Lecture 13

Rigid body kinetics: Inertial CS; Linear momentum balance; Angular momentum balance.

28 October, 2020

Inertial CS

I. The **laws of motion** are written in an *Inertial Coordinate System*.

II. **Definition**. An *inertial coordinate system* is one in which the laws of motion hold.

III. **No** inertial CS has yet been found.

1. Ref.: *Science of Mechanics* by *E. Mach*

IV. **However**, laws of mechanics hold excellently in a given CS, provided that the non-inertial nature of the CS is small compared to the motion being studied. For example, when studying a

1. satellite around the Earth ($a_{sat} \sim 1$ m-s^{-2}), we can ignore the Earth's acceleration around the sun (≈ 0.006 m-s^{-2}).

2. car ($a_{car} \sim 1$ m-s^{-2}), the Earth's centripetal acceleration (≈ 0.034 m-s^{-2}) due to its rotation may be ignored.

Balance laws

I. All mechanical system <u>must</u> satisfy

1. Mass balance.
2. Linear momentum balance (LMB)
3. Angular momentum balance (AMB)

II. **Two** ways to study a mechanical system:

1. *Control mass*: Follow evolution of the <u>same</u> set of bodies and material points.
   i. Good for systems with <u>solids</u>.
2. *Control volume*: Follow evolution of only those bodies and material points that occupy a given volume.
   i. Good for systems with <u>fluids</u>.

III. Rigid body system has <u>*no*</u> mass exchange

1. Follow a *control mass* analysis, i.e. focus on the <u>same</u> set of rigid bodies.
2. Mass balance is trivially true.

LMB for particles

वैशेषिक सूत्र (Vaiśeṣika Sūtra) — कणाद (Kaṇāda) in about 600 BC (300 years *before* Aristotle).

1. संयोगाभावे गुरुत्वात् पतनम् — *In the absence of conjunction, gravity [causes] fall.* [Newton's 1st]
2. नोदनविशेषाभावान्नोर्ध्वं न तिर्य्यग्गमनम् — *In the absence of a force, there is no upward motion, sideward motion or motion in general.*
3. नोदनादाद्यमिषोः कर्म तत्कर्मकारिताच्च संस्कारादुत्तरं तथोत्तरमुत्तरञ्च — *The initial pressure [on the bow] leads to the arrow's motion; from that motion is momentum, from which is the motion that follows and the next and so on similarly.* [$\leqslant$ Newton's 2nd]
4. कार्य्यविरोधि कर्म — *Action (kārya) is opposed by reaction (karman)* [Newton's 3rd]

Further reading

1. *Kaṇāda, Great Physicist and Sage of Antiquity* — Subhask Kak (Link)
2. *Matter and Mind: The Vaisheshika Sutra of Kanada* — Subhash Kak (Link to Amazon)

LMB

I. (**Newton's 2nd law, 1686 AD**) Rate of change of linear momentum of a particle equals the total applied force.

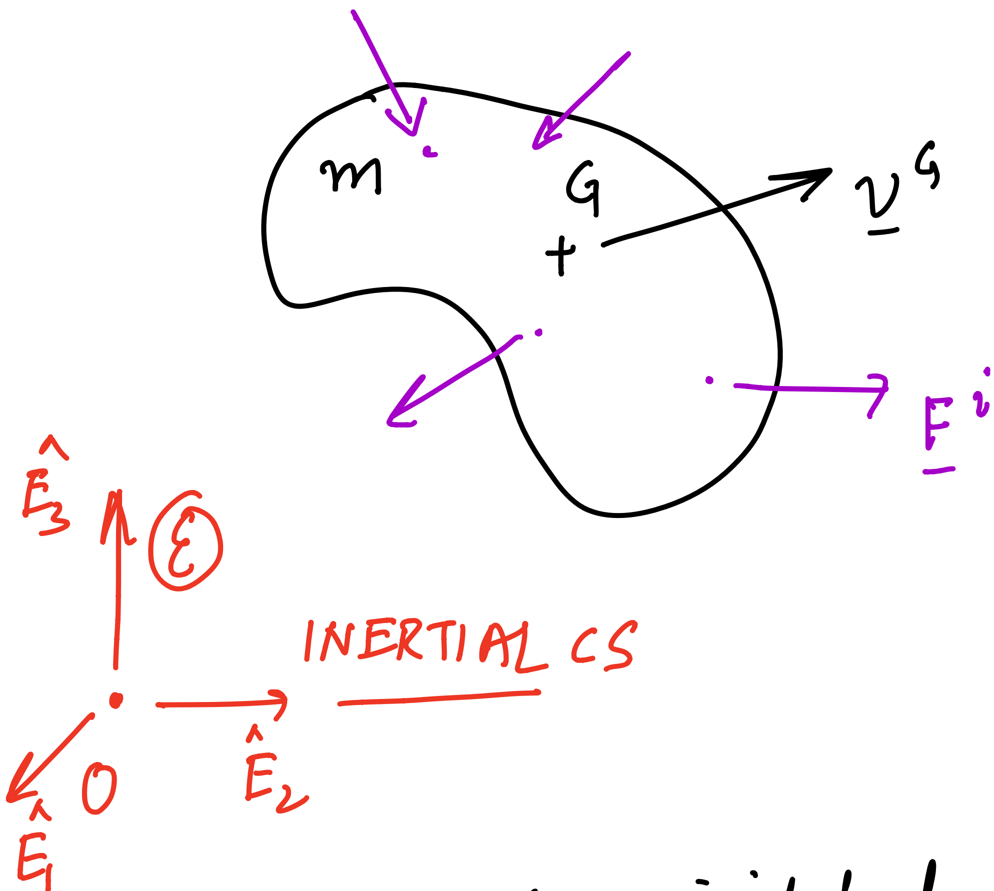
II. (**Euler's 1st law, ~1730 AD**) Rate of change of linear momentum of a rigid body equals the total applied force.

   1. Can<u>not</u> derive this from Newton's law

III. Linear momentum of a rigid body

$$\mathbf{p} = \int_V \rho(\mathbf{r})\mathbf{v}(\mathbf{r})\mathrm{d}V = m\mathbf{v}^G \,.$$

IV. Then, LMB gives: $\boxed{\sum \mathbf{F}^i = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\mathbf{a}^G \,.}$



Total force on the rigid body

$$\sum \underline{F}^i = \frac{d}{dt}\underline{p} = \frac{d}{dt}\left(m\underline{v}^G\right)$$

$$= m\,\underline{a}^G$$

1st law of rigid body mechanics

AMB

I. (**Euler's 2nd law, ~1730 AD**) Rate of change of angular momentum of a rigid body about a <u>non</u>-accelerating point P, i.e. $\mathbf{a}^P = \mathbf{0}$ in an <u>inertial</u> CS, equals the total applied moment about P.

1. Newton had <u>no</u> such law.
2. Can<u>not</u> derive from LMB.

II. *Angular momentum* about a point P :

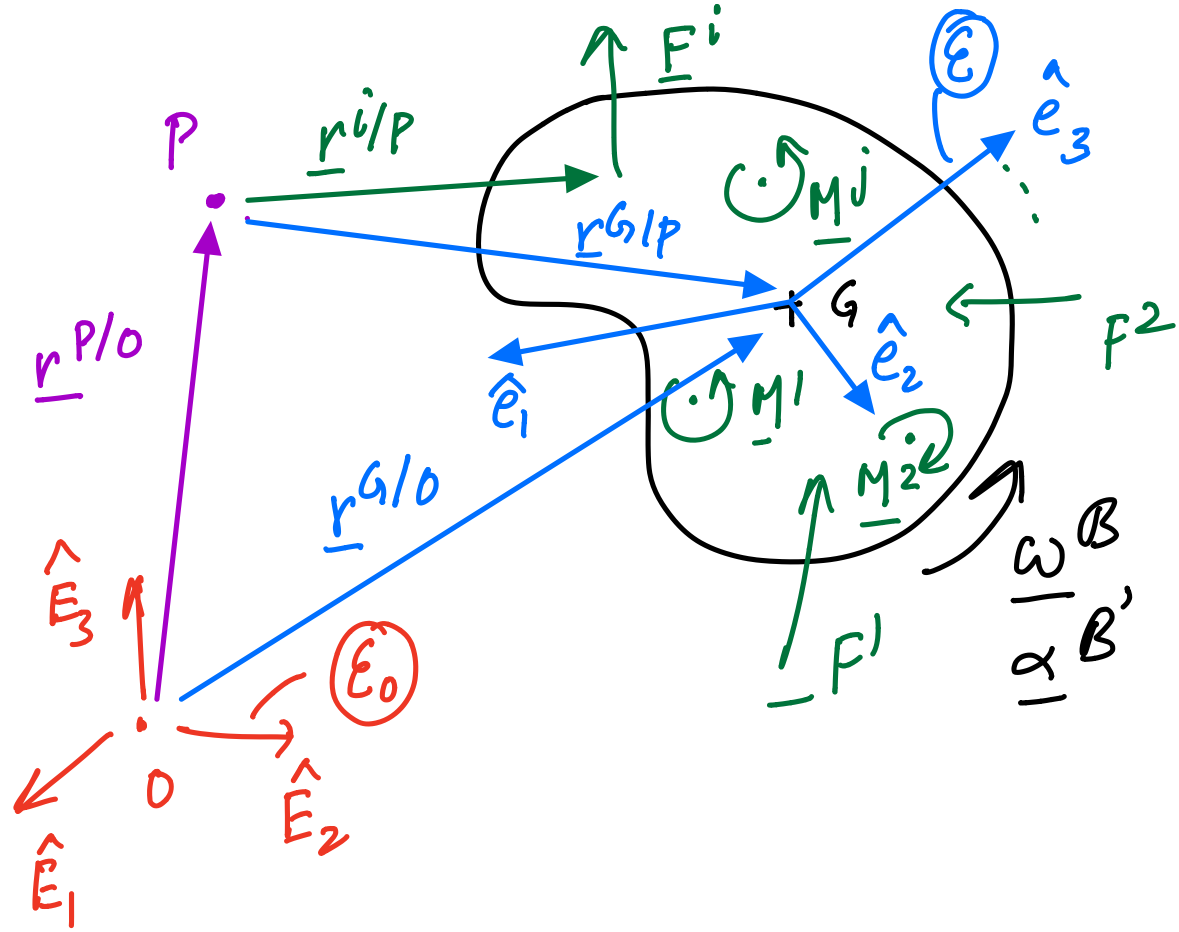
$$\mathbf{h}^P = \mathbf{r}^{G/P} \times m\mathbf{v}^G + \underbrace{\mathrm{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}}_{\mathbf{h}^G}.$$

III. Total moment about P:

$$\mathbf{M}^P = \sum \mathbf{r}^{i/P} \times \mathbf{F}^i + \sum \mathbf{M}^j$$

IV. AMB about P : $\mathbf{M}^P = \dfrac{\mathrm{d}\mathbf{h}^P}{\mathrm{d}t} \implies$

$$\boxed{\mathbf{M}^P = \mathbf{r}^{G/P} \times m\mathbf{a}^G + \boldsymbol{\omega}^{\mathscr{B}} \times \left(\mathrm{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}\right) + \mathrm{I}^G \cdot \boldsymbol{\alpha}^{\mathscr{B}}}$$



$$\frac{d\underline{b}}{dt} = \overset{o}{\underline{b}} + \underline{\omega}_{\mathcal{E}/\mathcal{E}_0} \times \underline{b}$$

Rate of change of $\underline{b}$ w.r.t $\mathcal{E}_0$

Rate of change w.r.t CS $\mathcal{E}$

$$\dot{\underline{b}} = \overset{o}{\underline{b}} + \underline{\omega}^{B} \times \underline{b}$$

$$\frac{d}{dt}\underline{h}^P = \dot{\underline{h}}^P = \frac{d}{dt}\left(\underline{r}^{G/P} \times m\underline{v}^G\right) + \frac{d\underline{h}^G}{dt}$$

$$= \frac{d\underline{r}^{G/P}}{dt} \times m\underline{v}^G + \underline{r}^{G/P} \times \frac{d\underline{v}^G}{dt} + \frac{d\underline{h}^G}{dt}$$

$$\frac{d\underline{r}^{G/P}}{dt} = \frac{d\underline{r}^{G/O}}{dt} + \cancelto{0}{\frac{d\underline{r}^{P/O}}{dt}} = \underline{v}^G$$

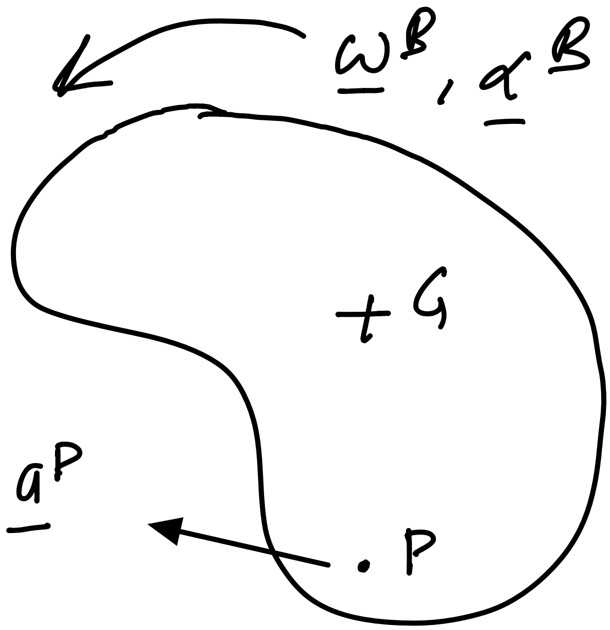
$$\frac{d\underline{h}^P}{dt} = \underline{r}^{G/P} \times m\,\underline{a}^G + \frac{d\underline{h}^G}{dt}$$

$$\frac{d\underline{h}^G}{dt} = \dot{\underline{h}}^G = \overset{\circ}{\underline{h}}^G + \underline{\omega}^B \times \underline{h}^G$$

But $\overset{\circ}{\underline{h}}^G = \underline{\underline{I}}^G \cdot \underline{\omega}^B$

$$= \underline{\underline{I}}^G \cdot \overset{\circ}{\underline{\omega}}^B$$

$$= \underline{\underline{I}}^G \cdot \dot{\underline{\omega}}^B = \underline{\underline{I}}^G \cdot \underline{\alpha}^B$$



AMB

I. AMB about <u>non</u>-accelerating P $\left(\mathbf{a}^P = \mathbf{0}\right)$:

$$\boxed{\mathbf{M}^P = \mathbf{r}^{G/P} \times m\mathbf{a}^G + \boldsymbol{\omega}^{\mathscr{B}} \times \left(\mathtt{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}\right) + \mathtt{I}^G \cdot \boldsymbol{\alpha}^{\mathscr{B}}}$$

II. **Special cases**.

1. $P = G$, and $\mathbf{a}^G$ arbitrary :

$$\boxed{\mathbf{M}^G = \boldsymbol{\omega}^{\mathscr{B}} \times \left(\mathtt{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}\right) + \mathtt{I}^G \cdot \boldsymbol{\alpha}^{\mathscr{B}}.}$$

2. P <u>lying</u> on rigid body, and $\mathbf{a}^P$ arbitrary:

$$\boxed{\mathbf{M}^P = \mathbf{r}^{G/P} \times m\mathbf{a}^P + \boldsymbol{\omega}^{\mathscr{B}} \times \left(\mathtt{I}^P \cdot \boldsymbol{\omega}^{\mathscr{B}}\right) + \mathtt{I}^P \cdot \boldsymbol{\alpha}^{\mathscr{B}}}$$

3. *2D rigid body kinetics*. If $\boldsymbol{\omega}^{\mathscr{B}}$ is <u>always</u> along (say) $\hat{\mathbf{e}}_3$ principal axis of $\mathtt{I}^G$:

$$\boxed{\mathbf{M}^P = \mathbf{r}^{G/P} \times m\mathbf{a}^G + I_3^G \boldsymbol{\alpha}^{\mathscr{B}}.}$$

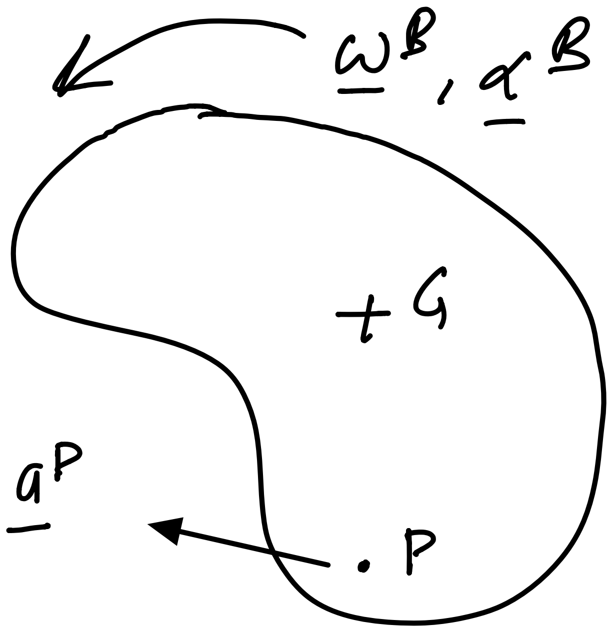
i. 2D kinematics $\supsetneq$ 2D kinetics.

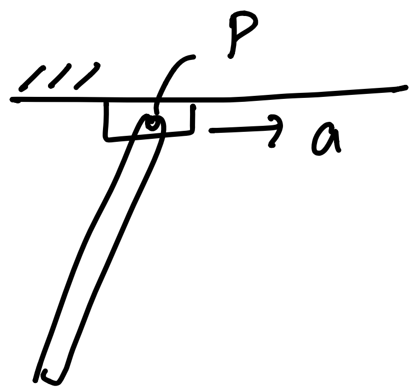
$$\frac{d\underline{h}^G}{dt} = \dot{\underline{h}}^G = \overset{\circ}{\underline{h}}{}^G + \underline{\omega}^B \times \underline{h}^G$$

$$\text{But } \overset{\circ}{\underline{h}}{}^G = \overset{\circ}{\overline{\underline{\underline{I}}^G \cdot \underline{\omega}^B}}$$

$$= \underline{\underline{I}}^G \cdot \overset{\circ}{\underline{\omega}}{}^B$$

$$= \underline{\underline{I}}^G \cdot \dot{\underline{\omega}}^B = \underline{\underline{I}}^G \cdot \underline{\alpha}^B$$





$$\underline{a}^P = \underline{a}^G + \underline{\omega}^B \times (\underline{\omega}^B \times \underline{r}^{P/G}) + \underline{\alpha}^B \times \underline{r}^{P/G}$$

$$\underline{\underline{I}}^G \cdot \underline{\omega}^B \parallel \underline{\omega}^B$$

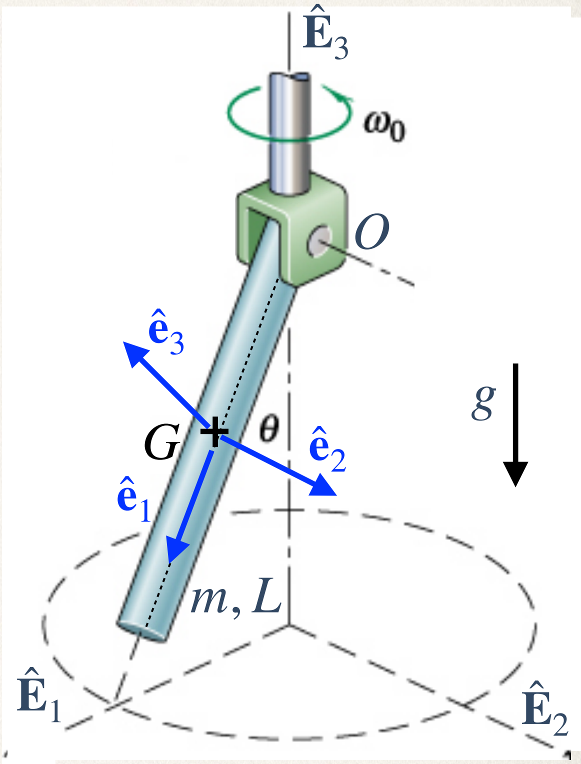
In 2D kinetics

$$\underline{\omega}^B = \omega \hat{e}_3$$

$$\Rightarrow \underline{\alpha}^B = \alpha \hat{e}_3$$

Example 1

Find angle θ given ω_0.



Important. The most crucial thing is to draw neat and correct *Free Body Diagrams* (FBD).

1. *LMB*: $\sum \mathbf{F}^i = m\mathbf{a}^G$
2. *AMB about O*: Point O lies on rod, so use

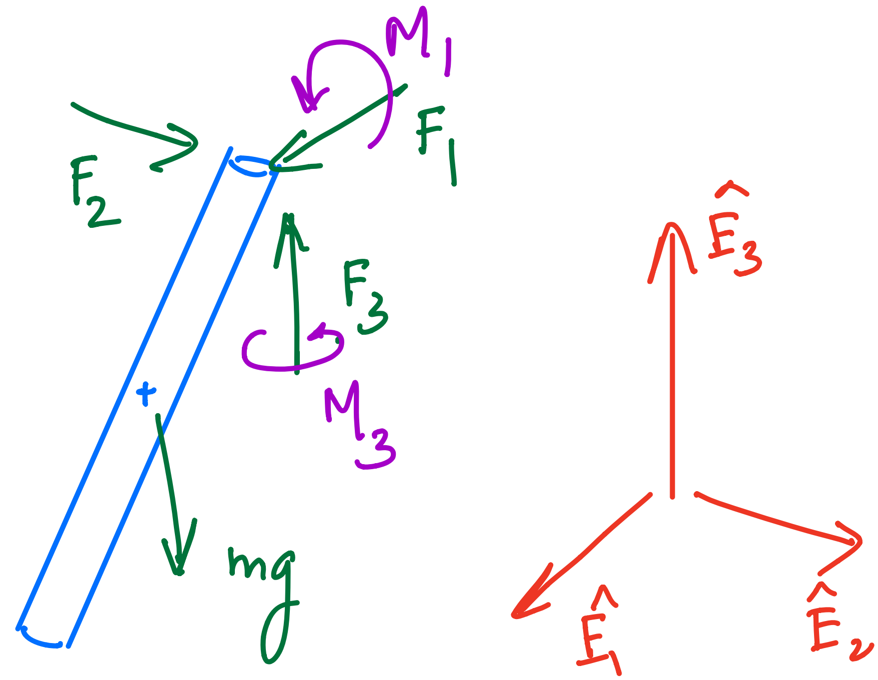
$$\mathbf{M}^O = \mathbf{r}^{G/O} \times m\mathbf{a}^O + \boldsymbol{\omega}^{\mathscr{B}} \times \left(\mathrm{I}^O \cdot \boldsymbol{\omega}^{\mathscr{B}}\right) + \mathrm{I}^O \cdot \boldsymbol{\alpha}^{\mathscr{B}}$$

3. *Kinematic analysis* $\implies$ $\boldsymbol{\omega}^{\mathscr{B}}, \boldsymbol{\alpha}^{\mathscr{B}}, \mathbf{a}^G$.
4. Use *parallel axis theorem* to get I^O:

$$\mathrm{I}^O = \mathrm{I}^G + m\left(\left|\mathbf{r}^{G/O}\right|^2 1 - \mathbf{r}^{G/O} \otimes \mathbf{r}^{G/O}\right).$$

5. **Answer**: $\cos\theta = 3g/\left(2\omega_0^2 L\right)$.
   i. When $\omega \to \infty, \theta \to 90^\circ$.
   ii. Non-zero θ only if $\omega_0 \geqslant \sqrt{3g/(2L)}$.

FBD



$$\text{LMB: } \overset{?}{F_1}\hat{E}_1 + \overset{?}{F_2}\hat{E}_2 + \overset{?}{F_3}\hat{E}_3 - mg E_3 = m\underline{a}^G \quad \text{— (1)}$$

$$\text{AMB}/_O\text{: } \overset{?}{M_1}\hat{E}_1 + \overset{?}{M_3}\hat{E}_3 + \underline{r}^{G/O}\times(-mg\hat{E}_3) = \underline{\omega}^B\times(\underline{\underline{I}}^O\cdot\underline{\omega}^B) + \underline{\underline{I}}^O\cdot\underline{\alpha}^B \quad \text{— (2)}$$

# of unknowns : ~~14~~ 5 + 1 α θ

# of equations : 6

KINEMATIC ANALYSIS

$$\underline{\omega}^B = \underline{\omega}_{\mathcal{E}/\mathcal{E}_0} = \omega_0 \hat{E}_3$$

$$\underline{\alpha}^B = \frac{d\underline{\omega}^B}{dt} = 0$$

(3)

$$\underline{a}^G = \cancel{\underline{a}^O}_{\to 0} + \underline{\omega}^B \times (\underline{\omega}^B \times \underline{r}^{G/O}) + \underline{\alpha}^B \times \underline{r}^{G/O}$$

$$\underline{r}^{G/O} = L/2 \ \hat{e}_1$$

$$\underline{a}^G = -\omega_0^2 \frac{L}{2} \sin\theta \ \hat{E}_1 \quad \text{—— (4)}$$

In $\mathcal{E}$: $\left[\underline{\underline{I}}^G\right]_{\mathcal{E}} = \dfrac{mL^2}{12} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left[\underline{\underline{I}}^{0}\right]_{\mathcal{E}} = \frac{mL^2}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (5)$$

Combining (1) – (5):

$$\text{LMB}: F_1\hat{E}_1 + F_2\hat{E}_2 + F_3\hat{E}_3 = -\frac{m\omega_0^2}{2}L\sin\theta\,\hat{E}_1$$

$$\text{AMB}/_{O}: M_1\hat{E}_1 + M_3\hat{E}_3 + \frac{mgL}{2}\sin\theta\,\hat{e}_2$$

$$= \omega_0\hat{E}_3 \times \left(\underline{\underline{I}}^{0}\cdot\omega_0\hat{E}_3\right)$$

$$\omega_0\hat{E}_3 = \omega_0\left(-\cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_3\right)$$

$$\left[\omega_0\hat{E}_3\right]_{\mathcal{E}} = \begin{bmatrix} -\omega_0\cos\theta \\ 0 \\ \omega_0\sin\theta \end{bmatrix}$$

$$\left[\underline{\underline{I}}^{0} \cdot \omega_0 \hat{E}_3\right]_{\mathcal{E}} = -\frac{mL^2\omega_0}{3}\begin{bmatrix} 0 \\ 0 \\ \sin\theta \end{bmatrix}$$

$$\Rightarrow \quad \underline{\underline{I}}^{0} \cdot \omega_0 \hat{E}_3 = -\frac{mL^2\omega_0}{3}\,\hat{e}_3$$

$$\therefore \ \omega_0 \hat{E}_3 \times \left(\underline{\underline{I}}^{0} \cdot \omega_0 \hat{E}_3\right)$$

sin θ

$$= -\frac{mL^2\omega_0^2}{3}\,\hat{E}_3 \times \hat{e}_3$$

sin θ

$$= -\frac{mL^2\omega_0^2}{3}\cos\theta\,\hat{e}_2$$

$$AMB/O: \quad M_1\hat{E}_1 + M_3\hat{E}_3 + \frac{mgL}{2}\sin\theta\,\hat{e}_2$$

$$= -\frac{mL^2\omega_0^2}{3}\cos\theta\,\hat{e}_2 \ \sin\theta$$

$$M_1 = M_3 = 0$$

$\hat{E}_2\ (\hat{e}_2)$ component $\Rightarrow$

$$\cancel{m}\frac{g\cancel{L}}{2}\cancel{\sin\theta} = -\frac{\cancel{m}L^{\cancel{2}}\omega_0^2}{3}\cos\theta\,\cancel{\sin\theta}$$

$$\cos\theta = \frac{3g}{2\omega_0^2 L}$$

$\omega_0 \longrightarrow \infty \qquad \cos\theta \longrightarrow 0$

$\Rightarrow \theta \rightarrow 90^\circ$

$$\cos\theta = \frac{3g}{2\omega_0^2 L} \leq 1 \Rightarrow \omega_0 \geq \sqrt{\frac{3g}{3L}}$$

Teaching western science to indian brains.

Babu Teacher. "Number One is called a 'right angle,' and you would naturally suppose that Number Two is a 'left angle.' But by order of Government of India Survey Department this is also a right angle."

Punch magazine 1924

Doesn’t work well.

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