DFA $M = (Q, \Sigma, \delta, s, F)$

$\delta : Q \times \Sigma \to Q$, $s \in Q$ $F \subseteq Q$.

$\hat{\delta} : Q \times \Sigma^* \to Q$

$x \in \Sigma^*$ is accepted by M if $\hat{\delta}(s, x) \in F$

" rejected " $\hat{\delta}(s, x) \notin F$.

$$L(M) = \{ x \in \Sigma^* \mid \hat{\delta}(s, x) \in F \}.$$

$A \subseteq \Sigma^*$ is regular if $A = L(M)$ for some DFA M.

$A, B \subseteq \Sigma^*$.

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$
$$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$$

$$\overline{A} = \{ x \in \Sigma^* \mid x \notin A \}$$

Question. Are regular sets closed under intersection?

if A and B which are regular then is $A \cap B$ regular?

For $A, B \subseteq \Sigma^*$, if A and B are regular then $A \cap B$ is regular.

Construct M_3 s.t $L(M_3) = A \cap B$.

$\exists\, M_1$ s.t $L(M_1) = A$.
$\exists\, M_2$ s.t $L(M_2) = B$.
To show.

M_3 runs over string in Σ^*.

$x \in \Sigma^*$, $x \in L(M_3)$

$M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1)$ $M_2 = (Q_2, \Sigma, \delta_2, s_2, F_2)$

$M_3 = (Q_3, \Sigma, \delta_3, s_3, F_3)$.

$Q_3 = Q_1 \times Q_2 = \{(q_1, q_2) \mid q_1 \in Q_1 \text{ and } q_2 \in Q_2\}$.

$s_3 = (s_1, s_2)$ $F_3 = F_1 \times F_2 = \{(q_1, q_2) \mid q_1 \in F_1 \text{ and } q_2 \in F_2\}$.

$\delta_3 : Q_3 \times \Sigma \to Q_3$

$$\delta_3(\underbrace{(q_1, q_2)}_{\in Q_3}, a) = \underbrace{(\overbrace{\delta_1(q_1, a)}^{\in Q_1}, \overbrace{\delta_2(q_2, a)}^{\in Q_2})}_{\in Q_3}$$

M_3 - product of M_1 & M_2.

$L(M_3) = A \cap B$ - To prove.

$L(M_3) = L(M_1) \cap L(M_2)$ - Theorem.

Lemma 1. For all $x \in \Sigma^*$,

$$\hat{\delta}_3(\underbrace{(q_1, q_2)}_{\in Q_3}, x) = \underbrace{(\overbrace{\hat{\delta}_1(q_1, x)}^{\in Q_1}, \overbrace{\hat{\delta}_2(q_2, x)}^{\in Q_2})}_{\in Q_3}$$

Proof. Induction on $|x|$.

Base Case $x = \epsilon$; $\hat{\delta}_3((q_1, q_2), \epsilon) = (q_1, q_2) =$
$(\hat{\delta}_1(q_1, \epsilon), \hat{\delta}_2(q_2, \epsilon))$

Induction step.

$$\hat{\delta}_3((q_1, q_2), xa) \underbrace{= (\hat{\delta}_1(q_1, xa), \hat{\delta}_2(q_2, xa))}_{\text{To prove}}$$

$= \delta_3(\hat{\delta}_3((q_1, q_2), x), a)$ - definition of $\hat{\delta}_3$

$= \delta_3((\hat{\delta}_1(q_1, x), \hat{\delta}_2(q_2, x)), a)$ - Induction hypothesis

$= (\delta_1(\hat{\delta}_1(q_1, x), a), \delta_2(\hat{\delta}_2(q_2, x), a))$

Definition of δ_3.

$= (\hat{\delta}_1(q_1, xa), \hat{\delta}_2(q_2, xa))$ Definition of $\hat{\delta}_1$ and $\hat{\delta}_2$.

Theorem. $L(M_3) = L(M_1) \cap L(M_2)$.

Proof. For all $x \in \Sigma^*$

$$x \in L(M_3) \iff \hat{\delta}_3(s_3, x) \in F_3 \quad \text{[acceptance.}$$

$$\iff \hat{\delta}_3((s_1, s_2), x) \in F_1 \times F_2$$

$$\iff (\hat{\delta}_1(s_1, x), \hat{\delta}_2(s_2, x)) \in F_1 \times F_2$$

[Lemma 1]

$$\iff \hat{\delta}_1(s_1, x) \in F_1 \text{ and } \hat{\delta}_2(s_2, x) \in F_2$$

[definition of set product]

$$\iff x \in L(M_1) \text{ and } x \in L(M_2). \quad \text{[Acceptance]}$$

$$\iff x \in L(M_1) \cap L(M_2) \quad \text{[defn of intersection]}$$

$x \in L(M_3)$ iff $x \in L(M_1) \cap L(M_2)$.

Question. Are regular sets closed under Complementation?

$A \subseteq \Sigma^*$. if $\underbrace{A \text{ is regular}}_{\exists M \text{ s.t } L(M) = A}$ then is $\underbrace{\bar{A} \text{ regular}}_{\exists \bar{M} \text{ s.t } L(\bar{M}) = \bar{A}}$? Yes

Interchange the set of accept states and non-accept states.

Question. if $A, B \subseteq \Sigma^*$ are regular then is $A \cup B$ regular? Yes

$$A \cup B = \overline{(\bar{A} \cap \bar{B})}$$

$A \to M_1$, $B \to M_2$ } Construct M_3 s.t $L(M_3) = L(M_1) \cup L(M_2)$

Intersection $F_3 = F_1 \times F_2$

Union $F_3 = (F_1 \times Q_2) \cup (Q_1 \times F_2)$

$$F_3 = \{ (q_1, q_2) \mid q_1 \in F_1 \text{ or } q_2 \in F_2 \}.$$

$AB = \{ xy \mid x \in A \text{ and } y \in B \}$

$A^* = A^0 \cup A^1 \cup A^2 \cup \cdots$

} Are regular sets Closed under Concatenation and $*$?