Introduction to Graphs

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Winter Semester,
3^{rd} May, 2023

Graphs

Why Graphs?

The Königsberg Bridge Problem:

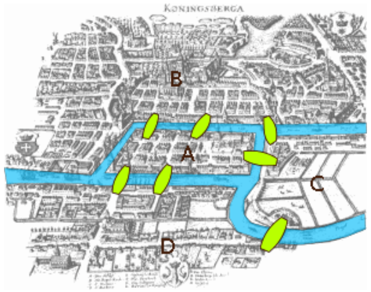


Figure: Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges (Courtesy: Wikipedia).

- Euler (1707-1782) became the father of graph theory as well as topology when in 1736 he settled a famous unsolved problem of his day called the Königsberg Bridge Problem.
- There were two islands linked to each other and to the banks of the Pregel River by seven bridges.

Why Graphs?

The Königsberg Bridge Problem:

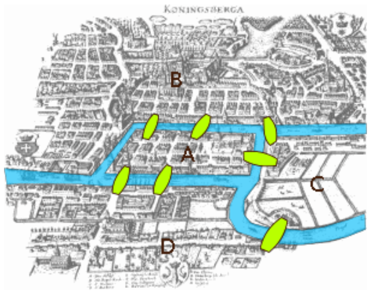


Figure: Map of Königsberg in Euler's time showing the actual layout of the seven bridges, highlighting the river Pregel and the bridges (Courtesy: Wikipedia).

- **Problem:** Begin at any of the four land areas, walk across each bridge exactly once and return to the starting point.
- One can easily try to solve this problem empirically.
- But according to Euler all attempts must be unsuccessful!

The Königsberg Bridge Problem: Euler's Modeling

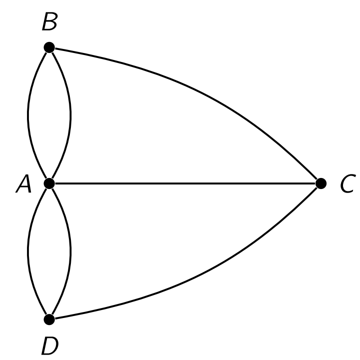


Figure: The graph of the Königsberg Bridge Problem.

- Replace each land area by a point.
- Replace each bridge by a line joining the corresponding points.
- Thereby producing a "graph".
- He generalized the problem.
- Developed a criterion for a given graph to be so traversable.
- **Criterion:** It must be connected and every point is incident with an even number of lines.
- Thereby solved the Königsberg Bridge Problem in the negative.

Electric Networks

- Kirchhoff developed the theory of trees in 1847.
- He was trying to solve the system of simultaneous linear equations which give the current in each branch and around each circuit of an electric network.
- He abstracted an electric network and replaced it by its corresponding combinatorial structure consisting only of points and lines.
- Thus, in effect, Kirchhoff replaced each electrical network by its underlying graph.

Electric Networks

- He showed that it is not necessary to consider every cycle in the graph of an electric network separately in order to solve the system of equations.
- He pointed this out by a simple but powerful construction, which has since become standard procedure.
- He showed that the independent cycles of a graph determined by any of its "spanning trees" suffices.

Electric Networks

N :

G :

T :

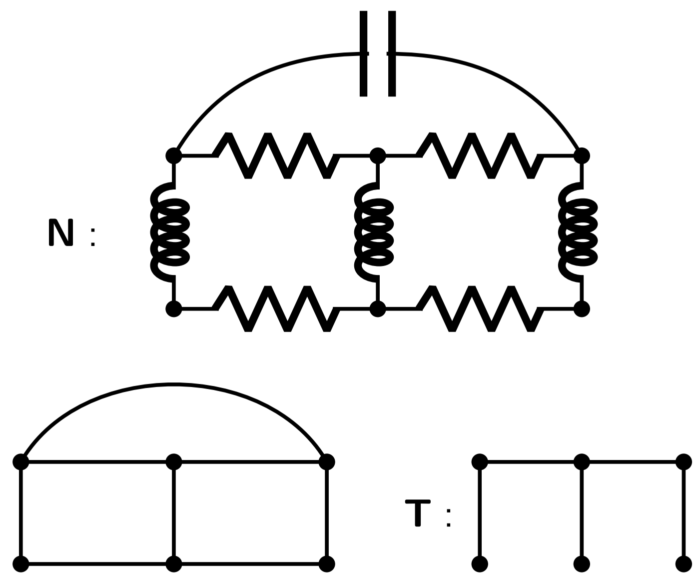


Figure: A network **N**, its underlying graph **G**, and a spanning tree **T**.

Colouring Maps

Figure: Map of India.

- Represent each state by a point.
- Connect two points if they have a common border.

Colouring Maps

Figure: Map of India.

Constraint: Two connected points should not get the same colour.

Colouring Maps (Cont.)

Consider the state West Bengal.

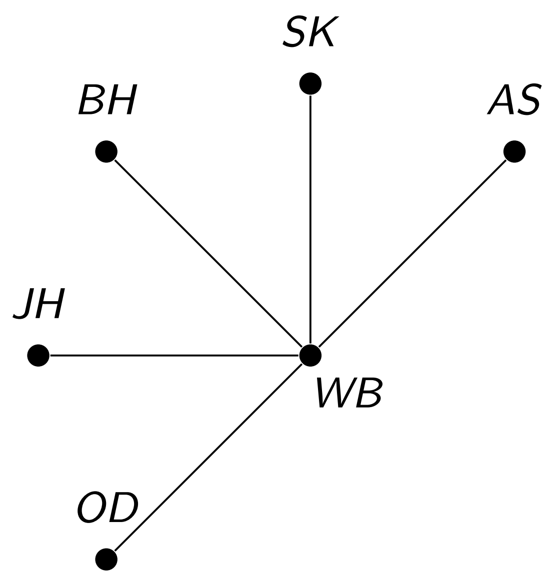


Figure: Graph denoting the neighbouring states of West Bengal.

- In general, this problem reduces to the famous 4-colouring conjecture, which was proved by Apple and Hacken in 1977.

Traveling Salesman Problem

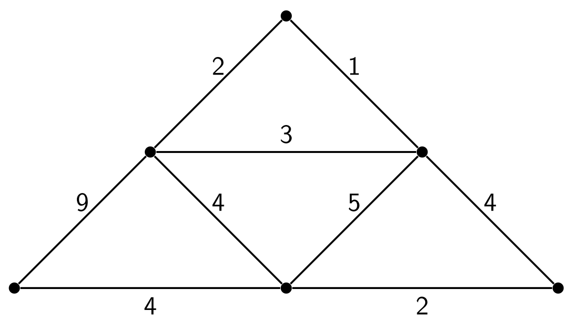


Figure: Graph denoting distances of highways connecting different cities.

- Cities of India are connected by highways.
- Suppose a salesperson wishes to visit all the cities exactly once and come back to his home city.
- The salesperson wishes to minimize the total distance traveled.
- This problem is famously known as the **Traveling Salesman Problem**.

Graphs: Definition

Definition (Simple Undirected Graph)

A **simple undirected graph** G is a pair (V, E), where V is a finite set of points (or *vertices*) and

$$E \subseteq \{ \underbrace{\{i,j\}:}_{\text{unordered pair}} \; i,j \in V, i \neq j\}.$$

The elements of E are called **lines** (or **edges**).

Abuse of notation: We will use (i,j) to denote $\{i,j\}$.

Graphs: Definition

Definition (Simple Directed Graph)

A **directed graph (or digraph)** G is a pair (V, E), where V is a finite set of points and

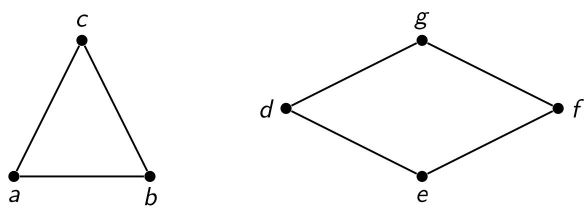
$$E \subseteq V \times V \setminus \{(i, i) : i \in V\}.$$

Elements of V are also called **nodes** and the elements of E are called **arcs**.

Note: Arc means direction but edges does not mean direction.

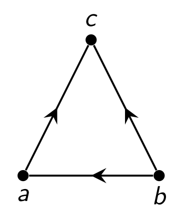
Examples

Simple Undirected Graphs:



$$
\begin{aligned}
V &= \{a, b, c, d, e, f, g\} \\
E &= \{\{a, b\}, \{a, c\}, \{b, c\}, \{d, e\}, \\
&\quad \{e, f\}, \{f, g\}, \{g, d\}\}
\end{aligned}
$$

Simple Directed Graphs:



$$
\begin{aligned}
V &= \{a, b, c\} \\
E &= \{(b, a), (a, c), (b, c)\}
\end{aligned}
$$

Walks, Trails, Paths and Cycles

Definition (Walk)

A **walk** is a finite alternating sequence of vertices and edges preserving adjacency, i.e,

$$v_0 x_1 v_1 x_2 v_2 \cdots v_{n-1} x_n v_n,$$

where $x_1 = \{v_0, v_1\}, x_2 = \{v_1, v_2\}, \ldots, x_n = \{v_{n-1}, v_n\}$. A walk is **closed** if $v_0 = v_n$.

Walks, Trails, Paths and Cycles

Definition (Trail)

A **trail** is a walk where the *edges are distinct*.

Definition (Closed Trail)

A trail is called **closed** is the first and the last vertex are equal.

Definition (Path)

A **path** is a walk where the *vertices are distinct*.

Walks, Trails, Paths and Cycles

Definition (Cycle)

A **cycle** is a closed walk where all the *vertices are distinct*.

Walks, Trails, Paths and Cycles

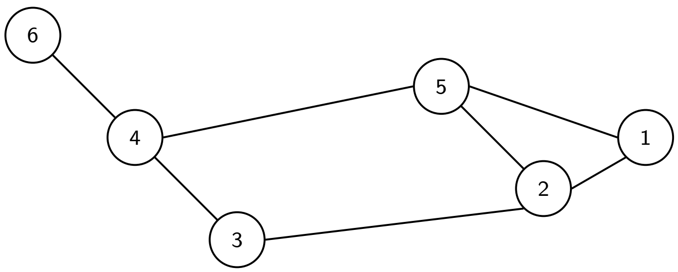
Let,

$$\begin{aligned} P_n : &\quad \text{path on } n \text{ vertices, and} \\ C_n : &\quad \text{cycle on } n \text{ vertices.} \end{aligned}$$

Then

1. Length of $P_n = n - 1$ and
2. Length of $C_n = n =$ number of edges.

Walks, Trails, Paths and Cycles: Example



- $< 1, 5, 4 >$: Is a walk from 1 to 4.
- $< 1, 3, 2, 5 >$: is not a walk.
- $< 1, 2, 5, 2, 3, 4, 5, 4, 6 >$: Is a walk from 1 to 6.
- $< 1, 2, 5, 4, 6 >$: Is a path from 1 to 6.
- $< 2, 3, 4, 5, 2 >$: Is a cycle.

More Definitions

Connectedness

Definition (Connected Vertices)

For a graph $G = (V, E)$ and $u, v \in V$, the vertices u and v are said to be **connected** if u and v are joined by a path.

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Define, a relation ρ called connectivity on $V \times V$ by

$$u \rho v, \text{ if and only if } u \text{ and } v \text{ are connected.}$$

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Define, a relation ρ called connectivity on $V \times V$ by

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Then

- ρ is an equivalence relation on V.
- ρ decomposes (partitions) V into *disjoint* subsets $V_1, V_2, \ldots, V_t$.

Components

Definition (Components)

The subgraph $G_1, G_2, \ldots, G_t$ induced by $V_1, V_2, \ldots, V_t$, respectively are called the **components** of G.

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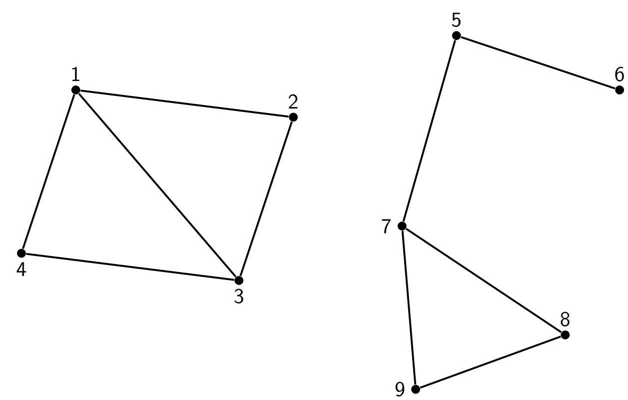
The subgraph $G_1, G_2, \ldots, G_t$ induced by $V_1, V_2, \ldots, V_t$, respectively are called the **components** of G.

$\therefore$ a component of a graph is a maximally connected subgraph.

Example: A Graph With Two Components

$$
\begin{aligned}
V &= \{1, 2, \ldots, 9\} \text{ and} \\
E &= \{(1,2), (1,3), (1,4), (2,3), (3,4), \\
&\quad (5,6), (5,7), (7,8), (7,9), (8,9)\}
\end{aligned}
$$

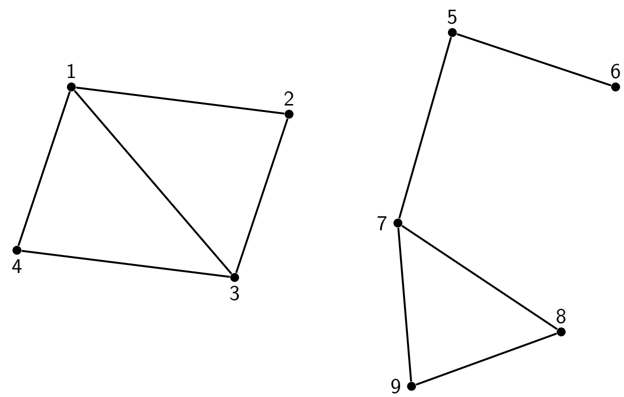


Note:

- Vertices $1, 2, 3, 4$ are all connected to each other.
- Similarly, vertices $5, 6, 7, 8, 9$ are also connected to each other.

Example: A Graph With Two Components

$$\begin{aligned} V &= \{1, 2, \ldots, 9\} \text{ and} \\ E &= \{(1,2), (1,3), (1,4), (2,3), (3,4), \\ &\quad (5,6), (5,7), (7,8), (7,9), (8,9)\} \end{aligned}$$



Note:

- 1 is not connected to 5, 2 is not connected to 5, . . .
- ρ decomposes (partitions) V into

$$V_1 = \{1, 2, 3, 4\} \quad \text{and} \quad V_2 = \{5, 6, 7, 8, 9\}.$$

Example: A Graph With Two Components

Component 1:

$$
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V_1 &= \{1, 2, 3, 4\} \text{ and} \\
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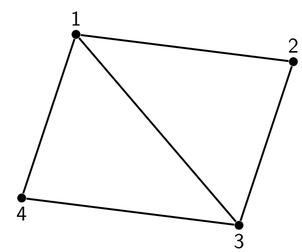
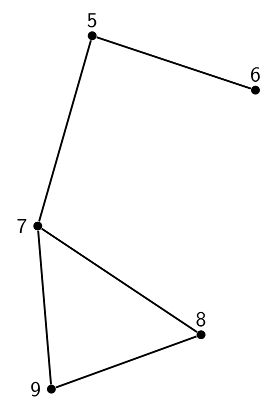


Example: A Graph With Two Components

Component 2:

$$
\begin{aligned}
V_2 &= \{5,6,7,8,9\} \text{ and} \\
E_2 &= \{(5,6),(5,7),(7,8),(7,9),(8,9)\}
\end{aligned}
$$



Definition: Weak and Strong Connectedness

Let $D = (V, A)$ be a digraph. There are three notions of connectivity for D.

1. D is connected as an undirected graph (i.e., connected if we ignore direction).

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3. D **is strongly connected:** For all $u, v \in V$, there is dipath from u to v and a dipath from v to u.
   - Partitions a digraph into **strongly connected components (s. c. c.)**.

Definition (Incident)

Let $G = (V, E)$ be a graph. Let $v \in V$ and $e \in E$. The edge e is said to be **incident on** v if one of the end points of e is v.

Incidence, Degree and Regular Graphs

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The **degree of a vertex** is the number of edges incident on it.

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A graph is said to be **regular** is all its vertices have the same degree.

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Definition (Weighted Graph)

A **weighted graph** is a graph where each edge has an associated **weight**, typically given by a weight function $w : E \to \mathbb{R}$.

Thank You for your kind attention!

Books and Other Materials Consulted

1. *Introduction to Algorithms* by Thomas H Cormen, Charles E Leiserson, Ronald L Rivest, Clifford Stein.

2. Graph Theory part taken from Discrete Mathematics Lecture Notes (M. Tech (CS), Monsoon Semester, 2007) taught by Prof. Palash Sarkar (ASU, ISI Kolkata).

Questions!!