Lecture 22

NPDA - a few remarks.

- $\perp$ is treated as any other stack symbol
  It can be pushed and popped at any time.
  It is useful to define the start configuration.
  It need not always stay at the bottom of the stack.

- Transition $((p, a, A), (q, \beta))$ does not apply unless A is on top of the stack.
  If stack is empty - no transition applies.
  PDA is stuck

- To accept by empty stack, everything must be popped and nothing pushed back.
  The stack must be empty in a configuration.

NPDAs and CFGs are equivalent is expressive power.

1. Given a CFG G we can construct an NPDA M s.t $L(G) = L(M)$.

2. Given an NPDA M we can construct a CFG G s.t $L(G) = L(M)$.

CFG $\Rightarrow$ NPDA.

Suppose $G = (N, \Sigma, P, S)$. To construct NPDA M s.t $L(M) = L(G)$.

Assume that all productions of G are of the form

$$\underbrace{A \to c B_1 B_2 \cdots B_k}_{\text{Greibach Normal Form.}} \quad , c \in \Sigma \cup \{\epsilon\}, \ k \geq 0.$$

Construct $M = (\{q\}, \Sigma, N, \delta, q, S, \phi)$

M has one state and accepts with empty stack

Σ - set of terminal symbols in G : input alphabet in M

N - Set of nonterminals of G is the stack alphabet of M.

S - Start symbol of G, the initial stack symbol of M.

Definition of δ:

For each production $A \to c B_1 B_2 \cdots B_k$, add the following to δ.

$$((q, c, A), (q, B_1 B_2 \cdots B_k))$$

$M = (\{q\}, \Sigma, N, \delta, q, S, \phi)$

δ: For each $A \to cB_1 B_2 \ldots B_k$, $((q, c, A), (q, B_1 B_2 \ldots B_k)) \in \delta$.

Example. Balanced Parentheses.

Production Rules in G	Transitions in M.
1. $S \to [BS$	$((q, [, S), (q, BS))$.
2. $S \to [B$	$((q, [, S), (q, B))$
3. $S \to [SB$	$((q, [, S), (q, SB))$
4. $S \to [SBS$	$((q, [, S), (q, SBS))$
5. $B \to]$	$((q,], B), (q, \epsilon))$

Leftmost derivation - productions are always applied to the leftmost nonterminal.

To show. Leftmost derivation of x in G corresponds to an accepting computation of M on input x.

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2. S → [B	((q, [, S), (q, B))
3. S → [SB	((q, [, S), (q, SB))
4. S → [SBS	((q, [, S), (q, SBS))
5. B →]	((q,], B), (q, ε))

Leftmost derivation - productions are always applied to the leftmost nonterminal.

To show. Leftmost derivation of x in G corresponds to an accepting computation of M on input x.

Example: Input $x =$ [[[]][]]

Rule		Configurations of M
	S	(q, [[[]][]], S)
3	[SB	(q, [[]][]], SB)
4	[[SBSB	(q, []][]], SBSB)
2	[[[BBSB	(q,]][]], BBSB)
5	[[[]BSB	(q,][]], BSB)
5	[[[]]SB	(q, []], SB)
2	[[[]][BB	(q,]], BB)
5	[[[]][]B	(q,], B)
5	[[[]][]]	(q, ε, ε).

Lemma 1. For any $z, y \in \Sigma^*$, $\gamma \in N^*$ and $A \in N$,

$A \xrightarrow[G]{n} z\gamma$ by a leftmost derivation iff $(q, zy, A) \xrightarrow[M]{n} (q, y, \gamma)$

Proof. Induction on n.

Base case, $n=0$ - easy.

Induction step.

Suppose $A \xrightarrow[G]{n+1} z\gamma$ using a leftmost derivation.

$c \in \Sigma \cup \{\epsilon\}, \beta \in N^*$

Suppose $B \to c\beta$ was the last production applied.

$A \xrightarrow[G]{n} uB\alpha \xrightarrow[G]{1} uc\beta\alpha = z\gamma$. / $z = uc$ and $\gamma = \beta\alpha$

By induction hypothesis, $(q, ucy, A) \xrightarrow[M]{n} (q, cy, B\alpha)$.

By definition of M, $((q, c, B), (q, \beta)) \in \delta$.

therefore, $(q, cy, B\alpha) \xrightarrow[M]{1} (q, y, \beta\alpha)$

Thus we have

$(q, zy, A) = (q, ucy, A) \xrightarrow[M]{n+1} (q, y, \beta\alpha) = (q, y, \gamma)$.

Conversly, Suppose $(q, zy, A) \xrightarrow[M]{n+1} (q, y, \gamma)$

let $((q, c, B), (q, \beta)) \in \delta$ be the last transition taken by M.

Then $z = uc$ for some $u \in \Sigma^*$, $\gamma = \beta\alpha$ for some $\alpha \in \Gamma^*$ and

$$(q, ucy, A) \xrightarrow[M]{n} (q, cy, B\alpha) \xrightarrow[M]{1} (q, y, \beta\alpha)$$

By induction hypothesis $A \xrightarrow[G]{n} uB\alpha$ by a leftmost derivation in G.

By definition of δ in M, $B \to c\beta$ is a production of G.

Then, $A \xrightarrow[G]{n} uB\alpha \xrightarrow[G]{1} uc\beta\alpha = z\gamma$ by a leftmost derivation.

Theorem. $L(G) = L(M)$.

Proof.

$x \in L(G)$ iff $S \xrightarrow[G]{*} x$ by a leftmost derivation [Defn of $L(G)$]

iff $(q, x, S) \xrightarrow[M]{*} (q, \epsilon, \epsilon)$ [Lemma 1]

iff $x \in L(M)$ [Definition of $L(M)$]