Formalization of Vector Spaces and the Cauchy-Schwarz Inequality

Mathematical Formalization

1 Foundational Definitions

1.1 Vector Space Structure

Definition 1 (Finite-Dimensional Real Vector Space). *Let $n \in \mathbb{N}$ be a natural number. A finite-dimensional real vector space of dimension n is defined as a function space $V = \{\vec{v} : \{0, 1, \ldots, n\} \to \mathbb{R}\}$, where each vector $\vec{v}$ is a mapping from a finite set of indices to real numbers.*

Definition 2 (Vector Addition). *For two vectors $\vec{u}, \vec{v} \in V$, the vector addition $\vec{u} + \vec{v}$ is defined pointwise as:*

$$(\vec{u} + \vec{v}) = \vec{u}_i + \vec{v}_i \quad \forall i \in \{0, 1, \ldots, n\}$$

Definition 3 (Scalar Multiplication). *For a scalar $a \in \mathbb{R}$ and a vector $\vec{v} \in V$, scalar multiplication $a\vec{v}$ is defined pointwise as:*

$$(a\vec{v})_i = a \cdot \vec{v}_i \quad \textit{for } i \in \{0, 1, \ldots, n\}$$

1.2 Inner Product and Norm

Definition 4 (Dot Product). *The dot product (inner product) of two vectors $\vec{u}, \vec{v} \in V$ is defined as:*

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i=0}^{n} \vec{u}_i \vec{v}_i$$

Definition 5 (Vector Norm). *The norm of a vector $\vec{v} \in V$ is defined as:*

$$|\vec{v}| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

1.3 Projection and Angle

Definition 6 (Projection). *The projection of a vector $\vec{u}$ onto $\vec{v}$ is defined as:*

$$proj_{\vec{v}}(\vec{u}) = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$$

This gives the component of $\vec{u}$ that lies along the direction of $\vec{v}$.

Definition 7 (Angle Between Vectors)**.** *The angle θ between two nonzero vectors $\vec{u}$ and $\vec{v}$ is defined as:*

$$\cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{|\vec{u}||\vec{v}|}.$$

The cosine function provides a measure of the alignment between the two vectors.

2 Cauchy-Schwarz Inequality

Theorem 1 (Cauchy-Schwarz Inequality)**.** *For any vectors $\vec{u}, \vec{v} \in V$:*

$$|\langle \vec{u}, \vec{v} \rangle| \leq |\vec{u}||\vec{v}|.$$

Proof. Consider the cases where $\|\vec{u}\| = 0$ or $\|\vec{v}\| = 0$. If either norm is zero, then $\vec{u}$ or $\vec{v}$ is the zero vector. In such cases:

$$\vec{u} \cdot \vec{v} = 0 \quad \text{and} \quad |\vec{u} \cdot \vec{v}| = 0 \leq \|\vec{u}\|\|\vec{v}\| = 0.$$

Hence, the inequality holds trivially.

Now, assume $\|\vec{u}\| > 0$ and $\|\vec{v}\| > 0$. Define:

$$\vec{x} = \frac{\vec{u}}{\|\vec{u}\|}, \quad \vec{y} = \frac{\vec{v}}{\|\vec{v}\|}.$$

Then $\|\vec{x}\| = 1$ and $\|\vec{y}\| = 1$, as they are unit vectors. Consider:

$$\|\vec{x} + \vec{y}\|^2 = (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \|\vec{x}\|^2 + 2(\vec{x} \cdot \vec{y}) + \|\vec{y}\|^2 = 2(1 + \vec{x} \cdot \vec{y}).$$

Similarly:

$$\|\vec{x} - \vec{y}\|^2 = (\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y}) = \|\vec{x}\|^2 - 2(\vec{x} \cdot \vec{y}) + \|\vec{y}\|^2 = 2(1 - \vec{x} \cdot \vec{y}).$$

Since norms are non-negative, we have $\|\vec{x} + \vec{y}\|^2 \geq 0$ and $\|\vec{x} - \vec{y}\|^2 \geq 0$. Therefore:

$$1 + \vec{x} \cdot \vec{y} \geq 0 \quad \text{and} \quad 1 - \vec{x} \cdot \vec{y} \geq 0.$$

These imply:

$$-1 \leq \vec{x} \cdot \vec{y} \leq 1.$$

Returning to the original vectors $\vec{u}$ and $\vec{v}$, observe that:

$$\vec{x} \cdot \vec{y} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}.$$

Thus:

$$-1 \leq \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \leq 1,$$

which implies:

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|.$$

□

3 Implementation Details and References

3.1 Implementation and Design Choices

The formalization of the Cauchy-Schwarz inequality required careful attention to mathematical rigor and type safety. The main considerations included:

- **Choice of Type Universes:** Real numbers $\mathbb{R}$ were chosen to model vector components. This made it easier to see what was going on and also given my subpar knowledge of Lean, made it easier to implement proofs for $\mathbb{R}^n$ instead of generalizing for abstract vector spaces which have different notations of the inner product than the one I defined.
- **Notation:** Although I defined the proper notation and instances for almost every definition, I decided to not use them in my proofs because they some times made it more difficult to see what was going on as well as typing the symbols being more cumbersome than just typing the name of the function (e.g. dotProduct u v instead of $\langle\langle u, v\rangle\rangle$)
- **Use of Type Classes:** When extending the proof to generalized vector spaces, type classes could provide abstractions for norms, inner products, and scalars. However, given the time constraints and my initial proposal, I decided to not generalize for abstract vector spaces and just stuck with $\mathbb{R}^n$.

3.2 References

This project relied on the following resources:

- **Mathlib**: For support of real numbers and vector spaces.
- **Textbooks:** Shifrin and Adams, "Linear Algebra: A Geometric Approach." Classic Math 221 textbook. Used their proof of the Cauchy-Schwarz Inequality.