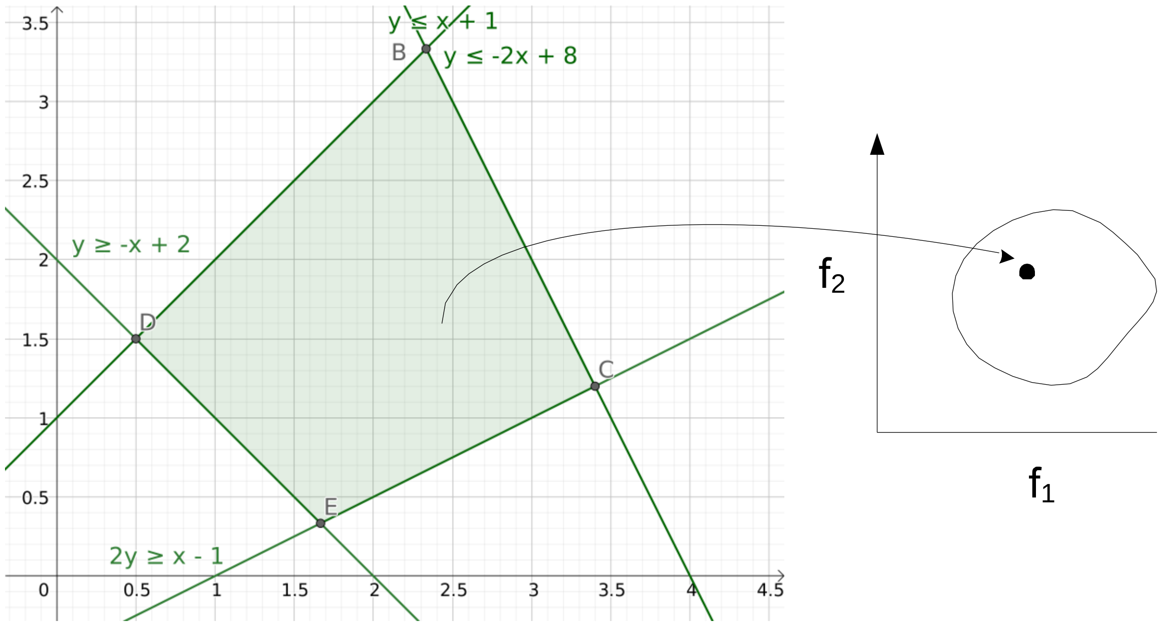
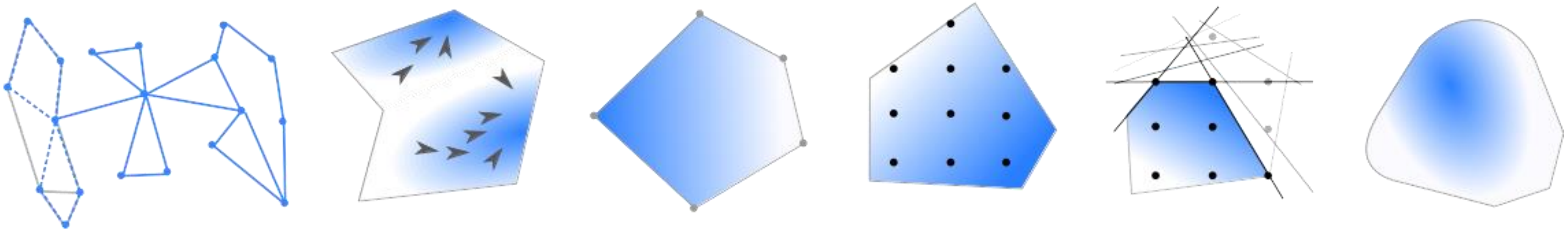
Multi-Objective Optimisation

COMP4691 / 8691





Motivation – Choosing a Car

- You are helping a friend to buy a new car and they want to take into consideration:

OR

	Criteria/Car	A	B	C	D	E	F
min	Price	16200	14900	**14000**	15200	17200	20000
min	Fuel Consumption	7.2	**7.0**	7.5	8.2	9.2	10
max	Power	66	62	55	**71**	51	40

- How to choose a car?

Choosing a Car – Ordered Preferences

	Criteria/Car	A	B	C	D	E	F
min	Price	16200	14900	**14000**	15200	17200	20000
min	Fuel Consumption	7.2	**7.0**	7.5	8.2	9.2	10
max	Power	66	62	55	**71**	51	40

- Suppose your friend can rank the criteria:
  1. Price (most important)
  2. Fuel Consumption
  3. Power
- How to solve it now?

Lexicographic Approach

- Given an **ordering** of criteria:
  1. $c_1 = dir_1\ f_1$ (e.g., min price)

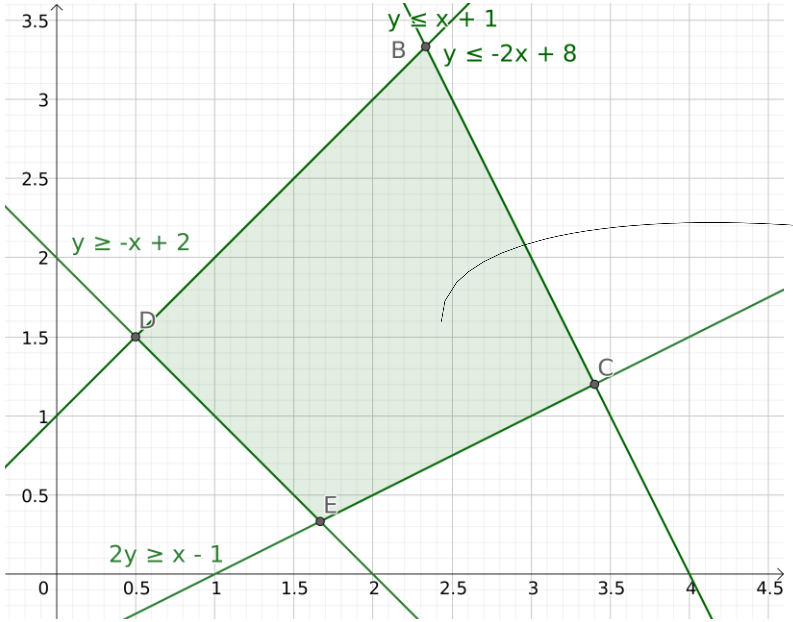
     ⋮

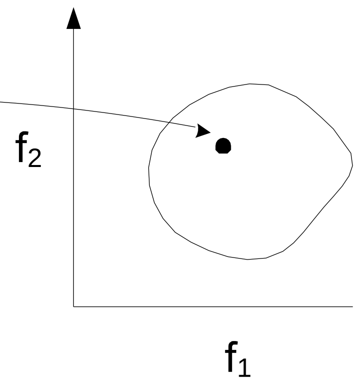
  n. $c_n = dir_n\ f_n$

- Solve (up to) n single-objective problems where i-th problem is
  – optimise $dir_i\ f_i(x)$
  – given its original constraints and
  – **$f_j(x) \leq f_j^*\ \ \forall j < i$** ← f_j^* are the solutions from previous problems
- What is problem here?
  – The trade-offs are resolved by the ordering
  – Finding such ordering can be hard (e.g., car example)

Outline

- Lexicographic Method
- **Dominance and Pareto Front and Pareto Set**
- Generative Approaches
- Analytical Approaches





Problem Definition

The problem

$$\underset{x \in \Omega}{\text{minimize}} \; \mathbf{f}(x) = [f_1(x), f_2(x), \ldots, f_m(x)]$$

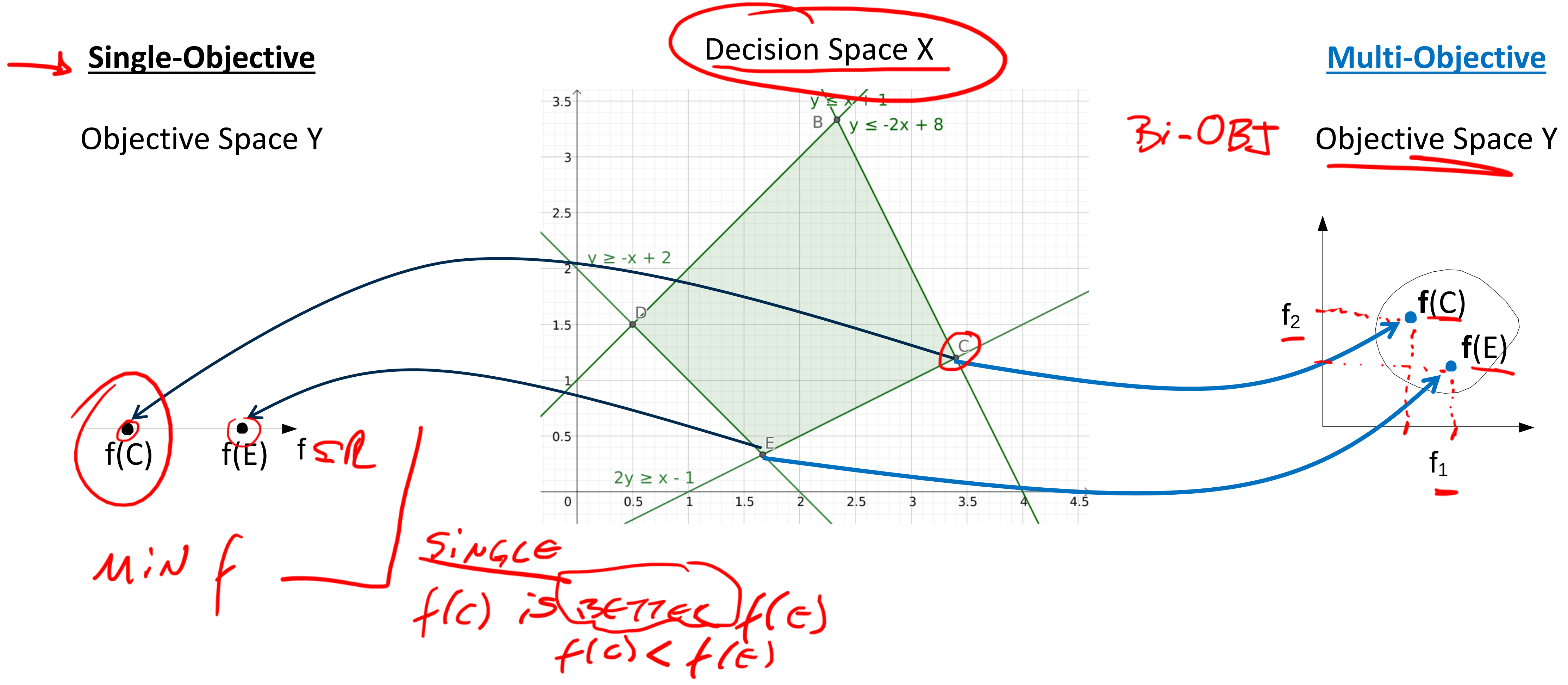
where:

- $\mathbf{f}: \Omega \to \mathbf{R}^m$ is the objective function, composed of $m \geq 2$ objective func.
- $\Omega \subseteq \mathbf{R}^n$ is the feasible space
  - Ω is defined through constraints
- $\mathbf{f}(\Omega)$ is the feasible objective space

- $\mathbf{R}^n$ is the **decision space**, $\mathbf{R}^m$ as the **objective space**.

Decision space and objective space

- Plots we have seen in the course are decision space plots



Pareto Dominance

x DOM y

MIN

Given two decision vectors x and y,

- **x dominates y** (denoted as $x \prec y$) if
  - $f_i(x) \le f_i(y)$ for all i = 1, 2, …, m, and
  - $\mathbf{f}(x) \neq \mathbf{f}(y)$

Examples: $\mathbf{f}(x) = [0, 1] \prec \mathbf{f}(y) = [2, 3]$

VECTOR

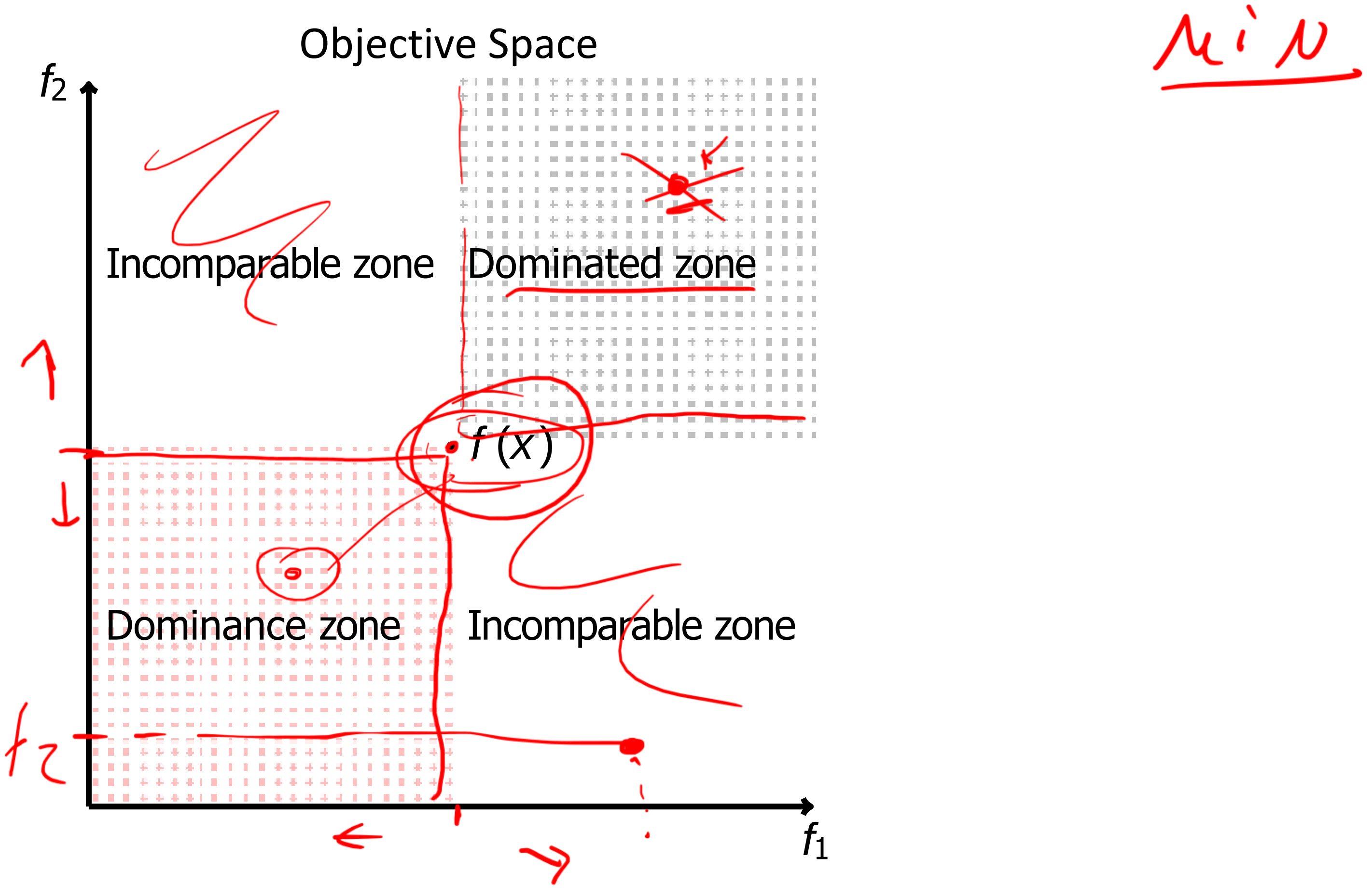
$$\begin{array}{ccc} y & & x \\ \begin{bmatrix}0\\1\\2\end{bmatrix} & \succ & \begin{bmatrix}0\\0\\2\end{bmatrix} \end{array}$$

- x **weakly dominates** y if $x \prec y$ or $\mathbf{f}(x) = \mathbf{f}(y)$ $\preceq$
- x and y are **incomparable** if
  - x does not weakly dominate y, and
  - y does not weakly dominate x

Equivalent

There exist i and j s.t.:
- $f_i(x) < f_i(y)$
- $f_j(x) > f_j(y)$

Dominance, Dominated and Indifferent Zones



Dominance – Car Example

	Criteria/Car	A	B	C	D	E	F
min	Price	16200	14900	**14000**	15200	17200	20000
min	Fuel Consumption	7.2	**7.0**	7.5	8.2	9.2	10
min	Negative Power	-66	-62	-55	**-71**	-51	-40

- Is **A** dominated by any other car?
  - No: it has better power than B, better fuel consumption then C, D, E and F
- Is **E** dominated by any other car?
  - Yes: A, B, C, and D
- Dominance is transitive
  - $D \prec E, E \prec F \rightarrow \boxed{D \prec F}$

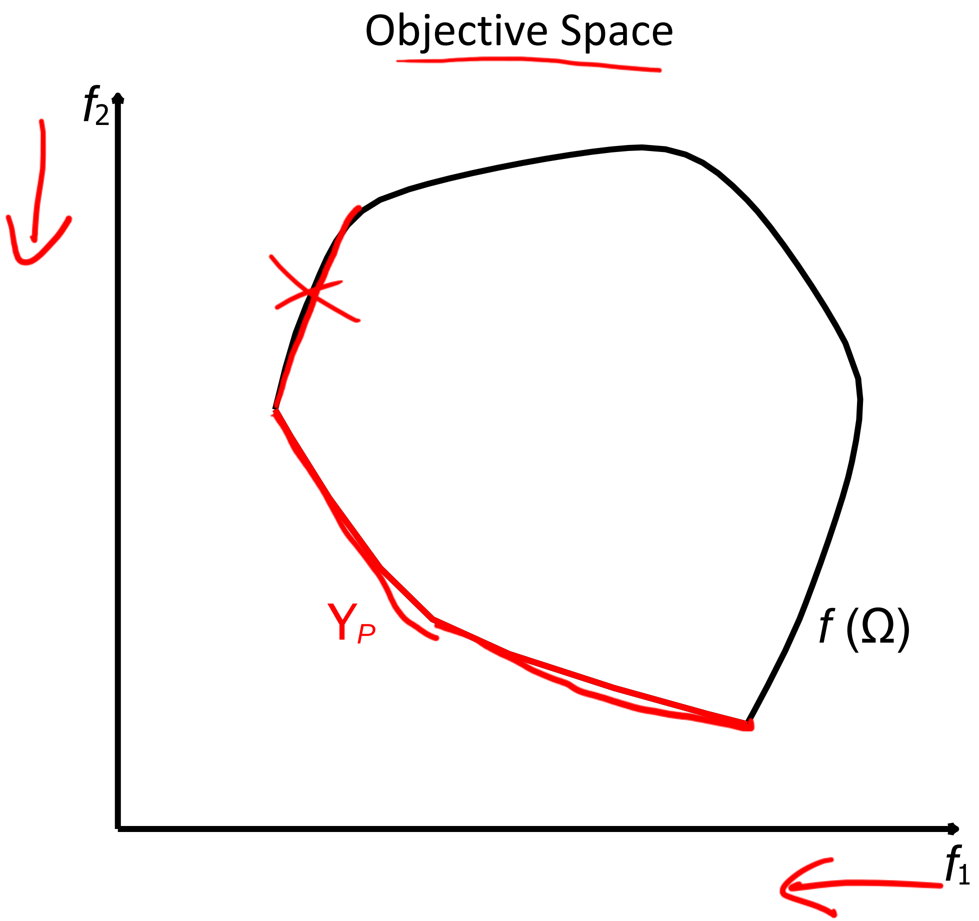
Pareto Optimal and Pareto Set

	Criteria/Car	A	B	C	D	E	F
min	Price	16200	14900	**14000**	15200	17200	20000
min	Fuel Consumption	7.2	**7.0**	7.5	8.2	9.2	10
min	Negative Power	-66	-62	-55	**-71**	-51	-40

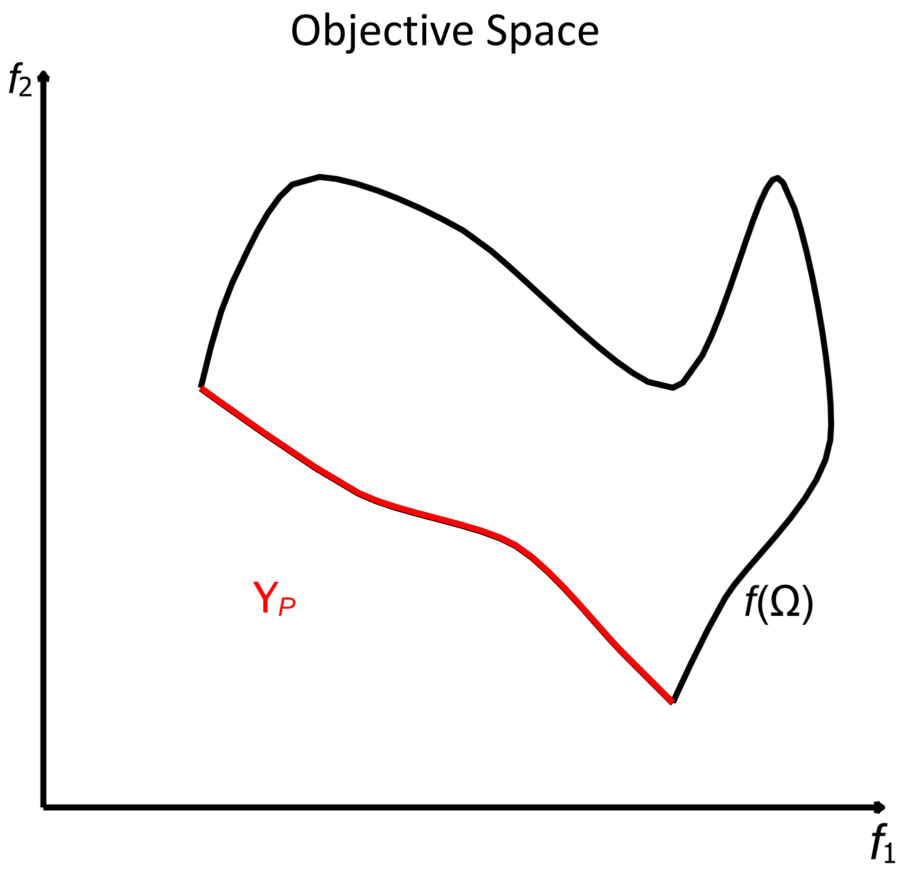
- $x^* \in \Omega$ is said to be **Pareto-optimal** **if there is no other** $x \in \Omega$ s.t. $x \prec x^*$
  - A is Pareto-optimal
- **Pareto Set:** the set of all Pareto-optimal solutions – denoted as X_P
  - $X_P = \{A, B, C, D\}$
- **Pareto Front:** image of the Pareto Set by the obj. func. – denoted as Y_P
  - $Y_P = \{[16200, 7.2, -66], [14900, 7.0, -62], [14000, 7.5, -55], [15200, 8.2, -71]\}$

Pareto Front (1)

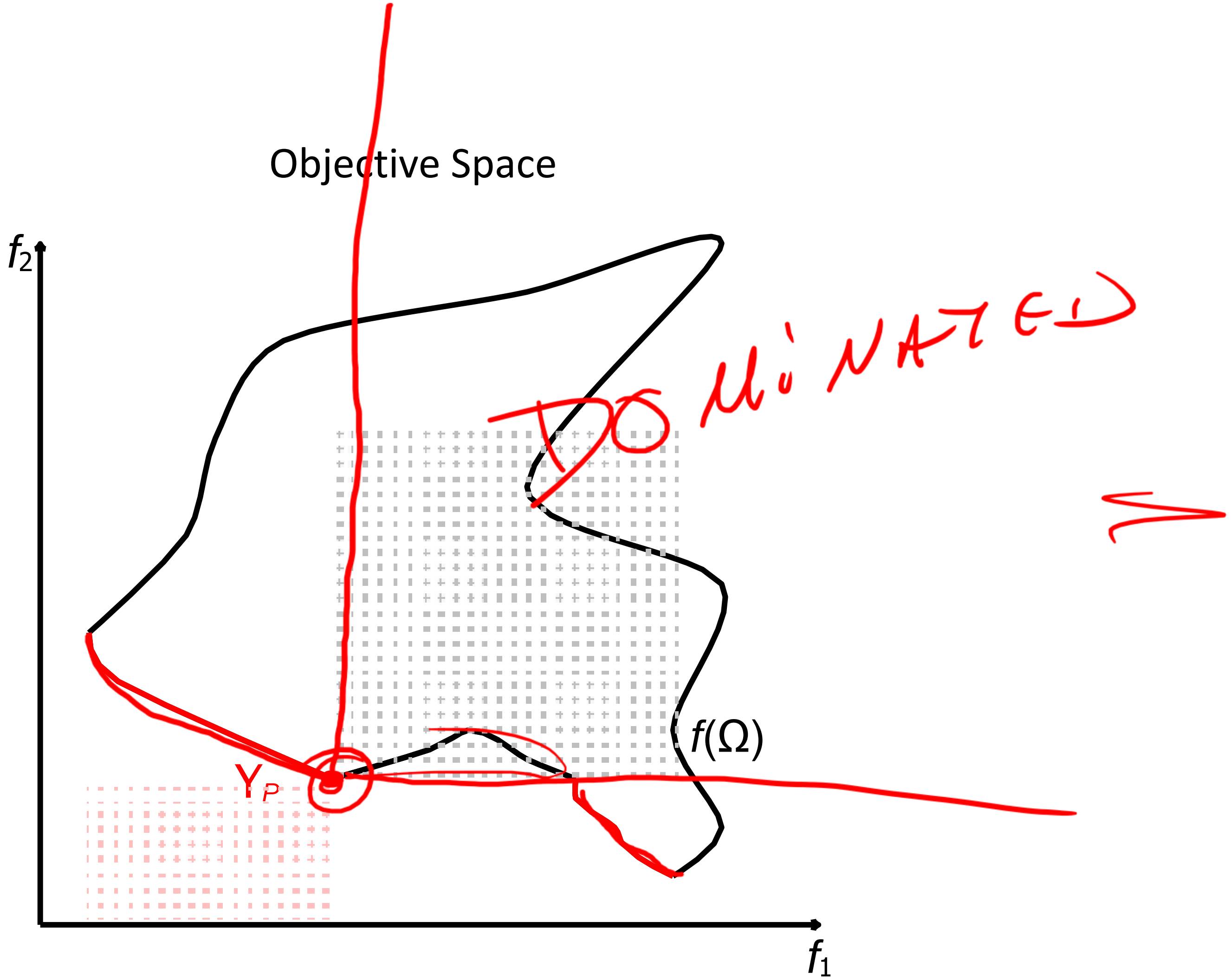
Min



Pareto Front (2)

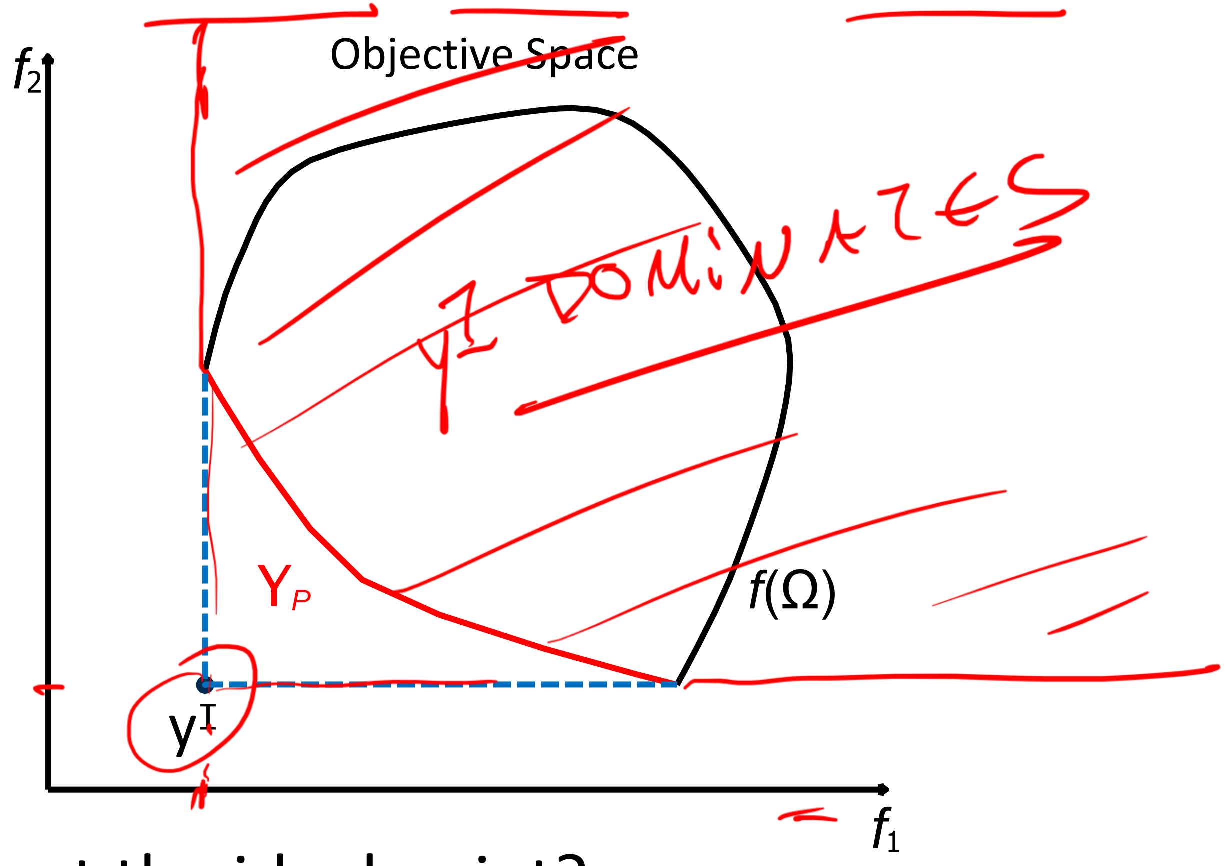


Pareto Front (3)



Ideal Point

- The ideal point is: $y^{I} = [\min_{x\in\Omega} f_1(x), \min_{x\in\Omega} f_2(x), \ldots, \min_{x\in\Omega} f_m(x)]$

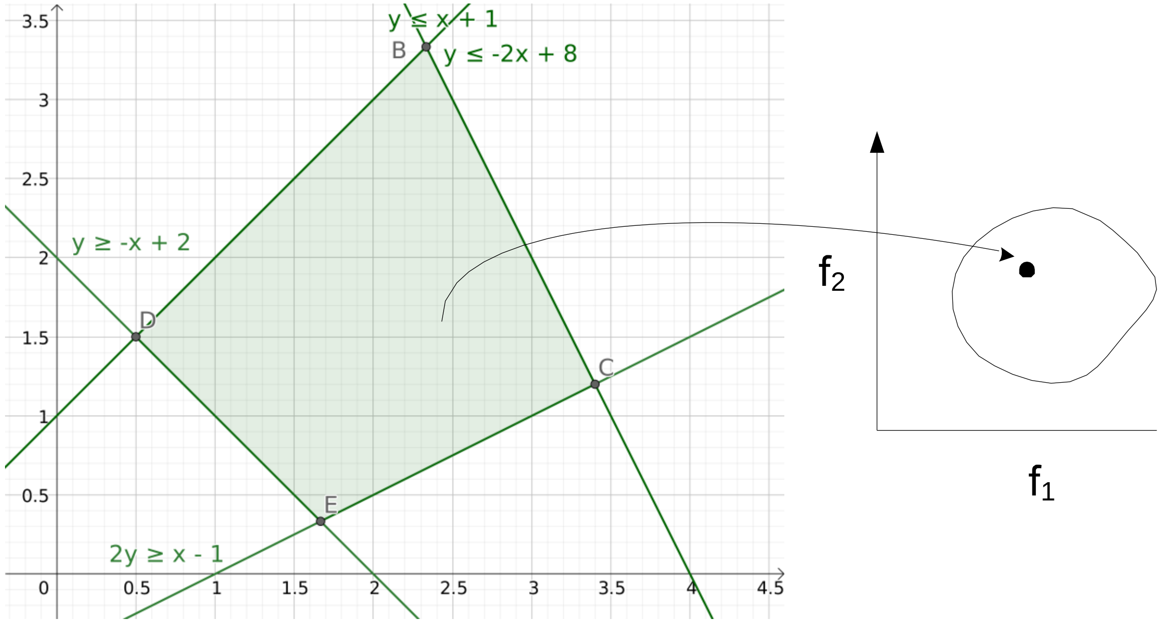


- What is special about the ideal point?

Outline

- Lexicographic Method
- Dominance and Pareto Front and Pareto Set
- **Generative Approaches**
  - Scalarization Methods
    - Weighted-sum method
    - ε-constraint method
  - Population Methods
- Analytical Approaches

Generate the points in the Pareto Front instead of analytically solving the problem



Scalarization Methods

- Main idea:
  - Convert the multi-objective optimization problem (MOP) into a series of parameterized single-objective subproblems (SOP_j)
- Goal:
  - The solution of each $\boxed{SOP_j}$ will generate a non-dominated point x_j

The Weighted-sum Scalarization Method

- Given non-negative weights $\lambda_1, \ldots, \lambda_m$ s.t. $\sum_{i=1}^{m} \lambda_i = 1$ solve the SOP:

$$\min_{x \in \Omega} \sum_{i=1}^{m} \lambda_i f_i(x)$$

CONVEX COMBO

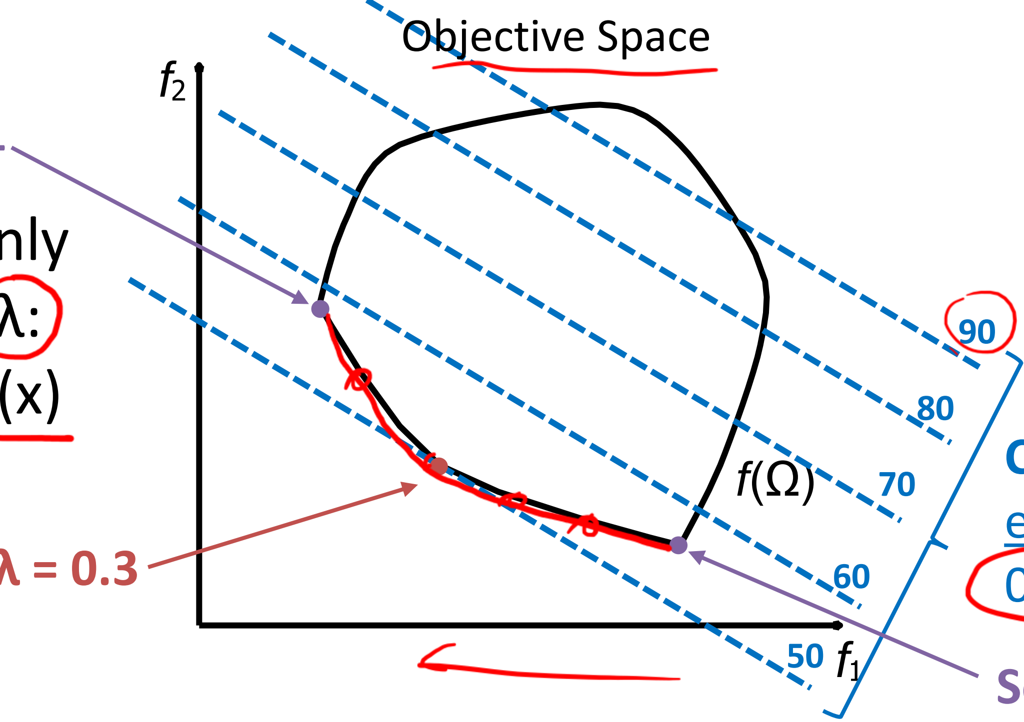
- Solve the SOP for multiple different sets of weights

$\lambda = 1$

For bi-objective we only need one parameter λ:

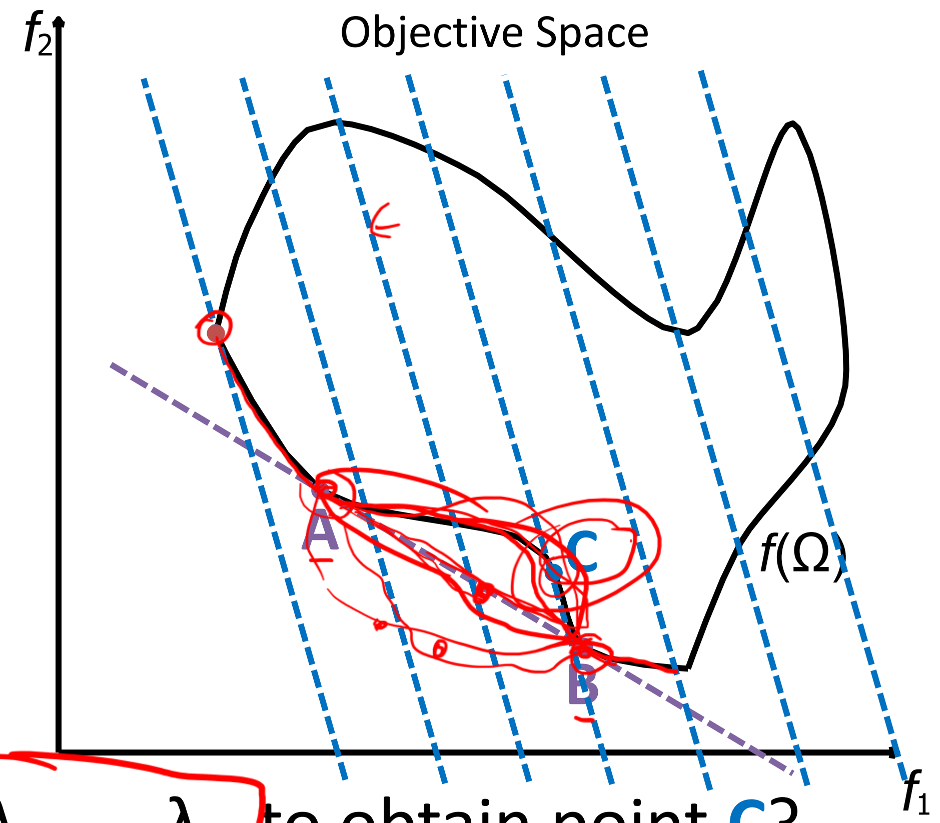
$\min \lambda f_1(x) + (1-\lambda) f_2(x)$

$\min f_2(x)$



Contour lines for $\lambda = 0.3$: each line as the same value of $0.3f_1(x) + 0.7f_2(x)$

Weighted-sum: Non-Convex Case



- Is there a value for $\lambda_1, \ldots, \lambda_m$ to obtain point **C**?
- Thm: weighted-sum method is
  - – **complete** for convex problems
  - – **incomplete** for non-convex problems

The ε-constraint Method

- Idea: optimise a single objective and constraint all others
- Given a vector $\boldsymbol{\varepsilon} = [\varepsilon_1, \ldots, \varepsilon_m]$ solve the SOP($\boldsymbol{\varepsilon}$,i)

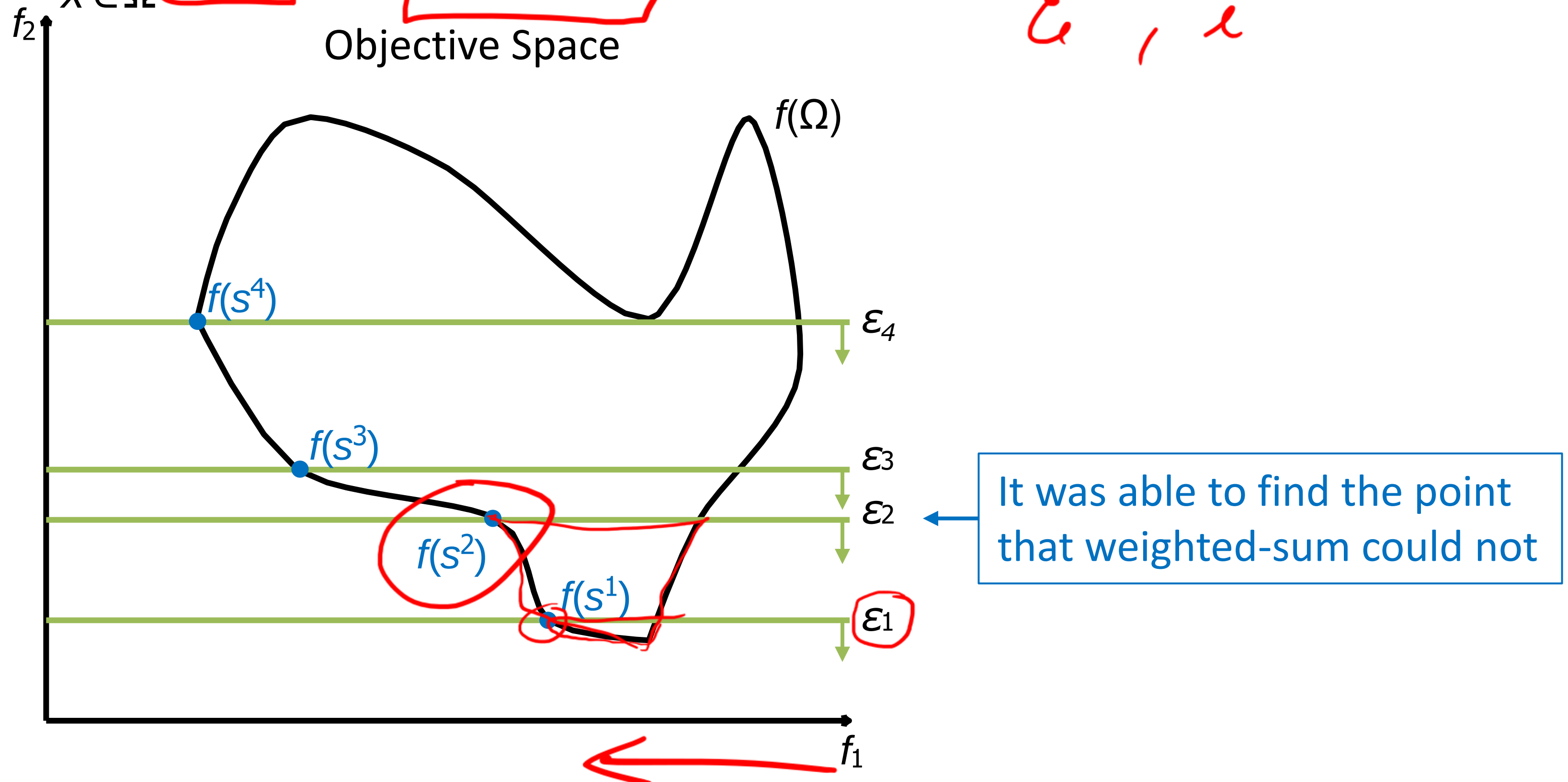
$$\min_{x \in \Omega} f_i(x)$$
$$\text{s.t. } \boxed{f_j(x) \leq \varepsilon_j} \text{ for all } j \neq i$$

- Solve the SOP($\boldsymbol{\varepsilon}$,i) for multiple $\boldsymbol{\varepsilon}$ and i

ε-constraint: illustration

- Bi-objective example: $\min_{x \in \Omega} f_1(x)$ s.t. $f_2(x) \leq \varepsilon_i$

$\vec{\varepsilon}, i$



- Thm: for any point found by weighted-sum there exist **ε** and i that returns the same point

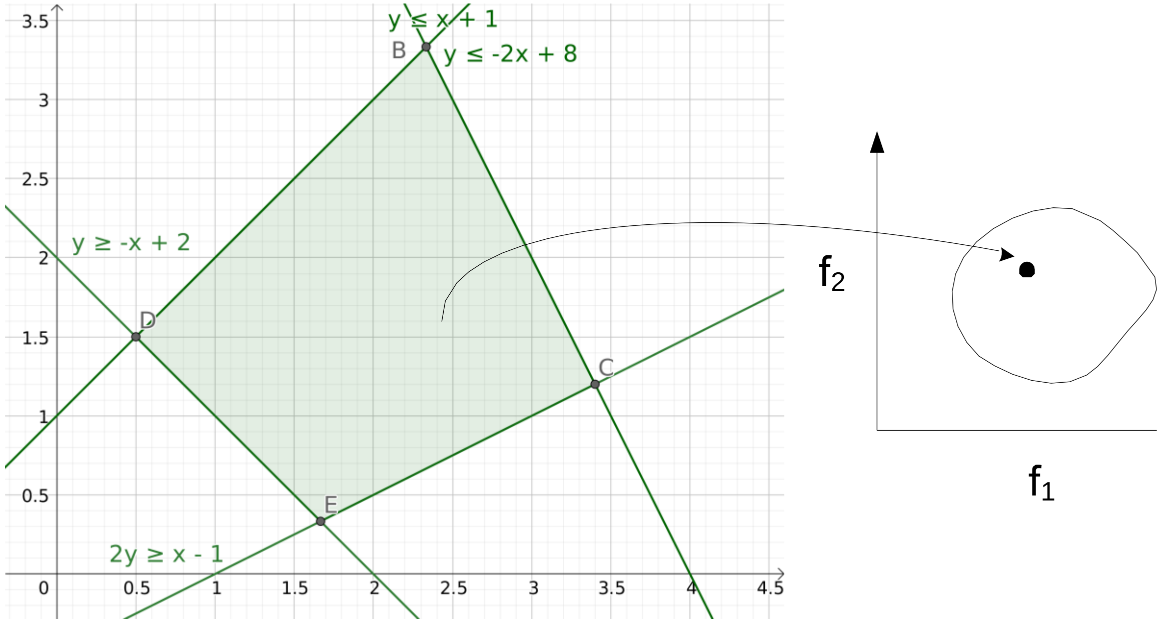
Population-based Algorithms: Overview

- **Intuition**
  - These algorithms already operate with a set of candidate solutions
  - Look at the non-dominated candidates
- **Examples**
  - Multi-objective Ant Colony Optimisation
  - Multi-objective Genetic Algorithms ← MOST POPULAR NOW
- **Key idea**: elitism
  - keep only the non-dominated candidates
  - possible for MOP – still not a good idea

FITNESS
DIVERSITY

Outline

- Lexicographic Method
- Dominance and Pareto Front and Pareto Set
- Generative Approaches
  - Scalarization Methods
  - Population Methods
- **Analytical Approaches**
  - Bi-objective LPs
  - Multi-objective LPs



Bi-Objective LPs: Intuition

- Scalarization can find all Pareto-optimal points for **bi-objective** LPs by solving for different λ:

$\lambda \in [0, 1]$

B.C. $\lambda \geq 0$, $\sum \lambda = 1$

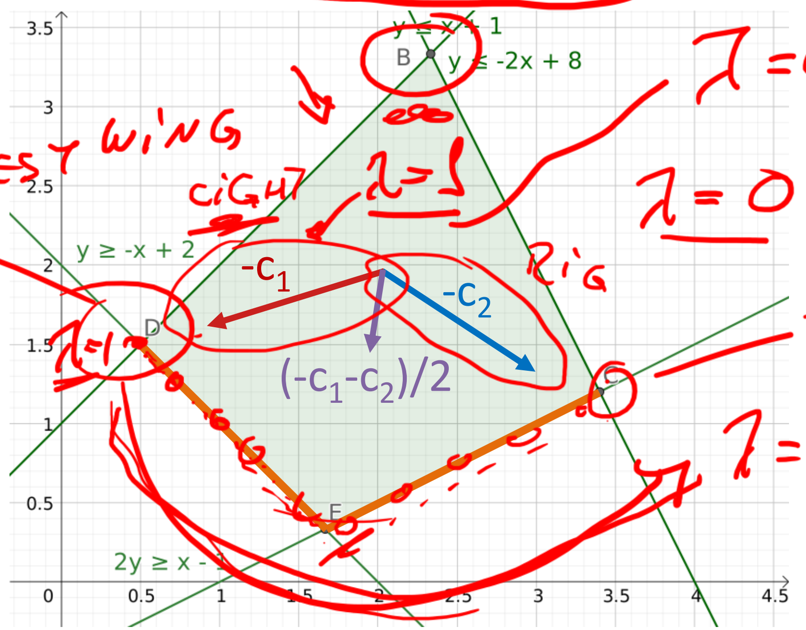
$$\rightarrow \min_{x\in\Omega} \lambda f_1(x) + (1-\lambda) f_2(x) = \min_{x\in\Omega} \lambda c_1^T x + (1-\lambda) c_2^T x$$

LINEAR CONSTR

- What point x is the optimal for
  - $\lambda = 1$? $\min f_1(x) = \min c_1^T x$
  - $\lambda = 0$? $0.5 c_1^T x + 0.5 c_2^T x$
  - $\boxed{\lambda = 0.5}$? $\min (c_1^T x + c_2^T x)/2$
  - $\lambda = 0.75$? 0.25
- What is the Pareto **Set**?
  - segments DE and EC → CONV COMB(E, C)

CONV COMBO (D, E)

Decision Space



LIGHTEST WING

LIGHT

λ = 1

λ = 0.5

λ = 0

RIG

RIGGEST WING

λ = 0

Bi-Objective Simplex: Overview

- As with scalarization, solve multiple LPs using simplex
- Take advantage of the LP geometry to update λ
- Phase 1: find a feasible solution (basis)
  - Do we need to care about λ here? No
- Phase 2: solve the LP for λ=1 using simplex and Phase 1's basis
- Phase 3:
  - while λ can be decreased: $\lambda \in [0, 1]$
    - decrease λ
    - save λ, and the updated solution (basis)
- Return the saved λs and solutions λ_i, x_i

Bi-Objective Simplex: Algorithm

REVISED SIMPLEX

$\lambda=1$, $\lambda=0.75$, $\lambda=0.5$ ...

$$\min_{x\in\Omega} \lambda c_1^T x + (1-\lambda) c_2^T x$$

BETTER

Algorithm 1 Parametric Simplex for bi-objective LPs

1: **Input:** Data A, b, C for a bi-objective LP
2: **Phase 2:** Solve the LP for $\lambda = 1$ starting from Phase 1's basis $\mathcal{B}$.
3: Compute $\tilde{A}$ and $\tilde{b}$.
4: **Phase 3:**
5: **while** $\mathcal{I} = \{i \in \mathcal{N} : \bar{c}_i^2 < 0, \bar{c}_i^1 \geq 0\} \neq \emptyset$ **do**
6: $\quad\lambda := \max_{i\in\mathcal{I}} \frac{-\bar{c}_i^2}{\bar{c}_i^1 - \bar{c}_i^2}$
7: $\quad s \in \arg\max\left\{i \in \mathcal{I} : \frac{-\bar{c}_i^2}{\bar{c}_i^1 - \bar{c}_i^2}\right\}$
8: $\quad r \in \arg\min\left\{j \in \mathcal{B} : \frac{b_j}{\tilde{A}_{sj}}, \tilde{A}_{sj} > 0\right\}$
9: $\quad$Let $\mathcal{B} := (\mathcal{B} \setminus \{r\}) \cup \{s\}$ and update $\tilde{A}$ and $\tilde{b}$.
10: **end while**
11: **Output:** Sequence of λ-values and sequence of optimal BFSs.

SWIPE FROM 1 TO 0

VAR W/ VALUE 0

Index of non-basic variables with:
- negative reduced cost wrt c_2
- non-neg. reduced cost wrt c_1

Largest λ s.t. object wrt c_2 increases

MOVE OUT

Regular Simplex rule for exiting variable

Adapted from: Multicriteria Optimization, 2007 – Matthias Ehrgott

Simplex for Multi-Objective LPs

- Multi-Objective LP Simplex exists – much more complicated!
- Multi-Objective Simplex, Bi-objective Simplex and most of the content of this lecture can be found in:
  - Multicriteria Optimization, 2007 – Matthias Ehrgott
  - Free access from ANU network
  - https://link.springer.com/book/10.1007/3-540-27659-9