Heaps, Binary Heaps and Heapsort

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Heapsort

Heapsort

- Like Merge-sort it's worst case time complexity is $\mathcal{O}(n \log n)$.
- Like Quick-sort it is an in place algorithm.
  - # of elements stored outside the input array at any time: $\mathcal{O}(1)$.
- Combines the better attributes of the two sorting algorithms.

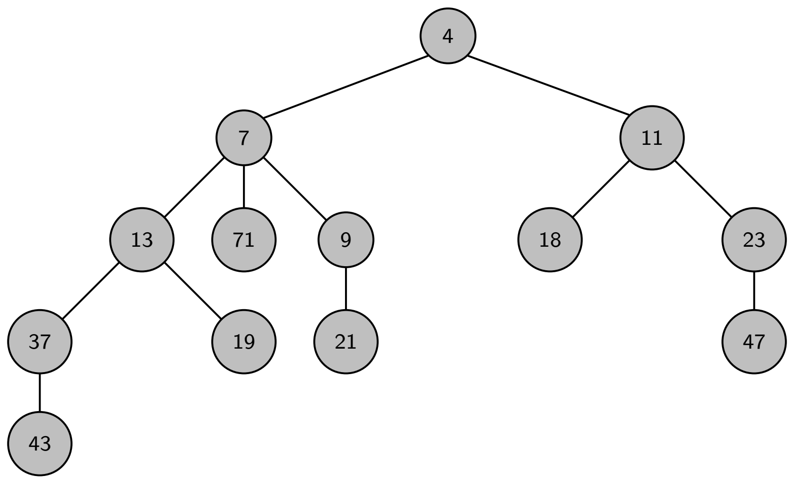
Heapsort (Cont.)

- Introduces a different algorithm design technique:
  - the use of a data structure to manage information during the execution of the algorithm.
  - This data structure is called a heap.
- Apart from heapsort it also makes an efficient priority queue.
- The term heap was originally coined in the context of heapsort.
- But it has since come to refer to as garbage-collection storage provided by programming languages like Lisp and Java.
- But heap data structure is not garbage-collected storage!

Heaps

Heap

A **Min-Heap** is a (rooted) tree data structure where the value stored in a node less than or equal to the value stored in each of its children.



Heap (Cont.)

- The lowest/highest priority element is always stored at the root.
- It is not a sorted structure.
- It can be regarded as being partially ordered.
- It is useful when it is necessary to repeatedly remove the object with the lowest/highest priority.

Basic Operations

Query Operations:

- FIND-MIN(H): Report the smallest key stored in the heap.

Modifying Operations:

- CREATEHEAP(H): Create an empty heap H.
- INSERT(x, H): Insert a new key with value x into the heap H.
- EXTRACT-MIN(H): Delete the smallest key from H.
- DECREASE-KEY(p, Δ, H): Decrease the value of the key p by amount Δ.
- MERGE(H_1, H_2): Merge two heaps H_1 and H_2.

Variants

- 2-3 heap
- B-heap
- Beap
- Binary heap
- Binomial heap
- Brodal queue
- d-ary heap
- Fibonacci heap
- K-D Heap
- Leaf heap
- Leftist heap
- Pairing heap
- Radix heap
- Randomized meldable heap
- Skew heap
- Soft heap
- Ternary heap
- Treap
- Weak heap

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Complete Binary Tree

Can we implement a binary tree using an array?

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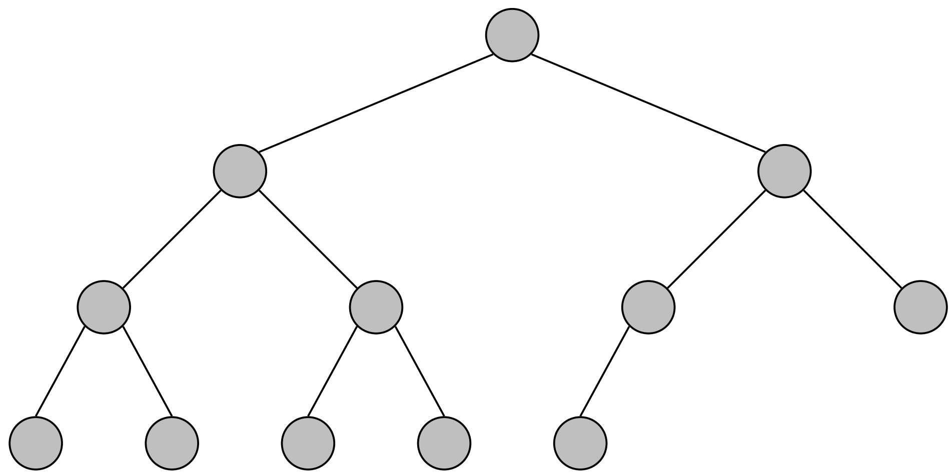
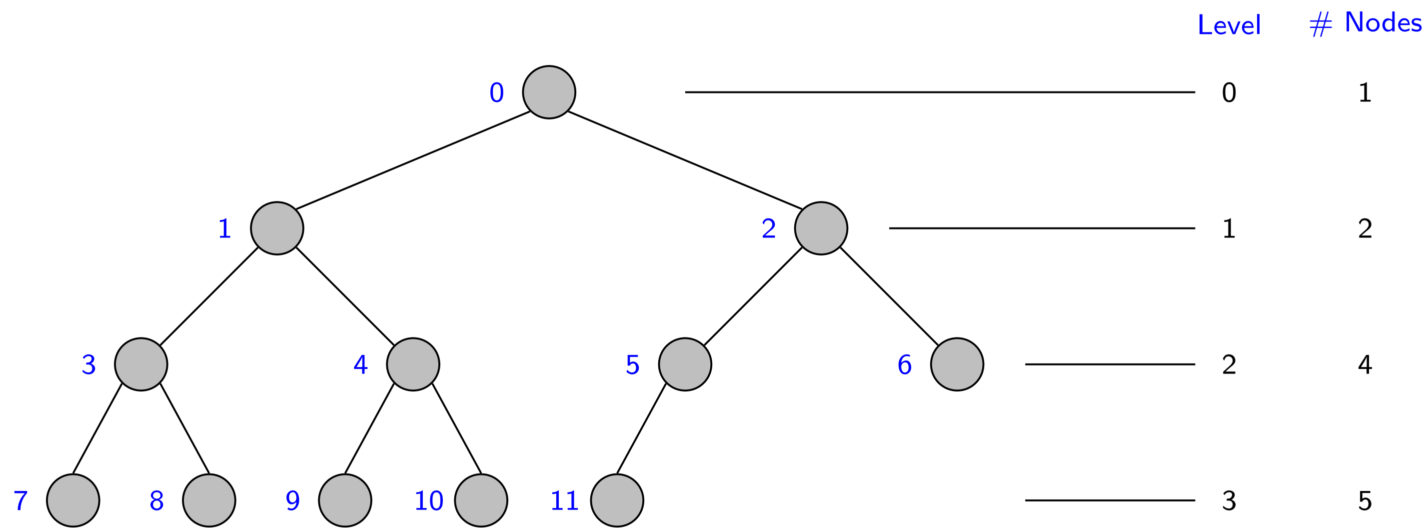
Yes, in some special cases.

A Complete Binary Tree



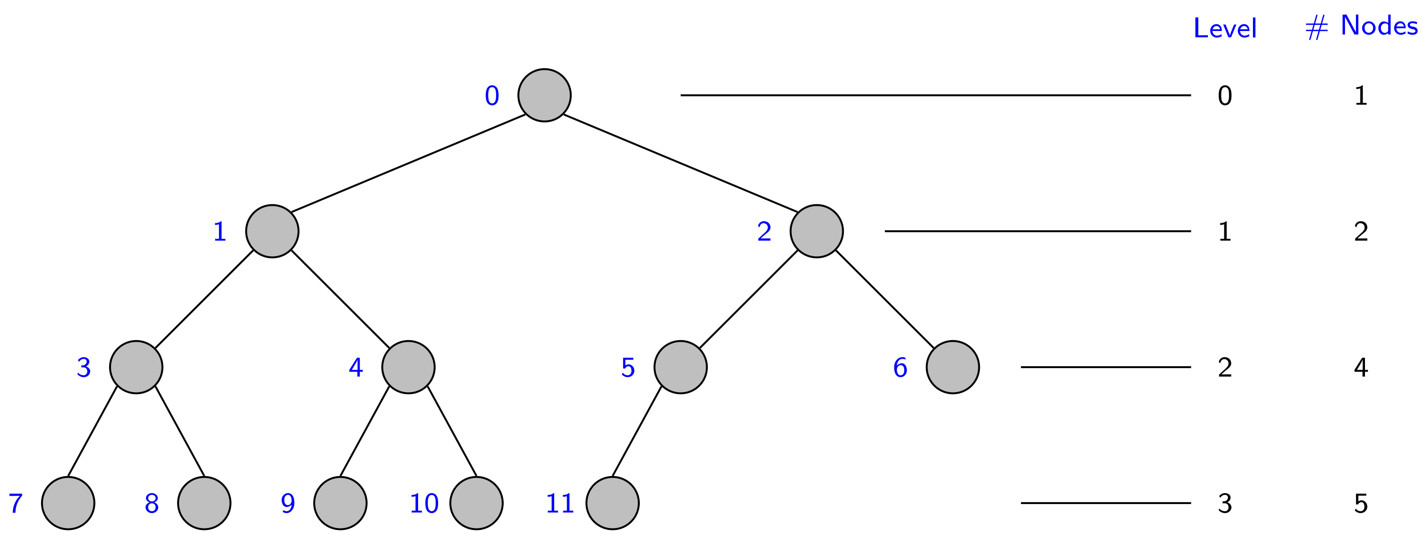
A complete binary of 12 nodes.

A Complete Binary Tree



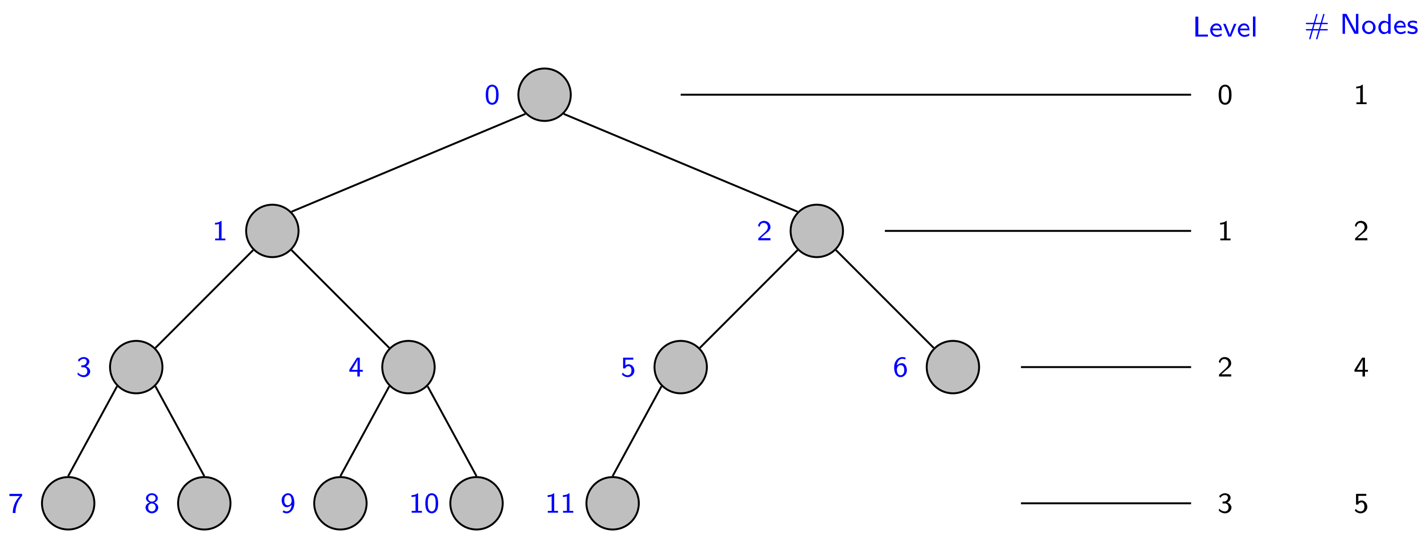
Can you see a relationship between label of a node and labels of its children?

A Complete Binary Tree



- The label of the **leftmost node** at level $i = 2^i - 1$.
- The label of a **node** v **at level** i **occurring at** $k^{\textbf{th}}$ **place from left** $= 2^i + k - 2$.
- The label of the **left** child of v is $= 2 \cdot (2^i + (k - 2)) + 1$.
- The label of the **right** child of v is $= 2 \cdot (2^i + (k - 2)) + 2$.

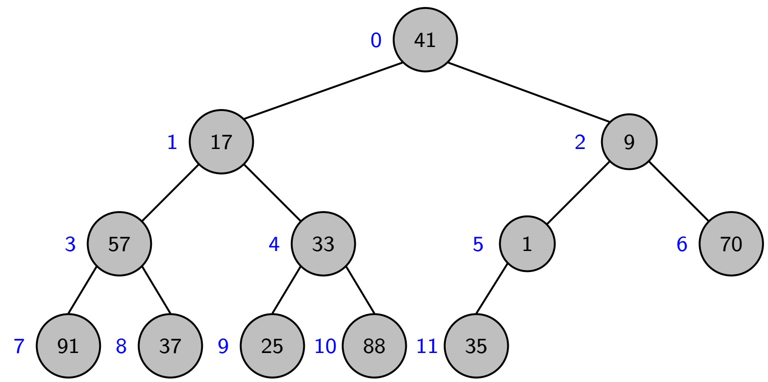
A Complete Binary Tree



- Let v be a node with label j.
- Label of **left child**$(v) = 2j + 1$.
- Label of **right child**$(v) = 2j + 2$.
- Label of **parent**$(v) = \lfloor (j-1)/2 \rfloor$.

A Complete Binary Tree and An Array

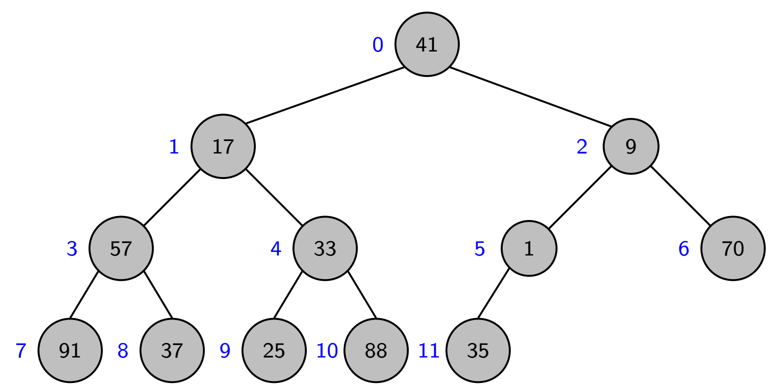
Can we implement a complete binary tree using an array?

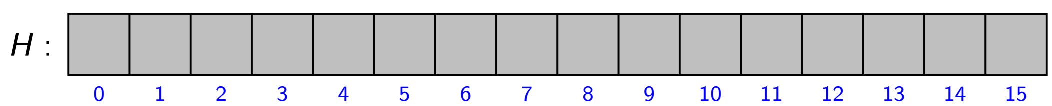


A Complete Binary Tree and An Array

Can we implement a complete binary tree using an array? Yes!

Advantage: It is the most compact representation.





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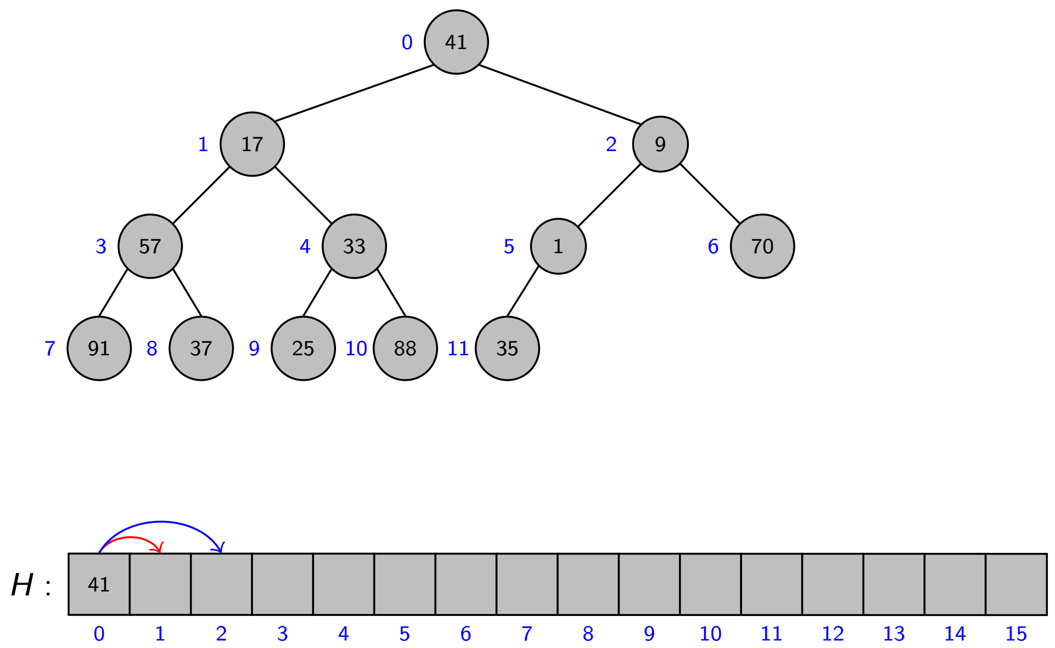

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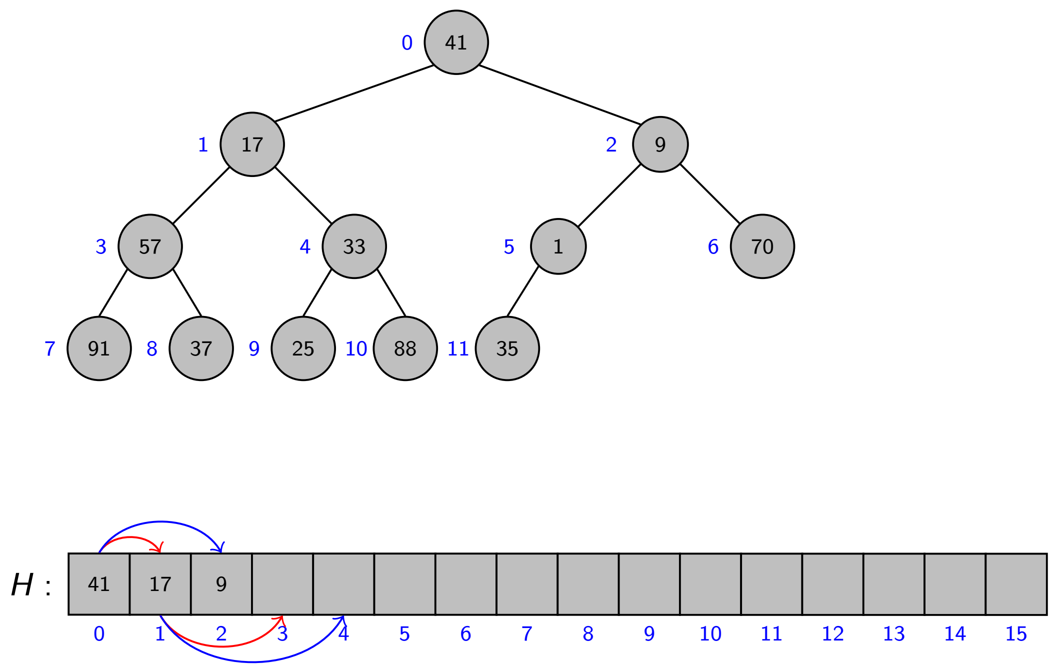

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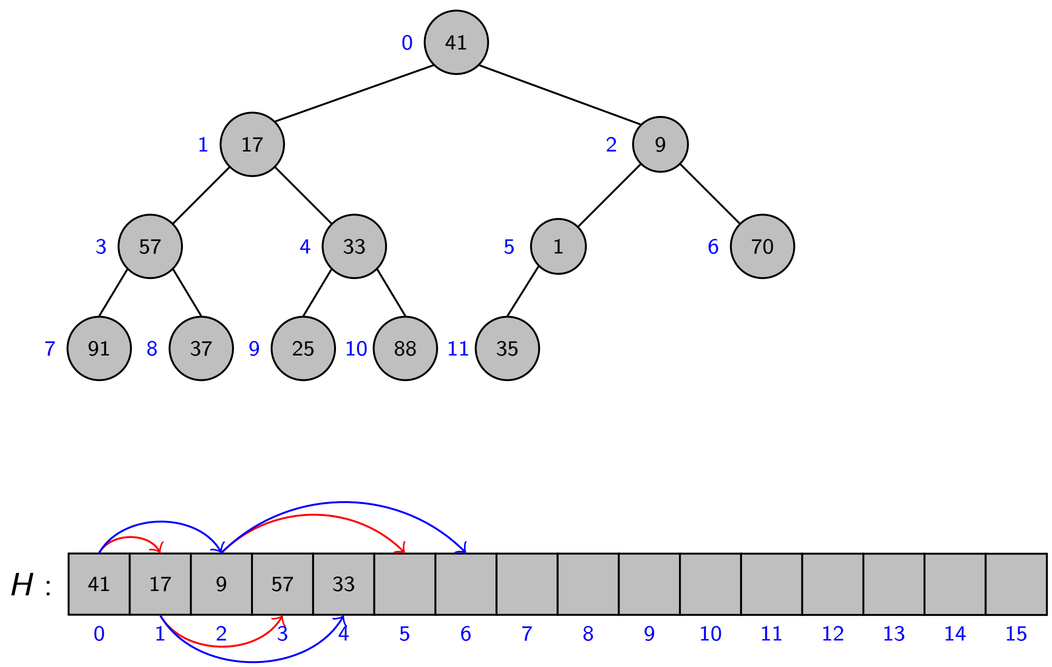

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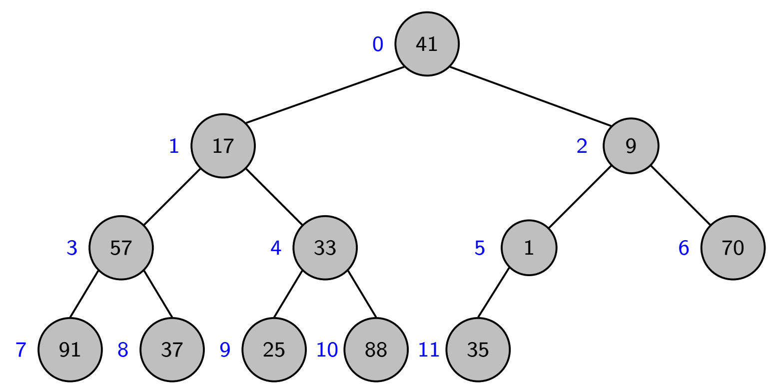

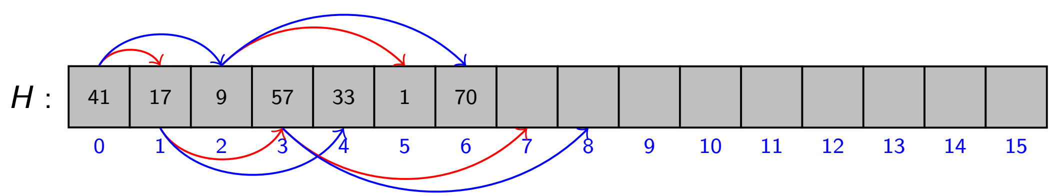

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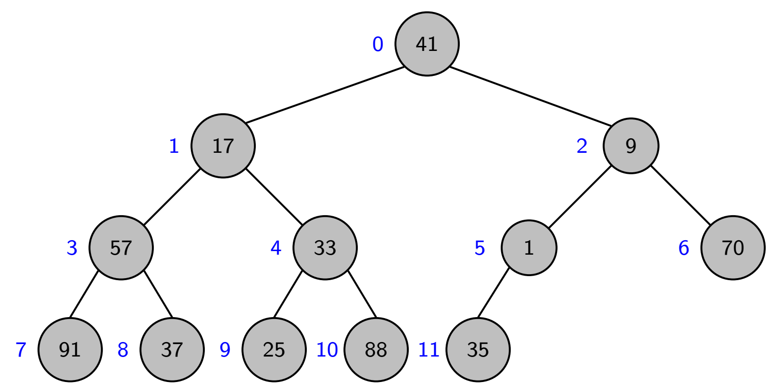

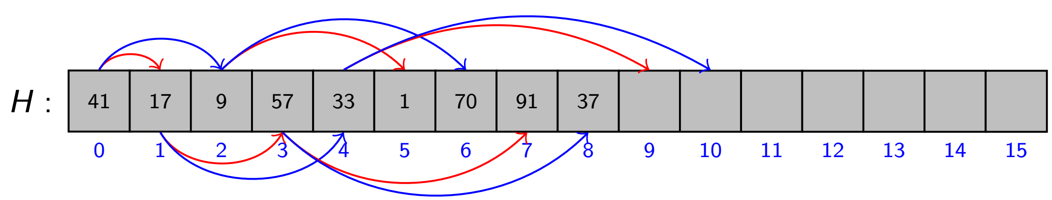

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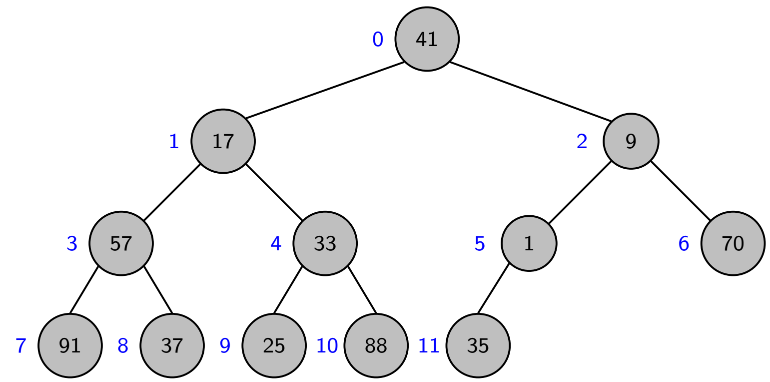

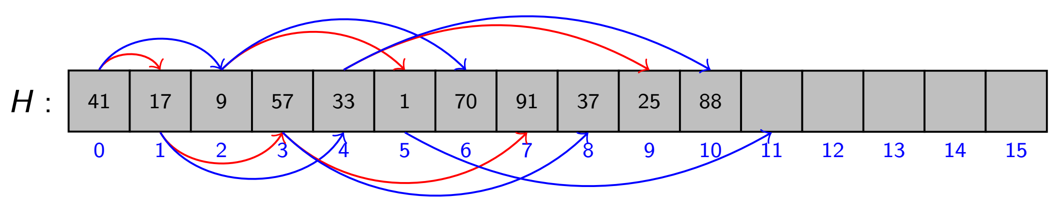

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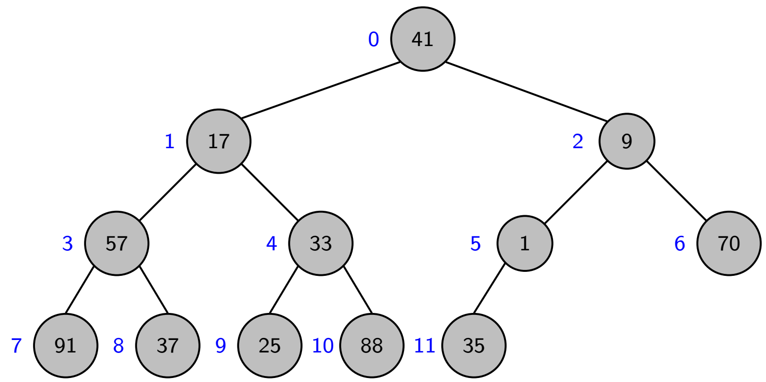
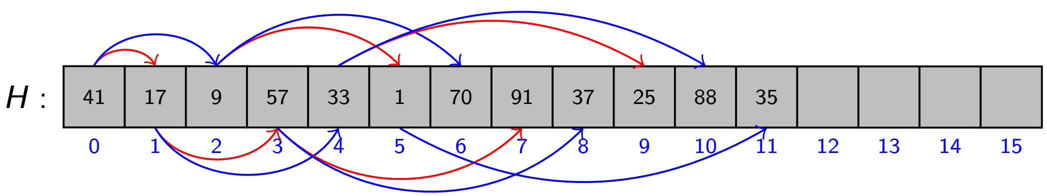
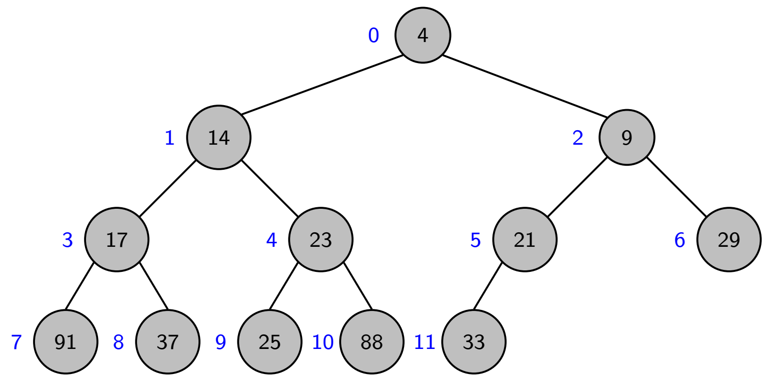




Binary Heap

Binary (Min) Heap

Definition: It is a complete binary tree satisfying the heap property at each node.



H :

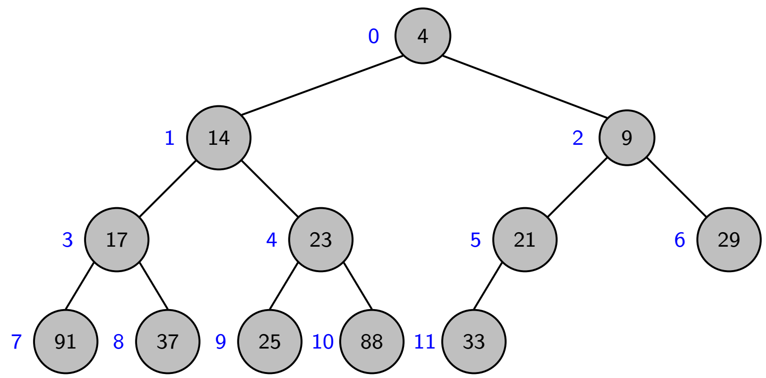
4	14	9	17	23	21	29	91	37	25	88	33				
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Implementation of a Binary Heap

H[ ] : An array of size n used for storing the binary heap.

size : A variable for the total number of keys currently in the heap.

Find-Min(H)



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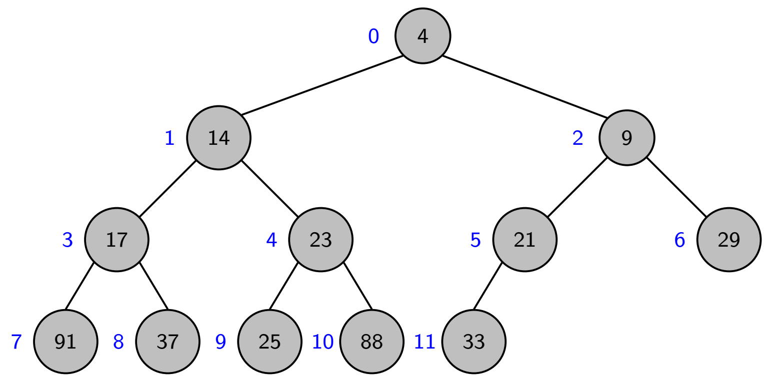
FIND-MIN(H)



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	4	14	9	17	23	21	29	91	37	25	88	33				
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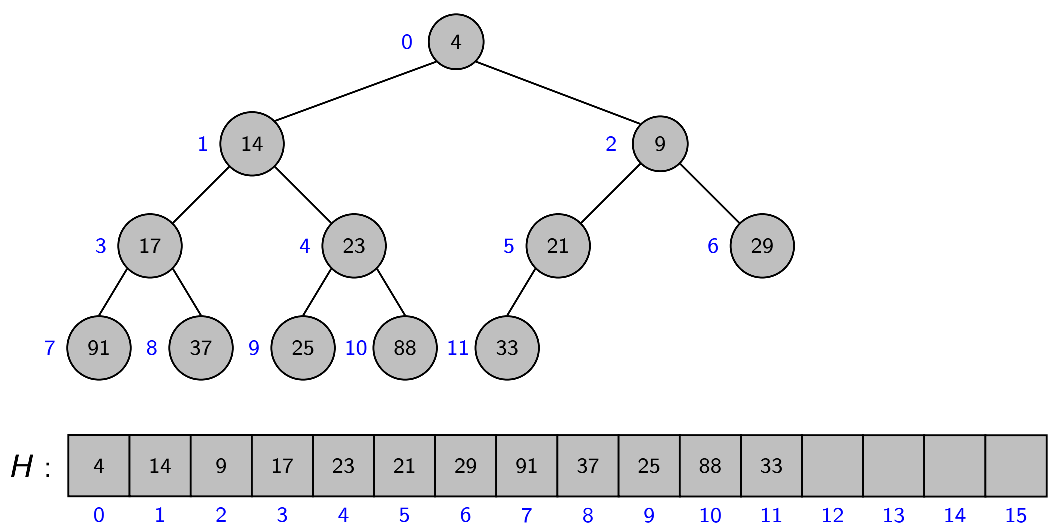
Return $H[0]$

EXTRACT-MIN(H)

Goal: Deletes the smallest key from H.

Challenge: Preserve the complete binary tree structure as well as the heap property!



Extract-Min(H)

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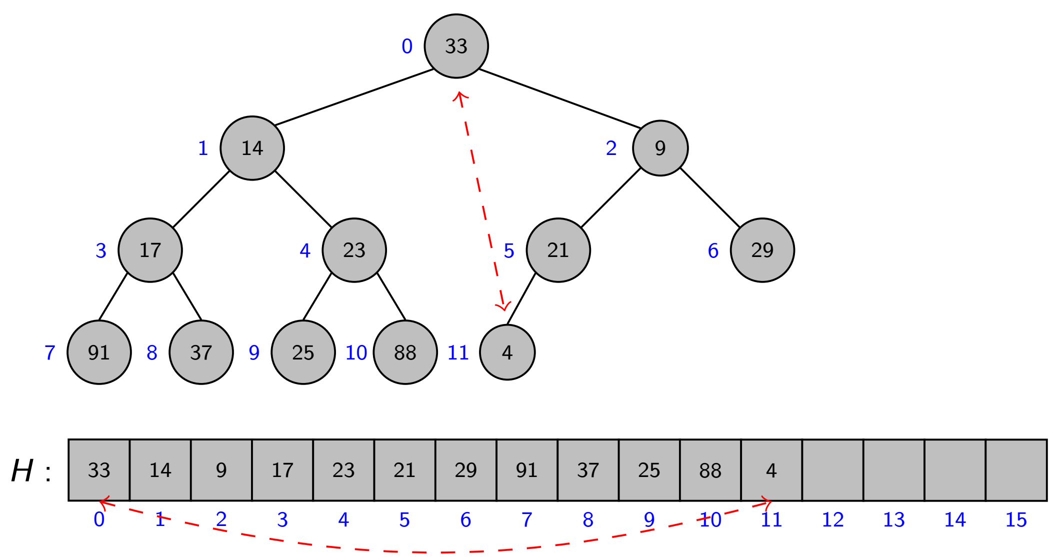

- *swap*(H[0], H[*size* − 1]).

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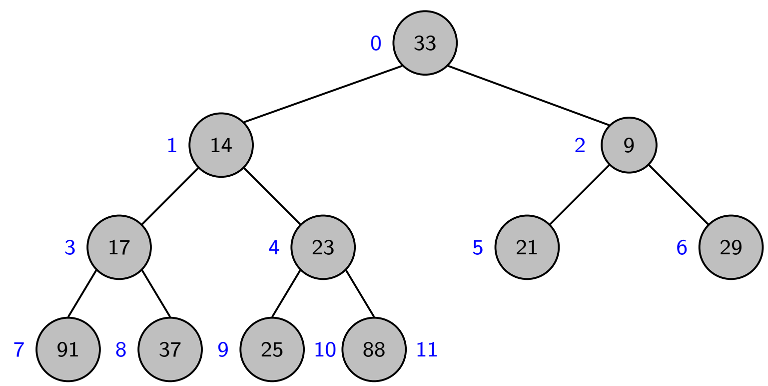

H :

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
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- $swap(H[0], H[size-1])$.
- $size = size - 1$.

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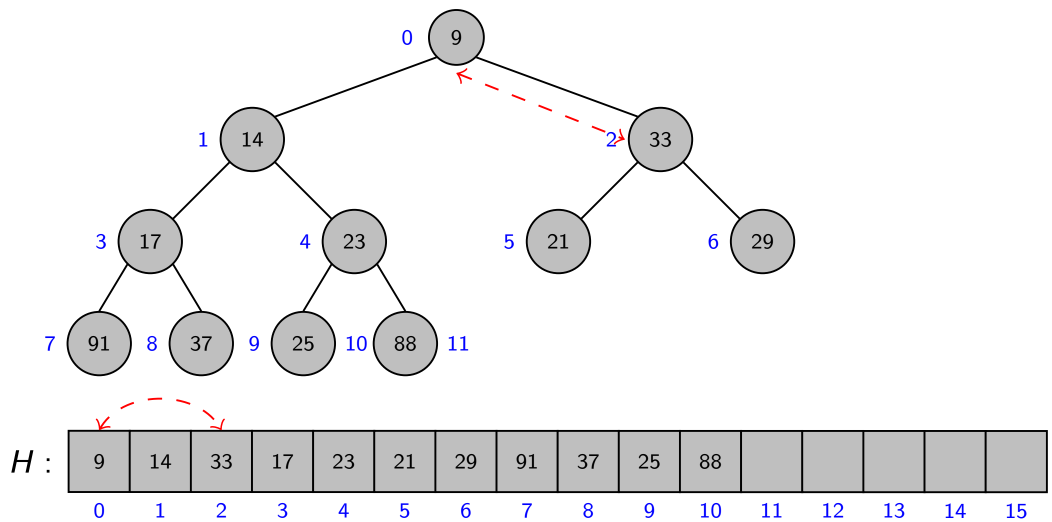

- $swap(H[0], H[size-1])$.
- $size = size - 1$.
- **While** $x > key[left[x]]$ or $x > key[right[x]]$, **then**
  - $swap(x, \min\{left[x], right[x]\})$.

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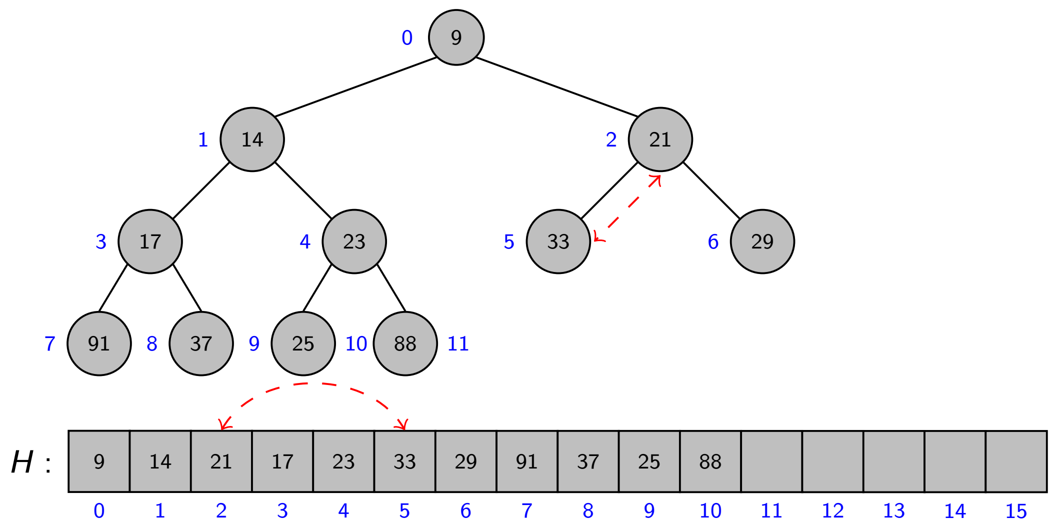

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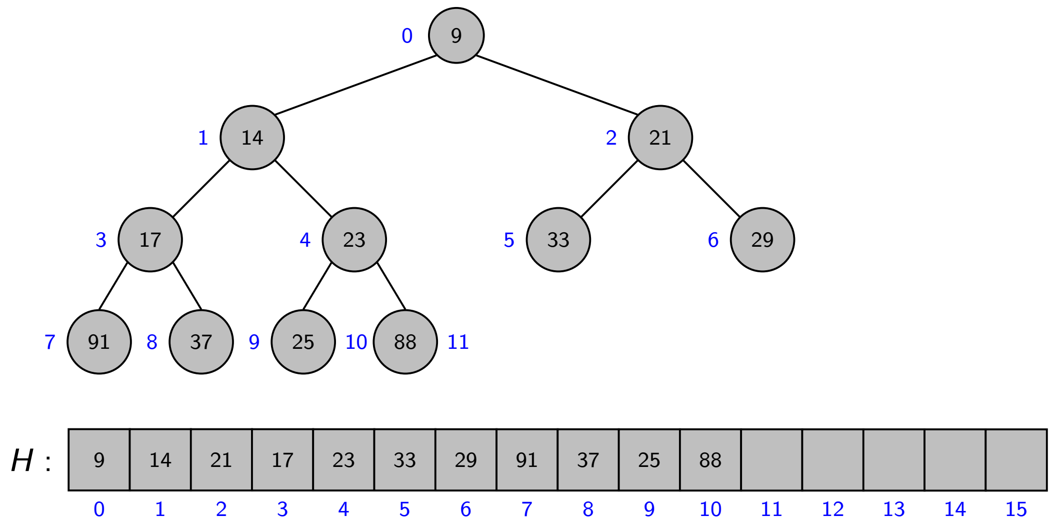
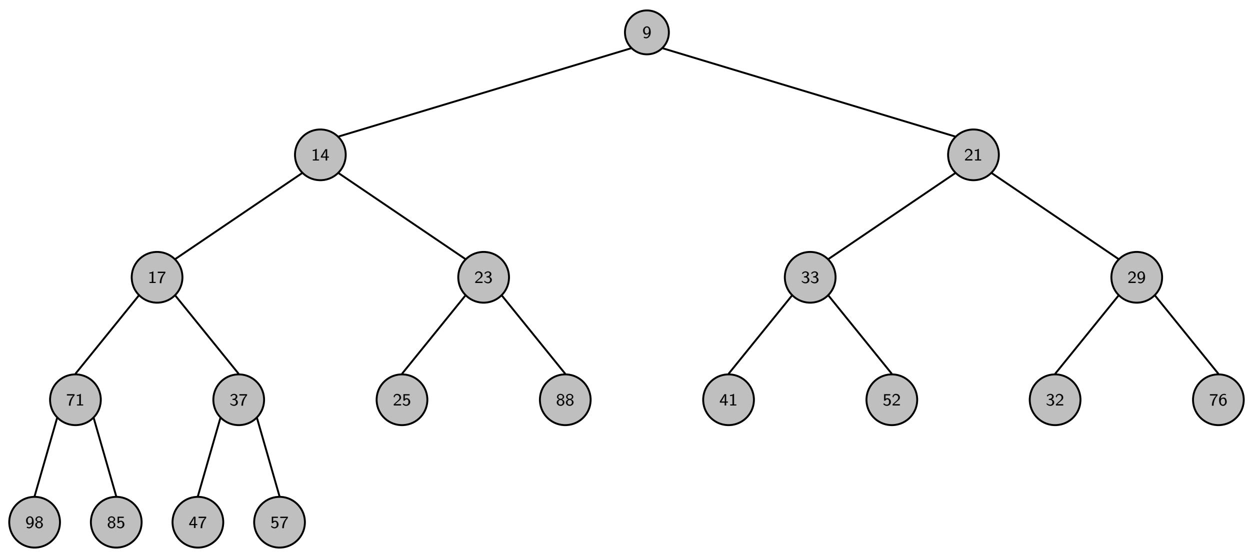


- $swap(H[0], H[size-1])$.
- $size = size - 1$.
- **While** $x > key[left[x]]$ or $x > key[right[x]]$, **then**
  - $swap(x, \min\{left[x], right[x]\})$.
- **Complexity:** $\#$ swaps $= \mathcal{O}(\#$ levels in binary heap$) = \mathcal{O}(\log n)$ (show it!).

INSERT(x, H)

Let $x = 11$.



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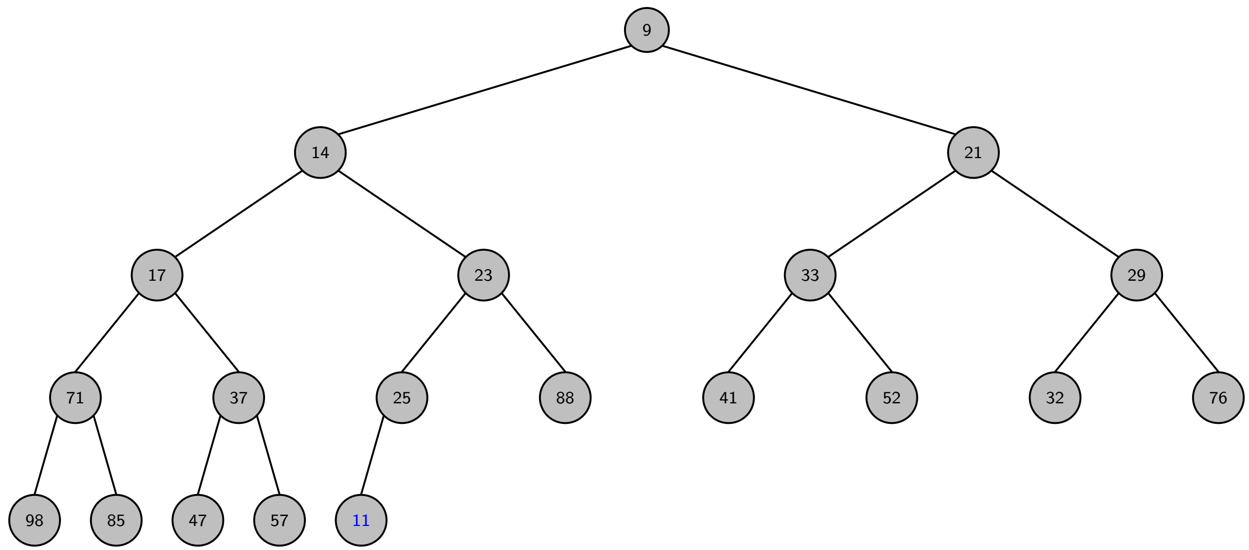

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- $H[size] = x$.
- $size = size + 1$

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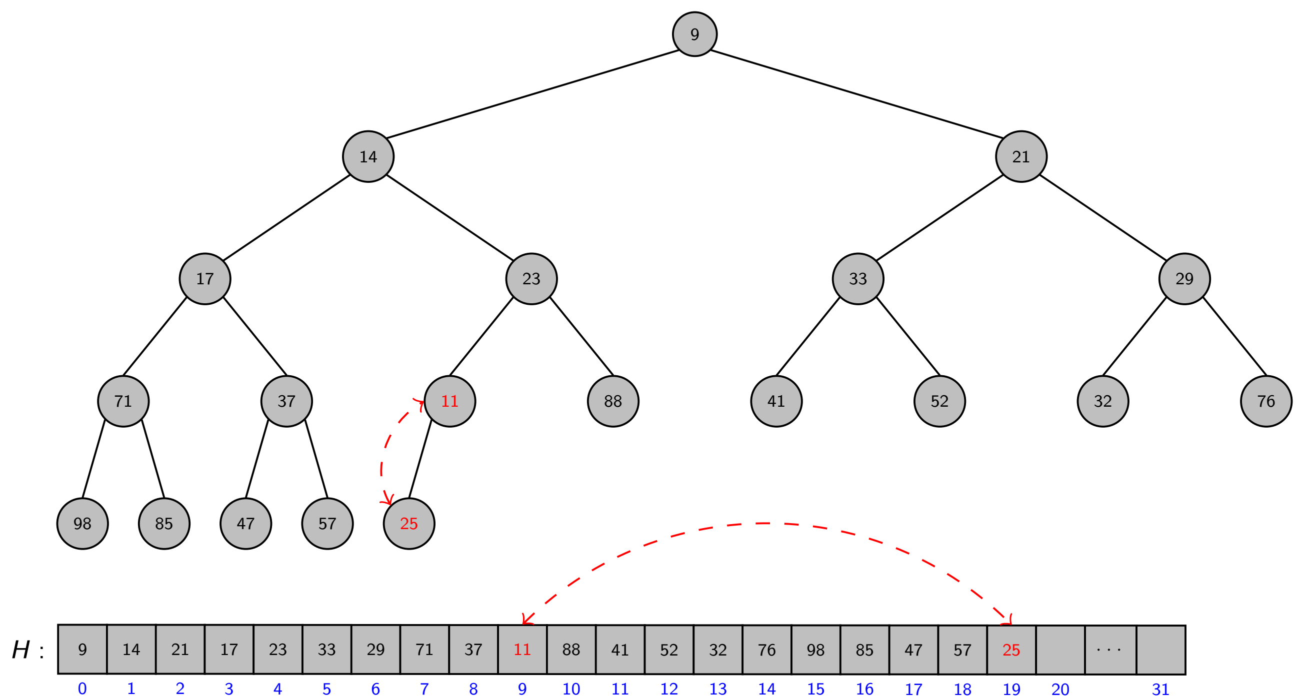

- $H[size] = x$.
- $size = size + 1$
- **While** $parent[x] > x$, **then**
  - $swap(x, parent[x])$.

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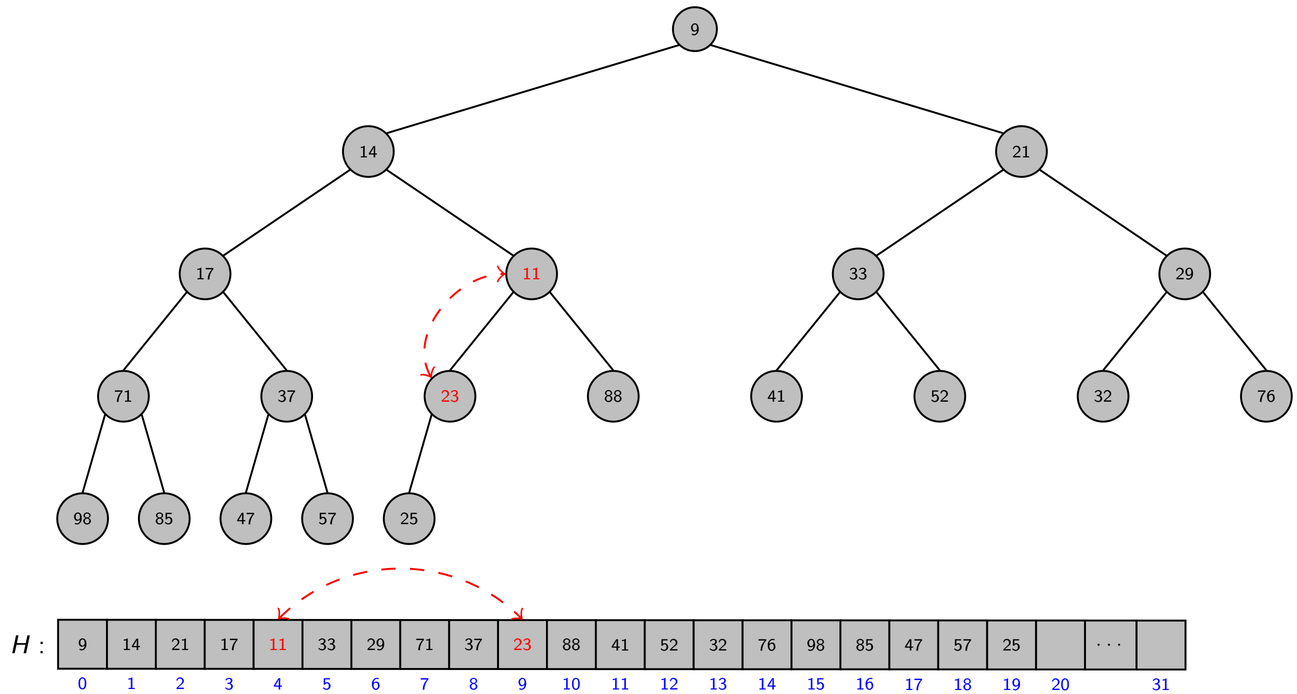

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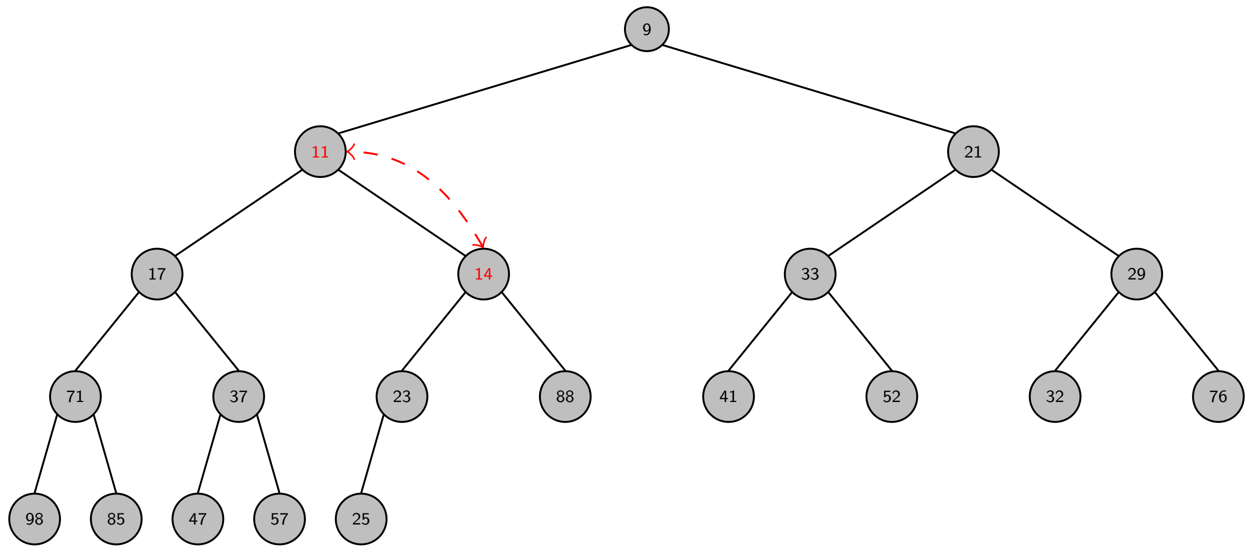

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INSERT(x, H) (Cont.)

Begin
$i \leftarrow size(H)$;
$H[size] \leftarrow x$;
$size(H) \leftarrow size(H) + 1$;

while $(i > 0$ and $H[i] < H[\lfloor (i-1)/2 \rfloor])$
$swap(H[i], H[\lfloor (i-1)/2 \rfloor])$;
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End

Complexity?

$\textsc{Insert}(x, H)$ (Cont.)

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End

Complexity: $\mathcal{O}(\log n)$.

The Remaining Operations on Binary Heap

- $\textsc{Decrease-Key}(p, \Delta, H)$: Decrease the value of the key p by amount Δ.
  - Similar to $\textsc{Insert}(x, H)$.
  - Do it as an exercise!
  - **Complexity?**

- $\textsc{Merge}(H_1, H_2)$: Merge two heaps H_1 and H_2.

The Remaining Operations on Binary Heap

- $\textsc{Decrease-Key}(p, \Delta, H)$: Decrease the value of the key p by amount Δ.
  - Similar to $\textsc{Insert}(x, H)$.
  - Do it as an exercise!
  - **Complexity:** $\mathcal{O}(n)$.
  - **Note:** Searching for p takes $\mathcal{O}(n)$
  - Can you do it in $\mathcal{O}(\log n)$?

- $\textsc{Merge}(H_1, H_2)$: Merge two heaps H_1 and H_2.

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  - Do it as an exercise!
  - **Complexity:** $\mathcal{O}(n)$.
  - Needs some additional information called $\textsc{Map}$!
  - $\textsc{Map}$: Stores the index corresponding to each key.

- $\textsc{Merge}(H_1, H_2)$: Merge two heaps H_1 and H_2.

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- $\textsc{Merge}(H_1, H_2)$: Merge two heaps H_1 and H_2.
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The Remaining Operations on Binary Heap

- DECREASE-KEY(p, Δ, H): Decrease the value of the key p by amount Δ.
  - Similar to INSERT(x, H).
  - Do it as an exercise!
  - **Complexity:** $\mathcal{O}(n)$.
  - Needs some additional information called MAP!
  - MAP: Stores the index corresponding to each key.

- MERGE(H_1, H_2): Merge two heaps H_1 and H_2.
  - **Complexity:** $\mathcal{O}(n \log n)$
  - Can you do it in $\mathcal{O}(n)$?

Thank You for your kind attention!

Books and Other Materials Consulted

1. *Introduction to Algorithms* by Thomas H Cormen, Charles E Leiserson, Ronald L Rivest, Clifford Stein.

2. Taken from Prof. Surendar Baswana (CSE, IIT Kanpur) lecture slides.

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Questions!!