Module 4 :

→ Principles of counting - II

- Principle of Inclusion and Exclusion.
- Derangements.
- Rook polynomials.

→ Recurrence Relations

- first order linear Recurrence Relation.
- second order linear Homogenous Recurrence relⁿ with constant coefficient

# Recurrence Relations

* **First order Recurrence relations:**

A I order recurrence relation with constant coefficients is of the form

$$a_n = c\, a_{n-1} + f(n) \rightarrow (1) \quad \text{for } n \geq 1.$$

where $c \rightarrow$ a known constant.

$f(n) \rightarrow$ a known function of $f$.

If $f(n) = 0$, then (1) is called a homogenous recurrence relation; otherwise it is called a non-homogenous recurrence relation.

Note:- 1) The General soln of a homogenous R.R is $a_n = c^n a_0$ for $n \geq 1$.

2) The General soln of a non-homogenous R.R of order 1 is given by $a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k)$, for $n \geq 1$.

Problems:-

1) Solve the recurrence relation $a_n = 7a_{n-1}$, where $n \geq 1$ given that $a_2 = 98$.

Soln:- $a_n = 7\, a_{n-1} \rightarrow (1)$ is a homogenous $1^{st}$ order recurrence relation.

General soln is given by

$$a_n = c^n a_0.$$

$$a_n = 7^n a_0 \rightarrow (2)$$

for $n = 2$; $a_2 = 7^2 a_0 \Rightarrow a_2 = 49 a_0$

$\Rightarrow 98 = 49 a_0 \Rightarrow \boxed{a_0 = 2}$

Sub in (2), $\boxed{a_n = 2 \cdot 7^n}$ is the General soln.

2) Solve the recurrence relation $a_n = n\, a_{n-1}$ for $n \geq 1$

given that $a_0 = 1$.

Sol$^n$:- $a_n = n\, a_{n-1}$

$n=1$; $a_1 = 1 \times a_0$

$n=2$; $a_2 = 2 \times a_1 = (2 \times 1) \times a_0$

$n=3$; $a_3 = 3 \times a_2 = (3 \times 2 \times 1) \times a_0$.

$n=4$; $a_4 = 4 \times a_3 = (4 \times 3 \times 2 \times 1) \times a_0$. and so on.

$\therefore$ General sol$^n$ is

$a_n = n!\, a_0$ for $n \geq 1$.

using $a_0 = 1$ $\Rightarrow$ $\boxed{a_n = n!}$ is the required sol$^n$.

3) Solve the recurrence relation $a_n - 3a_{n-1} = 5 \times 3^n$ for $n \geq 1$,

given that $a_0 = 2$.

Sol$^n$:- Given $a_n = 3a_{n-1} + (5 \times 3^n) \rightarrow (1)$ is a non-homo.

relation with $c = 3$, $f(n) = 5 \times 3^n$.

General sol$^n$ is given by

$$a_n = 3^n a_0 + \sum_{k=1}^{n} 3^{n-k} f(k)$$

$$a_n = 3^n a_0 + 3^{n-1} f(1) + 3^{n-2} f(2) + \dots + 3^0 f(n).$$

$$\Rightarrow a_n = 3^n \times 2 + 3^{n-1}(5 \times 3^1) + 3^{n-2}(5 \times 3^2) + \dots + 3^0 \times (5 \times 3^n)$$

$$= 2 \times 3^n + 5 \times [3^n + 3^n + \dots + 3^n] \text{ (n times)}$$

$$= 2 \times 3^n + 5 \times n \times 3^n$$

$\boxed{a_n = 3^n(2+5n)}$ is the required sol$^n$.

4) Solve the recurrence relation $a_n - 3a_{n-1} = 5 \times 7^n$, for $n \geq 1$,

given that $a_0 = 2$.

P.T.O.

Soln :- Given : $a_n = 3a_{n-1} + (5 \times 7^n) \rightarrow (1)$ is a non-homo. recurrence relation with $c = 3$, $f(n) = 5 \times 7^n$.

The general soln is given by

$$a_n = 3^n a_0 + \sum_{k=1}^{n} 3^{n-k} f(k).$$

$$= 3^n \times 2 + \sum_{k=1}^{n} 3^{n-k} \times (5 \times 7^k)$$

$$= 2 \times 3^n + (5 \times 3^n) \sum_{k=1}^{n} 3^{-k} . 7^k .$$

$$= 2 \times 3^n + (5 \times 3^n) \sum_{k=1}^{n} \left(\frac{7}{3}\right)^k .$$

$$= 2 \times 3^n + (5 \times 3^n) \left[ \frac{7}{3} + \left(\frac{7}{3}\right)^2 + \dots + \left(\frac{7}{3}\right)^n \right]$$

$$= 2 \times 3^n + (5 \times 3^n) \times \frac{7}{3} \left[ 1 + \left(\frac{7}{3}\right) + \left(\frac{7}{3}\right)^2 + \dots + \left(\frac{7}{3}\right)^{n-1} \right]$$

$$a + ar + ar^2 + \dots = \frac{a(r^n - 1)}{r - 1} ; r > 1$$

here $a = 1$, $r = 7/3$

$$\therefore a_n = 2 \times 3^n + (35 \times 3^{n-1}) \left[ \frac{\left(\frac{7}{3}\right)^n - 1}{\left(\frac{7}{3} - 1\right)} \right]$$

$$= (2 \times 3^n) + (35 \times \cancel{3^{n-1}}) \times \frac{\cancel{3}}{4} \left[ \frac{7^n - 3^n}{\cancel{3^n}} \right]$$

$$= (2 \times 3^n) + \left(\frac{35}{4}\right) (7^n - 3^n)$$

$$= 3^n \left[ 2 - \frac{35}{4} \right] + \frac{35}{4} . 7^n$$

$$= -3^n \times \frac{27}{4} + \frac{5 \times 7 \times 7^n}{4}$$

$$= -\frac{1}{4} . 3^{n+3} + \frac{5}{4} . 7^{n+1}$$

$$\boxed{a_n = \frac{1}{4} \left[ 5 \times 7^{n+1} - 3^{n+3} \right]}$$ is the required soln.

5) Solve the recurrence relation

$a_n = 2a_{n/2} + (n-1)$ for $n = 2^k$, $k \geq 1$, given $a_1 = 0$.

Soln:. $a_n = 2a_{n/2} + (n-1)$

$\Rightarrow$ $a_n - 2a_{n/2} = (n-1)$.

we obtain the following successive eqns

$$a_{n/2} - 2a_{n/4} = \left(\frac{n}{2} - 1\right)$$

$$a_{n/4} - 2a_{n/8} = \left(\frac{n}{4} - 1\right)$$

$$\vdots$$

$$a_{n/2^{k-1}} - 2a_{n/2^k} = \left(\frac{n}{2^{k-1}} - 1\right)$$

These can be written as

$$a_n - 2a_{n/2} = (n-1).$$

$$2a_{n/2} - 2^2 a_{n/4} = (n-2)$$

$$2^2 a_{n/4} - 2^3 a_{n/8} = (n-2^2)$$

$$\vdots$$

$$2^{k-1} a_{n/2^{k-1}} - 2^k \cdot a_{n/2^k} = (n - 2^{k-1}).$$

adding these, we get

$$a_n - 2^k \cdot a_{n/2^k} = (n-1) + (n-2) + (n-2^2) + \dots + (n-2^{k-1})$$

since $n = 2^k$, $a_{n/2^k} = a_1 = 0$ (given)

$$\therefore \quad a_n = (n+n+\dots+n)_{k \text{ times}} - (1+2+2^2+\dots+2^{k-1})$$

$a + ar + ar^2 + \dots = \dfrac{a(r^k - 1)}{r-1}$, $r > 1$

$a = 1$, $r = 2$

$$= kn - \frac{(1)(2^k - 1)}{2-1}$$

$$= kn - (2^k - 1) = kn - (n-1) \quad (\because n = 2^k)$$

$$= 1 + (k-1)n$$

$$\boxed{a_n = 1 + [\log_2 n - 1]n}$$

$(\because$ $2^k = n$.
$\log 2^k = \log n$
$k \cdot \log 2 = \log n$.
$k = \dfrac{\log_e n}{\log_e 2} = \log_2 n)$

6) Find the recurrence relation and the initial condⁿ for
QP the sequence 0, 2, 6, 12, 20, 30, 42 .... Hence find the general term of the sequence.

Soln:-

Given $a_0 = 0, \quad a_1 = 2, \quad a_2 = 6, \quad a_3 = 12, \quad a_4 = 20 \ldots$

Consider
$$a_1 - a_0 = 2 - 0 = 2 = 2 \times 1$$
$$a_2 - a_1 = 6 - 2 = 4 = 2 \times 2$$
$$a_3 - a_2 = 12 - 6 = 6 = 2 \times 3$$
$$a_4 - a_3 = 20 - 12 = 8 = 2 \times 4$$
$$\vdots$$
$a_n - a_{n-1} = 2 \times n$ is the R.R with the initial condⁿ $a_0 = 0$.

adding all these,
$$a_n - a_0 = (2\times1) + (2\times2) + (2\times3) + (2\times4) + \ldots + (2\times(n-1)) + (2\times n)$$
$$a_n - 0 = 2\left[1+2+3+\ldots+n\right]$$
$$a_n = 2\,\frac{n(n+1)}{2}$$
$$\boxed{a_n = n(n+1)}$$

7) The number of virus affected files in a system is 1000 (to
QP start with) and this increases 250% every 2 hours. Use a recurrence relation to determine the no. of virus affected files in the system after one day.

Soln:- Let $a_0 = 1000$

Let $a_n$ denote the no. of virus affected files after $2n$ hours.

It is given that the no. increases by 250% every 2 hours.

$$\therefore \quad a_1 = a_0 + 250\%\, a_0$$
$$a_2 = a_1 + 250\%\, a_1$$
$$\vdots$$
$$a_n = a_{n-1} + 250\%\, a_{n-1}$$

$$\therefore \quad a_n = a_{n-1}\left[1+250\%\right]$$
$$= a_{n-1}\left[1+\frac{250}{100}\right]$$
$$= a_{n-1}(1+2.5)$$
$$\boxed{a_n = 3.5\, a_{n-1}} \quad \forall\ n \geq 1.$$

This is the recurrence relation for the no. of virus affected files.

$\therefore$ General soln of the recurrence relation is given by

$$a_n = c^n a_0.$$
$$a_n = (3.5)^n \times 1000.$$

This gives the no. of virus affected files after 2n hours.

$\therefore$ No. of virus affected files after 24 hrs (1 day) (when $n=12$) is

$$a_{12} = (3.5)^{12} \times 1000. = 3379220508.$$

8) A person invests Rs. 10,000 at 10.5% interest (per year) compounded monthly. Find and solve the recurrence relation for the value of the investment at the end of n months. what is the value of the investment at the end of the I year? How long will it take to double the investment?

Soln :- Let $a_0$ denote the initial investment.

Let $a_1, a_2, \ldots a_n$ denote the investments after $1, 2, 3 \ldots n$ months respectively.

Given : annual rate of interest = 10.5%.

$\therefore$ Monthly rate of interest $= \frac{10.5\%}{12} = 0.875\%$.

Thus $a_0 = 10000$

$$a_1 = a_0 + (0.875\%)a_0.$$
$$a_2 = a_1 + (0.875\%)a_1$$
$$\vdots$$
$$a_n = a_{n-1} + (0.875\%)a_{n-1}$$
$$a_n = a_{n-1}\left[1+0.875\%\right]$$

$$\Rightarrow a_n = a_{n-1}\left[1 + \frac{0.875}{100}\right]$$

$$\boxed{a_n = 1.00875\, a_{n-1}}, \quad \forall\, n \geq 1.$$

This is the Recurrence relation at the end of 'n' months.

$\therefore$ General solⁿ of the recurrence relation is given by

$$a_n = c^n a_0.$$

$$a_n = (1.00875)^n \times 10000$$

$\therefore$ Investment at the end of first year is $(n = 12)$

$$a_{12} = (1.00875)^{12} \times 10000$$

$$= 11102.03$$

$$\underline{a_{12} \cong 11{,}102}$$

Next, to find $n$ given that

$$a_n = 2a_0.$$

$$\Rightarrow (1.00875)^n \times \cancel{10000} = 2 \times \cancel{10000}$$

$$\Rightarrow (1.00875)^n = 2$$

$$\Rightarrow n \log_e(1.00875) = \log_e 2$$

$$\Rightarrow n = \frac{\log_e 2}{\log_e(1.00875)} = 79.56.$$

$$\boxed{n \cong 80}.$$

Thus the investment will be doubled in about 80 months time ie 6 years and 8 months.

9) A bank pays a certain % of annual interest on deposits, compounding the interest once in 3 months. If a deposit doubles in 6 years and 6 months, what is the annual % of interest paid by the bank?.

Solⁿ: Let the annual rate of interest be $x\%$.

$\therefore$ Quarterly rate of interest is $\left(\frac{x}{4}\right)\%$.

Let $a_0$ be the initial deposit and $a_n$ be the deposit ~~after~~ at the end of nᵗʰ quarter.

$a_1 = a_0 + \left(\frac{x}{4}\%\right) a_0$

$a_2 = a_1 + \left(\frac{x}{4}\%\right) a_1$

$\vdots$

$a_n = a_{n-1} + \left(\frac{x}{4}\%\right) a_{n-1}$

$= a_{n-1}\left[1 + \frac{x}{4}\%\right]$

$\boxed{a_n = a_{n-1}\left(1+\frac{x}{400}\right)} \rightarrow (1)$ is the recurrence relation.

General solⁿ of (1) is

$a_n = c^n a_0$

$\Rightarrow a_n = \left(1+\frac{x}{400}\right)^n a_0$

Given that the deposit doubles in 6 yrs, 6 months (ie 78 months) ie deposit doubles in 26 quarters $\left(\because \frac{78}{3} = 26\right)$

$\therefore n = 26$.

we have $a_n = 2a_0$.

ie $a_{26} = 2a_0$.

$\Rightarrow \left(1+\frac{x}{400}\right)^{26} \cancel{a_0} = 2\cancel{a_0}$.

$\Rightarrow 26 \log_e\left(1+\frac{x}{400}\right) = \log_e 2$

$\Rightarrow \log_e\left(1+\frac{x}{400}\right) = 0.0266595$

$\Rightarrow 1+\frac{x}{400} = e^{0.0266595} = 1.027$

$\Rightarrow \frac{x}{400} = 0.027 \Rightarrow \boxed{x = 10.8}$

Thus the annual rate of interest paid by the bank is 10.8% (compounding the interest once in 3 months).

10) A bank pays 6% interest compound quarterly. If Laura invests Rs. 100 then how many months must she wait for her money to double?

HW Q

HINT :- 3 months – 6% interest
1 months – ?
$\frac{1\times 6\%}{3} = 2\%$.

$a_0 = 100$.
$a_n = a_{n-1}(1+2\%)$
$a_n = 1.02\, a_{n-1}$

$a_n = 2a_0$.
$\Rightarrow n = 35$ months.

## Second Order homogenous Recurrence Relation

A second order homogenous recurrence relation is of the form $c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0 \quad \forall n \geq 2 \rightarrow (1)$

where $c_n, c_{n-1}, c_{n-2}$ are real constants.

The auxilary equation of eqⁿ (1) is given by

$$c_n K^2 + c_{n-1} K + c_{n-2} = 0$$

Suppose $k_1$ and $k_2$ are the roots of A.E

Case (i): If $k_1$ and $k_2$ are real and distinct, then general solⁿ of eqⁿ (1) is given by

$$a_n = A k_1^n + B k_2^n \text{; where } A \text{ \& } B \text{ are arbitrary constants}$$

Case (ii): If $k_1 = k_2 = k$, then general solⁿ of eqⁿ (1) is given by

$$a_n = (A + Bn) k^n.$$

Case (iii): If $k_1$ and $k_2$ are imaginary i.e if $k_1 = p + iq$, and $k_2 = p - iq$, then the general solⁿ of eqⁿ (1) is given by

$$a_n = r^n [A \cos n\theta + B \sin n\theta] \quad \text{where } r = \sqrt{p^2 + q^2}, \quad \theta = \tan^{-1}\left(\frac{q}{p}\right).$$

**Solve the following Recurrence relations:**

1) $a_n + a_{n-1} - 6 a_{n-2} = 0 \quad \forall n \geq 2$ given $a_0 = -1, a_1 = 8$. $\rightarrow (1)$

Q1

Solⁿ:- comparing with $c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} = 0$, we have

$c_n = 1, \; c_{n-1} = 1, \; c_{n-2} = -6.$

A.E is $c_n K^2 + c_{n-1} K + c_{n-2} = 0.$

$\Rightarrow K^2 + K - 6 = 0$

$\Rightarrow (K + 3)(K - 2) = 0.$

$\Rightarrow$ Roots are $\boxed{k_1 = -3, \; k_2 = 2}$ real & distinct roots.

$\therefore$ General solⁿ of (1) is given by

$$a_n = A \cdot (-3)^n + B \cdot 2^n \rightarrow (2).$$

Given $a_0 = -1,\ a_1 = 8$.

sub $n=0$ in ②,

$$a_0 = A(-3)^0 + B(2)^0$$

$$-1 = A + B \rightarrow ③$$

sub $n=1$ in ②

$$a_1 = A(-3)^1 + B(2)^1$$

$$8 = -3A + 2B \rightarrow ④$$

Solving ③ & ④, $\boxed{A = -2,\ B = 1}$

sub in ②, $a_n = -2(-3)^n + 1\cdot(+2)^n$

ie $a_n = 2^n - 2(-3)^n$

2) $2a_n = 7a_{n-1} - 3a_{n-2}$, $n \geq 2$, $a_0 = 2,\ a_1 = 5$.

soln:- $2a_n - 7a_{n-1} + 3a_{n-2} = 0. \rightarrow ①$.

AE; $2k^2 - 7k + 3 = 0$.

Roots are $k_1 = 3,\ k_2 = 1/2$. (real & distinct)

General soln is given by

$$a_n = A\cdot 3^n + B\cdot\left(\frac{1}{2}\right)^n \rightarrow ②$$

given $a_0 = 2,\ a_1 = 5$

sub $n=0$ in ②, $a_0 = A + B$.

$\Rightarrow 2 = A + B \rightarrow ③$

sub $n=1$ in ②, $a_1 = 3A + \frac{1}{2}B$

$\Rightarrow 5 = \frac{6A + B}{2}$ (or) $6A + B = 10 \rightarrow ④$

solving ③ & ④, $\boxed{A = 8/5}$, $\boxed{B = 2/5}$.

sub in ②, $a_n = \left(\frac{8}{5}\right)3^n + \left(\frac{2}{5}\right)\left(\frac{1}{2}\right)^n$

3) $a_n - 6a_{n-1} + 9a_{n-2} = 0$, $n \geq 2$, $a_0 = 5,\ a_1 = 12$.
$\rightarrow ①$

soln:-

AE: $k^2 - 6k + 9 = 0$.

$\Rightarrow \boxed{K = 3, 3}$

Roots are real & repeated.

$\therefore$ General soln of ① is

$$a_n = (A + Bn)\, 3^n \longrightarrow ②$$

Given $a_0 = 5$, $a_1 = 12$

sub $n = 0$ in ② $\Rightarrow$ $a_0 = A \cdot 3^0$

$$\boxed{5 = A}$$

sub $n = 1$ in ② $\Rightarrow$ $a_1 = (A + B)\, 3^1$

$$12 = (5 + B)3$$
$$12 = 15 + 3B$$
$$3B = -3 \Rightarrow \boxed{B = -1}$$

sub in ①, $\boxed{a_n = (5 - n)\, 3^n}$

4) $4a_n + 2a_{n-1} + a_{n-2} = 0$.

Soln: AE: $4k^2 + 2k + 1 = 0$.

$$k = \frac{-2 \pm \sqrt{4 - 16}}{8} = \frac{-2 \pm \sqrt{-12}}{8} = \frac{-2 \pm 2i\sqrt{3}}{8}$$

$$\boxed{k = \frac{-1 \pm \sqrt{3}\, i}{4}}$$

Roots are $k_1 = -\frac{1 + \sqrt{3}i}{4}$, $k_2 = -\frac{1 - \sqrt{3}i}{4}$ (imaginary roots)

comparing with $p \pm iq$, $p = -\frac{1}{4}$, $q = \frac{\sqrt{3}}{4}$.

$$\therefore\ r = \sqrt{p^2 + q^2} = \sqrt{\frac{1}{16} + \frac{3}{16}} = \sqrt{\frac{4}{16}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\theta = \tan^{-1}\left(\frac{q}{p}\right) = \tan^{-1}\left(\frac{\sqrt{3}/4}{-1/4}\right) = \tan^{-1}(-\sqrt{3}) = -60^\circ = -\pi/3$$

$\therefore$ General soln of ① is

$$a_n = r^n \left[A \cos n\theta + B \sin n\theta\right]$$

$$a_n = \left(\frac{1}{2}\right)^n \left[A \cos\left(-\frac{n\pi}{3}\right) + B \sin\left(-\frac{n\pi}{3}\right)\right].$$

5) $a_n = 2(a_{n-1} - a_{n-2})$, for $n \geq 2$ given that $a_0 = 1$ & $a_1 = 2$.

HW

Ans: $a_n = (\sqrt{2})^n \left[\cos \frac{n\pi}{4} + \sin \frac{n\pi}{4}\right]$.

6) $D_n = bD_{n-1} - b^2 D_{n-2}$ for $n \geq 3$, given $D_1 = b > 0$, $D_2 = 0$

Soln:- $D_n - bD_{n-1} + b^2 D_{n-2} = 0 \rightarrow (1)$

AE: $k^2 - bk + b^2 = 0$

$$k = \frac{b \pm \sqrt{b^2 - 4b^2}}{2} = \frac{b \pm \sqrt{-3b^2}}{2} = \frac{b \pm i\sqrt{3}\,b}{2}$$

$\therefore k_1 = \frac{b}{2} + i\frac{\sqrt{3}b}{2}$ and $k_2 = \frac{b}{2} - \frac{i\sqrt{3}b}{2}$ (imaginary roots)

$\therefore$ General soln for $D_n$ is

$$D_n = r^n [A \cos n\theta + B \sin n\theta] \rightarrow (2)$$

where A and B are arbitrary constants.

$p = \frac{b}{2}, q = \frac{\sqrt{3}b}{2}$

$$r = \sqrt{p^2 + q^2} = \sqrt{\frac{b^2}{4} + \frac{3b^2}{4}} = \sqrt{\frac{4b^2}{4}} = b.$$

$$\theta = \tan^{-1}\left(\frac{q}{p}\right) = \tan^{-1}\left(\frac{\sqrt{3}b/2}{b/2}\right) = \tan^{-1}\sqrt{3} = \pi/3.$$

(2) $\Rightarrow$ $D_n = b^n\left[A \cos \frac{n\pi}{3} + B \sin \frac{n\pi}{3}\right] \rightarrow (3)$ is the G. Soln.

given $D_1 = b$, $D_2 = 0$.

Put $n=1$ in (3), $D_1 = b\left[A \cos\frac{\pi}{3} + B \sin \frac{\pi}{3}\right]$

$$\cancel{b} = \cancel{b}\left[A \cdot \frac{1}{2} + B \cdot \frac{\sqrt{3}}{2}\right]$$

$$\Rightarrow \boxed{1 = \frac{1}{2}A + \frac{\sqrt{3}}{2}B.} \rightarrow (4)$$

Put $n=2$ in (3), $D_2 = b^2\left[A \cos \frac{2\pi}{3} + B \sin \frac{2\pi}{3}\right]$

$$0 = b^2\left[A \cos(180^\circ - 60^\circ) + B \sin(180^\circ - 60^\circ)\right]$$

$$\Rightarrow 0 = -A \cos 60^\circ + B \sin 60^\circ$$

$$\Rightarrow \boxed{0 = -\frac{1}{2} \cdot A + \frac{\sqrt{3}}{2}B} \rightarrow (5)$$

solving (4) & (5), $\boxed{A = 1, \quad B = 1/\sqrt{3}}$

Sub in (3),

$$D_n = b^n\left[\cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}} \sin\frac{n\pi}{3}\right].$$

7) $F_{n+2} = f_{n+1} + f_n$ for $n \geq 0$, given $F_0 = 0$, $F_1 = 1$.

Q1.

soln:- Rewriting as $f_n = f_{n-1} + f_{n-2}$

$$\Rightarrow f_n - F_{n-1} - F_{n-2} = 0 \quad \text{for } n \geq 2. \rightarrow (1)$$

AE: $k^2 - k - 1 = 0$.

$$k = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$ (real & distinct roots)

$$k_1 = \frac{1+\sqrt{5}}{2}, \quad k_2 = \frac{1-\sqrt{5}}{2}$$

General soln of (1) is

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n \rightarrow (2)$$, where A & B are arbitrary constants.

Given $F_0 = 0$, $f_1 = 1$.

sub $n = 0$ in (2), $f_0 = A + B$

$$\boxed{0 = A + B} \rightarrow (3)$$

sub $n = 1$ in (2), $f_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$

$$\Rightarrow \boxed{1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)} \rightarrow (4)$$

Eqn (3) $\times \left(\frac{1+\sqrt{5}}{2}\right)$ gives $0 = \left(\frac{1+\sqrt{5}}{2}\right)A + \left(\frac{1+\sqrt{5}}{2}\right)B \rightarrow (5)$

(4) − (5) $\Rightarrow$ $1 = B\left(\frac{1-\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$

$$1 = B\left(-\frac{2\sqrt{5}}{2}\right) \Rightarrow \boxed{B = -1/\sqrt{5}}$$

from (3), $A = -B \Rightarrow \boxed{A = 1/\sqrt{5}}$

sub in (2),

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

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