Lecture 12

Rigid body kinetics: Rigid body motion; Kinetic quantities; Moment of inertia tensor.

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21 October, 2020

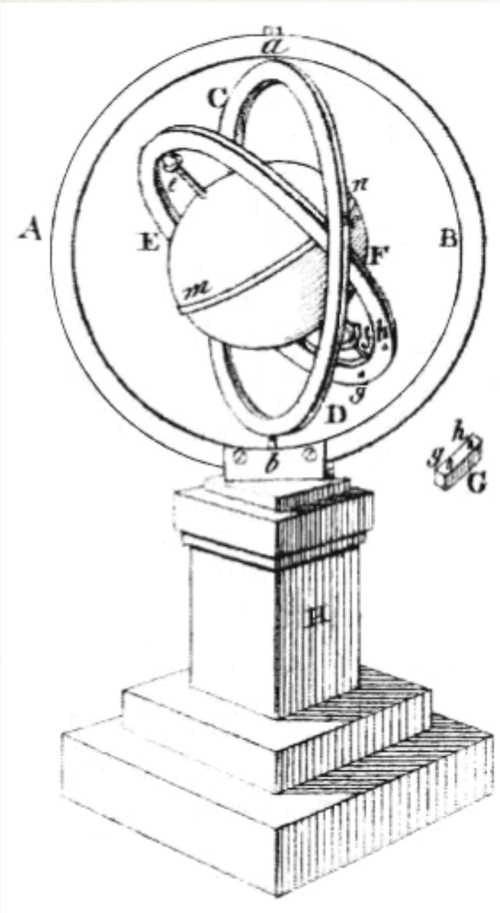
Rigid body dynamics

KINEMATICS

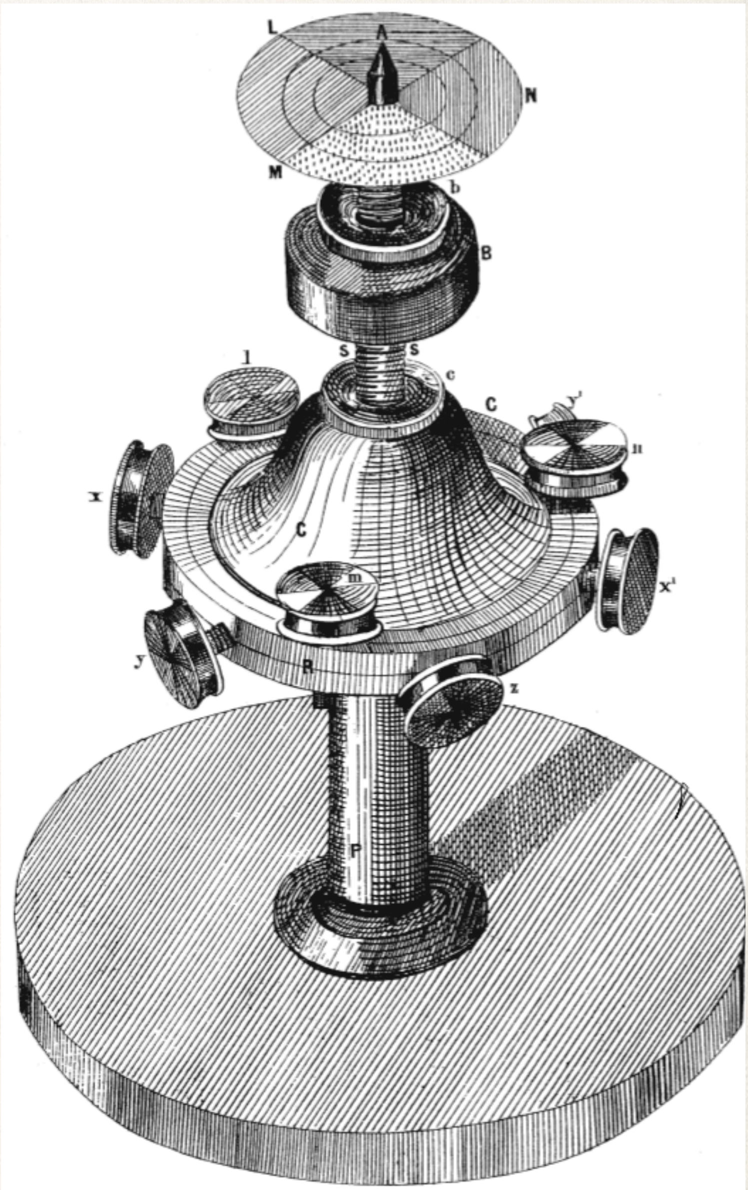
CONSTITUTIVE LAW

KINETICS

Rigid body motion



Bohenenberger's machine to show Earth's rotation.



Maxwell's top: Visualises path of instantaneous rotation axis. Further reading ; Video link .

Rigid body motion

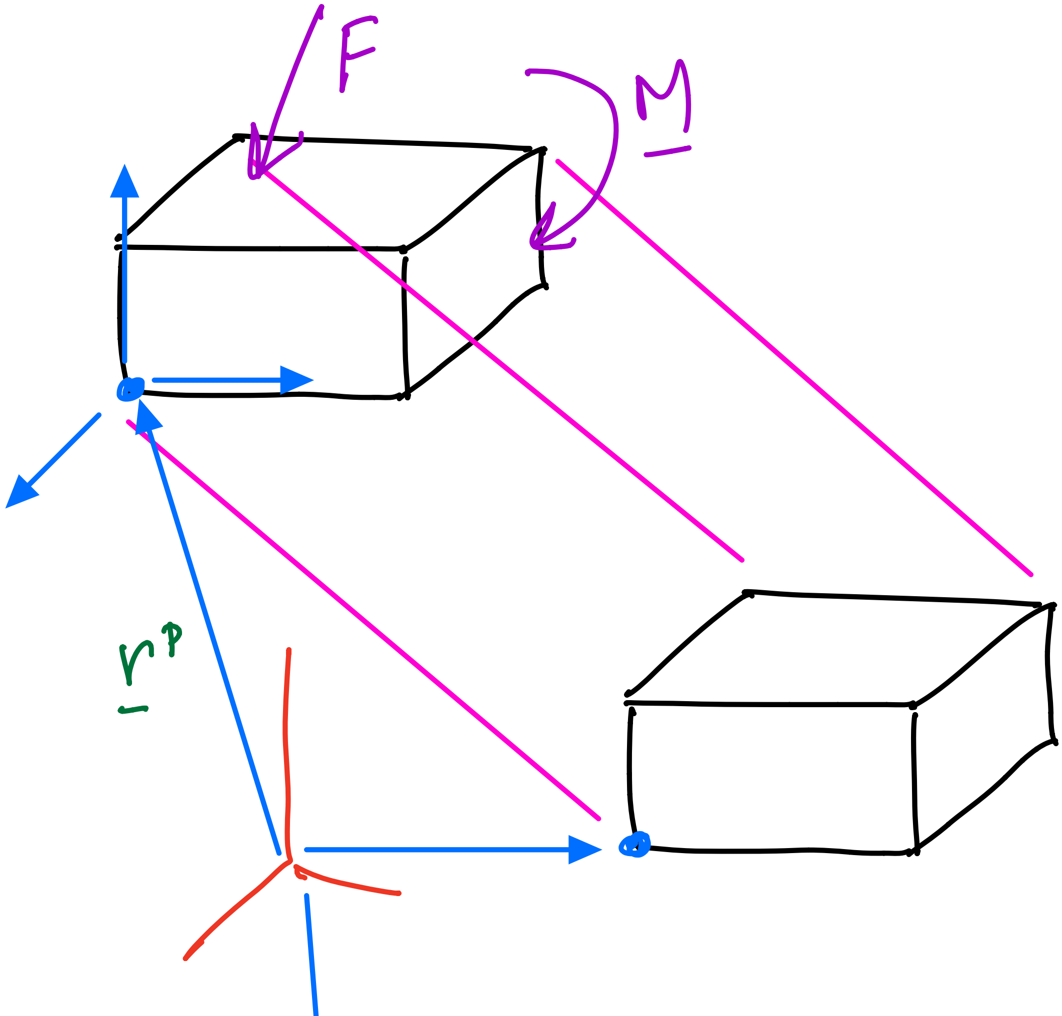
I. **General rigid body motion**.

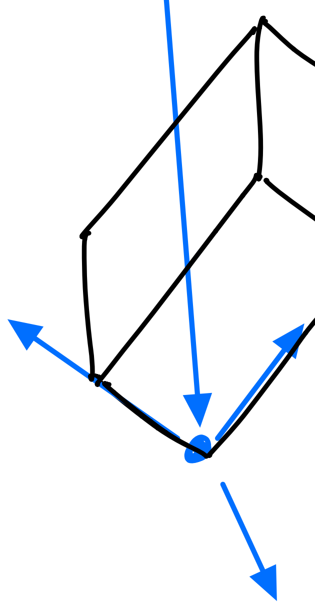
   1. A rigid body can *translate* and *rotate*.
   2. To track *translation*: Follow location of a point on the rigid body.
   3. To track *rotation*: Generally use an Euler angle sequence.

II. To find location of a point needs its velocity as function of time. To find rigid body's orientation need its angular velocity as a function of time.

III. To find velocity and angular velocity integrate, respectively, acceleration and angular acceleration.

IV. Accelerations and angular accelerations depend on applied forces and moments through the **laws of motion**.





$$\dot{\underline{r}}^P(t) = \underline{v}^P(t)$$

$$\underline{\omega}(t)$$

$$\underline{a}^P = \frac{d\underline{v}^P}{dt}, \quad \underline{\alpha} = \frac{d}{dt}\underline{\omega}$$

Rigid body: Kinetic quantities

Laws of motion are in terms of the following :

I. *Mass*: $m = \int_V \rho(\mathbf{r})\mathrm{d}V.$

II. *Center of mass*: $\mathbf{r}^{G/O} = \dfrac{1}{m}\int_V \rho(\mathbf{r})\,\mathbf{r}\,\mathrm{d}V\,.$

III. *Linear momentum*: $\mathbf{p} = \int_V \rho(\mathbf{r})\mathbf{v}(\mathbf{r})\mathrm{d}V = m\mathbf{v}^G.$

IV. *Angular momentum* <u>about</u> a point P :

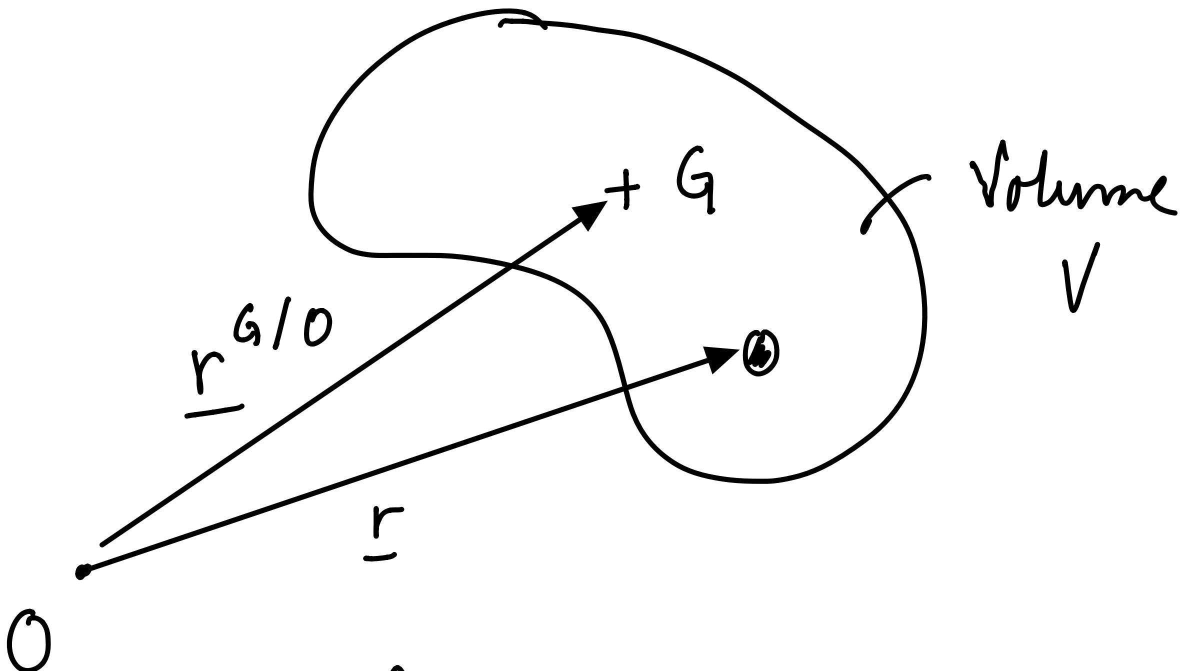
$$\mathbf{h}^P = \int_V \mathbf{r}^{/P}\rho(\mathbf{r})\mathbf{v}(\mathbf{r})\mathrm{d}V = \mathbf{r}^{G/P} \times m\mathbf{v}^G + \underbrace{\mathbb{I}^G \cdot \boldsymbol{\omega}^{\mathscr{B}}}_{\mathbf{h}^G},$$

where, $\quad \mathbb{I}^G = \int_V \rho(\mathbf{r})\left(\left|\mathbf{r}^{/G}\right|^2 \mathbb{1} - \mathbf{r}^{/G} \otimes \mathbf{r}^{/G}\right)\mathrm{d}V$

is the *Moment of Inertia* <u>about</u> G of the body.

V. *Kinetic energy*: $E_K = \dfrac{1}{2}\int_V \rho(\mathbf{r})\left|\mathbf{v}(\mathbf{r})\right|^2 \mathrm{d}V$

$$= \frac{1}{2}m\left|\mathbf{v}^G\right|^2 + \frac{1}{2}\mathbb{I}^G\left|\boldsymbol{\omega}^{\mathscr{B}}\right|^2$$



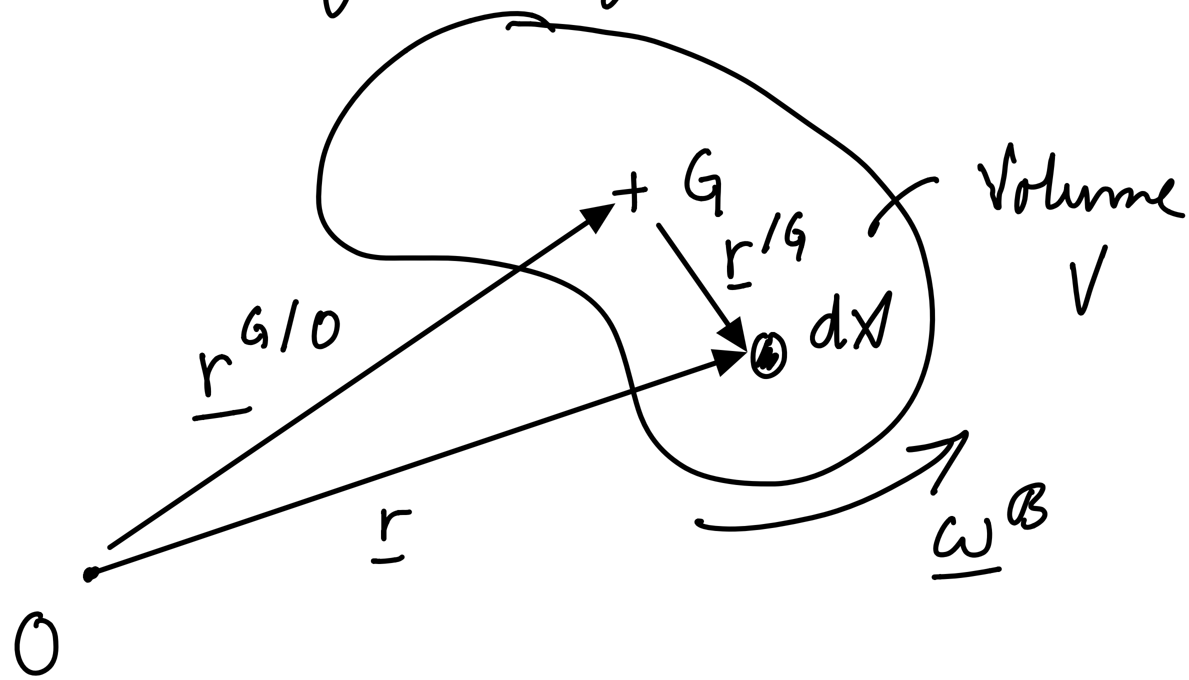
$$m = \int_V \rho(\underline{r})\, dV$$

$$\frac{\sum m_i \underline{r}_i}{\sum m_i} = \underline{r}_{CM}$$

$$\int_{\cancel{V}} \frac{\rho(\underline{r})\, \underline{r}\, d\cancel{V}}{m}$$

$$\underline{p} = \sum m_i \underline{v}_i$$

For a rigid body



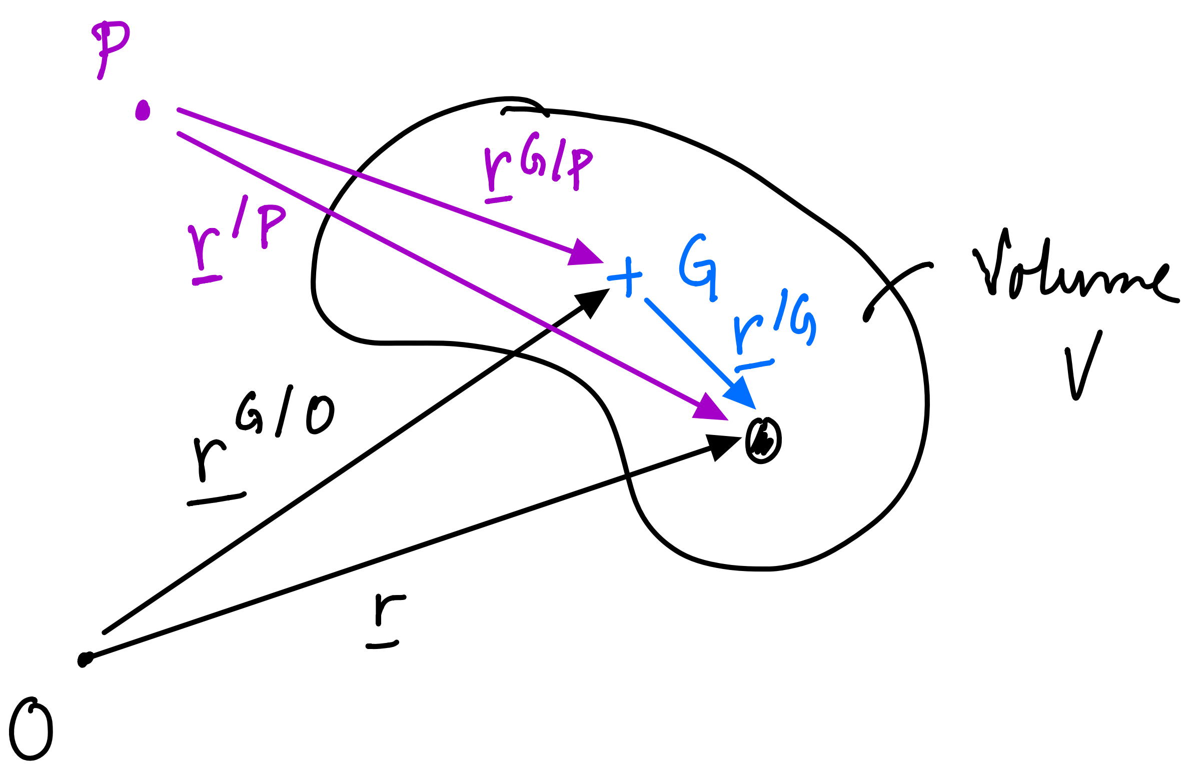
$$\underline{p} = \int_V \rho(r)\, \underline{v}(\underline{r})\, dV$$

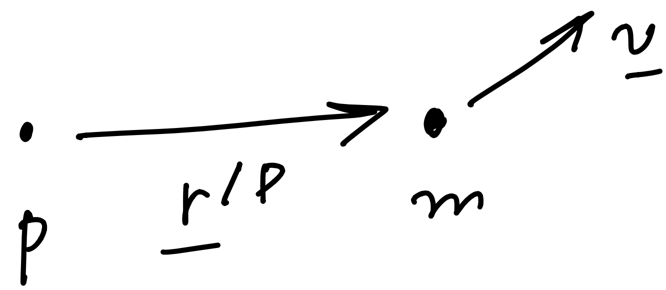
$$\underline{v}(\underline{r}) = \underline{v}^{G} + \underline{\omega}^{B} \times \underline{r}^{/G}$$

$$\underline{p} = \int_V \rho(\underline{r})\left(\underline{v}^G + \underline{\omega}^B \times \underline{r}^{/G}\right) dV$$

$$= m\underline{v}^G + \underline{\omega}^B \times \underbrace{\int_V \rho(\underline{r})\,\underline{r}^{/G}\, dV}_{0}$$

$$\underline{p} = m\underline{v}^G$$





$$\underline{h}^P = \underline{r}^{/P} \times m\underline{v}$$

The ang. mom. of a rigid body about a point P

$$\underline{h}^P = \int_V \underline{r}^{/P} \times \left(\rho(\underline{r})\, dV\right) \underline{v}(\underline{r})$$

$$= \int \underline{r}^{/P} \times \rho(\underline{r})\, \underline{v}(\underline{r})\, dV$$

$$\underline{v}(\underline{r}) = \underline{v}^G + \underline{\omega}^B \times \underline{r}^{/G}$$

$$\underline{r}^{/P} = \underline{r}^{/G} + \underline{r}^{G/P}$$

$$\underline{h}^P = \int_V \underline{r}^{G/P} \times \left(\underline{v}^G + \underbrace{\underline{\omega}^B \times \underline{r}^{/G}}_{0}\right) \rho(\underline{r})\, dV + \int_V \underline{r}^{/G} \times \left(\underbrace{\underline{v}^G}_{0} + \underline{\omega}^B \times \underline{r}^{/G}\right) \rho(\underline{r})\, dV$$

$$= \underline{r}^{G/P} \times m\, \underline{v}^G + \int_V \underline{r}^{/G} \times \left(\underline{\omega}^B \times \underline{r}^{/G}\right) \rho(\underline{r})\, dV$$

$$\underline{r}^{/G} \times \left(\underline{\omega}^B \times \underline{r}^{/G}\right) = \left|\underline{r}^{/G}\right|^2 \underline{\omega}^B - \left(\underline{r}^{/G} \cdot \underline{\omega}^B\right) \underline{r}^{/G}$$

$$\left|\underline{r}^{/G}\right|^2 \underline{\omega}^B = \left|\underline{r}^{/G}\right|^2 \underline{\underline{1}} \cdot \underline{\omega}^B$$

$$\left(\underline{r}^{/G} \cdot \underline{\omega}^B\right) \underline{r}^{/G} = \left(\underline{r}^{/G} \otimes \underline{r}^{/G}\right) \cdot \underline{\omega}^B$$

$$\int \underline{r}^{/G} \times (\underline{\omega}^{B} \times \underline{r}^{/G})\, \rho(\underline{r})\, dV$$

$$= \underbrace{\int \left\{ |\underline{r}^{/G}|^2 \underline{\underline{1}} - \underline{r}^{/G} \otimes \underline{r}^{/G} \right\} \rho(\underline{r})\, dV}_{\underline{\underline{I}}^{G}} \cdot \underline{\omega}^{B}$$

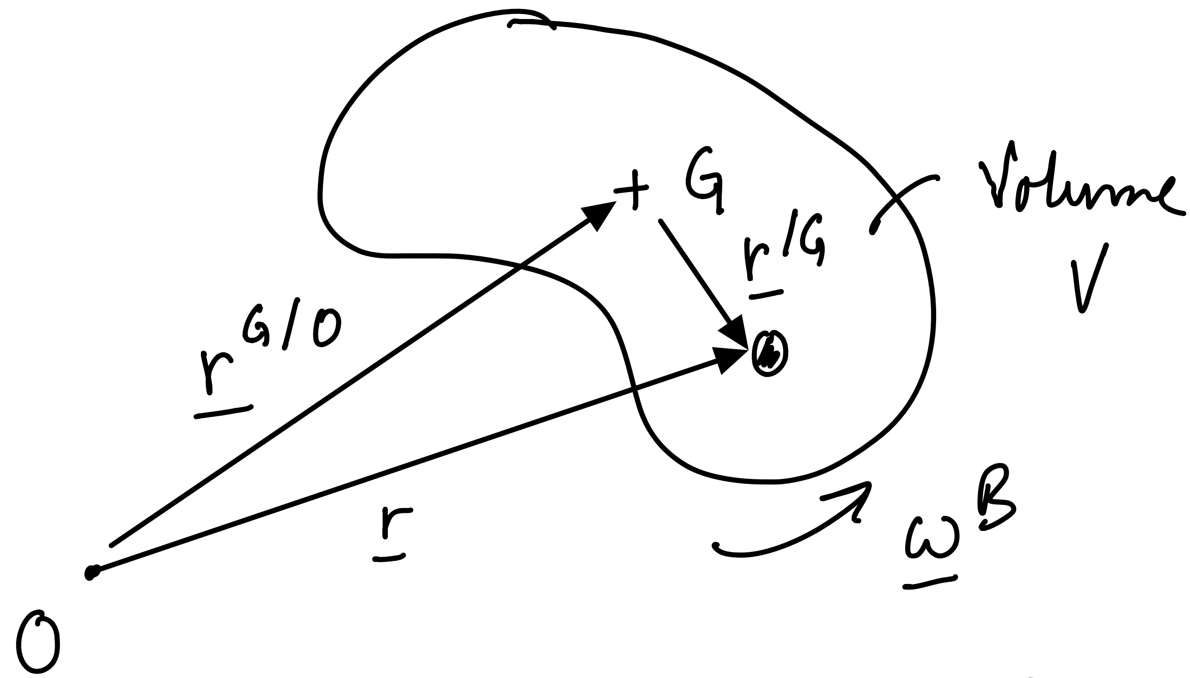
$$\underline{h}^{P} = \underline{r}^{G/P} \times m \underline{v}^{G}$$

$$+ \underline{\underline{I}}^{G} \cdot \underline{\omega}^{B} \sim \underline{h}^{G}$$

is the ang. mom. of the rigid body about its CM.

If $G = P$

$$\underline{h}^{P} = \underline{h}^{G} = \underline{\underline{I}}^{G} \cdot \underline{\omega}^{B}$$

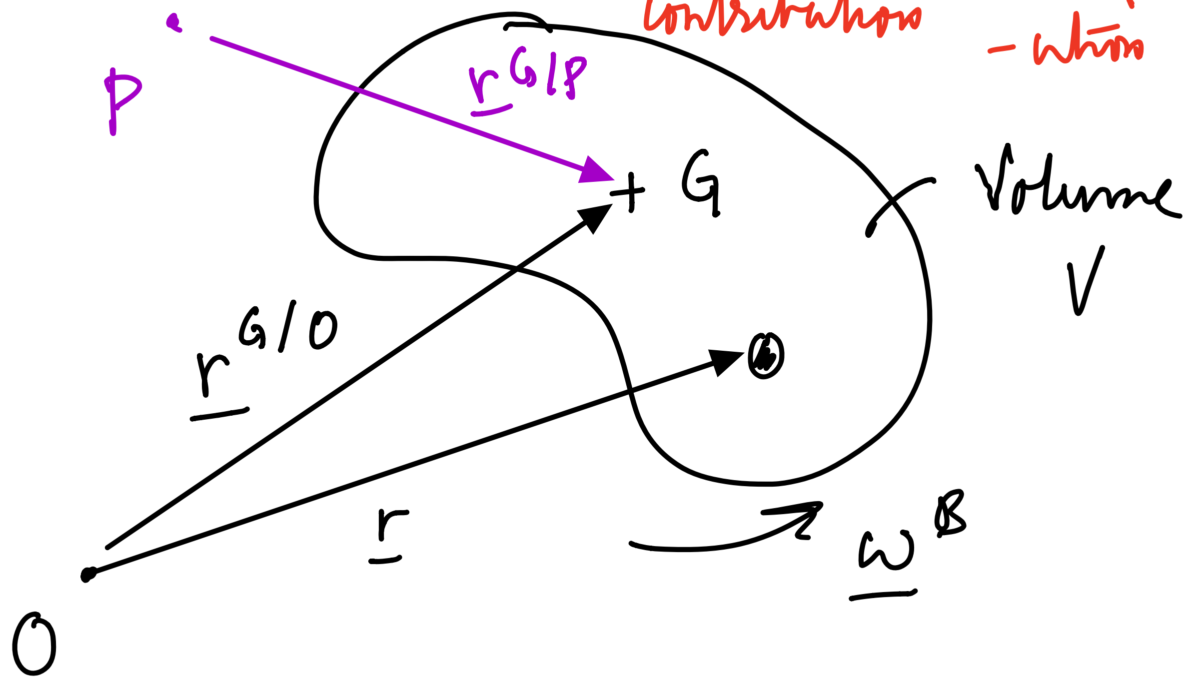


$$E_K = \int_V \frac{1}{2}\left(\rho(\underline{r})\, dV\right) |\underline{v}(\underline{r})|^2$$

$$= \frac{1}{2}\int_V \rho(\underline{r})\, |\underline{v}(\underline{r})|^2\, dV$$

$$\underline{v}(\underline{r}) = \underline{v}^G + \underline{\omega}^B \times \underline{r}^{/G}$$

$$\underline{h}^P = \underbrace{\underline{r}^{G/P} \times m\,\underline{v}^G}_{\text{translational contribution}} + \underbrace{\underline{\underline{I}}^G \cdot \underline{\omega}^B}_{\text{rotational contribution}}$$



Moment of Inertia

I. **Definition**. *Moment of Inertia Tensor* about a point P of a rigid body is given by

$$\mathbf{I}^P = \int_V \rho(\mathbf{r})\left(\left|\mathbf{r}^{/P}\right|^2 \mathbb{1} - \mathbf{r}^{/P} \otimes \mathbf{r}^{/P}\right) \mathrm{d}V \,.$$

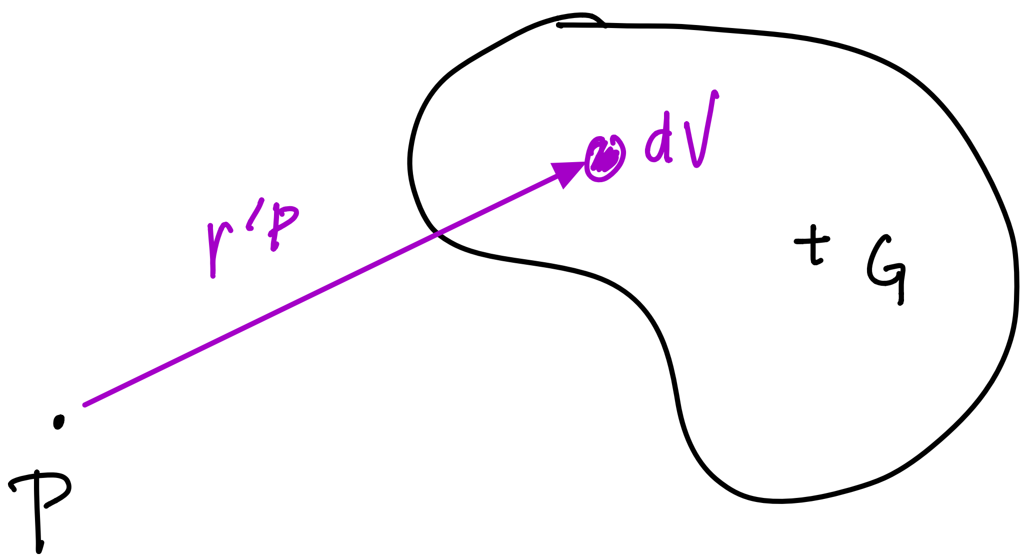
1. *Symmetric, positive definite* tensor.
2. Three *real, positive* principal values.
3. Three *orthogonal* principal axes which define the rigid body's ***principal CS***.
4. Moment of inertia changes with P.
5. Estimates mass distribution *w.r.t.* P .

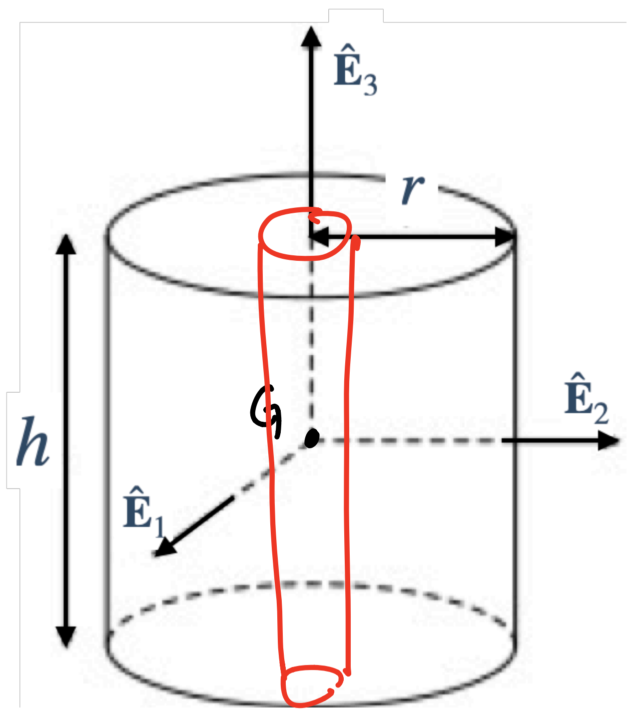
II. In CS $\{\mathscr{E}, P, \hat{\mathbf{E}}_i\}$, let $\mathbf{I}^P = I_{ij}^P \hat{\mathbf{E}}_i \otimes \hat{\mathbf{E}}_j$, then

$$I_{11}^P = \int_V \rho(\mathbf{r})\left(X_2^2 + X_3^2\right)\mathrm{d}V,\; I_{22}^P = \int_V \rho(\mathbf{r})\left(X_3^2 + X_1^2\right)\mathrm{d}V$$

$$I_{33}^P = \int_V \rho(\mathbf{r})\left(X_2^2 + X_3^2\right)\mathrm{d}V,\; I_{12}^P = I_{21}^P = -\int_V \rho(\mathbf{r})X_1X_2\mathrm{d}V,$$

$$I_{23}^P = I_{32}^P = -\int_V \rho(\mathbf{r})X_2X_3\mathrm{d}V,\; I_{31}^P = I_{13}^P = -\int_V \rho(\mathbf{r})X_3X_1\mathrm{d}V \,.$$





$\rho(\underline{r}) = \rho$

$m = \rho \pi r^2 h$

$\{\mathcal{E}, G, \hat{E}_i\}$

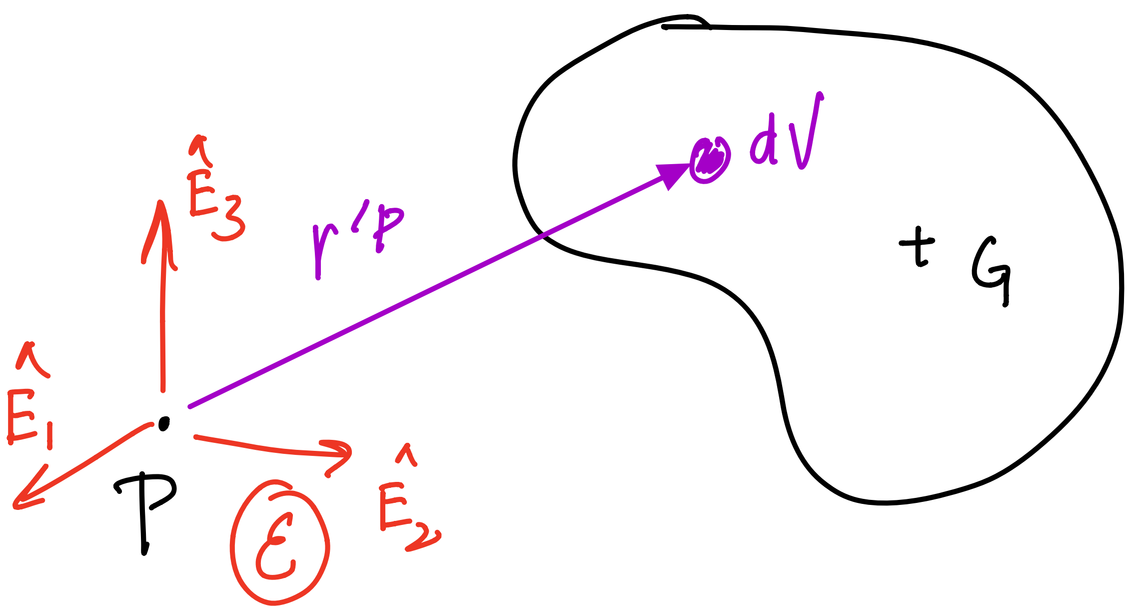
$P = G$

Can compute $\underline{\underline{I}}^G$

$$\left[\underline{\underline{I}}^G\right]_{\mathcal{E}} = \frac{mr^2}{2}\begin{bmatrix} 1/2 + h^2/6r^2 & 0 & 0 \\ 0 & 1/2 + h^2/6r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose $r \to 0$, then

$$\left[\underline{\underline{I}}^G\right]_{\mathcal{E}} \longrightarrow \frac{mh^2}{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$\underline{r}^{/P} = X_i \hat{E}_i$$

$$|\underline{r}^{/P}| = X_1^2 + X_2^2 + X_3^2$$

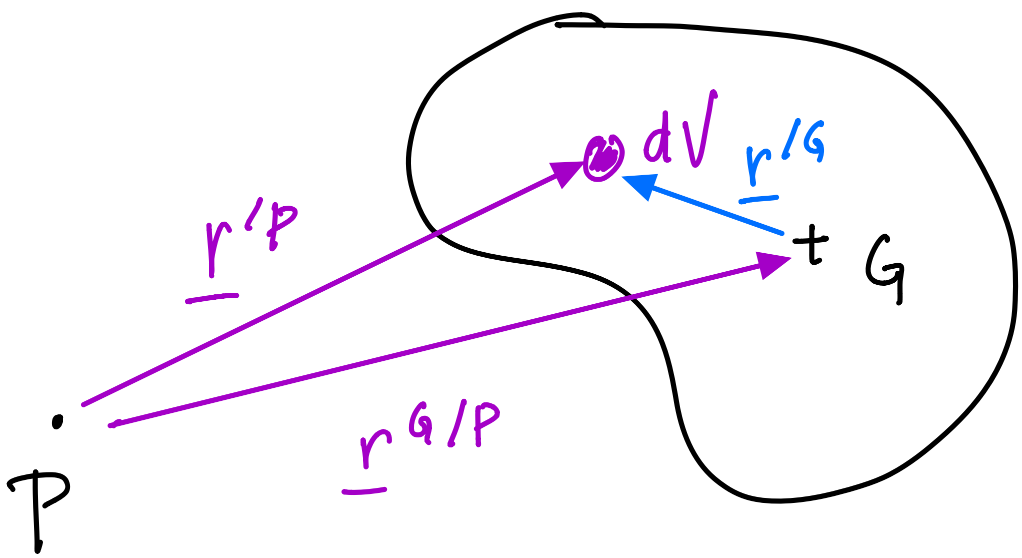
$$\underline{r}^{/P} \otimes \underline{r}^{/P} = X_i X_j \hat{E}_i \otimes \hat{E}_j$$

$$\underline{\underline{I}}^P = \int_V \rho(\underline{r}) \left\{ \left(X_1^2 + X_2^2 + X_3^2\right) \delta_{ij} \hat{E}_i \otimes \hat{E}_j - X_i X_j \hat{E}_i \otimes \hat{E}_j \right\} dV$$

$$= \int_V \rho(\underline{r}) \left\{ \left(X_1^2 + X_2^2 + X_3^2\right) \delta_{ij} - X_i X_j \right) dV \; \hat{E}_i \otimes \hat{E}_j$$

$$I_{11}^{P} = \int \rho(\underline{r})\left(x_2^2 + x_3^2\right) dV$$

Cross product of inertia



$$\underline{\underline{I}}^{P} = \int_V \rho(\underline{r})\left(|\underline{r}^{/P}|^2 \underline{\underline{1}} - \underline{r}^{/P} \otimes \underline{r}^{/P}\right) dV$$

$$\underline{r}^{/P} = \underline{r}^{/G} + \underline{r}^{G/P}$$

Moment of Inertia

I. **Parallel axes theorem**. Let CM be at G:

$$\mathbf{I}^P = \mathbf{I}^G + m\left(\left|\mathbf{r}^{G/P}\right|^2 \mathbf{1} - \mathbf{r}^{G/P} \otimes \mathbf{r}^{G/P}\right).$$

II. In CS $\{\mathscr{E}, P, \hat{\mathbf{E}}_i\}$, let $\mathbf{r}^{P/G} = d\hat{\mathbf{n}}^{GP} = dn_i^{GP}\hat{\mathbf{E}}_i$:

$$I_{kk}^P = I_{kk}^G + md^2 n_k^{GP} n_k^{GP} \quad (\text{no sum on } k),$$

$$I_{ij}^P = I_{ij}^G - md^2 n_i^{GP} n_j^{GP} \quad (i \neq j)\,.$$

III. **Application**. If a rigid body has a *fixed point* C , *i.e.* $\mathbf{v}^C = \mathbf{0}$, then

$$\mathbf{h}^C = \mathbf{I}^C \cdot \boldsymbol{\omega}^{\mathscr{B}}.$$

IV. **Perpendicular axes theorem**. If body is ***planar*** with normal $\hat{\mathbf{n}}$, then

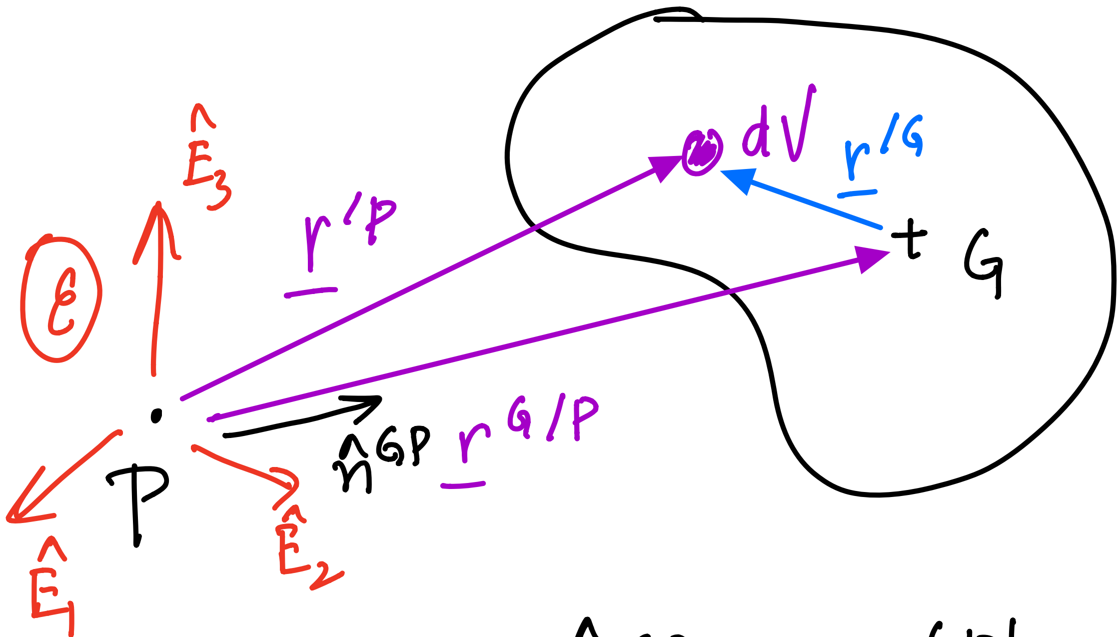
$$\mathbf{I}^G = I_n^G\, \hat{\mathbf{n}} \otimes \hat{\mathbf{n}} + I_{a_1}^G \hat{\mathbf{a}}_1 \otimes \hat{\mathbf{a}}_1 + I_{a_1}^G \hat{\mathbf{a}}_2 \otimes \hat{\mathbf{a}}_2\,,$$

with $I_n^G = I_{a_1}^G + I_{a_2}^G$, and $\left\{I_n^G, \hat{\mathbf{n}}\right\}$ and $\left\{I_{a_i}^G, \hat{\mathbf{a}}_i\right\}$ being the *principal pairs* of $\mathbf{I}^G$.

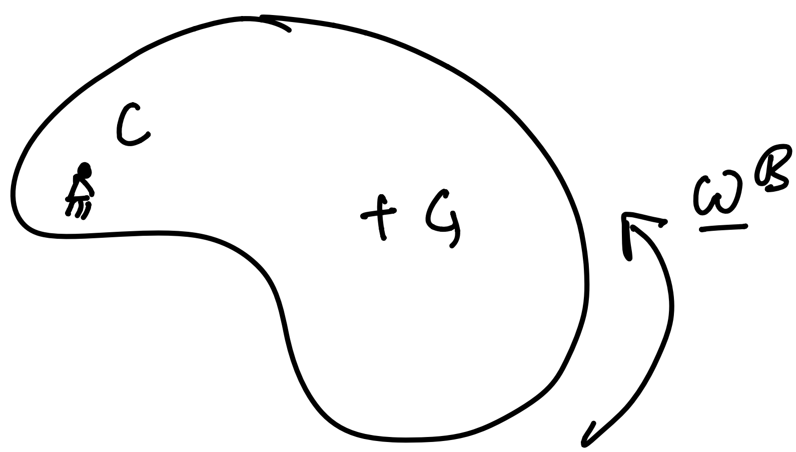
$$|\underline{r}^{/P}|^2 = |\underline{r}^{/G}|^2 + |\underline{r}^{G/P}|^2 + 2\,\underline{r}^{/G}\cdot\underline{r}^{G/P}$$

$$\underline{r}^{/P}\otimes\underline{r}^{/P} = \underline{r}^{/G}\otimes\underline{r}^{/G} + \underline{r}^{G/P}\otimes\underline{r}^{G/P} + \underline{r}^{/G}\otimes\underline{r}^{P/G} + \underline{r}^{P/G}\otimes\underline{r}^{/G}$$

$$\underline{\underline{I}}^{P} = \int_V \rho(\underline{r})\left(|\underline{r}^{/G}|^2\,\underline{\underline{1}} - \underline{r}^{/G}\otimes\underline{r}^{/G}\right)dV + \int_V \rho(\underline{r})\,dV\left(|\underline{r}^{G/P}|^2\,\underline{\underline{1}} - \underline{r}^{G/P}\otimes\underline{r}^{G/P}\right) + \cancel{\int_V \rho(\underline{r})\left(2\,\underline{r}^{/G}\cdot\underline{r}^{G/P}\,\underline{\underline{1}} + \underline{r}^{/G}\otimes\underline{r}^{G/P} + \underline{r}^{G/P}\otimes\underline{r}^{/G}\right)dV} \rightarrow 0$$



$$\underline{r}^{G/P} = d\hat{n}^{GP} \qquad |\underline{r}^{G/P}| = d$$



$$\underline{h}^{C} = \underline{r}^{G/C} \times m\underline{v}^{G} + \underline{\underline{I}}^{G} \cdot \underline{\omega}^{B}$$

$$\underline{v}^G = \cancel{\underline{v}^C}^{\,0} + \underline{\omega}^B \times \underline{r}^{G/C}$$

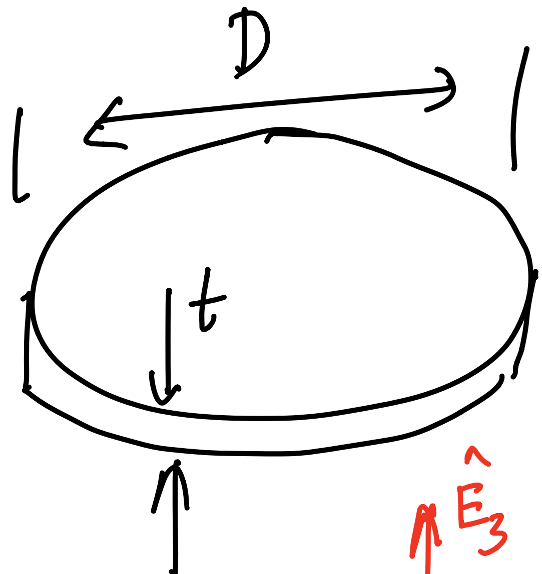
$$= \underline{\omega}^B \times \underline{r}^{G/C}$$

$$\underline{h}^G = m\,\underline{r}^{G/C} \times \left(\underline{\omega}^B \times \underline{r}^{G/C}\right) + \underline{\underline{I}}^G \cdot \underline{\omega}^B$$

$$\underline{r}^{G/C} \times \left(\underline{\omega}^B \times \underline{r}^{G/C}\right) = \left|\underline{r}^{G/C}\right|^2 \underline{\underline{1}} \cdot \underline{\omega}^B - \left(\underline{r}^{G/C} \otimes \underline{r}^{G/C}\right) \cdot \underline{\omega}^B$$

$$\underline{h}^C = \left\{ \underline{\underline{I}}^G + m\left(\left|\underline{r}^{G/C}\right|^2 \underline{\underline{1}} - \underline{r}^{G/C} \otimes \underline{r}^{G/C}\right)\right\} \cdot \underline{\omega}^B$$

$$= \underline{\underline{I}}^C \cdot \underline{\omega}^B$$



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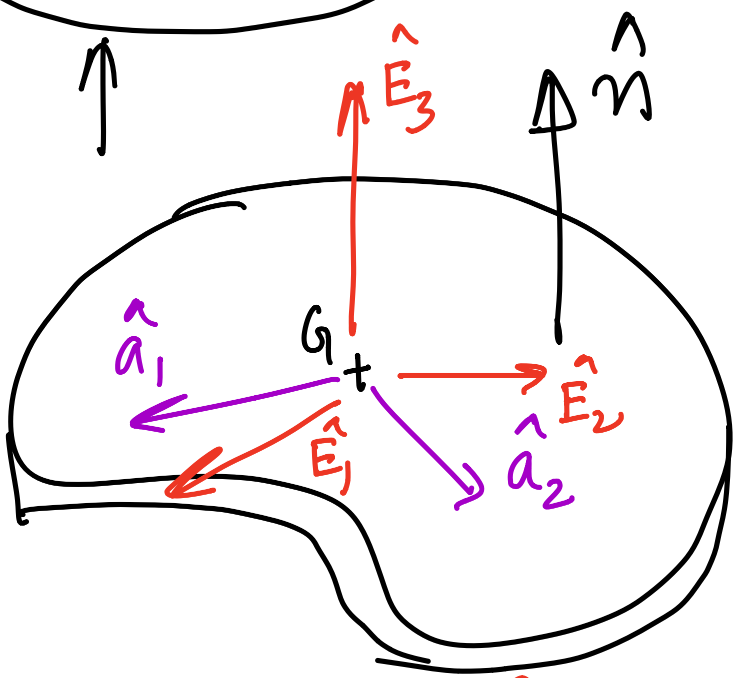


$\{\mathcal{E}, G, \hat{E}_i\}$

$[\underline{\underline{I}}^G]_{\mathcal{E}}$

$$I_{11}^G = \int \rho \left(x_2^2 + \cancelto{0}{x_3^2} \right) dV$$

$$I_{22}^G = \int \rho \left(\cancelto{0}{x_3^2} + x_1^2 \right) dV$$

$$I_{33}^G = \int \rho \left(x_1^2 + x_2^2 \right) dV$$

$$I_{33}^G = I_{11}^G + I_{22}^G$$

$$I_{13}^G = -\int \rho \, x_1 \cancelto{0}{x_3} \, dV$$

$$I_{23}^G = -\int \rho \, x_2 \cancelto{0}{x_3} \, dV$$

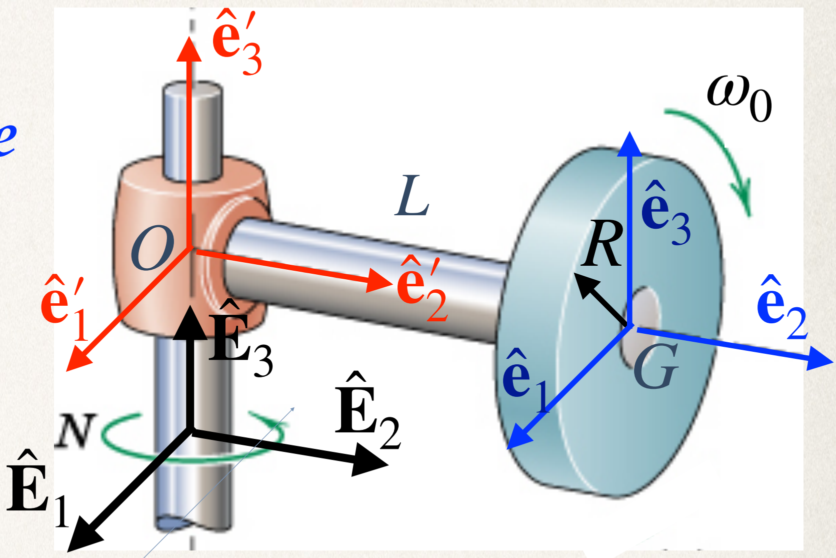
$$I_{12}^G = -\int \rho \, x_1 \, x_2 \, dV$$

$$\left[\underline{\underline{I}}^G\right]_E = \begin{bmatrix} I_{11}^G & I_{12}^G & 0 \\ I_{12}^G & I_{22}^G & 0 \\ 0 & 0 & I_{11}^G + I_{22}^G \end{bmatrix}$$

$$\left[\underline{\underline{I}}^G\right]_{\{\hat{a}_1 - \hat{a}_2 - \hat{E}_3\}} = \begin{bmatrix} I_{a_1}^G & 0 & 0 \\ 0 & I_{a_2}^G & 0 \\ 0 & 0 & I_{a_1}^G + I_{a_2}^G \end{bmatrix}$$

Example 1

Find the angular momentum of the system about G and O. The arm OG is massless.



I. $\{\mathscr{E}_0, \hat{\mathbf{E}}_i\} \xrightarrow{\mathsf{R}(\hat{\mathbf{E}}_3,\, \varphi_N)} \{\mathscr{E}', \hat{\mathbf{e}}_i'\} \xrightarrow{\mathsf{R}(\hat{\mathbf{e}}_2',\, \varphi_{\omega_0})} \{\mathscr{E}, \hat{\mathbf{e}}_i\}$

II. $\mathbf{h}^G_{sys} = \mathbf{h}^G_{disk} + \cancelto{0}{\mathbf{h}^G_{arm}} = \mathrm{I}^G_{disk} \cdot \boldsymbol{\omega}^{disk}$

III. $\left[\mathrm{I}^G_{disk}\right]_{\mathscr{E}} = \dfrac{mR^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

IV. $\mathbf{h}^G_{sys} = mR^2\left(-2\omega_0 \hat{\mathbf{E}}_2 + N\hat{\mathbf{E}}_3\right)/4$

V. $\mathbf{h}^O_{disk} = \mathbf{r}^{G/O} \times m\mathbf{v}^G + \mathbf{h}^G_{disk}$

VI. $\mathbf{h}^O_{sys} = -\dfrac{m\omega_0 R^2}{2}\hat{\mathbf{E}}_2 + \dfrac{mN\left(R^2 + 4L^2\right)}{4}\hat{\mathbf{E}}_3$

$$\underline{\omega}^{DISK} = \underline{\omega}_{\mathcal{E}/\mathcal{E}_0} = \underline{\omega}_{\mathcal{E}/\mathcal{E}'} + \underline{\omega}_{\mathcal{E}'/\mathcal{E}_0}$$

$$= -\omega_0 \hat{e}_2' + N\hat{E}_3$$

THIN DISK

mass/area, const.

$$I_{22}^{G} = \int_A \rho \, (x_1^2 + x_3^2) \, dA$$

$$= \int_0^{2\pi} \int_0^{R} \rho \, r^2 \, r \, dr \, d\theta$$

$$= 2\pi \rho \, R^4 / 4$$

$$= mR^2/2 \quad \text{as } m = \pi R^2 \cdot \rho$$

From symmetry $I^G_{11} = I^G_{33}$

$\perp$ axis theorem

$$I^G_{11} + I^G_{33} = I^G_{22}$$

$$\Rightarrow I^G_{11} = I^G_{33} = mR^2/4$$

$$I^G_{13} = -\int_A \rho\, x_1 x_3 \, dA = 0$$

BFCS $\mathcal{E} \iff$ Principal CS of $\underline{\underline{I}}^G$

$$\underline{h}^O_{sys} = \underline{h}^O_{DISK} + \cancelto{0}{\underline{h}^O_{arm}}$$

$$\underline{r}^{G/O} = L\hat{e}_2'$$

$$\underline{v}^G = \underline{v}^O + \underline{\omega}_{arm} \times r^{G/O}$$

$$= N\hat{E}_3 \times L\hat{e}_2'$$

$$= -NL\hat{E}_1$$

$$\underline{r}^{G/O} \times m v^G = mL\hat{e}_2' \times (-NL\hat{E}_1)$$

$$= mNL^2\hat{E}_3$$

$$\underline{\omega}^{sys} = N\hat{E}_3$$

$$\underline{\underline{I}}^G_{sys} = \underline{\underline{I}}^G_{DISK}$$

$$\underline{r}^{G/O} = L\hat{e}_2'$$

$$\left[\underline{r}^{G/O} \otimes \underline{r}^{G/O}\right]_{\mathcal{E}'} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & L^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

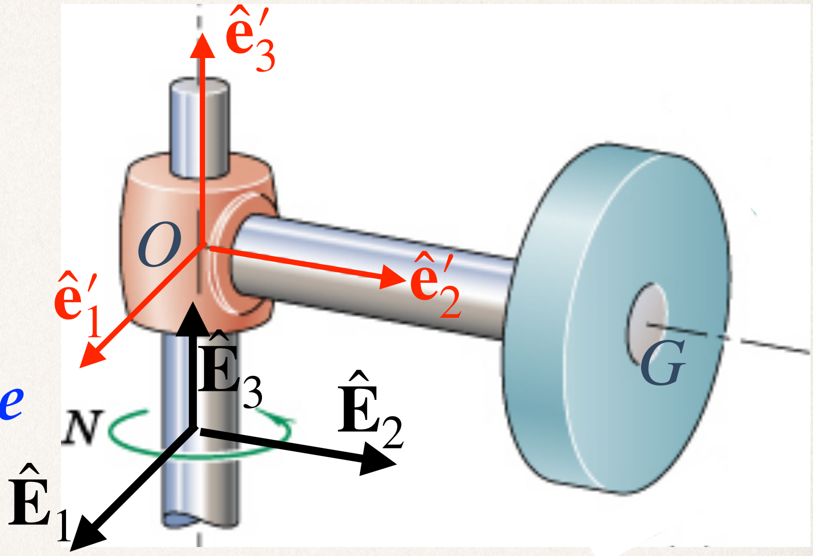
$$\left|\underline{r}^{G/O}\right|^2 \underline{\underline{1}} = \begin{bmatrix} L^2 & 0 & 0 \\ 0 & L^2 & 0 \\ 0 & 0 & L^2 \end{bmatrix}$$

$$\left[\underline{\underline{I}}^{O}_{sys}\right] = \left[\underline{\underline{I}}^{G}_{sys}\right] + \begin{bmatrix} L^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L^2 \end{bmatrix}$$

$$\underline{h}^{O}_{sys} = \frac{m\,(R^2 + 4L^2)N}{4}\,\hat{\underline{E}}_3$$

Example 2

The disc is now welded to the massless arm. Find the angular momentum of the system about G and O.



I. $\{\mathscr{E}_0, \hat{\mathbf{E}}_i\} \xrightarrow{\mathsf{R}(\hat{\mathbf{E}}_3, \varphi_N)} \{\mathscr{E}', \hat{\mathbf{e}}_i'\}$

II. $\mathbf{h}^G_{sys} = \mathbb{I}^G_{sys} \cdot \boldsymbol{\omega}^{sys} = mR^2 N \hat{\mathbf{E}}_3/4$

III. $\mathbf{h}^O_{sys} = \mathbb{I}^O_{sys} \cdot \boldsymbol{\omega}^{sys}$

IV. $\mathbb{I}^O = \mathbb{I}^G + m\left(\left|\mathbf{r}^{G/O}\right|^2 \mathbb{1} - \mathbf{r}^{G/O} \otimes \mathbf{r}^{G/O}\right)$

$$\left[\mathbb{I}^O_{disk}\right]_{\mathscr{E}'} = \frac{m}{4}\begin{pmatrix} R^2 + 4L^2 & 0 & 0 \\ 0 & 2R^2 & 0 \\ 0 & 0 & R^2 + 4L^2 \end{pmatrix}$$

V. $\mathbf{h}^O_{sys} = \dfrac{mN\left(R^2 + 4L^2\right)}{4}\hat{\mathbf{E}}_3$

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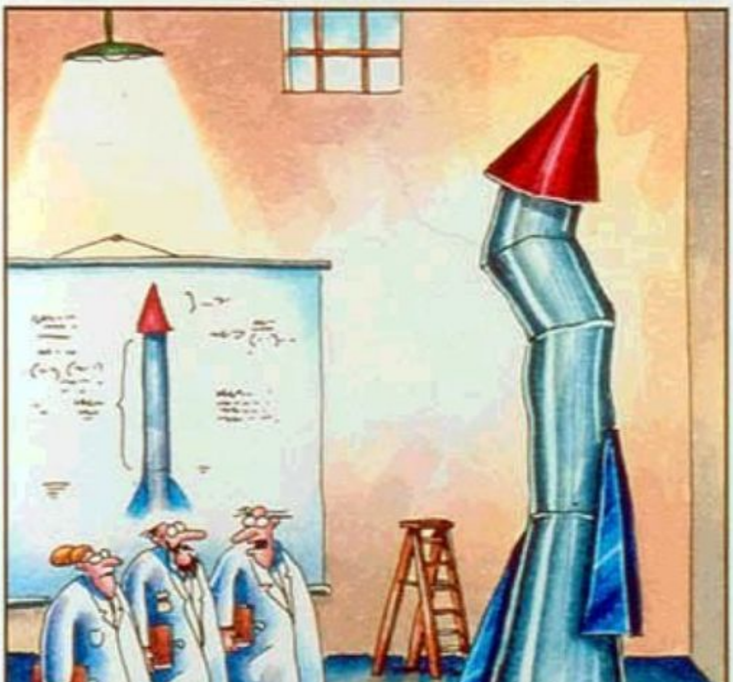

"It's time we face reality, my friends... We're not exactly rocket scientists."