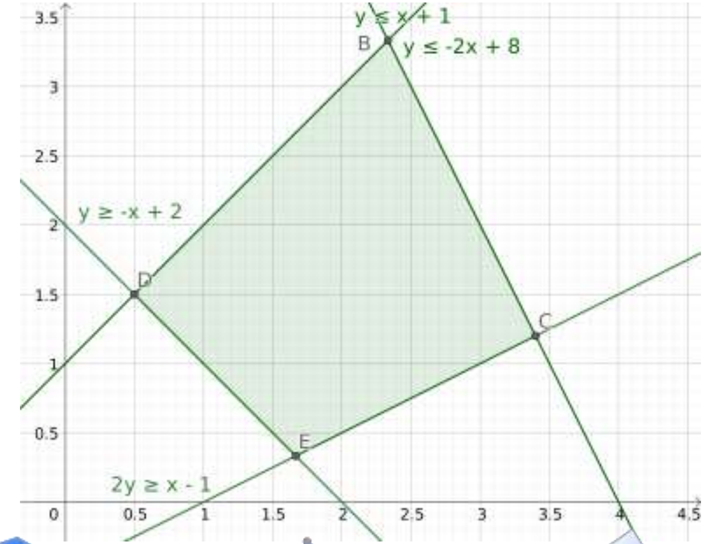
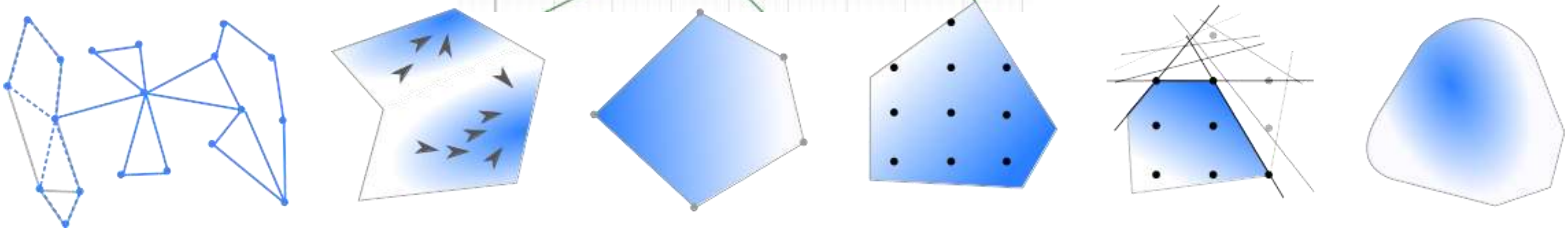
Linear Programming 2

COMP4691 / 8691





LP Topic Outline

- LP Introduction
- Modelling and solving
- **Feasible region and convexity**
  - Feasibility
  - Feasible region
  - Convex sets
  - Optimality of LP
  - Network flow example
- Simplex algorithm
- Relaxations and approximations
- The dual of a linear program

Feasibility

$$\min_{x \in \mathbb{R}^n} \quad c^\intercal x$$
$$\text{s.t. } Ax \leq b$$

A **feasible point** or **feasible solution** is a point $x^* \in \mathbb{R}^n$ that satisfies all constraints

If no such point exists then the **problem** is infeasible

Feasible region / **solution space** = the set of all feasible points

A feasible solution might be **optimal** or **suboptimal** for a given objective

Feasibility: Equalities

LP feasibility is somewhat analogous to the possible outcomes of solving a *system of linear equations.*

$$Ax = b$$

m linearly independent **equations** with **n variables**:

$m < n$ ⇒ Underdetermined, infinite solutions $\quad x + y = 5 \ (m<n)$

$m = n$ ⇒ Single unique solution $\quad x - y = 5 \ (m=n)$

$m > n$ ⇒ Overdetermined, no solution $\quad 3x + y = 0 \ (m>n)$

Inequalities further complicate things.

Feasibility: Inequalities

An inequality defines a **half-space.** A half-space is a hyperplane partitioning space into two regions

$$Ax \leq b$$

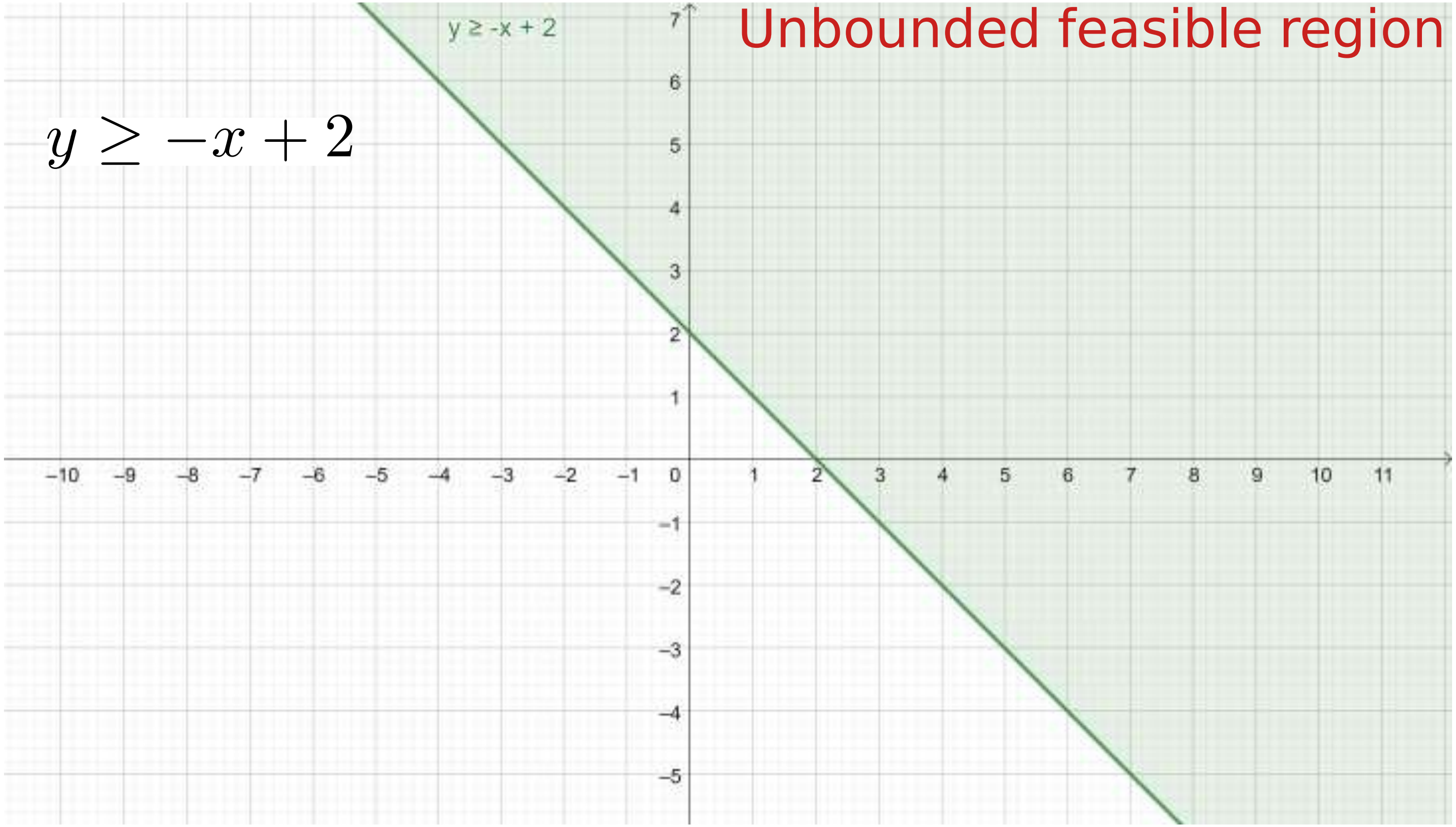
A single inequality defines an unbounded feasible region.

Multiple inequalities can define:

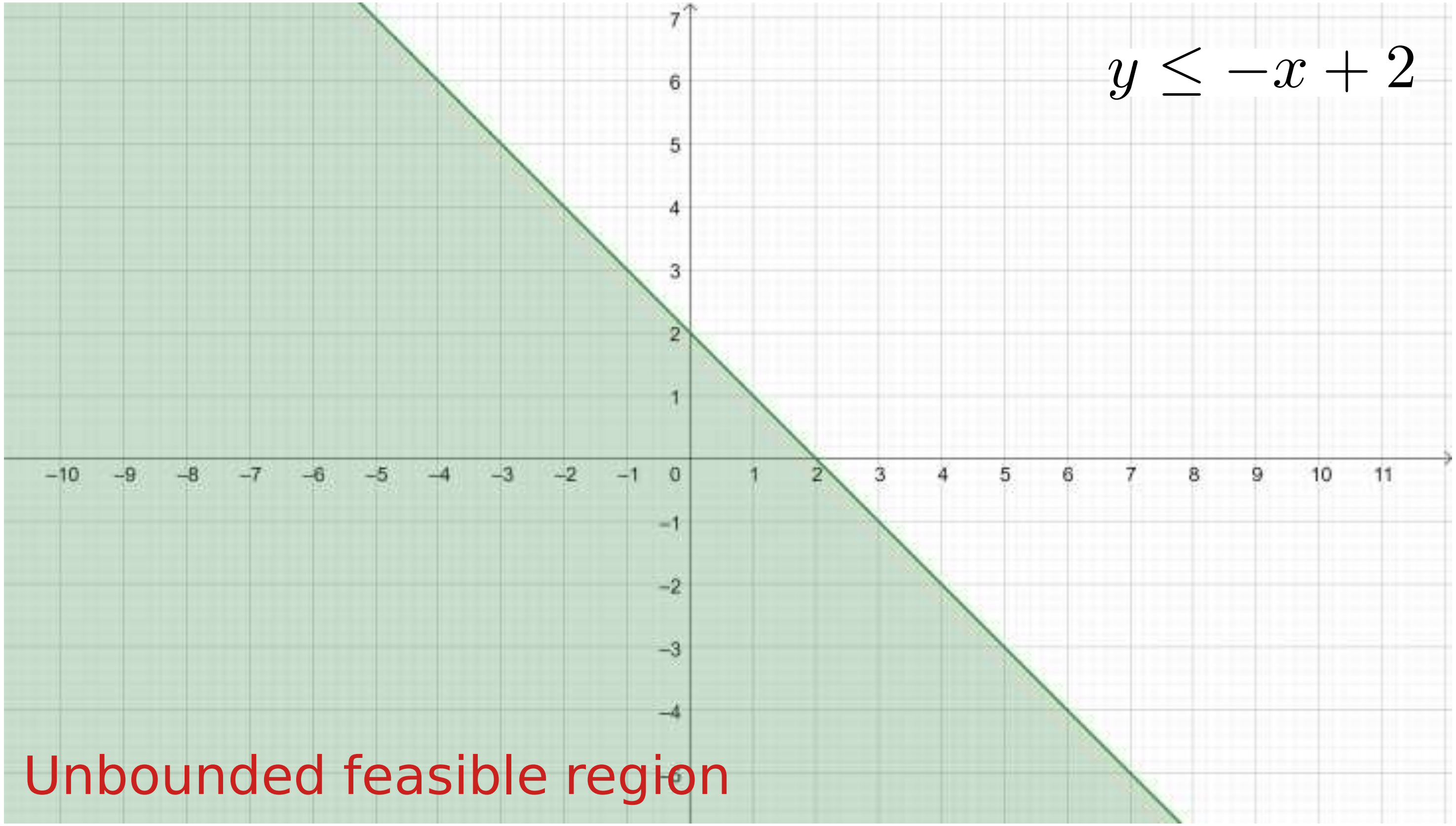
- bounded feasible regions
- unbounded feasible regions
- an empty region, i.e. an infeasible problem

Let's plot some examples of feasible regions.

Feasibility: Inequalities



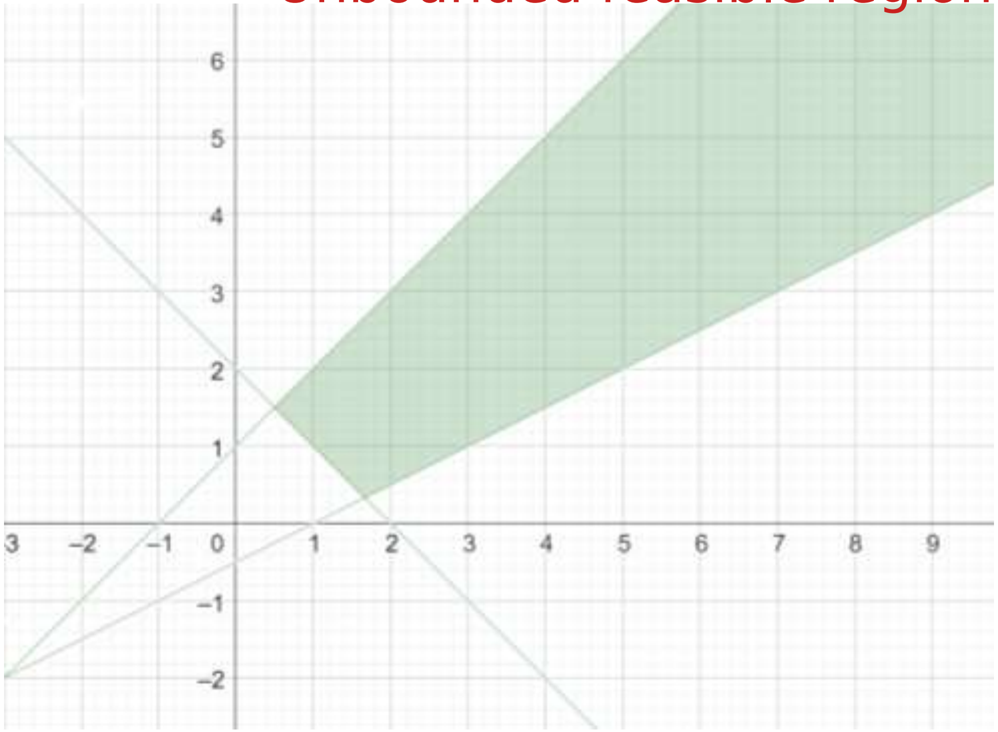
Feasibility: Inequalities



Feasibility: Inequalities

$$y \geq -x + 2$$
$$2y \geq x - 1$$
$$y \leq x + 1$$

Unbounded feasible region

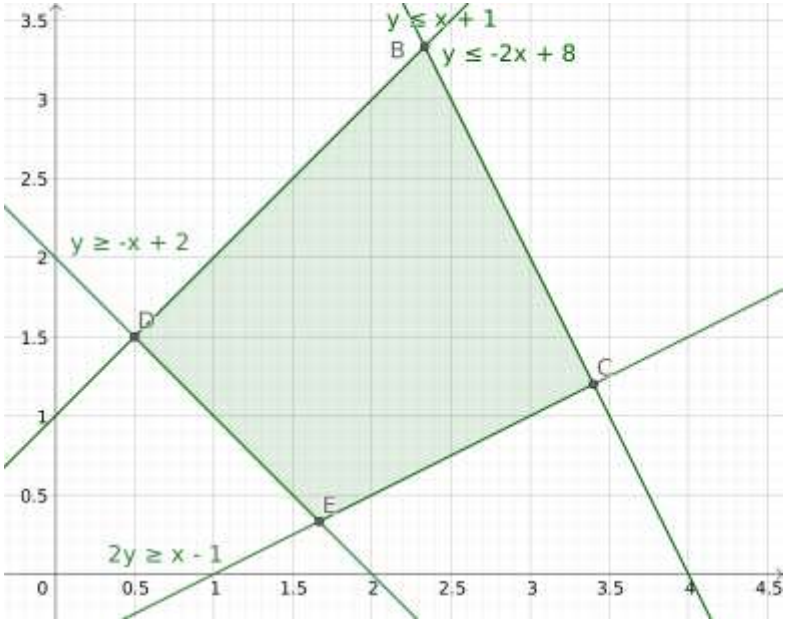


The intersection of the inequalities (half-spaces) gives the feasible region

Feasibility: Inequalities

$$y \geq -x + 2$$
$$2y \geq x - 1$$
$$y \leq x + 1$$
$$y \leq -2x + 8$$

Bounded feasible region



Iterative example here:

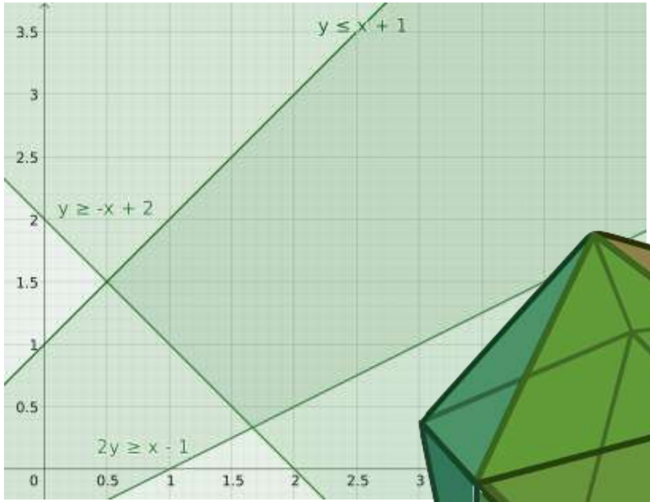
https://www.geogebra.org/graphing/bhrfuq27

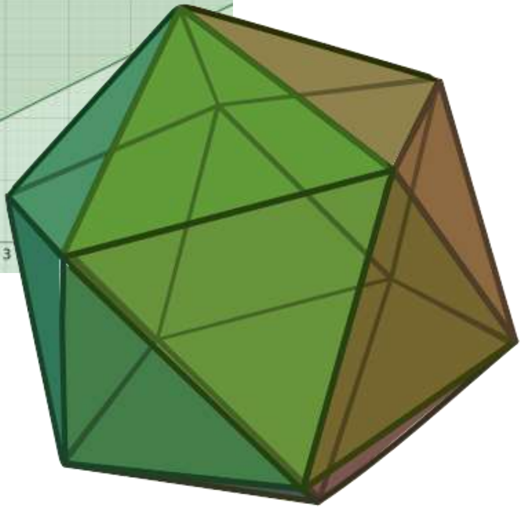
Feasible Region

The intersection of half-spaces gives us a **convex** , possibly unbounded, **polyhedron***.

<u>**When bounded**</u>**:**

- 2-D Convex Polygon
- 3-D Convex Polyhedron
- n-D Convex Polytope





Linear programming is a special case of the more general convex optimisation

* In the mathematical optimisation community polyhedron is commonly used for the arbitrary dimension possibly unbounded object, with polytope reserved for the bounded case.

Convex Sets

A convex set or region in a vector space over the real numbers is a subset:

$$\mathcal{R} \subseteq \mathbb{R}^n$$

which must satisfy the following condition:

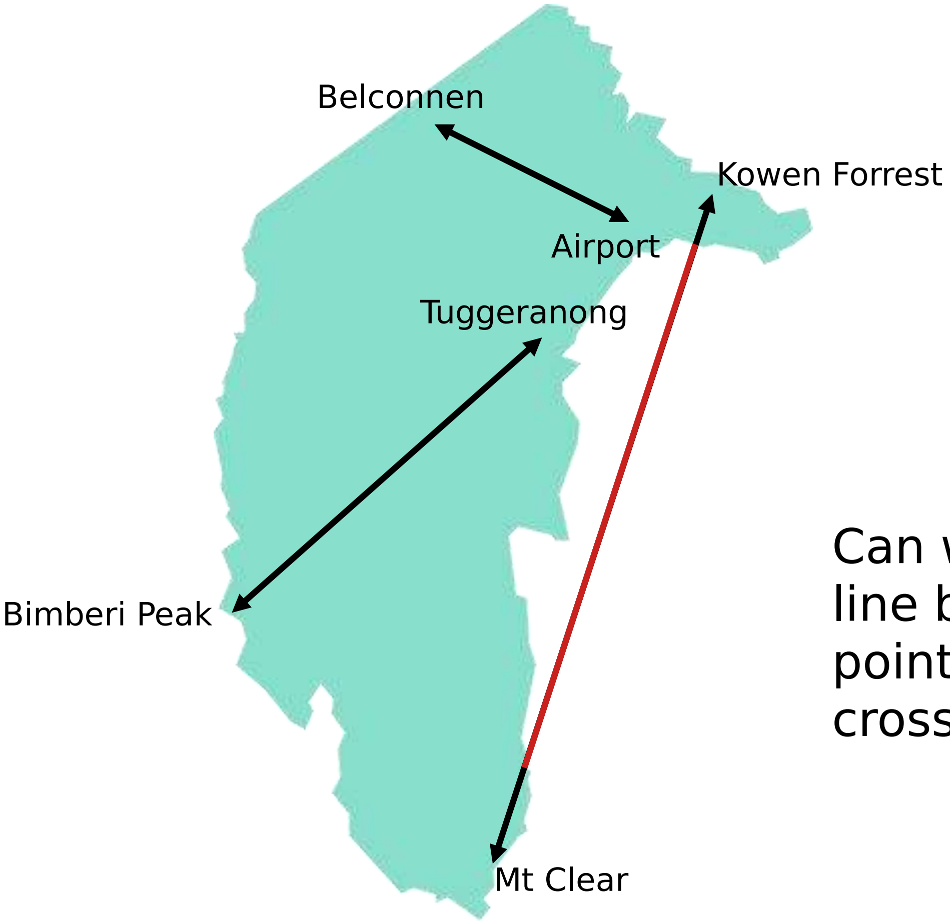
A line segment between any two points in the region must also lie within the region:

$$\forall x, y \in \mathcal{R} : \quad \{x + \alpha(y - x) | \alpha \in [0, 1]\} \subseteq \mathcal{R}$$

i.e., direct line of sight to every other point in region!

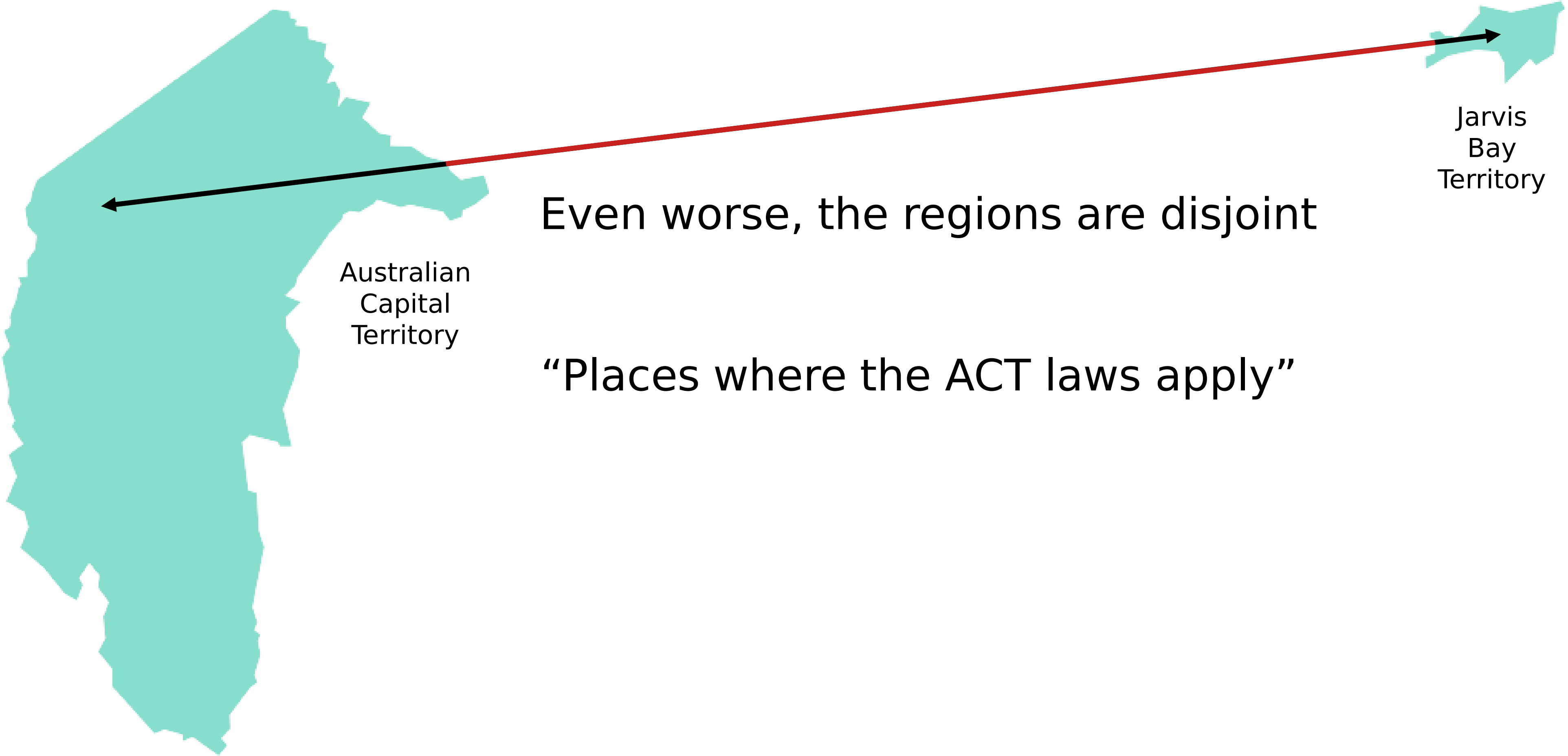
Why might this be a useful property in optimisation?

Game of Convex or Non-Convex



Can we walk in a straight line between any two points in the ACT without crossing the border?

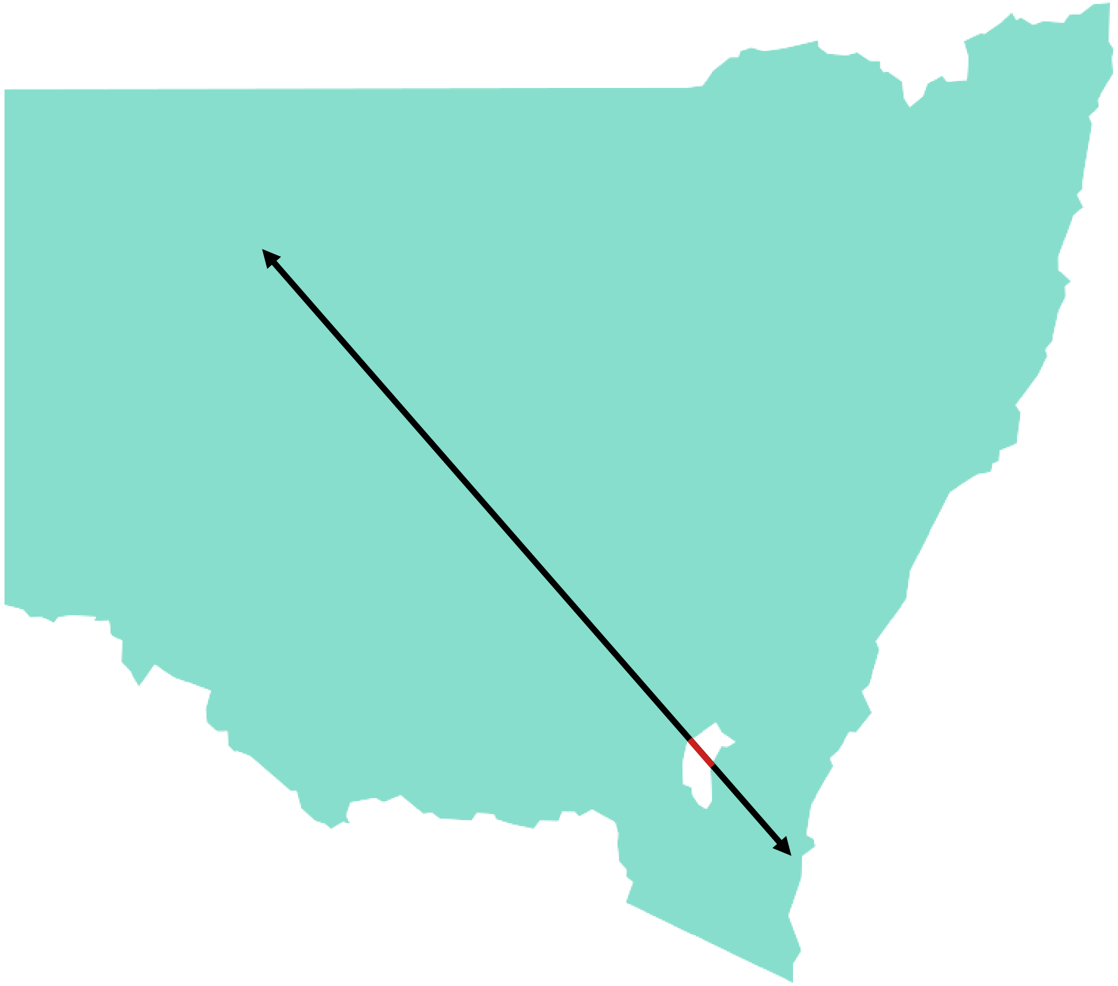
Game of Convex or Non-Convex



Even worse, the regions are disjoint

“Places where the ACT laws apply”

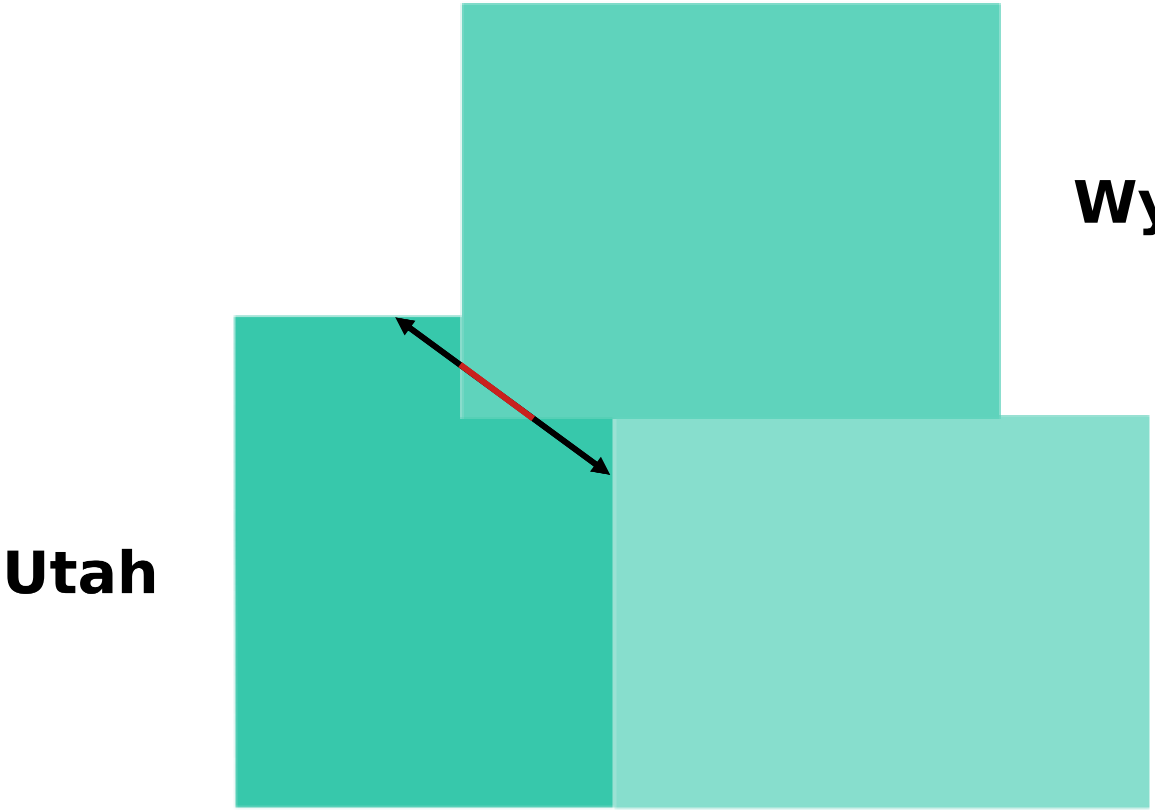
Game of Convex or Non-Convex



Game of Convex or Non-Convex

Colorado

Game of Convex or Non-Convex



Convex Sets

An important property:

The intersection of convex sets is a convex set

Using this:

- Linear inequality constraints define half-spaces, and
- half-spaces define convex regions, therefore
- their intersection (combination of all constraints) is a convex region.

Ta-dah!

(Note that the empty set is convex)

Intersection of Convex Sets

Pretty intuitive, and the proof is also straight-forward.

Given two convex sets in A and B in $\mathbb{R}^n$

The definition of convexity:

$$\forall x, y \in A : \quad \{x + \alpha(y - x) | \alpha \in [0, 1]\} \subseteq A$$

$$\forall x, y \in B : \quad \{x + \alpha(y - x) | \alpha \in [0, 1]\} \subseteq B$$

Any points in both A and B will also have their line segment in A and B, and so it will also be in the intersection of A and B:

$$\forall x, y \in A \cap B : \quad \{x + \alpha(y - x) | \alpha \in [0, 1]\} \subseteq A \cap B$$

Game of C or NC Part II

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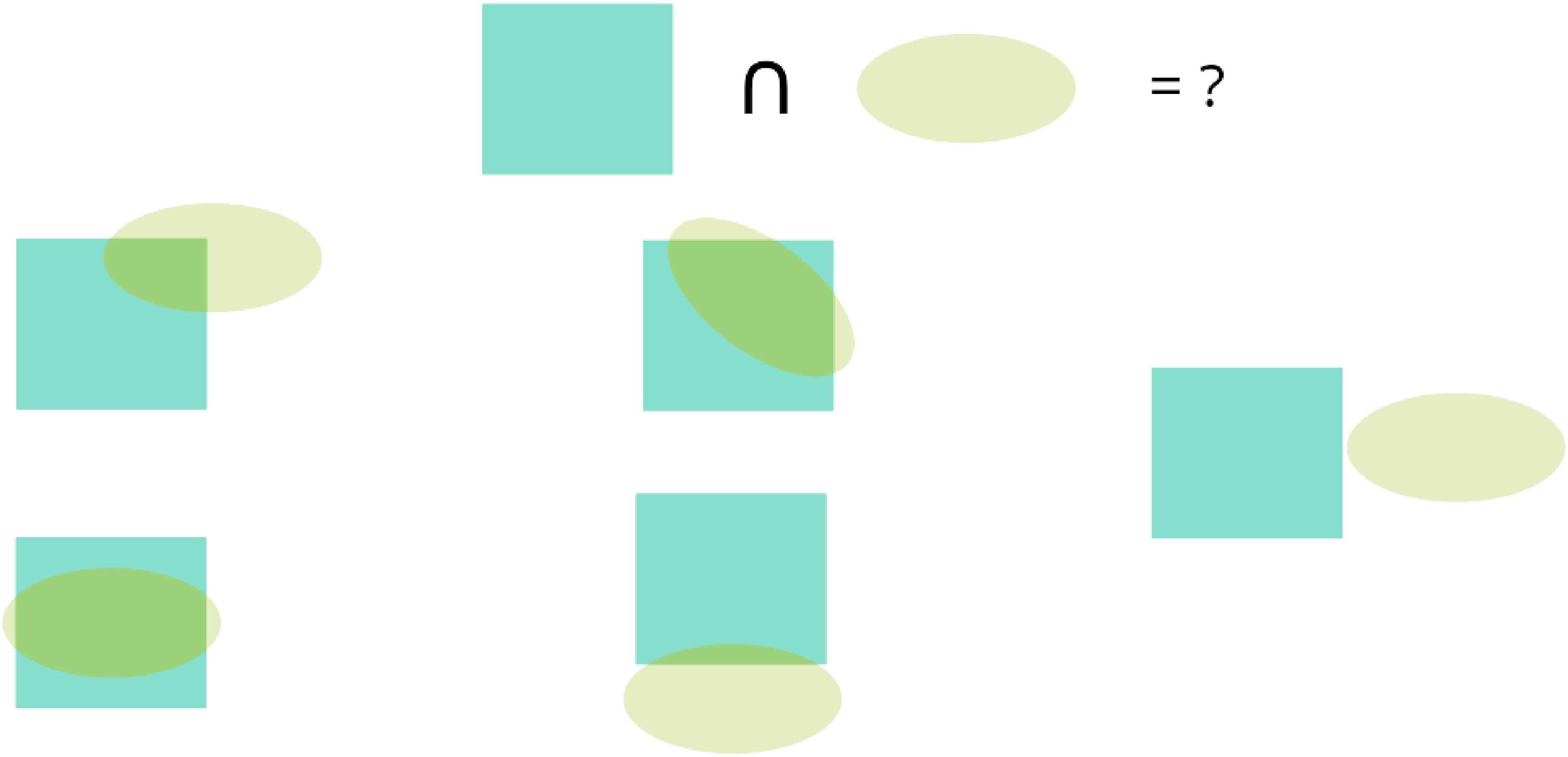
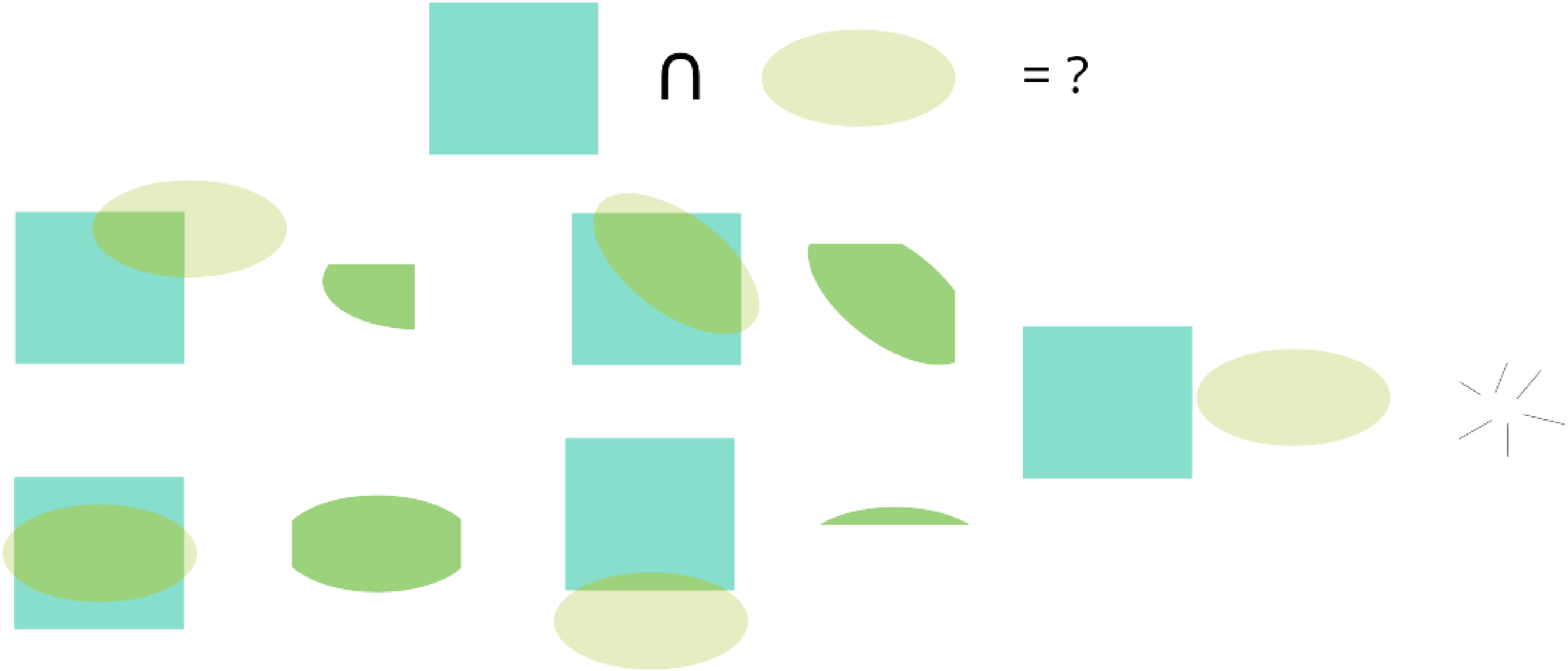
Game of C or NC Part II



Game of C or NC Part II



Game of C or NC Part II

What about the **union** of regions?

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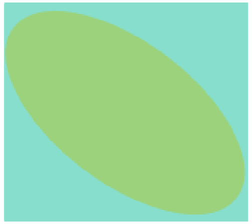

Game of C or NC Part II

What about the **union** of regions?

Game of C or NC Part II

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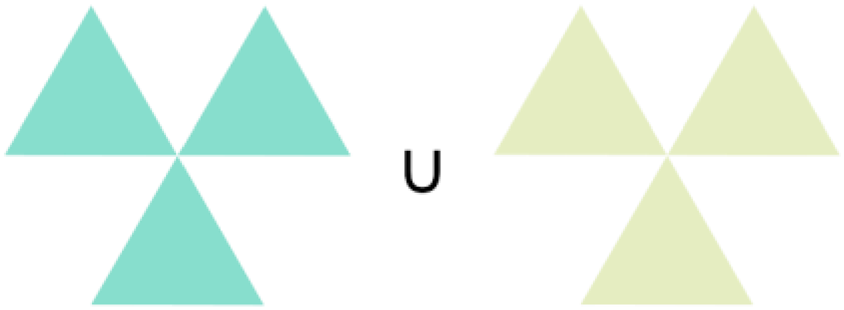
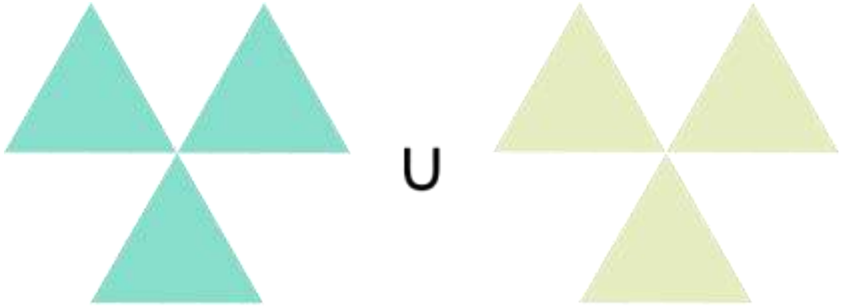
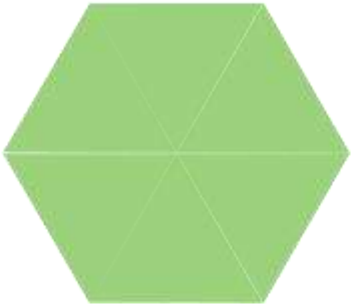


Game of C or NC Part II

What about the **union** of regions?





Convexity

Strict convexity : all open line segments (excluding end points) are in the interior of the region.

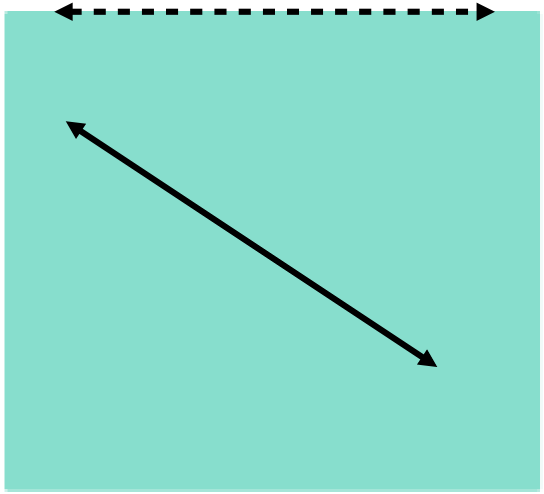
The **convex hull** of a set S is the smallest possible convex set that contains S.

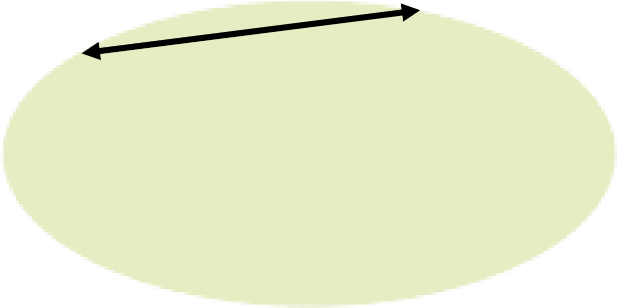
An **extreme point** of a convex set S does not belong to an open line segment for **any** two points in S.

We will come back to convexity in the convex optimisation part of the course. In particular with defining what we mean by *convex functions* and some of their useful properties. But for now (well after a few more examples), that is enough convexity!

Strict Convexity

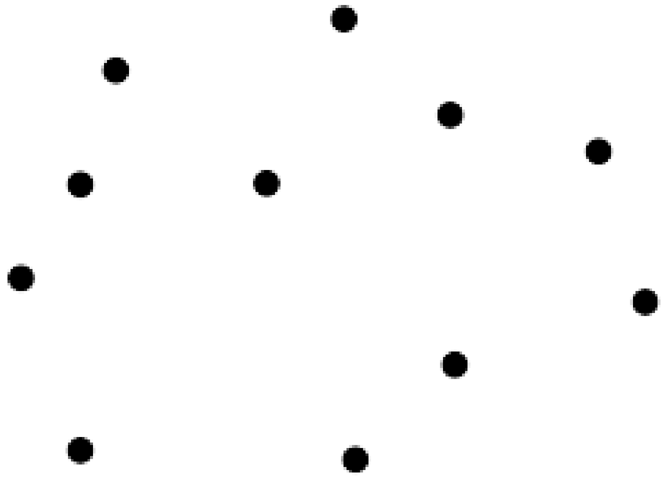
This line segment has values (other than endpoints) that are on the boundary: therefore the region is not strictly convex



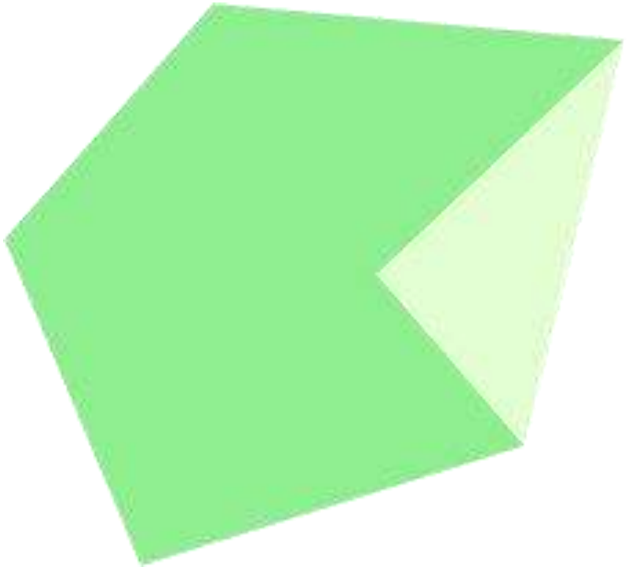


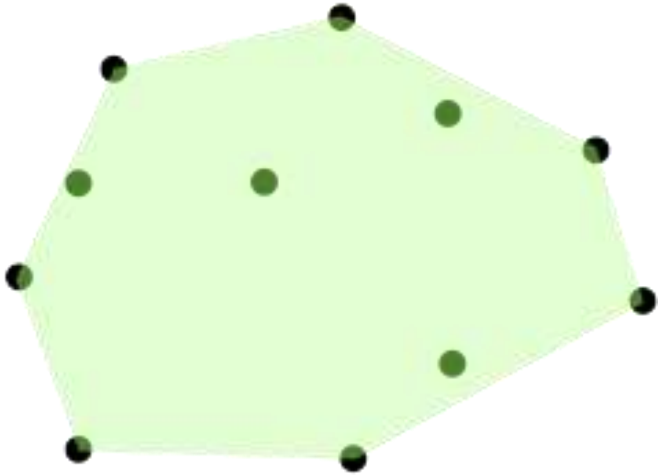
A strictly convex region.

Convex Hull



Convex Hull





Convex Hull

Extreme Points

For a convex polyhedron, extreme points are vertices and vice versa.

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Optimisation

Decision problems in computer science look for the presence of a solution that satisfies some conditions (any value in the feasible region), **optimisation problems** search for the "best" solution among these according to some objective.

When we combine the notions of feasibility and optimality we get the following possible outcomes for an LP.

Optimality of an LP

Unbounded – feasible region must be unbounded

- The problem is probably modelled incorrectly, rarely in the real world can we generate an infinite amount of value from something, we normally hit some limit at some point

A single optimal solution – feasible region may be unbounded, bounded or a single point

- Solution lies at a vertex

An infinite number of optimal solutions – feasible region may be unbounded or bounded

- Optimal solutions lie along an edge / … / facet of the feasible region polyhedron (or the entire regions itself), including the relevant vertices

Infeasible – feasible region must be empty

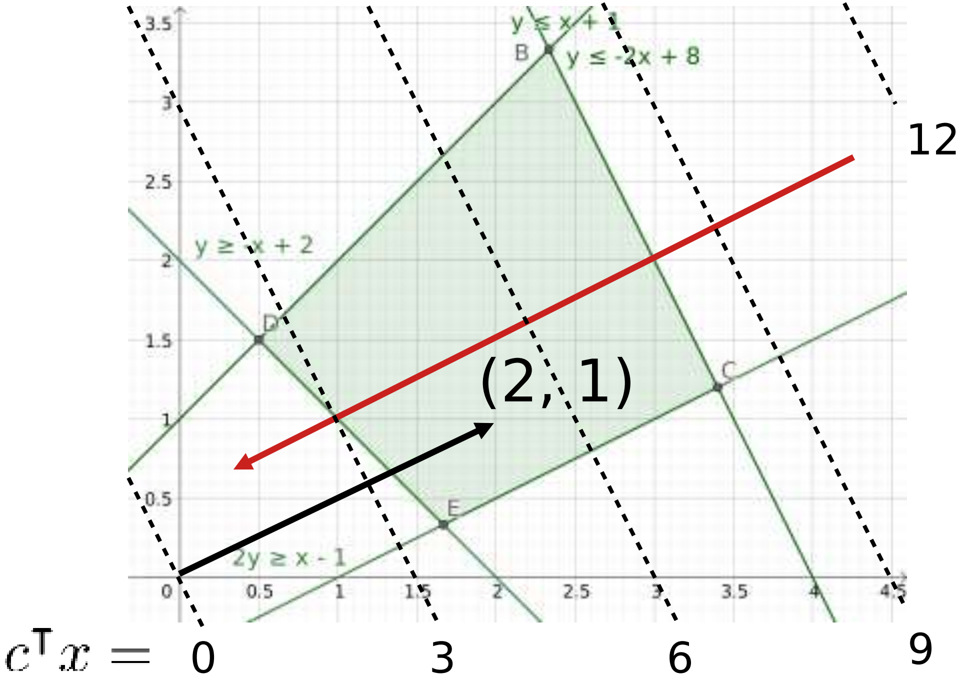
- It might not be possible to do what the problem states, or an error in the modelling

Objective Visualisation

$$\underset{x,y}{\boxed{\min}}\ 2x + y$$

$$\text{s.t. } y \geq -x + 2 \quad 2y \geq x - 1$$

$$y \leq x + 1 \quad y \leq -2x + 8$$



Let's visualise the objective, by plotting the vector "c".

$$\min_{x \in \mathbb{R}^n} \ c^\intercal x$$

$$\text{s.t. } Ax \leq b$$

Which direction gives better solutions?

For min, want to move in opposite direction to c.

Does the magnitude of c matter?

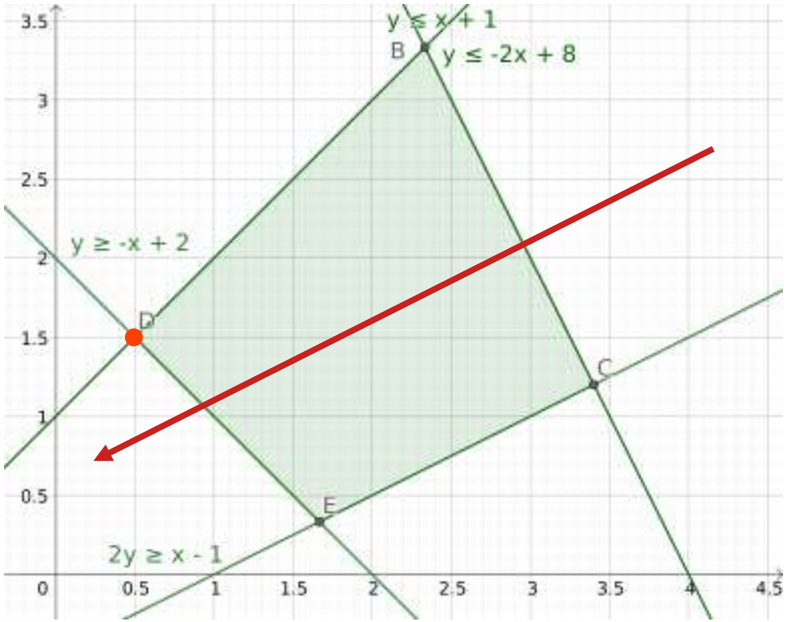
Can multiply the objective by a scalar k > 0 and will get same solution.

What is the optimal solution?

At vertex "D".

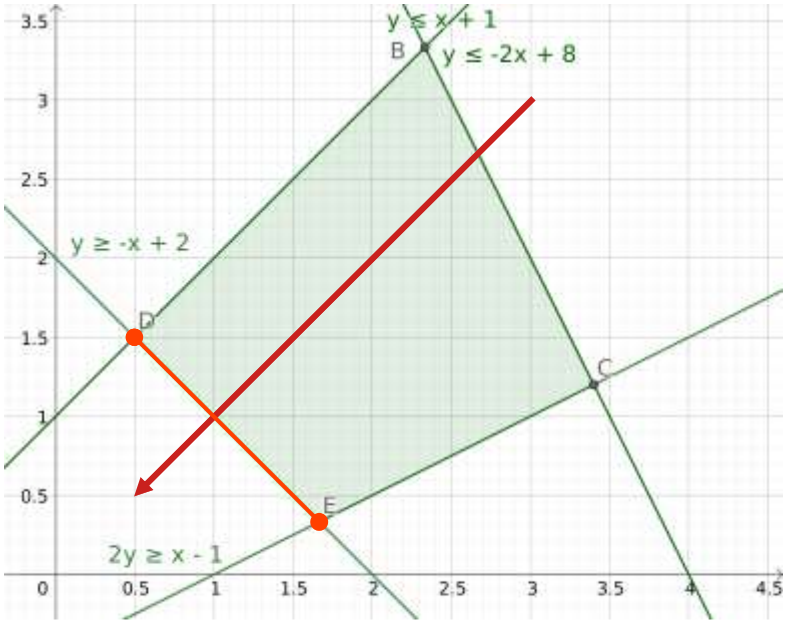
------- **Level set** of objective / contours of equal objective value

Optimality Examples



How many optimal solutions?

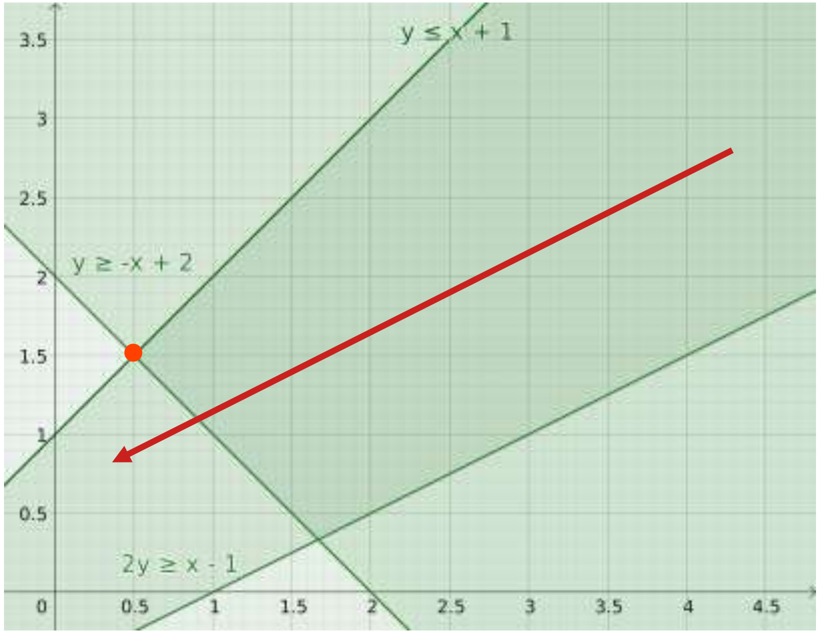
Single unique optimum



How many optimal solutions?

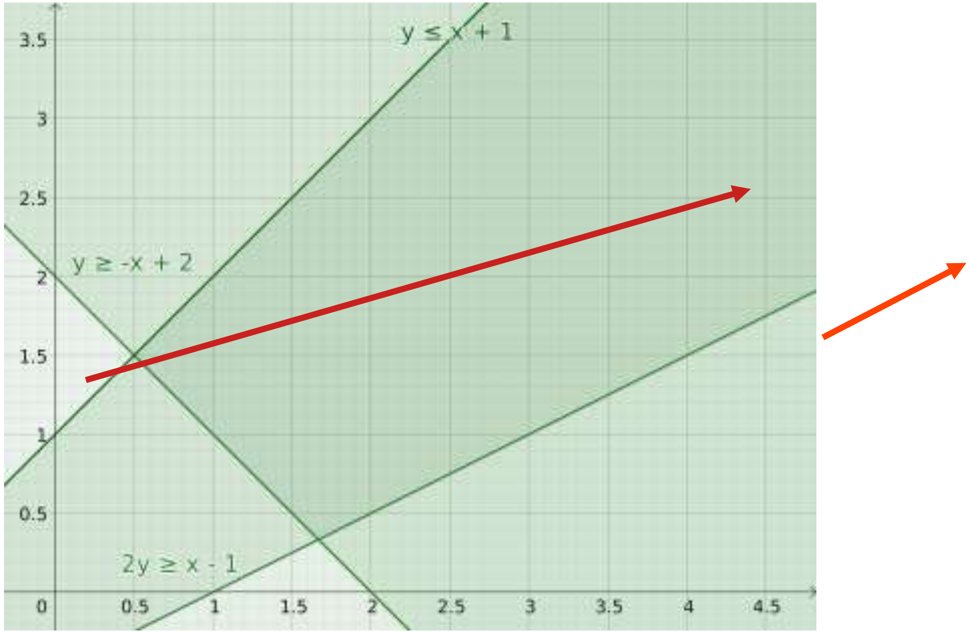
Infinite number of optimal solutions

Optimality Examples



Can we find an optimal solution?

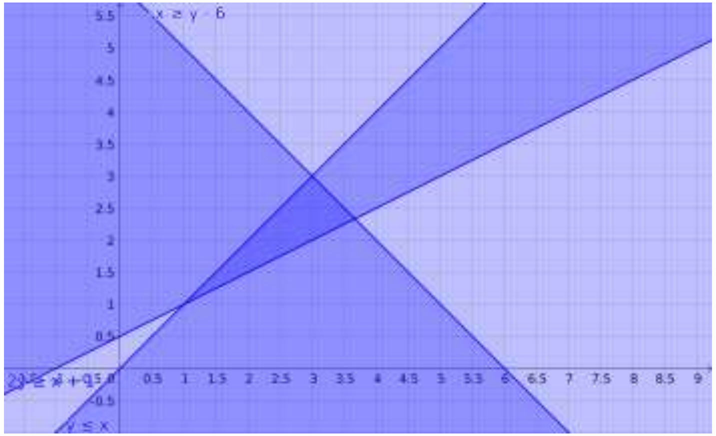
- Unbounded feasible region
- Single unique solution



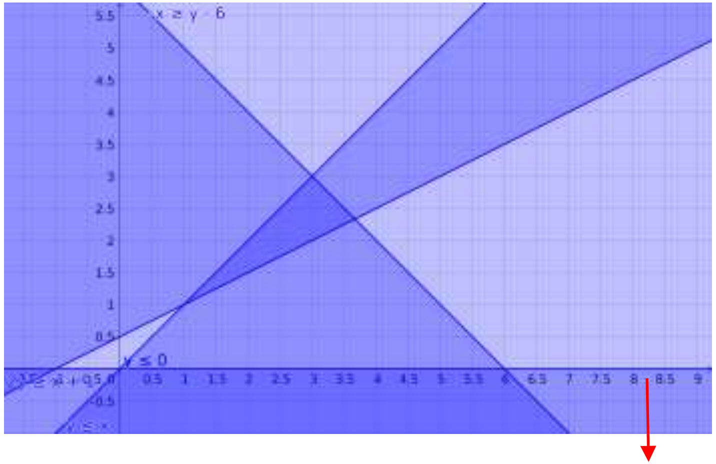
Can we find an optimal solution?

- Unbounded feasible region
- No solution

Optimality Examples

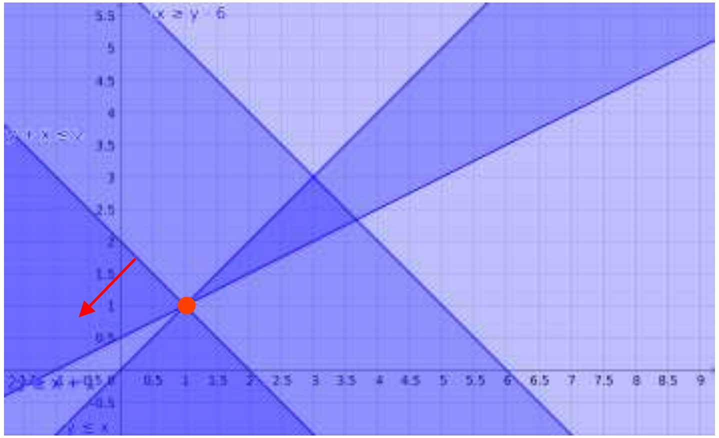


$y \leq 0$



Infeasible problem

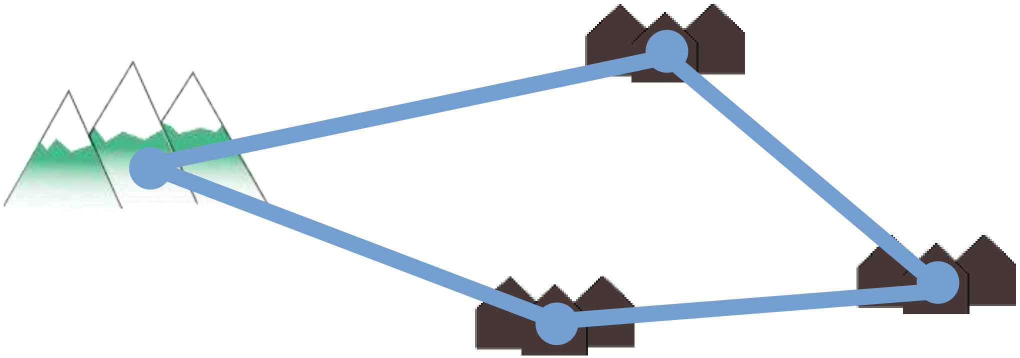
$x + y \leq 2$



Single feasible point and optimal solution

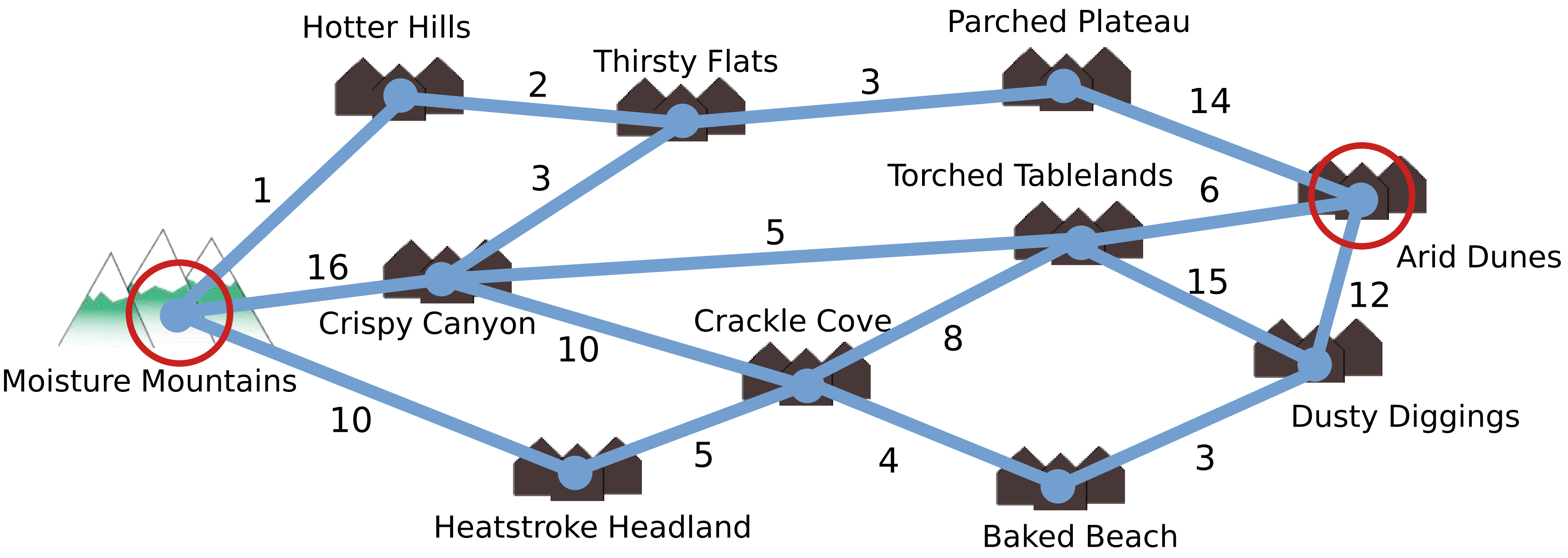
Water Supply Examples

To demonstrate these points on the optimality of LPs, we will look the task of **distributing water** in a network of pipes.



These are examples of **“network flow”** type problems.

Maximum Flow



Maximum flow between a source and sink where edges have limited capacities. What would be the problem formulation?

Maximum Flow

Directed graph: $G := (V, E)$ Maximum flow between two vertices: (s, n)

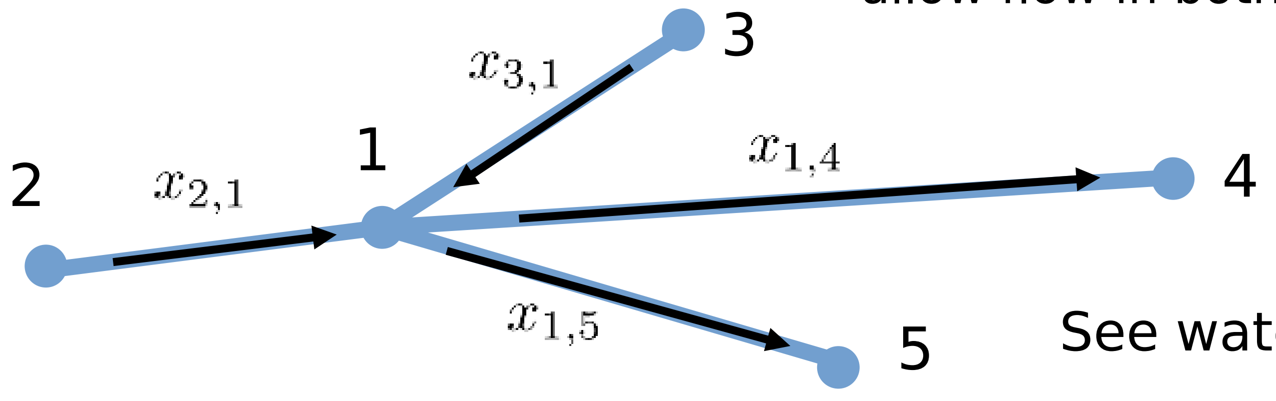
Source Sink

$$\max_x \sum_{(j,n)\in E} x_{j,n} - \sum_{(n,i)\in E} x_{n,i}$$

$$\text{s.t.} \sum_{(i,k)\in E} x_{i,k} - \sum_{(k,j)\in E} x_{k,j} = 0 \quad \forall k \in V \setminus \{s, n\}$$

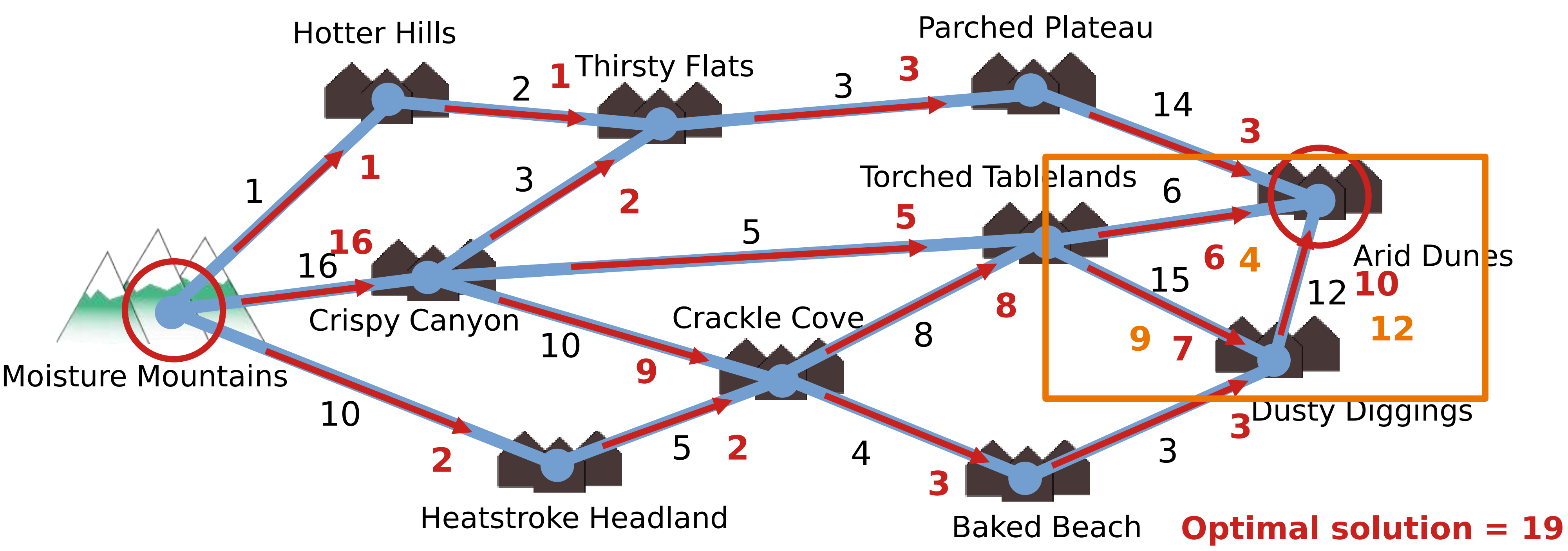
$$-c_{i,j} \le x_{i,j} \le c_{i,j} \quad \forall (i, j) \in E$$

Graph is directed, but we allow flow in both directions.



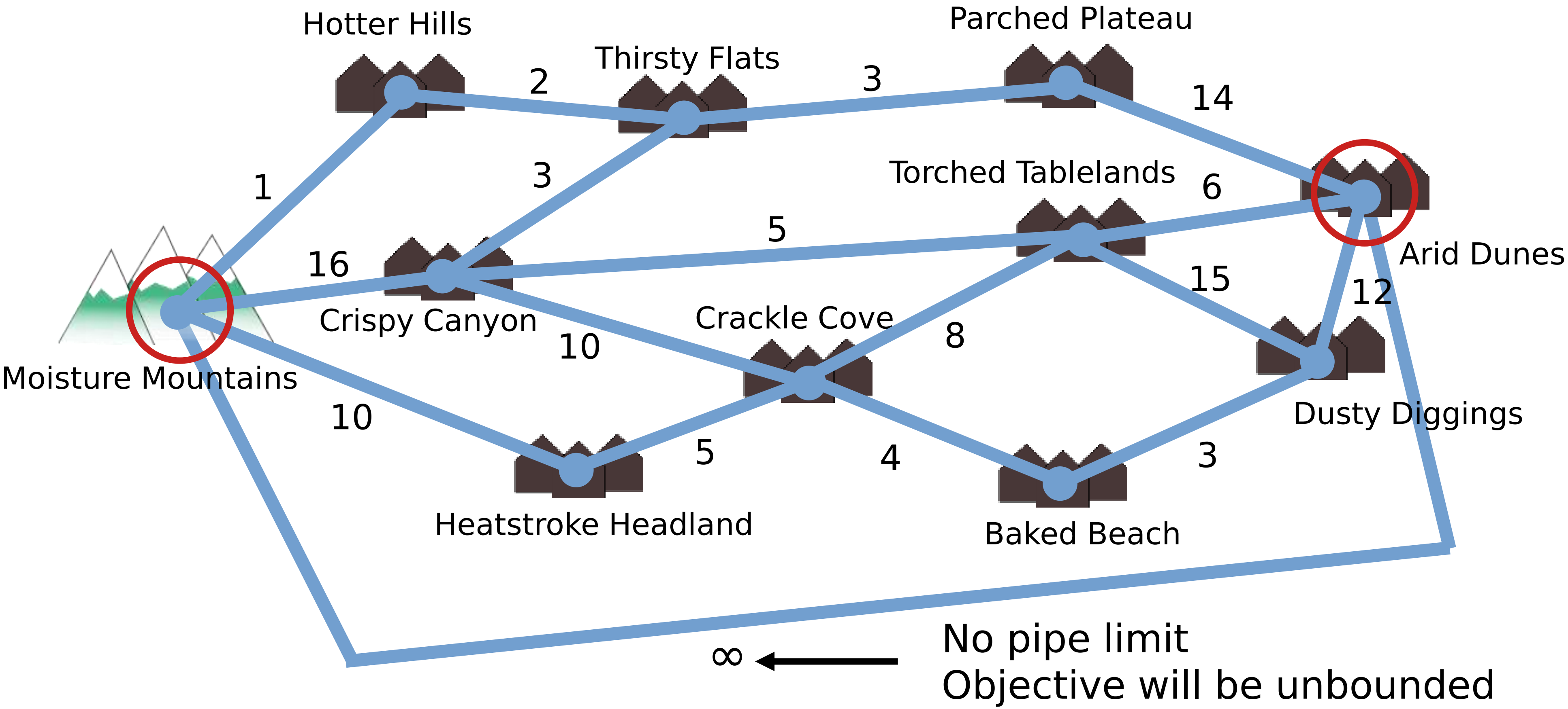
See water_max_flow.py

Maximum Flow

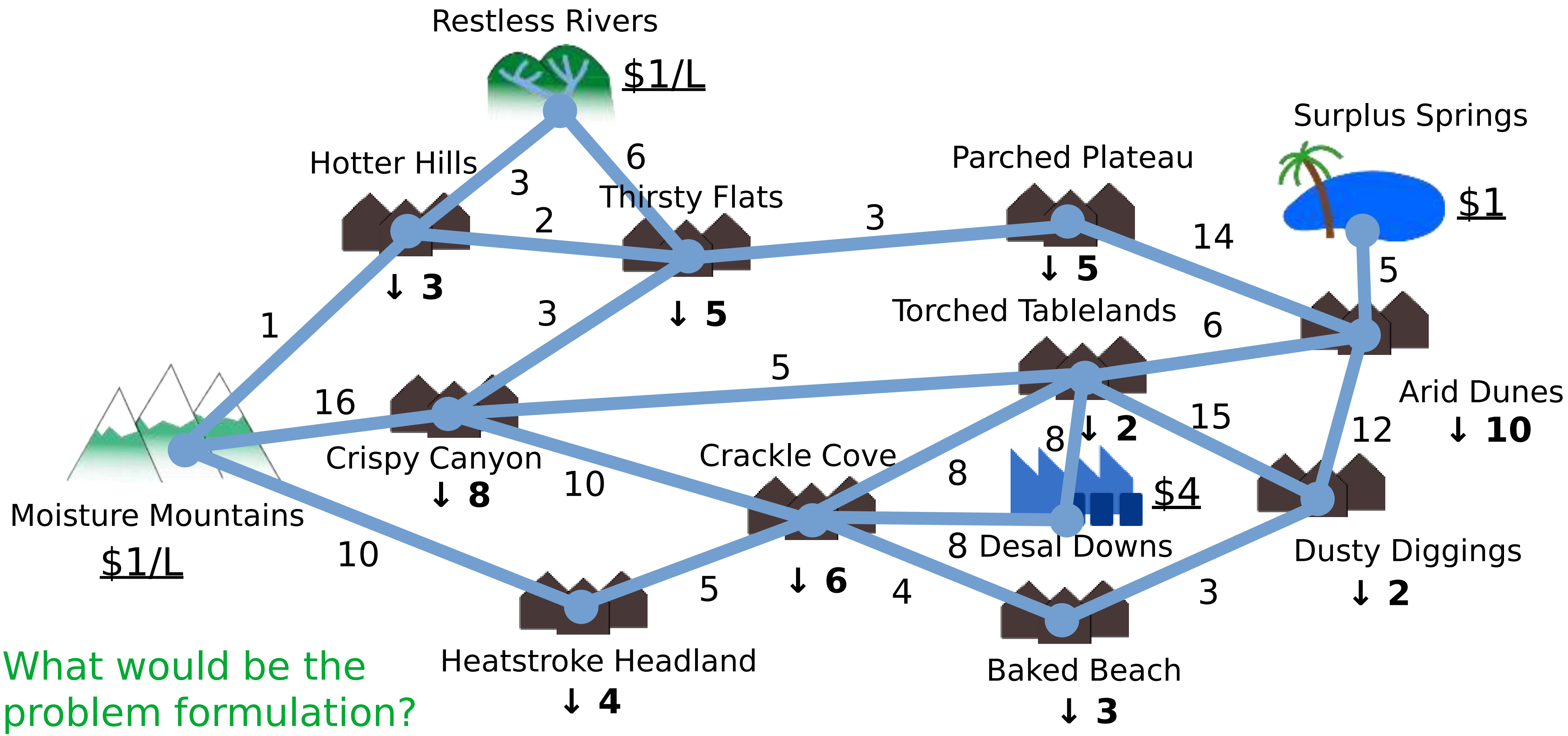


Multiple optimal solutions

Unbounded Problem



Minimum Cost Supply



What would be the problem formulation?

Minimum Cost Supply

supply price

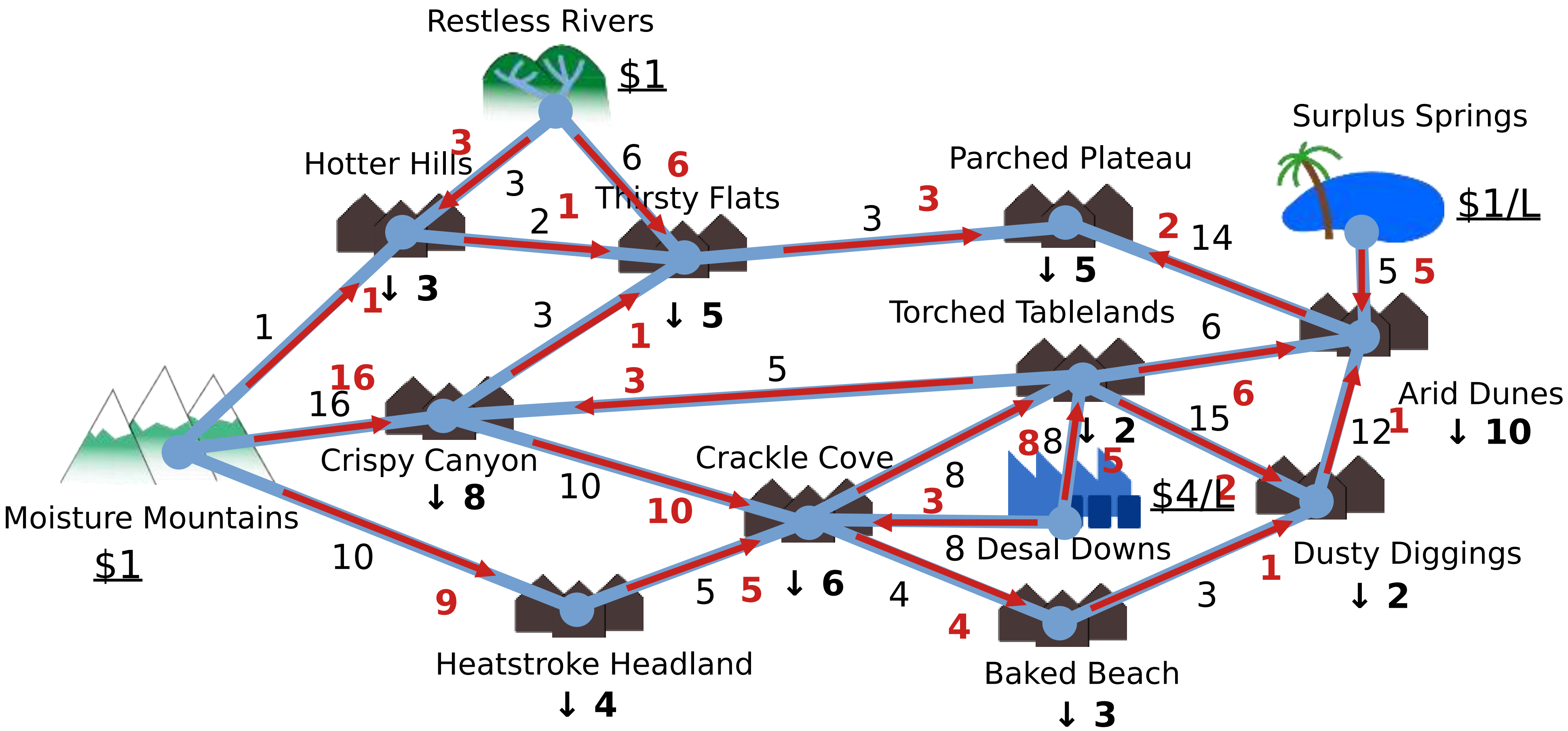
supply

demand

$$
\begin{aligned}
\min_{x,y} \quad & \sum_{k \in V} p_k y_k \\
\text{s.t.} \quad & \sum_{(i,k) \in E} x_{i,k} - \sum_{(k,j) \in E} x_{k,j} + y_k - d_k = 0 \quad \forall k \in V \\
& -c_{i,j} \leq x_{i,j} \leq c_{i,j} \quad \forall (i,j) \in E \\
& y_k \in [0, \overline{y}_k] \quad \forall k \in V
\end{aligned}
$$

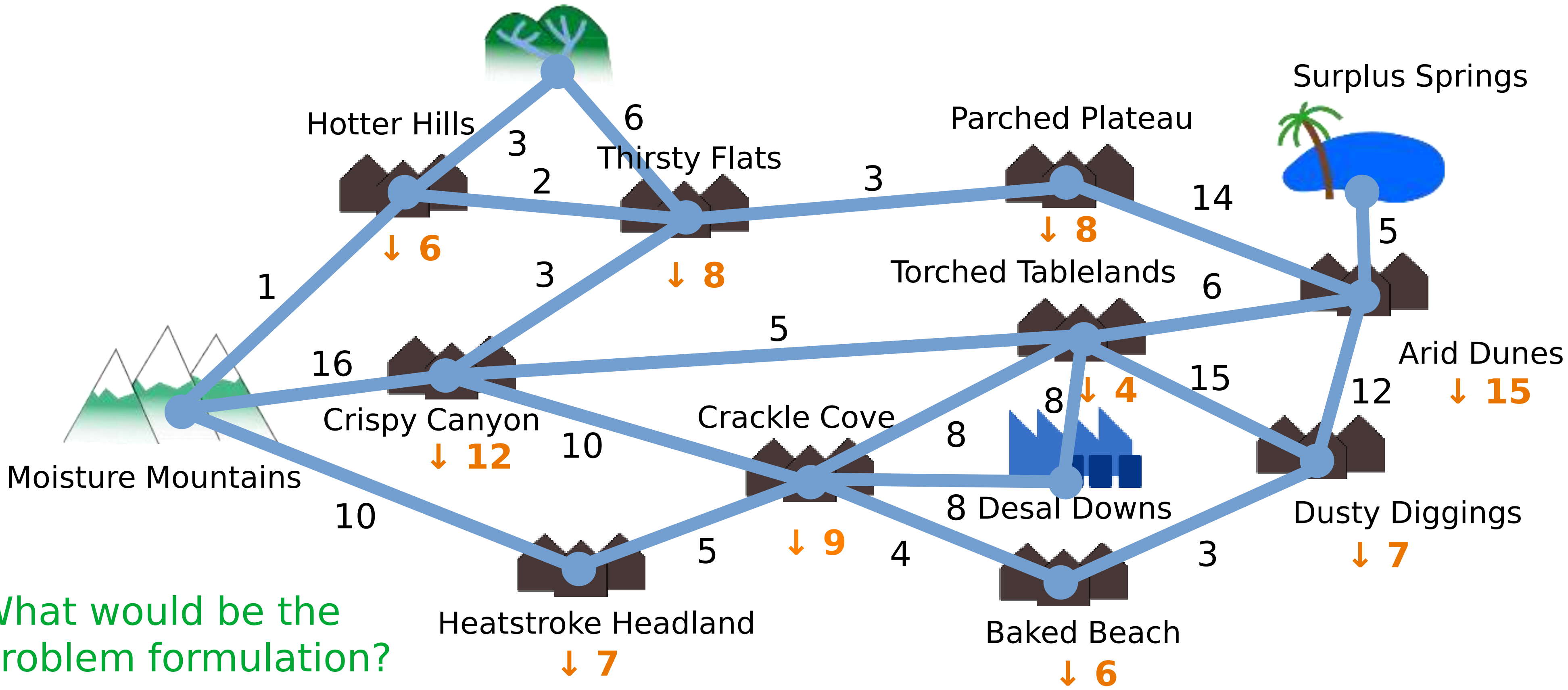
See water_supply.py

Minimum Cost Supply



Minimum Pipe Upgrades

Infeasible with existing line capacities



What would be the problem formulation?

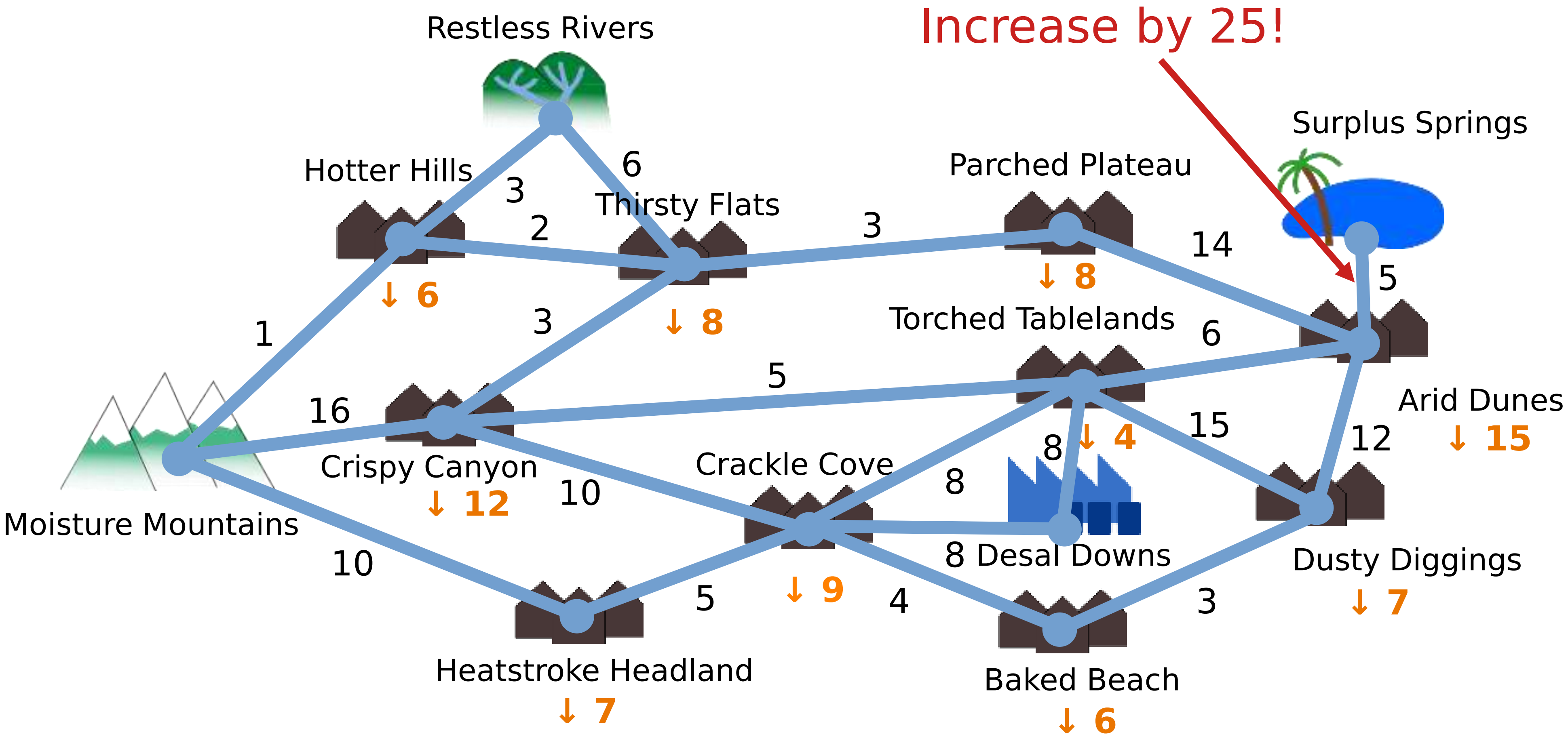
Minimum Pipe Upgrades

$$\begin{aligned}
\min_{x,y,z} \quad & \sum_{(i,j)\in E} z_{i,j} \\
\text{s.t.} \quad & \sum_{(i,k)\in E} x_{i,k} - \sum_{(k,j)\in E} x_{k,j} + y_k - d_k = 0 \quad \forall k \in V \\
& -c_{i,j} - z_{i,j} \le x_{i,j} \le c_{i,j} + z_{i,j} \quad \forall (i,j) \in E \\
& y_k \in [0, \overline{y}_k] \quad \forall k \in V \\
& z_{i,j} \ge 0 \quad \forall (i,j) \in E
\end{aligned}$$

Upgrade of pipe capacities

One of the lab implementation problems

Minimum Pipe Upgrades



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