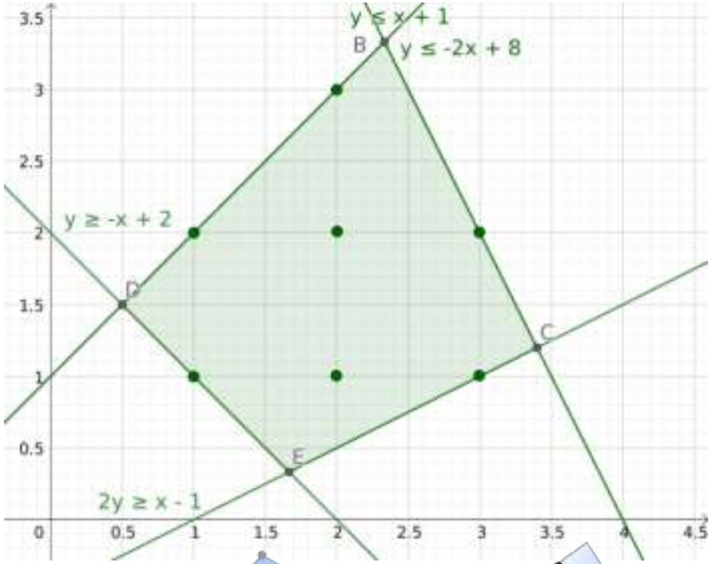
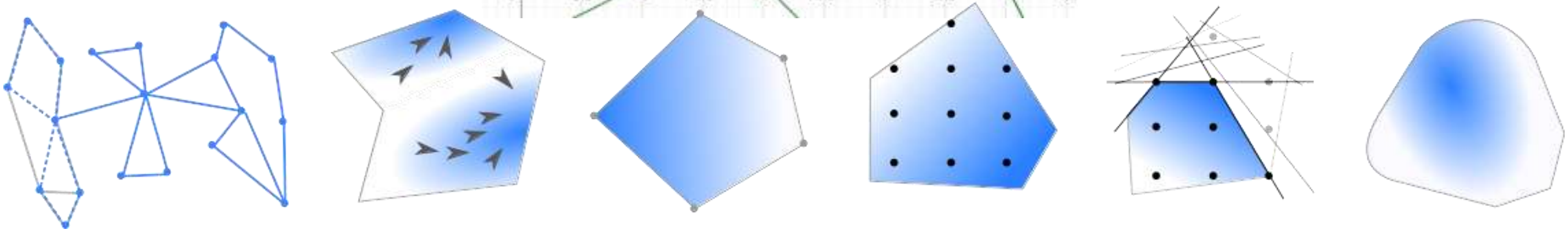
Mixed-Integer Programming 4

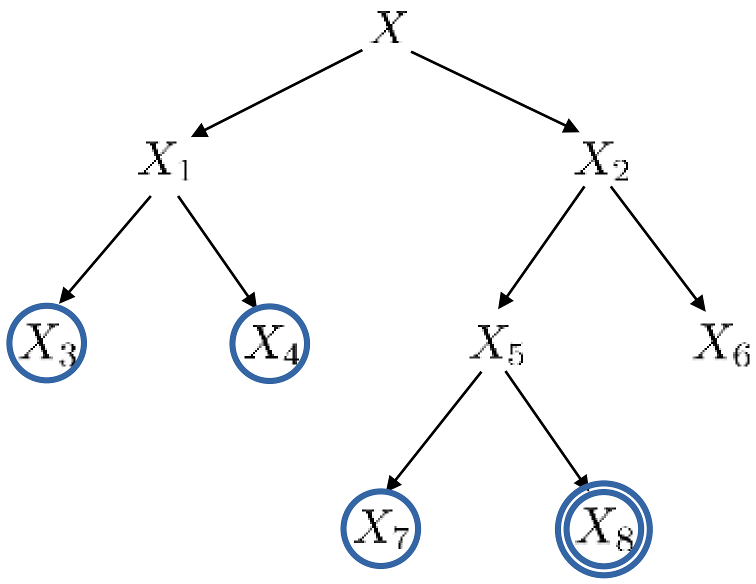
COMP4691/8691





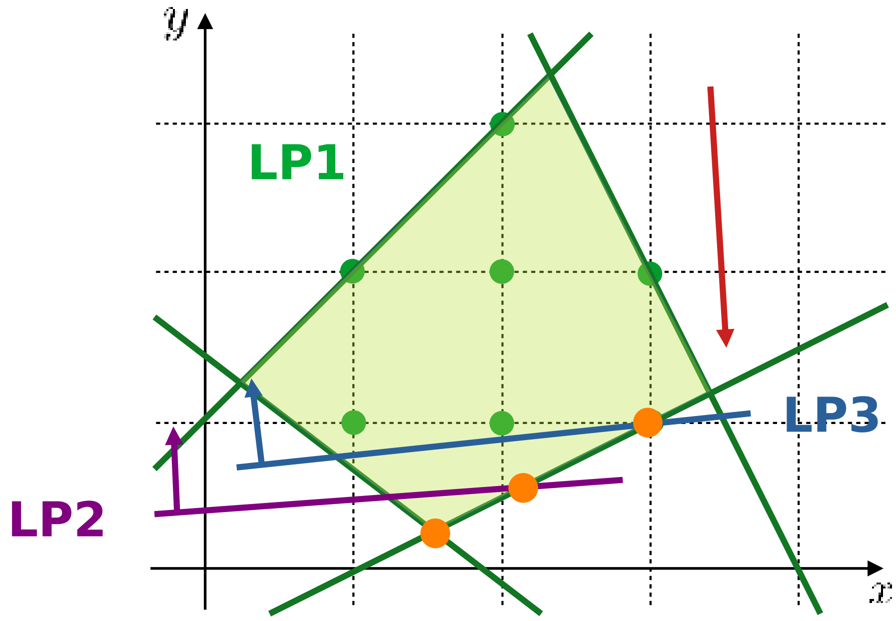
MILP Topic Outline

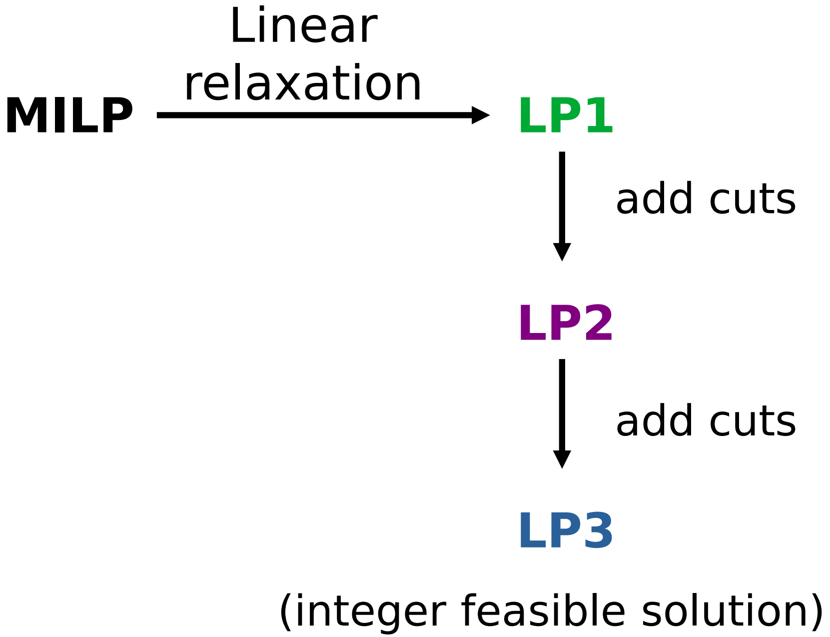
- Linear relaxation
- Modelling and solving
- Branch and bound
- **Cutting plane methods**
    - Gomory cuts
- **Duality for ILPs**



Cutting Plane Methods

Iteratively add cuts (<u>linear inequalities</u>) to the problem until the solution to the linear relaxation becomes integer feasible (have captured the **convex hull** of integer feasible points **near the optimal**).





Cutting Plane Methods

Iteratively add cuts (linear inequalities) to the problem until the solution to the linear relaxation becomes integer feasible (have captured the **convex hull** of integer feasible points **near the optimal**).

Can solve ILPs on their own, or the techniques can be used in conjunction with branch and bound to improve relaxations.

Different types of cuts with varying complexity can be added.

We will focus on the **Gomory cutting plane algorithm**, developed by Ralph Gomory in the 1950s (served in the US Navy), which makes use of **Gomory cuts**.

Recap: Revised Simplex

$$\min_x \; c^\intercal x$$
$$\text{s.t. } Ax = b$$
$$x \geq 0$$
$$x \in \mathbb{Z}^n$$
$$A \in \mathbb{R}^{m \times n}$$

We work with variables instead of constraints.

Basic feasible solution (BFS): just another way of saying a vertex of the feasible polyhedron.

BFS: B_k ⟶ Pick entering variable ⟶ Pick leaving variable ⟶ BFS: B_{k+1}

Variables can be **basic** or **non-basic** :

basis

Basic variables: $x_j \geq 0 \quad \forall j \in B_k$ (m)

Non-basic variables: $x_j = 0 \quad \forall j \notin B_k$ (n-m)

Gomory Cuts

$$\begin{aligned} \min_x \quad & c^\intercal x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \\ & x \in \mathbb{Z}^n \end{aligned}$$

$$A \in \mathbb{R}^{m \times n}$$

Let x^* be an optimal BFS of the linear relaxation with basis B.

Splitting into **basic** $\bar{A} \in \mathbb{R}^{m \times m}$, $\bar{x} \in \mathbb{R}^m$ and **non-basic parts**: $\tilde{A} \in \mathbb{R}^{m \times (n-m)}$, $\tilde{x} \in \mathbb{R}^{n-m}$

For the linear relaxation optimal:

$$\bar{A}\bar{x}^* = b \qquad \tilde{x}^* = 0$$

Rewrite constraint:

$$Ax = \bar{A}\bar{x} + \tilde{A}\tilde{x} = b$$

$\bar{A}\bar{x} = b$ has a solution so we can invert:

$$\bar{x} + \underbrace{\bar{A}^{-1}\tilde{A}}_{\hat{A}}\tilde{x} = \underbrace{\bar{A}^{-1}b}_{\hat{b}}$$

$$\bar{x} + \hat{A}\tilde{x} = \hat{b}$$

Gomory Cuts

$$\bar{x} + \hat{A}\tilde{x} = \hat{b}$$

Element-wise floor operator

Since $x \geq 0$:

$$\bar{x} + \lfloor \hat{A} \rfloor \tilde{x} \leq \bar{x} + \hat{A}\tilde{x} = \hat{b}$$

Haven't done anything significant yet, as it still holds for all feasible solutions to linear relaxation

For our original problem all x should be integer, and the coefficients on the LHS are now integer ⇒ we can tighten the inequality without losing any integer feasible solutions:

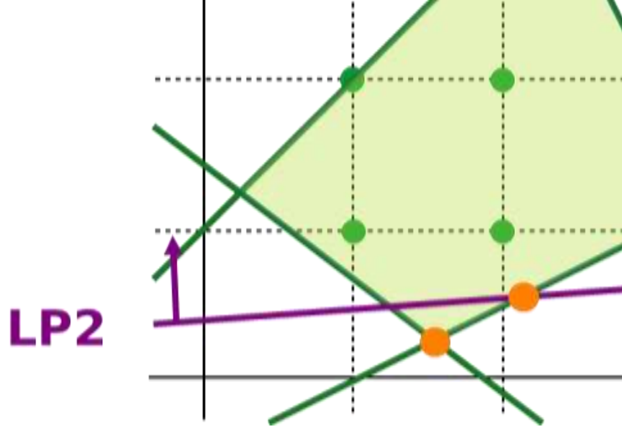
$$\bar{x} + \lfloor \hat{A} \rfloor \tilde{x} \leq \lfloor \hat{b} \rfloor$$

$$x_i + \sum_{j \notin B} \lfloor \hat{A}_{i,j} \rfloor x_j \leq \lfloor \hat{b}_i \rfloor \quad \forall i \in B$$

These are **Gomory cuts**.

Gomory cuts can also be derived for the **mixed-integer case.**

Gomory Cuts



$$x_i + \sum_{j \notin B} \lfloor \hat{A}_{i,j} \rfloor x_j \leq \lfloor \hat{b}_i \rfloor \quad \forall i \in B$$

If all basic variables are integral, we have an optimal solution to original problem, otherwise if one x_i^* is fractional:

$$x_i + \sum_{j \notin B} \lfloor \hat{A}_{i,j} \rfloor x_j \leq \lfloor \hat{b}_i \rfloor < x_i^*$$

Noting: $\bar{x}^* = \bar{A}^{-1} b = \hat{b}$

$\tilde{x}^* = 0$

If added to the problem, the cut will ensure that the optimal for the LP relaxation is no longer feasible.

Gomory Cut Example

$$\begin{aligned} \min_x \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 - 4x_3 + x_4 = 5 \\ & x_1 + 3x_3 - 2x_4 = 6 \\ & x \geq 0 \\ & x \in \mathbb{Z} \end{aligned}$$

Solve linear relaxation:

Objective = 39/7
x = (39/7, 0, 1/7, 0)

Which variables are basic?

x_1 and x_3 are basic.

$$\bar{A} = \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$$

Gomory Cut Example

$$\hat{A} = \bar{A}^{-1}\tilde{A} = \begin{bmatrix} \frac{6}{7} & -\frac{5}{7} \\ -\frac{2}{7} & -\frac{3}{7} \end{bmatrix} \qquad \hat{b} = \bar{A}^{-1}b = \bar{x}^* = \begin{bmatrix} \frac{39}{7} \\ \frac{1}{7} \end{bmatrix}$$

Objective = 39/7
x = (39/7, 0, 1/7, 0)

$$x_i + \sum_{j \notin B} \lfloor \hat{A}_{i,j} \rfloor x_j \leq \lfloor \hat{b}_i \rfloor \quad \forall i \in B$$

Small abuse of notation for $\hat{A}_{i,j}$

- i is the index associated with x_i not the i-th row
- same for columns and non-basic vars

i = 1: $x_1 + \left\lfloor \frac{6}{7} \right\rfloor x_2 + \left\lfloor -\frac{5}{7} \right\rfloor x_4 \leq \left\lfloor \frac{39}{7} \right\rfloor$

$$x_1 - x_4 \leq 5$$

Verify LP BFS no longer valid

i = 3: $x_3 + \left\lfloor -\frac{2}{7} \right\rfloor x_2 + \left\lfloor -\frac{3}{7} \right\rfloor x_4 \leq \left\lfloor \frac{1}{7} \right\rfloor$

$$x_3 - x_2 - x_4 \leq 0$$

Gomory Cut Example

Previous LP relaxation optimal:

Objective = 5.571428
x = (5.571428, 0, 0.142857, 0)

New LP relaxation optimal after **adding Gomory cuts:**

Objective = 6.3333333
x = (5.0, 0.66666667, 0.33333333, 0.0)

More basic variables than before?

The **final optimal solution to the ILP:**

Objective = 7
x = (7, 0, 1, 2)

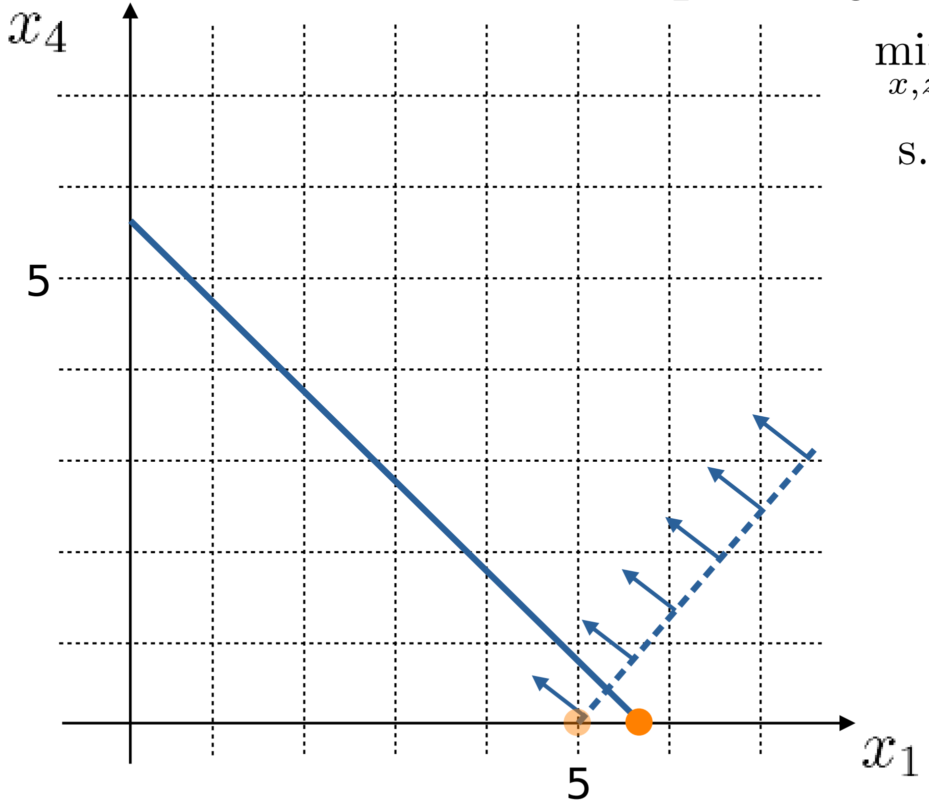
$$\begin{aligned} \min_x \;\; & c^\intercal x \\ \text{s.t.} \;\; & Ax = b \\ & x \geq 0 \\ & x \in \mathbb{Z}^n \end{aligned}$$

In equality form we have 2 more constraints and 2 more slack variables (implicit here)

Gomory Cut Example

Let's try to visualise this by plotting the feasible region of just the first constraint, and where $x_2 = 0$, $x_3 = 1/7$.

x = (39/7, 0, 1/7, 0)



$$\min_{x,z} \; x_1 + 2x_2$$

$$\text{s.t.} \; x_1 + 2x_2 - 4x_3 + x_4 = 5$$

$$x_1 + 3x_3 - 2x_4 = 6$$

$$x \geq 0$$

$$x_1 + 2 \cdot 0 - 4 \cdot \frac{1}{7} + x_4 = 5$$

$$\implies x_4 = \frac{39}{7} - x_1$$

One of our cuts:

$$x_1 - x_4 \leq 5$$

Next solution is out of plane:

x = (5, 2/3, 1/3, 0)

Gomory Cutting Plane Method

Iteratively solve linear relaxation, **adding one or more Gomory cuts** each time for any fractional basic variables.

Once an integer solution is obtained for the linear relaxation, return optimal for original problem.

The number of constraints increases over time, and in equality form, the number of slack variables.

This terminates in a finite number of iterations.

Cuts gradually improve over time until they become **facet-defining** for the convex hull.

Branch and Cut

Combine branch and bound with cut generation at nodes, which will improve lower bounds and improve the ability to find integer feasible solutions.

Gomory cuts are one option, but there are multitude of different types of cuts that can work well in particular settings. Let’s take a look at the output of CBC on one of our problems…

https://github.com/coin-or/Cbc
https://github.com/coin-or/Cgl (cut generation library)

These cuts are also known as **valid inequalities**: they are valid for all ILP feasible point (i.e. don’t remove any ILP feasible points, but hopefully tighten up the linear relaxation)

Understanding CBC Messages

The code to the left of the printed output are defined in `CbcMessage.cpp` of the CBC source. Some of the common ones `CbcXXXXI:`

0010 Overall status update
0012 Summary of after finding integer feasible solution

0013 Cuts improved root node
0014 Cut generation output

0031 Status on cuts
0032 Status on branching

0035 Branching depth

0038 Feasibility pump output

CBC vs …

Other MILP solvers:
http://plato.asu.edu/ftp/milp.html

CP solvers (e.g., Geocode/Chuffed):
https://www.minizinc.org/challenge.html

Note that the problems and timescales in this challenge are really geared towards comparing the strengths of CP solvers: very hard feasibility problems with a focus on getting a solution quickly. No mixed (integer and continuous) problems either.

Duality for ILPs

There are a number of different duality theories for ILPs.
Some of them (e.g., Superadditive Duality) lead to **strong duality,** like for LPs, but result in problems that have exponential-sized formulations.

The duality theory we have explored for LPs is **Lagrangian Duality**, which can be applied to ILPs also. However, while weak duality holds, strong duality doesn’t for the Lagrangian dual of an ILP.

A key result is that the **Lagrangian dual** provides a lower bound that is at least as strong as the linear relaxation, and can often be stronger.

This process is called **Lagrangian Relaxation.** We can solve for a feasible point of the dual, it doesn’t need to be optimal (weaker relaxation).