Suppose A satisfies the following:

(¬P) For all $k \geq 0$ there exists strings x, y, z such that $xyz \in A$, $|y| \geq k$ and for all u, v, w with $y = uvw$ and $v \neq \epsilon$, there exists $i \geq 0$ s.t $xuv^iwz \notin A$.

Then A is not regular.

$$A = \{ww \mid w \in \{a,b\}^*\}.$$

Claim. A is not regular.

Proof. Consider any $k \geq 0$. Let

$x = \epsilon$, $y = a^k$, $z = ba^kb$. Then $xyz \in A$.

Consider any split of y; $y = uvw$ s.t $v \neq \epsilon$.

Say $y = a^j a^m a^n$ where $k = j+m+n$; $m > 0$

Let $i = 2$. Then $xuv^2wz = a^j a^m a^m a^n b a^k b$

$$= a^{k+m} b a^k b$$

Since $m > 0$, $a^{k+m} b a^k b \notin A$.

$\therefore$ A is not regular.

Use of closure properties.

$$A = \{ x \in \{a,b\}^* \mid \#a(x) = \#b(x) \}.$$

$\#a(x)$: number of a's in the string x.

Claim: A is not regular.

Proof.

Suppose not, Assume A is regular.

$$A' = A \cap L(a^* b^*).$$

1) $L(a^* b^*)$ is regular.

2) If A is regular and $L(a^*b^*)$ is regular then A' is regular. Since regular sets are closed under intersection.

$$A' = A \cap L(a^* b^*) = \underbrace{\{a^n b^n \mid n \geq 0\}}_{\text{is not regular.}}$$

This gives a contradiction.

$\therefore$ A is not regular.

For a string $x = a_1 a_2 a_3 \dots a_n$, let

$rev(x) = a_n a_{n-1} \cdots a_2 a_1$.

$A \subseteq \Sigma^*$, $rev(A) = \{ rev(x) \mid x \in A \}$.

Theorem. If A is regular then $rev(A)$ is also regular.

Let DFA $M = (Q, \Sigma, \delta, s, F)$ where $L(M) = A$.

To construct N s.t $L(N) = rev(L(M))$.

$$N = (Q \cup \{s'\}, \Sigma, \Delta, s', \{s\})$$

$$\Delta(s', \epsilon) = F$$

$$\Delta(s', a) = \phi \qquad \forall a \in \Sigma$$

$$\Delta(p, a) = \{ q \mid \delta(q, a) = p \} \; \forall p \in Q, a \in \Sigma$$

Claim. if $x \in L(M)$. then $rev(x) \in L(N)$.

Claim. if $rev(x) \in L(N)$ then $x \in L(M)$.

$A = \{a^n b^m \mid n \geq m\}$ Claim. A is not regular.

Suppose A is regular. Then rev(A) is regular.

$rev(A) = \{b^m a^n \mid n \geq m\}$.

Consider the homomorphism h where $h(a)=b$ & $h(b)=a$

Let $A' = h(rev(A))$. Since rev(A) is regular, h(rev(A)) is also regular.

$A' = \{a^m b^n \mid n \geq m\}$.

$$A \cap A' = \{\underbrace{a^n b^n \mid n \geq 0}_{\text{is not regular}}\}.$$

– Contradiction

$\therefore$ A is not regular.