Σ, Γ - finite sets.

Homomorphism $h : \Sigma^* \to \Gamma^*$ such that $\forall x, y \in \Sigma^*$

$$h(xy) = h(x)\,h(y) \qquad \text{— (H1)}$$

$$h(\epsilon) = \epsilon \qquad \text{— (H2)}$$

Any homomorphism defined on Σ^* is uniquely determined by its values on Σ.

$\Sigma = \{a, b\}$ $\quad \Gamma = \{c, d, e\}$ $\quad h(a) = cde,\ h(b) = dd$

$h(aab) = cde\ cde\ dd = h(a)\,h(a)\,h(b)$

Any function $h : \Sigma \to \Gamma^*$ extends uniquely (by induction) to a homomorphism defined on Σ^*.

For $A \subseteq \Sigma^*$ let $h(A) = \{ h(x) \mid x \in A \} \subseteq \Gamma^*$

For $B \subseteq \Gamma^*$ let $h^{-1}(B) = \{ x \mid h(x) \in B \} \subseteq \Sigma^*$

$h(A)$ - image of A under h.

$h^{-1}(B)$ - preimage of B under h.

Theorem 1. Let $h : \Sigma^* \to \Gamma^*$ be a homomorphism.
if $A \subseteq \Sigma^*$ is regular then $h(A)$ is regular

Theorem 2. Let $h : \Sigma^* \to \Gamma^*$ be a homomorphism.
if $B \subseteq \Gamma^*$ is regular then $h^{-1}(B)$ is regular.

Proof of Theorem 2. Let $M = (Q, \Gamma, \delta, s, F)$ - DFA
Such that $L(M) = B$. To show: $\exists M'$ over Σ s.t
$L(M') = h^{-1}(B)$.

Definition of M': $M' = (Q, \Sigma, \delta', s, F)$

$$\underset{\in \Sigma}{\delta'(q, \underset{\downarrow}{a})} = \hat{\delta}(q, \underbrace{h(a)}_{\in \Gamma^*})$$

Lemma 1. $\hat{\delta}'(q, x) = \hat{\delta}(q, h(x))$.
Induction on $|x|$.
Base Case: $x = \epsilon$ - Trivial.

$$\hat{\delta}'(q, xa) = \delta'(\hat{\delta}'(q, x), a) \quad [\text{Def of } \hat{\delta}']$$
$$= \delta'(\hat{\delta}(q, h(x)), a) \quad [IH]$$
$$= \hat{\delta}(\hat{\delta}(q, h(x)), h(a)) \quad [\text{Def of } \delta']$$
$$= \hat{\delta}(q, h(x)h(a))$$
$$= \hat{\delta}(q, h(xa)) \quad [\text{property } (H1)]$$

Lemma 1. $\hat{\delta}'(q, x) = \hat{\delta}(q, h(x))$

To show $L(M') = h^{-1}(B)$

$L(M') = h^{-1}(L(M))$

For any $x \in \Sigma^*$,

$x \in L(M')$ iff $\hat{\delta}'(s, x) \in F$ [Defn. of acceptance]

iff $\hat{\delta}(s, h(x)) \in F$ [Lemma 1].

iff $h(x) \in L(M)$ [Defn. of acceptance]

iff $x \in h^{-1}(L(M))$ [Defn. of $h^{-1}(L(M))$].

$L(M') = h^{-1}(B)$

Theorem 1. Let $h : \Sigma^* \to \Gamma^*$ be a homomorphism.

if $A \subseteq \Sigma^*$ is regular then $h(A)$ is regular

Proof. α is a regular expression s.t $L(\alpha) = A$.
over Σ.

To show: Construct α' s.t $L(\alpha') = h(A)$.

α' - Replace each letter (or symbol) $a \in \Sigma$ in α with the string $h(a) \in \Gamma^*$.

Definition of α': By induction on the structure of α.

$$a' = h(a) \quad , a \in \Sigma.$$

$$\phi' = \phi$$

$$(\beta + \gamma)' = \beta' + \gamma'$$

$$(\beta\gamma)' = \beta' . \gamma'$$

$$(\beta^*)' = \beta'^*$$

Claim: For any regular expression β (over Σ.) : $L(\beta') = h(L(\beta))$

$\Rightarrow L(\alpha') = h(A)$.

Lemma 2. For $C, D \subseteq \Sigma^*$, $h(CD) = h(C)\,h(D)$.

Lemma 3. For a family of subsets $C_i \subseteq \Sigma^*$, $i \in I$,

we have $h\left[\bigcup_{i \in I} C_i\right] = \bigcup_{i \in I} h(C_i)$

Proof of Lemma 2 & 3 - Exercise.

For any β, $L(\beta') = h(L(\beta))$ - To prove

Base Case.

$$L(a') = L(h(a)) = \{h(a)\} = h(\{a\}) = h(L(a))$$

$$L(\phi') = L(\phi) = \phi = h(\phi) = h(L(\phi))$$

Induction Step. Operators: $+, \cdot, *$.

$$L((\beta+\gamma)') = L(\beta'+\gamma') \quad [\text{Defn of } ']$$

$$= L(\beta') \cup L(\gamma') \quad [\text{Defn. of } +]$$

$$= h(L(\beta)) \cup h(L(\gamma)) \quad [\text{Induction hypothesis}]$$

$$= h(L(\beta) \cup L(\gamma)) \quad [\text{Lemma 3}]$$

$$= h(L(\beta+\gamma)) \quad [\text{Definition of } +].$$

For concatenation proof is similar to $+$. Use Lemma 2

$L(\beta^{*\prime})$

$= L(\beta^{\prime *})$ [Definition of ′]

$= L(\beta')^*$ [Defn. of regular expression operator *].

$= h(L(\beta))^*$ [Induction hypothesis.]

$= \bigcup_{n \geq 0} h(L(\beta))^n$ [Defn. of set operator *]

$= \bigcup_{n \geq 0} h(L(\beta)^n)$ [Lemma 2].

$= h\left(\bigcup_{n \geq 0} L(\beta)^n\right)$ [Lemma 3]

$= h(L(\beta)^*)$ [Defn of set operator *]

$= h(L(\beta^*))$ [Defn of regular expression operator *].