Module 5:

Introduction to graph theory.

→ Definitions and Examples.

→ Subgraphs.

→ complement of a graph.

→ Graph Isomorphism

→ vertex degree.

→ Euler trails and circuits.

Graphs:- A Graph is a pair (V,E), where V is a non-empty set and E is a set of unordered pairs of elements taken from the set V.

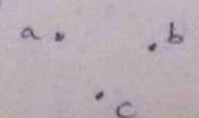
The elements of V are called Vertices and the set V is called Vertex set. The elements of E are called undirected edges (or) just edges and the set E is called Edge set.

Note:- 1) The Vertex set of a graph/digraph has to be non-empty but the edge set can be empty.

2) (A,B) denote an ordered pair of A & B (or) a directed edge. $\{A,B\}$ denote an unordered pair of A & B (or) an undirected edge.

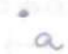
Null Graph:- A graph/digraph containing no edges is called a Null Graph.

Ex:-



$V = \{a, b, c\}$, $E = \phi$ (or) $\{\ \}$

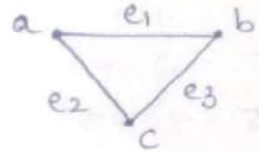
Trivial Graph :- A null graph with only one Vertex is Called a trivial Graph.

Ex :- 

$V = \{a\}$ & $E = \{\ \}$

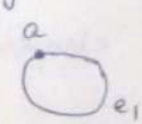
Finite Graph/digraph :- A graph (or) digraph with only a finite no of Verticies and edges is Called a finite graph/digraph.

End Verticies :- If V_i and V_j denote two verticies of a graph and if e_k denotes the edge joining V_i and V_j, then V_i & V_j are Called the End Verticies of e_k.

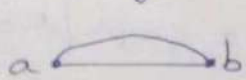
Ex :- 

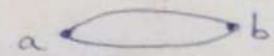
a & b are end Verticies of e_1
a & c " ———— " e_2
b & c " ———— " e_3

Loop :- An edge whose end Verticies are Same is Called a Loop

Ex :- 

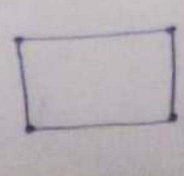
Multiple edges :- Two (or) more edges having Same end Verticies are Called Multiple edges. 

Parallel edges :- Two edges having Same end verticies are Called parallel edges 

(or) 

Simple Graph :- A graph which does not Contain loops and multiple edges is Called Simple Graph.

Loop-free :- A graph which does not Contain loop, is Called Loop-free.

Ex :- 

Simple Graph / Loop-free

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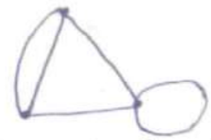
Multigraph :- A graph which contains multiple edges but no loops is called Multi-graph.

General Graph :- A graph which contains multiple edges (or) loops (ON both is called a General graph.

Multigraph



General Graph

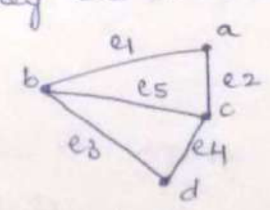
Incidence :- If 'V' is an end vertex of an edge 'e' in a graph G then the edge 'e' is incident on (to) the vertex 'V'.

Note :- 1) Every edge is incident on two verticies one at each end.
2) The two end verticies are coincident if the edge is a loop.

Adjacent Verticies :- Two verticies are said to be adjacent verticies if there is an edge joining them.

Adjacent edges :- Two non-parallel edges are said to be adjacent edges if they are incident on a common vertex.

Ex :-



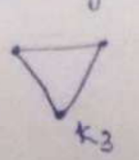
Adj - verticies :- (a,b) (a,c) (b,c) (b,d) (c,d)

Adj - edges :- (e_1,e_5) (e_1,e_2) (e_2,e_5) (e_2,e_4) (e_4,e_5) (e_3,e_1) (e_3,e_4) (e_5,e_3)

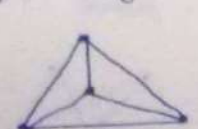
Complete Graph :- A simple graph of order ≥ 2 (ie no of verticies ≥ 2) in which there is an edge b/w every pair of verticies is called a Complete graph / full graph. It is denoted by K_n.

Ex :-

K_2



K_3



K_4

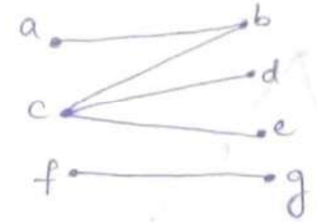
K_5

K_5 is called the Kuratowski's first graph.

Bipartite Graph :- A simple Graph G in which its vertex set V is the union of two of its mutually disjoint non-empty subsets V_1 and V_2 and each edge in G joins a vertex in V_1 and a vertex in V_2, then G is said to be a bipartite graph.

Ex :-

G :



In the above Graph G,

$$V = \{a, b, c, d, e, f, g\}$$

$$E = \{ab, cb, cd, ce, fg\}$$

V is the union of two of its subsets $V_1 = \{a, c, f\}$ $V_2 = \{b, d, e, g\}$ which are such that

(i) V_1 & V_2 are disjoint $[\Rightarrow V_1 \cap V_2 = \phi]$

(ii) every edge in G joins a vertex in V_1 & a vertex in V_2.

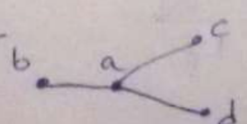
& (iii) G contains no edge that joins two vertices both of which are in V_1 (or) V_2.

Thus G is a bipartite Graph with V_1 & V_2 as bipartites (or partitions).

Complete Bipartite Graph :- A bipartite graph $G = (V_1, V_2; E)$ is said to be a complete bipartite graph if there is an edge b/w every vertex in V_1 & V_2. It is denoted by $K_{r,s}$ where r is the no of vertices in V_1 & s is the no of Vertices in V_2, with $r \leq s$.

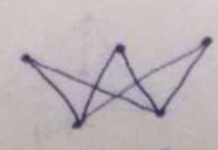
Thus, $K_{r,s}$ has $r+s$ vertices and rs edges.

Ex :-

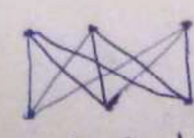


$K_{1,3}$

$V_1 = \{a\}$, $V_2 = \{b, c, d\}$



$K_{2,3}$



~~$K_{3,3}$~~ $K_{3,3}$

$K_{3,3}$ is called as Kuratowski's 2^{nd} Graph.

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Problems :-

1) If $G = G(V,E)$ is a simple Graph, prove that $2|E| \le |V|^2 - |V|$

Soln :- In a simple Graph, there are no multiple edges.

Each edge of a graph is determined by a pair of verticies.

The no of edges cannot exceed the no of Pairs of verticies.

The no of pair of verticies that can be choosen from n verticies is

$$^nC_2 = \frac{n!}{(n-2)!\,2!} \qquad \left[\because {}^nC_r = \frac{n!}{(n-r)!\,r!}\right]$$

$$= \frac{n(n-1)(n-2)!}{2\,(n-2)!} = \frac{n(n-1)}{2}$$

Thus for a simple Graph with $n (\geq 2)$ verticies, the no of edges cannot exceed $\frac{1}{2}n(n-1)$.

Thus if a graph $G = G(V,E)$ has n-verticies & m edges, then

$$m \le \frac{1}{2}n(n-1)$$

$$2m \le n^2 - n$$

ie $2|E| \le |V|^2 - |V|$.

Note :- In a simple Graph, for a pair of verticies, we can have only one edge
ie 2 verticies $\longleftrightarrow$ 1 edge
Hence no of edges cannot exceed no of pair of verticies.

2) Show that a complete graph with n verticies, namely K_n has $\frac{1}{2}n(n-1)$ edges.

Soln :- In a complete graph, there exists exactly one edge b/w every pair of verticies.

$\therefore$ the no of ~~ver~~ edges in a complete graph = the no of pair of verticies.

If the no of verticies is n, then no of pair of verticies is

$$^nC_2 = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)(n-2)!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$

Thus in a complete Graph with n verticies, no of edges, $m = \frac{1}{2}n(n-1)$

3> Show that a simple Graph of order $n=4$ & size $m=7$ & a complete graph of order $n=4$ & size $m=5$ do not exist.

Soln :- For a simple Graph,

$$2m \leq n^2 - n.$$

$$2(7) \leq 16-4$$

$$14 \leq 12 \;\#\; \text{not true.}$$

for a complete graph,

$$m = \frac{1}{2}n(n-1)$$

$$5 = \frac{1}{\cancel{2}} \cdot \cancel{4}^{2} \times 3$$

$$5 = 6 \;\#\; \text{not true.}$$

4> (a) How many verticies and how many edges are there in the complete bipartite graphs $K_{4,7}$ & $K_{7,11}$?

(b) If the graph $K_{r,12}$ has 72 edges, what is r?

Soln :- A complete Bipartite Graph $K_{r,s}$ has $r+s$ verticies and rs edges.

(a) The Graph $K_{4,7}$ has $4+7=11$ verticies and $4 \times 7 = 28$ edges

& the Graph $K_{7,11}$ has $7+11 = 18$ verticies & $7 \times 11 = 77$ edges

(b) If the graph $K_{r,12}$ has 72 edges, then

$$12r = 72 \Rightarrow \boxed{r=6}$$

5> Let $G=(V,E)$ be a simple graph of order $|V|=n$ & size $|E| = m$. If G is a bipartite graph, P.T $4m \leq n^2$.

Soln :- Let V_1 and V_2 be bipartites of G, with $|V_1| = r$ & $|V_2| = s$.

Since for the Given graph, $|V| = n$, we should have $r+s=n$

so that ~~r=n-s~~ ~~and~~ $s = n-r$.

The graph G has the maximum no of edges when each of the r-verticies in V_1 is joined by an edge to each of the s-verticies in V_2 & this maximum is equal to rs.

$\Rightarrow \quad |E| = m \leq rs$.

ie $\quad m \leq rs = r(n-r)$

$= rn - r^2$

$= -(r^2 - nr)$

by completing the square, we have

$rs = -\left[\left(r - \frac{1}{2}n\right)^2 - \left(\frac{1}{2}\cdot n\right)^2\right]$

$rs = -\left[\left(r - \frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2\right]$

(or) $\quad rs = \left(\frac{n}{2}\right)^2 - \left(r - \frac{n}{2}\right)^2$, this is maximum when $r = \frac{n}{2}$.

Thus when $r = \frac{n}{2}$, $rs = \frac{n^2}{4}$

Hence $\quad m \leq rs = \frac{n^2}{4}$

$\Rightarrow \quad m \leq \frac{n^2}{4} \quad \Rightarrow \quad 4m \leq n^2$

6) Show that a simple graph of order $n=4$ & size $m=5$ cannot be a bipartite graph.

Soln :- for a bipartite graph, $4m \leq n^2$. $\qquad 4(5) \leq 4^2$

$20 \leq 16$ #

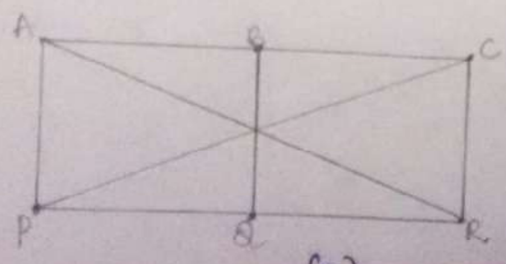
Here $4m = 4(5) = 20$ & $n^2 = 4^2 = 16$.

$4m > 16$ #

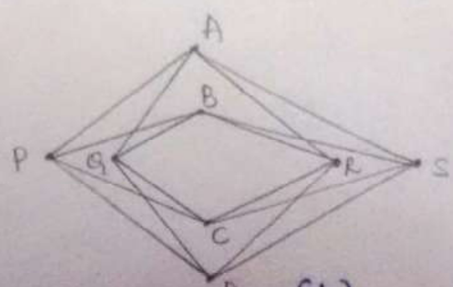
Hence not a bipartite Graph.

Hw

1) Verify that the foll are bipartite Graphs? what are their bipartites?



(a)



(b)

2) Company X has offices in cities B, D & K; Company Y in cities B & M; Company Z in cities C & M. Represent this situation by a bipartite Graph. is this a complete bipartite Graph?

3) State whether the foll Graphs can exist or cannot exist:

✓ (a) Simple Graph of order 3 & size 2
× (b) " ——— " —— 5 & " 12
✓ (c) Complete " ——— " 5 & size 10
✓ (d) Bipartite " ——— " 4 & " 3
× (e) " ——— " —— 3 & " 4
✓ (f) complete Bipartite Graph of order 4 & size 4.

✓ → exist
× → not

Note :- 1) for a simple Graph, $2|E| \leq |V|^2 - |V|$

2) for a complete Graph, $m = \frac{1}{2} n(n-1)$.

3) for a complete Bipartite Graph, $K_{r,s}$, $r+s$ vertices & rs edges exists.

4) for a Bipartite Graph, $4m \leq n^2$.

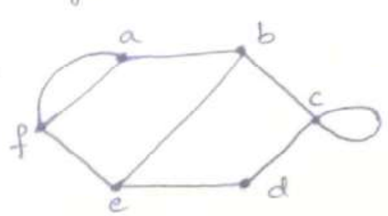
Vertex Degree :-

Let G be a graph & v be a vertex of G. Then the no of edges of G that are incident on v with the loops counted twice is called the degree of the vertex v and is denoted by $\deg(v)$ (or) $d(v)$.

Degree Sequence :- The degrees of the verticies of a graph arranged in non-decreasing order is called degree sequence.

Degree of the graph :- The minimum of the degrees of verticies of a graph is called the degree of the graph.

Ex :-



$d(a) = 3$ $\quad$ $d(e) = 3$
$d(b) = 3$ $\quad$ $d(f) = 3$
$d(c) = 4$
$d(d) = 2$

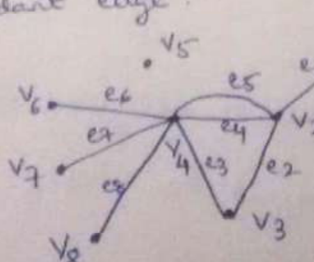
$\therefore$ The degree sequence is $2, 3, 3, 3, 3, 4$ and the degree of the graph is '2'.

Isolated Vertex :- A vertex in a graph which is not an end vertex of any edge of the graph is called an isolated vertex.
ie a vertex is an isolated vertex if & only if its degree is zero.

Pendant Vertex :- A vertex of degree 'one' is called a pendant vertex.

Pendant edge :- An edge incident on a pendant vertex is called a pendant edge.

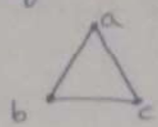
Ex :-



$v_5 \to$ isolated vertex.
v_1, v_6, v_7, v_8 are pendant verticies.
e_1, e_6, e_7, e_8 are pendant edges.

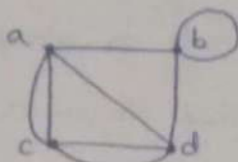
Regular Graph :- A graph in which all the verticies are of same degree 'K' is called a regular graph of degree 'K' (or) a K-regular Graph.

Ex :-



$d(a) = d(b) = d(c) = 2$

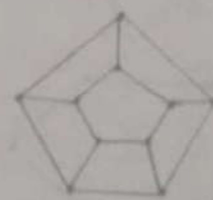
$\Rightarrow$ 2-regular Graph.



$d(a) = 4 = d(b) = d(c) = d(d)$

Petersen Graph :- A 3-regular Graph which contains 10 verticies and 15 edges is called petersen Graph.

(or)



Three-dimensional hypercube :-

A cubic graph with $8 = 2^3$ verticies is called 3-dimensional hypercube & is denoted by Q_3

In General, for any +ve integer k, a loop-free K-regular graph with 2^k verticies is called the k-dimensional hypercube & is denoted by Q_k.

Ex :-

3- dimensional hypercube

Hand shaking Property :- The sum of the degrees of all verticies in a graph is an even no, and this no is equal to twice the no of edges in the Graph.

ie for a graph $G = (V, E)$, $\sum_{v \in V} deg(v) = 2|E|$

Proof :- The above property is obvious from the fact that while counting the degrees of verticies, each edge is counted twice (once at each end).

(Just say this
Note :- The name "Hand shaking property" because "if several people shake hands, then the total no of hands shaken must be even, since two hands are involved in each handshake".)

Theorem :- In every graph, the no of verticies of odd degree is Even.

Proof :- Consider a graph with 'n' verticies.

Suppose 'k' of these verticies are of odd degree then the remaining n-k verticies are of even degree.

Let $v_1, v_2 \dots v_k$ be the verticies of odd degree and $v_{k+1}, v_{k+2} \dots v_n$ be " ________ even degree.

then $$\sum_{i=1}^{n} \deg(v_i) = \sum_{i=1}^{k} \deg(v_i) + \sum_{i=k+1}^{n} \deg(v_i) \longrightarrow (1)$$

In view of hand shaking property, the sum on the LHS of the above expression is equal to twice the no of edges in the graph. This sum is even.

The second sum in the RHS, is the sum of the degree of verticies with even degree. This sum is also even.

$\therefore$ the 1st sum in the RHS must also be even.

ie $d(v_1) + d(v_2) + \dots + d(v_k) = \text{Even} \longrightarrow (2)$.

but each of $d(v_1), d(v_2) \dots d(v_k)$ is odd.

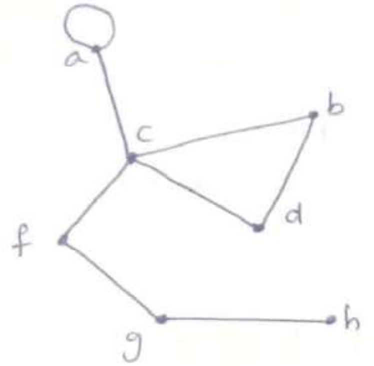
$\therefore$ the no of terms in the LHS of (2) must be even [$\because$ odd no's added even no of times, the result is Even]

ie k is Even.

Hence the proof.

Problems :-

1) For the given graph, indicate the degree of each vertex & verify the handshaking property.



Soln :- deg (a) = 3, deg (b) = 2, deg (c) = 4, deg (d) = 2
d(e) = 0 , deg (f) = 2, deg (g) = 2 , deg (h) = 1

The sum of the degrees of verticies = 16.

The graph has 8 edges.

Thus the sum of the degrees of verticies = twice the no of edges.

16 = 2 (8) True.

Hand shaking property is verified.

2) Can there be a graph with 12 verticies such that two of the verticies have degree 3 each & the remaining 10 verticies have degree 4 each ? If so, find its size ie $|E|$?

Soln :- Sum of the deg of verticies = $(2\times3) + (10\times4)$
$= 6 + 40$
$= 46$, which is Even.

By Hand Shaking property, sum of the deg of verticies = twice the no of edges.

ie $46 = 2|E| \Rightarrow |E| = 23$ ~~property holds.~~

Hence there can be a graph of the desired type (whose size is 23)

3) In a graph $G = (V, E)$, what is the largest possible value for $|V|$ if $|E| = 19$ and $\deg(v) \geq 4$ for all $v \in V$?

Soln :- Given :- all verticies are of degree greater than (or) equal to 4.

$\therefore$ the sum of the degrees of verticies $\geq 4n$

where $n = |V|$ & this sum = twice the no of edges.

$$\therefore \quad 2|E| \geq 4n$$
$$2 \times 19 \geq 4n \quad \Rightarrow \quad n \leq \frac{38}{4} = 9.5 < 10$$

Thus the largest possible value of $|V| = 9$

ie given graph can have atmost 9 verticies.

4) Prove that the hypercube Q_n has $n2^{n-1}$ edges.
Determine the no of edges in Q_8.

Soln :- In the hypercube Q_n, the no of verticies is 2^n and each vertex is of degree 'n'.

$\therefore$ the sum of the degrees of verticies of Q_n is $n \times 2^n$.

By Hand shaking property, we have

Sum of the deg of verticies $= 2 \times$ no of edges

ie $\quad n \times 2^n = 2|E|$

$$\Rightarrow \quad |E| = \frac{1}{2} \times n \times 2^n = n2^{n-1}$$

$\therefore$ Q_n has $n2^{n-1}$ no of edges.

Also, No of edges in $Q_8 = 8 \times 2^7 = 1024$.

Do't 5) (a) what is the dimension of the hypercube with 524288 edges?

(b) How many verticies are there in a hypercube with 4980736 edges?

P.T.O.

Soln:- for the k-dimensional hypercube Q_k, the no of verticies is 2^k & no of edges is $k.2^{k-1}$.

(a) To find k, given $k.2^{k-1} = \cancel{54}\,524288$

For (b), keep dividing by 2, until you get a no not divisible by 2. whatever asked, express in terms of 2 power

ie $k.2^{k-1} = 2^{19}$

$= 2^4 \times 2^{15}$

$k.2^{k-1} = 16 \times 2^{15}$ of the form $k.2^{k-1}$

$\Rightarrow \boxed{k = 16}$

Thus the dimension of the hypercube with 524288 edges is

$k = 16$

(b) we have $4980736 = 19 \times 2^{18}$ of the form $k.2^{k-1}$,

which indicates that Q_k has 4980736 edges when $k = 19$.

$\therefore$ In this hypercube, no of verticies is $2^k = 2^{19} = 524288$.

6) (a) If k is odd, show that the no of verticies in a k-regular graph is even

(b) Show that it is not possible to have a set of nine people at a party such that each one knows exactly five of the others in the party.

Soln:-(a) In a k-regular graph, the degree of each vertex is k.

$\therefore$ if such a graph has n verticies, then the sum of degrees is nk, and this has to be an Even no (by handshaking property)

If k is odd, n must be Even to satisfy the above.

Hence the proof

(b) Let G be a graph with 9 verticies, each vertex representing a person in the given set, and each edge representing an acquiantance (given situation).

P.T.O

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If each person in the set (party) knows exactly five other persons (10)
in the set (party), then there will be exactly five edges incident on each vertex & the Graph G will be 5-regular.

This is not possible b'coz G has an odd no of verticies. (Since by (a), if k is odd, n must be Even).

Hence the graph G of the desired type does not exist.

7)(a) If a graph with n verticies & m edges is k-regular, Show that $m = kn/2$.

(b) Does there exist a cubic graph with 15 verticies?

(c) " ——————— " 4-regular graph with 15 edges?

Soln :- (a) Given :- The Graph G is k-regular.

$\Rightarrow$ the degree of every vertex is k.

$\therefore$ if G has n verticies, then the sum of deg of verticies is nk.
By Hand Shaking property, this must be equal to 2m (if G has m edges)

ie $nk = 2m \Rightarrow m = \frac{nk}{2}$.

(b) If there is a cubic graph (3-regular graph) with 15 verticies, the no of edges it should have is $m = \frac{kn}{2}$

$m = \frac{3 \times 15}{2} = \frac{45}{2}$ (not an integer ~~whole no~~)

Thus the graph of desired type does not exist.

(c) If there is a 4-regular graph with 15 edges (ie $k = 4$, $m = 15$), the no of ~~edges~~ verticies it should have is $n = \frac{2m}{k}$.

$\Rightarrow n = \frac{2 \times 15}{4} = \frac{30}{4}$ (not an ~~whole no~~ integer)

Thus the graph of desired type does not exist.

8) (a) Show that in a complete Graph of n verticies (namely K_n), the degree of every vertex is $(n-1)$ & that the total no of edges is $\frac{n(n-1)}{2}$.

(b) If K_n has 'm' edges, S.T $n(n+1) = 2(n+m)$

Soln :- A complete graph is a simple graph in which every ~~edge~~ vertex is joined with every other vertex through exactly one edge.

∴ If there are 'n' verticies, each vertex is joined to $(n-1)$ verticies through exactly one edge.

Hence there occur $(n-1)$ edges at every vertex

$\Rightarrow$ degree of every vertex is $(n-1)$.

∴ sum of degree of verticies is $n(n-1)$, this sum must be equal to $2m$ (by H.S.P)

i.e $n(n-1) = 2m \Rightarrow m = \frac{1}{2}n(n-1)$.

Thus K_n has $\frac{1}{2}n(n-1)$ edges.

(b) If K_n has m edges, then

$$m = \frac{1}{2}n(n-1)$$

add n on b.s

$$n+m = n + \frac{1}{2}n(n-1)$$

$$2(n+m) = 2n + n^2 - n = n^2 + n$$

$$2(n+m) = n(n+1)$$

9) Show that there is no graph with 12 verticies & 28 edges in the foll cases :-

(a) The degree of a vertex is either 3 (or) 4

(b) The degree of a " ______ " 3 (or) 6.

Soln :- Suppose there is a graph with 28 edges & 12 verticies, of which k verticies are of deg 3 (each), then :

(a) If all the remaining $(12-k)$ verticies have degree 4, (11)

then $3k + 4(12-k) = 2\times 28$

$3k + 48 - 4k = 56$

$-k = 8 \Rightarrow k = -8$ #

(b) If all the remaining $(12-k)$ verticies have degree 6,

then $3k + 6(12-k) = 56$

$3k + 72 - 6k = 56.$

$-3k = -16 \Rightarrow k = \frac{16}{3}$ #

Hence in both cases, the graph of the desired type cannot exist.

10) Determine the order $|V|$ of the graph $G=(V,E)$ in the foll...

Cases :- (a) G is a cubic graph with 9 edges

(b) G is regular with 15 edges

(c) G has 10 edges with 2 verticies of deg 4 & all others of deg 3.

Soln :- (a) Suppose the order of G is n,

Since G is a cubic graph, all verticies of G have deg 3.

$\therefore$ Sum of degrees of verticies is $3n$.

Since G has 9 edges, by HSP,

$3n = 2\times 9 = 18$

$\Rightarrow n = 6 \Rightarrow |V| = 6.$

(b) Given : G is a regular graph

$\Rightarrow$ all verticies of G must be of same degree, say k.

Let G be of order 'n', then the sum of the degree of verticies is nk.

Since G has 15 edges, by HSP

$nk = 2\times 15 \Rightarrow n = \frac{30}{k}$

Since k is a +ve integer, it follows that 'n' must be a divisor of 30.

ie n must be $1, 2, 3, 5, 6, 10, 15$ and 30 (possible orders of G)

(c) Suppose the order of G is 'n'.

Since 2 verticies of G are of deg 4 & all others are of deg 3, sum of the degrees of verticies is

$$(2\times4)+(n-2)3 = 8+3(n-2).$$

By HSP,

$$8+3(n-2) = 2\times10$$
$$3n-6 = 20-8 = 12$$
$$3n = 18 \Rightarrow \boxed{n=6}$$
$$\text{ie } |V| = 6.$$

Hw

1) Consider a graph having n verticies & m edges. If p no of verticies are of deg k & the remaining verticies are of deg $k+1$, P.T $p = (k+1)n - 2m$.

2) P.T there is no simple graph with 7 verticies, one of which has deg 2, 2 have deg 3, 3 have deg 4 & the remaining vertex has deg 5.

×

11) For a graph with 'n' vertices and 'm' edges, if δ is the minimum, Δ is the maximum of degree of vertices, show that $\delta \leq \frac{2m}{n} \leq \Delta$.

Sol^n:- Let $d_1, d_2 \ldots d_n$ be the degrees of $1^{st}, 2^{nd} \ldots n^{th}$ vertex resp, then

$$\Sigma \deg(v) = d_1+d_2+\ldots+d_n$$

by HSP, $\Sigma \deg(v) = 2|E| = 2m$

$$\therefore d_1+d_2+\ldots+d_n = 2m \rightarrow (1).$$

Given that δ is min of $d_1, d_2 \ldots d_n$

ie $\delta \leq d_1, \delta \leq d_2, \ldots \delta \leq d_n$.

$$\therefore \delta+\delta+\ldots+\delta_{(n\text{ times})} \leq d_1+d_2+\ldots+d_n$$
$$\Rightarrow n\delta \leq 2m \quad (\text{by } (1))$$
$$\Rightarrow \delta \leq \frac{2m}{n} \rightarrow (2)$$

Also given that Δ is max of $d_1, d_2 \ldots d_n$

ie $\Delta \geq d_1, \Delta \geq d_2 \ldots \Delta \geq d_n$

$$\Rightarrow \Delta+\Delta+\ldots+\Delta_{(n\text{ times})} \geq d_1+d_2+\ldots+d_n$$
$$\Rightarrow n\Delta \geq 2m \quad (\text{by } (1))$$
$$\therefore \Delta \geq \frac{2m}{n} \quad (\text{or}) \quad \frac{2m}{n} \leq \Delta \rightarrow (3)$$

from (2) & (3),

$$\delta \leq \frac{2m}{n} \leq \Delta.$$

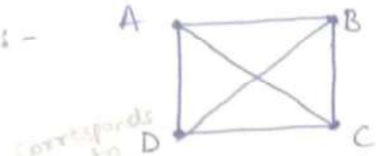
Isomorphism :-

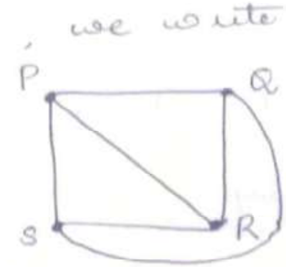
Two graphs G and G' are said to be isomorphic if there is a one-one correspondance b/w their verticies & b/w their edges such that the adjacency of verticies is preserved.

Such graphs will have the same structure, differing only in the way their verticies & edges are labelled (or) only in the way they are represented geometrically.

If G & G' are isomorphic, we write $G \cong G'$.

Ex :-





$A \leftrightarrow P$, $B \leftrightarrow Q$, $C \leftrightarrow R$, $D \leftrightarrow S$ — 1-1 correspondance b/w verticies.

$\{A, B\} \leftrightarrow \{P, Q\}$ $\quad$ $\{A, C\} \leftrightarrow \{P, R\}$
$\{B, C\} \leftrightarrow \{Q, R\}$ $\quad$ $\{B, D\} \leftrightarrow \{Q, S\}$
$\{C, D\} \leftrightarrow \{R, S\}$
$\{A, D\} \leftrightarrow \{P, S\}$

1-1 Correspondence b/w edges

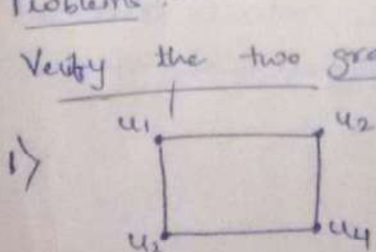
and the adjacency of verticies is preserved.

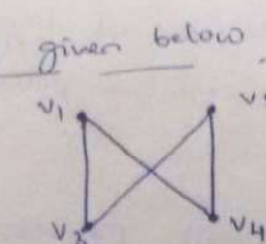
Two digraphs D_1 & D_2 are said to be Isomorphic if there is a one-one correspondence b/w their verticies & b/w their edges such that adjacency of verticies along with the direction is preserved.

Problems :-

Verify the two graphs given below are isomorphic :-

1)

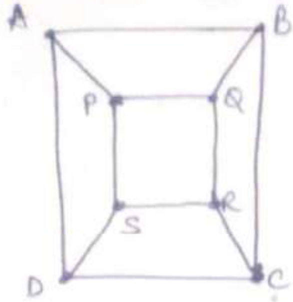


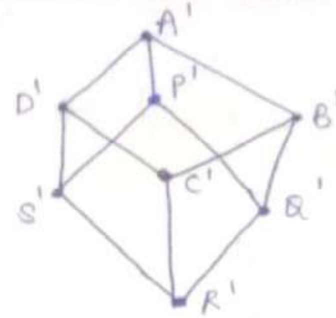


$u_1 \leftrightarrow v_1$ $\quad$ $\{u_1, u_2\} \leftrightarrow \{v_1, v_4\}$
$u_2 \leftrightarrow v_4$ $\quad$ $\{u_2, u_4\} \leftrightarrow \{v_4, v_2\}$
$u_3 \leftrightarrow v_3$ $\quad$ $\{u_3, u_4\} \leftrightarrow \{v_3, v_2\}$
$u_4 \leftrightarrow v_2$ $\quad$ $\{u_1, u_3\} \leftrightarrow \{v_1, v_3\}$

1-1 correspondence b/w the edges & verticies of 2 graphs exist. adjacent verticies in the 1st graph correspond to adjacent verticies in the 2nd graph & vice-versa. Hence it is Isomorphic

2)





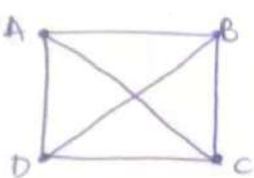
$A \longleftrightarrow A'$, $B \longleftrightarrow B'$, $C \longleftrightarrow C'$, $D \longleftrightarrow D'$,
$P \longleftrightarrow P'$, $Q \longleftrightarrow Q'$, $R \longleftrightarrow R'$, $S \longleftrightarrow S'$.
$\{A, B\} \longleftrightarrow \{A', B'\}$, $\{B, C\} \longleftrightarrow \{B', C'\}$ $\{C, D\} \longleftrightarrow \{C', D'\}$

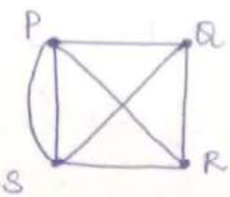
etc ie 1-1 correspondence b/w the verticces & b/w the edges exist.

Also adjacency of verticies is preserved.

$\Rightarrow$ Isomorphic graphs.

3)



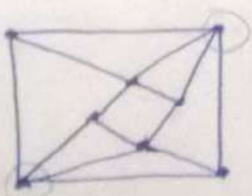


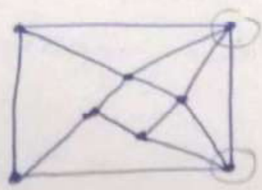
first Graph has 4 verticies & 6 edges ;
Second " ——— " ——— " but 7 edges.

Thus the 1-1 correspondence b/w the edges is not possible.

$\Rightarrow$ Not Isomorphic

4)



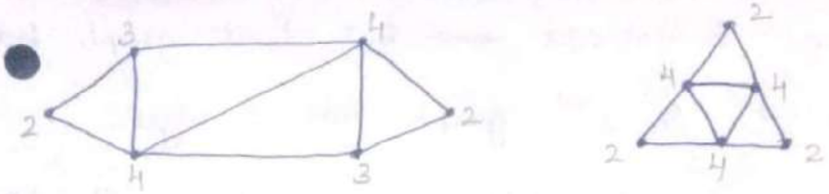


First Graph has a pair of verticies of degree 4, which are not adjacent whereas second graph has a pair of verticies of degree 4, which are adjacent.

$\therefore$ Adjacency of verticies is not preserved.

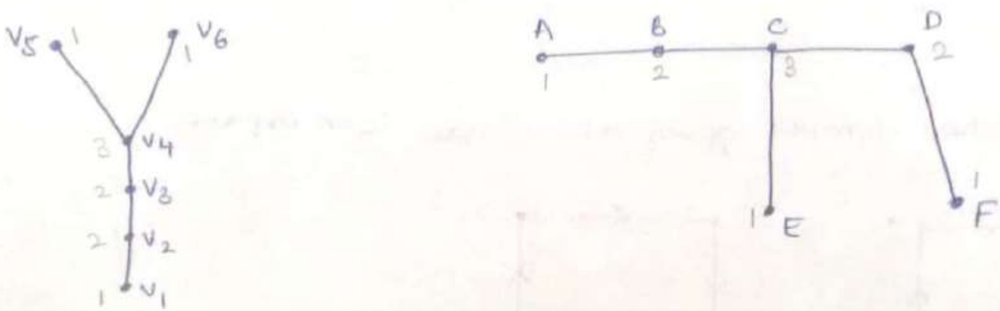
$\Rightarrow$ Not Isomorphic.

5>



Both the Graphs has 6 verticies and 9 edges.
But first Graph has 2 verticies of deg 4 & whereas 2nd Graph has 3 verticies of deg 4.
∴ there cannot be one-one Correspondence b/w the verticies & b/w the edges of the 2 graphs, which preserves the adjacency of verticies.
=> Not Isomorphic.

6>

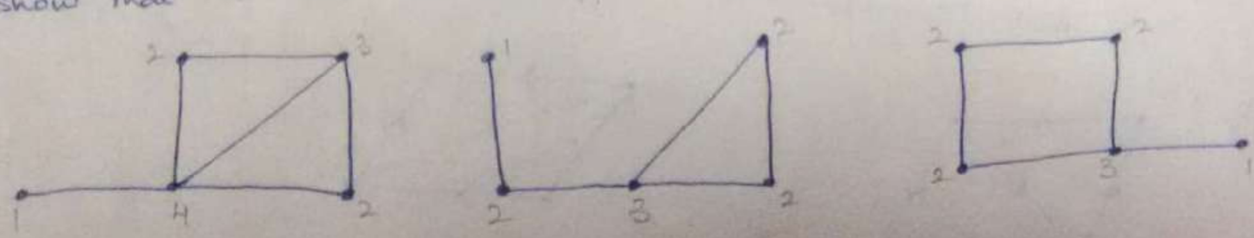


Both the Graphs has 6 verticies and 5 edges.

$V_1 \leftrightarrow A$, $V_2 \leftrightarrow B$, $V_3 \leftrightarrow D$, $V_4 \leftrightarrow C$, $V_5 \leftrightarrow E$, $V_6 \leftrightarrow F$

In the first Graph, vertex V_4 (which is of deg 3) is adjacent to verticies V_5, V_6, V_3 which are of deg 1, 1, 2 resp whereas in the second Graph, vertex C (which is of deg 3) is adjacent to verticies B, D, E which are of deg 2, 2, 1 resp.
∴ Adjacency of verticies is not preserved.
=> Not Isomorphic.

7> Show that no two of the foll 3 graphs are isomorphic:



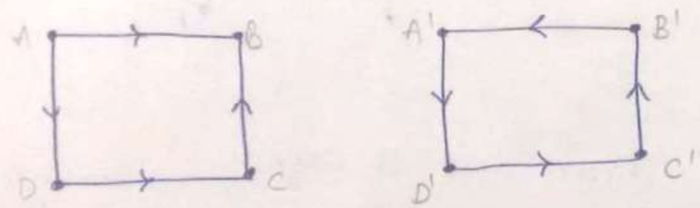
All the 3 graphs has 5 vesticies, ~~and~~ but first graph has 6 edges ~~and~~ whereas 2^{nd} & 3^{rd} graphs has 5 edges.

$\therefore$ 1-1 correspondence b/w the edges does not exist for 1^{st} & 2^{nd} graph, as wel as, 1^{st} and 3^{rd} graphs.

In the 2^{nd} Graph, a vertex of deg 3 is adjacent to 3 Verticies of deg 2, whereas in the 3^{rd} Graph, vertex of deg 3 is adjacent to 2 Verticies of deg 2 and 1 Vertex of deg 1.

$\therefore$ Adjacency of verticies is not preserved b/w 2^{nd} & 3^{rd} graphs.

Hence No two Graphs given above are isomorphic.

8) 

The two digraphs have same no of verticies ie 4 and same no of directed edges ie 4.

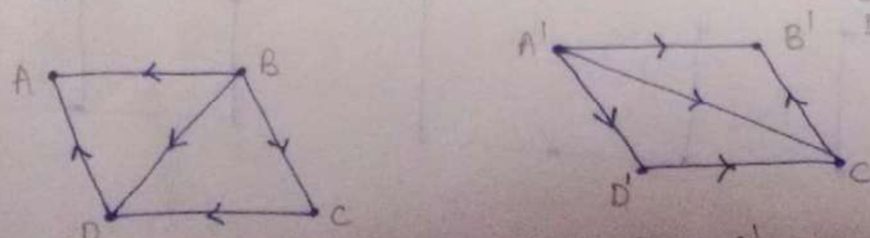
~~Here A ⟺ A'~~

we observe that, vertex A of the 1^{st} graph has 2 out degrees and no indegree, whereas there is no such ~~grap~~ vertex in the 2^{nd} graph.

$\therefore$ there is no 1-1 correspondence b/w the verticies of the 2 digraphs which preserves the direction of edges.

$\Rightarrow$ Not Isomorphic.

A → 2 in, 0 out
B → 0 in, 3 out
C → 1 in, 1 out
D → 2 in, 1 out

9) 

A' → 0 in, 3 out
B' → 2 in, 0 out
C' → 2 in, 1 out
D' → 1 in, 1 out

$\therefore$ A ⟷ B', B ⟷ A', C ⟷ D', D ⟷ C'

$A \leftrightarrow B'$, $B \leftrightarrow A'$, $C \leftrightarrow D'$, $D \leftrightarrow C'$

we observe that there exists 1-1 correspondence b/w the verticies of the given digraphs.

Also $(B, A) \leftrightarrow (A', B')$, $(B, D) \longleftrightarrow (A', C')$, $(D, A) \longleftrightarrow (C', B')$

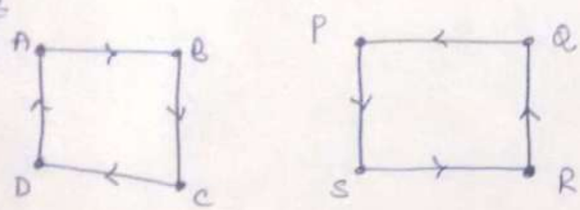
$(B, C) \longleftrightarrow (A', D')$, $(C, D) \longleftrightarrow (D', C')$.

ie 1-1 correspondence b/w the edges of the given digraph exist, preserving the adjacency of verticies including directions of the edges.

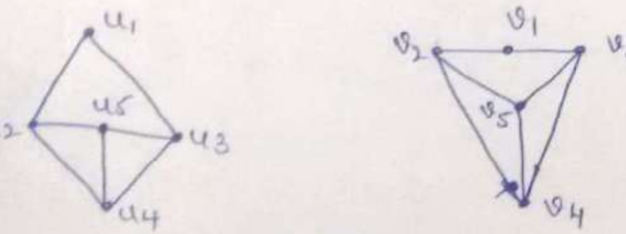
$\Rightarrow$ Isomorphic.

Hw

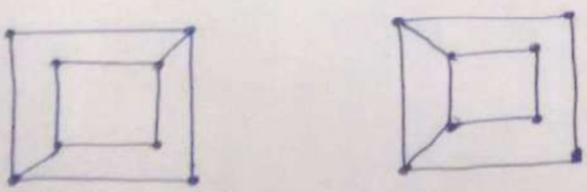
1)



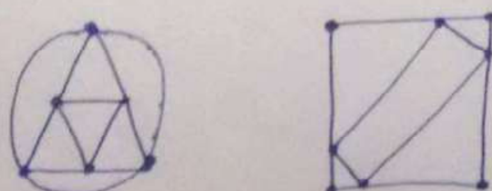
2)



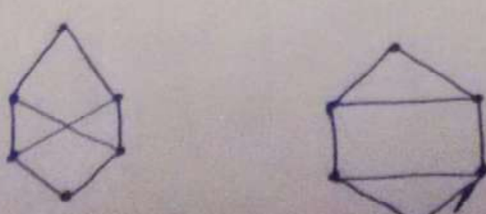
3)



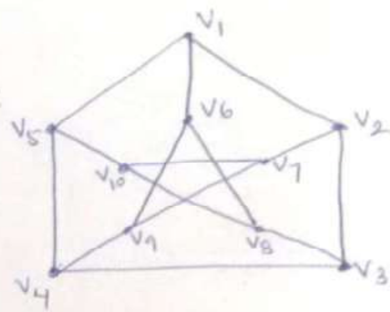
4)



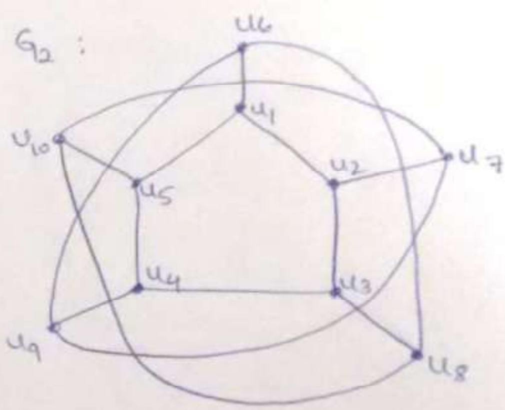
5)



10) G_1 :



G_2 :



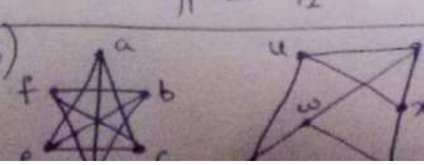
$v_i \leftrightarrow u_i \quad \forall\ i = 1, 2, 3 \ldots 10$

Both G_1 & G_2 have 10 vertices & 15 edges and degree of each vertex in both G_1 and $G_2 = 3$

with this correspondence, $\exists$ 1-1 correspondence b/w ~~vertices &~~ the edges of G_1 & G_2 which preserves adjacency of vertices.

$\therefore G_1 \cong G_2$.

6)



Subgraphs :-

Given two graphs G and G_1, we say G_1 is a subgraph of G if the following conditions hold :

(i) All the verticies and all the edges of G_1 are in G.

(ii) Each edge of G_1 has the same end verticies in G as in G_1.

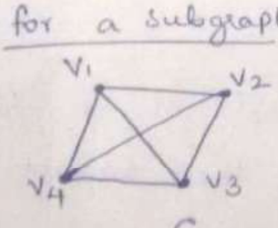
→ Give Ex Here.

Note :- Any graph isomorphic to a subgraph of a graph G is also a subgraph of G.

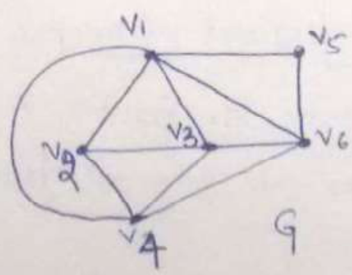
** Consequences of the defn of a Subgraph :-

1) Every graph is a subgraph itself.
2) Every simple graph of n-verticies is a subgraph of the complete graph K_n.
3) If G_1 is a subgraph of a graph G_2 and G_2 is a subgraph of a graph G, then G_1 is a subgraph of G.
4) A single vertex in a graph G is a subgraph of G.
5) A single edge in a graph G, together with its end verticies is a subgraph of G.

Ex :- for a subgraph :-



G_1



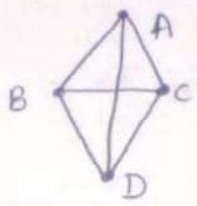
G

Spanning Subgraph :-

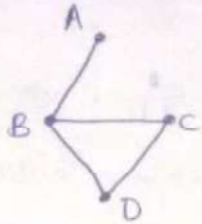
Given a graph $G = (V, E)$, if there is a subgraph $G_1 = (V_1, E_1)$ of G such that $V_1 = V$, then G_1 is called a spanning subgraph of G. [ie all the verticies of G should exist in G_1]

Note :- Every graph is its own spanning subgraph.

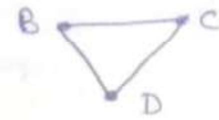
Ex :-



Graph G



Spanning Subgraph



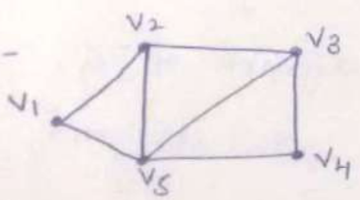
A subgraph, but not Spanning Subgraph

Induced Subgraph :-

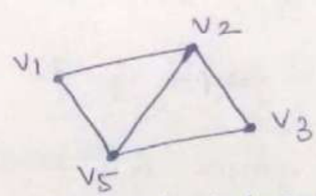
Given a graph $G = (V, E)$, suppose there is a subgraph $G_1 = (V_1, E_1)$ of G such that every edge $\{A, B\}$ of G, where $A, B \in V_1$ is an edge of G_1 also. Then G_1 is called an induced subgraph of G and is denoted by $\langle V_1 \rangle$

ie a subgraph G_1 of the graph $G = (V, E)$ is called an induced subgraph if $\exists$ $V_1 \subseteq V$ and contains all the edges from G, denoted by $\langle V_1 \rangle$

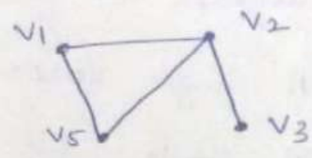
Ex :-



G



induced subgraph



Not an induced Subgraph

Edge-disjoint and Vertex-disjoint Subgraphs :-

Let G be a graph and G_1, G_2 be 2 subgraphs of G, then

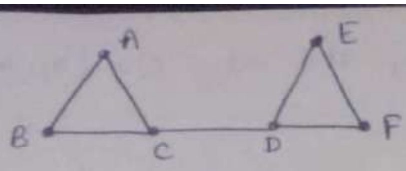
(i) G_1 & G_2 are said to be edge-disjoint if they do not have any common edge

(ii) G_1 & G_2 are said to be vertex-disjoint if they do not have any common edge & any common vertex.

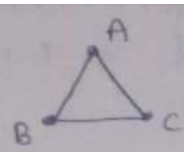
Note :- Subgraphs that have no vertex in common cannot have edges in common.

ie two vertex-disjoint subgraphs must be edge-disjoint but the converse is not true.

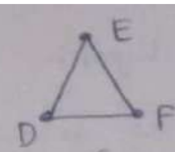
Ex: 1)



G



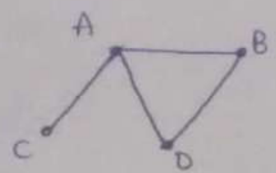
G_1



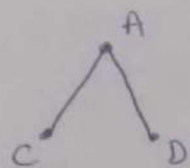
G_2

G_1 & G_2 are Vertex-disjoint subgraphs, also edge-disjoint subgraphs.

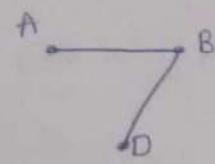
2)



G



G_1



G_2

G_1 and G_2 are edge-disjoint, but not vertex-disjoint.

Problems :-

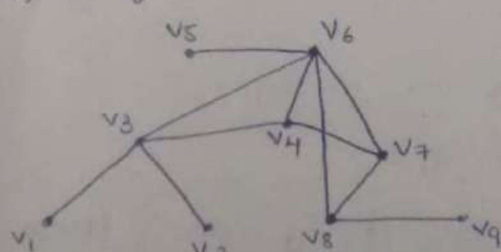
1) Consider the Graph G shown below :

(a) verify that the graph G_1 is an induced subgraph of G.

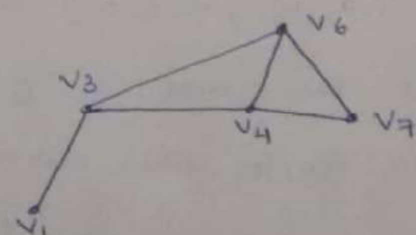
Is this a spanning subgraph of G ?

(b) Draw the subgraph G_2 of G induced by the set $V_2 = \{v_3, v_4, v_6, v_8, v_9\}$.

Soln :-



G



G_1

Soln :- (a) The vertex set of the graph G_1, namely $V_1 = \{v_1, v_3, v_4, v_6, v_7\}$ is a subset of the vertex set $V = \{v_1, v_2 \dots v_9\}$ of G.

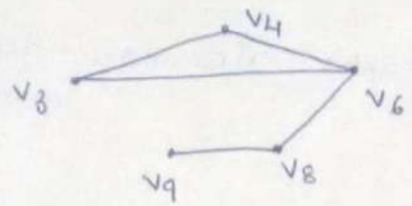
Also, all the edges of G_1 are in G. Each edge in G_1 has the Same end verticies in G as in G_1.

$\therefore$ G_1 is a Subgraph of G.

Every edge $\{v_i, v_j\}$ of G where $v_i, v_j \in V_1$, is an edge of G_1

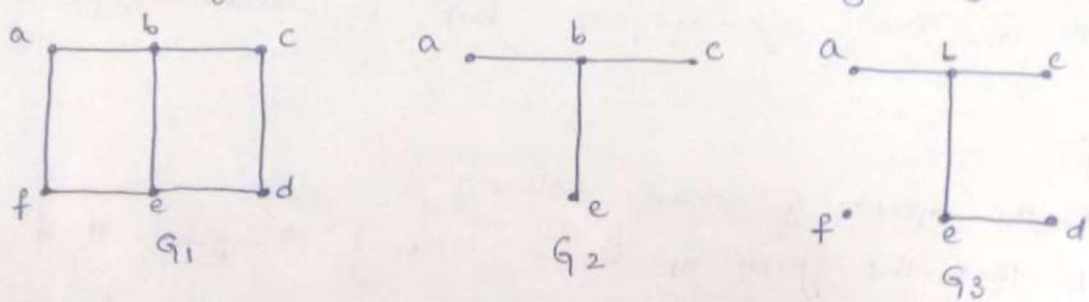
$\therefore$ G_1 is an induced subgraph of G, but not a spanning subgraph of G since $V_1 \neq V$.

(b) Subgraph G_2 of G induced by the set $V_2 = \{v_3, v_4, v$ [illegible] $v_8, v_9\}$ is as below:



$G_2 = \langle V_2 \rangle$

2) Three graphs G_1, G_2, G_3 are shown below: Are G_2 and G_3 induced subgraphs of G_1? Are they spanning subgraphs?



Soln:- The vertex set of G_2 & G_3, namely $V_2 = \{a, b, c, e\}$ and $V_3 = \{a, b, c, d, e, f\}$ are subsets of the vertex set $V_1 = \{a, b, c, d, e, f\}$ of G_1.

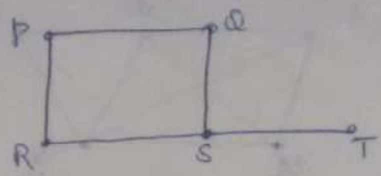
Also all the edges of G_2 & G_3 are in G_1.

$\therefore$ G_2, G_3 are subgraphs of G_1.

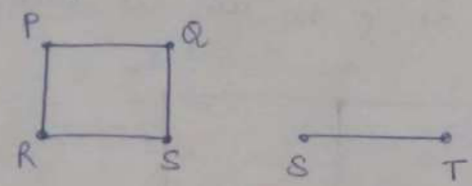
Since every edge of ~~G_1~~ G_2 is an edge of ~~G_2~~ G_1 also. Hence G_2 is an induced subgraph of G_1, but it is not a spanning subgraph of G_1 since $V_2 \neq V_1$.

The graph G_3 does not contain all the edges of G_1. Hence G_3 is not an induced subgraph of G_1, but it is a spanning subgraph of G_1 since $V_3 = V_1$.

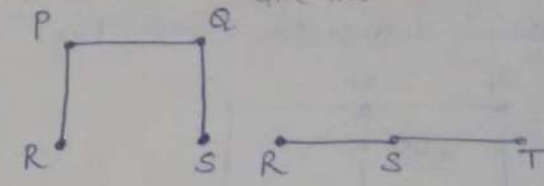
3) For the graph shown below, find two edge-disjoint subgraphs and two vertex-disjoint subgraphs:



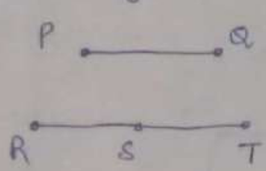
Soln :- Edge-disjoint subgraphs are as below :- Give this



(or)



Vertex-disjoint subgraphs are as below :-



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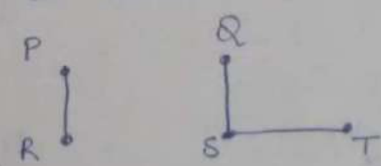


Operations on Graphs :—

Consider 2 graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, then

(i) The Graph whose vertex set is $V_1 \cup V_2$ & the edge set is $E_1 \cup E_2$ is called the union of G_1 & G_2 and it is denoted by $G_1 \cup G_2$.

Thus $G_1 \cup G_2 = (V_1 \cup V_2 , E_1 \cup E_2)$

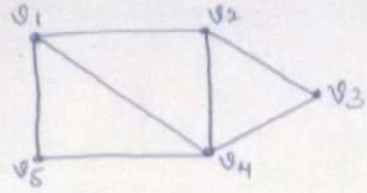
(ii) If $V_1 \cap V_2 \neq \phi$, the graph whose vertex set is $V_1 \cap V_2$ & the edge set is $E_1 \cap E_2$ is called the intersection of G_1 and G_2 & is denoted by $G_1 \cap G_2$.

Thus $G_1 \cap G_2 = (V_1 \cap V_2 , E_1 \cap E_2)$ if $V_1 \cap V_2 \neq \phi$

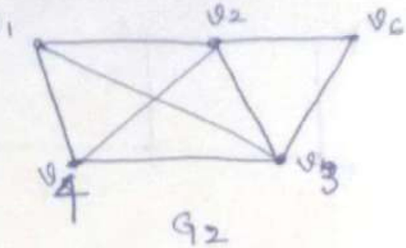
(iii) The graph whose vertex set is $V_1 \cup V_2$ & the edge set is $E_1 \Delta E_2$, where $E_1 \Delta E_2$ is the symmetric difference of E_1 & E_2. This graph is called the ring sum of G_1 & G_2 and is denoted by $G_1 \Delta G_2$. Thus $G_1 \Delta G_2 = (V_1 \cup V_2 , E_1 \Delta E_2)$

Note :- $E_1 \Delta E_2$ denotes set of all edges, which are in E_1 (or) E_2, but not in both. ie $E_1 \Delta E_2 = (E_1 \cup E_2) - (E_1 \cap E_2)$

Ex :-

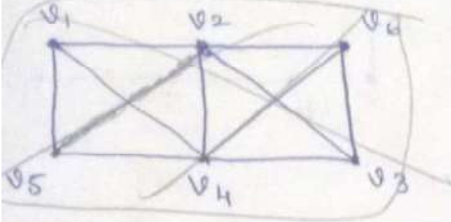


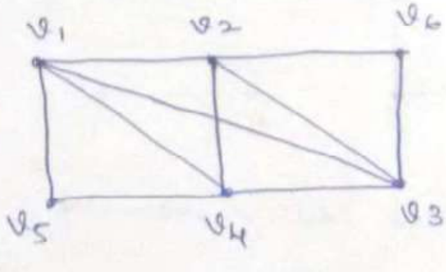
G_1



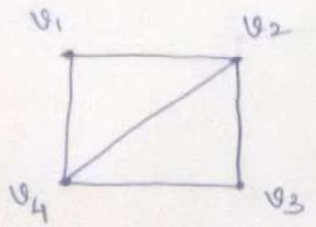
G_2

Union, Intersection and Ring sum of G_1 & G_2 are as below :-

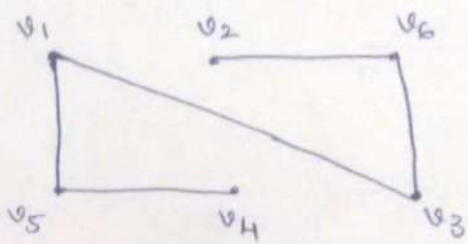




$G_1 \cup G_2$



$G_1 \cap G_2$



$G_1 \Delta G_2$

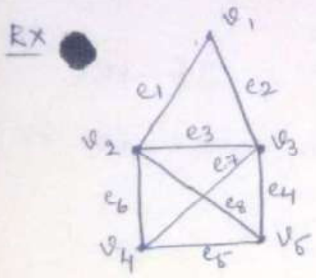
<u>Decomposition</u> :- The graph G is decomposed into 2 subgraphs G_1 and G_2 if $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \phi$

<u>Deletion</u> :- If v is a vertex in a graph G, then $G-v$ denotes the subgraph of G obtained by deleting v and all edges incident on v from G. This subgraph $G-v$ is called <u>vertex-deleted subgraph</u> of G.

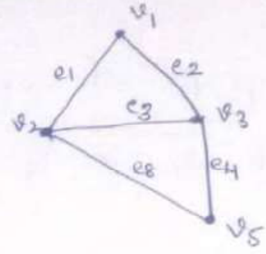
Clearly $G-v$ is the subgraph of G <u>induced by</u> $V_1 = V - \{v\}$

If 'e' is on edge in the graph G, then $G-e$ denotes the subgraph of G obtained by deleting the edge 'e' from G. This graph $G-e$ is refered to as <u>edge-deleted subgraph</u> of G. The deletion of an edge does not alter the <u>no</u> of vertices. Thus an edge deleted subgraph of a graph G is a <u>spanning subgraph</u>.

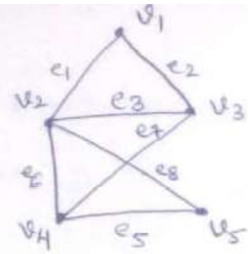
EX •



G



$G - \{v_4\}$



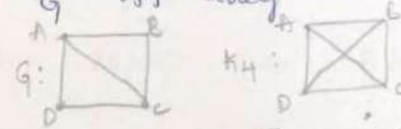
$G - e_4$

Complement of a Simple Graph :-

Complement of a simple graph G, denoted by $\overline{G}$, is the graph obtained by deleting those edges which are in G and adding the edges which are not in G. (Give Ex Here)

Note :- 1> G and $\overline{G}$ have the same vertex set.

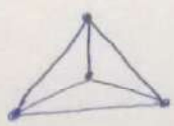
2> Two verticies are adjacent in G iff they are not adjacent in $\overline{G}$.



3> $\overline{\overline{G}} = G$ 4> $\overline{G} = K_n - G$. then $\overline{G} =$

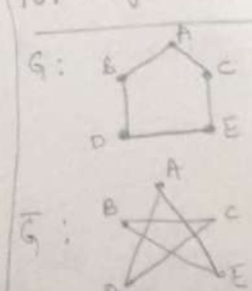
5) complement of K_n is a null graph.

$K_n = G = K_4$ (say)

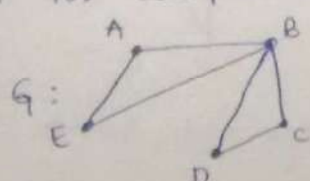


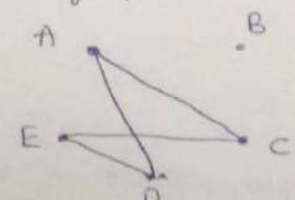
, then $\overline{G} = \{ \}$ ie

for self-complementary



Ex :- for complement of a simple graph

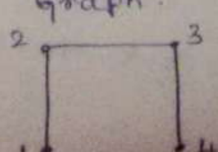


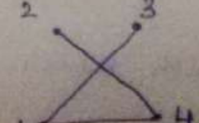


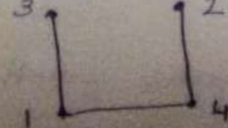
Self-Complementary Graphs :-

A Simple graph G which is isomorphic to its complement $\overline{G}$ is called a Self-complementary Graph.

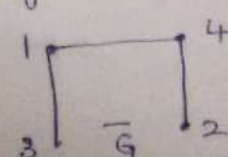
Ex :-







(or)



$G \cong \overline{G}$ ∴ self-complementary.

Examples :-

1) Let G be a simple graph of order 'n'. If the no. of edges in G is 56 & in $\bar{G}$ is 80. what is 'n'?

Soln :- $\bar{G} = K_n - G$

$\therefore$ No. of edges in $\bar{G}$ = no. of edges in K_n − no. of edges in G

ie $56 = \frac{1}{2} n(n-1) - 80$

$(56+80)2 = n^2 - n$

ie $n^2 - n - 272 = 0$.

$n^2 - 17n + 16n - 272 = 0$

$n(n-17) + 16(n-17) = 0$

$(n-17)(n+16) = 0$

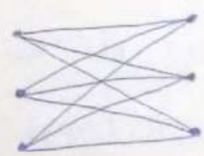
$n = 17$ (or) $n = -16$ (not possible)

$$= \frac{1 \pm \sqrt{1-4(1)(-272)}}{2} = \frac{1 \pm \sqrt{1+1088}}{2} = \frac{1 \pm 33}{2} = 17, -16$$

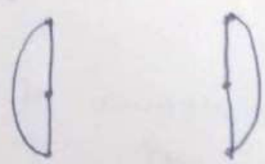
$\therefore n = 17$ ie order of $G = 17$.

2) Find the complement of the complete Bipartite graph $K_{3,3}$.

Soln :- we have $K_{3,3}$ given by :-

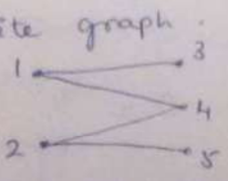


Complement of $K_{3,3}$ is as shown :



3) Show that the complement of a bipartite graph need not be a bipartite graph.

Soln :



G

$\bar{G}$

Here G is a bipartite graph with order 5, but its complement $\bar{G}$ is not a bipartite graph.

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Module 5:

Introduction to graph theory.

→ Definitions and Examples.

→ Subgraphs.

→ complement of a graph.

→ Graph Isomorphism

→ vertex degree.

→ Euler trails and circuits.

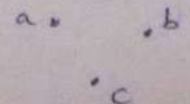
Graphs :- A Graph is a pair (V, E), where V is a non-empty Set and E is a set of unordered pairs of elements taken from the set V.

The elements of V are Called Vertices and the Set V is Called Vertex Set. The elements of E are Called undirected edges (or) just edges and the set E is Called Edge set.

Note :- 1) The Vertex Set of a graph/digraph has to be non-empty but the edge set Can be empty.

2) (A, B) denote an ordered pair of A & B (or) a directed edge. $\{A, B\}$ denote an unordered pair of A & B (or) an undirected edge.

Null Graph :- A graph/digraph containing no edges is called a Null Graph.

Ex:- 

$V = \{a, b, c\}$, $E = \phi$ (or) $\{ \}$

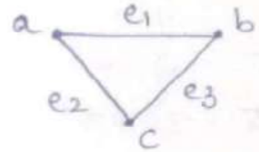
Trivial Graph :- A null graph with only one Vertex is Called a trivial Graph.

Ex :- $\cdot a$ $\quad V = \{a\}$ & $E = \{\ \}$

Finite Graph / digraph :- A graph (or) digraph with only a finite no of Verticies and edges is Called a finite graph/digraph.

End Verticies :- If v_i and v_j denote two verticies of a graph and if e_k denotes the edge joining v_i and v_j, then v_i & v_j are called the End Verticies of e_k.

Ex :-



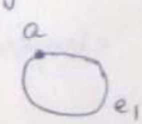
a & b are end Verticies of e_1

a & c " ——— " e_2

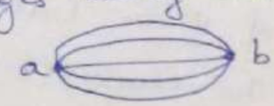
b & c " ——— " e_3

Loop :- An edge whose end Verticies are Same is Called a Loop

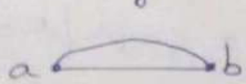
Ex :-



Multiple edges :- Two (or) more edges having Same end Verticies are Called Multiple edges.



Parallel edges :- Two edges having Same end verticies are called parallel edges



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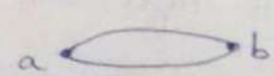
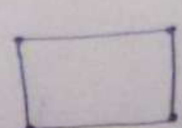


Simple Graph :- A graph which does not Contain loops and multiple edges is Called Simple Graph.

Loop-free :- A graph which does not contain loop, is Called Loop-free.

Ex :-



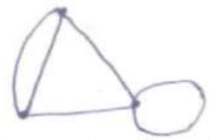
Simple Graph / Loop-free

Page 2,3 is not there

<u>Multigraph</u> :- A graph which contains multiple edges but no loops is called Multi-graph.

<u>General Graph</u> :- A graph which Contains multiple edges (or) loops (or) both is called a General graph.

Multigraph



General Graph

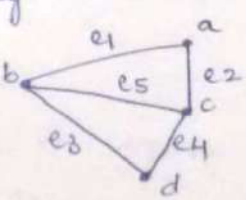
<u>Incidence</u> :- If 'V' is an end vertex of an edge 'e' in a graph G then the edge 'e' is incident on (to) the Vertex 'V'.

<u>Note</u> :- 1) Every edge is incident on two verticies one at each end.
2) The two end Verticies are Coincident if the edge is a loop.

<u>Adjacent Verticies</u> :- Two Verticies are Said to be adjacent verticies if there is an edge joining them.

<u>Adjacent edges</u> :- Two non-Parallel edges are said to be adjacent edges if they are incident on a common vertex.

<u>Ex</u> :-



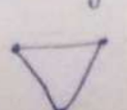
<u>Adj-verticies</u> :- (a, b) (a, c) (b, c) (b, d) (c, d)

<u>Adj-edges</u> :- (e_1, e_5) (e_1, e_2) (e_2, e_5) (e_2, e_4) (e_4, e_5) (e_3, e_1) (e_3, e_4) (e_5, e_3)

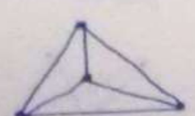
<u>Complete Graph</u> :- A simple graph of order ≥ 2 (<u>ie</u> no of verticies ≥ 2) in which there is an edge b/w every pair of verticies is called a Complete graph/full graph. It is denoted by K_n.

<u>Ex</u> :-

K_2



K_3



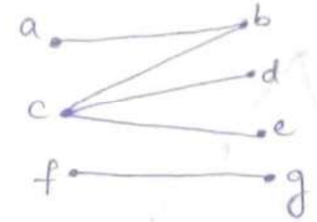
K_4

K_5

K_5 is called the Kuratowski's first graph.

<u>Bipartite Graph</u> :- A simple Graph G in which its vertex set V is the union of two of its mutually disjoint non-empty subsets V_1 and V_2 and each edge in G joins a vertex in V_1 and a vertex in V_2, then G is said to be a bipartite graph.

Ex :-



In the above Graph G,

$$V = \{a, b, c, d, e, f, g\}$$

$$E = \{ab, cb, cd, ce, fg\}$$

V is the union of two of its subsets $V_1 = \{a, c, f\}$ $V_2 = \{b, d, e, g\}$

which are such that

(i) V_1 & V_2 are disjoint $[\Rightarrow V_1 \cap V_2 = \phi]$

(ii) every edge in G joins a vertex in V_1 & a vertex in V_2.

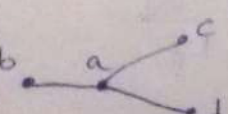
& (iii) G contains no edge that joins two vertices both of which are in V_1 (or) V_2.

Thus G is a bipartite Graph with V_1 & V_2 as bipartites (or partitions).

<u>Complete Bipartite Graph</u>:- A bipartite graph $G = (V_1, V_2; E)$ is said to be a complete bipartite graph if there is an edge b/w every vertex in V_1 & V_2. It is denoted by $K_{r,s}$ where r is the no of vertices in V_1 & s is the no of vertices in V_2, with $r \leq s$.

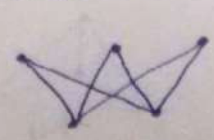
Thus, $K_{r,s}$ has $r+s$ vertices and rs edges.

Ex:-

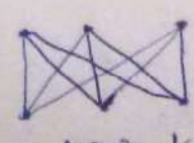


$K_{1,3}$

$V_1 = \{a\}$, $V_2 = \{b, c, d\}$



$K_{2,3}$



~~$K_{3,3}$~~ $K_{3,3}$

$K_{3,3}$ is called as Kuratowski's 2nd Graph.

Problems :-

1) If $G = G(V, E)$ is a simple Graph, prove that $2|E| \leq |V|^2 - |V|$

Soln :- In a simple Graph, there are no multiple edges.

Each edge of a graph is determined by a pair of verticies.

The no of edges cannot exceed the no of Pairs of verticies.

The no of pair of verticies that can be choosen from n verticies

is

$$^nC_2 = \frac{n!}{(n-2)!\,2!} \qquad \left[\because\ ^nC_r = \frac{n!}{(n-r)!\,r!}\right]$$

$$= \frac{n(n-1)(n-2)!}{2\,(n-2)!} = \frac{n(n-1)}{2}$$

Thus for a simple Graph with $n (\geq 2)$ verticies, the no of edges cannot exceed $\frac{1}{2}n(n-1)$.

Thus if a graph $G = G(V, E)$ has n-verticies & m edges, then

$$m \leq \frac{1}{2}n(n-1)$$

$$2m \leq n^2 - n$$

ie $2|E| \leq |V|^2 - |V|$.

Note :- In a simple Graph, for a pair of verticies, we can have only one edge
ie 2 verticies $\longleftrightarrow$ 1 edge
Hence no of edges cannot exceed no of pair of verticies.

2) Show that a complete graph with n verticies, namely K_n has $\frac{1}{2}n(n-1)$ edges.

Soln :- In a complete graph, there exists exactly one edge b/w every pair of verticies.

$\therefore$ the no of ~~ver~~ edges in a complete graph = the no of pair of verticies.

If the no of verticies is n, then no of pair of verticies is

$$^nC_2 = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)(n-2)!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$

Thus in a complete Graph with n verticies, no of edges, $m = \frac{1}{2}n(n-1)$

3) Show that a simple Graph of order $n=4$ & Size $m=7$ & a complete graph of order $n=4$ & size $m=5$ do not exist.

Soln :- For a simple Graph,

$$2m \le n^2 - n$$
$$2(7) \le 16-4$$
$$14 \le 12 \;\#\; \text{not true}$$

for a complete graph,

$$m = \frac{1}{2}n(n-1)$$
$$5 = \frac{1}{\cancel{2}}\cdot \cancel{4}^{2}\times 3$$
$$5 = 6 \;\#\; \text{not true}$$

4) (a) How many verticies and how many edges are there in the complete bipartite graphs $K_{4,7}$ & $K_{7,11}$?

(b) If the graph $K_{r,12}$ has 72 edges, what is r?

Soln :- A complete Bipartite Graph $K_{r,s}$ has $r+s$ verticies and rs edges.

(a) The Graph $K_{4,7}$ has $4+7=11$ verticies and $4\times 7 = 28$ edges

& the Graph $K_{7,11}$ has $7+11 = 18$ verticies & $7\times 11 = 77$ edges

(b) If the graph $K_{r,12}$ has 72 edges, then

$$12r = 72 \Rightarrow \boxed{r=6}$$

5) Let $G=(V,E)$ be a simple graph of order $|V|=n$ & size $|E|=m$. If G is a bipartite graph, P.T $4m \le n^2$.

Soln :- Let V_1 and V_2 be bipartites of G, with $|V_1| = r$ & $|V_2| = s$.

Since for the Given graph, $|V| = n$, we should have $r+s=n$ so that ~~r=n-s~~ and $s = n-r$.

The graph G has the maximum no of edges when each of the r-verticies in V_1 is joined by an edge to each of the s-verticies in V_2 & this maximum is equal to rs.

$\Rightarrow \quad |E| = m \leq rs$.

ie $\quad m \leq rs = r(n-r)$

$= rn - r^2$

$= -(r^2 - nr)$

by completing the square, we have

$rs = -\left[\left(r - \frac{1}{2}n\right)^2 - \left(\frac{1}{2}.n\right)^2\right]$

$rs = -\left[\left(r - \frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2\right]$

(or) $\quad rs = \left(\frac{n}{2}\right)^2 - \left(r - \frac{n}{2}\right)^2$, this is maximum when $r = \frac{n}{2}$.

Thus when $r = \frac{n}{2}$, $rs = \frac{n^2}{4}$

Hence $\quad m \leq rs = \frac{n^2}{4}$

$\Rightarrow \quad m \leq \frac{n^2}{4} \Rightarrow 4m \leq n^2$

6) Show that a simple graph of order $n=4$ & size $m=5$ cannot be a bipartite graph.

Soln :- for a bipartite graph, $4m \leq n^2$.

$4(5) \leq 4^2$

$20 \leq 16$ #

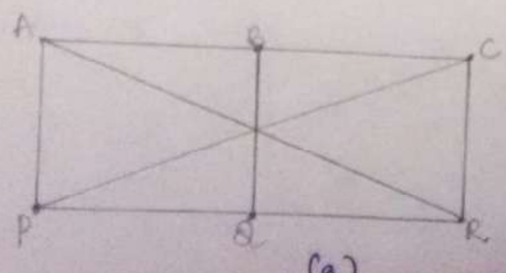
Here $4m = 4(5) = 20$ & $n^2 = 4^2 = 16$.

$4m > 16$ #

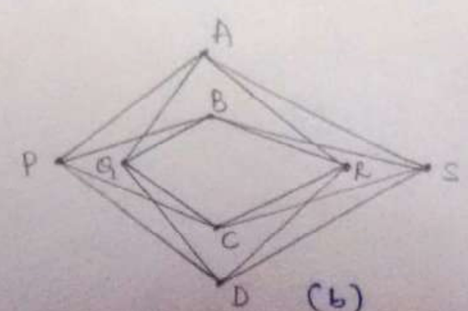
Hence not a bipartite Graph.

Hw

1) Verify that the foll are bipartite Graphs? what are their bipartites?



(a)



(b)

2) Company X has offices in cities B, D & K; Company Y in cities B & M; Company Z in cities C & M. Represent this situation by a bipartite Graph. is this a complete bipartite Graph?

3) State whether the foll Graphs can exist or cannot exist:

× → No
✓ → exist

✓(a) Simple Graph of order 3 & size 2

× (b) " ——— " —— 5 & " 12

✓(c) complete " ——— " 5 & size 10

✓(d) Bipartite " ——— " 4 & " 3

× (e) " ——— " —— 3 & " 4

✓ (f) complete Bipartite Graph of order 4 & size 4.

Note :- 1) for a simple Graph, $2|E| \leq |V|^2 - |V|$

2) for a complete Graph, $m = \frac{1}{2} n(n-1)$.

3) for a complete Bipartite Graph, $K_{r,s}$, $r+s$ vertices & rs edges exists.

4) for a Bipartite Graph, $4m \leq n^2$.

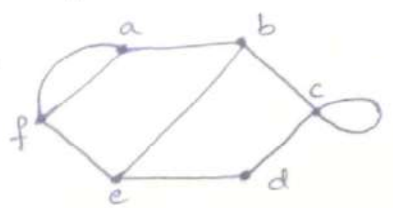
Vertex Degree :-

Let G be a graph & v be a vertex of G. Then the no of edges of G that are incident on v with the loops counted twice is called the degree of the vertex v and is denoted by $deg(v)$ (or) $d(v)$.

Degree Sequence :- The degrees of the verticies of a graph arranged in non-decreasing order is called degree sequence.

Degree of the graph :- The minimum of the degrees of verticies of a graph is called the degree of the graph.

Ex :-



$d(a) = 3$
$d(b) = 3$
$d(c) = 4$
$d(d) = 2$
$d(e) = 3$
$d(f) = 3$

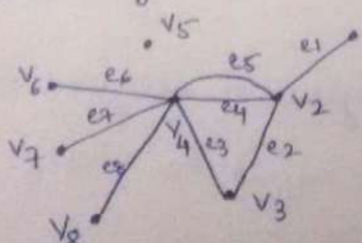
∴ The degree sequence is 2, 3, 3, 3, 3, 4 and the degree of the graph is '2'.

Isolated Vertex :- A vertex in a graph which is not an end vertex of any edge of the graph is called an isolated vertex. ie a vertex is an isolated vertex if & only if its degree is zero.

Pendant Vertex :- A vertex of degree 'one' is called a pendant vertex.

Pendant edge :- An edge incident on a pendant vertex is called a pendant edge.

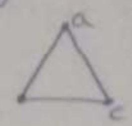
Ex :-



$v_5 \to$ isolated vertex.
v_1, v_6, v_7, v_8 are pendant verticies.
e_1, e_6, e_7, e_8 are pendant edges.

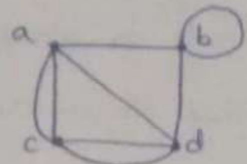
Regular Graph :- A graph in which all the verticies are of same degree 'K' is called a regular graph of degree 'K' (or) a K-regular Graph.

Ex :-



$d(a) = d(b) = d(c) = 2$

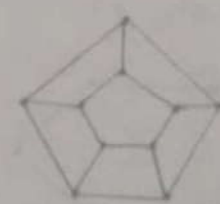
$\Rightarrow$ 2-regular Graph.



$d(a) = 4 = d(b) = d(c) = d(d)$

Petersen Graph :- A 3-regular Graph which contains 10 verticies and 15 edges is called petersen Graph.

(or)

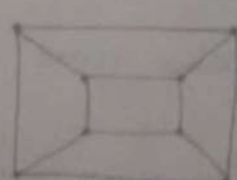


Three-dimensional hypercube :-

A cubic graph with $8 = 2^3$ verticies is called 3-dimensional hypercube & is denoted by Q_3

In General, for any +ve integer k, a loop-free K-regular graph with 2^k verticies is called the k-dimensional hypercube & is denoted by Q_k.

Ex :-



3- dimensional hypercube

Handshaking Property :- The sum of the degrees of all verticies in a graph is an even no, and this no is equal to twice the no of edges in the Graph.

ie for a graph $G = (V, E)$, $\sum_{v \in V} deg(v) = 2|E|$

Proof :- The above property is obvious from the fact that while counting the degrees of verticies, each edge is counted twice (once at each end).

Just say this
(Note :- The name "Hand shaking property" because "if several people shake hands, then the total no of hands shaken must be even, since two hands are involved in each handshake".)

Theorem :- In every graph, the no of verticies of odd degree is Even.

Proof :- Consider a graph with 'n' verticies.

Suppose 'K' of these verticies are of odd degree then the remaining n−K verticies are of even degree.

Let $V_1, V_2 \ldots V_K$ be the verticies of odd degree and $V_{K+1}, V_{K+2} \ldots V_n$ be " ______ even degree.

then $$\sum_{i=1}^{n} \deg(v_i) = \sum_{i=1}^{k} \deg(v_i) + \sum_{i=k+1}^{n} \deg(v_i) \longrightarrow (1)$$

In view of hand shaking property, the sum on the LHS of the above expression is equal to twice the no of edges in the graph.

This sum is even.

The second sum in the RHS, is the sum of the degree of verticies with even degree. This sum is also even.

$\therefore$ the 1^{st} sum in the RHS must also be even.

ie $d(v_1) + d(v_2) + \ldots + d(v_K) = \text{Even} \longrightarrow (2)$.

but each of $d(v_1), d(v_2) \ldots d(v_K)$ is odd.

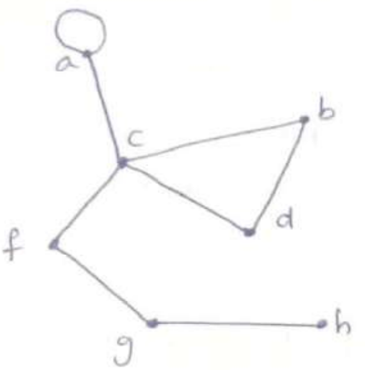
$\therefore$ the no of terms in the LHS of (2) must be even [$\because$ odd no's added even no of times, the result is Even]

ie K is Even.

Hence the proof.

Problems:-

1) For the given graph, indicate the degree of each vertex & verify the handshaking property.



Soln:- $deg(a) = 3$, $deg(b) = 2$, $deg(c) = 4$, $deg(d) = 2$
$d(e) = 0$, $deg(f) = 2$, $deg(g) = 2$, $deg(h) = 1$

The Sum of the degrees of vertices $= 16$.

The graph has 8 edges.

Thus the Sum of the degrees of vertices = twice the no of edges.

$16 = 2(8)$ True.

Hand shaking property is verified.

2) Can there be a graph with 12 vertices such that two of the vertices have degree 3 each & the remaining 10 vertices have degree 4 each? If so, find its size ie $|E|$?

Soln:- Sum of the deg of vertices $= (2\times3) + (10\times4)$
$= 6 + 40$
$= 46$, which is Even.

By Hand Shaking property, sum of the deg of vertices = twice the no of edges.

ie $46 = 2|E| \Rightarrow |E| = 23$ ~~property holds~~.

Hence there can be a graph of the desired type (whose Size is 23)

3) In a graph $G = (V, E)$, what is the largest possible value for $|V|$ if $|E| = 19$ and $\deg(v) \geq 4$ for all $v \in V$?

Soln :- Given :- all verticies are of degree greater than (or) equal to 4.

$\therefore$ the sum of the degrees of verticies $\geq 4n$

where $n = |V|$ & this sum = twice the no of edges.

$$\therefore \quad 2|E| \geq 4n$$
$$2 \times 19 \geq 4n \quad \Rightarrow \quad n \leq \frac{38}{4} = 9.5 < 10$$

Thus the largest possible value of $|V| = 9$

ie given graph can have atmost 9 verticies.

4) Prove that the hypercube Q_n has $n2^{n-1}$ edges.

Determine the no of edges in Q_8.

Soln :. In the hypercube Q_n, the no of verticies is 2^n and each vertex is of degree 'n'.

$\therefore$ the sum of the degrees of verticies of Q_n is $n \times 2^n$.

By Hand shaking property, we have

Sum of the deg of verticies = 2 × no of edges

ie $\quad n \times 2^n = 2|E|$

$$\Rightarrow \quad |E| = \frac{1}{2} \times n \times 2^n = n2^{n-1}$$

$\therefore$ Q_n has $n2^{n-1}$ no of edges.

Also, No of edges in $Q_8 = 8 \times 2^7 = 1024$.

5) (a) what is the dimension of the hypercube with 524288 edges?

(b) How many verticies are there in a hypercube with 4980736 edges?

P.T.O.

Soln:- for the k-dimensional hypercube Q_k, the no of verticies is 2^k & no of edges is $k.2^{k-1}$.

(a) To find k, given $k.2^{k-1} = \cancel{54}\ 524288$

For (b), keep dividing by 2, until you get a no not divisible by 2. whatever asked, express in terms of 2 power

ie $k.2^{k-1} = 2^{19}$

$= 2^4 \times 2^{15}$

$k.2^{k-1} = 16 \times 2^{15}$ of the form $k.2^{k-1}$

$\Rightarrow \boxed{k = 16}$

Thus the dimension of the hypercube with 524288 edges is $k=16$

(b) we have $4980736 = 19 \times 2^{18}$ of the form $k.2^{k-1}$, which indicates that Q_k has 4980736 edges when $k=19$.

$\therefore$ In this hypercube, no of verticies is $2^k = 2^{19} = 524288$.

6) (a) If k is odd, show that the no of verticies in a k-regular graph is even

(b) Show that it is not possible to have a set of nine people at a party such that each one knows exactly five of the others in the party.

Soln:- (a) In a k-regular graph, the degree of each vertex is k.

$\therefore$ if such a graph has n verticies, then the sum of degrees is nk, and this has to be an Even no (by handshaking property)

If k is odd, n must be Even to satisfy the above.

Hence the proof

(b) Let G be a graph with 9 verticies, each vertex representing a person in the given set, and each edge representing an acquiantance (given situation).

P.T.O

If each person in the set (party) knows exactly five other persons (10)
in the set (party), then there will be exactly five edges incident on each vertex & the Graph G will be 5-regular.

This is not possible b'coz G has an odd no of verticies. (since by (a), if k is odd, n must be Even).

Hence the graph G of the desired type does not exist.

7) (a) If a graph with n verticies & m edges is k-regular, Show that $m = kn/2$.

(b) Does there exist a cubic graph with 15 verticies?

(c) " ——————— " 4-regular graph with 15 edges?

Soln :- (a) Given :- The Graph G is k-regular.

$\Rightarrow$ the degree of every vertex is k.

$\therefore$ if G has n verticies, then the sum of deg of verticies is nk.

By Hand Shaking property, this must be equal to $2m$ (if G has m edges)

$$\text{ie } nk = 2m \Rightarrow m = \frac{nk}{2}.$$

(b) If there is a cubic graph (3-regular graph) with 15 verticies, the no of edges it should have is $m = \frac{kn}{2}$

$$m = \frac{3 \times 15}{2} = \frac{45}{2} \text{ (not an integer)}$$

Thus the graph of desired type does not exist.

(c) If there is a 4-regular graph with 15 edges (ie $k = 4$, $m = 15$), the no of ~~edges~~ verticies it should have is $n = \frac{2m}{k}$.

$$\Rightarrow n = \frac{2 \times 15}{4} = \frac{30}{4} \text{ (not an integer)}$$

Thus the graph of desired type does not exist.

8) (a) Show that in a complete Graph of n verticies (namely K_n), the degree of every vertex is $(n-1)$ & that the total no of edges is $\frac{n(n-1)}{2}$.

(b) If K_n has 'm' edges, S.T $n(n+1) = 2(n+m)$

Soln :- A complete graph is a simple graph in which every ~~edge~~ vertex is joined with every other vertex through exactly one edge.

∴ If there are 'n' verticies, each vertex is joined to $(n-1)$ verticies through exactly one edge.

Hence there occur $(n-1)$ edges at every vertex

$\Rightarrow$ degree of every vertex is $(n-1)$.

∴ sum of degree of verticies is $n(n-1)$, this sum must be equal to $2m$ (by H.S.P)

i.e $n(n-1) = 2m \Rightarrow m = \frac{1}{2}n(n-1)$.

Thus K_n has $\frac{1}{2}n(n-1)$ edges.

(b) If K_n has m edges, then

$$m = \frac{1}{2}n(n-1)$$

add n on b.s

$$n+m = n + \frac{1}{2}n(n-1)$$

$$2(n+m) = 2n + n^2 - n = 2n^2 + n$$

$$2(n+m) = n(n+1)$$

9) Show that there is no graph with 12 verticies & 28 edges in the foll cases :-

(a) The degree of a vertex is either 3 (or) 4

(b) The degree of a " ——— " 3 (or) 6.

Soln :- Suppose there is a graph with 28 edges & 12 verticies, of which k verticies are of deg 3 (each), then :

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(a) If all the remaining $(12-k)$ verticies have degree 4,

then $3k + 4(12-k) = 2\times 28$

$3k + 48 - 4k = 56$

$-k = 8 \Rightarrow k = -8$ #

(b) If all the remaining $(12-k)$ verticies have degree 6,

then $3k + 6(12-k) = 56$

$3k + 72 - 6k = 56.$

$-3k = -16 \Rightarrow k = \frac{16}{3}$ #

Hence in both cases, the graph of the desired type cannot exist.

10) Determine the order $|V|$ of the graph $G=(V,E)$ in the foll

Cases :- (a) G is a cubic graph with 9 edges

(b) G is regular with 15 edges

(c) G has 10 edges with 2 verticies of deg 4 & all others of deg 3.

Soln :- (a) Suppose the order of G is n,

Since G is a cubic graph, all verticies of G have deg 3.

$\therefore$ Sum of degrees of verticies is $3n$.

Since G has 9 edges, by HSP,

$3n = 2\times 9 = 18$

$\Rightarrow n = 6 \Rightarrow |V| = 6.$

(b) Given : G is a regular graph

$\Rightarrow$ all verticies of G must be of same degree, say k.

Let G be of order 'n', then the sum of the degree of verticies is nk.

Since G has 15 edges, by HSP

$nk = 2\times 15 \Rightarrow n = \frac{30}{k}$

Since k is a +ve integer, it follows that 'n' must be a divisor of 30.

ie n must be 1, 2, 3, 5, 6, 10, 15 and 30 (possible orders of G)

(c) suppose the order of G is 'n'.

Since 2 verticies of G are of deg 4 & all others are of deg 3, sum of the degrees of verticies is

$$(2\times4)+(n-2)3 = 8+3(n-2).$$

By HSP,

$$8+3(n-2) = 2\times10$$
$$3n-6 = 20-8 = 12$$
$$3n = 18 \Rightarrow \boxed{n=6}$$

ie $|V| = 6$.

HW

1) Consider a graph having n verticies & m edges. If p no of verticies are of deg k & the remaining verticies are of deg $k+1$, P.T $p=(k+1)n-2m$.

2) P.T there is no simple graph with 7 verticies, one of which has deg 2, 2 have deg 3, 3 have deg 4 & the remaining vertex has deg 5.

×

11) For a graph with 'n' vertices and 'm' edges, if δ is the minimum, Δ is the maximum of degree of vertices, show that $\delta \le \frac{2m}{n} \le \Delta$.

Solⁿ:- Let $d_1, d_2 \dots d_n$ be the degrees of $1^{st}, 2^{nd} \dots n^{th}$ vertex resp, then

$\sum \deg(v) = d_1+d_2+\dots+d_n$

by HSP, $\sum \deg(v) = 2|E| = 2m$

$\therefore d_1+d_2+\dots+d_n = 2m \rightarrow (1)$.

Given that δ is min of $d_1, d_2 \dots d_n$

ie $\delta \le d_1, \delta \le d_2, \dots \delta \le d_n$.

$\therefore \delta+\delta+\dots+\delta_{(n\,times)} \le d_1+d_2+\dots+d_n$

$\Rightarrow n\delta \le 2m$ (by (1))

$\Rightarrow \delta \le \frac{2m}{n} \rightarrow (2)$

Also given that Δ is max of $d_1, d_2 \dots d_n$

ie $\Delta \geq d_1, \Delta \geq d_2 \dots \Delta \geq d_n$

$\Rightarrow \Delta+\Delta+\dots+\Delta_{(n\,times)} \geq d_1+d_2+\dots+d_n$

$\Rightarrow n\Delta \geq 2m$ (by (1))

$\therefore \Delta \geq \frac{2m}{n}$ (or) $\frac{2m}{n} \le \Delta \rightarrow (3)$

from (2) & (3),

$$\delta \le \frac{2m}{n} \le \Delta.$$

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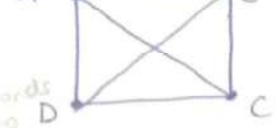
Isomorphism :-

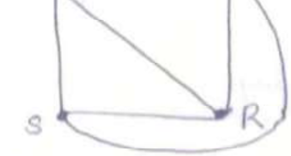
Two graphs G and G' are said to be isomorphic if there is a one-one correspondance b/w their verticies & b/w their edges such that the adjacency of verticies is preserved.

Such graphs will have the same structure, differing only in the way their verticies & edges are labelled (or) only in the way they are represented geometrically.

If G & G' are isomorphic, we write $G \cong G'$.

Ex :-





corresponds to

$A \leftrightarrow P$, $B \leftrightarrow Q$, $C \leftrightarrow R$, $D \leftrightarrow S$ – 1-1 correspondance b/w verticies.

$\{A, B\} \leftrightarrow \{P, Q\}$
$\{B, C\} \leftrightarrow \{Q, R\}$
$\{C, D\} \leftrightarrow \{R, S\}$
$\{A, D\} \leftrightarrow \{P, S\}$

$\{A, C\} \leftrightarrow \{P, R\}$
$\{B, D\} \leftrightarrow \{Q, S\}$

1-1 Correspondence b/w edges

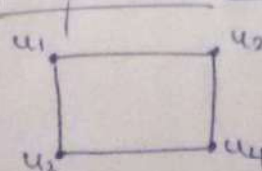
and the adjacency of verticies is preserved.

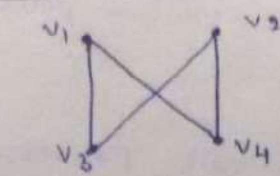
Two digraphs D_1 & D_2 are said to be Isomorphic if there is a one-one correspondence b/w their verticies & b/w their edges such that adjacency of verticies along with the direction is preserved.

Problems :-

Verify the two graphs given below are isomorphic :-

1)



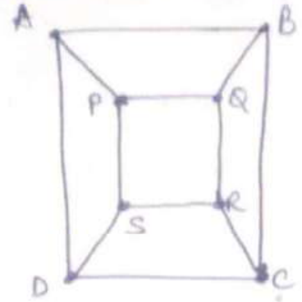


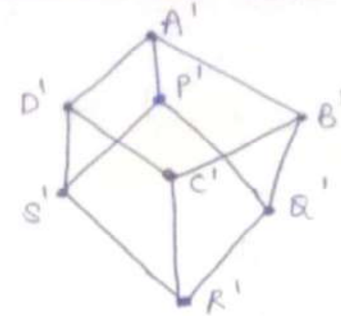
$u_1 \leftrightarrow v_1$
$u_2 \leftrightarrow v_4$
$u_3 \leftrightarrow v_3$
$u_4 \leftrightarrow v_2$

$\{u_1, u_2\} \leftrightarrow \{v_1, v_4\}$
$\{u_2, u_4\} \leftrightarrow \{v_4, v_2\}$
$\{u_3, u_4\} \leftrightarrow \{v_3, v_2\}$
$\{u_1, u_3\} \leftrightarrow \{v_1, v_3\}$

1-1 correspondence b/w the edges & verticies of 2 graphs exist. adjacent verticies in the 1st graph correspond to adjacent verticies in the 2nd graph & vice-versa. Hence it is Isomorphic

2)





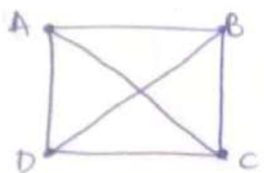
$A \longleftrightarrow A'$, $B \longleftrightarrow B'$, $C \longleftrightarrow C'$, $D \longleftrightarrow D'$,
$P \longleftrightarrow P'$, $Q \longleftrightarrow Q'$, $R \longleftrightarrow R'$, $S \longleftrightarrow S'$.
$\{A, B\} \longleftrightarrow \{A', B'\}$, $\{B, C\} \longleftrightarrow \{B', C'\}$ $\{C, D\} \longleftrightarrow \{C', D'\}$

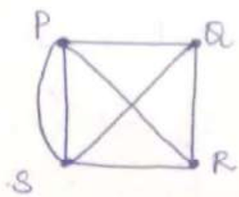
etc ie 1-1 correspondence b/w the verticies & b/w the edges exist.

Also adjacency of verticies is preserved.

$\Rightarrow$ Isomorphic graphs.

3)



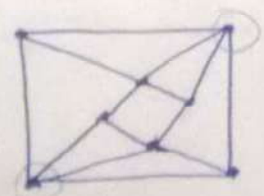


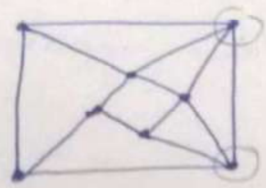
first Graph has 4 verticies & 6 edges ;
Second " ——— " ——— " but 7 edges.

Thus the 1-1 correspondence b/w the edges is not possible.

$\Rightarrow$ Not Isomorphic

4)





First Graph has a pair of verticies of degree 4, which are not adjacent whereas second graph has a pair of verticies of degree 4, which are adjacent.

$\therefore$ Adjacency of verticies is not preserved.

$\Rightarrow$ Not Isomorphic.

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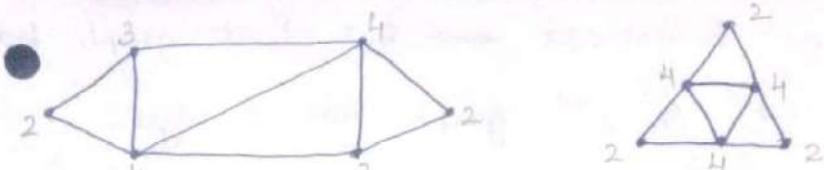
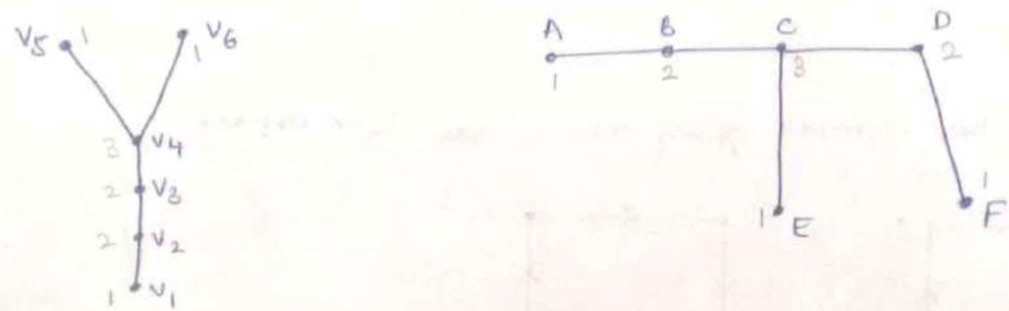
5)



Both the Graphs has 6 verticies and 9 edges.
But first Graph has 2 verticies of deg 4 whereas 2nd Graph has 3 verticies of deg 4.
∴ there cannot be one-one Correspondence b/w the verticies & b/w the edges of the 2 graphs, which preserves the adjacency of verticies.
⇒ Not Isomorphic.

6)

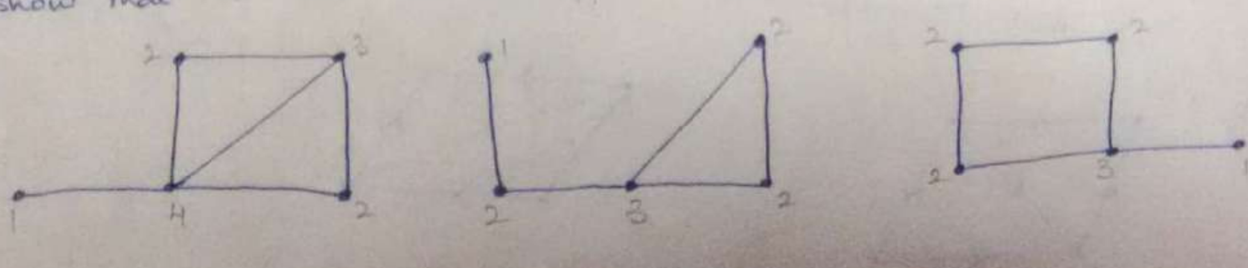


Both the Graphs has 6 verticies and 5 edges.

$V_1 \leftrightarrow A$, $V_2 \leftrightarrow B$, $V_3 \leftrightarrow D$, $V_4 \leftrightarrow C$, $V_5 \leftrightarrow E$, $V_6 \leftrightarrow F$

In the first Graph, vertex V_4 (which is of deg 3) is adjacent to verticies V_5, V_6, V_3 which are of deg 1, 1, 2 resp whereas in the second Graph, vertex C (which is of deg 3) is adjacent to verticies B, D, E which are of deg 2, 2, 1 resp.
∴ Adjacency of verticies is not preserved.
⇒ Not Isomorphic.

7) Show that no two of the foll 3 graphs are isomorphic:



All the 3 graphs has 5 vesticies, ~~and~~ but first graph has 6 edges ~~and~~ whereas 2^{nd} & 3^{rd} graphs has 5 edges.

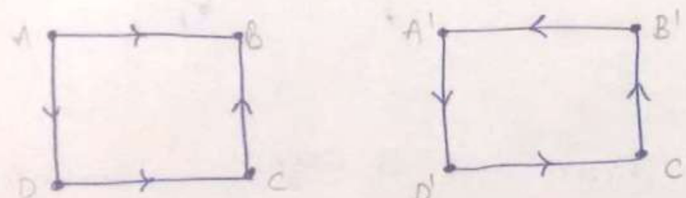
$\therefore$ 1–1 correspondence b/w the edges does not exist for 1^{st} & 2^{nd} graph, as wel as, 1^{st} and 3^{rd} graphs.

In the 2^{nd} Graph, a vertex of deg 3 is adjacent to 3 Verticies of deg 2, whereas in the 3^{rd} Graph, vertex of deg 3 is adjacent to 2 Verticies of deg 2 and 1 Vertex of deg 1.

$\therefore$ Adjacency of verticies is not preserved b/w 2^{nd} & 3^{rd} graphs.

Hence No two Graphs given above are isomorphic.

8)



The two digraphs have same no of verticies ie 4 and same no of directed edges ie 4.

~~Here A ⇔ A'~~

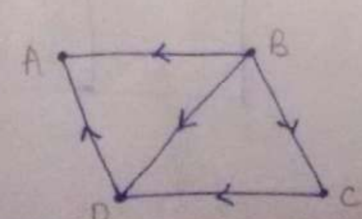
we observe that, vertex A of the 1^{st} graph has 2 out degrees and no indegree, whereas there is no such ~~grap~~ vertex in the 2^{nd} graph.

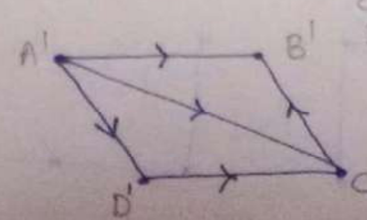
$\therefore$ there is no 1–1 correspondence b/w the verticies of the 2 digraphs which preserves the direction of edges.

$\Rightarrow$ Not Isomorphic.

A → 2 in, 0 out
B → 0 in, 3 out
C → 1 in, 1 out
D → 2 in, 1 out

9)





A' → 0 in, 3 out
B' → 2 in, 0 out
C' → 2 in, 1 out
D' → 1 in, 1 out

$\therefore$ A ⟷ B', B ⟷ A', C ⟷ D', D ⟷ C'

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- $A \leftrightarrow B'$, $B \leftrightarrow A'$, $C \leftrightarrow D'$, $D \leftrightarrow C'$

we observe that there exists 1-1 correspondence b/w the verticies of the given digraphs.

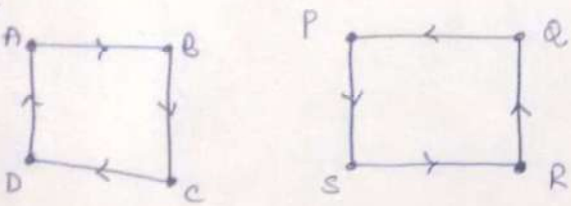
Also $(B, A) \leftrightarrow (A', B')$, $(B, D) \leftrightarrow (A', C')$, $(D, A) \leftrightarrow (C', B')$

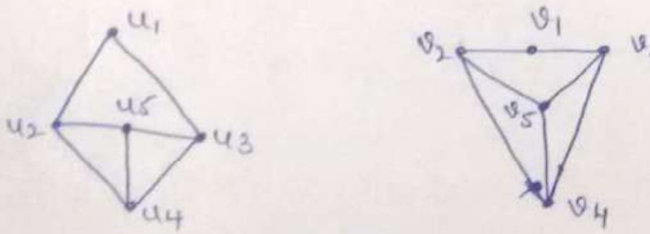
$(B, C) \leftrightarrow (A', D')$, $(C, D) \leftrightarrow (D', C')$.

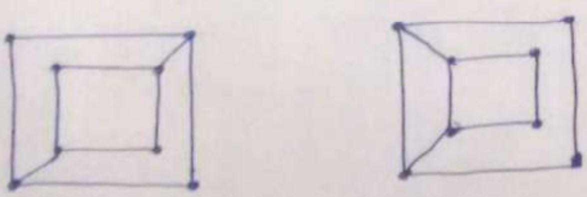
ie 1-1 correspondence b/w the edges of the given digraph exist, preserving the adjacency of verticies including directions of the edges.

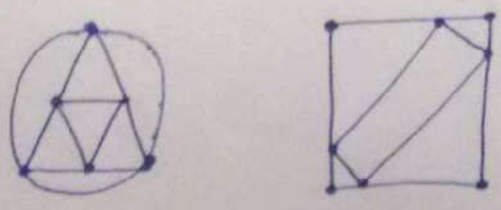
$\Rightarrow$ Isomorphic.

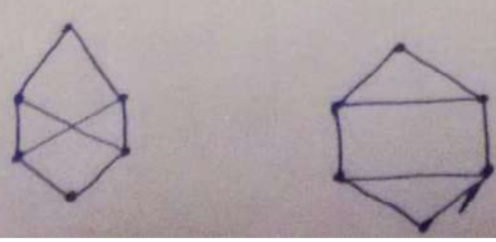
Hw

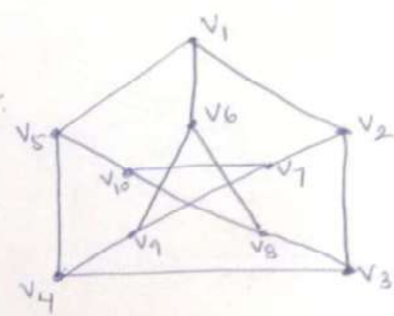
1) 

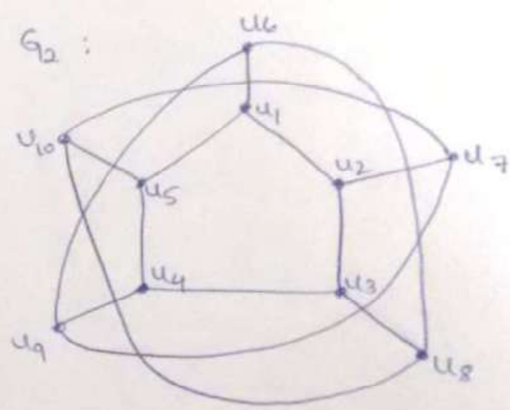
2) 

3) 

4) 

5) 

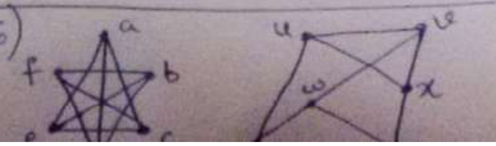
10) G1 : 

G2 : 

$v_i \leftrightarrow u_i \quad \forall\ i = 1, 2, 3 \ldots 10$

Both G_1 & G_2 have 10 vertices & 15 edges and degree of each vertex in both G_1 and $G_2 = 3$ with this correspondence, $\exists$ 1-1 correspondence b/w ~~vertices &~~ the edges of G_1 & G_2 which preserves adjacency of vertices.

$\therefore G_1 \cong G_2$.

6) 

Subgraphs :-

Given two graphs G and G_1, we say G_1 is a subgraph of G if the following conditions hold :

(i) All the verticies and all the edges of G_1 are in G.

(ii) Each edge of G_1 has the same end verticies in G as in G_1.

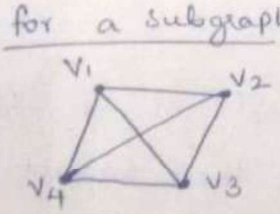
⟶ Give Ex Here.

Note :- Any graph isomorphic to a subgraph of a graph G is also a subgraph of G.

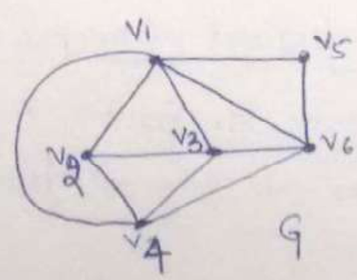
** Consequences of the defn of a Subgraph :-

1) Every graph is a subgraph itself.
2) Every simple graph of n-verticies is a subgraph of the complete graph K_n.
3) If G_1 is a subgraph of a graph G_2 and G_2 is a subgraph of a graph G, then G_1 is a subgraph of G.
4) A single vertex in a graph G is a subgraph of G.
5) A single edge in a graph G, together with its end verticies is a subgraph of G.

Ex :- for a subgraph:-



G_1



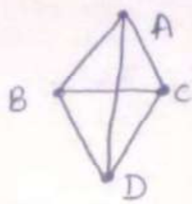
G

Spanning Subgraph :-

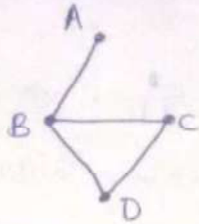
Given a graph $G = (V, E)$, if there is a subgraph $G_1 = (V_1, E_1)$ of G such that $V_1 = V$, then G_1 is called a spanning subgraph of G. [ie all the verticies of G should exist in G_1]

Note :- Every graph is its own spanning subgraph.

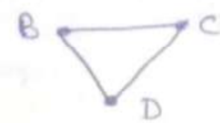
Ex :-



Graph G



Spanning Subgraph



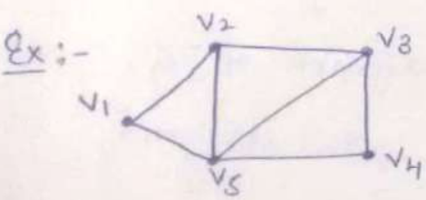
A subgraph, but not Spanning Subgraph

Induced Subgraph :-

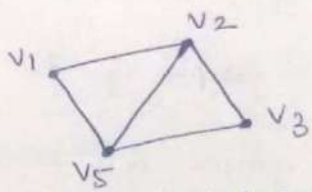
Given a graph $G = (V, E)$, suppose there is a subgraph $G_1 = (V_1, E_1)$ of G such that every edge $\{A, B\}$ of G, where $A, B \in V_1$ is an edge of G_1 also. Then G_1 is called an induced subgraph of G and is denoted by $\langle V_1 \rangle$

ie a subgraph G_1 of the graph $G = (V, E)$ is called an induced subgraph if $\exists$ $V_1 \subseteq V$ and contains all the edges from G, denoted by $\langle V_1 \rangle$

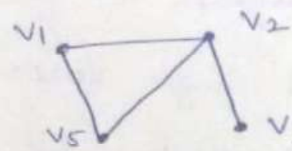
Ex :-



G



induced Subgraph



Not an induced Subgraph

Edge-disjoint and Vertex-disjoint Subgraphs :-

Let G be a graph and G_1, G_2 be 2 subgraphs of G, then

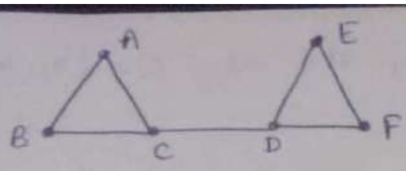
(i) G_1 & G_2 are said to be edge-disjoint if they do not have any common edge

(ii) G_1 & G_2 are said to be vertex-disjoint if they do not have any common edge & any common vertex.

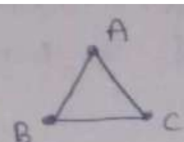
Note :- Subgraphs that have no vertex in common cannot have edges in common.

ie two vertex-disjoint subgraphs must be edge-disjoint but the converse is not true.

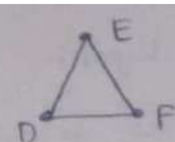
Ex :- 1)



G



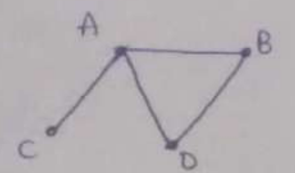
G_1



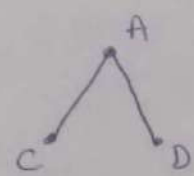
G_2

G_1 & G_2 are vertex-disjoint subgraphs, also edge-disjoint subgraphs.

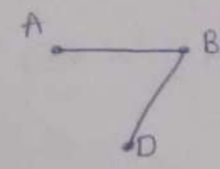
2)



G



G_1



G_2

G_1 and G_2 are edge-disjoint, but not vertex-disjoint.

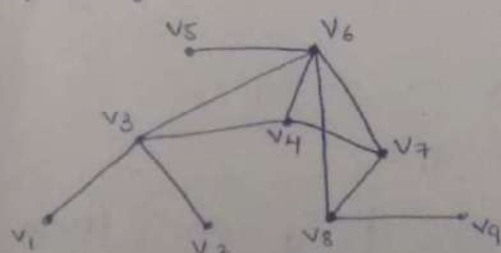
Problems :-

1) Consider the Graph G shown below :

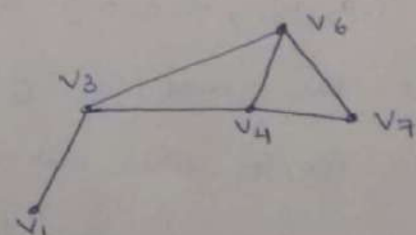
(a) verify that the graph G_1 is an induced subgraph of G.
Is this a spanning subgraph of G ?

(b) Draw the subgraph G_2 of G induced by the set $V_2 = \{v_3, v_4, v_6, v_8, v_9\}$.

Soln :-



G



G_1

Soln :- (a) The vertex set of the graph G_1, namely $V_1 = \{v_1, v_3, v_4, v_6, v_7\}$ is a subset of the vertex set $V = \{v_1, v_2 \ldots v_9\}$ of G.

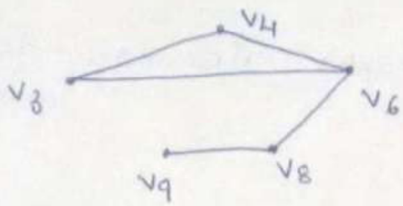
Also, all the edges of G_1 are in G. Each edge in G_1 has the Same end vertices in G as in G_1.

$\therefore$ G_1 is a Subgraph of G.

Every edge $\{v_i, v_j\}$ of G where $v_i, v_j \in V_1$, is an edge of G_1.

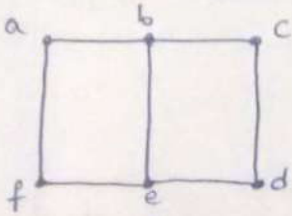
$\therefore$ G_1 is an induced subgraph of G, but not a Spanning Subgraph of G since $V_1 \neq V$.

(b) Subgraph G_2 of G induced by the set $V_2 = \{v_3, v_4, v_6, v_8, v_9\}$ is as below:

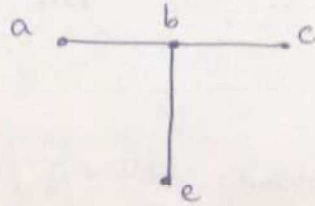


$G_2 = \langle V_2 \rangle$

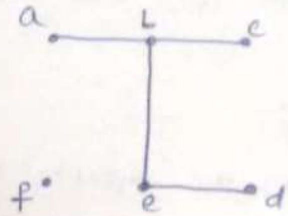
2) Three graphs G_1, G_2, G_3 are shown below: Are G_2 and G_3 induced subgraphs of G_1? Are they spanning subgraphs?



G_1



G_2



G_3

Soln:- The vertex set of G_2 & G_3, namely $V_2 = \{a, b, c, e\}$ and $V_3 = \{a, b, c, d, e, f\}$ are subsets of the vertex set $V_1 = \{a, b, c, d, e, f\}$ of G_1.

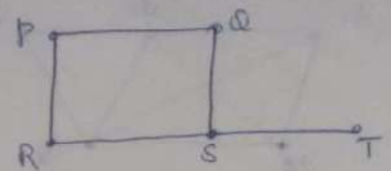
Also all the edges of G_2 & G_3 are in G_1.

$\therefore$ G_2, G_3 are subgraphs of G_1.

Since every edge of $\cancel{G_1}$ G_2 is an edge of $\cancel{G_2}$ G_1 also. Hence G_2 is an induced subgraph of G_1, but it is not a spanning subgraph of G_1 since $V_2 \neq V_1$.

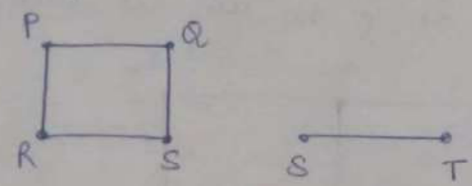
The graph G_3 does not contain all the edges of G_1. Hence G_3 is not an induced subgraph of G_1, but it is a spanning subgraph of G_1 since $V_3 = V_1$.

3) For the graph shown below, find two edge-disjoint subgraphs and two vertex-disjoint subgraphs:

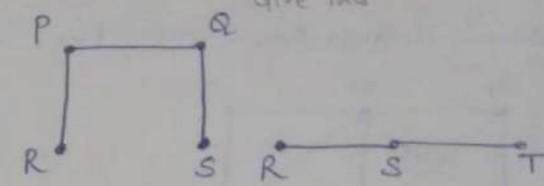


Soln :- Edge-disjoint subgraphs are as below :-

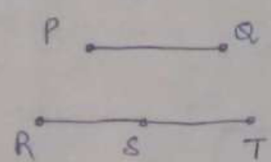
Give this



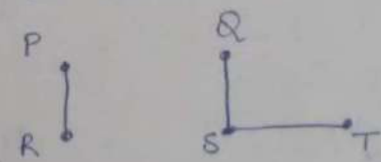
(or)



Vertex-disjoint subgraphs are as below :-



(or)



Operations on Graphs :—

Consider 2 graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, then

(i) The Graph whose vertex set is $V_1 \cup V_2$ & the edge set is $E_1 \cup E_2$ is called the union of G_1 & G_2 and it is denoted by $G_1 \cup G_2$.

Thus $G_1 \cup G_2 = (V_1 \cup V_2 , E_1 \cup E_2)$

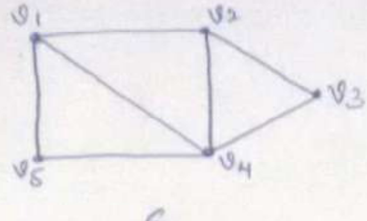
(ii) If $V_1 \cap V_2 \neq \phi$, the graph whose vertex set is $V_1 \cap V_2$ & the edge set is $E_1 \cap E_2$ is called the intersection of G_1 and G_2 & is denoted by $G_1 \cap G_2$.

Thus $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ if $V_1 \cap V_2 \neq \phi$

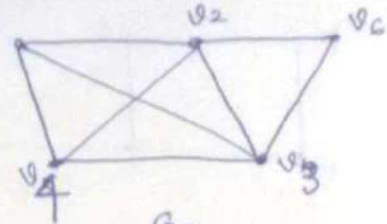
(iii) The graph whose vertex set is $V_1 \cup V_2$ & the edge set is $E_1 \Delta E_2$, where $E_1 \Delta E_2$ is the symmetric difference of E_1 & E_2. This graph is called the ring sum of G_1 & G_2 and is denoted by $G_1 \Delta G_2$. Thus $G_1 \Delta G_2 = (V_1 \cup V_2 , E_1 \Delta E_2)$

Note :- $E_1 \Delta E_2$ denotes set of all edges, which are in E_1 (or) E_2, but not in both. ie $E_1 \Delta E_2 = (E_1 \cup E_2) - (E_1 \cap E_2)$

Ex :-

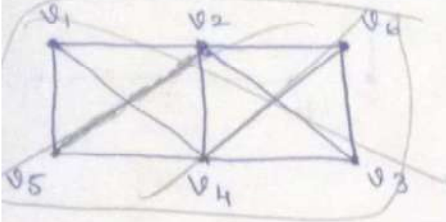


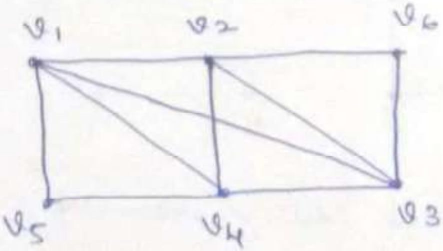
G_1



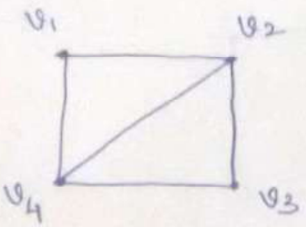
G_2

Union, Intersection and Ring sum of G_1 & G_2 are as below :-

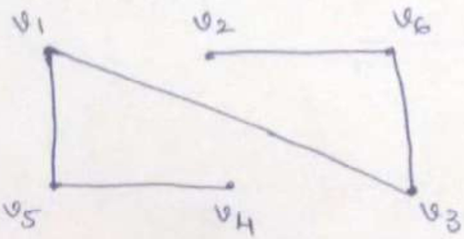




$G_1 \cup G_2$



$G_1 \cap G_2$



$G_1 \Delta G_2$

<u>Decomposition</u> :- The graph G is decomposed into 2 subgraphs G_1 and G_2 if $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \phi$

<u>Deletion</u> :- If v is a vertex in a graph G, then $G-v$ denotes the subgraph of G obtained by deleting v and all edges incident on v from G. This subgraph $G-v$ is called <u>vertex-deleted subgraph</u> of G.

Clearly $G-v$ is the subgraph of G <u>induced by</u> $V_1 = V - \{v\}$

If 'e' is an edge in the graph G, then $G-e$ denotes the subgraph of G obtained by deleting the edge 'e' from G. This graph $G-e$ is referred to as <u>edge-deleted subgraph</u> of G. The deletion of an edge does not alter the <u>no</u> of vertices. Thus an edge deleted subgraph of a graph G is a <u>spanning subgraph</u>.

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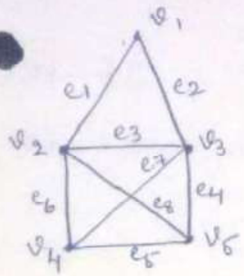
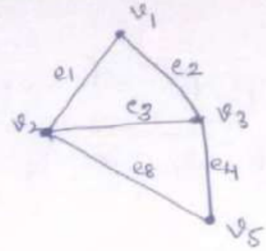
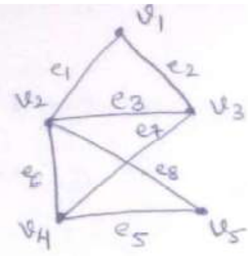
Ex •



G



$G - \{v_4\}$



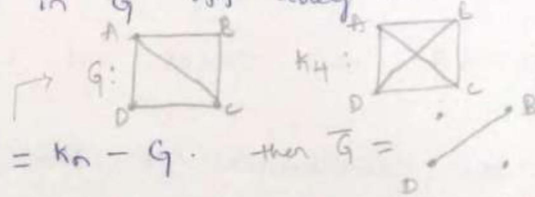
$G - e_4$

Complement of a Simple Graph :-

Complement of a simple graph G, denoted by $\overline{G}$, is the graph obtained by deleting those edges which are in G and adding the edges which are not in G. (Give Ex Here)

Note :- 1) G and $\overline{G}$ have the same vertex set.

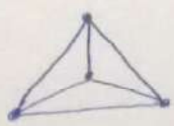
2) Two verticies are adjacent in G iff they are not adjacent in $\overline{G}$.



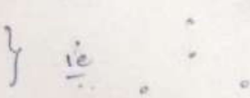
3) $\overline{\overline{G}} = G$ 4) $\overline{G} = K_n - G$.

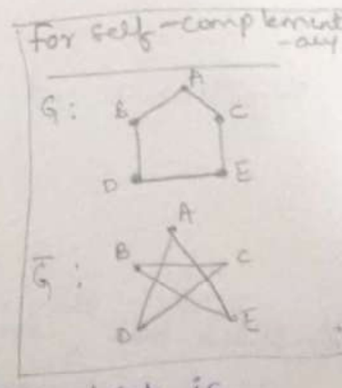
5) complement of K_n is a null graph.

$K_n = G = K_4$ (say)

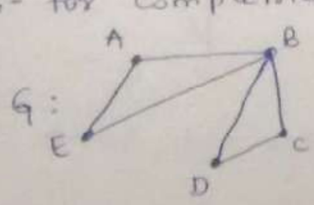


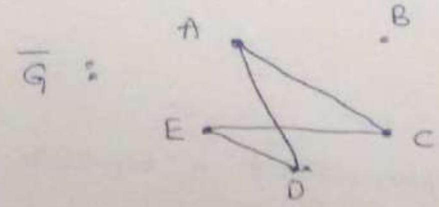
, then $\overline{G} = \{\ \}$ i.e.





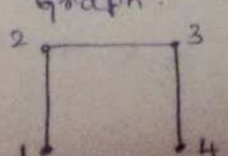
Ex :- for complement of a simple graph

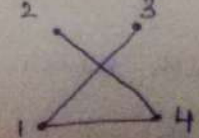


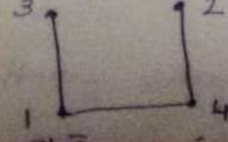


Self-Complementary Graphs :- A simple graph G which is isomorphic to its complement $\overline{G}$ is called a self-complementary Graph.

Ex :-

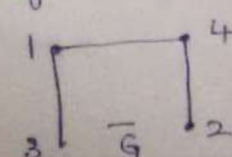






$G \cong \overline{G}$

(or)



∴ self-complementary.

Examples :-

1) Let G be a simple graph of order 'n'. If the no of edges in G is 56 & in $\bar{G}$ is 80. what is 'n'?

Soln :- $\bar{G} = K_n - G$

$\therefore$ No of edges in $\bar{G}$ = no of edges in K_n − no of edges in G

ie $56 = \frac{1}{2}n(n-1) - 80$

$(56+80)2 = n^2 - n$

ie $n^2 - n - 272 = 0$.

$n^2 - 17n + 16n - 272 = 0$

$n(n-17) + 16(n-17) = 0$

$(n-17)(n+16) = 0$

$n = 17$ (or) $n = -16$ (not possible)

$= \frac{1 \pm \sqrt{1-4(1)(-272)}}{2}$

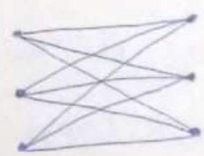
$= \frac{1 \pm \sqrt{1+1088}}{2} = \frac{1 \pm 33}{2}$

$= 17, -16$

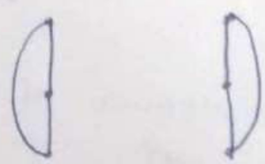
$\therefore$ $n = 17$ ie order of $G = 17$.

2) Find the complement of the complete Bipartite graph $K_{3,3}$.

Soln :- we have $K_{3,3}$ given by :-

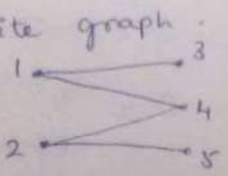


Complement of $K_{3,3}$ is as shown :

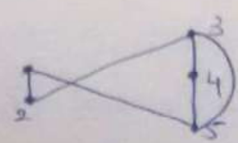


3) Show that the complement of a bipartite graph need not be a bipartite graph.

Soln :



G



$\bar{G}$

Here G is a bipartite graph with order 5, but its complement $\bar{G}$ is not a bipartite graph.

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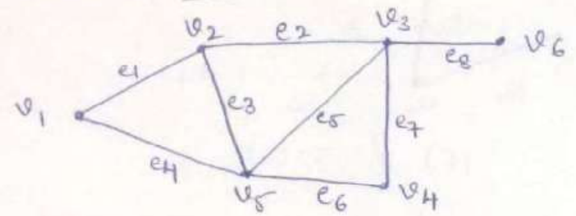
Walk and their classification :-

Walk :- A finite alternating sequence of verticies and edges is called a walk.

Ex:-



$v_1\, e_1\, v_2\, e_2\, v_3\, e_5\, v_5\, e_6\, v_4\, e_7\, v_3\, e_8 v_6$ is a walk.

Note :-

1) A walk begins and ends with verticies.

2) An edge in the sequence is incident on the verticies preceeding and following it in the sequence.

3) In a walk, a vertex (or) an edge (or both) can appear more than once.

4) The no of edges present in a walk is called its Length.

5) The vertex with which a walk begins is called the initial vertex (or the origin) of the walk and the vertex with which a walk ends is called the final vertex (or the terminus) of the walk. 6) The initial & final vertex of a walk are together called its terminal verticies. The terminal verticies of a walk need not be distinct (same).

7) Non-terminal verticies of a walk are called its Internal verticies

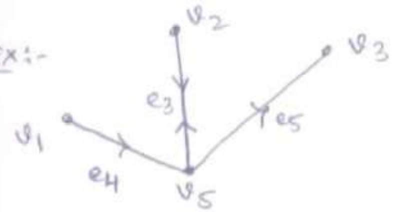
8) A walk having 'u' as the initial vertex and 'v' as the final vertex is called a walk from u to v (or) a u-v walk.

9) A walk that begins and ends at the same vertex is called a closed walk.

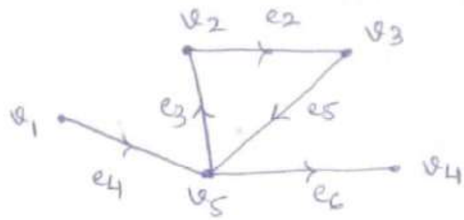
10) A walk which is not closed is called an open walk.

Trail :- An open walk in which no edge appears more than once is called a Trail.

Ex:-



(a) Not a trial



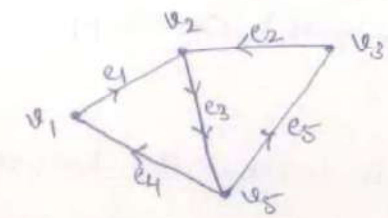
(b) A Trial

(a) $v_1 e_4 v_5 e_3 v_2 e_3 v_5 e_5 v_3$ is not a trial since the edge e_3 is repeated twice.

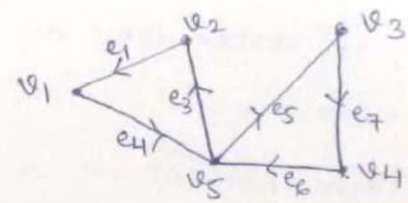
(b) $v_1 e_4 v_5 e_3 v_2 e_2 v_3 e_5 v_5 e_6 v_4$ is a trial

Circuit :- A closed walk in which no edge appears more than once is called a circuit.

Ex:-



(a) Not a circuit



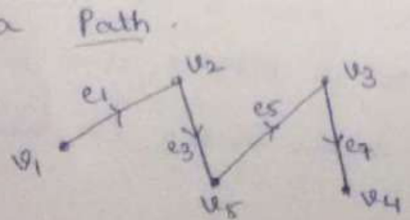
(b) A circuit

(a) $v_1 e_1 v_2 e_3 v_5 e_5 v_3 e_2 v_2 e_3 v_5 e_4 v_1$ is not a circuit since the edge e_3 is repeated twice

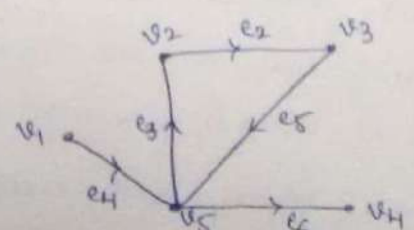
(b) $v_5 e_3 v_2 e_1 v_1 e_4 v_5 e_5 v_3 e_7 v_4 e_6 v_5$ is a circuit.

Path :- A trial in which no vertex appears more than once is called a Path.

EX :-



(a) path



(b) Not a path.

(a) $v_1 e_1 v_2 e_3 v_5 e_5 v_3 e_7 v_4$ is a path.

(b) $v_1 e_4 v_5 e_3 v_2 e_2 v_3 e_5 v_5 e_6 v_4$ is not a path since the vertex v_5 is repeated twice.

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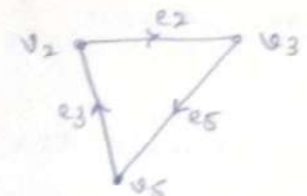
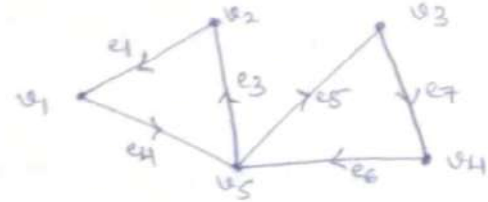
Cycle :- A circuit in which the terminal vertex does not appear as an internal vertex and no internal vertex is repeated is called a cycle.

Ex :-



(a) cycle



(b) Not a cycle

(a) $v_2 e_2 v_3 e_5 v_5 e_3 v_2$ is a cycle.

(b) $v_2 e_1 v_1 e_4 v_5 e_5 v_3 e_7 v_4 e_6 v_5 e_3 v_2$ is not a cycle since the internal vertex v_5 is repeated twice.

(or)

$\underline{v_5} e_3 v_2 e_1 v_1 e_4 \underline{v_5} e_5 v_3 e_7 v_4 e_6 \underline{v_5}$ is not a cycle since the terminal vertex (v_5) appears as an internal vertex

Note / observations :-

1) A walk can be open (or) closed. In a walk, a vertex and/or an edge can appear more than once.

2) A trial is an open walk in which a vertex can appear more than once but an edge cannot appear more than once.

3) A circuit is a closed walk in which a vertex can appear more than once but an edge cannot appear more than once.

4) A path is an open walk in which neither a vertex nor an edge can appear more than once.

5) Every path is a ~~trial~~ trail; but a ~~trial~~ trail need not be a path.

6) A cycle is a closed walk in which neither a vertex nor an edge can appear more than once.

7) Every cycle is a circuit; but a circuit need not be a cycle.

8) If a cycle contains only one edge, it has to be a loop.

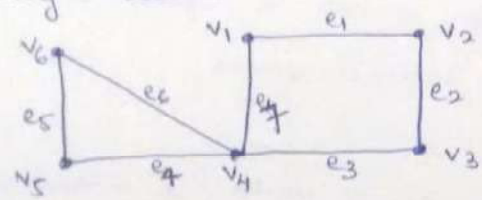
9) Two parallel edges (when they occur) form a cycle.

10) In a simple graph, a cycle must have atleast 3 edges.

Note :- In case of digraphs, the walks, ~~trails~~ trails, circuits, paths and cycles become directed walks, directed trails, directed circuits, directed paths and directed cycles.

Problems :-

1) For the graph given below, indicate the nature of the following walks:



(i) $v_1 e_1 v_2 e_2 v_3 e_2 v_2$ → Open walk which is not a trial since edge e_2 is repeated.

(ii) $\underline{v_4} e_7 v_1 e_1 v_2 e_2 v_3 e_3 \underline{v_4} e_4 v_5$ → Trial which is not a path since vertex v_4 is repeated.

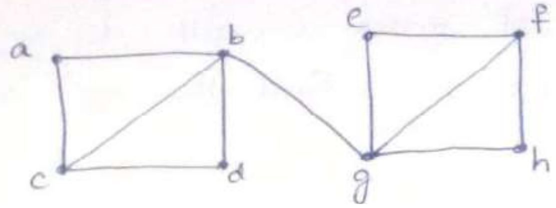
(iii) $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5$ → Trial which is a path.

(iv) $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_7 v_1$ → closed walk which is a cycle.

(v) $v_6 e_5 v_5 e_4 \underline{v_4} e_3 v_3 e_2 v_2 e_1 v_1 e_7 \underline{v_4} e_6 v_6$ → closed walk which is a circuit, but not a cycle, since vertex v_4 is repeated.

2) In the graph shown below, how many paths are there from a to h? How many of these paths have a length 5?

P.T.O



The following are the paths from a to h :—

abgh , acbgh , acdbgh ,

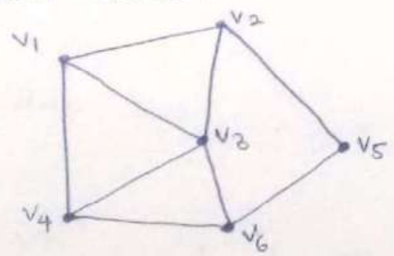
abgefh , abgfh , acbgefh , acbgfh,

acdbgefh , acdbgfh , whose lengths are respectively

3 , 4 , 5 , 5 , 4 , 6 , 5 , 7 , 6 .

Thus the total no of paths from a to h are Nine and the paths of length 5 are three in number.

3) Determine the no of different paths of length 2 in the graph shown below:



Soln :- The no of paths of length 2 that pass through the vertex v_1 is the no of pairs of edges incident on v_1. $= {}^3C_2$ ($\because$ 3 edges are incident on v_1)
since 3 edges are incident on v_1, this number $= {}^3C_2 = 3$.
lly the no of paths of length 2 that pass through the verticies v_2, v_3, v_4, v_5, v_6 resp are :

${}^3C_2 = 3$, ${}^4C_2 = 6$, ${}^3C_2 = 3$, ${}^2C_2 = 1$, ${}^3C_2 = 3$

Total no of paths of length 2 in the above graph is
$3 + 3 + 6 + 3 + 1 + 3 = 19$.

P.T.O.

4) If G is a simple graph of order n with d_i as the degree of a vertex v_i for $i = 1, 2, \ldots n$, find the no of paths of length 2 in G.

Soln:- we have $\deg(v_i) = d_i$, the no of edges incident on v_i is exactly d_i.

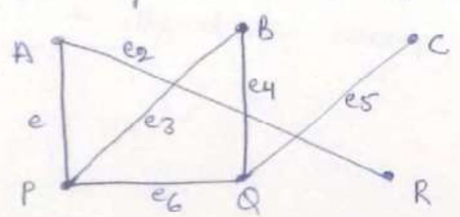
In this, every two edges give a path of length 2 which contains v_i.

combination

$\therefore \exists\ C(d_i, 2)$ paths containing $v_i\ \forall\ i = 1, 2 \ldots n$

$\therefore$ total no of paths of length 2 in G is $\sum_{i=1}^{n} C(d_i, 2)$.

5) Find all the cycles in the graph shown below:



Soln:- It is clear that, no cycle begin & end with the vertices A, C and R.

The cycles beginning & ending with B, P, Q are:

$Be_3Pe_6Qe_4B$, $Pe_6Qe_4Be_3P$, $Qe_4Be_3Pe_6Q$.

But all of these represent ~~one & only the~~ same cycle.

Thus there is only one cycle for the given graph.

6) If G is a bipartite graph, show that G has no cycle of odd length.

Soln:- Since G is Bipartite, the vertex set V is partitioned into two disjoint sets V_1 & V_2.

Let $v_0 v_1 v_2 \ldots v_m v_0$ be a cycle in G.

Assume that v_0 is in V_1, v_1 is in V_2, v_2 is in V_1, v_3 is in V_2 and so on.

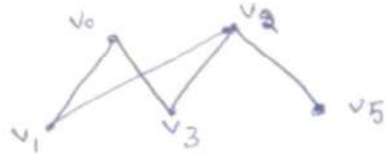
Thus the verticies in the cycle belong to V_1, V_2 alternately. (22)

Since the terminal vertex of the cycle is v_0 & it is in V_1, the no of edges that belong to the cycle cannot be 3 or 5 or 7 or any odd number.

Thus G has no cycle of odd length.

Ex:- 1)

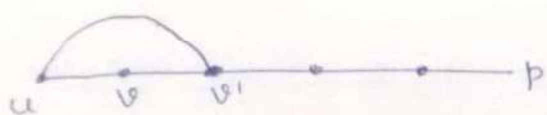
2)



only $v_0v_1v_2v_3v_0$ is a cycle, of even length not the whole graph.

7) If G is a simple graph with no cycles, prove that G has atleast one pendant vertex.

Soln:- Consider a path p in G which has a maximum no of verticies. Let 'u' be an end vertex of p, then every neighbour of u belongs to p.



If u has atleast 2 neighboure say v & v', then v & v' both belong to p and then the edges (u,v) (v,v') (v',u) form a cycle. This is not possible, because G has no cycles.

Hence u can have only one neighbour.

$\Rightarrow$ u is a pendant vertex.

Thus G has atleast one pendant vertex.

8) Prove the following:-

(i) A path with n verticies is of length $(n-1)$

(ii) If a cycle has n-verticies, then it has n edges.

(iii) The degree of every vertex in a cycle is two.

P.T.O

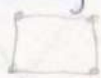
Soln :-

(i) In a path, every vertex except the last vertex followed by precisely one edge.

∴ If a path has n-verticies, it must have n-1 edges.

Hence its length is n-1.

(ii) In a cycle, every vertex is followed by precisely one edge.



∴ If a cycle has n verticies, it must have n edges.

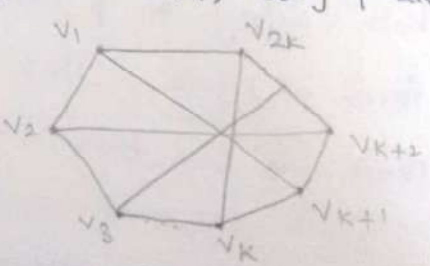
(iii) In a cycle, exactly two edges are incident on every vertex [one edge through which we enter the vertex & one edge through which we leave the vertex]

∴ degree of every vertex in a cycle is two.

9) Show that, for any +ve integer $K \geq 2$, $\exists$ a simple cubic graph of order $2K$.

Soln:. Consider a set of points $v_1, v_2 \ldots v_{2K}$ and the cycle made up of the foll $2K$ edges:

$\{v_1, v_2\}\ \{v_2, v_3\} \ldots \{v_{K-1}, v_K\}\ \{v_K, v_{K+1}\} \ldots \{v_{2K-1}, v_{2K}\}\ \{v_{2K}, v_1\}$



To this cycle, add K-edges

$\{v_1, v_{K+1}\}\ \{v_2, v_{K+2}\} \ldots \{v_K, v_{2K}\}$

Thus the ~~generating~~ resulting graph is simple and contains $2K + K = 3K$ edges.

In this graph, exactly 3 edges are incident on every vertex v_i, namely the edges $\{v_{i-1}, v_i\}$ & $\{v_i, v_{i+1}\}$ which belong to the original cycle and the edge $\{v_i, v_{K+i}\}$ which has been added to the original cycle. Thus the simple graph constructed is of order $2K$ in which the deg of every vertex is 3. Thus the desired graph exists.

10) Prove that in a graph, there is a u-v trial iff there is a u-v path. (23)

Soln: <u>Necessary</u>:

Given :- There is a u-v trial

To prove : ~~There is a u-v path~~. u-v trial is a path.

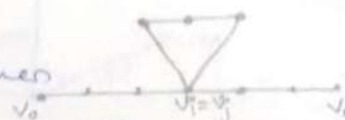
Let u-v be trial in G.

Among these trials, choose a trial of minimum length and let this be $v_0 v_1 v_2 \ldots v_n \rightarrow (1)$

where $v_0 = u$ & $v_n = v$ & the edges ~~of~~ joined between.

~~case (i) If there is only one u-v trial, then it will be the only trial of minimum length.~~

case (ii) If in ①, no vertex repeats, then it is a path from u to v & hence the proof.



Case (iii) If a vertex repeats say $v_i = v_j$ then

① $\Rightarrow v_0 v_1 v_2 \ldots v_{i-1} v_i v_{i+1} \ldots v_{j-1} v_j v_{j+1} \ldots v_n$.

$v_0 v_1 v_2 \ldots v_{i-1} v_i \; v_{j+1} \ldots v_n \rightarrow (2)$ is the trial got by skipping the verticies $v_{i+1}, v_{i+2} \ldots v_{j-1} v_j$ together with all the edges preceeding them.

Thus ② is the trial which is shorter than that of ①.

This is a contradiction since we have assumed ① to be of shorter length.

Thus a u-v trial with minimum length is a u-v path ~~is a path~~.

<u>Sufficiency</u>:

Given : there is a u-v path

To prove : ~~there is a~~ u-v path is a trial.

WKT Every path is a trial.

Hence if there is a u-v path, then it is a u-v trial.

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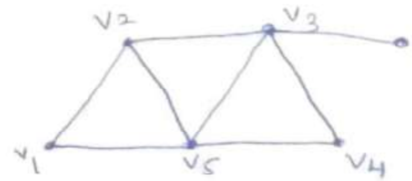
Hw

1) for the Graph shown below, find the nature of the foll walks:

(a) $v_1 v_2 v_5 v_3 v_4 v_5 v_1$ (b) $v_1 v_2 v_3 v_5 v_1$



2) for the graph shown below, determine:

(i) a walk from v_2 to v_4 which is not a trail.

(ii) a $v_2 - v_4$ trail which is not a path.

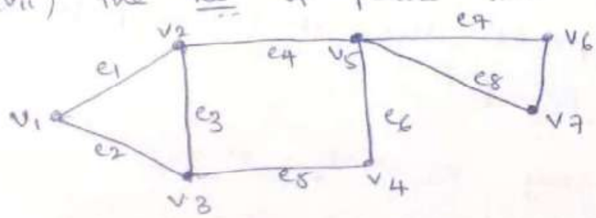
(iii) a path from v_2 to v_4.

(iv) a closed walk from v_2 to v_2 which is not a circuit.

(v) a circuit from v_2 to v_2 which is not a cycle.

(vi) a cycle from v_2 to v_2

(vii) the no of paths from v_2 to v_6.



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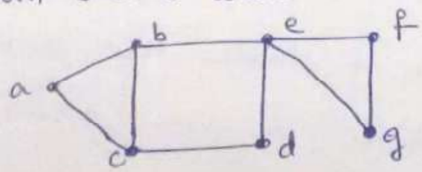
11) In the graph shown below, determine i) a walk from b to d that is not a trail. ii) b-d trail that is not a path iii) a path from b to d. iv) a closed walk from b to b that is not a circuit v) a circuit from b to b that is not a cycle vi) a cycle from b to b.

QP



Soln:. i) b-c-a-b-c-d is a walk, but not a trail since the edge {b, c} is repeated.

ii) b-a-c-b-e-d is an open walk which is a trail, but not a path since vertex b is repeated.

iii) b-c-d is a path from b to d.

iv) b-e-f-g-e-b is a closed walk from b to b, which is not a circuit since the edge {b, e} is repeated.

v) b-c-d-e-g-f-e-b is a circuit from b to b, which is not a cycle since the internal vertex e is repeated.

vi) b-a-c-b is a cycle from b to b.

Connected and Disconnected Graphs :-

A Graph G is said to be

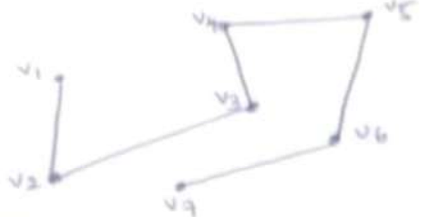
(i) Connected if there is atleast one path b/w every two distinct vertices in G. (OR)**

(ii) disconnected if G has atleast one pair of distinct verticies b/w which there is no path.

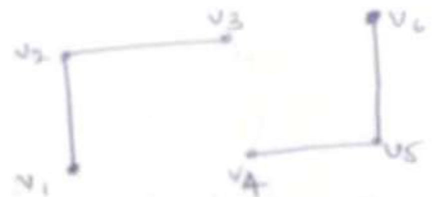
**(OR)

Two verticies in a graph G are said to be connected if there is atleast one path from one vertex to the other

Ex :-



Connected



disconnected

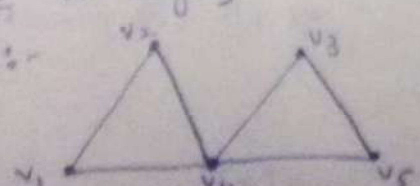
Component :- Every non-trivial graph G consists of one or more connected graphs. Each such connected graph is a subgraph of G and is called a component of G.

Note : 1) A connected graph has only one component, where as a disconnected graph has two (or) more components.

2) The No of components of a graph G is denoted by K(G). In above Ex, K(G) = 1 & K(G) = 2 resp.

3) If u & v are 2 verticies in a connected graph, then the length of the shortest path [ie the path containing least no of edges] is called the distance b/w u & v.

Ex :-



Distance from v_1 to v_3 is 2

" ——— " v_2 to v_5 is 2

Theorem 1 :- If a graph has exactly two verticies of odd degree, then there must be a path connecting these verticies.

Proof :- Let v_1 & v_2 be two verticies of a graph G with odd degrees.

Suppose there is no path b/w v_1 & v_2, then the graph G is disconnected and v_1 & v_2 belong to two different components say H_1 & H_2 of G.

$\Rightarrow$ Each of H_1 & H_2 contains only one vertex of odd degree.

This is a contradiction because H_1 & H_2 are graphs and WKT in a graph, the no of verticies of odd degrees is always Even [by Thm in Hand shaking property].

Hence there must be a path connecting v_1 & v_2.

Hence the proof.

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Theorem 2 :- A simple graph with 'n' verticies & 'K' components can have atmost $(n-K)(n-K+1)/2$ edges.

Proof :- Given G is a simple graph.

$\Rightarrow$ each of the components of G is a simple graph.

Let n_1 be the no of verticies in the 1^{st} component

n_2 " ——— " ——— 2^{nd} "

⋮

n_K " ——— " ——— K^{th} "

$\therefore$ max no of edges in the i^{th} component is $\frac{1}{2} n_i(n_i - 1)$

Let N be the max no of edges

$$\therefore \ N \leq \frac{1}{2} \sum_{i=1}^{K} n_i(n_i-1) \leq \frac{1}{2}\left[\sum_{i=1}^{K} n_i^2 - \sum_{i=1}^{K} n_i\right] \rightarrow (1)$$

(since G has K components)

Also $n_1 + n_2 + \dots + n_K = n$

$$\Rightarrow \sum_{i=1}^{K} n_i = n \longrightarrow (2)$$

Now, $(n_1-1)+(n_2-1)+\dots+(n_K-1) = (n_1+n_2+\dots+n_K) - (1+1+\dots K \text{ times})$

$= n - K$.

Squaring on b.s

$$(n_1-1)^2 + (n_2-1)^2 + \dots + (n_K-1)^2 + S = (n-K)^2 \rightarrow (3)$$

where S is the sum of products of the form $2(n_i-1)(n_j-1)$,

$i = 1,2 \dots K$, $j = 1,2\dots K$, $i \neq j$.

Since each of $n_1, n_2, \dots n_K \geq 1$, we have $S \geq 0$

$$\therefore (3) \Rightarrow (n_1-1)^2 + (n_2-1)^2 + \dots + (n_K-1)^2 \leq (n-K)^2$$

$$\Rightarrow n_1^2 + n_2^2 + \dots + n_K^2 + (1+1+\dots K \text{ times}) - 2(n_1+n_2+\dots+n_K) \leq (n-K)^2$$

$$\Rightarrow n_1^2 + n_2^2 + \dots + n_K^2 + K - 2n \leq (n-K)^2$$

$$\Rightarrow n_1^2 + n_2^2 + \dots + n_K^2 \leq (n-K)^2 - K + 2n = n^2 + K^2 - 2nK - K + 2n$$

$$\leq n^2 + \{K(K-2n) - 1(K-2n)\}$$

$$\leq n^2 + (K-1)(K-2n)$$

(or) $$\sum_{i=1}^{K} n_i^2 \leq n^2 - (K-1)(2n-K) \longrightarrow (4)$$

Sub (2) & (4) in (1),

$$N \leq \frac{1}{2}\left[n^2 - (K-1)(2n-K) - n\right] = \frac{1}{2}\left[\underbrace{n^2 - 2nK + K^2} + 2n - K - n\right]$$

$$N \leq \frac{1}{2}\left[(n-K)^2 + (n-K)\right] = \frac{1}{2}(n-K)(n-K+1)$$

ie $N \leq \frac{1}{2}(n-K)(n-K+1)$.

Thus the no of edges in G cannot exceed $\frac{1}{2}(n-K)(n-K+1)$.

Theorem 3 :- A connected graph with 'n' verticies has atleast (n-1) edges

Proof: Since the graph is Connected, $n \geq 2$.

If 'm' denote the no of edges, we have to prove $m \geq n-1$.

The proof is by mathematical Induction.

when $n = 2$

$\Rightarrow$ there are two verticies in the graph and since the graph is connected, $\exists$ atleast one edge joining these verticies

i.e $m \geq 1 = (2-1) = (n-1)$

$\Rightarrow m \geq (n-1)$.

This verifies the result for $n=2$.

Assume that the result is true for $n=k$, with $k \geq 2$.

i.e $m_K \geq K-1$ where m_K is the no of edges in G_K

To prove: the result is true for $n = k+1$

Let G_{K+1} be the graph with $(K+1)$ no of verticies.

Let G_K be the graph obtained by deleting an edge from G_{K+1} for which v is an end vertex.

Thus G_K is a connected graph with K- verticies.

Let m_K be the no of edges in G_K, then from the assumption made above,

$$m_K \geq k-1$$

$$\Rightarrow m_K + 1 \geq (k-1)+1 \quad \text{i.e} \quad m_K + 1 \geq (K+1) - 1$$

But $m_K + 1$ is the no of edges in G_{K+1} & $K+1$ is the no of verticies in G_{K+1}.

Thus the result $m \geq (n-1)$ holds for $n = K+1$.

Hence the proof.

Theorem 4 :- A Graph G is disconnected iff its vertex set
V can be partitioned into 2 non-empty disjoint subsets V_1 & V_2 such that ∃ no edge in G whose one end vertex is in V_1 and the other is in V_2.

Proof :- Necessary :

Given : G is disconnected.

Consider a vertex v in G. Let V_1 be the set of all verticies of G that are connected to v. Since G is disconnected, V_1 does not include all verticies of G

$\Rightarrow V_1$ is a proper subset of V.

Let $V_2 = V - V_1$ then $V_1 \cap V_2 = \phi$, $V = V_1 \cup V_2$ and no vertex in V_1 is connected to any vertex in V_2.

Hence V_1 & V_2 form a partition of V of desired type.

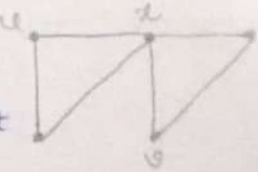
Sufficiency :

To prove : G is disconnected.

Consider 2 arbitrary verticies v & u in G such that $v \in V_1$ & $u \in V_2$ $\Rightarrow$ ∃ no path b/w v & u (given)

Hence G is not connected.

EX 1 :- If G is a simple graph with 'n' verticies in which the deg of every vertex is atleast $\frac{n-1}{2}$, P.T G is connected



Soln :- Take any 2 verticies u & v of G.

then they are either adjacent (or) not adjacent

If they are adjacent then G is connected.

otherwise, each vertex has atleast $\frac{n-1}{2}$ neighbours since the degree at each vertex is $\frac{n-1}{2}$.

∴ u & v taken together have atleast $(n-1)$ neighbours.

×(Since G has total of n-verticies, the total no of neighbours which u & v together can have is only $n-2$)×

~~∴ atleast one vertex say~~ Let x be a neighbour of both u & v.

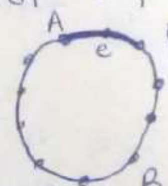
$\Rightarrow$ there is an edge b/w u & x & b/w x & v.

$\Rightarrow$ there is a path b/w u & v.

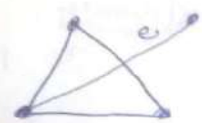
$\Rightarrow$ G is connected.

Ex 2 :- Prove that a connected graph G remains connected after removing an edge 'e' from G iff e is part of some cycle of G.

Soln :- Suppose 'e' is a part of some cycle 'C' in G then the removal of 'e' from G will not affect the connectivity of G since after the removal of 'e', the end verticies of 'e' remain connected.



Suppose 'e' is not a part of any cycle, then the removal of 'e' from G disconnects these end points.



Thus $G-e$ is a disconnected graph.

Hence the proof.

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Euler Circuits and Euler Trails :-

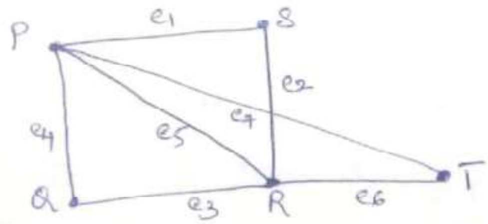
Euler Circuit :- A circuit in a connected graph G which Contains all the edges of G is called an Euler circuit

Euler Trail :- A Trail in a connected graph G which contains all the edges of G is called an Euler Trail.

A Graph which contains Euler circuit is called Euler graph.

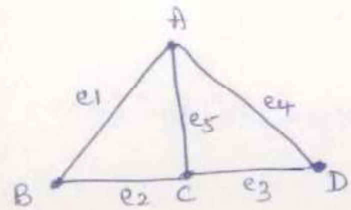
A Graph which contains Euler trail is called Semi-Euler graph

Ex :- 1)



The closed walk $P e_1 S e_2 R e_3 Q e_4 P e_5 R e_6 T e_7 P$ is an Euler circuit. ∴ This graph is an Euler graph.

2)



The trail $A e_1 B e_2 C e_3 D e_4 A e_5 C$ is an Euler trail.

∴ the graph is a Semi-Euler graph.

just say [This graph is not Euler graph since it ~~is not a~~ does not contain any Euler circuit. B'coz every sequence of edges which starts & ends with the same vertex & which includes all edges will contain atleast one repeated edge]

Theorem 1 :- A Connected graph G has an Euler circuit if & only if all verticies of G are of Even degree.

Proof :- Necessary :

Given :- A connected graph G has an Euler circuit

T.P.T : All verticies of G are of Even degree.

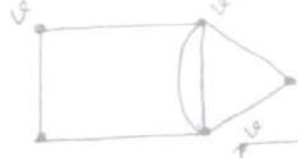
while tracing this circuit, every time the circuit meets a vertex 'v' it goes through two edges incident on 'v'

This is true for all verticies that belong to the circuit.

Since the circuit is an Euler circuit, it contains all the edges of G & meets all the verticies atleast once.

Thus the degree of every vertex is a multiple of two.

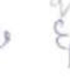
=> Every vertex is of Even degree.



Sufficiency :-

Given :- All verticies of G are of Even degree.

T.P.T :- G has an Euler circuit.



Construct a circuit starting at an arbitrary vertex v & going through the edges of G such that no edge is repeated

Since every vertex is of Even degree, we can depart from every vertex we enter and the tracing cannot stop at any vertex other than v. Thus a circuit 'q' is obtained having 'v' as the initial & final vertex.

If this circuit contains all the edges in G, then the circuit 'q' is an Euler circuit.

Otherwise consider the subgraph H obtained by removing all the edges that belong to q from G. The degrees of verticies in this subgraph are also Even.

Since G is connected, the circuit 'q' & the subgraph 'H' must have atleast one vertex in common (say v').

Starting from v', construct a circuit q' in H.

The two circuits q & q' together constitutes a circuit which starts & ends at 'v', and has more edges than 'q'.

If this circuit contains all the edges in G, then the circuit is an Euler circuit, otherwise repeat the process until we get a circuit that starts from v & ends at v containing all the edges in G. In this way, we obtain an Euler circuit in G. Hence the proof.

Theorem 2 :- A connected graph G has an Euler circuit iff G can be decomposed into edge disjoint cycles. (28)

Proof :-

Necessary: Given :- G has an Euler circuit

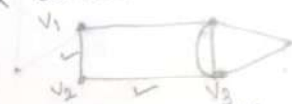
T.P.T : G can be decomposed into edge disjoint cycles.

G has an Euler circuit $\Rightarrow$ every vertex in G is of Even degree [by Thm 1]

Consider a vertex v_1 in G. Since v_1 is of even degree, there is atleast 2 edges incident on v_1. choose one of these edges. let v_2 be the other end vertex of this (chosen) edge.

Since v_2 is also of Even degree, $\Rightarrow$ there must be atleast one other edge incident on v_2. choose one of such edges and let v_3 be the other end vertex.

Proceeding like this, we get a vertex which has been traversed, thus forming a cycle C_1.



Remove C_1 from G. All the verticies in the resulting graph must also be of even degree, and in this graph, construct a cycle C_2 as previously done. Remove the cycle C_2 and proceed as above. The process ends when no edges are left.

Thus we get a sequence of cycles whose union is G and intersection is a null graph. Hence G has been decomposed into edge-disjoint cycles.

Sufficiency :

Given :- G can be decomposed (partitioned) into edge-disjoint cycles.

T.P.T :- G has an Euler circuit

~~Since~~ the degree of every vertex in a cycle is two,

$\Rightarrow$ Every vertex in G is of even degree.

$\Rightarrow$ G has Euler circuit [by Thm 1]

Problems :-

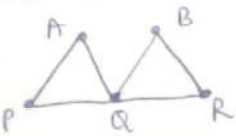
1) for what value of 'n', does K_n has an Euler trail, but not an Euler circuit? find all positive integers $n (\geq 2)$ for which the complete graph K_n contains an Euler circuit.

Soln:- for $n=2$, the graph K_n contains exactly one edge. This edge together with its end verticies constitutes an Euler trail. In this case, K_n cannot have an Euler circuit.

for $n \geq 3$, K_n contains an Euler circuit iff $n-1$ [ie deg of every vertex in K_n] is Even; ie iff n is odd.

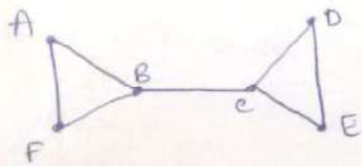
2) find the Euler Circuit in the following :-

(i)



Euler circuit
→ APQRBQA.

(ii)

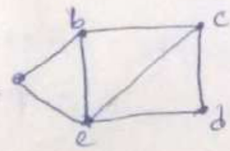


No Euler circuit

An Euler trail exists

ie BAFBCEDC

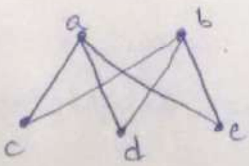
(iii)



No Euler circuit

Euler trail → baecbedc.

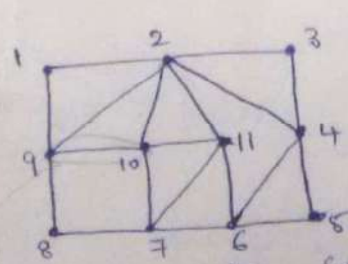
(iv)



No Euler circuit

Euler trail → acbeadb

(v)
QP



Euler circuit ↓

12 3 4 5 6 7 8 9 10 7 11 6 4 2 11 10 2 9 1

Remove the edge {9,10} & find an Euler trail.

Ans:- 10 7 11 6 4 2 11 10 2 9 1 2 3 4 5 6 7 8 9

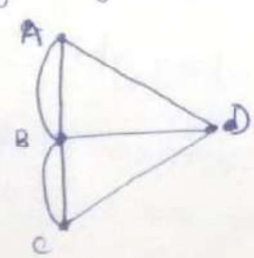
Konigsberg Bridge Problem :- → a city in Europe

QP The city of Konigsberg was divided into 4 parts by Pregel river and seven bridges connected these regions.

The citizens of the city tried to find a way to walk along the city so as to cross each bridge <u>exactly once</u> & then return to the starting point. This problem is now known as Konigsberg Bridge problem.

Euler analysed the problem with the help of graph and gave the solution.

Euler represented the 4 regions by 4 verticies & the 7 bridges by edges as shown below:



WKT the existence of Euler circuit depends on the degrees of verticies of the graph.

from the fig,

$deg(A) = 3 = deg(C) = deg(D)$; $deg(B) = 5$

All the 4 verticies are of odd degree.

Also a graph G has Euler circuit iff every vertex in G has even degree.

Thus for the above graph, Euler circuit does not exist.

=> it is not possible to walk over each of the seven bridges exactly once & return to the starting point.

Thus the soln to the Konigsberg Bridge problem is <u>negative</u>.