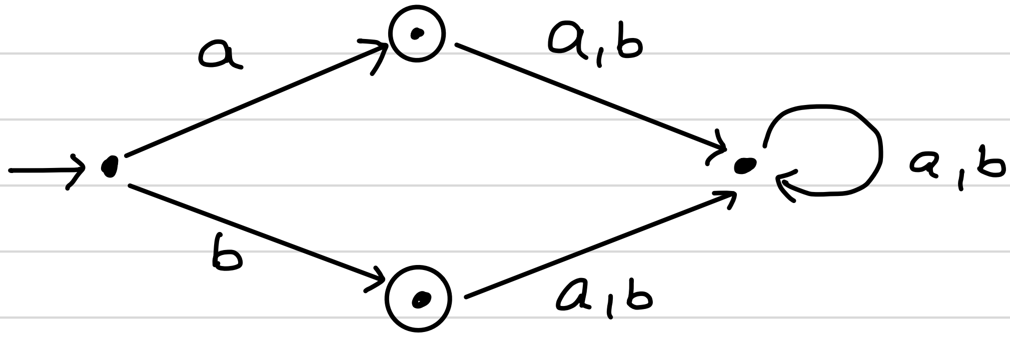
State Minimization - DFA.

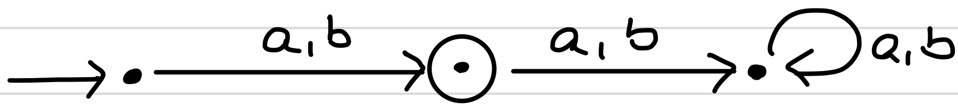
Example 1.

M_1 :



$L(M_1) = \{a, b\}$

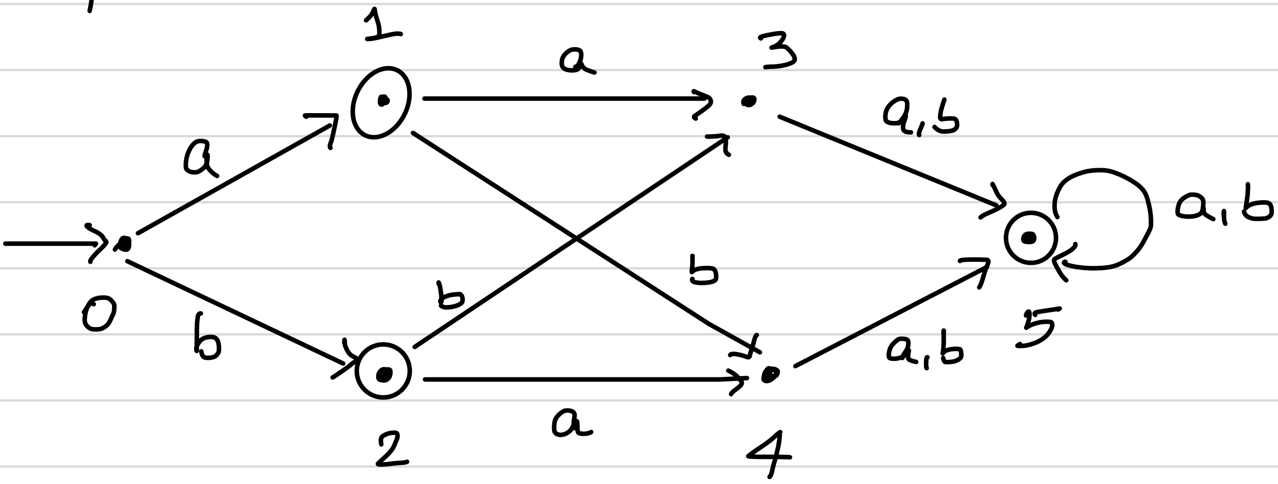
M_2 :



$L(M_1) = L(M_2)$

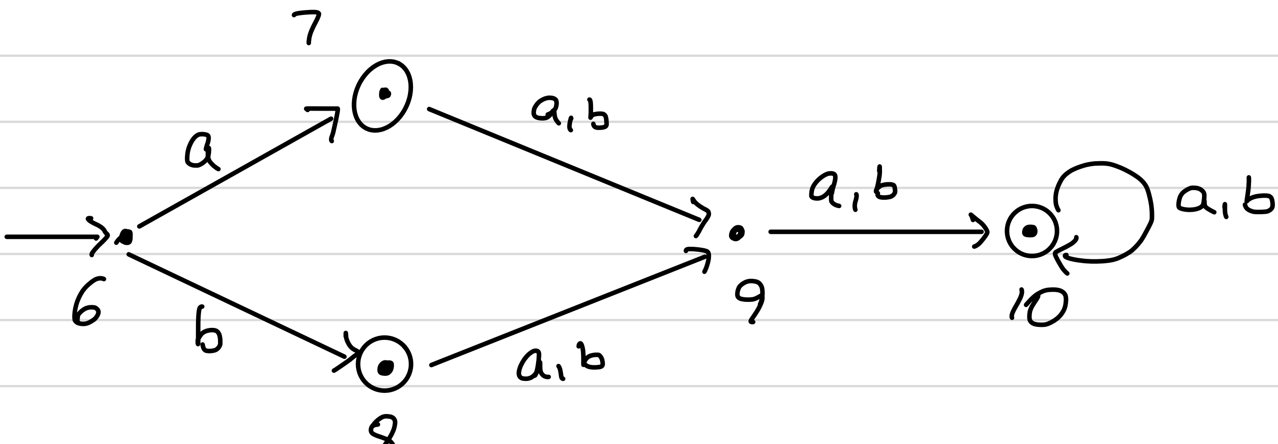
Exemple 2.

M_1

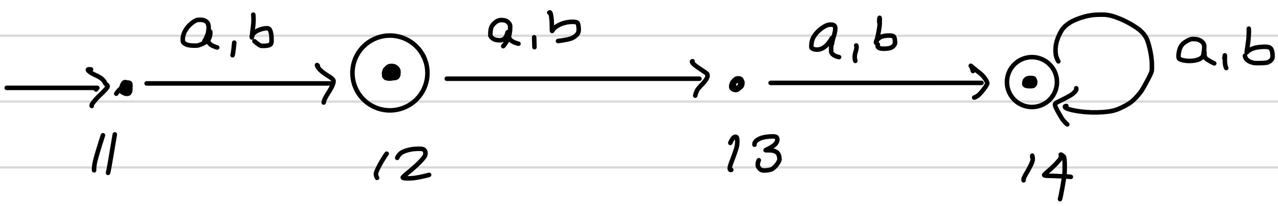


$L(M_1) = \{a,b\} \cup \{\text{strings of length at least } 3\}.$

M_2

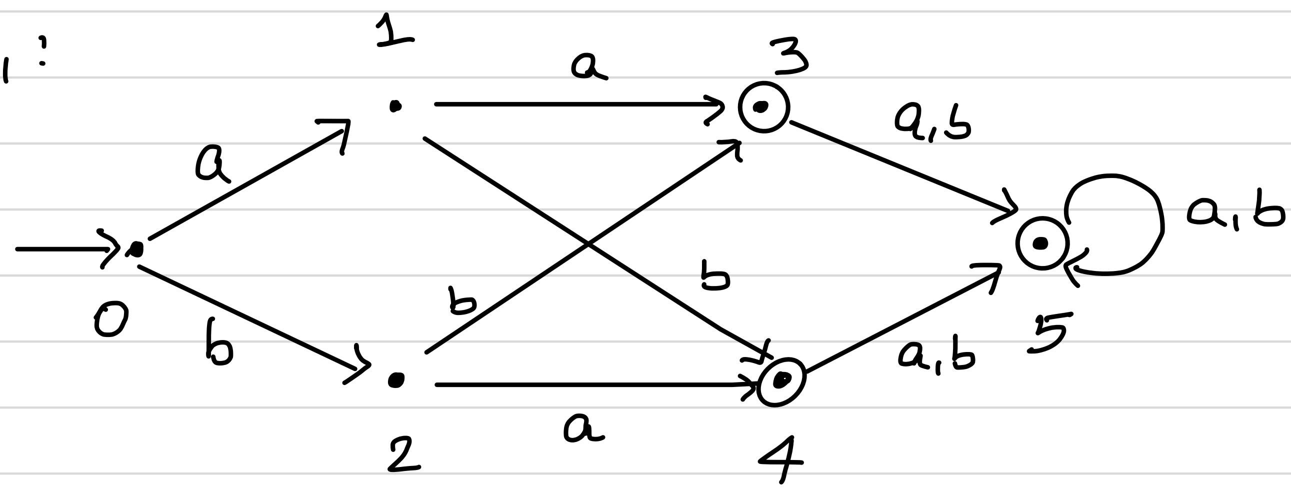


M_3:



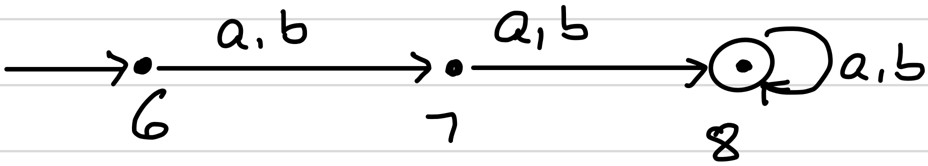
Example 3.

M_1:



$L(M_1) = \{\text{strings of length at least } 2\}$

M_2



For a DFA $M=(Q,\Sigma,\delta,s,F)$ the state minimization process consists of

1. Remove inaccessible states.
2. Collapse "equivalent" states.

- We do not want to collapse an final state p and a non-final state q.

  if $p=\hat{\delta}(s,x)$ and $q=\hat{\delta}(s,y)$ then

  $x \in L(M)$ and $y \notin L(M)$.

- if we collapse states p & q then we should also collapse $\delta(p,a)$ and $\delta(q,a)$ Otherwise the resulting automaton will not be deterministic.

Definition of an equivalence relation on Q.

$p \approx q$ iff $\forall x \in \Sigma^* \, (\hat{\delta}(p,x) \in F$ iff $\hat{\delta}(q,x) \in F)$

Claim. $\approx$ is an equivalence relation.

$\approx$ partitions Q into a set of equivalence classes.

$[p] = \{q \mid q \approx p\}$.

Easy to verify that $p \approx q$ iff $[p] = [q]$.

Quotient Automata.

DFA $M = (Q, \Sigma, \delta, s, F)$ $\quad M/\approx \; = (Q', \Sigma, \delta', s', F')$

$Q' = \{[p] \mid p \in Q\}$ - states of $M/\approx$ are the $\approx$-equivalence classes.

$\delta'([p], a) = [\delta(p, a)]$

$s' = [s]$, $F' = \{[p] \mid p \in F\}$

To show. δ' is well defined.

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$s' = [s]$, $F = \{[p] \mid p \in F\}$

To show. δ' is well defined.

Lemma 1. if $p \approx q$ then $\delta(p,a) \approx \delta(q,a)$

That is, if $[p] = [q]$ then $[\delta(p,a)] = [\delta(q,a)]$

Proof. Suppose $p \approx q$. For $a \in \Sigma$ and $y \in \Sigma^*$

$\hat{\delta}(\delta(p,a), y) \in F$ iff $\hat{\delta}(p, ay) \in F$

iff $\hat{\delta}(q, ay) \in F$ [Since $p \approx q$]

iff $\hat{\delta}(\delta(q,a), y) \in F$.

Since the above holds for all $y \in \Sigma^*$, $\delta(p,a) \approx \delta(q,a)$ [By defn of $\approx$].

DFA $M = (Q, \Sigma, \delta, s, F)$ $\quad M/\approx \, = (Q', \Sigma, s', \delta', F')$

Lemma 2. $p \in F$ iff $[p] \in F'$.

Proof. $\Rightarrow$ Follows from the definition of F'

$\Leftarrow$ if $p \approx q$ and $p \in F$ then $q \in F$. That is,

every $\approx$-equivalence class is either a subset of F or disjoint from F.

Follows by taking $x = \epsilon$ in the definition of $p \approx q$.

Lemma 3. For all $x \in \Sigma^*$, $\hat{\delta}'([p], x) = [\hat{\delta}(p, x)]$

Proof. By induction on $|x|$.

Base case: $x = \epsilon$. $\hat{\delta}'([p], \epsilon) = [p] = [\hat{\delta}(p, \epsilon)]$

Induction Step.

$$\begin{aligned}
\hat{\delta}'([p], xa) &= \delta'(\hat{\delta}'([p], x), a) && \text{[defn of } \hat{\delta}'\text{]} \\
&= \delta'([\hat{\delta}(p, x)], a) && \text{[Induction Hypothesis]} \\
&= [\delta(\hat{\delta}(p, x), a)] && \text{[Defn. of } \delta'\text{]} \\
&= [\hat{\delta}(p, xa)] && \text{[Defn of } \hat{\delta}\text{]}
\end{aligned}$$

Lemma 3. For all $x \in \Sigma^*$, $\hat{\delta}'([p], x) = [\hat{\delta}(p, x)]$

Lemma 2. $p \in F$ iff $[p] \in F'$.

Theorem. $L(M/\approx) = L(M)$.

Proof. For $x \in \Sigma^*$,

$x \in L(M/\approx)$ iff $\hat{\delta}'(s', x) \in F'$

iff $\hat{\delta}'([s], x) \in F'$ [Defn. of s']

iff $[\hat{\delta}(s, x)] \in F'$ [Lemma 3]

iff $\hat{\delta}(s, x) \in F$ [Lemma 2]

iff $x \in L(M)$ [Defn. of acceptance]

What if you do the quotient construction again on $M/\approx$?

$[p] \sim [q]$ iff $\forall x \in \Sigma^* \left(\hat{\delta}'([p], x) \in F' \text{ iff } \hat{\delta}'([q], x) \in F' \right)$

Use $\sim$ to denote the equivalenc relation on Q' to distinguish it from the relation $\approx$ on Q.

$[p] \sim [q] \Rightarrow \forall x \left(\hat{\delta}'([p], x) \in F' \text{ iff } \hat{\delta}'([q], x) \in F' \right)$

[Defn. of $\sim$]

$\Rightarrow \forall x \left([\hat{\delta}(p, x)] \in F' \text{ iff } [\hat{\delta}(q, x)] \in F' \right)$

[Lemma 3]

$\Rightarrow \forall x \left(\hat{\delta}(p, x) \in F \text{ iff } \hat{\delta}(q, x) \in F \right)$

[Lemma 2]

$\Rightarrow p \approx q \Rightarrow [p] = [q]$

$\therefore$ Two equivalent states of $M/\approx$ are equal and $\sim \subseteq Q' \times Q'$ is the identity relation.