Linear Programming Problems

COMP4691-8691
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1 Problem Transformation

Implement and solve the following problem in PuLP (you can use the `pulp_template.py` file as a starting point):

$$\begin{aligned}
\min_{x,y,z} \quad & x + 2y - 5z \\
\text{s.t.} \quad & x + 2y + 3z \geq 2 \\
& y \geq x + 2z \\
& z \leq 6
\end{aligned}$$

Convert the above problem to an equality and non-negative variable form (only equality constraints and non-negative variables). Implement the transformed form in PuLP and verify you get the same solution.

2 Visualising Constraints

Write out the inequalities that represent the feasible region in figure 1. Write two objective functions for this feasible region, one where the problem is bounded and one where it is unbounded.

Add the following constraints to the figure:

$$\begin{aligned}
x &\geq 1 \\
4y &\geq 5x - 15 \\
7y + 5x &\leq 35
\end{aligned}$$

Is the modified problem feasible, infeasible, bounded and / or unbounded?

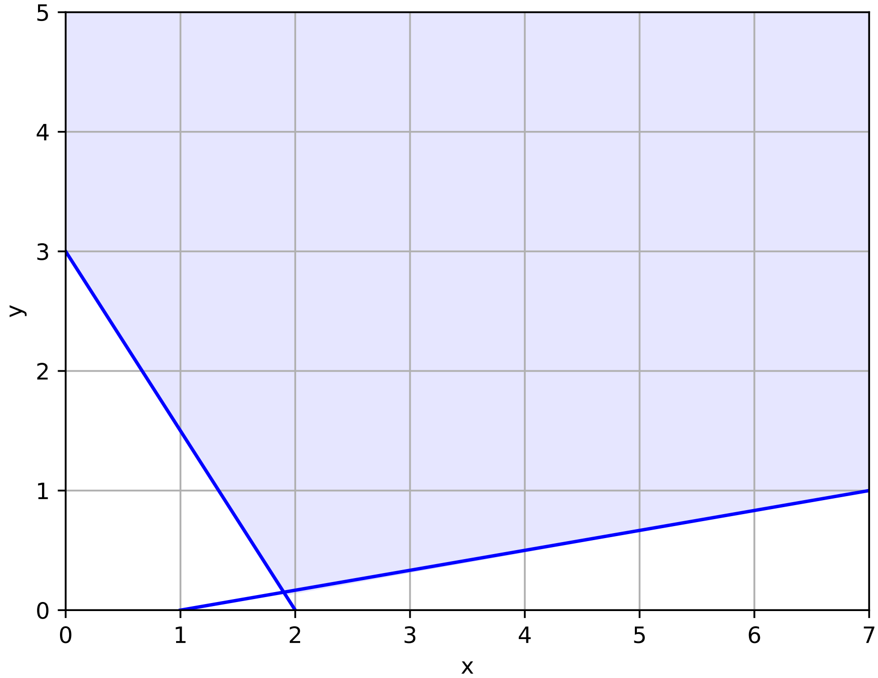

Figure 1: Initial feasible region.

3 Convex Hull

Draw the convex hull of the feasible region in figure 2.

4 Degeneracy

Explain what degenerate vertices are. Why do they matter for the simplex algorithm?

5 Starting Solution

Explain how a starting feasible solution is obtained for the simplex algorithm.

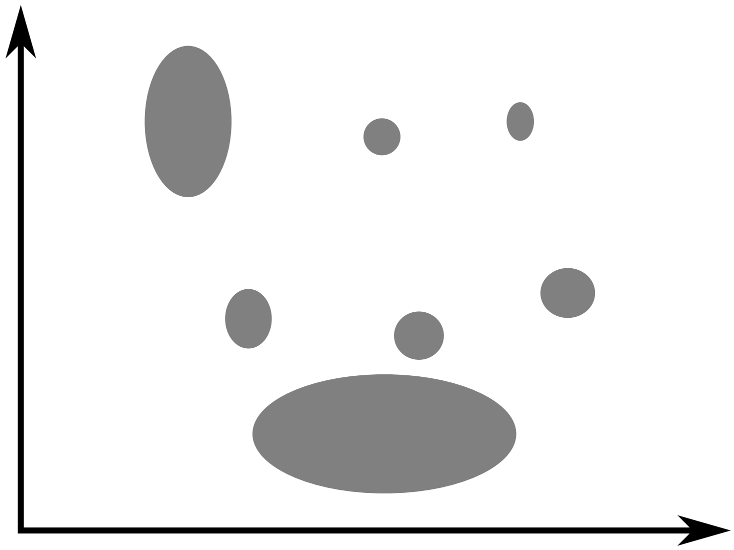

Figure 2: Feasible region.

6 Simplex Pivoting

Explain how a constraint is selected to enter the active set in a pivot step of the "alternative" simplex algorithm. Alternatively, for the standard / revised simplex algorithm explain the analogous operation where a basic variable is chosen to exit the basis.

7 Exact Relaxation

What does it mean for a relaxation of a problem to be exact?

8 Convex Relaxation

Derive a linear convex relaxation for the following constraints, using at most 3 inequalities:

$$y = ax^2$$
$$0 \leq x \leq \overline{x}$$

where x and y are variables, and $a > 0$. Plot an example of the relaxation. How do you expect the worst-case error in this relaxation to change with $\overline{x}$.

Bonus: derive the worst-case error for your relaxation.

Explain how the relaxation could be improved given more inequalities to work with.

9 Optimality Gap

For a maximisation problem the following values were discovered:

1. A proven lower bound on the objective of 20.
2. Known feasible solutions with objective 10, 15, and 30.
3. A proven upper bound on the objective of 40.

What is the optimality gap for this problem at this point in time?

10 Forming the Dual

Transform problem from question 1 into its dual: Implement the dual version in PuLP and verify it gives the same objective value as the primal. Give two other names for dual variables.

11 Pipe Upgrades

Implement the water pipe upgrade problem presented at the end of the LP 3 Optimality lecture slides, using `water_upgrades.py` as a starting point. Confirm the solution for the high demand instance with that presented in the lecture slides.

12 Optimal HVAC Operation

Optimising the operation of heating, ventilation and air conditioning (HVAC) in a building can help reduce energy consumption and electricity costs. This problem is similar to the optimal battery scheduling problem we did in lectures, but where the state variables are the room temperatures and the decision variables are how much to heat each room at each time step over the forward horizon. The objective is to minimise the electricity bill, where power prices can vary over time. Rooms must be kept within the comfortable temperature range when they are occupied (e.g., due to scheduled meeting). A further complication is that heat transfers occur between rooms when there is a temperature difference, and also with the external environment.

We will consider buildings made up of n by n identically sized cube-shaped rooms, e.g., as shown for $n = 3$ in figure 3. The thermal power that flows through a wall with thermal conductance σ is:

$$P_{i,j} = \sigma(T_i - T_j)$$

All the internal walls have identical conductances, which can be different from the external wall conductance that insulates the building form the environment. The floor and ceiling heat transfers are ignored.

Each room has a heater (we will focus only on heating), which can be operated from zero up to a maximum limit.

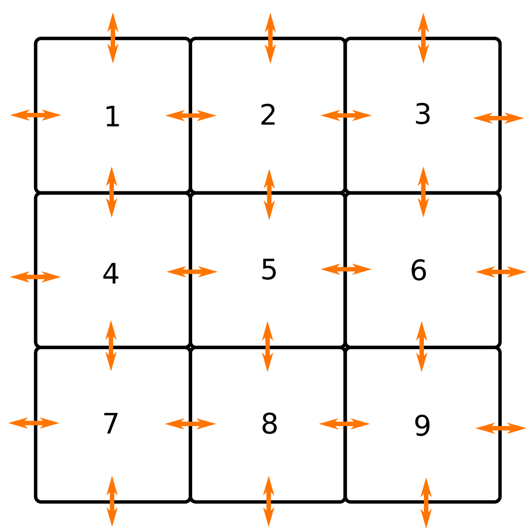

Figure 3: Top-down view of a 3 by 3 building showing wall heat transfers.

The change in room temperature depends on the net thermal power transfers (sum of power from all walls and from room heater) and the heat capacity C of the room, and is given by:

$$P_{i,net}\Delta t = C(T_{i,t} - T_{i,t-1})$$

where Δt is the time step length in hours.

The cost of operating the heaters is equal to the energy consumed (the heater power multiplied by the time step length) in each time slot, multiplied by the electricity price for that time step.

Code up your model in `hvac_solver.py`, referring to the files `hvac.py` for the input and output types and `hvac_instances.py` for the two instances.

12.1 Implementation

a) Implement the LP for this HVAC operation problem based on the above description. What are the electricity bills for the two instances?

b) Calculate the dollar savings relative to just keeping all rooms warm (i.e. by forcing temperature limits after first hour).

c) Can you identify any preheating of rooms prior to the periods of occupation? Why might this be occurring?

d) What happens if the heater capacity is limited to 2 kW?

12.2 Soft Constraints

The current implementation allows any temperatures between the upper and lower limits when the room is occupied. The occupants have complained that this often leads to temperatures sitting at the lower limit, when in practice occupants prefer the temperature to be in the middle of the range (unless there is a *strong enough* economic / environmental reason to deviate).

This preference can be implemented as a soft constraint. After conducting a survey, it has become apparent that occupants would like a saving of $0.12 per hour for every degree variation in temperature either side of 20°C in each occupied room. The existing hard limits are still in place. Management have agreed to reward occupants by contributing half of the soft constraint cost to the yearly Christmas party.

a) Implement the above soft constraint. What are the electricity bill and the soft constraint cost components for the two instances?

b) Why might management have come up with this way of rewarding customers and will it make sense on their balance sheet?

c) Assuming this day is typical, estimate how much money management will end up contributing to the office Christmas party.

13 Optimal Power Flow

The optimal power flow (OPF) problem is to determine the cheapest dispatch of generators to meet estimated power consumption, while taking into account the physical flow of power on the network and various operating limits. Electrical flows can be thought of as a type type of *potential* network flow problem, where a difference in potential between two nodes drives the flow.

The equations that govern the flow of power on AC electricity networks are non-linear and even non-convex in nature. However, to a first approximation they can be linearised, which can work reasonably well at the high voltage transmission level:

$$P_{i,j} = -b_{i,j}(\theta_i - \theta_j)$$

Where $P_{i,j}$ represents the power flowing from node i to node j in the network along a transmission line, and θ_i represents the voltage phase angle (in radians) at node i (think of it as a potential). $b_{i,j}$ is a (typically negative) parameter of a line known as the susceptance (similar to conductance).

Each node may have a load and / or generator connected.

Just like regular network flow, the sum of power entering a node (including the power from any generators and loads present at the node) must be equal to zero.

Other constraints include power limits for each line $-\overline{P}_{i,j} \leq P_{i,j} \leq \overline{P}_{i,j}$, and a power limit for each generator $\overline{G}_i$. Other parameters include the load at each node, and the price of operating each generator.

One of the nodes is labelled the reference node, where the voltage phase angle is fixed to zero: $\theta_1 = 0$.

The objective is to minimise the cost of generation.

a) Write an OPF LP based on the above problem description, first mathematically, and then implemented in PuLP using `opf_solver.py`.