pi . \ln 2$$

$$\therefore \int_0^\infty \frac{\ln(x^2+1) . dx}{(x^2+1)} = \pi . \ln 2 \quad \text{[Showed]}$$

② Show that $\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

$\Rightarrow$ Now, consider, $\int_c \frac{dz}{z^4+1}$



For pole, $z^4+1=0$

$\Rightarrow z^4 = -1$

$\therefore z = (-1)^{1/4}$

Here, $x=-1$

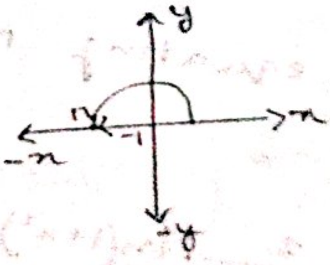
$y=0$

$\therefore r = \sqrt{x^2+y^2} = 1$

$\therefore \theta = \tan^{-1}(y/x)$

$= \tan^{-1}(0)$

$= \tan^{-1}(\tan\pi)$



$\therefore \theta = \pi$

$\therefore (-1) = r \cdot e^{i\cdot(\theta+2n\pi)}$

$\Rightarrow (-1) = e^{i(\pi+2n\pi)}$

$\Rightarrow (-1)^{1/4} = e^{i\cdot(\pi+2n\pi)\cdot\frac{1}{4}}$

$\therefore (-1)^{1/4} = e^{i\cdot\left(\frac{\pi}{4}+\frac{n\pi}{2}\right)}$; $n = 0, 1, 2, 3$.

Now,

$n=0, \quad z_1 = e^{i\left(\frac{\pi}{4}\right)} = e^{i\left(\frac{\pi}{4}\right)}$

$= \cos\left(\frac{\pi}{4}\right) + i \cdot \sin\left(\frac{\pi}{4}\right)$

$= \left(\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)$

$\therefore n=1, \quad z_2 = e^{i\left(\frac{\pi}{4} + \frac{\pi}{2}\right)} = e^{i\left(\frac{3\pi}{4}\right)}$

$= \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)$

$= -\left(\frac{1}{\sqrt{2}}\right) + i \frac{1}{\sqrt{2}}$

$\therefore n=2, \quad z_3 = e^{i \cdot \left(\frac{\pi}{4} + \pi\right)} = e^{i\left(\frac{5\pi}{4}\right)}$

$= \cos\left(\frac{5\pi}{4}\right) + i \cdot \sin\left(\frac{5\pi}{4}\right)$

$= \left(-\frac{1}{\sqrt{2}} - i \cdot \frac{1}{\sqrt{2}}\right)$

$\therefore n=3, \quad z_4 = e^{i\left(\frac{\pi}{4} + \frac{3\pi}{2}\right)} = e^{i \cdot \left(\frac{7\pi}{4}\right)}$

$= \cos\left(\frac{7\pi}{4}\right) + i \cdot \sin\left(\frac{7\pi}{4}\right)$

$= \left(\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}\right)$

$\therefore$ So, we have poles, $z = \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)$ lie inside the c.

Now, Residue at $z = \left(\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)$

Here, $a = \left(\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)$

$m = 1$

$$\therefore R_1 = \lim_{z \to a} \cdot \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \cdot \left\{ (z-a)^m \cdot f(z) \right\}$$

$$= \lim_{z \to \left(\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right)} \cdot \frac{1}{0!} \cdot 1 \cdot \cancel{\left(z - \frac{1}{\sqrt{2}} - i \cdot \frac{1}{\sqrt{2}}\right)} \cdot \frac{1}{\cancel{\left(z - \frac{1}{\sqrt{2}} - i \cdot \frac{1}{\sqrt{2}}\right)} \cdot \left(z + \frac{1}{\sqrt{2}} \cdots\right) \cdot \left(z - \frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}} \cdots\right)}$$

$$= \frac{1}{\left(\frac{1}{\sqrt{2}} + \cancel{i \cdot \frac{1}{\sqrt{2}}} + \frac{1}{\sqrt{2}} - \cancel{i \cdot \frac{1}{\sqrt{2}}}\right) \cdot \left(\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}\right) \cdot \left(\cancel{\frac{1}{\sqrt{2}}} + i \cdot \frac{1}{\sqrt{2}} - \cancel{\frac{1}{\sqrt{2}}} + i \cdot \frac{1}{\sqrt{2}}\right)}$$

$$= \frac{1}{\left(\frac{2}{\sqrt{2}}\right) \cdot \left(\frac{2}{\sqrt{2}} + \frac{2i}{\sqrt{2}}\right) \left(\frac{2i}{\sqrt{2}}\right)}$$

$$= \frac{1}{\frac{8i}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{1}{\frac{8i}{2} \cdot \left(\frac{1+i}{\sqrt{2}}\right)}$$

$$= \frac{1}{\frac{8i\,(1+i)}{2\sqrt{2}}}$$

$$= \frac{2\sqrt{2}}{8i.\,(1+i)}$$

$$= \frac{2\sqrt{2}.(1-i)}{8i\cdot(1-i^2)}$$

$$= \frac{2\sqrt{2}.\,(1-i)}{8i\,.\,2}$$

$$\therefore R_1 = \frac{\sqrt{2}(1-i)}{8i}$$

Now, Residue, $z = \left(-\frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}\right)$

$$\therefore R_2 = \lim_{z \to \left(-\frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}\right)} \cdot \left(z + \frac{1}{\sqrt{2}} - i.\frac{1}{\sqrt{2}}\right).\, f(z).$$

$$\therefore R_2 = \frac{\sqrt{2}(1+i)}{8i}$$

$$\therefore \int_c f(z).dz = 2\pi i.(R_1 + R_2) = 2\pi i.\left[\frac{\sqrt{2}.(1-i)}{8i} + \frac{\sqrt{2}.(1+i)}{8i}\right]$$

$$= 2\pi i \cdot \frac{2\sqrt{2}}{8i}$$

$$= \frac{\sqrt{2}\pi}{2}$$

$$\therefore \int_C f(z).dz = \left(\frac{\pi}{\sqrt{2}}\right)$$

Now

$$\int_C f(z).dz = \int_{-R}^{R} f(x).dx + \int_{C_R} f(z).dz$$

$$\Rightarrow \frac{\pi}{\sqrt{2}} = \lim_{R\to\alpha} \int_{-R}^{R} f(x).dx + \lim_{R\to\alpha} \int_{C_R} f(z).dz$$

$$= \int_{-\alpha}^{\alpha} \frac{1}{x^4+1}.dx + 0$$

$$\Rightarrow \left(\frac{\pi}{\sqrt{2}}\right) = \int_{-\alpha}^{0} \frac{dx}{(x^4+1)} + \int_{0}^{\alpha} \frac{dx}{(x^4+1)} \quad \text{——— } (1)$$

Now,

$$\int_{-\infty}^{0} \frac{dx}{(x^4+1)}$$

Replacing x by $(-x)$

$$\int_{\infty}^{0} \frac{-dx}{(x^4+1)} = \int_{0}^{\infty} \frac{dx}{(x^4+1)}$$

$\therefore$ ① $\Rightarrow$

$$\left(\frac{\pi}{\sqrt{2}}\right) = \int_{0}^{\infty} \frac{dx}{(x^4+1)} + \int_{0}^{\infty} \frac{dx}{(x^4+1)}$$

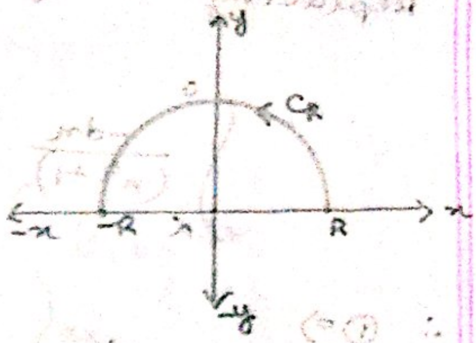
$$\Rightarrow \quad 2.\int_{0}^{\infty} \frac{dx}{(x^4+1)} = \frac{\pi}{\sqrt{2}}$$

$$\therefore \quad \int_{0}^{\infty} \frac{dx}{(x^4+1)} = \frac{\pi}{2\sqrt{2}}$$

[Showed]

(3) Show that $\displaystyle\int_0^{\infty} \frac{1}{(x^4+x^2+1)} \cdot dx = \frac{\pi\sqrt{3}}{6}$.

$\Rightarrow$ Now, consider, $\displaystyle\int_c \frac{1}{z^4+z^2+1} \cdot dz$



Here, $f(z) = \dfrac{1}{z^4+z^2+1}$

$\therefore$ for poles, $z^4+z^2+1=0$

$\Rightarrow z^4+2z^2+1-z^2=0$

$\Rightarrow (z^2+1)^2 - z^2 = 0$

$\Rightarrow (z^2+z+1) \cdot (z^2-z+1) = 0$

$\therefore (z^2+z+1) = 0$ or $(z^2-z+1) = 0$

$\Rightarrow z = \dfrac{-1 \pm \sqrt{3}i}{2}$	$\Rightarrow z = \dfrac{1 \pm i \cdot \sqrt{3}}{2}$
$\therefore z = -\left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)$	$\therefore z = \left(\frac{1}{2}\right) + i \cdot \left(\frac{\sqrt{3}}{2}\right)$
$\therefore z = -\left(\frac{1}{2}\right) - i\left(\frac{\sqrt{3}}{2}\right)$	$\therefore z = \left(\frac{1}{2}\right) - i \cdot \left(\frac{\sqrt{3}}{2}\right)$

$\therefore$ So, we have poles, $z = \left(-\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}\right), \left(\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}\right)$ lie inside c.

Now Residue at $z = -\left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)$; $m = 1$, $a = -\left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)$.

$$\therefore R_1 = \lim_{z \to a} \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

$$= \lim_{z \to -\left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)} \frac{1}{0!} \cdot 1 \left(\cancel{z + \frac{1}{2} - i\frac{\sqrt{3}}{2}}\right) \cdot \frac{1}{\left(\cancel{z + \frac{1}{2} - i\frac{\sqrt{3}}{2}}\right)\left(z - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(z + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(z - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(-\cancel{\frac{1}{2}} + i\frac{\sqrt{3}}{2} + \cancel{\frac{1}{2}} + i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{-1\left(i\sqrt{3}\right)\left(-1 + i\sqrt{3}\right)}$$

$$= \frac{1}{i\sqrt{3}\left(1 - i\sqrt{3}\right)}$$

$$= \frac{\left(1 + i\sqrt{3}\right)}{i\sqrt{3}\left(1 - 3i^2\right)}$$

$$= \frac{\left(1 + i\sqrt{3}\right)}{i\sqrt{3}\left(1+3\right)}$$

$$\therefore R_1 = \left(\frac{1 + i\sqrt{3}}{4i\sqrt{3}}\right)$$

Now Residue, $z = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$$\therefore R_2 = \lim_{z \to \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)} \left(z - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right) f(z)$$

$$\therefore R_2 = \left(\frac{1 - i\sqrt{3}}{4i\sqrt{3}}\right)$$

$$\therefore \int_c f(z)\,.\,dz = 2\pi i\,.\,(R_1+R_2)$$

$$= 2\pi i\,.\left[\frac{1+i\sqrt{3}}{4i\sqrt{3}} + \frac{1-i\sqrt{3}}{4i\sqrt{3}}\right]$$

$$= 2\pi i\left(\frac{2}{4i\sqrt{3}}\right)$$

$$\therefore \int_c f(z)\,.\,dz = \frac{\pi}{\sqrt{3}}\,.$$

Now,

$$\int_c f(z)\,.\,dz = \int_{-R}^{R} f(x)\,.\,dx + \int_{C_R} f(z)\,.\,dz$$

$$\Rightarrow \frac{\pi}{\sqrt{3}} = \int_{-\alpha}^{\alpha} \frac{1}{x^4+x^2+1}\,.\,dx + 0$$

$$\Rightarrow \frac{\pi}{\sqrt{3}} = 2\,.\int_{0}^{\alpha} \frac{dx}{x^4+x^2+1}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{(x^4+x^2+1)} = \frac{\pi}{2\sqrt{3}}.$$

$$\therefore \int_0^{\infty} \frac{dx}{(x^4+x^2+1)} = \frac{\pi\sqrt{3}}{6}$$

[showed]

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+x^2+1} = \int_{-\infty}^{0} \frac{dx}{x^4+x^2+1} + \int_{0}^{\infty} \frac{dx}{x^4+x^2+1}$$

Now, $\displaystyle\int_{-\infty}^{0} \frac{dx}{x^4+x^2+1} = \int_{\infty}^{0} \frac{(-)dx}{x^4+x^2+1}$ [By replacing x by $(-x)$]

$$= \int_0^{\infty} \frac{dx}{x^4+x^2+1}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{x^4+x^2+1} = 2\int_0^{\infty} \frac{dx}{x^4+x^2+1}$$

***(4) show that, $\int_0^{\alpha} \frac{\cos 2\pi x}{x^4+x^2+1} \cdot dx = \frac{-\pi}{2\sqrt{3}} \cdot e^{-\pi \cdot \sqrt{3}}$

$\Rightarrow$ Now. consider, $\int_C \frac{e^{i \cdot 2\pi z}}{z^4+z^2+1} \cdot dz$

$\therefore f(z) = \frac{e^{i 2\pi z}}{z^4+z^2+1}$

for poles, $(z^4+z^2+1)=0$

$\Rightarrow z^4+2z^2+1-z^2=0$

$\Rightarrow (z^2+1)^2 - z^2 = 0$

$\Rightarrow (z^2+z+1)\cdot(z^2-z+1)=0$

$\therefore (z^2+z+1)=0$

$\Rightarrow z = \frac{-1 \pm i\sqrt{3}}{2}$

$\therefore z = -\left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)$

$\therefore z = -\left(\frac{1}{2}\right) - i\left(\frac{\sqrt{3}}{2}\right)$

or $(z^2-z+1)=0$

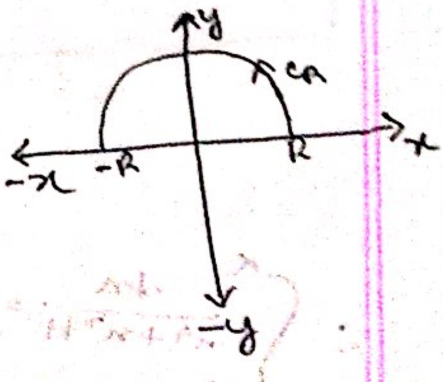
$\Rightarrow z = \frac{-(-1) \pm \sqrt{(-1)^2-4.1.1}}{2.1}$

$\Rightarrow z = \frac{1 \pm \sqrt{-3}}{2}$

$\Rightarrow z = \frac{1 \pm i\sqrt{3}}{2}$

$\therefore z = \left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)$

$\therefore z = \left(\frac{1}{2}\right) - i\left(\frac{\sqrt{3}}{2}\right)$



So, we have poles $z = -\left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)$; $+\left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)$.

Now,

Residue at $z = -\left(\frac{1}{2}\right) + i\cdot\left(\frac{\sqrt{3}}{2}\right)$

$\therefore a = -\left(\frac{1}{2}\right) + i\cdot\left(\frac{\sqrt{3}}{2}\right)$

$\therefore m = 1$

$$\therefore R_1 = \lim_{z \to a} \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \cdot \left\{ (z-a)^m \cdot f(z) \right\}$$

$$= \lim_{z \to \left(-\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right)} \frac{1}{0!} \cdot 1 \cdot \cancel{\left(z + \frac{1}{2} - i\cdot\frac{\sqrt{3}}{2}\right)} \cdot \frac{e^{i\cdot 2\pi z}}{\cancel{\left(z + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)}\cdot\left(z + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(z - \frac{1}{2} - i\cdot\frac{\sqrt{3}}{2}\right)\cdot\left(z - \frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right)}$$

$$= \lim_{z \to \left(-\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right)} \frac{e^{i\cdot 2\pi z}}{\left(z + \frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right)\cdot\left(z - \frac{1}{2} - i\cdot\frac{\sqrt{3}}{2}\right)\cdot\left(z - \frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{e^{i\cdot 2\pi\cdot\left(-\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right)}}{\left(\cancel{-\frac{1}{2}} + i\cdot\frac{\sqrt{3}}{2} + \cancel{\frac{1}{2}} + i\cdot\frac{\sqrt{3}}{2}\right)\cdot\left(-\frac{1}{2} + \cancel{i\cdot\frac{\sqrt{3}}{2}} - \frac{1}{2} - \cancel{i\cdot\frac{\sqrt{3}}{2}}\right)\cdot\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{e^{i\cdot 2\pi\cdot\left(-\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right)}}{i\sqrt{3}\cdot(-1)\cdot(-1 + i\cdot\sqrt{3})}$$

$$= \frac{e^{i(-\pi + \pi i\cdot\sqrt{3})}}{(1 - i\cdot\sqrt{3})\cdot i\cdot\sqrt{3}} = \frac{e^{-i\pi}\cdot e^{-\sqrt{3}\pi}\cdot(1 + i\cdot\sqrt{3})}{i\cdot\sqrt{3}\cdot(1 - i^2\cdot 3)}$$

$$= \frac{(-1).e^{-\sqrt{3}\pi}.(1+i\sqrt{3})}{4.i.\sqrt{3}}$$

$$e^{-i\pi} = \cos\pi - i.\sin\pi$$

$$= (-1) - 0$$

$$\therefore \boxed{e^{-i\pi} = (-1)}$$

$$\therefore R_1 = \frac{e^{-\sqrt{3}.\pi}.(-1-i.\sqrt{3})}{4.i.\sqrt{3}}$$

Now,

Residue at $z = \left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)$

$$\therefore R_2 = \lim_{z \to \left(\frac{1}{2}+i.\frac{\sqrt{3}}{2}\right)} \left(z - \frac{1}{2} - i.\frac{\sqrt{3}}{2}\right).f(z)$$

$$\therefore R_2 = \frac{e^{-\sqrt{3}\pi}.(-1+i.\sqrt{3})}{4i.\sqrt{3}}$$

Now,

$$\int_c f(z).dz = 2\pi i.(R_1+R_2)$$

$$= 2\pi i.\left[\frac{e^{-\sqrt{3}.\pi}.(-1-i.\sqrt{3})}{4.i.\sqrt{3}} + \frac{e^{-\sqrt{3}\pi}.(-1+i.\sqrt{3})}{4.i.\sqrt{3}}\right]$$

$$= 2\pi i \frac{-e^{-\sqrt{3}\pi} - \cancel{i.\sqrt{3}.e^{-\sqrt{3}\pi}} - e^{-\sqrt{3}\pi} + \cancel{i.\sqrt{3}.e^{-\sqrt{3}\pi}}}{4i.\sqrt{3}}$$

$$= 2\pi i \frac{-2.e^{-\sqrt{3}\pi}}{4i\sqrt{3}}$$

$$\therefore \int_C f(z).dz = -\pi \frac{e^{-\sqrt{3}\pi}}{\sqrt{3}}$$

Here,

$$\int_C f(z).dz = \int_{-R}^{R} f(x).dx + \int_{C_R} f(z).dz$$

$$\Rightarrow -\pi . \frac{e^{-\sqrt{3}\pi}}{\sqrt{3}} = \int_{-R}^{R} \frac{e^{i.2\pi x}}{x^4+x^2+1} dx + \int_{C_R} f(z).dz$$

$$\Rightarrow -\frac{\pi}{\sqrt{3}}.e^{-\pi.\sqrt{3}} = \lim_{R\to\infty} . \int_{-R}^{R} \frac{\cos 2\pi x + i \sin 2\pi x}{x^4+x^2+1} dx + \lim_{R\to\infty} \int_{C_R} f(z).dz$$

$$\Rightarrow -\frac{\pi}{\sqrt{3}}.e^{-\pi\sqrt{3}} = \int_{-\infty}^{\infty} \frac{\cos 2\pi x + i \sin 2\pi x}{x^4+x^2+1} dx + 0$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos 2\pi x + i . \sin 2\pi x}{x^4+x^2+1} dx = -\frac{\pi}{\sqrt{3}}.e^{-\pi\sqrt{3}}$$

Equating real parts;

$$\int_{-\alpha}^{\alpha} \frac{\cos 2\pi x}{x^4+x^2+1} \cdot dx = -\frac{\pi}{\sqrt{3}} \cdot e^{-\sqrt{3}\pi}$$

$$\Rightarrow \int_{-\alpha}^{0} \frac{\cos 2\pi x}{x^4+x^2+1} \cdot dx + \int_{0}^{\alpha} \frac{\cos 2\pi x}{x^4+x^2+1} \cdot dx = -\frac{\pi}{\sqrt{3}} \cdot e^{-\sqrt{3}\pi} \quad \text{——(1)}$$

Now,

$$\int_{-\alpha}^{0} \frac{\cos 2\pi x}{x^4+x^2+1} \cdot dx$$

Here replacing x by $(-x)$

$$\therefore \int_{-\alpha}^{0} \frac{\cos 2\pi x}{x^4+x^2+1} \cdot dx = \int_{\alpha}^{0} \frac{\cos 2\pi(-x)}{x^4+x^2+1} \cdot (-) \cdot dx$$

$$= \int_{0}^{\alpha} \frac{\cos 2\pi x}{x^4+x^2+1} \cdot dx$$

$\therefore$ ① $\Rightarrow$

$$2\cdot\int_0^{\infty}\frac{\cos 2\pi x}{x^4+x^2+1}\cdot dx=-\frac{\pi}{\sqrt{3}}\cdot e^{-\pi\cdot\sqrt{3}}$$

$$\therefore \int_0^{\infty}\frac{\cos 2\pi x}{x^4+x^2+1}\cdot dx=-\frac{\pi}{2\sqrt{3}}\cdot e^{-\pi\cdot\sqrt{3}}$$

[showed]

---x---

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Laplace Transformation:

Here,

$F(t), \ t>0$

$$\therefore \mathcal{L}\{F(t)\} = f(s) = \int_0^\alpha e^{-st} . f(t).dt$$

If, $F(t) = e^{at}$, $\mathcal{L}\{F(t)\} - \mathcal{L}\{e^{at}\}$

$$= \int_0^\alpha e^{-st} . e^{at} . dt$$

$$= \left(\frac{1}{s-a}\right)$$

Inverse Laplace Transformation:

$F(t) = \mathcal{L}^{-1}.\{f(s)\}$ is the inverse Laplace Transformation of $\mathcal{L}.\{F(t)\} = f(s)$

Some Information about Laplace Transformation: The **Laplace transform** is a widely used integral transform with many applications in physics and engineering. Denoted $\mathcal{L}\{f(t)\}$, it is a linear operator of a function $F(t)$ with a real argument t ($t \geq 0$) that transforms $F(t)$ to a function $f(s)$ with complex argument s. This transformation is bijective for the majority of practical uses; the most-common pairs of $F(t)$ and $f(s)$ are often given in tables for easy reference. The Laplace transform has the useful property that many relationships and operations over the original $F(t)$ correspond to simpler relationships and operations over its image $f(s)$. It is named after Pierre-Simon Laplace, who introduced the transform in his work on probability theory.

The Laplace transform is related to the Fourier transform, but whereas the Fourier transform expresses a function or signal as a series of modes of vibration (frequencies), the Laplace transform resolves a function into its moments. Like the Fourier transform, the Laplace transform is used for solving differential and integral equations. In physics and engineering it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems. In such analyses, the Laplace transform is often interpreted as a transformation from the *time-domain*, in which inputs and outputs are functions of time, to the *frequency-domain*, where the same inputs and outputs are functions of complex angular frequency, in radians per unit time. Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional description that often simplifies the process of analyzing the behavior of the system, or in synthesizing a new system based on a set of specifications.

Source :< http://en.wikipedia.org/wiki/Laplace_transform>

Laplace Transformation of some elementary functions:

	$F(t)$	$L\{F(t)\} = f(s)$
1.	1	$\frac{1}{s}$, s>0
2.	t	$\frac{1}{s^2}$, s>0
3.	t^n n= 0, 1, 2, ...	$\frac{n!}{s^{n+1}}$, s>0
4.	e^{at}	$\frac{1}{s-a}$, $s > a$
5.	$sinat$	$\frac{a}{s^2+a^2}$, $s > 0$
6.	$cosat$	$\frac{s}{s^2+a^2}$, $s > 0$
7.	$sinhat$	$\frac{a}{s^2-a^2}$, $s > \lvert a \rvert$

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8.	$coshat$	$\frac{s}{s^2-a^2}, s > \lvert a \rvert$

Inverse Laplace Transformation

	$f(s)$	$L^{-1}\{f(s)\} = F(t)$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s^2}$	t
3.	$\frac{1}{s^{n+1}}$, $n = 0,1,2,\ldots$	$\frac{t^n}{n!}$
4.	$\frac{1}{s-a}$	e^{at}
5.	$\frac{1}{s^2+a^2}$	$\frac{sinat}{a}$
6.	$\frac{s}{s^2+a^2}$	$cosat$
7.	$\frac{1}{s^2-a^2}$	$\frac{sinhat}{a}$
8.	$\frac{s}{s^2-a^2}$	$coshat$
9.	$\frac{1}{s^{n+1}}, n > -1$	$\frac{t^n}{(n+1)!}$, $n > -1$

*** 1st Transformation or shifting property:

If, $\mathcal{L}\{f(t)\} = f(s)$ then $\boxed{\mathcal{L}.\{e^{at} f(t)\} = f(s-a)}$

** ① Find Laplace Transformation of the following functions:

① $3e^{-2t}$

$\therefore \mathcal{L}\{3.e^{-2t}\}$

$= 3.\mathcal{L}.\{e^{-2t}\}$

$= 3.\dfrac{1}{s-(-2)}$

$= \dfrac{3}{(s+2)}$ (ans)

② $4t^3 - e^{-t}$

$\therefore \mathcal{L}\{4t^3 - e^{-t}\}$

$= \mathcal{L}.\{4t^3\} - \mathcal{L}.\{e^{-t}\}$

$= 4.\mathcal{L}.\{t^3\} - \mathcal{L}.\{e^{-t}\}$

$= 4.\dfrac{3!}{s^{3+1}} - \dfrac{1}{s-(-1)}$

$= \dfrac{24}{s^4} - \dfrac{1}{(s+1)}$

(ans)

③ $7 \cdot \sin 2t - 3 \cdot \cos 2t$

$\therefore \mathcal{L}\{7 \sin 2t - 3 \cdot \cos 2t\}$

$= \mathcal{L}\{7 \sin 2t\} - \mathcal{L}\{3 \cdot \cos 2t\}$

$= 7 \cdot \mathcal{L}\{\sin 2t\} - 3 \cdot \mathcal{L}\{\cos 2t\}$

$= 7 \cdot \dfrac{2}{(s^2+2^2)} - 3 \cdot \dfrac{s}{(s^2+2^2)}$

$= \dfrac{14}{(s^2+4)} - \dfrac{3s}{(s^2+4)}$

$= \dfrac{(14-3s)}{(s^2+4)}$ (ans)

④ $(t^2+1)^2$

$\therefore \mathcal{L}\{(t^2+1)^2\}$

$= \mathcal{L}\{t^4+2t^2+1\}$

$= \mathcal{L}\{t^4\} + 2 \cdot \mathcal{L}\{t^2\} + \mathcal{L}\{1\}$

$= \dfrac{4!}{s^{4+1}} + 2 \cdot \dfrac{2!}{s^{2+1}} + \dfrac{1}{s}$

$= \left(\dfrac{24}{s^5} + \dfrac{4}{s^3} + \dfrac{1}{s}\right)$ (ans)

(5) $(4e^{2t}-2)^3$

$\therefore \mathcal{L}\{(4.e^{2t}-2)^3\}$

$= \mathcal{L}\{(4.e^{2t})^3 - 3.(4e^{2t})^2.2 + 3.4.e^{2t}.2^2 - 2^3\}$

$= \mathcal{L}\{64.e^{6t} - 96.e^{4t} + 48.e^{2t} - 8\}$

$= \mathcal{L}\{64.e^{6t}\} - \mathcal{L}\{96.e^{4t}\} + \mathcal{L}\{48.e^{2t}\} - \mathcal{L}\{8\}$

$= 64.\mathcal{L}\{e^{6t}\} - 96.\mathcal{L}\{e^{4t}\} + 48.\mathcal{L}\{e^{2t}\} - \mathcal{L}\{8\}$

$= 64.\frac{1}{s-6} - 96.\frac{1}{s-4} + 48.\frac{1}{s-2} - \frac{8}{s}$

$= \frac{64}{(s-6)} - \frac{96}{(s-4)} + \frac{48}{(s-2)} - \frac{8}{s}$

(Ans)

***② Evaluate Each of the following:

(i) $\mathcal{L}\{t^3 . e^{-3t}\}$

$= f(s+3)$

$= \frac{6}{(s+3)^4}$

(Ans)

Here, $f(s) = \mathcal{L}\{t^3\}$

$= \frac{3!}{s^{3+1}}$

$= \frac{6}{s^4}$

$\therefore f(s-(-3)) = f(s+3) = \frac{6}{(s+3)^4}$

***(ii) $\mathcal{L}.\{5 . e^{3t} . \sin 4t\}$

$= 5 . \mathcal{L}\{e^{3t} . \sin 4t\}$

$= 5 . f(s-3)$

$= 5 . \frac{4}{(s-3)^2 + 16}$

$= \frac{20}{s^2 - 6s + 9 + 16}$

Here, $f(s) = \mathcal{L}\{\sin 4t\}$

$= \frac{4}{s^2 + 4^2}$

$\therefore f(s) = \frac{4}{s^2 + 16}$

$\therefore f(s-3) = \frac{4}{(s-3)^2 + 16}$

$$= \frac{20}{s^2 - 6s + 25}$$

(Ans)

(iii) $\mathcal{L}\{(t+2)^2.e^t\}$

$$= \mathcal{L}\{(t^2+2t+4).e^t\}$$

$$= \mathcal{L}\{e^t.t^2\} + 2.\mathcal{L}\{e^t.t\} + 4.\mathcal{L}.\{e^t\}$$

$$= \frac{2}{(s-1)^3} + 2.\frac{1}{(s-1)^2} + 4.\frac{1}{(s-1)}$$

$$= \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{4}{(s-1)}$$

(Ans)

Here,

(i) $f(s) = \mathcal{L}.\{t^2\}$

$= \frac{2}{s^{2+1}}$

$= \frac{2}{s^3}$

$\therefore f(s-1) = \frac{2}{(s-1)^3}$

(ii) $f(s) = \mathcal{L}.\{t\}$

$\therefore f(s) = \frac{1}{s^2}$

$\therefore f(s-1) = \frac{1}{(s-1)^2}$

(iv) $\mathcal{L}.\{e^{-t}.(3.\sinh 2t - 5.\cosh 2t)\}$

$= \mathcal{L}\{e^{-t}.(3.\sinh 2t)\} - \mathcal{L}.\{e^{-t}.(5\cosh 2t)\}$

$= 3.\mathcal{L}.\{e^{-t}.\sinh 2t\} - 5.\mathcal{L}\{e^{-t}.\cosh 2t\}$

$= 3.\dfrac{2}{(s+1)^2-4} - 5.\dfrac{(s+1)}{(s+1)^2-4}$

$= \dfrac{6-5s-5}{(s+1)^2-4}$

$= \dfrac{(1-5s)}{(s^2+2s-3)}$

(Ans)

Here,

(i) $\mathcal{L}.\{.\sinh 2t\} = f(s)$

$\Rightarrow f(s) = \dfrac{2}{s^2-4}$

$\therefore f(s+1) = \dfrac{2}{(s+1)^2-4}$

(ii) $f(s) = \mathcal{L}\{\cosh 2t\}$

$\Rightarrow f(s) = \dfrac{s}{(s^2-4)}$

$\therefore f(s+1) = \dfrac{(s+1)}{(s+1)^2-4}$

***(v) $\mathcal{L}\{e^{-4t}\cosh 2t\}$ ***

Here,

$$f(s) = \mathcal{L}\{\cosh 2t\}$$

$$\Rightarrow f(s) = \frac{s}{(s^2-4)}$$

$$\therefore f(s-(-4)) = f(s+4) = \frac{(s+4)}{(s+4)^2-4}$$

$$\therefore \mathcal{L}\{e^{-4t}\cosh 2t\} = \frac{s+4}{(s+4)^2-4}$$

$$= \frac{s+4}{(s^2+8s+12)}$$

***(iv) $\mathcal{L}\{e^{2t}(3\cdot\sin 4t - 4\cdot\cos 4t)\}$

$= \mathcal{L}\{e^{2t}\cdot 3\cdot\sin 4t\} - \mathcal{L}\{e^{2t}\cdot 4\cdot\cos 4t\}$

$= 3\cdot\mathcal{L}\{e^{2t}\cdot\sin 4t\} - 4\mathcal{L}\{e^{2t}\cdot\cos 4t\}$

Here,

① $f(s) = \mathcal{L}\{\sin 4t\}$

$\Rightarrow f(s) = \dfrac{4}{s^2+16}$

$\therefore f(s-2) = \dfrac{4}{(s-2)^2+16}$

② $f(s) = \mathcal{L}\{\cos 4t\}$

$\Rightarrow f(s) = \dfrac{s}{s^2+16}$

$\therefore f(s-2) = \dfrac{(s-2)}{(s-2)^2+16}$

Now,

$3\cdot\mathcal{L}\{e^{2t}\cdot\sin 4t\} - 4\cdot\mathcal{L}\{e^{2t}\cdot\cos 4t\}$

$= 3\cdot\dfrac{4}{(s-2)^2+16} - 4\cdot\dfrac{(s-2)}{(s-2)^2+16}$

$= \dfrac{12-4s+8}{(s-2)^2+16}$

$= \dfrac{(20-4s)}{(s^2-4s+20)}$ (ans)

***③ Determine each of the followings:

(i) $\mathcal{L}^{-1}\left\{\frac{12}{4-3s}\right\}$

$= \mathcal{L}^{-1}\cdot\left\{\frac{12}{4\cdot(1-\frac{3}{4}s)}\right\}$

$= \mathcal{L}^{-1}\cdot\left\{\frac{3}{(1-\frac{3s}{4})}\right\}$

$= \mathcal{L}^{-1}\cdot\left\{\frac{3}{-\frac{3}{4}\cdot(s-\frac{4}{3})}\right\}$

$= -\frac{4}{3}\cdot 3\cdot \mathcal{L}^{-1}\left\{\frac{1}{s-\frac{4}{3}}\right\}$

$= -3\cdot e^{+(\frac{4}{3})t}$ $\left[\because e^{at} = \frac{1}{(s-a)}\right]$

(ans)

(ii) $\mathcal{L}^{-1}\left\{\dfrac{2s-5}{s^2-9}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{2s}{s^2-9} - \dfrac{5}{s^2-9}\right\}$

$= \mathcal{L}^{-1}.\left\{\dfrac{2s}{s^2-9}\right\} - \mathcal{L}^{-1}.\left\{\dfrac{5}{s^2-9}\right\}$

$= 2.\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-3^2}\right\} - 5.\mathcal{L}^{-1}.\left\{\dfrac{1}{s^2-3^2}\right\}$

$= 2.\cosh 3t - 5.\dfrac{1}{3}.\mathcal{L}^{-1}.\left\{\dfrac{3}{s^2-3^2}\right\}$ $\left[\because \dfrac{s}{s^2-a^2} = \cosh at\right]$

$= 2.\cosh 3t - \dfrac{5}{3}.\sinh 3t$ $\left[\because \dfrac{a}{s^2-a^2} = \sinh at\right]$

(Ans)

** (iii) $\mathcal{L}^{-1}\left\{\frac{23s-15}{s^2+8}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{23s}{s^2+8} - \frac{15}{s^2+8}\right\}$

$= 23.\mathcal{L}^{-1}.\left\{\frac{s}{s^2+(2\sqrt{2})^2}\right\} - \frac{15}{2\sqrt{2}}.\mathcal{L}^{-1}.\left\{\frac{2\sqrt{2}}{s^2+(2\sqrt{2})^2}\right\}$

$= 23.\cos 2\sqrt{2}t - \frac{15}{2\sqrt{2}}\sin 2\sqrt{2}\,t$ $\left[\because \frac{s}{s^2+a^2} = \cos at;\ \frac{a}{s^2+a^2} = \sin at\right]$

** (iv) $\mathcal{L}^{-1}.\left\{\frac{1}{s^{3/2}}\right\}$

$= \mathcal{L}^{-1}.\left\{\frac{1}{s^{1/2+1}}\right\}$

$= \frac{t^{1/2}}{\Gamma(1/2+1)}$ $\left[\because \frac{1}{s^{n+1}};\ n>-1 = \frac{t^n}{(n+1)!}\right]$

$= \frac{t^{1/2}}{1/2\,\Gamma(1/2)}$

$= 2.\frac{t^{1/2}}{\sqrt{\pi}}$

$= 2.\sqrt{\frac{t}{\pi}}$

(v) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^{4/3}}\right\}$

$$= \mathcal{L}^{-1}\cdot\left\{\frac{s}{s^{4/3}} + \frac{1}{s^{4/3}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^{1/3}}\right\} + \mathcal{L}^{-1}\cdot\left\{\frac{1}{s^{4/3}}\right\}$$

$$= \mathcal{L}^{-1}\cdot\left\{\frac{1}{s^{-2/3+1}}\right\} + \mathcal{L}^{-1}\cdot\left\{\frac{1}{s^{1/3+1}}\right\}$$

$$= \frac{t^{-2/3}}{(-2/3+1)!} + \frac{t^{1/3}}{(1/3+1)!} \qquad \left[\because \frac{1}{s^{n+1}} ; n > -1 = \frac{t^n}{(n+1)!}\right]$$

$$= \frac{t^{-2/3}}{\Gamma 1/3} + \frac{t^{1/3}}{\Gamma 4/3} \quad \text{(ans)}$$

*** (vi) $\mathcal{L}^{-1}\cdot\left\{\frac{1}{s^4}\right\}$

$= \mathcal{L}^{-1}\cdot\left\{\frac{1}{s^{3+1}}\right\}$

$= \frac{t^3}{3!}$

$= \frac{t^3}{6}$ (Ans)

*** (vii) $\mathcal{L}^{-1}\cdot\left\{\frac{1}{\sqrt{2s+3}}\right\}$

$= \mathcal{L}^{-1}\cdot\left\{\frac{1}{\sqrt{2(s+3/2)}}\right\}$

$= \frac{1}{\sqrt{2}}\cdot\mathcal{L}^{-1}\cdot\left\{\frac{1}{(s+3/2)^{1/2}}\right\}$

$= \frac{1}{\sqrt{2}}\cdot e^{-3/2\cdot t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}}\right\}$

$= \frac{1}{\sqrt{2}}\cdot e^{-3/2\cdot t}\cdot\mathcal{L}^{-1}\cdot\left\{\frac{1}{s^{-1/2+1}}\right\}$

$$= \frac{1}{\sqrt{2}} \cdot e^{-\frac{3}{2}\cdot t} \cdot \frac{t^{-1/2}}{(-1/2+1)!}$$

$$= \frac{1}{\sqrt{2}} \cdot e^{-3/2\cdot t} \cdot \frac{t^{-1/2}}{(1/2)!}$$

$$= \frac{1}{\sqrt{2}} \cdot e^{-3/2\cdot t} \cdot \frac{t^{-1/2}}{\sqrt{\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-3/2\cdot t} \cdot t^{-1/2}$$

(Ans)

(viii) $\mathcal{L}^{-1} \cdot \left\{ \frac{1}{(s+4)^{5/2}} \right\}$

$$= e^{-4t} \cdot \mathcal{L}^{-1} \cdot \left\{ \frac{1}{s^{5/2}} \right\}$$

$$= e^{-4t} \cdot \mathcal{L}^{-1} \cdot \left\{ \frac{1}{s^{3/2+1}} \right\}$$

$$= e^{-4t} \cdot \frac{t^{3/2}}{(3/2+1)!}$$

$$= e^{-4t} \cdot \frac{t^{3/2}}{\Gamma 5/2}$$

$$= e^{-4t} \cdot \frac{t^{3/2}}{\Gamma 3/2+1}$$

$$= e^{-4t} \cdot \frac{t^{3/2}}{3/2 \, \Gamma 3/2}$$

$$= e^{-4t} \cdot \frac{t^{3/2}}{3/2 \cdot 1/2 \cdot \Gamma 1/2}$$

$$= \frac{4}{3} \cdot e^{-4t} \cdot \frac{t^{3/2}}{\sqrt{\pi}}$$

(Ans)

**(4) Evaluate using partial fractions:

(i) $\mathcal{L}^{-1}.\left\{\dfrac{6s-4}{s^2-4s+20}\right\}$

$= \mathcal{L}^{-1}.\left\{\dfrac{6s-4}{s^2-2.2.s+4+16}\right\}$

$= \mathcal{L}^{-1}.\left\{\dfrac{6(s-2)+8}{(s-2)^2+16}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{6(s-2)}{(s-2)^2+4^2}\right\} + \mathcal{L}^{-1}.\left\{\dfrac{8}{(s-2)^2+4^2}\right\}$

$= 6.\mathcal{L}^{-1}\left\{\dfrac{(s-2)}{(s-2)^2+4^2}\right\} + \dfrac{8}{4}\mathcal{L}^{-1}.\left\{\dfrac{4}{(s-2)^2+4^2}\right\}$

$= 6.e^{2t}.\cos 4t + 2.e^{2t}.\sin 4t$

(Ans)

---x---

***(ii)** $\mathcal{L}^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{4s+12}{s^2+2.4.s+4^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{4.(s+4)-4}{(s+4)^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{4(s+4)}{(s+4)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{4}{(s+4)^2}\right\}$

$= 4.\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - 4.\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}$

$= 4.e^{-4t} - 4.e^{-4t}.t \quad \left[\because \frac{1}{s-a} = e^{at};\ \frac{1}{s^2} = t\right]$

$= 4.e^{-4t}.(1-t)$ (ans)

(iii) $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)\cdot(s-2)\cdot(s-3)}\right\}$

Now,

$$\frac{2s^2-4}{(s+1)\cdot(s-2)\cdot(s-3)} = \frac{A}{(s+1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$\Rightarrow (2s^2-4) = A\cdot(s-2)\cdot(s-3) + B\cdot(s+1)\cdot(s-3) + C(s+1)\cdot(s-2)$$

when, $s=-1$,

$(2-4)=(-3)\cdot(-4)\cdot A$

$\Rightarrow 12A = -2$

$\therefore A = -\left(\frac{1}{6}\right)$

$s=2$,

$(8-4)= 3\cdot(-1)\cdot B$

$\Rightarrow -3B = 4$

$\therefore B = -\left(\frac{4}{3}\right)$

$s=3$,

$(18-4)= 4\cdot 1\cdot C$

$\Rightarrow 4C = 14$

$\therefore C = \left(\frac{7}{2}\right)$

$$\therefore \frac{2s^2-4}{(s+1)\cdot(s-2)\cdot(s-3)} = \frac{-1/6}{(s+1)} + \frac{-4/3}{(s-2)} + \frac{7/2}{(s-3)}$$

$$= -\frac{1}{6}\cdot\frac{1}{(s+1)} - \frac{4}{3}\cdot\frac{1}{(s-2)} + \frac{7}{2}\cdot\frac{1}{(s-3)}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)\cdot(s-2)\cdot(s-3)}\right\} = -\frac{1}{6}\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} - \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\} + \frac{7}{2}\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s-3)}\right\}$$

$$= -\left(\frac{1}{6}\right)\cdot e^{-t} - \left(\frac{4}{3}\right)\cdot e^{2t} + \left(\frac{7}{2}\right)\cdot e^{3t}$$

(Ans)

***** (iv)** $\mathcal{L}^{-1}\left\{\frac{5s^2-15s-11}{(s+1).(s-2)^3}\right\}$

$\Rightarrow$ NOW,

$$\frac{5s^2-15s-11}{(s+1).(s-2)^3} = \frac{A}{(s+1)} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$$

$$\Rightarrow (5s^2-15s-11) = A.(s-2)^3 + B.(s+1).(s-2)^2 + C(s+1).(s-2) + D.(s+1)$$

$$= A(s^3 - 3.s^2.2 + 3s.2^2 - 2^3) + B.(s+1).(s^2-4s+4) + C(s^2-2s+s-2) + D(s+1)$$

$$= A.(s^3 - 6s^2 + 12s - 8) + B(s^3 - 4s^2 + \cancel{4s} + s^2 - \cancel{4s} + 4) + C(s^2-s-2) + D(s+1)$$

$$= A.s^3 - 6A.s^2 + 12As - 8A + Bs^3 - 3Bs^2 + 4B + Cs^2 - Cs - 2C + Ds + D$$

$$\therefore (5s^2-15s-11) = s^3.(A+B) + s^2.(-6A-3B+C) + s(+12A-C+D) + (-8A+4B-2C+D)$$

∴ Equating both sides we have,

$(A+B)=0$ —— (i)

$(-6A-3B+C)=5$ —— (ii)

$(+12A-C+D)=-15$ —— (iii)

$(-8A+4B-2C+D)=-11$ —— (iv)

Now, (i) $\Rightarrow$ $B=(-A)$ —— (v)

(ii) $\Rightarrow$ $-6A-3(-A)+C=5$

$\Rightarrow -6A+3A+C=5$

$\therefore C=(3A+5)$ —— (vi)

(iii) $\Rightarrow$ $+12A-C+D=-15$

$\Rightarrow +12A-(3A+5)+D=-15$

$\Rightarrow +9A+D=-10$

$\therefore D=+(-9A-10)$ —— (vii)

$\therefore$ (iv) $\Rightarrow$ $-8A+4(-A)-2(3A+5)+(-9A-10)=-11$

$\Rightarrow -8A-4A-6A-9A-10-10=-11$

$\Rightarrow -27A=9$

$\therefore \boxed{A = \left(\frac{-1}{3}\right)}$

$\therefore$ (v) $\Rightarrow \boxed{B = \left(\frac{1}{3}\right)}$

$\therefore$ (vi) $\Rightarrow \boxed{C = 4}$

$\therefore$ (vii) $\Rightarrow \boxed{D = -7}$

$$\therefore \frac{5s^2-15s-11}{(s+1)\cdot(s-2)^3} = \frac{-1/3}{(s+1)} + \frac{1/3}{(s-2)} + \frac{4}{(s-2)^2} + \frac{-7}{(s-2)^3}$$

$$= -\frac{1}{3}\cdot\frac{1}{(s+1)} + \frac{1}{3}\cdot\frac{1}{(s-2)} + 4\cdot\frac{1}{(s-2)^2} - 7\cdot\frac{1}{(s-2)^3}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{5s^2-15s-11}{(s+1)\cdot(s-2)^3}\right\} = -\frac{1}{3}\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} + \frac{1}{3}\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\}$$

$$+ 4\cdot\mathcal{L}^{-1}\cdot\left\{\frac{1}{(s-2)^2}\right\} - 7\cdot\mathcal{L}^{-1}\cdot\left\{\frac{1}{(s-2)^3}\right\}$$

$$= -\frac{1}{3}\cdot e^{-t} + \frac{1}{3}\cdot e^{2t} + 4\cdot e^{2t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$-7\cdot e^{2t}\cdot\mathcal{L}^{-1}\cdot\left\{\frac{1}{s^3}\right\}$$

$$= -\left(\frac{1}{3}\right)\cdot e^{-t} + \left(\frac{1}{3}\right)\cdot e^{2t} + 4\cdot t\cdot e^{2t} - 7\cdot e^{2t}\cdot\mathcal{L}^{-1}\left\{\frac{1}{s^{2+1}}\right\}$$

$$= -\left(\frac{1}{3}\right)\cdot e^{-t} + \left(\frac{1}{3}\right)\cdot e^{2t} + 4\cdot t\cdot e^{2t} - \left(\frac{7}{2}\right)t^2\cdot e^{2t}$$

Ans

**(v) $\mathcal{L}^{-1}\left\{\dfrac{3s+1}{(s^2+1)\cdot(s-1)}\right\}$

$$\therefore \frac{3s+1}{(s^2+1)\cdot(s-1)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+1)}$$

$$\Rightarrow 3s+1 = A(s^2+1) + (Bs+C)\cdot(s-1)$$

$$= A(s^2+1) + Bs^2 - Bs + Cs - C$$

$$\therefore (3s+1) = s^2(A+B) + s\cdot(C-B) + (A-C)$$

$\therefore$ Equating the both sides we have,

$(A+B)=0$ —— (i)

$(C-B)=3$ —— (ii)

$(A-C)=1$ —— (iii)

$\therefore$ (i) $\Rightarrow$ $A=(-B)$ —— (vi)

$\therefore$ (ii) $\Rightarrow$ $C-B=3$

$\Rightarrow C+A=3$

$\therefore C=(3-A)$ —— (v)

$\therefore$ (iii) $\Rightarrow$ $A-(3-A)=1$

$\Rightarrow 2A=4$

$\therefore$ $\boxed{A=2}$

$\therefore$ (vi) $\Rightarrow$ $\boxed{B=-2}$

$\therefore$ (v) $\Rightarrow$ $\boxed{C=1}$

$$\therefore \frac{3s+1}{(s^2+1)\cdot(s-1)} = \frac{2}{(s-1)} + \frac{-2s+1}{(s^2+1)}$$

$$\therefore \mathcal{L}^{-1}\cdot\left\{\frac{3s+1}{(s^2+1)\cdot(s-1)}\right\} = \mathcal{L}^{-1}\cdot\left\{\frac{2}{(s-1)} + \frac{-2s+1}{(s^2+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{(s-1)}\right\} + \mathcal{L}^{-1}\cdot\left\{\frac{-2s+1}{(s^2+1)}\right\}$$

$$= 2.\mathcal{L}^{-1}\cdot\left\{\frac{1}{(s-1)}\right\} + \mathcal{L}^{-1}\cdot\left\{\frac{-2s}{s^2+1}\right\} + \mathcal{L}^{-1}\cdot\left\{\frac{1}{s^2+1}\right\}$$

$$= 2.e^t - 2.\mathcal{L}^{-1}\cdot\left\{\frac{s}{s^2+1^2}\right\} + \mathcal{L}^{-1}\cdot\left\{\frac{1}{s^2+1^2}\right\}$$

$$= 2.e^t - 2.\cos t + \sin t$$

(ans)

(vi) $\mathcal{L}^{-1}\left\{\dfrac{s^2+2s+3}{(s^2+2s+2)\cdot(s^2+2s+5)}\right\}$

$\Rightarrow$ Now,

$$\frac{s^2+2s+3}{(s^2+2s+2)\cdot(s^2+2s+5)} = \frac{As+B}{(s^2+2s+2)} + \frac{Cs+D}{(s^2+2s+5)}$$

$$\Rightarrow s^2+2s+3 = (As+B)\cdot(s^2+2s+5) + (Cs+D)\cdot(s^2+2s+2)$$

$$= As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B + Cs^3 + 2Cs^2 + 2Cs + Ds^2 + 2Ds + 2D$$

$$\therefore (s^2+2s+3) = s^3\cdot(A+C) + s^2(2A+B+2C+D) + s(5A+2B+2C+2D) + (5B+2D)$$

$\therefore$ Equating both sides we have,

$(A+C)=0$ —— (i)

$(2A+2B+2C+D)=1$ —— (ii)

$(5A+2B+2C+2D)=2$ —— (iii)

$(5B+2D)=3$ —— (vi)

Now,

$(i) \Rightarrow A = (-C)$ —— (v)

$(vi) \Rightarrow (5B+2D) = 3$

$\Rightarrow 2D = 3 - 5B$

$\therefore D = \left(\frac{3-5B}{2}\right)$ —— (vi)

$\therefore (ii) \Rightarrow$

$2.(-C) + B + 2C + \left(\frac{3-5B}{2}\right) = 1$

$\Rightarrow \frac{2B + 3 - 5B}{2} = 1$

$\Rightarrow -3B = -1$

$\therefore \boxed{B = \left(\frac{1}{3}\right)}$

$\therefore (vi) \Rightarrow D = \frac{(3 - 5/3)}{2}$

$= \frac{\frac{4}{3}}{2}$

$= \frac{4}{3} \times \frac{1}{2}$

$\therefore \boxed{D = \left(\frac{2}{3}\right)}$

Now,

(iii) $\Rightarrow$ $5A + \frac{2}{3} + 2C + 2\frac{2}{3} = 2$

$\Rightarrow 5(-C) + \frac{2}{3} + 2C + \frac{4}{3} = 2$

$\Rightarrow -3C + \frac{6}{3} = 2$

$\Rightarrow -3C = 0$

$\therefore \boxed{C = 0}$

(v) $\Rightarrow$ $\therefore \boxed{A = 0}$

$$\therefore \frac{(s^2+2s+3)}{(s^2+2s+2)\cdot(s^2+2s+5)} = \frac{0+\frac{1}{3}}{(s^2+2s+2)} + \frac{\frac{2}{3}}{(s^2+2s+5)}$$

$$= \frac{1}{3}\cdot\frac{1}{(s^2+2s+1+1)} + \frac{2}{3}\cdot\frac{1}{(s^2+2s+1+4)}$$

$$= \frac{1}{3}\cdot\frac{1}{(s+1)^2+1^2} + \frac{2}{3}\cdot\frac{1}{(s+1)^2+2^2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)\cdot(s^2+2s+5)}\right\} = \frac{1}{3}\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\}$$

$$+\frac{2}{3}\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3} \cdot e^{-t} \cdot \sin t + \frac{1}{3} \cdot \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3} \cdot e^{-t} \cdot \sin t + \frac{1}{3} \cdot e^{-t} \cdot \sin 2t$$

$$= \frac{1}{3} \cdot e^{-t} \cdot (\sin t + \sin 2t)$$

(Ans)

$\therefore \boxed{D = \left(\frac{2}{3}\right)}$

***Formula of Laplace transformation of nth derivative:**

$$\boxed{\mathcal{L}\{F^n(t)\} = s^n . f(s) - s^{n-1} . F(0) - s^{n-2} F'(0) \ldots - s F^{n-2}(0) - F^{n-1}(0)}$$

⑤ Solve the Differential Equations

① Solve:

$$Y''' - 3Y'' + 3Y' - Y = e^t . t^2 \; ; \; Y(0) = 0 \; , \; Y'(0) = 0 \; , \; Y''(0) = -2.$$

$\Rightarrow$ Let, $\mathcal{L}\{Y\} = y$.

Now,

$$\mathcal{L}\{Y'''\} - 3\mathcal{L}\{Y''\} + 3\mathcal{L}\{Y'\} - \mathcal{L}\{Y\} = \mathcal{L}.\{e^t . t^2\}$$

$$\Rightarrow \left[s^3 . \mathcal{L}\{Y\} - s^2 Y(0) - s . Y'(0) - Y''(0)\right] - 3.\left[s^2 . \mathcal{L}\{Y\} - s . Y(0) - Y'(0)\right]$$

$$+ 3 . \left[s . \mathcal{L} . \{Y\} - Y(0)\right] - \mathcal{L}.\{Y\} = \frac{2!}{(s-1)^3}$$

$$\Rightarrow \left[s^3 y - 0 - 0 + 2\right] - 3\left[s^2 . y - 0 - 0\right] + 3\left[s . y - 0\right] - y = \frac{2}{(s-1)^3}$$

$$\Rightarrow s^3 y + 2 - 3 . s^2 . y + 3 . s . y - y = \frac{2}{(s-1)^3}$$

$$\Rightarrow y . (s^3 - 3s^2 + 3s - 1) = \frac{2}{(s-1)^3} - 2$$

$$\Rightarrow y . (s-1)^3 = \frac{2}{(s-1)^3} - 2$$

$$\Rightarrow y = \frac{1}{(s-1)^3}\cdot\left\{\frac{2}{(s-1)^3} - 2\right\}$$

$$\Rightarrow y = 2\frac{1}{(s-1)^6} - 2\cdot\frac{1}{(s-1)^3}$$

$$\Rightarrow \mathcal{L}\{Y\} = \frac{2}{(s-1)^6} - \frac{2}{(s-1)^3}$$

$$\Rightarrow Y = \mathcal{L}^{-1}\cdot\left\{\frac{2}{(s-1)^6} - \frac{2}{(s-1)^3}\right\}$$

$$= 2\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^6}\right\} - 2\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

$$= 2\cdot e^t\cdot\frac{t^5}{5!} - 2\cdot e^t\cdot\frac{t^2}{2!}$$

$$\therefore Y = \frac{e^t\cdot t^5}{60} - e^t\cdot t^2$$

(Ans)

*** ② Solve, $(Y''+9Y) = \cos 2t$; $Y(0)=1$, $Y(\pi/2) = -1$

$\Rightarrow$ Let $\mathcal{L}.\{Y\} = y$

Now,

$$\mathcal{L}\{Y''\} + 9\mathcal{L}.\{Y\} = \mathcal{L}.\{\cos 2t\}$$

$$\Rightarrow \left[s^2.\mathcal{L}\{Y\} - s.Y(0) - Y'(0)\right] + 9\left[\mathcal{L}\{Y\}\right] = \frac{s}{s^2+2^2}$$

$$\Rightarrow s^2.y - 1.s - c + 9y = \frac{s}{s^2+2^2}$$

$$\Rightarrow y.(s^2+9) = \frac{s}{s^2+4} + c + s$$

$$\Rightarrow y = \frac{1}{s^2+9}.\left[\frac{s}{s^2+4} + c + s\right]$$

$$\Rightarrow \mathcal{L}\{Y\} = \frac{s}{(s^2+9).(s^2+4)} + \frac{c}{(s^2+9)} + \frac{s}{(s^2+9)}$$

$$\Rightarrow Y = \mathcal{L}^{-1}.\left\{\frac{s}{(s^2+9).(s^2+4)} + \frac{c}{(s^2+9)} + \frac{s}{(s^2+9)}\right\}$$

$$= \mathcal{L}^{-1}.\left\{\frac{s}{5.(s^2+4)} - \frac{s}{5(s^2+9)} + \frac{s}{(s^2+9)} + \frac{c}{(s^2+9)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{5\cdot(s^2+4)} + \frac{4}{5}\cdot\frac{s}{(s^2+9)} + \frac{c}{s^2+9}\right\}$$

$$= \frac{1}{5}\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \frac{4}{5}\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + c\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$\therefore\ Y(t) = \frac{1}{5}\cdot\cos 2t + \frac{4}{5}\cdot\cos 3t + c\cdot\frac{\sin 3t}{3} \quad \text{①}$$

$$\therefore\ Y(\pi/2) = \frac{1}{5}\cdot\cos\left(2\cdot\frac{\pi}{2}\right) + \frac{4}{5}\cdot\cos\left(\frac{3\pi}{2}\right) + \frac{c}{3}\cdot\sin\left(\frac{3\pi}{2}\right)$$

$$\Rightarrow -1 = \frac{1}{5}\cdot(-1) + \frac{4}{5}\cdot(0) + \frac{c}{3}(-1)$$

$$\Rightarrow -1 = -\frac{1}{5} - \frac{c}{3}$$

$$\therefore\ c = \frac{12}{5}$$

$\therefore$ ① $\Rightarrow$

$$Y = \frac{1}{5}\cdot\cos 2t + \frac{4}{5}\cdot\cos 3t + \frac{12}{15}\cdot\sin 3t$$

(Ans)

$$\therefore Y = \frac{1}{5}\cdot\cos 2t + \frac{4}{5}\cdot\cos 3t + \frac{4}{5}\cdot\sin 3t.$$

***③ $Y'' - 3Y' + 2Y = 4e^{2t}$; $Y(0) = -3$, $Y'(0) = 5$

$\Rightarrow$ Let, $\mathcal{L}\{Y\} = y$

Now,

$$\mathcal{L}.\{Y''\} - 3.\mathcal{L}.\{Y'\} + 2.\mathcal{L}.\{Y\} = 4\mathcal{L}.\{e^{2t}\}$$

$$\Rightarrow \left[s^2.\mathcal{L}\{Y\} - s.Y(0) - Y'(0)\right] - 3.\left[s.\mathcal{L}.\{Y\} - Y(0)\right] + 2.y = 4.\frac{1}{s-2}$$

$$\Rightarrow s^2.y + 3.s - 5 - 3.\left[s.y + 3\right] + 2y = \frac{4}{s-2}$$

$$\Rightarrow s^2.y + 3s - 5 - 3sy - 9 + 2y = \frac{4}{s-2}$$

$$\Rightarrow y.(s^2 - 3s + 2) = \frac{4}{(s-2)} - 3s + 5 + 9$$

$$\Rightarrow y = \frac{4}{(s^2-3s+2).(s-2)} - \frac{3s}{(s^2-3s+2)} + \frac{14}{(s^2-3s+2)}$$

$$= \frac{4}{(s^2-3s+2).(s-2)} + \frac{14-3s}{(s^2-3s+2)}$$

$$= \frac{4 + (14-3s).(s-2)}{(s^2-3s+2).(s-2)}$$

$$= \frac{4 + (14s - 28 - 3s^2 + 6s)}{\{(s^2-2s+1)-(s-1)\}.(s-2)}$$

$$= \frac{(4+14s-28-3s^2+6s)}{\{(s-1)^2-(s-1)\}(s-2)}$$

$$= \frac{-3s^2+20s-24}{(s-1)(s-1-1).(s-2)}$$

$$\therefore y = \frac{-3s^2+20s-24}{(s-1).(s-2)^2} \quad \text{———} (i)$$

Now,

$$\frac{-3s^2+20s-24}{(s-1).(s-2)^2} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2} \quad \text{———} (ii)$$

$$\Rightarrow -3s^2+20s-24 = A.(s-2)^2 + B.(s-1).(s-2) + C(s-1) \quad \text{—} (iii)$$

Here, $s=2$,

$$\therefore -3.2^2+20\times2-24 = A(2-2)^2 + B(2-1).(2-2) + C(2-1)$$

$$\therefore C = 4$$

Here, $s=1$

$$\therefore -3+20-24 = A$$

$$\therefore A = -7$$

Here, $s=0$

$\therefore -24 = 4A + 2B + (-C)$

$\Rightarrow -24 = 4\times(-7) + 2B - 4$

$\Rightarrow -24 = -32 + 2B$

$\Rightarrow B = \frac{8}{2}$

$\therefore B = 4$

$\therefore$ (11) $\Rightarrow$

$$\frac{-3s^2+20s-24}{(s-1).(s-2)^2} = \frac{-7}{(s-1)} + \frac{4}{(s-2)} + \frac{4}{(s-2)^2}$$

$\therefore$ (1) $\Rightarrow$

$$y = -\frac{7}{(s-1)} + \frac{4}{(s-2)} + \frac{4}{(s-2)^2}$$

$$\Rightarrow \mathcal{L}\{Y\} = -\frac{7}{(s-1)} + \frac{4}{(s-2)} + \frac{4}{(s-2)^2}$$

$$\Rightarrow Y = \mathcal{L}^{-1}\left\{\frac{-7}{(s-1)} + \frac{4}{(s-2)} + \frac{4}{(s-2)^2}\right\}$$

$$= -7\mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\} + 4.\mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\} + 4.\mathcal{L}^{-1}.\left\{\frac{1}{(s-2)^2}\right\}$$

$$\therefore Y = \left(-7.e^{t} + 4.e^{2t} + 4.t.e^{2t}\right)$$ (Ans)

---x---

*** (4) solve, $(Y'' + 2Y' + 5Y) = e^{-t}.\sin t$; $Y(0) = 0$, $Y'(0)$

$\Rightarrow$ Let, $\mathcal{L}\{Y\} = y$

Now,

$$\mathcal{L}\{Y''\} + 2.\mathcal{L}\{Y'\} + 5\mathcal{L}\{Y\} = \mathcal{L}\{e^{-t}.\sin t\}$$

$$\Rightarrow \left[s^2.\mathcal{L}\{Y\} - s.Y(0) - Y'(0)\right] + 2.\left[s.\mathcal{L}\{Y\} - Y(0)\right] + 5$$

$$= \frac{1}{(s+1)^2 + 1^2}$$

$$\Rightarrow \left[s^2.y - 0 - 1\right] + 2.\left[sy - 0\right] + 5y = \frac{1}{s^2 + 2s + 1 + 1}$$

$$\Rightarrow s^2.y - 1 + 2sy + 5y = \frac{1}{(s^2 + 2s + 2)}$$

$$\Rightarrow y.(s^2 + 2s + 5) = \frac{1}{(s^2 + 2s + 2)} + 1$$

$$\Rightarrow y = \frac{1}{(s^2 + 2s + 2).(s^2 + 2s + 5)} + \frac{1}{(s^2 + 2s + 5)}$$

$$\therefore y = \frac{(s^2 + 2s + 3)}{(s^2 + 2s + 2).(s^2 + 2s + 5)} \quad (1)$$

Now,

$$\frac{(s^2+2s+3)}{(s^2+2s+2)\cdot(s^2+2s+5)} = \frac{(As+B)}{(s^2+2s+2)} + \frac{(Cs+D)}{(s^2+2s+5)} \quad \text{--- (ii)}$$

$$\Rightarrow (s^2+2s+3) = (As+B)\cdot(s^2+2s+5) + (Cs+D)\cdot(s^2+2s+2)$$

$$= As^3 + 2As^2 + 5As + Bs^2 + 2sB + 5B + Cs^3 + 2Cs^2 + 2Cs + Ds^2 + 2Ds + 2D$$

$$\therefore (s^2+2s+3) = s^3\cdot(A+C) + s^2\cdot(2A+B+2C+D) + s(5A+2B+2C+2D) + (5B+2D)$$

Now, equating both sides we have,

$(A+C)=0$ --- (iii)

$(2A+B+2C+D)=1$ --- (iv)

$(5A+2B+2C+2D)=2$ --- (v)

$(5B+2D)=3$ --- (vi)

Now,

(iii) $\Rightarrow$ $A=(-C)$ --- (vii)

(vi) $\Rightarrow$ $5B+2D=3$

$\Rightarrow B=\left(\frac{3-2D}{5}\right)$ --- (viii)

$\therefore$ (iv) $\Rightarrow$ $2\cdot(-C) + \frac{3-2D}{5} + 2C + D = 1$

$\Rightarrow \frac{3-2D}{5} + D = 1$ (ii)

$\Rightarrow 3-2D+5D = 5$

$\Rightarrow 3D = 2$

$\therefore D = (2/3)$

$\therefore$ (viii) $\Rightarrow B = \frac{3 - 2/3 \cdot 2}{5}$

$\Rightarrow B = \frac{3 - \frac{4}{3}}{5}$

$= \frac{9-4}{3} \times \frac{1}{5}$

$\therefore B = (1/3)$

$\therefore$ (v) $\Rightarrow$

$5(-C) + 2 \cdot 1/3 + 2C + 2 \cdot 2/3 = 2$

$\Rightarrow -3C + \frac{2}{3} + \frac{4}{3} = 2$

$\Rightarrow 3C = \frac{2+4}{3} - 2$

$\Rightarrow 3C = \frac{2+4-6}{3}$

$\Rightarrow 3C = 0/3$

$\Rightarrow 3C = 0$

$\therefore C = 0$

$\therefore A = 0$

∴ (11) $\Rightarrow$

$$\frac{s^2+2s+3}{(s^2+2s+2)\cdot(s^2+2s+5)} = \frac{0.5+1/3}{(s^2+2s+2)} + \frac{0.5+2/3}{(s^2+2s+5)}$$

∴ (1) $\Rightarrow$

$$y = \frac{1}{3}\cdot\frac{1}{(s^2+2s+2)} + \frac{2}{3}\cdot\frac{1}{(s^2+2s+5)}$$

$$\Rightarrow \mathcal{L}\{Y\} = \frac{1}{3}\cdot\frac{1}{s^2+2s+1+1} + \frac{2}{3}\cdot\frac{1}{s^2+2s+1+4}$$

$$\Rightarrow Y = \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{1}{(s+1)^2+1^2} + \frac{2}{3}\cdot\frac{1}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3}\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{1}{3}\cdot\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}$$

$$= \frac{1}{3}\cdot e^{-t}\cdot\sin t + \frac{1}{3}\cdot e^{-t}\cdot\sin 2t$$

$$\therefore Y = \frac{1}{3}\cdot e^{-t}\cdot(\sin t + \sin 2t)$$ (Ans)

⊛⊛⊛ Solve the system:

① $\frac{dx}{dt} = (2x-3y)$

$; \quad x(0)=8, \quad y(0)=3$

$\frac{dy}{dt} = (-2x+y)$

$\Rightarrow$ Let, $\mathcal{L}\{x\} = X$ and $\mathcal{L}\{y\} = Y$

$\therefore \mathcal{L}^{-1}\{X\} = x \qquad \therefore \mathcal{L}^{-1}\{Y\} = y$

Now,

$\mathcal{L}\{\frac{dx}{dt}\} = \mathcal{L}\{ \qquad \} = \mathcal{L}\{2x-3y\}$

$\therefore \left[s.\mathcal{L}\{x\} - x(0)\right] = 2\mathcal{L}\{x\} - 3\mathcal{L}\{y\}$

$\Rightarrow sX - 8 = 2X - 3Y$