# 1. Overview of Mixed-Integer Programming (MIP)

Integer Linear Programming (ILP): LP where all variables are integers.

Mixed-Integer Linear Programming (MILP): LP where variables are either integers or continuous.

Complexity: MILP is generally NP-hard, meaning it is computationally intensive, but clever formulations and solvers can manage the complexity for specific instances.

# 2. Linear Relaxation

Concept: Relaxing the integer constraints in an ILP/MILP to get an LP (Linear Programming) problem.

Importance: Solving the LP relaxation gives a bound on the ILP/MILP objective, which is useful in methods like Branch and Bound.

Convex Hull: The convex hull of ILP feasible points is the tightest possible linear relaxation. The closer the relaxation to the convex hull, the stronger the formulation.

Formulas:

- Relaxed LP Objective: Z = c1x1 + c2x2 + ... + cnxn

Steps for Linear Relaxation:

1. Remove the integer constraints from the ILP/MILP problem.

2. Solve the resulting LP problem using an LP solver.

3. If the solution to the LP is integer, it is the optimal solution to the ILP/MILP.

4. If the solution is not integer, further methods (e.g., Branch and Bound, Cutting Planes) are applied.

# 3. Branch and Bound

Concept: A divide-and-conquer approach to solving MIP problems by splitting the feasible region into smaller partitions (branches) and calculating bounds (upper and lower) to prune the solution space.

Key Steps:

1. Branching: Splitting the problem into subproblems.

2. Bounding: Calculating upper and lower bounds to check feasibility and optimality of partitions.

3. Pruning: Discarding subproblems that cannot improve the current best solution.

Formulas:

- Upper Bound: Z\_upper - obtained from the best feasible solution.

- Lower Bound: Z\_lower - obtained from LP relaxation.

Steps for applying Branch and Bound.

# 4. Cutting Plane Methods

Concept: Adding linear inequalities (cuts) iteratively to the problem to tighten the LP relaxation until an integer solution is found.

Gomory Cuts: A specific type of cutting plane introduced by Ralph Gomory. These cuts are derived from the optimal solution of the LP relaxation and help in eliminating fractional solutions.

Formulas:

- Gomory Cut: Start with the LP solution, zj - cj = (cbxj) - cj.

- Fractionalization: xb = [xbi] + fbi, where fbi is the positive fractional part.

Steps for applying Cutting Plane Methods:

1. Solve the LP relaxation of the ILP/MILP.

2. Identify any fractional variables in the solution.

3. Generate a cutting plane to eliminate the fractional solution.

4. Add the cutting plane to the problem and resolve the LP relaxation.

5. Repeat the process until an integer solution is found.

# 5. Constraint Types in MIP

Common Constraints:

- Choice Constraints: Binary variables representing choices.

- Logical Constraints: Represent logical relationships.

- On-Off Constraints: Constraints activated based on variable states.

Formulas:

- Big-M Formulation: M \* yi >= xi, where M is a large positive constant.

- Indicator Constraints: xi = 0 or 1 based on logical conditions.

# 6. Feasibility Heuristics

Purpose: To find good feasible solutions without necessarily solving the entire problem to optimality. This can reduce the amount of branching needed.

Feasibility Pump: A heuristic used to quickly find feasible solutions, but not guaranteed to work every time. It can be applied at the root node or at intermediate nodes.

Steps for Feasibility Pump:

1. Start with a fractional solution obtained from the LP relaxation.

2. Round the solution to the nearest integer.

3. Project the rounded solution back to the feasible region of the LP.

4. Repeat until a feasible solution is found or a maximum number of iterations is reached.

# 7. Scheduling Problems in MIP

Definition: Scheduling involves allocating resources over time to tasks, such as job shop scheduling, project planning, or timetabling.

Key Modelling Choices:

- Time Variables: Represent the start time of each job.

- Indicator Variables: Represent whether a job is active during a specific time period.

Formulas:

- Makespan: Makespan = max(completion time of all tasks).

Steps for solving scheduling problems.

# 8. Duality in ILPs

Lagrangian Relaxation: This technique provides a lower bound on the ILP objective and can sometimes offer a stronger bound than the linear relaxation. Formulas:

- Lagrangian Dual: Z = max(lambda >= 0) [ min\_x { c^T x + lambda (Ax - b) } ].

Explanation of duality in ILPs.