Limitations of Finite Automata.

Canonical example

$$B = \{a^n b^n \mid n \geq 0\} = \{\epsilon, ab, aabb, aaabbb, \ldots\}$$

$$aaaaaa\ aaaa\ bbbbbbbbbb$$

Suppose B is regular. There is a DFA M s.t $L(M) = B$. Let k be the number of states of M.

Consider the string $a^n b^n$ where $n > k$.

$$\overbrace{aaaaaaaaaa}^{n}\overbrace{bbbbbbbbbb}^{n}$$

$$\underset{s}{\uparrow}\underbrace{aaaaaa}_{u}\underset{p}{\downarrow}\underbrace{aa}_{v}\underset{p}{\downarrow}\underbrace{aabbbbbbbbbb}_{w}\underset{r\in F}{\downarrow}$$

$\exists$ a state p in M s.t.

$|v| = j > 0.$ $\qquad \hat{\delta}(s, u) = p \quad \hat{\delta}(p, v) = p$

String $- uw$ $\qquad \hat{\delta}(p, w) = r \in F$

$$\hat{\delta}(s, uw) = \hat{\delta}(\hat{\delta}(s, u), w) = \hat{\delta}(p, w) = r \in F$$

$uw \in L(M)$ $\qquad uw = a^{n-|v|} b^n \notin B$

$$uv^2w = a^{n+|v|} b^n \in L(M)$$

$$\begin{aligned}\hat{\delta}(s, uvvw) &= \hat{\delta}(\hat{\delta}(\hat{\delta}(\hat{\delta}(s,u), v), v), w) \\ &= \hat{\delta}(\hat{\delta}(\hat{\delta}(p, v), v), w) \\ &= \hat{\delta}(\hat{\delta}(p, v), w) \\ &= \hat{\delta}(p, w) = r \in F.\end{aligned}$$

$a^{n+|v|} b^n \notin B$.

Contradicts our assumption that $L(M) = B$.

Pumping Lemma.

Let A be a regular set. Then the following property holds of A.

(P) There exists $k \geq 0$ such that for any string x, y, z with $xyz \in A$ and $|y| \geq k$, there exists strings u, v, w s.t $y = uvw$, $v \neq \epsilon$ and for all $i \geq 0$, the string $xuv^iwz \in A$.

Contrapositive: Suppose A satisfies the following:

($\neg$P) For all $k \geq 0$ there exists strings x, y, z such that $xyz \in A$, $|y| \geq k$ and for all u, v, w with $y = uvw$ and $v \neq \epsilon$, there exists $i \geq 0$ s.t $xuv^iwz \notin A$.

Then A is not regular.

Suppose A satisfies the following:

$(\neg P)$ For all $k \geq 0$ there exists strings x, y, z such that $xyz \in A$, $|y| \geq k$ and for all u, v, w with $y = uvw$ and $v \neq \epsilon$, there exists $i \geq 0$ s.t. $xuv^iwz \notin A$.

Then A is not regular.

Claim: A is not regular

Example. $A = \{a^n b^m \mid n \geq m\}$. To show A satisfies $\neg P$

Consider any $k \geq 0$, let $x = a^k$ $y = b^k$ & $z = \epsilon$.

Then $xyz \in A$. Consider any split of $y = uvw$ with $v \neq \epsilon$. Say $y = \underbrace{b^j}_{u} \underbrace{b^m}_{v} \underbrace{b^n}_{w}$ $\therefore k = j + m + n$

Let $i = 2$. $xuv^2wz = a^k b^j b^m b^m b^n$

$$= a^k b^{j+2m+n}$$

$$= a^k b^{k+m} \notin A$$