M accepts $x \in \Sigma^*$ if $(s, \vdash x \sqcup^{\omega}, 0) \xrightarrow[M]{*} (t, y, n)$

$$L(M) = \{x \in \Sigma^* \mid M \text{ accepts } x\}$$

M rejects $x \in \Sigma^*$ if $(s, \vdash x \sqcup^{\omega}, 0) \xrightarrow[M]{*} (r, y, n)$

M **halts** on input $x \in \Sigma^*$ if it either accepts x or rejects x.

M **loops** on input $x \in \Sigma^*$ if M neither accepts nor rejects x.

Total TM. A Turing machine that halts on all input is called a total TM.

$A \subseteq \Sigma^*$ is **recursive** if $A = L(M)$ for a total TM M.

$A \subseteq \Sigma^*$ is **recursively enumerable (r.e)** if $A = L(M)$ for some TM M (need not be a total TM).

Ex. $A = \{a^n b^n c^n \mid n \geq 0\}$ is recursive.

Example. $A = \{ww \mid w \in \{a,b\}^*\}$ — not a CFL

A is recursive since we can define a total TM M s.t $L(M) = A$.

Working of M on input x.

- Scan (left → right) till first ⊔-symbol). replace ⊔ with ⊣ and check that number of symbols in x is even.
- In each pass from right to left, it marks the first unmarked a, b with $\acute{a}, \acute{b}$
- In each pass from left to right, it marks the first unmarked a, b with $\grave{a}, \grave{b}$

Ex. Tape Content $\Gamma = \{a, b, \vdash, \sqcup, \dashv, \grave{a}, \grave{b}, \acute{a}, \acute{b}\}$

$\vdash \; a \; a \; b \; b \; b \; a \; a \; b \; b \; b \; \sqcup \; \sqcup \; \sqcup \; \sqcup$

$\vdash \; \grave{a} \; \grave{a} \; \grave{b} \; \grave{b} \; \grave{b} \; \acute{a} \; \acute{a} \; \acute{b} \; \acute{b} \; \acute{b} \; \dashv \; \sqcup \; \sqcup \; \sqcup$

$\vdash \; \sqcup \; \sqcup \; \grave{b} \; \grave{b} \; \grave{b} \; \sqcup \; \sqcup \; \acute{b} \; \acute{b} \; \acute{b} \; \dashv \; \sqcup \; \sqcup \; \sqcup$

Recursive and recursively enumerable (r.e) Sets.

- Every recursive set is r.e
- Not every TM is equivalent to a total TM.

Recursive sets are closed under complementation.

Suppose $A \subseteq \Sigma^*$ is recursive. Then there exists a total TM M s.t $L(M) = A$.

- Switch the accept and reject states.
Resulting M': $L(M') = \Sigma^* - A$.

This construction does not work for r.e. sets.

M' will still loop on the strings that M loops on.

Such strings are not accepted or rejected by either machines.

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Claim. if both A and $\bar{A}$ are r.e then A is recursive

Decidable / Semi-decidable.

A property P of strings is decidable if the set of all strings having property P is a recursive set.

P is decidable iff $\{x \mid P(x)\}$ is recursive.

$A \subseteq \Sigma^*$ is recursive iff $x \in A$ is decidable

P is semidecidable iff $\{x \mid P(x)\}$ is r.e

$A \subseteq \Sigma^*$ is r.e iff $x \in A$ is semidecidable.

Example of a property: String x is of the form ww.