Theorem. Let $A \subseteq \Sigma^*$. The following statements are equivalent.

1) $A$ is regular. $\exists$ a finite automaton $M$ s.t $L(M) = A$.

2) $A = L(\alpha)$ for some pattern $\alpha$.

3) $A = L(\alpha)$ for some regular expression $\alpha$.

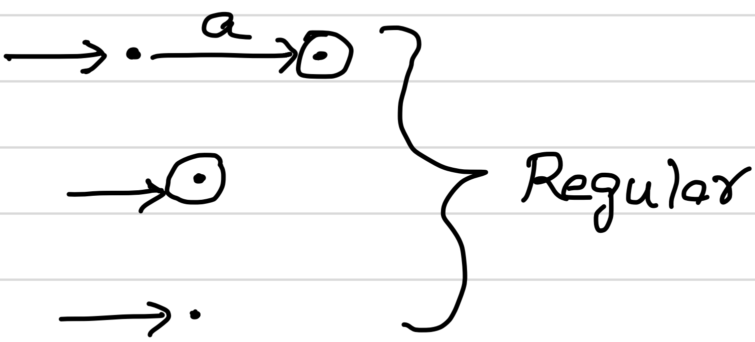
Proof of $2 \Rightarrow 1$. $\quad 3 \Rightarrow 2 \quad 1 \Rightarrow 3$

| Atomic | Compound. |
|---|---|
| $a \in \Sigma$ | $\beta + \gamma$ |
| $\epsilon$ | $\beta \cap \gamma$ |
| $\phi$ | $\beta\gamma$ |
| | $\beta^*$ |
| # | $\beta^+$ |
| @ | $\neg\beta$ |

$a \in \Sigma \quad L(a) = \{a\}$

$\epsilon : L(\epsilon) = \{\epsilon\}$

$\phi : L(\phi) = \phi$



$\beta + \gamma$ $\quad L(\beta+\gamma) = L(\beta) \cup L(\gamma)$

By IH $L(\beta)$ is regular and $L(\gamma)$ is regular

Regular sets are closed under union

$\therefore$ $L(\beta+\gamma)$ is regular.

$\beta \cap \gamma$ $\quad L(\beta \cap \gamma) = L(\beta) \cap L(\gamma)$.

By IH $L(\beta)$ is regular, $L(\gamma)$ is regular

Regular sets are closed under intersection

$\therefore$ $L(\beta \cap \gamma)$ is regular.

$\beta\gamma$ By definition, $L(\beta\gamma) = L(\beta)L(\gamma)$

By IH $L(\beta)$, $L(\gamma)$ are regular.

Regular sets are closed under concatenation

$\beta^*$. $\quad L(\beta^*) = L(\beta)^*$ $\quad L(\beta)$ is regular (IH)

Regular sets are closed under $*$.

$\neg\beta$ $\quad L(\neg\beta) = \overline{L(\beta)}$ $\quad L(\beta)$ is regular (IH)

Regular sets are closed under complementation

$\therefore$ $L(\neg\beta)$ is regular.

$AB = \{xy \mid x \in A \text{ and } y \in B\}$.

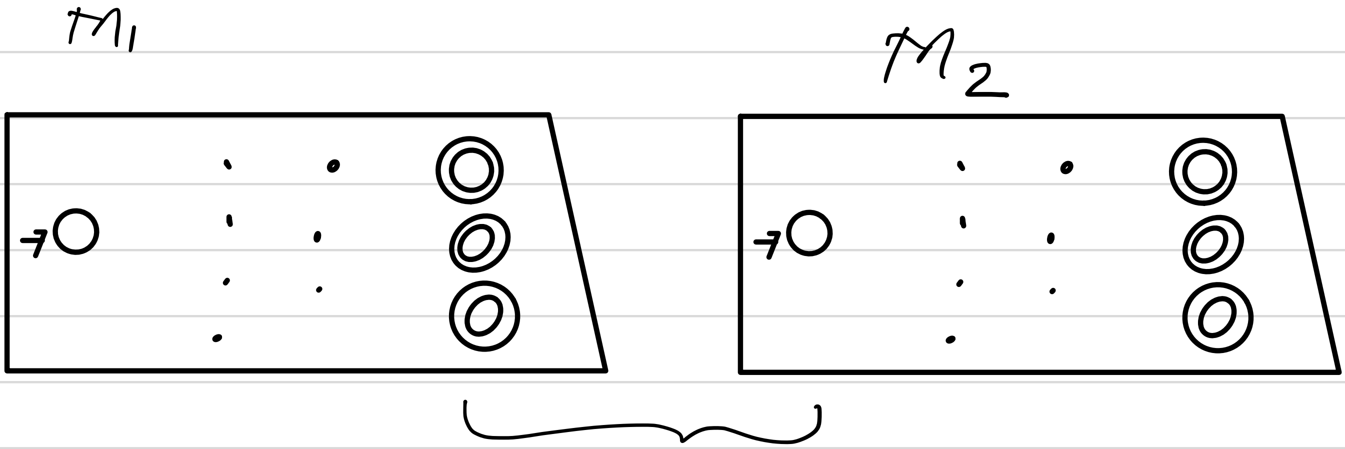
if $A, B \subseteq \Sigma^*$ are regular then $AB$ is regular.

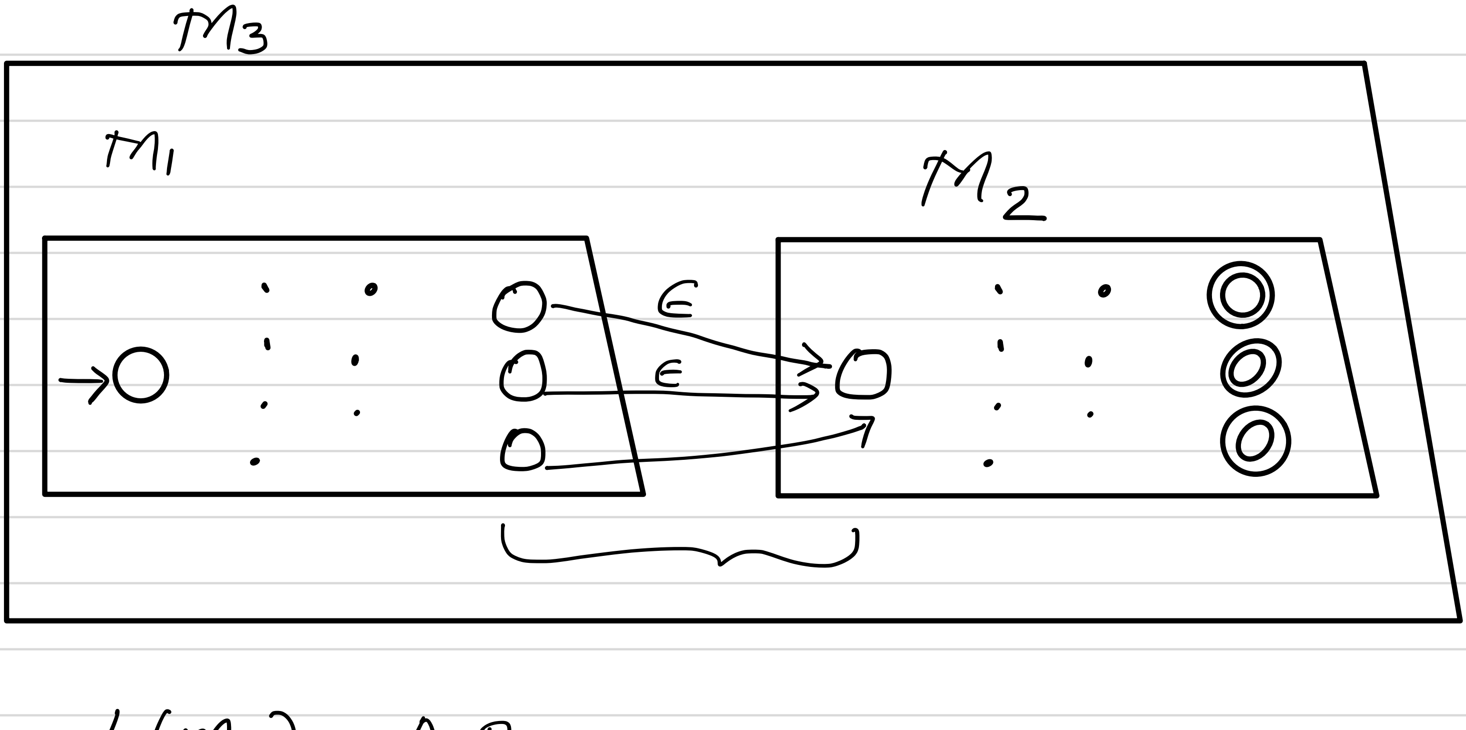
$A \longrightarrow \exists \overset{DFA}{M_1}$ s.t $L(M_1) = A$; $B \longrightarrow \exists \overset{DFA}{M_2}$ s.t $L(M_2) = B$

To Construct $M_3$ s.t $L(M_3) = AB$

$\downarrow$

$\epsilon$-NFA




$L(M_3) = AB$.

$M_1 = (Q_1, \Sigma, \Delta, s_1, F_1)$ $\qquad$ $M_2 = (Q_2, \Sigma, \Delta_2, s_2, F_2)$

$L(M_1) = A$ $\qquad$ $L(M_2) = B$

$M = (Q, \Sigma, \Delta, s, F)$ s.t $L(M) = AB$

$Q = Q_1 \cup Q_2$ $\qquad$ $s = \{s_1\}$ $\qquad$ $F = F_2$

Transition function $\Delta$:

$$\Delta(q,a) = \begin{cases} \Delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \Delta_1(q,a) & q \in F_1 \text{ and } a \neq \epsilon \\ \Delta_1(q,a) \cup \{s_2\} & q \in F_1 \text{ and } a = \epsilon \\ \Delta_2(q,a) & q \in Q_2 \end{cases}$$

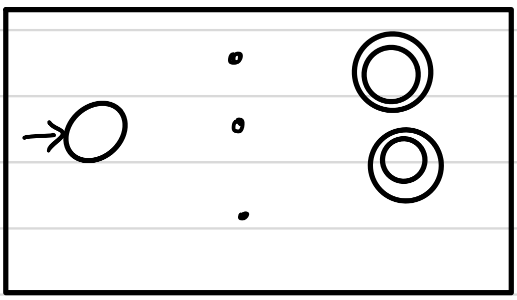
$A^* = \{x_1 x_2 \ldots x_n \mid n \geq 0$ and $x_i \in A,\ 1 \leq i \leq n\}.$

$= \{\epsilon\} \cup A \cup A^2 \cup \ldots$

if $\underbrace{A \text{ is regular}}$ then $A^*$ is regular.

$\exists$ DFA $M_1$ s.t $L(M_1) = A$.

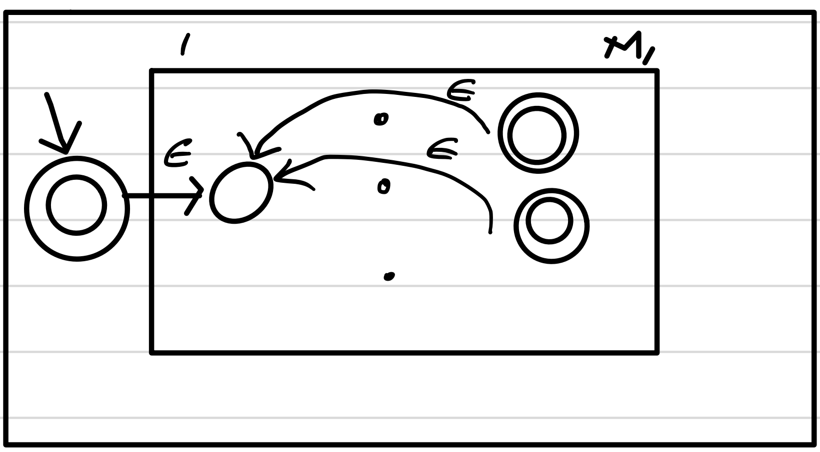
$M_1$



To construct $M_2$ s.t $L(M_2) = A^*$.

$\hookrightarrow$ NFA with $\epsilon$-transitions.

$M_2$




$L(M_2) = A^*$.

$A \subseteq \Sigma^*$, $\quad M_1 = (Q_1, \Sigma, \Delta_1, s_1, F_1) \quad L(M_1) = A$.

Construct $M = (Q, \Sigma, \Delta, s, F)$ s.t $L(M) = A^*$

$Q = \{s_0\} \cup Q_1 \qquad s = \{s_0\} \qquad F = \{s_0\} \cup F_1$

Transition function $\Delta$.

$$\Delta(q,a) = \begin{cases} \Delta_1(q,a) & \text{if } q \in Q_1 \text{ and } q \notin F_1 \\ \Delta_1(q,a) & \text{if } q \in F_1 \text{ and } a \neq \epsilon \\ \Delta_1(q,a) \cup \{s_1\} & \text{if } q \in F_1 \text{ and } a = \epsilon \\ \{s_1\} & \text{if } q = s_0 \text{ and } a = \epsilon \\ \phi & \text{if } q = s_0 \text{ and } a \neq \epsilon. \end{cases}$$