EE2703: Assignment 5

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March 23, 2021

Laplace Equation: Resistor problem

1 Abstract

In this report we wish to discuss about,

- the distribution of potential over a resistor(a copper plate)
- solving for the currents in the resistor
- finding the hottest part of the resistor
- analyse the errors using semilog and loglog plots
- using Least Squares to model the errors
- stopping condition after analysing the errors

2 Introduction

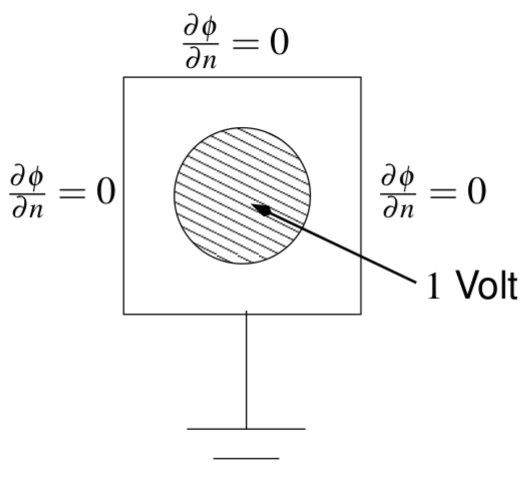


Figure 1: The copper plate

In this assignment, we are given a 1cm x 1cm square copper plate which is the resistor. A wire is soldered to its centre and its voltage is held at 1 volt. The bottom side of the plate is grounded whereas the other sides are floating.
In order to solve the assignment , we use the following list of equations:

- Conductivity Equation:
$$\vec{J} = \sigma \vec{E} \tag{1}$$
- Relation between Electric Field and Potential
$$\vec{E} = -\nabla \phi \tag{2}$$
- Continuity equation of charge
$$\nabla . \vec{J} = -\frac{\partial \rho}{\partial t} \tag{3}$$

- Combining the above equations above, we get
$$\nabla . (-\sigma \nabla \phi) = -\frac{\partial \rho}{\partial t} \tag{4}$$
- Let us assume that the plate has constant conductivity. Then we have,

$$\nabla^2\phi = \frac{1}{\sigma}\frac{\partial\rho}{\partial t} \tag{5}$$

- For DC currents, the right side is zero, and we obtain

$$\nabla^2\phi = 0 \tag{6}$$

- Here we use a 2-D plate so the Numerical solutions in 2D can be easily transformed into a difference equation. The equation can be written out in

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{7}$$

$$\frac{\partial\phi}{\partial x}_{(x_i,y_j)} = \frac{\phi(x_{i+1/2},y_j) - \phi(x_{i-1/2},y_j)}{\Delta x} \tag{8}$$

$$\frac{\partial^2\phi}{\partial x^2}_{(x_i,y_j)} = \frac{\phi(x_{i+1},y_j) - 2\phi(x_i,y_j) + \phi(x_{i-1},y_j)}{(\Delta x)^2} \tag{9}$$

- Using above equations we get

$$\phi_{i,j} = \frac{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1}}{4} \tag{10}$$

→ Therefore, we see that the potential at any point should be the average of its four neighbours(i.e) top,bottom,left and right. So the solution process is to take each point and replace the potential by the average of its neighbours. Keep iterating till the solution converges .
→ At boundaries where the electrode is present, just put the value of potential itself. At boundaries where there is no electrode(top, right and left), the current should be tangential because charge can't leap out of the material into air. Since current is proportional to the Electric Field, what this means is the gradient of ϕ should be tangential. This is implemented by requiring that ϕ should not vary in the normal direction.
→We then solve for the currents in the plate and then find the most heated part of the plate.

3 Visualisation of the problem with python:

We have a few parameters that define the resistor and also define the iteration process of finding the potential ϕ. We take this parameter from the user using the command line (sys.argv()), but also have some default value to them. The default values taken here are $N_x = 25$ and $N_y = 25$ and Number of iterations as 1500, and the radius as 0.35 cm.
We now define the potential array with all the elements initialized to zero with the size of N_x and N_y.
Now we find the indices of the points that lie within the radius of 0.35 cm from the centre of the square (this can be obtained by selecting all the points within the magnitude of 0.35 cm from the centre).
To implement this, we use the `where()` function in python and obtain the coordinates of all such points satisfying the following condition:

$$X^2 + Y^2 \leq 0.35^2 \tag{11}$$

After finding the set of coordinates satisfying the above condition, we apply the potential at those points to be 1V. After doing this we plot the contour plot of the potential before starting to solve the Laplace equation.
We can find that the contour plot becomes smoother (nearly circular) as we increase N_x and N_y, because there are more number of points.

The contour plot is as shown below:

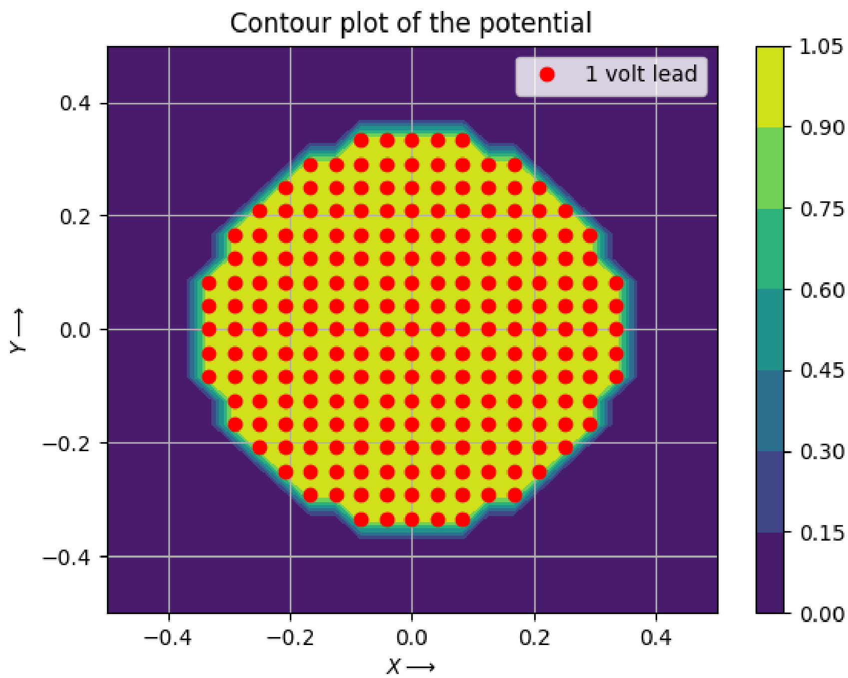


Figure 2: Contour plot of initial potential

4 Updating the plate potential:

We will use the below equation as reference to update the potential at each point on the plate. Each point's potential is the average of its 4 neighbours.

$$\phi_{i,j} = \frac{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1}}{4} \tag{12}$$

We have already seen how to assign the boundary condition for each side of the plate. We will also calculate the error for each iteration as,

$$error_k = \phi - \phi_{old} \tag{13}$$

where $error_k$ is the error for the k^{th} iteration. We will use Vectorised code to do the iterations.
Python Code:

```
errors=np.zeros(Niter)      #This block updates the value of potential at each point in the plate and
                            #keeps track of the error in each iteration
for i in range(Niter):
    oldphi= phi.copy()
    phi[1:-1,1:-1]= 0.25*(phi[1:-1,0:-2]+phi[1:-1,2:]+phi[0:-2,1:-1]+phi[2:,1:-1])
    phi[1:-1,0]= phi[1:-1,1]
    phi[1:-1,-1]= phi[1:-1,-2]
    phi[0,:]= phi[1,:]
    phi[ii]= 1.0
    errors[i]= (abs(oldphi-phi)).max()
```

5 Analysing the errors:

Using the error obtained by running the above code, we plot semilog and log-log plots for analysis. For better visualization, we plot every 50^{th} point.
The plots are given below.

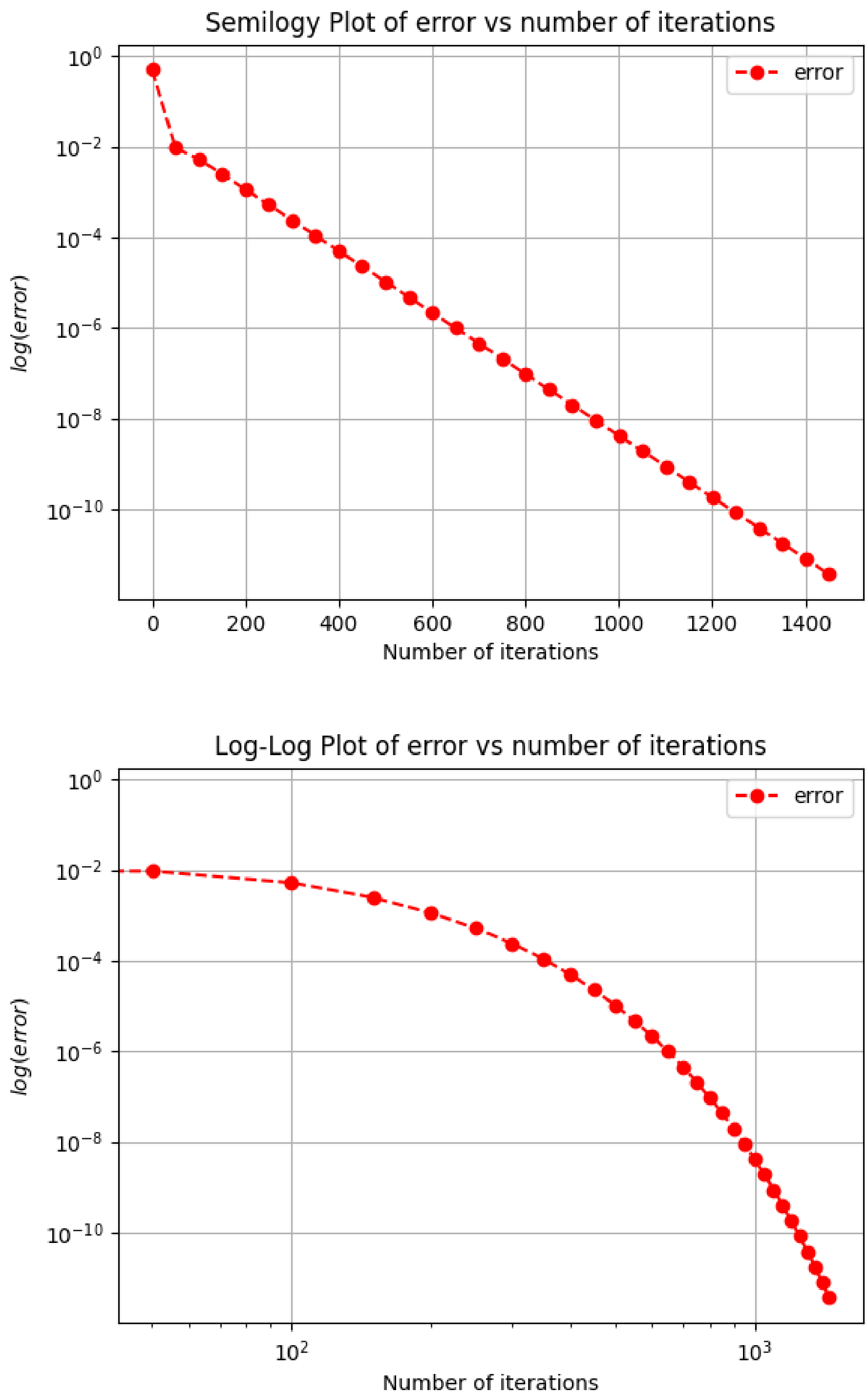


Figure 3: Semilog and Log-Log plots of Error vs No.of Iterations

- As we observe the semilog plot that error decreases linearly for higher no of iterations,so from this we conclude that for large iterations error decreases exponentially with No of iterations i.e it follows Ae^{Bx} as it is a semilog plot
- And if we observe loglog plot the error is almost linearly decreasing for smaller no of iterations so it follows a^x form since it is loglog plot and follows some other pattern at larger iterations.
- So to conclude the error follows Ae^{Bx} for higher no of iterations($\approx$ 500) and it follows a^x form for smaller

iterations which can be seen from the plots given above.

6 Least Squares Fit:

→To find the fit using Least squares for all iterations named as **fit1** and for iterations greater than 500 named as **fit2** separately and compare them.

→ As we know that error follows Ae^{Bx} at large iterations, we use equation given below to fit the errors using least squares

$$logy = logA + Bx \tag{14}$$

where A and B are constants.
We can arrive at the fit for the default values as,

- Fit1 : A = 0.026628934257051245 , B = -0.01564482721966434
- Fit2 : A = 0.026454455729215034 , B = -0.01563763735036898

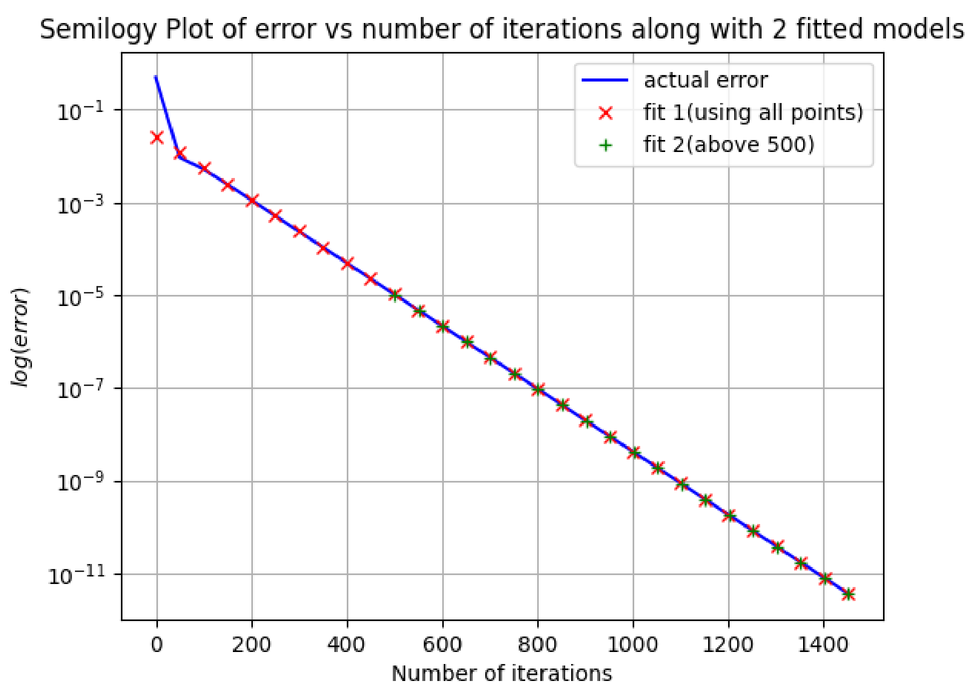
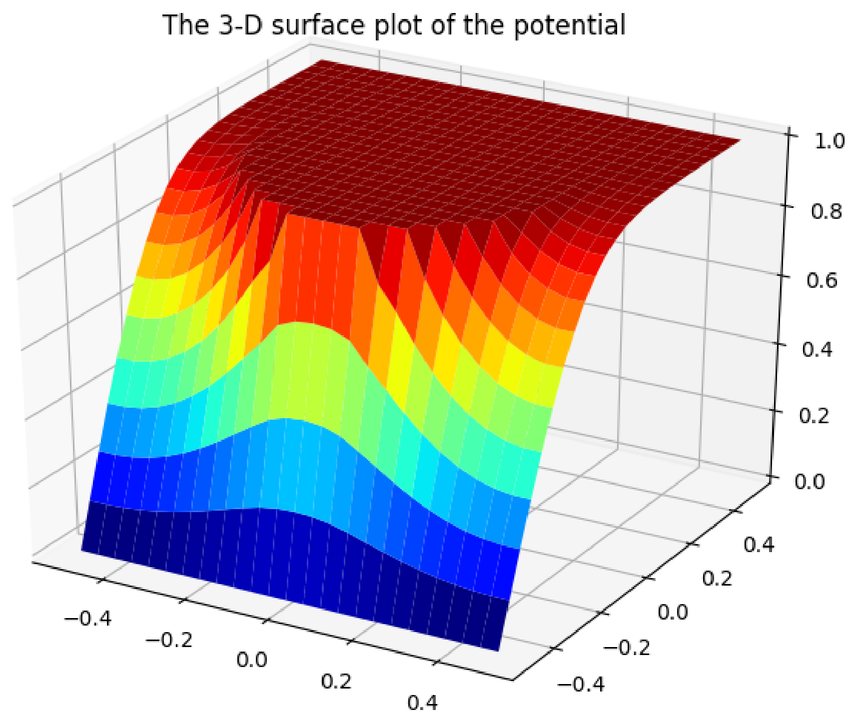


Figure 4: Semilog plot of Error vs No.of Iterations

7 Surface and Contour Plot of the Potential

Let us plot the 3-D surface plot and the contour plot of the potential after updating its values all over the plate.

→ As we observe that the surface plot we conclude that after updating the potential,the potential gradient is higher in down part of the plate since, the down side is grounded and the electrode is at 1 V,so there is high potential gradient from electrode to grounded plate.
→ And the upper part of the plate is almost 1 V since they didnt have forced Voltage and their's were floating,so while applying updating we replaced all points by average of surrounding points so the potential is almost 1 V in the upper region of the plate.
→ Same observation we see using contour plot in 2 dimensions, we note that there are gradients in down part of the plate and almost negligible gradient in upper part of the plate.
The plots are given below.



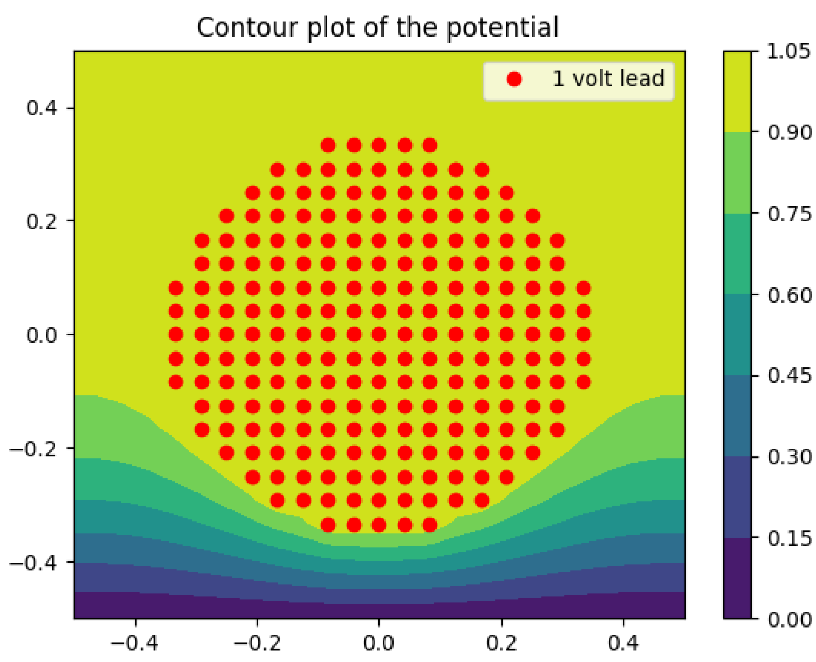


Figure 5: 3-D Surface potential plot and Contour plot of potential

8 Vector Plot of Currents:

We now further plot the current, which can be obtained simply by taking the gradient of the potential in both the directions as per the following equation:

$$J_x = -\frac{\partial \phi}{\partial x} \tag{15}$$

$$J_y = -\frac{\partial \phi}{\partial y} \tag{16}$$

We calculate this in our program as per the following equations:

$$J_{x,ij} = \frac{1}{2}(\phi_{i,j-1} - \phi_{i,j+1}) \tag{17}$$

$$J_{y,ij} = \frac{1}{2}(\phi_{i-1,j} - \phi_{i+1,j}) \tag{18}$$

We plot the vector currents of the potential using the `quiver()` plot. The plot is given below.

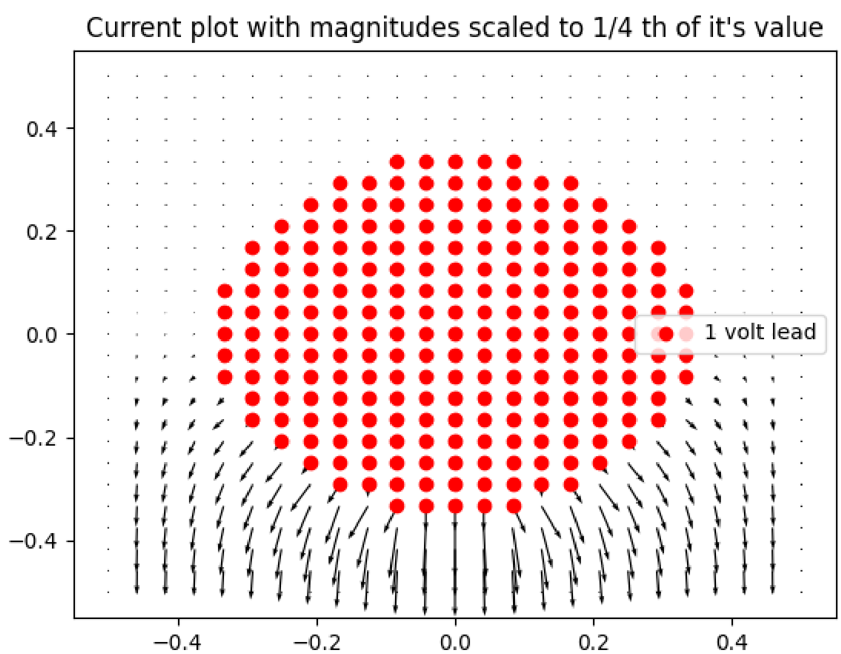


Figure 6: Vector plot of current flow

We can very well see that the current is high in the bottom region, close to the grounded side. This occurs because the current is directly proportional to the gradient of the potential, which is high in the bottom part.

$$\vec{E} = -\nabla \phi \tag{19}$$

$$\vec{J} = \sigma \vec{E} \tag{20}$$

- So $\vec{J}$ is higher and perpendicular to equipotential electrode region i.e "Red dotted region" so the current is larger in down part of the plate and perpendicular to the red dotted electrode region since $I = \vec{J}.\vec{A}$
- So because of this most of the current flows from electrode to the bottom plate which is grounded because of higher potential gradient.
- And there is almost zero current in upper part of the plate since there is not much potential gradient as we observed from the surface and contour plot of the potential ϕ

9 Visualizing the Temperature with plots:

We know that heat generated is from $\vec{J}.\vec{E}$ (ohmic loss) so since $\vec{J}$ and $\vec{E}$ are higher in the bottom region of the plate, there will more heat generation and temperature rise will be present.
The variation of temperature in the plate can be seen in the below 3-D surface plot.

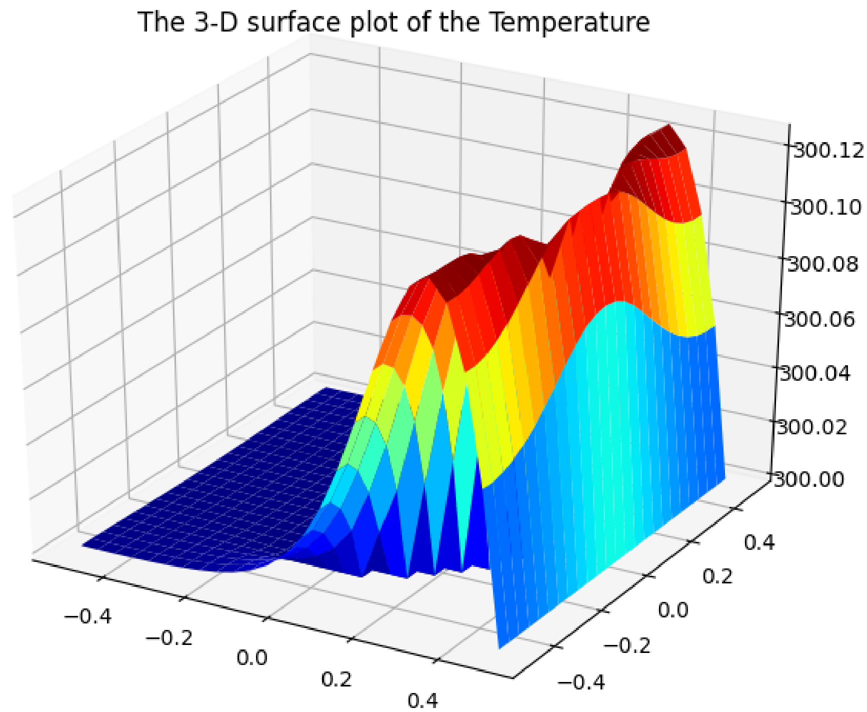


Figure 7: Variation of temperature in the plate

10 Conclusion :

1. We have seen that most of the current is in the narrow region at the bottom. So that portion of the plate is what will get strongly heated.

2. Since there is almost no current in the upper region of plate,the bottom part of the plate gets hotter and temperature increases in down region of the plate.

3. And we know that heat generated is from $\vec{J}.\vec{E}$ (ohmic loss) so since $\vec{J}$ and $\vec{E}$ are higher in the bottom region of the plate, there will more heat generation and temperature rise will be present.

4. We saw how currents can be modelled using python in this assignment.

5. With the increase in N_x and N_y, the contour plots become smoother as there are more number of points, thereby, getting a better circular picture for the wire.