

Limitations of Finite Automata.

Canonical example

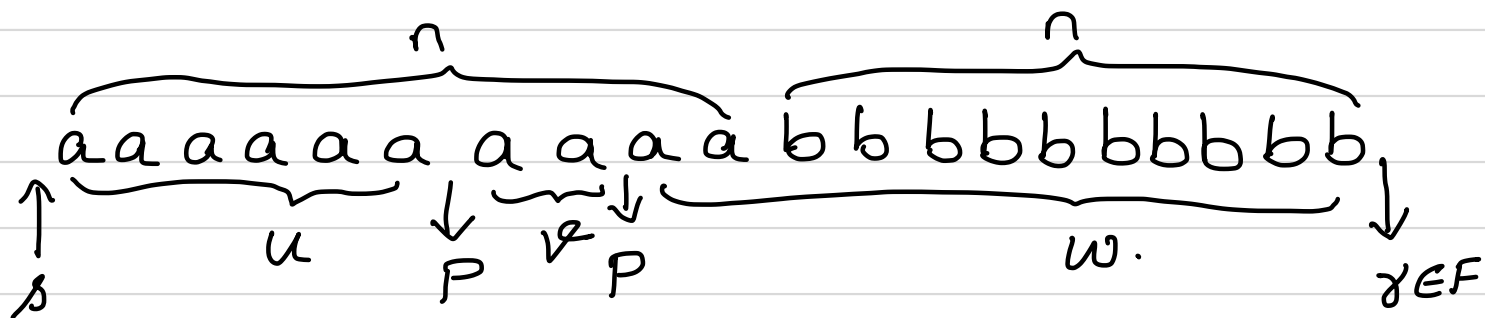
$$B = \{a^n b^n \mid n \geq 0\} = \{\epsilon, ab, aabb, aaabbb, \dots\}$$

aaaaaaaaa a a a a b b b b b b b b b b

Suppose B is regular. There is a DFA M

s.t. $L(M) = B$. Let k be the number of states of M .

Consider the string $a^n b^n$ where $n > k$.



\exists a state P in M s.t.

$$|v| = j > 0. \quad \hat{S}(s, u) = P \quad \hat{S}(P, v) = P$$

$$\text{String} - uw \quad \hat{S}(P, w) = x \in F$$

$$\hat{S}(s, uw) = \hat{S}(\hat{S}(s, u), w) = \hat{S}(P, w) = x \in F$$
$$uw \in L(M) \quad uw = a^{n-|v|} b^n \notin B$$

$$u v^2 w = a^{n+|v|} b^n \in L(M)$$

$$\begin{aligned} \hat{S}(B, u v^2 w) &= \hat{S}(\hat{S}(\hat{S}(\hat{S}(B, u), v), v), w) \\ &= \hat{S}(\hat{S}(\hat{S}(P, v), v), w) \\ &= \hat{S}(\hat{S}(P, v), w) \\ &= \hat{S}(P, w) = x \in F. \end{aligned}$$

$$a^{n+|v|} b^n \notin B.$$

Contradicts our assumption that $L(M) = B$.

Pumping Lemma.

Let A be a regular set. Then the following property holds of A .

(P) There exists $k \geq 0$ such that for any string x, y, z with $xyz \in A$ and $|y| \geq k$, there exists strings u, v, w s.t. $y = uvw$, $v \neq \epsilon$ and for all $i \geq 0$, the string $xu v^i w z \in A$.

Contrapositive: Suppose A satisfies the following:

(¬P) For all $k \geq 0$ there exists strings x, y, z such that $xyz \in A$, $|y| \geq k$ and for all u, v, w with $y = uvw$ and $v \neq \epsilon$, there exists $i \geq 0$ s.t. $xu v^i w z \notin A$.

Then A is not regular.

Suppose A satisfies the following:

($\neg P$) For all $k \geq 0$ there exists strings x, y, z such that
 $xyz \in A$, $|y| \geq k$ and for all u, v, w
with $y = uvw$ and $v \neq \epsilon$, there exists
 $i \geq 0$ s.t. $xuv^i w z \notin A$.

Then A is not regular.

\Rightarrow Claim: A is not regular

Example. $A = \{a^n b^m \mid n \geq m\}$. To show A satisfies $\neg P$

Consider any $k \geq 0$, let $x = a^k$, $y = b^k$ & $z = \epsilon$.

Then $xyz \in A$. Consider any split of $y = uvw$
with $v \neq \epsilon$. Say $y = \underbrace{b^j}_u \underbrace{b^m}_v \underbrace{b^n}_w$. $\therefore k = j + m + n$

$$\begin{aligned} \text{Let } i=2. \quad xuv^2wz &= a^k b^j b^m b^m b^n \\ &= a^k b^{j+2m+n} \\ &= a^k b^{k+m} \notin A \end{aligned}$$