Review

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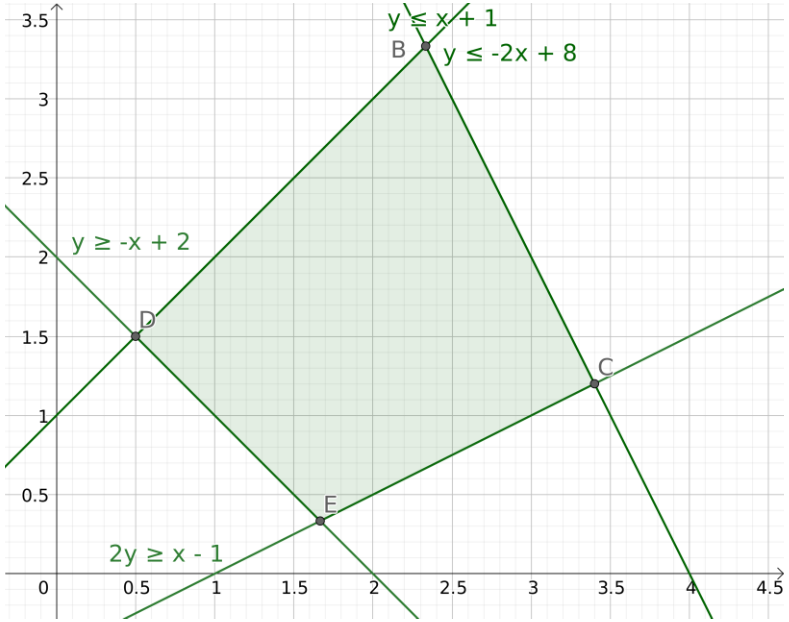

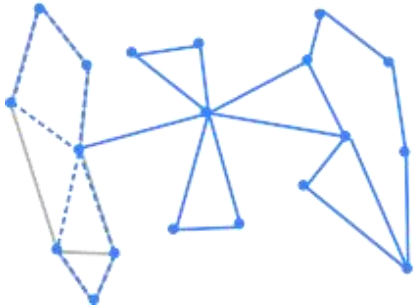

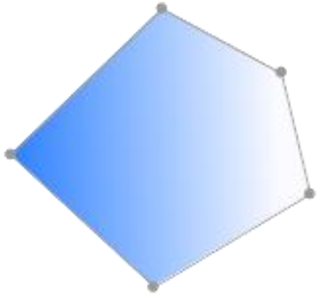
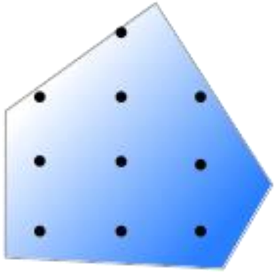
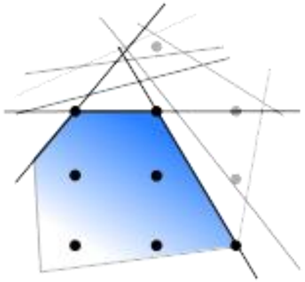
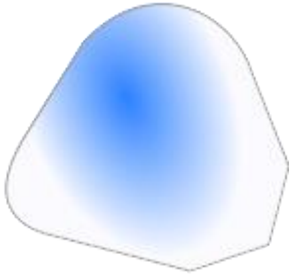

Outline

- Topics for this week's quiz
  - Multi-Objective Opt
  - Stochastic Opt
- Exam details
- Ask me questions

Multi-Objective Optimisation

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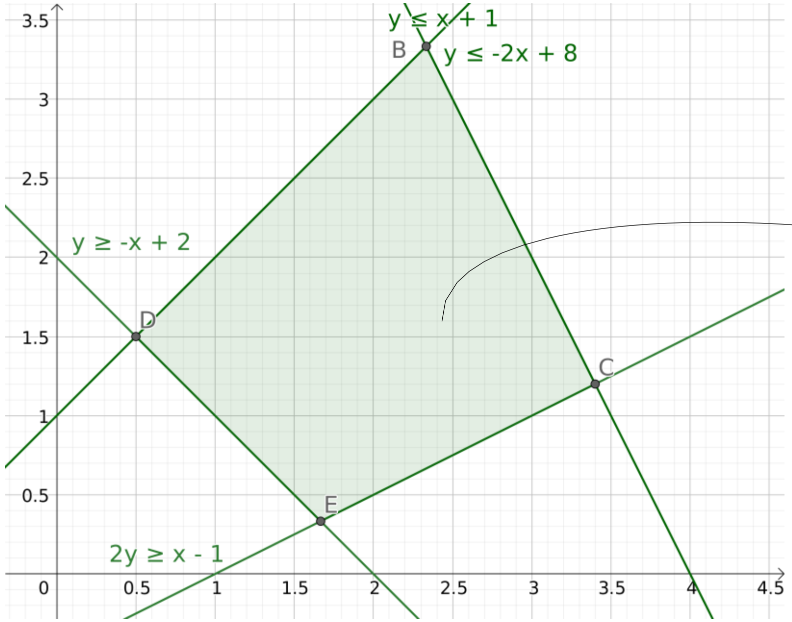
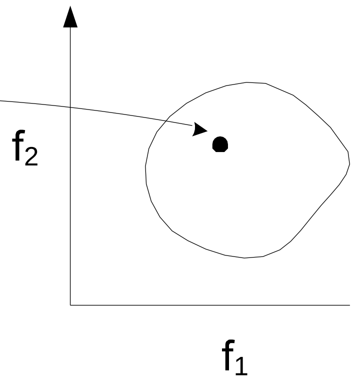
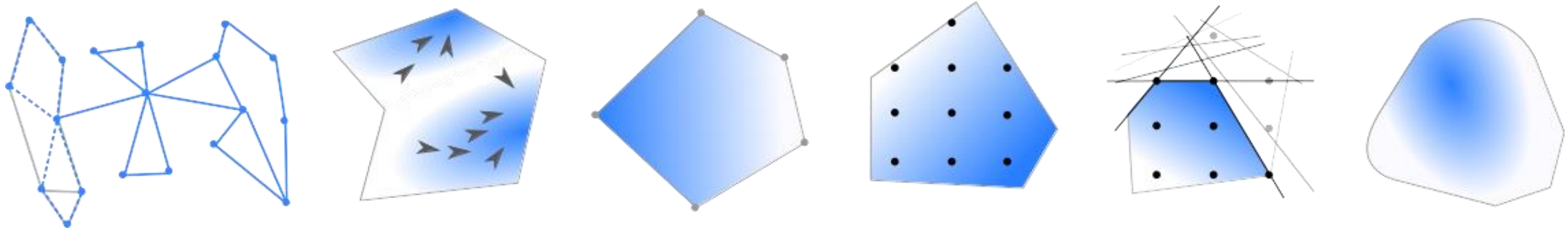






Problem Definition

The problem

$$\underset{x \in \Omega}{\text{minimize}} \quad \mathbf{f}(x) = [f_1(x), f_2(x), \ldots, f_m(x)]$$

VEC OF OBJ FUNC

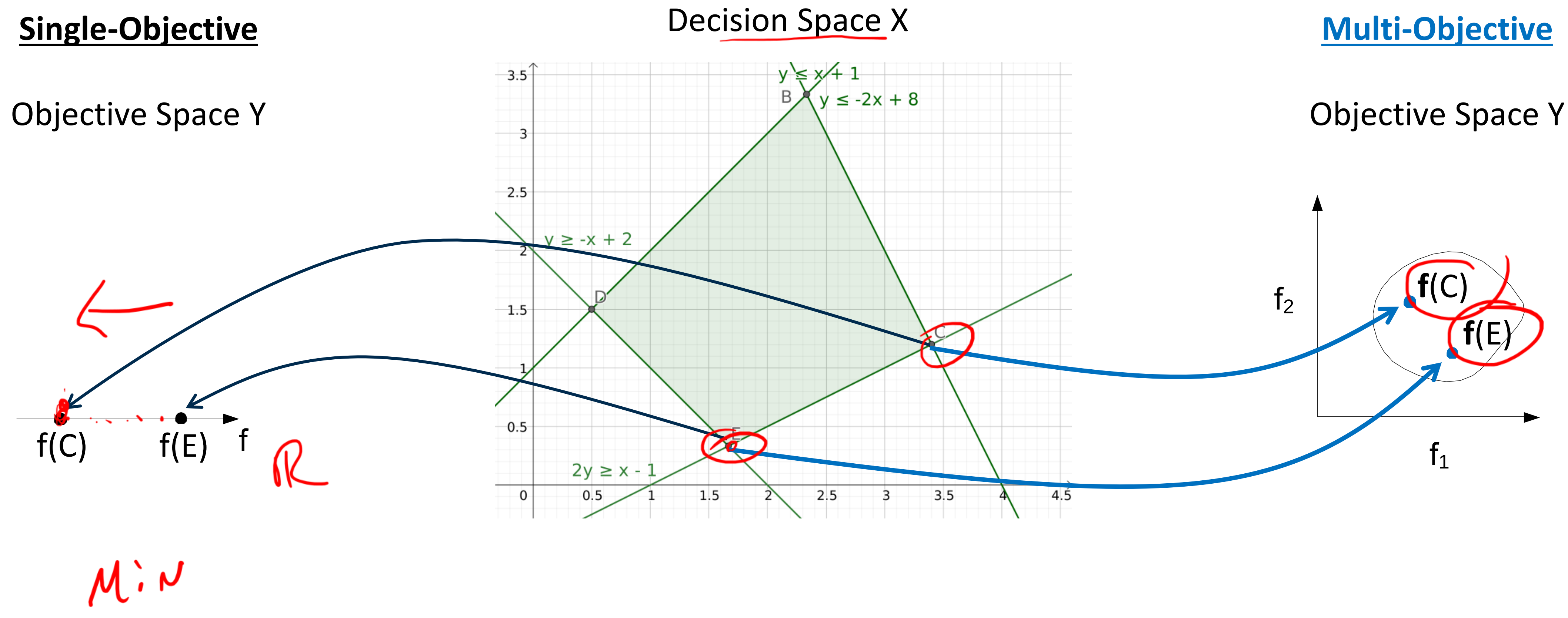
where:

- $\mathbf{f}: \Omega \rightarrow \mathbf{R}^m$ is the objective function, composed of $m \geq 2$ objective func.
- $\Omega \subseteq \mathbf{R}^n$ is the feasible space
  - Ω is defined through constraints
- $\mathbf{f}(\Omega)$ is the feasible objective space

- $\mathbf{R}^n$ is the **decision space**, $\mathbf{R}^m$ as the **objective space**.

Decision space and objective space

- Plots we have seen in the course are decision space plots



Pareto Dominance

$\prec$ $<$

Given two decision vectors x and y,

- **x dominates y** (denoted as $\boxed{x \prec y}$) if
  - $f_i(x) \le f_i(y)$ for all $i = 1, 2, \ldots, m$, and
  - $\mathbf{f}(x) \neq \mathbf{f}(y)$

min

$\exists j \in \{1, \ldots, m\}$ $f_j(x) < f_j(y)$

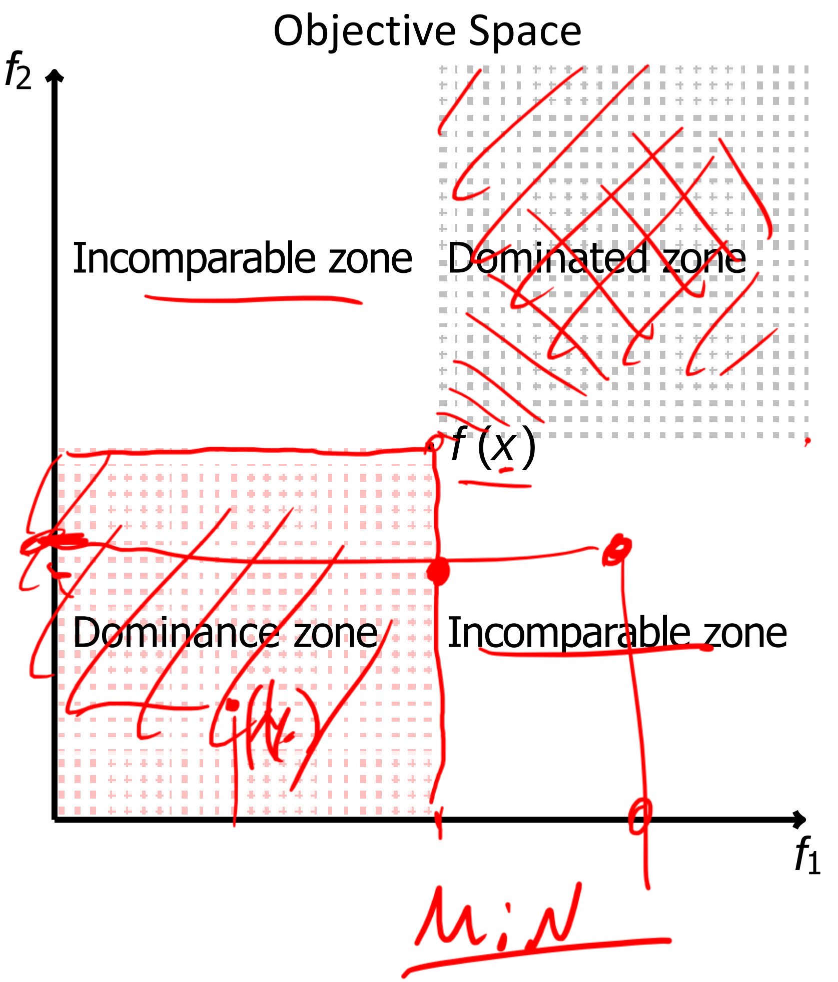
Examples: $\mathbf{f}(x) = [0, 1] \prec \mathbf{f}(y) = [2, 3]$

- x **weakly dominates** y if $x \prec y$ or $\mathbf{f}(x) = \mathbf{f}(y)$
- x and y are **incomparable** if
  - x does not weakly dominate y, and
  - y does not weakly dominate x

Equivalent

There exist i and j s.t.:
- $f_i(x) < f_i(y)$
- $f_j(x) > f_j(y)$

Dominance, Dominated and Indifferent Zones



Pareto Optimal and Pareto Set

$\lambda_i \geq 0$

$\sum \lambda_i = 1$

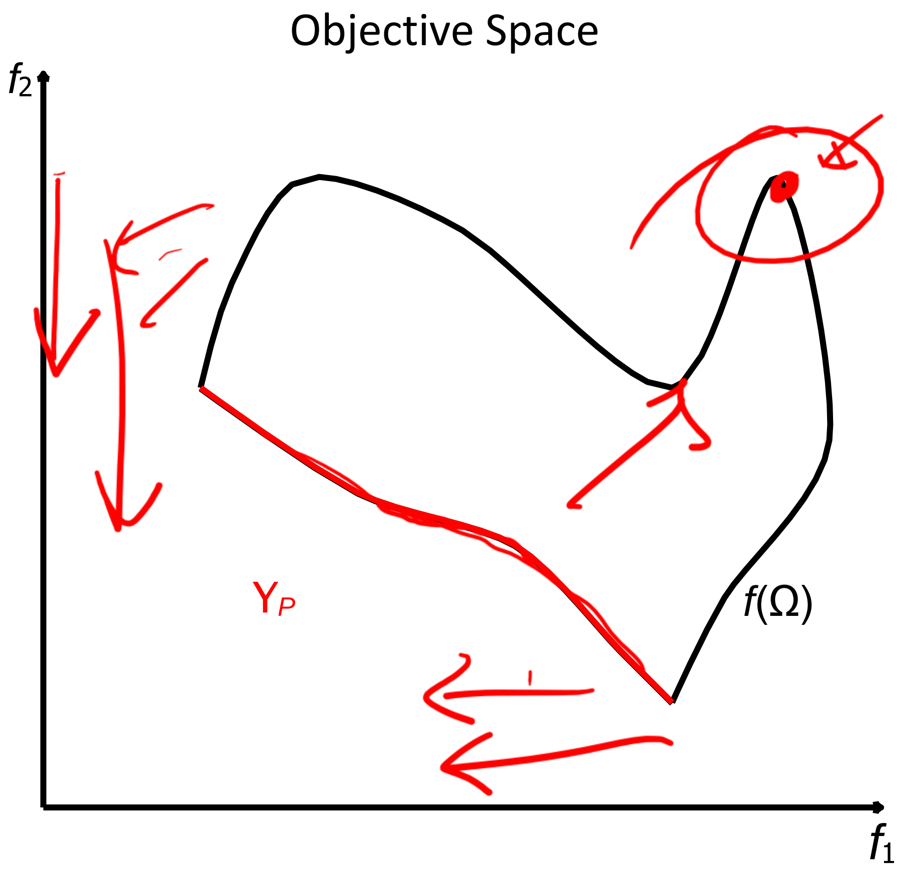
WEIGHTED SUM → $\lambda_1, \lambda_2, \lambda_3$

OPT IF THIS CONVEX

	Criteria/Car	A	B	C	D	E	F
min	Price	16200	14900	**14000**	15200	17200	20000
min	Fuel Consumption	2ND BEST 7.2	**7.0**	7.5	8.2	9.2	10
min	Negative Power	-66	-62	-55	**-71**	-51	-40

- $x^* \in \Omega$ is said to be **Pareto-optimal if there is no other** $x \in \Omega$ s.t. $x \prec x^*$
  - **A** is Pareto-optimal
- **Pareto Set:** the set of all Pareto-optimal solutions – denoted as X_P
  - $X_P = \{A, B, C, D\}$
- **Pareto Front:** image of the Pareto Set by the obj. func. – denoted as Y_P
  - $Y_P = \{[16200, 7.2, -66], [14900, 7.0, -62], [14000, 7.5, -55], [15200, 8.2, -71]\}$

Pareto Front (2)

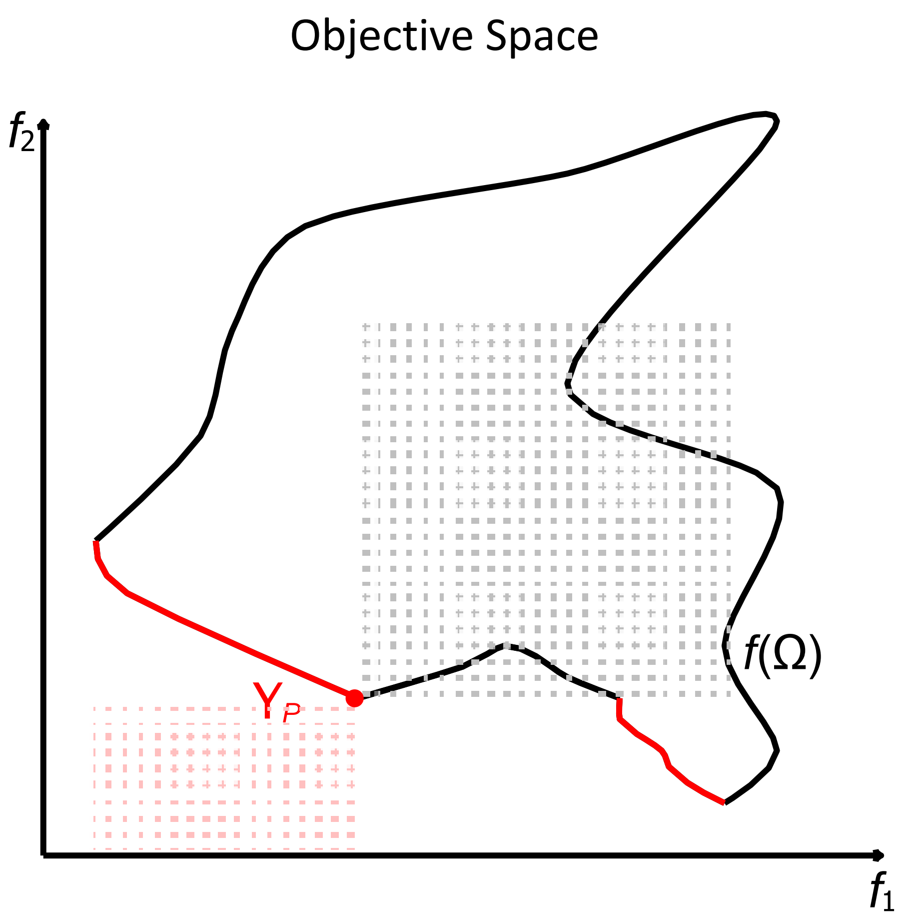


BEST SOLUTION USING ONLY $\lambda_i \geq 0$

$\sum \lambda_i = 1$

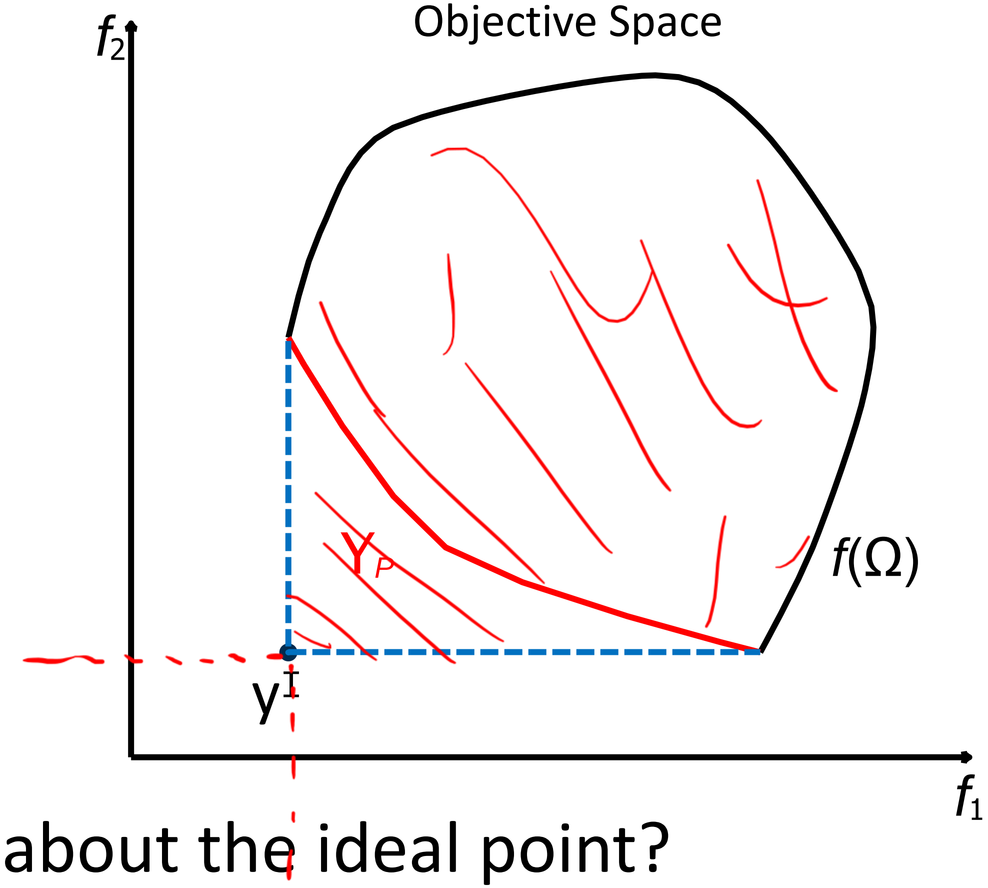
⚠ NON-CONVEX

Pareto Front (3)



Ideal Point

- The ideal point is: $y^I = [\min_{x\in\Omega} f_1(x), \min_{x\in\Omega} f_2(x), \ldots, \min_{x\in\Omega} f_m(x)]$



- What is special about the ideal point?

GENERATING METHOD

The Weighted-sum Scalarization Method

$= 1 - \lambda_1 - \lambda_2 \cdots \lambda_{m-1}$

- Given non-negative weights $\lambda_1, \ldots, \lambda_m$ s.t. $\sum_{i=1}^{m} \lambda_i = 1$ solve the SOP:

SINGLE OBJ PROB

$\lambda_1 = 1$

$\lambda_2 = 1$

$$\min_{x \in \Omega} \sum_{i=1}^{m} \lambda_i f_i(x)$$

- Solve the SOP for multiple different sets of weights

ANY PROBLEM

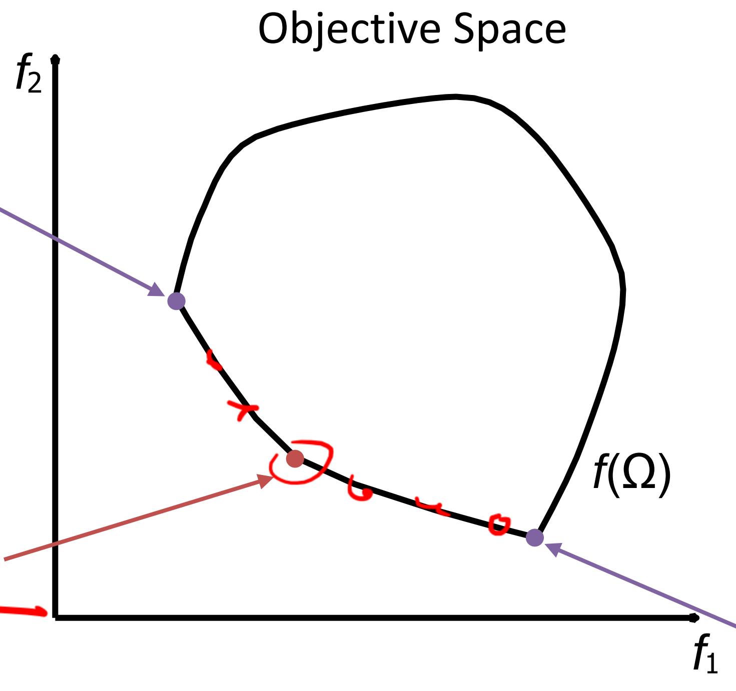
CVX ✓

N CVX ← NOT COMPLETE

Solution of SOP for **λ = 1**

For bi-objective we only need one parameter λ:

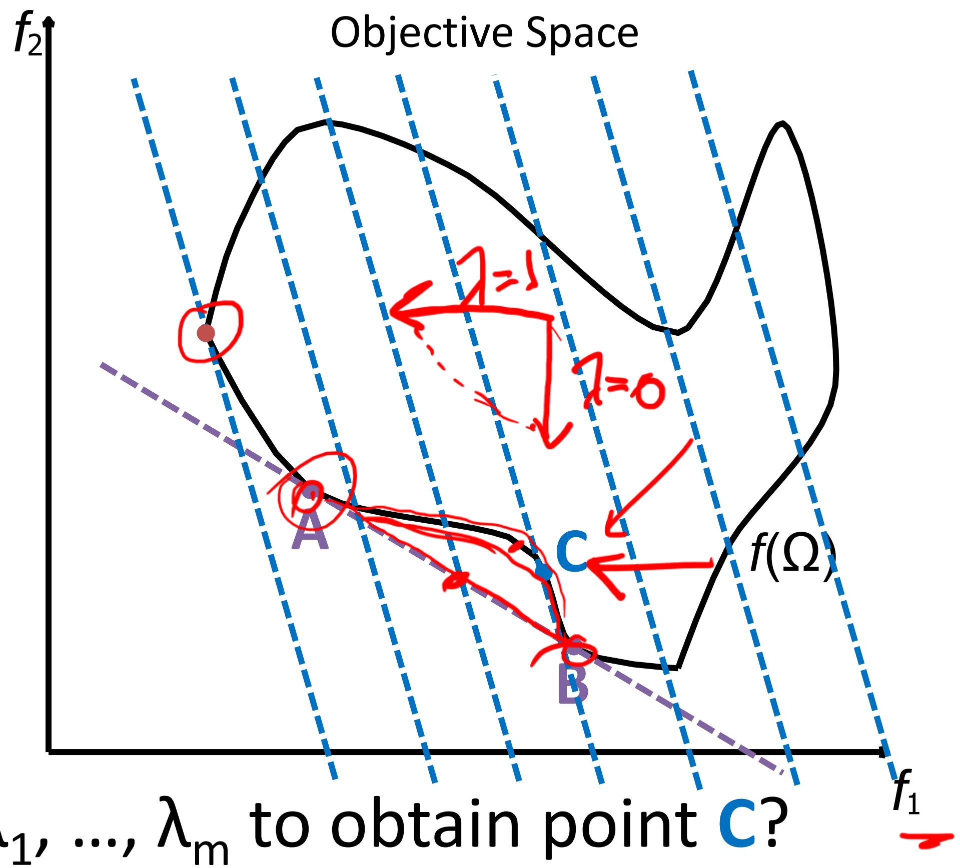
$\min \lambda f_1(x) + (1-\lambda) f_2(x)$



Solution of SOP for λ = 0.3

Solution of SOP for λ = 0

Weighted-sum: Non-Convex Case



- Is there a value for $\lambda_1, \ldots, \lambda_m$ to obtain point **C**?
- Thm: weighted-sum method is
  - **complete** for convex problems
  - **incomplete** for non-convex problems

The ε-constraint Method

- Idea: optimise a single objective and constraint all others
- Given a vector $\boldsymbol{\varepsilon} = [\varepsilon_1, \ldots, \varepsilon_m]$ solve the SOP($\boldsymbol{\varepsilon}$,i)

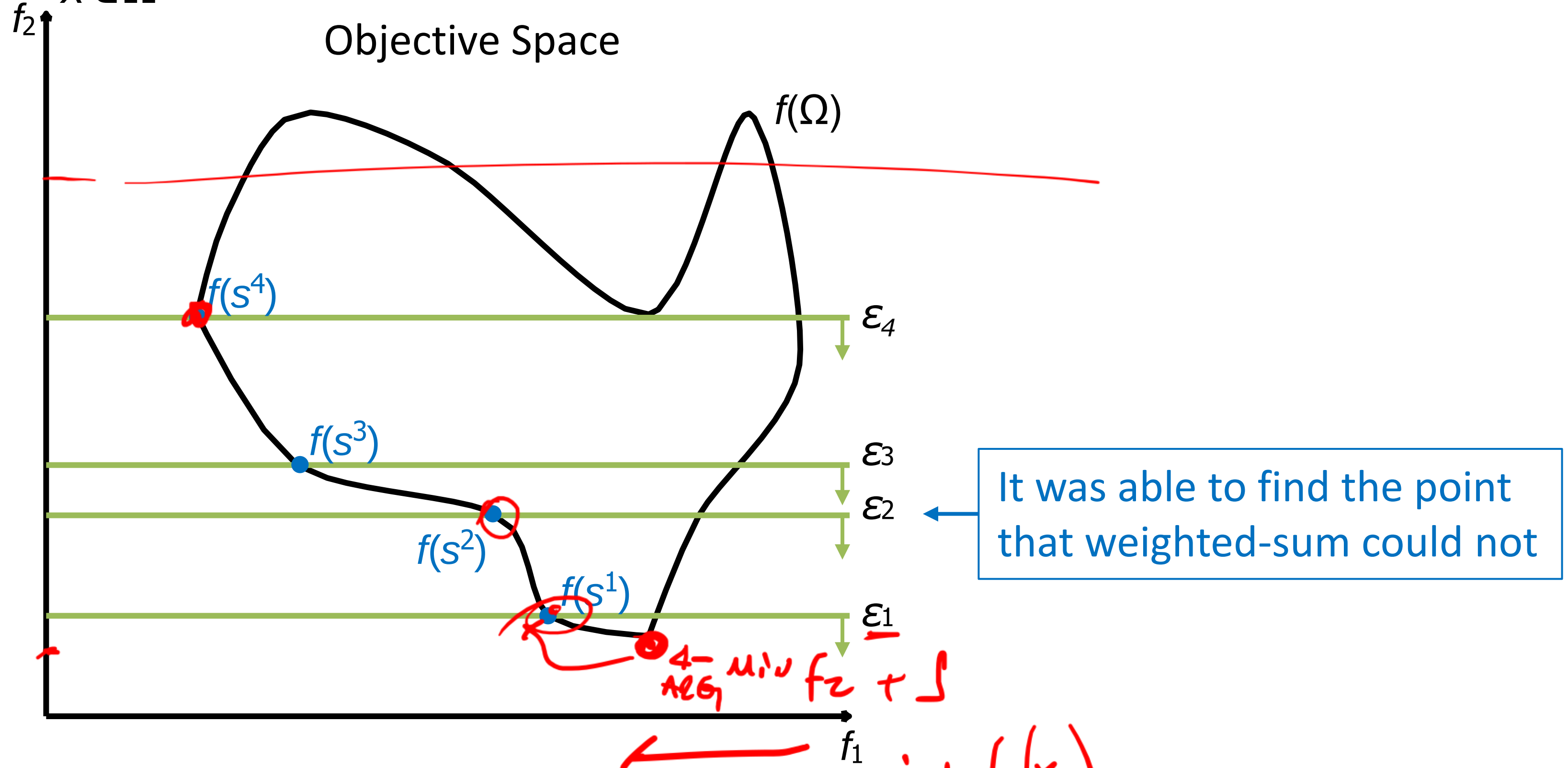
$$\min_{x \in \Omega} f_i(x)$$
$$\text{s.t. } f_j(x) \leq \varepsilon_j \text{ for all } j \neq i$$

- Solve the SOP($\boldsymbol{\varepsilon}$,i) for multiple $\boldsymbol{\varepsilon}$ and i

ε-constraint: illustration

- Bi-objective example: $\min_{x \in \Omega} f_1(x)$ s.t. $f_2(x) \le \varepsilon_i$

$\lambda_i = \mathrm{RAND}\left(0, 1-\sum_{j=1}^{i-1} \lambda_j\right)$



OBS min f1(x)

- Thm: for any point found by weighted-sum there exist ε and i that returns the same point

Bi-Objective LPs: Intuition

- Scalarization can find all Pareto-optimal points for **bi-objective** **LPs** by solving for different λ:

$$\min_{x\in\Omega} \lambda f_1(x) + (1-\lambda) f_2(x) = \min_{x\in\Omega} \lambda c_1^T x + (1-\lambda) c_2^T x$$

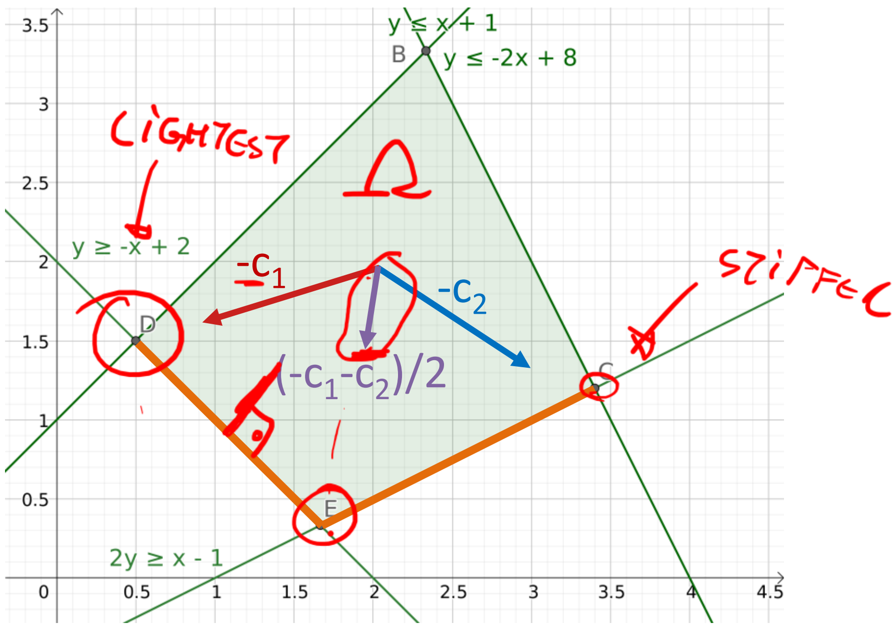
PARAMETRIZE LP (λ)

- What point x is the optimal for
  - $\lambda = 1$?
  - $\lambda = 0$?
  - $\lambda = 0.5$? $\min (c_1^T x + c_2^T x)/2$
  - $\lambda = 0.75$?

λ=0.99
.98

- What is the Pareto **Set**?
  - segments DE and EC

Decision Space



Bi-Objective Simplex: Overview

- As with scalarization, solve multiple LPs using simplex
- Take advantage of the LP geometry to update λ
- <u>Phase 1</u>: find a feasible solution (basis)
  - Do we need to care about λ here?

Ω

- <u>Phase 2</u>: solve the LP for λ=1 using simplex and Phase 1's basis
- <u>Phase 3:</u>
  - while λ can be decreased:
    - decrease λ
    - save λ, and the updated solution (basis)
- Return the saved λs and solutions

Bi-Objective Simplex: Algorithm

$$\min_{x \in \Omega} \lambda c_1^T x + (1-\lambda) c_2^T x$$

Algorithm 1 Parametric Simplex for bi-objective LPs

1: **Input:** Data A, b, C for a bi-objective LP

2: **Phase 2:** Solve the LP for $\lambda = 1$ starting from Phase 1's basis $\mathcal{B}$.

3: Compute $\tilde{A}$ and $\tilde{b}$.

4: **Phase 3:**

Index of non-basic variables with:

- negative reduced cost wrt c_2

NON BASIC VARS

RED. COST

MIN

OBJ FUNC IDX

$$\bar{c}_i^{\lambda} = \lambda \bar{c}_i^{1} + (1-\lambda) \bar{c}_i^{2} \geq 0$$

SOLVE FOR λ

$$\lambda(\bar{c}_i^{1} - \bar{c}_i^{2}) + \bar{c}_i^{2} = 0$$

Adapted from: Multicriteria Optimization, 2007 – Matthias Ehrgott

Stochastic Programming

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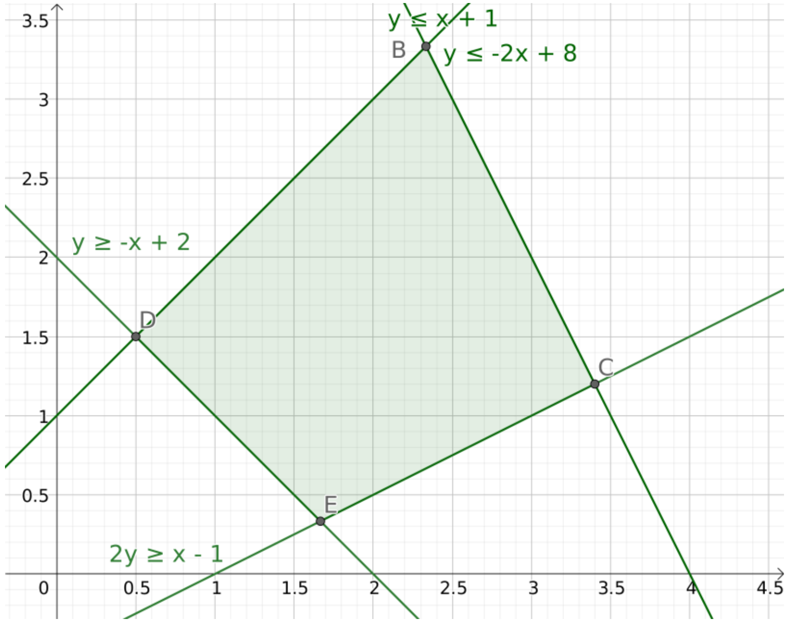
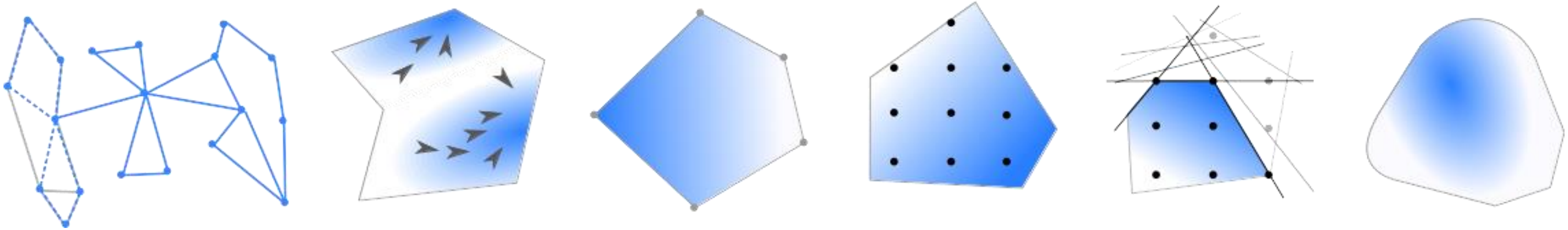




Farmer’s Example – Data

	Wheat	**Corn**	**Beets**
Yield (T/acre)	2.5	3	20
Plating cost ($/acre)	150	230	260
Selling price ($/T)	170	150	36 under 6000 T 10 above 6000 T
Purchase price ($/T)	238	210	–
Min. requirement (T)	200	240	–

- All this values comes from historical data the farmer collected over the years

Farmer's Example – Variables

	Wheat	Corn	Beets
Acres allocated	x_1	x_2	x_3
Amount sold	w_1	w_2	w_3 for under 6000T w_4 for over 6000T
Amount purchased	y_1	y_2	–
	240 T	200 T	FOR CATTLE

- Goal: minimize loss (negative loss == profit)

Famer's Problem LP

COST TO PROD

$$\min\ 150x_1 + 230x_2 + 238x_3$$
$$+\ 238y_1 + 210y_2$$

← WHY BUY

REVENUE

$$-\ 170w_1 - 150w_2 - 36w_3 - 10w_4$$

ALLOC

x_1, x_2, x_3

$\vec{y}, \vec{w}$] RECOURSE

$$\text{s.t.}\quad x_1 + x_2 + x_3 \le 500$$

SIZE OF THE FARM

$$2.5x_1 + y_1 - w_1 \ge 200$$
$$3x_2 + y_2 - w_2 \ge 240$$
$$w_3 + w_4 \le 20x_3$$
$$w_3 \le 6000$$

x_i: land allocated
w_i: amount sold
y_i: amount purchased
wheat → 1
corn → 2
beets → 3 (up to quota)
4 (above)

- Recall: minimize loss == maximize profit (negative loss)

The Effect of the Weather

- Consider 2 scenarios: -20% and +20% change in yield due weather
- Opt. solution for each one of the cases and the previous average case:

	min $g(x, y)$ -20% yield			min $g(x, E(\theta))$ average yield			+20% yield		
	Wheat	**Corn**	**Beets**	**Wheat**	**Corn**	**Beets**	**Wheat**	**Corn**	**Beets**
Acres allocated	100	25	375	120	80	300	183	66	300 ~~250~~
Amount sold	0	0	6000	300	240	6000	500 ~~350~~	0	7000 6000
Amount purchased	0	180	–	0	0	–	0	0	–
Total Profit			$59,950			$118,600			$167,667

- In each solution, we allocate enough to sell 6000 T of beets, then handle wheat and buy corn if needed
- Can we use these 3 solutions?

Two-stage Stochastic Program with Recourse

- The farmer's problem has:
  - **two-stages**: decide the land allocation (x) **<u>then</u>** we observe the weather/yield, and
  - **recourse**: we can buy the missing (y) wheat or corn if our production is below the requirements
- This is the class of problems we will focus in this lecture
- Expected profit if scenario has probability 1/3
  - Using the oracle: \$115,406 ← $\mathbb{E}\left[\min_x g(x,\theta)\right]$
  - Using the average case solution: \$107,240 ←
    - Under produces beets in the -20% case: \$86,600 ←
    - Over produces beets in the +20% case: \$107,683 ← $\min_x g(x, \mathbb{E}(\theta))$
- Can we do better than using the average case in all weathers?

Recourse-Problem LP

$$
\begin{aligned}
\min \quad & 150x_1 + 230x_2 + 238x_3 \\
& + 1/3 * (238y_{11} + 210y_{21} - \underset{156}{170w_{11}} - \underset{120}{150w_{21}} - 36w_{31} - 10w_{41}) \\
& + 1/3 * (238y_{12} + 210y_{22} - 170w_{12} - 150w_{22} - 36w_{32} - 10w_{42}) \\
& + 1/3 * (238y_{13} + 210y_{23} - 170w_{13} - 150w_{23} - 36w_{33} - 10w_{43})
\end{aligned}
$$

s.t. $x_1 + x_2 + x_3 \leq 500$

6000T

Scenario 1 (+20%)	**Scenario 2 (avg)**	**Scenario 1 (-20%)**
$3x_1 + y_{11} - w_{11} \geq 200$	$2.5x_1 + y_{12} - w_{12} \geq 200$	$2x_1 + y_{13} - w_{13} \geq 200$
$3.6x_2 + y_{21} - w_{21} \geq 240$	$3x_2 + y_{22} - w_{22} \geq 240$	$2.4x_2 + y_{23} - w_{23} \geq 240$
$w_{31} + w_{41} \leq 24x_3$	$w_{32} + w_{42} \leq 20x_3$	$w_{33} + w_{43} \leq 16x_3$
$w_{31} \leq 6000$	$w_{32} \leq 6000$	$w_{33} \leq 6000$

x_i: land allocated
w_{ik}: amount sold
y_{ik}: amount purchased
wheat → 1
corn → 2
beets → 3 (up to quota)
4 (above)
k → scenario index

Recourse-Problem Solution

		Wheat	Corn	Beets
First Stage (x)	**Acres allocated**	**170**	**80**	**250**
scenario 1 (+20% yield)	Yield (T)	510	288	6000
	Sold/Purchased (T)	310	48	6000
scenario 2 (avg yield)	Yield (T)	425	240	5000
	Sold/Purchased (T)	225	0	5000
scenario 3 (-20% yield)	Yield (T)	340	192	4000
	Sold/Purchased (T)	140	-48	4000

Total Profit: $108,390

- Key differences:
  - Allocate land for **beets to reach quota at best case**
  - Allocate land for corn to meet constraint in the average case
  - Left over land for wheat

Comparing Solutions

Wait-and-See (WS)		Recourse-Problem (RP)		Expected-Value Prob.
$E_\varepsilon[\min_x g(x,\varepsilon)]$	$\leq$	$\min_x E_\varepsilon[g(x,\varepsilon)]$	$\leq$	$\min_x g(x, \boxed{E_\varepsilon[\varepsilon]})$
-$115,406		-$108,390		-$107,240

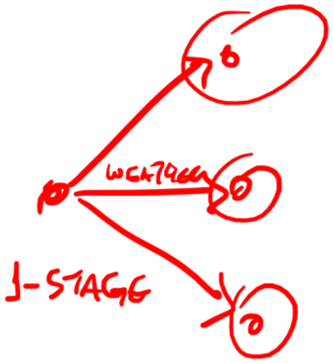
- How much should we pay for a perfect prediction of the future?
  - WS - RP = -115,406 -(-108,390) = -$7,016
  - Known as the Expected Value of Perfect Information (EVPI)
- How much is Stochastic Programming helping us?
  - RP - E[EV] = -108,390 - (-107,240) = -$1,150
  - Known as the Value of the Stochastic Solution (VSS)
- Stochastic Programming leverages information about the distribution of the uncertain outcomes to improve the quality of the solution

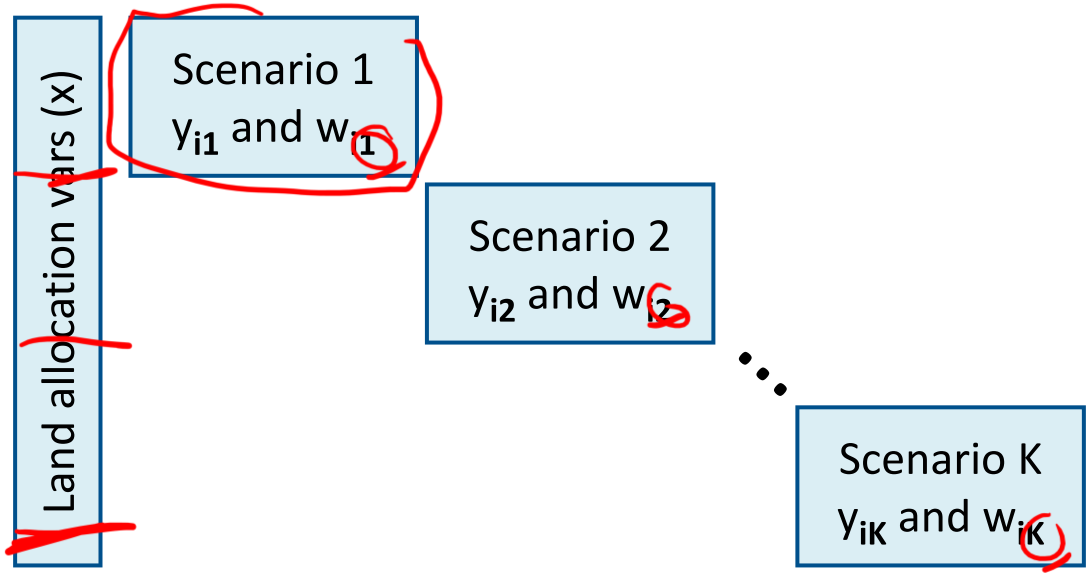
Sampling

- What if
  1) we have a complicated or black-box model, e.g., weather forecast?
  2) we have a continuous distribution?
- **Sampling can solve both**
  - In some cases, (2) can be solved analytically
- The samples is treated as scenarios of equal probability
  - Referred as the **sample average approximation (SAA)**
- Better results with more samples
- Different sampling methods can also improve the solution:
  - Importance sampling
  - Quasi-Monte Carlo
  - Conditional Sampling

Handling Large Problem

- If you have several scenarios or samples in the farmer's example, the LHS of our constraints will look like this:





x_i: land allocated for i
w_{ik}: amount sold
y_{ik}: amount purchased
k → scenario index
i → wheat, corn, beets

- Each scenario is only connected through the first-stage variables (x)
- Where have we seen this before?

L-Shaped Method (aka Benders Decomp)

- Use Benders Decomposition
  - Instead of one big sub-problem, we have K independent small subproblems
- From Benders Lecture adapted notation:

General Recourse Problem

$$
\begin{aligned}
\min_{x,z} \quad & f^\intercal x + \sum_{k=1}^{K} c_k^\intercal z_k \\
\text{s.t.} \quad & Ax = b \\
& \boxed{B_k x} + \boxed{D z_k} = d_k \quad \forall k \in \{1, \dots, K\} \\
& x, z \geq 0
\end{aligned}
$$

Y_{ik}, W_{ik}

z_k is THE RECOURSE FOR SCENARIO K

RMP

Benders Reduced Master Problem

$$
\begin{aligned}
\min_{x,\eta} \quad & f^\intercal x + \eta \\
\text{s.t.} \quad & Ax = b \\
& \eta \geq \pi_e^\intercal (d_k - B_k x) \quad \forall k, \pi_e \in \bar{E}_k \\
& 0 \geq r_q^\intercal (d_k - B_k x) \quad \forall k, r_q \in \bar{Q}_k \\
& x \geq 0
\end{aligned}
$$

K sub-problems

(Primal) Sub-Problem k

$$
\min_{z_k \geq 0} \{ c_k^\intercal z_k : D z_k = d_k - B_k \hat{x} \}
$$

L-Shaped Method applied to Farmer's Example

General Recourse Problem

$$
\begin{aligned}
\min_{x,z} \quad & f^{\mathsf{T}}x + \sum_{k=1}^{K} c_k^{\mathsf{T}} z_k \\
\text{s.t.} \quad & \boxed{\begin{aligned} Ax &= b \\ B_k x + D z_k &= d_k \end{aligned}} \quad \forall k \in \{1, \ldots, K\} \\
& x, z \geq 0
\end{aligned}
$$

EXP. LARGE

Benders Reduced Master Problem

$$
\begin{aligned}
& \boxed{\min_{x,\eta} \quad f^{\mathsf{T}}x + \eta} \\
\text{s.t.} \quad & Ax = b \\
& \boxed{\eta \geq \pi_e^{\mathsf{T}}(d_k - B_k x)} \quad \forall k, \pi_e \in \bar{E}_k \\
& \cancel{0 \geq r_q^{\mathsf{T}}(d_k - B_k x) \quad \forall k, r_q \in \bar{Q}_k} \\
& x \geq 0
\end{aligned}
$$

← NEG → PROFIT

FIXING THE RECOURSE

NOT NEEDED w/ TOTAL RECOURSE

ND CO

- The matrix A contains a single constraint: $x_1+x_2+x_3 \leq 500$
- $f^t x$ in the objective function is: $150x_1 + 230x_2 + 238x_3$
- Now, let's look into the sub-problem k

COST OF EACH CROP

Sub-Problem for Farmer's Example

- From the recourse-problem LP:

COMPLICATING VARS

Scenario 1 (+20%)

$$3x_1 + y_{11} - w_{11} \geq 200$$
$$3.6x_2 + y_{21} - w_{21} \geq 240$$
$$w_{31} + w_{41} \leq 24x_3$$
$$w_{31} \leq 6000$$

Making yield a parameter

Scenario/Sample k for yield γ

$$\begin{array}{llll} \gamma_1 x_1 & + y_{1k} - w_{1k} & \geq 200 \\ \gamma_2 x_2 & + y_{2k} - w_{2k} & \geq 240 \\ \gamma_3 x_3 & - w_{3k} - w_{4k} & \geq 0 \\ & - w_{3k} & \geq -6000 \end{array}$$

$\leftarrow d_k$

$D z_k$

SUB PROBLEM

(Primal) Sub-Problem k

$$\min_{z_k \geq 0} \{ c_k^T z_k : D z_k = d_k - B_k \hat{x} \}$$

Sub-Problem k (γ)

$$\begin{array}{lll} y_{1k} - w_{1k} & \geq 200 - \gamma_1 \hat{x}_1 \\ y_{2k} - w_{2k} & \geq 240 - \gamma_2 \hat{x}_2 \\ - w_{3k} - w_{4k} & \geq 0 \quad - \gamma_3 \hat{x}_3 \\ - w_{3k} & \geq -6000 \end{array}$$

RMP: $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

$z_k^* = y_{ik}^*, w_{ik}^*$

CORRESPONDING OPT DUAL VALUES π_e

x_i: land allocated for i
w_{ik}: amount sold
y_{ik}: amount purchased
k → scenario index
i → wheat, corn, beets

Chance Constraints

- So far in the course, we have seen:
  - **(hard) constraints**: must be satisfied
  - **soft constraints**: penalize if not satisfied
- **Chance constraints**: a probabilistic constraint

$$\boxed{P(a^T x \le b) \ge \alpha}$$

where either a or b depends on a random variable

- Famer’s problem example:
  - P(producing less than 6000 T of beets) ≤ 0.25
  - P(buy 20 T or less of corn and wheat) ≥ 0.8

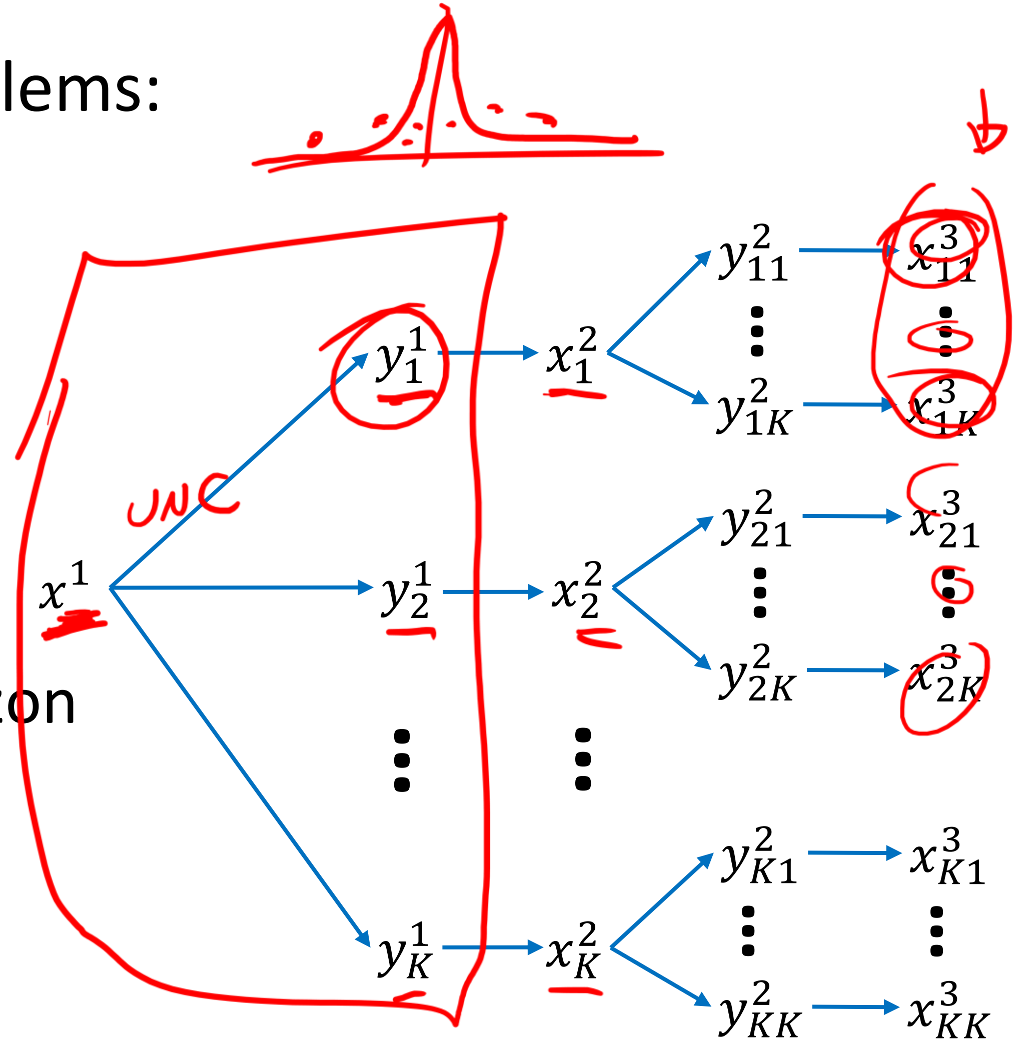
Modeling Chance Constraints

- For discrete distributions and sampling:
  1. use binary variables to count the constraint violations
  2. constraint the sum of scenario probability where violation occurred
- Famer's problem example: P(buy 20 T or less of corn and wheat) ≥ 0.8
  1. for each scenario k:

     $z_k \in \{0,1\}$: **constraint violated implies $z_k = 1$**
     - What is the maximum amount of corn and wheat needed?
       440 from the specification (200 wheat, 240 corn)
     - Use this tight upper bound for a Big-M constraint
       $\mathbf{y_{1k} + y_{2k} \leq 20 + 420z_k}$
  2. in the main problem: $\mathbf{\Sigma_k p_k z_k \leq 0.2}$
     - Note that we modeled the complement, i.e., 1 - P(buy 20T or less) ≤ 1 - 0.8

Multi-Stage Stochastic Programming

- Multi-stage is a series of two-stage problems:
  - Superscript denotes discrete time step
- In the farmer's example:
  - crop rotation: rotate field every year t
  - beets production quota over multiple seasons
- Issue: **curse of dimensionality**
  - exponential growth of scenarios wrt horizon
- Key techniques:
  - Nested Benders Decomposition
  - Better Sampling 4 ←



Outline

- Topics for this week's quiz
- Exam details
- Ask me questions

Exam Details

- **Type**: wattle exam
  - Very little to no multiple-choice questions
  - No negative marks
  - No internet nor calculators
- **Material allowed**: one A4 page with notes on both sides
- **Hurdle**: 40% of the total exam points
- **Assessable material**: all contents of the lectures with exception of:
  - i-dual algorithm (Felipe's guest lecture)
  - Charles' guest lecture
- **No coding questions**

x^1_3

$x \boxed{13}$

$x_{1,3}$

Outline

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Dual LPs – Rules

The dual of a **min**imisation problem:

$$
\begin{array}{lcl}
\min_x c^\intercal x & \longleftrightarrow & \max_y b^\intercal y \\
x \geq 0 & \longleftrightarrow & A^\intercal y \leq c \\
x \leq 0 & \longleftrightarrow & A^\intercal y \geq c \\
x \in \mathbb{R}^n & \longleftrightarrow & A^\intercal y = c \\
Ax \leq b & \longleftrightarrow & y \leq 0 \\
Ax \geq b & \longleftrightarrow & y \geq 0 \\
Ax = b & \longleftrightarrow & y \in \mathbb{R}^m
\end{array}
$$

swap ineq. (variable constraints)

same ineq. (constraints)

The dual of a **max**imisation problem:

$$
\begin{array}{lcl}
\max_x c^\intercal x & \longleftrightarrow & \min_y b^\intercal y \\
x \geq 0 & \longleftrightarrow & A^\intercal y \geq c \\
x \leq 0 & \longleftrightarrow & A^\intercal y \leq c \\
x \in \mathbb{R}^n & \longleftrightarrow & A^\intercal y = c \\
Ax \leq b & \longleftrightarrow & y \geq 0 \\
Ax \geq b & \longleftrightarrow & y \leq 0 \\
Ax = b & \longleftrightarrow & y \in \mathbb{R}^m
\end{array}
$$

same ineq.

swap ineq.

- Keep in mind:
  - Strong duality: primal and dual has the same optimal objective value
  - Dual of the dual of an LP is the primal

Dual LPs – Example

Primal problem:

$$\begin{aligned} \min_{x} \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & 2x_1 + 9x_2 \geq 14 \\ & 2x_1 + 3x_2 \geq 10 \\ & x_1 + 2x_2 \geq 1 \\ & x \geq 0 \end{aligned}$$

Dual problem:

$$\begin{aligned} \max_{y} \quad & 14y_1 + 10y_2 + y_3 \\ \text{s.t.} \quad & 2y_1 + 2y_2 + y_3 \leq 1 \\ & 9y_1 + 3y_2 + 2y_3 \leq 3 \\ & y \geq 0 \end{aligned}$$

$\mathbb{R}^2$

$$\begin{bmatrix} 2 & 9 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\begin{array}{ccc} 14 & 10 & 1 \end{array}$$

$$\begin{bmatrix} 2 & 2 & 1 \\ 9 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

y