Context Free Grammar: $G = (N, \Sigma, P, S)$

Language generated by G.

$$L(G) = \{x \in \Sigma^* \mid S \xrightarrow[G]{*} x\}$$

$B \subseteq \Sigma^*$ is a context free language (CFL) if

$B = L(G)$ for some CFG G.

Balanced Parenthesis

G: $S \to [S] \mid SS \mid \epsilon$ $\qquad \Sigma = \{[,]\}$

$N = \{S\}$

Claim. $L(G) = \{x \in \{[,]\}^* \mid x$ is balanced$\}$.

$L(x) = \#[(x)$: Number of [in string x.

$R(x) = \#](x)$: "] "

$x \in \Sigma^*$ is balanced iff

1. $L(x) = R(x)$

2. $\forall$ prefix y of x, $L(y) \geq R(y)$

x is balanced iff
1. $L(x) = R(x)$
2. $L(y) \geq R(y)$ $\forall$ prefix y of x.

$G: S \to [S] \mid SS \mid \epsilon$

Theorem. $L(G) = \{ x \in \{[,]\}^* \mid x$ satisfies ① & ② $\}$.

if $S \xrightarrow[G]{*} x$ then x satisfies 1 & 2.

Induction on the length of derivation [of x from S]

For any $\alpha \in (N \cup \Sigma)^*$, if $S \xrightarrow[G]{*} \alpha$ then α satisfies (1) & (2).

Base Case: if $S \xrightarrow[G]{0} \alpha$ then $\alpha = S$

Induction Step: Suppose $S \xrightarrow[G]{n+1} \alpha$ / $S \xrightarrow[G]{n} \beta \xrightarrow[G]{1} \alpha$

By IH, β satisfies (1) & (2)

$S \to \epsilon$, $S \to SS$ / $\exists \beta_1, \beta_2 \in (N \cup \Sigma)^*$ s.t.

$\beta = \beta_1 S \beta_2$ $\quad\quad$ $\alpha = \begin{cases} \beta_1 \beta_2 & \text{if } S \to \epsilon \\ \beta_1 SS \beta_2 & \text{if } S \to SS \end{cases}$

if $S \to [S]$. $\exists\ \beta_1, \beta_2 \in (N \cup \Sigma)^*$, $\beta = \beta_1 S \beta_2$

$\alpha = \beta_1 [S] \beta_2$.

By IH, β satisfies (1) & (2)

$$L(\alpha) = L(\beta) + 1 = R(\beta) + 1 = R(\alpha).$$

To show: α satisfies (2). Let γ be an arbitrary prefix of α.

To show: $L(\gamma) \geq R(\gamma)$

- γ is a prefix of $\beta_1 \Rightarrow \gamma$ is a prefix of β.

- γ is a prefix of $\beta_1 [S$ but not of β_1

$$L(\gamma) = L(\beta_1) + 1 \geq R(\beta_1) + 1 \quad [\text{IH, Since } \beta_1 \text{ is a prefix of } \beta].$$

$$> R(\beta_1) = R(\gamma)$$

- $\gamma = \beta_1 [S] \delta$ - where δ is a prefix of β_2.

$$L(\gamma) = L(\beta_1 S \delta) + 1$$

$$\geq R(\beta_1 S \delta) + 1 \qquad [IH].$$

$$= R(\gamma).$$

Thus in all the cases we have shown that $L(\gamma) \geq R(\gamma)$.

if x satisfies (1) & (2) then $S \xrightarrow[G]{*} x$. / Induction on $|x|$.

Base Case $|x| = 0$

Induction Step: if $|x| > 0$ – consider 2 cases.

a. $\exists$ a proper prefix y of x satisfying (1) & (2).
$\hookrightarrow (0 < |y| < |x|)$.

b. no such prefix exists.

a. $x = yz$ for some z, $0 < |z| < |x|$.
Claim. z satisfies (1) & (2).

$L(z) = L(x) - L(y) = R(x) - R(y) = R(z)$.

For any prefix w of z $\quad L(w) = L(yw) - L(y)$

$\geq R(yw) - R(y)$ [Since yw is a prefix of x and $L(y) = R(y)$.

$= R(w)$.

By induction hypothesis $S \xrightarrow[G]{*} y$, $S \xrightarrow[G]{*} z$

$$S \xrightarrow[G]{1} SS \xrightarrow[G]{*} yS \xrightarrow[G]{*} yz = x.$$

Case b. $x = [z]$ for some z.

Claim: z satisfies (1) & (2).

$L(z) = L(x) - 1 = R(x) - 1 = R(z)$.

$\forall$ non-null prefixes u of z,

$L(u) - R(u) = L([u) - 1 - R([u) \geq 0$

Since $L([u) - R([u) \geq 1$.

By induction hypothesis $S \xrightarrow[G]{*} z$

$S \xrightarrow[G]{1} [S] \xrightarrow[G]{*} [z] = x$.