EE2703: Assignment 7: The Laplace Transform

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1 Abstract:

In this assignment, we will,

1. solve for the time response of a spring with different decays and frequencies
2. solve for a coupled spring problem
3. obtain the magnitude and phase response of the Steady State Transfer function of a two-port network
4. obtain the $v_o(t)$ for the said two-port network

2 Introduction:

In this assignment, we will make use of Laplace Transform to solve 'continuous time' systems. We will analyse the LTI systems using the python library, `scipy.signal`.

3 Time Response of a Spring:

We need to find the response of a spring, whose equation is given by,

$$\ddot{x} + 2.25x = f(t)$$

Here f(t) is given by,

$$f(t) = e^{-at}cos(\omega t)u(t)$$

where, $'a'$ represents decay and takes values 0.5 and 0.05. ω takes values 1.4, 1.45, 1.5, 1.55 and 1.6 .
We will use the **Laplace Transform** to solve the problem.
The Laplace transform of $f(t) = e^{-at}cos(\omega t)u(t)$ is given as:

$$\mathcal{L}\{f(t)\} = \frac{s+a}{(s+a)^2+\omega^2}$$

From the property of Laplace transforms, we know:

$$\begin{aligned} x(t) &\longleftrightarrow \mathcal{X}(s) \\ \implies \dot{x}(t) &\longleftrightarrow \ s\mathcal{X}(s) - x(0^-) \\ \implies \ddot{x}(t) &\longleftrightarrow \ s^2\mathcal{X}(s) - sx(0^-) - \dot{x}(0^-) \end{aligned}$$

From the above equations, we get, for $a = 0.5$ and $\omega = 1.5$:

$$F(s) = \frac{s+0.5}{(s+0.5)^2+2.25}$$

Substituting the given ICs $x(0)$ and $\dot{x}(0)$ are 0, we get:

$$X(s) = \frac{s+0.5}{((s+0.5)^2+2.25)(s^2+2.25)}$$

Using `scipy.signal.impulse` to find the x(t) by the following code, and plotting it (for $0 < t < 50s$), we get:

Python Code:

```
F_num= np.poly1d([1, gamma1])        #defining numerator of function  with decay as 0.5
FX_den= np.polymul([1, 2*gamma1, gamma1**2 + w**2], [1, 0, 2.25])
X= sgnl.lti(F_num,FX_den)
t,x= sgnl.impulse(X, None, np.linspace(0, 50, 1000))     #Getting the solution for x(t)
```

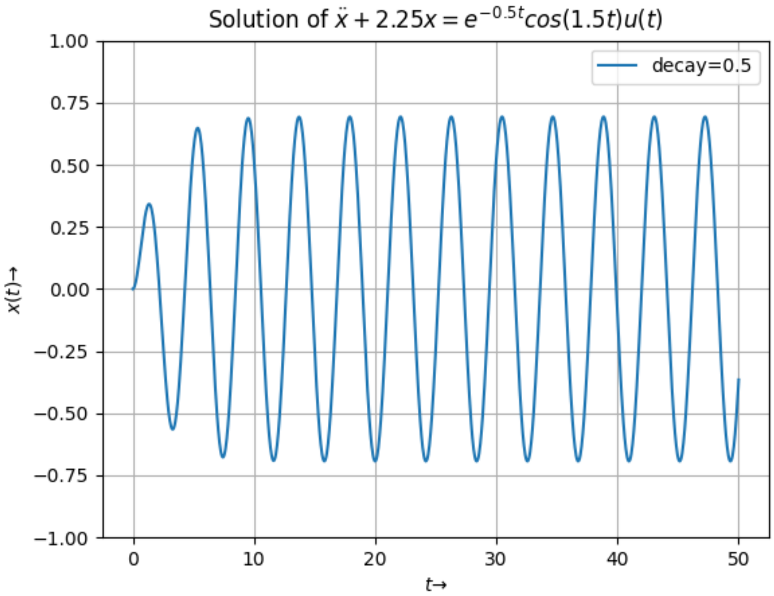

Figure 1: $x(t)$ for $a = 0.5$ and $\omega = 1.5$

Now, we use a smaller decay of $a = 0.05$, we get:

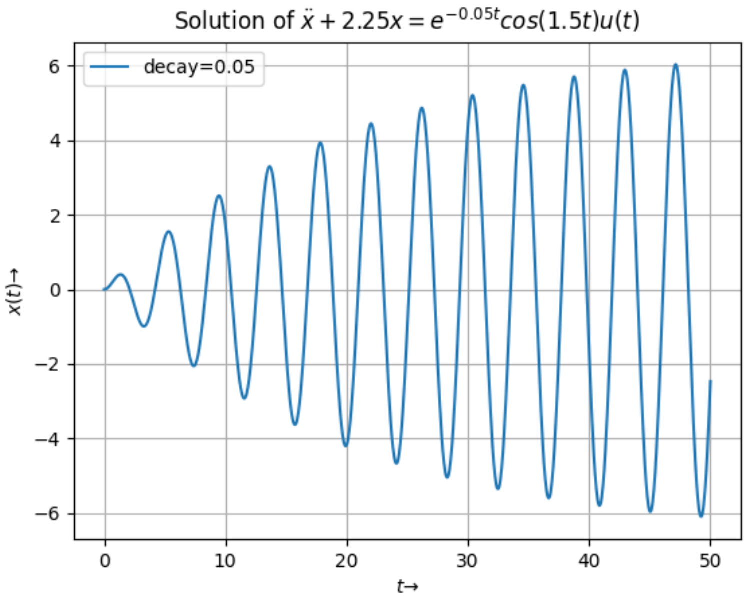

Figure 2: $x(t)$ for $a = 0.05$ and $\omega = 1.5$

Now, we obtain the response for different frequencies, using `signal.lsim`,

Python Code:

```python
H1_s = sgnl.lti([1], [1, 0, 2.25])           # Transfer function H1(s)= X(s)/F(s)
plt.figure(3)
for w in np.arange(1.4, 1.6, 0.05):
    T = np.linspace(0, 50, 1000)
    t, y, rest = sgnl.lsim(H1_s,U= np.exp(-0.05*t)*np.cos(w*t), T=T)
    plt.plot(t,y, label='$w = {} rad/s$'.format(w))
    plt.legend()
plt.grid()
plt.show()
```

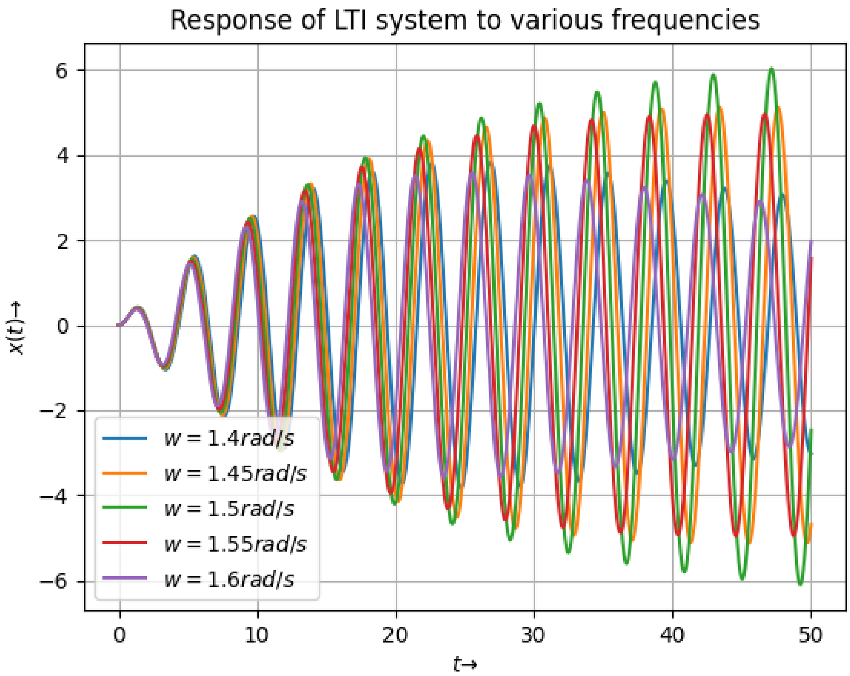

Figure 3: $x(t)$ for $a = 0.05$ and varying ω

We can see the maximum amplitude of oscillations is obtained when the frequency of $f(t)$ is 1.5 rad/s . We can also see that $\omega = 1.5$ rad/s is the natural frequency. Therefore, at $\omega = 1.5$ rad/s, it is in resonance.

4 Coupled Spring Problem:

We have to solve the Coupled equations given by,

$$\ddot{x} + (x - y) = 0$$
$$\ddot{y} + 2(y - x) = 0$$

Given the ICs $x(0) = 1$ and $\dot{x}(0) = y(0) = \dot{y}(0) = 0$. After solving ,we get the X(s) and Y(s) as,

$$X(s) = \frac{s^2 + 3}{s^3 + 3s}$$
$$Y(s) = \frac{2}{s^3 + 3s}$$

Now, we get the graph of x(t) and y(t) by using `signal.impulse` .

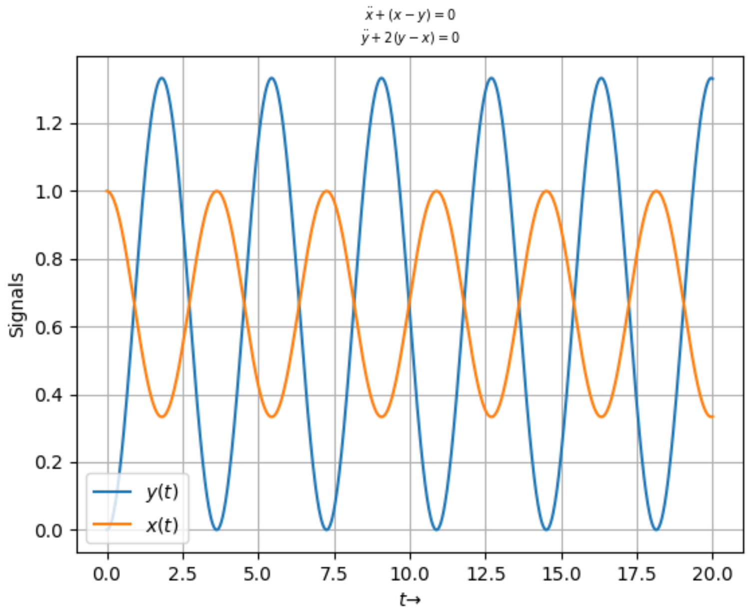

Figure 4: $x(t)$ and $y(t)$ for the coupled spring problem

5 Two-port Network:

The transfer function of the given two-port network can be written as:

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{s^2LC + sRC + 1}$$

The Bode magnitude and phase plots can be found using the method `scipy.signal.bode()`. The plots are shown below:

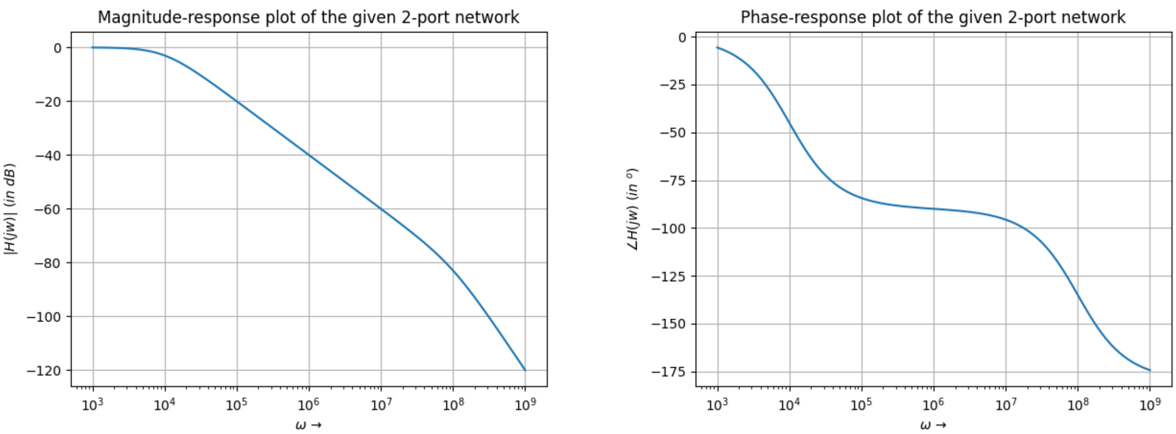

Figure 5: Bode Plots of the RLC Network's Transfer function

Now, when the input to the network, $v_i(t)$ is $cos(10^3t)u(t) - cos(10^6t)u(t)$, the output is given by:

$$V_o(s) = V_i(s).H(s)$$

We will use `scipy.signal.lsim` to find $v_o(t)$ from $V_o(s)$. Plotting the obtained $v_o(t)$, we get:

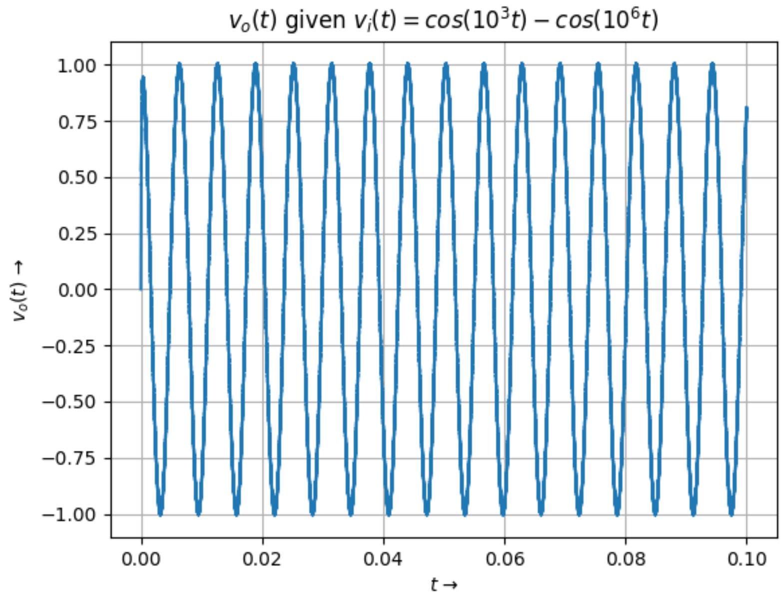

Figure 6: $v_o(t)$ of the RLC network, when $v_i(t) = (cos(10^3t) - cos(10^6t))u(t)$

We can see it to be varying as a sinusoid of frequency approximately 160 Hz, which is expected, as the RLC network acts as a low pass filter - it allows low frequencies to pass through unchanged, while damping high frequencies to huge extent.

We notice that in the initial variation, When zoomed in, for $0 < t < 30\ us$, we get:

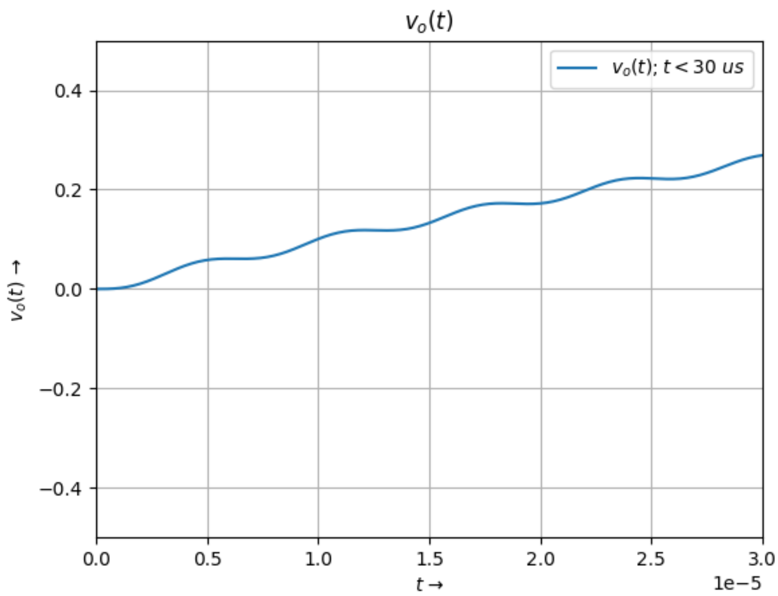

Figure 7: Initial transients

This is due to application of the step input, i.e., the input is suddenly turned on at $t = 0$.

6 Conclusion

In this assignment, we saw the power of Laplace Transform for the analysis of LTI systems. We used the `scipy.signal` python library to obtain various signals from functions. We solved for the time response of a spring, coupled spring equations, and a two-port problem.