Decision Problem. Function with a one-bit Output: "Yes" or "No"

Set of possible inputs to a decision problem

Set of finite length strings over some fixed finite alphabet.

Alphabet - finite set - denoted by Σ

Ex. $\{0,1,2,\dots,9\}$ - decimal numbers.

$\{0,1\}$ -

$a, b \in \Sigma$

Strings over Σ - finite length sequence of elements of Σ.

Ex. $\Sigma = \{a,b\}$ abbaa - String of length 5

x, y, z - denote Strings.

length of a string x - $|x|$ - number of Symbols in x.

Unique string of length 0 - null string.

ϵ [epsilon].

$|\epsilon| = 0$.

$a \in \Sigma$, a^n - string of a's of length n.

$a^0 = \epsilon$ $\quad a^{n+1} = a^n a$

Set of all strings over an alphabet Σ - denoted by Σ^*.

Ex. $\{a, b\}^* = \{\epsilon, a, b, aa, ab, ba, bb, aaa, \ldots\}$.

$\{a\}^* = \{\epsilon, a, aa, aaa, \ldots\}$.

$= \{a^n \mid n \geq 0\}$

$\phi^* = \{\epsilon\}$

String and sets are not the same.

$\{a, b\} = \{b, a\}$ / $ab \neq ba$

$\{a, a, b\} = \{a, b\}$ / $aab \neq ab$

ϕ - empty set

ϵ - null string.

$\{\epsilon\}$ - Set with one element - null string.

Operations on Strings

Concatenation. 2 string x & y and creates a new string xy

Note. $xy \neq yx$ are in general different

- Concatenation is associative $(xy)z = x(yz)$
- null string ϵ is identity for concatenation

$$\epsilon x = x\epsilon = x$$

- $|xy| = |x| + |y|$

Set Σ^* with concatenation as a binary operator and ϵ as identity is a monoid.

$x \in \Sigma^*$, $\quad x^n$

$x^0 = \epsilon$, $\quad x^{n+1} = x^n x$

Operations on sets. : subsets of Σ^*

A, B - sets of strings.

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

Complement $\bar{A} = \{x \in \Sigma^* \mid x \notin A\}$.

Set concatenation

$AB = \{xy \mid x \in A \text{ and } y \in B\}$.

$\{a, ab\}\{b, ba\} = \{ab, aba, abb, abba\}$

Note: In general AB and BA are different.

A^n - inductively

$A^0 = \{\epsilon\} \qquad A^{n+1} = A A^n$

$\{ab, aab\}^0 = \{\epsilon\}$
$\{ab, aab\}^1 = \{ab, aab\}$.
$\{ab, aab\}^2 = \{abab, abaab, aabab, aabaab\}$.

$\{a,b\}^n = \{x \in \{a,b\}^* \mid |x| = n\}$.

$A^* = \bigcup_{n \geq 0} A^n = A^0 \cup A^1 \cup A^2 \cup \ldots$

Σ

$A^* = \{x_1 x_2, \ldots x_n \mid n \geq 0 \text{ and } x_i \in A, 1 \leq i \leq n\}$

$n = 0$ $\quad\epsilon$ is in A^* for any A.

$A^+ = AA^* = \bigcup_{n \geq 1} A^n$

Properties of Set operations

$$\left.\begin{array}{l}(A \cup B) \cup C = A \cup (B \cup C) \\ (A \cap B) \cap C = A \cap (B \cap C)\end{array}\right\}$$

$(AB)C = A(BC)$

Concatenation is not Commutative

$\{\epsilon\}$ - Identity for Concatenation.

$\{\epsilon\}A = A\{\epsilon\} = A$

$$A\phi = \phi A = \phi$$

* satisfies the following properties:

$$A^* A^* = A^*$$

$$A^{**} = A^*$$

$$\phi^* = \{\epsilon\}$$