EE2703 Assignment 4

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10^{th} March,2021

Fourier Approximations

1 Abstract

In this assignment, we will try to analyse 2 functions, e^x and $cos(cos(x))$ over the interval -2π to 4π using two methods, (i) By direct Integration , (ii) By Least Squares method.

2 Introduction

The Fourier Series of a function $f(x)$ with period 2π is computed as follows:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\{a_n cos(nx) + b_n sin(nx)\} \tag{1}$$

where ,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$
$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) * \cos(nx)dx$$
$$b_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) * \sin(nx)dx$$

Note that we treat exp(x) as a 2π-periodic function while calculating the coefficients

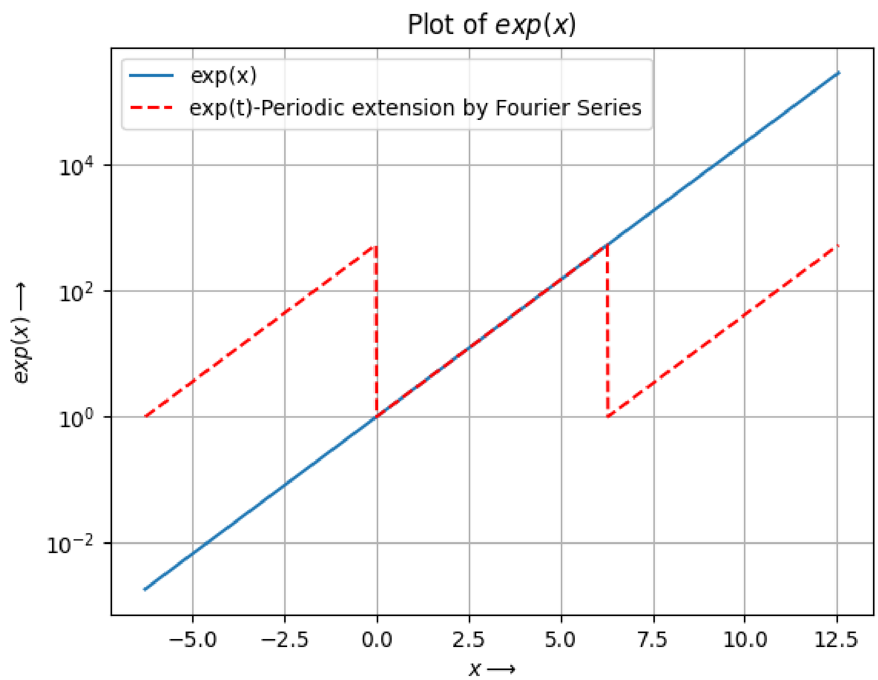
3 Answers for questions in assignment

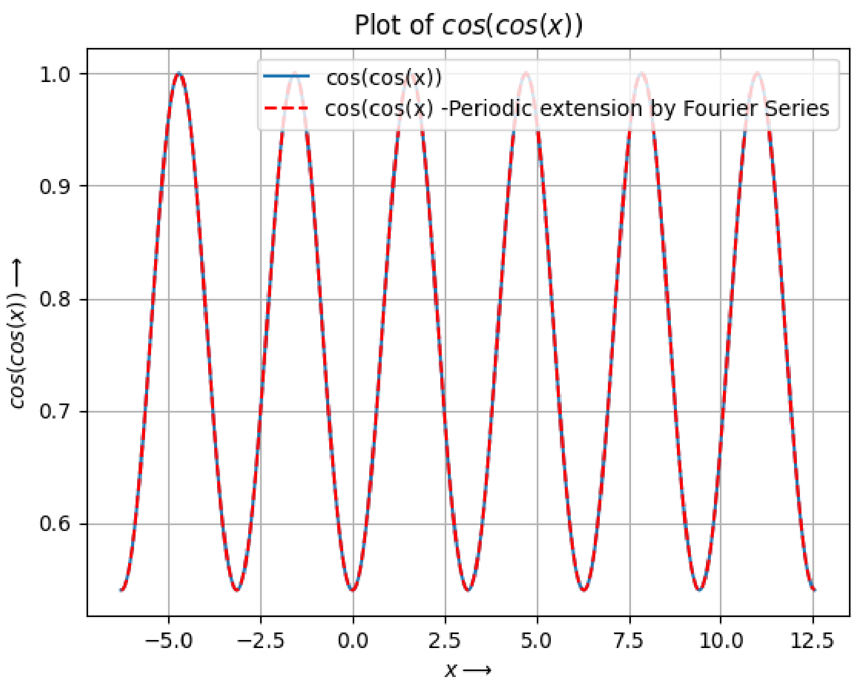
3.1 Creating the Python functions

We know that $cos(cos(x))$ is a periodic function with period 2π whereas e^x is non-periodic. Therefore, the functions that will be generated from the Fourier series are $cos(cos(x))$ and $e^{x\%(2\pi)}$[Periodic extension of exp(x)] respectively.

```
#function that returns exp(x)
def exp_func(x):
    return(np.exp(x))
#function that returns cos(cos(x))
def cos_cos_func(x):
    return np.cos(np.cos(x))
```

Below are the plots of actual function along with the periodic extension of the same. Note that the cos(cos(x)) graphs coincides while the exp(x) does not.





3.2 Generating Fourier Coefficients

We generate the first 51 coefficients using the **scipy.integrate.quad** and the equations mentioned in the introduction section.They are saved in the following form as required by part 3:

$$\begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ \dots \\ a_{25} \\ b_{25} \end{bmatrix}$$

3.3 Analysing the Fourier Coefficients with plots

The semilogy and log-log plots of the Coefficients obtained are plotted below.
(a) We can see that $\|b_n\|$ for cos(cos(x)) is almost 0. This is because cos(cos(x)) is an even function, whereas the integral involved in finding the b_n coefficient is odd in nature. b_n is not exactly 0 because of error in computational accuracy .
(b) The coefficients for e^x decay slower than that of $cos(cos(x))$ because rate of decay is directly proportional to the smoothness of the function.

(c) The Log-log plot for Fourier coefficients of e^x is nearly linear because the integrals,

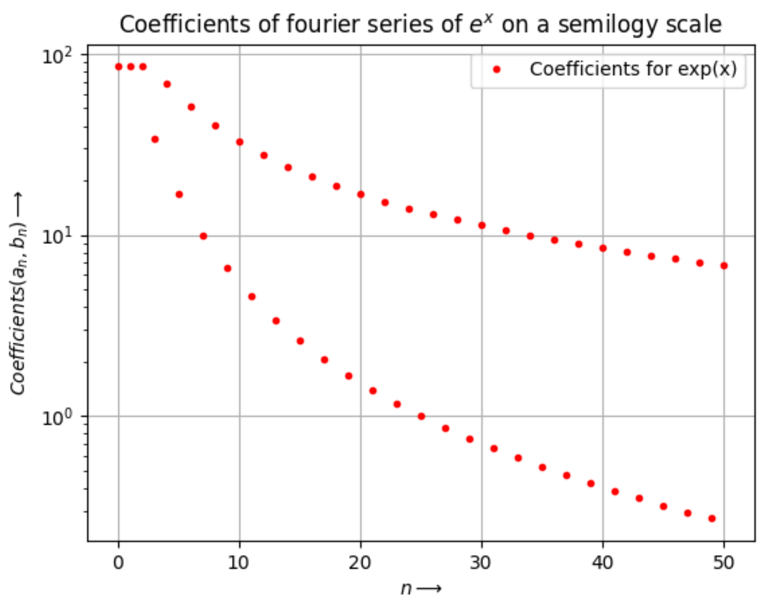
$$\int_0^{2\pi} e^x cos(kx)dx = \frac{(e^{2\pi}-1)}{(k^2+1)} \tag{2}$$

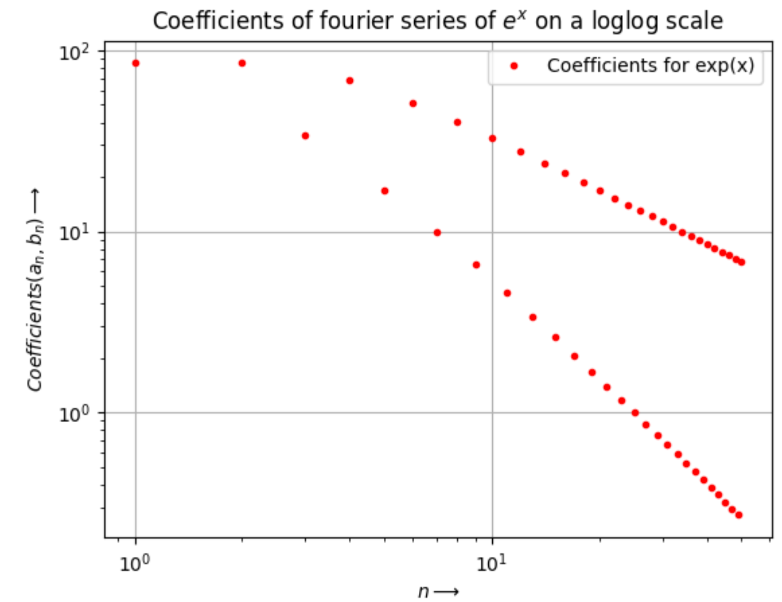
and

$$\int_0^{2\pi} e^x sin(kx)dx = \frac{(-ke^{2\pi}+k)}{(k^2+1)} \tag{3}$$

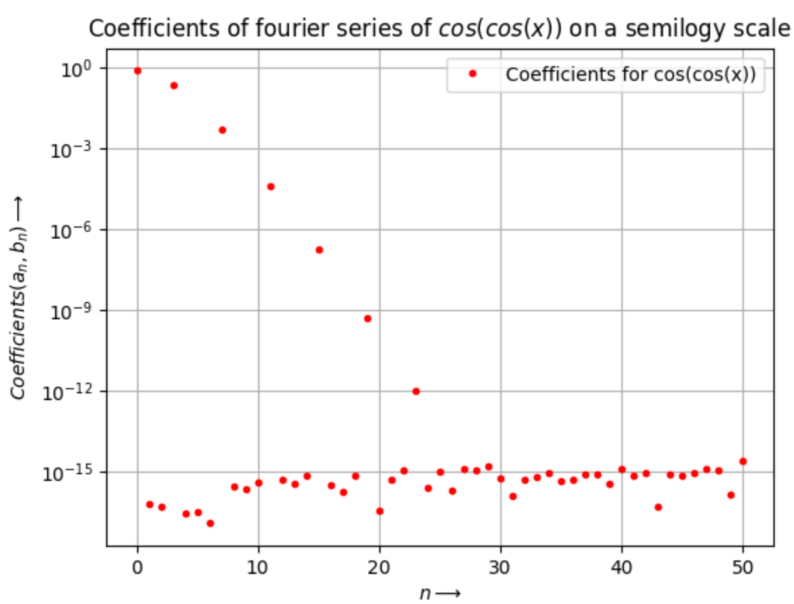
have $1/n^2$ relation . The log-log plots of these functions are linear.

The semi-logy plot for Fourier Coefficients of $cos(cos(x))$ is linear as the integral converges to a Linear Combination of Bessel functions which are proportional to e^x.

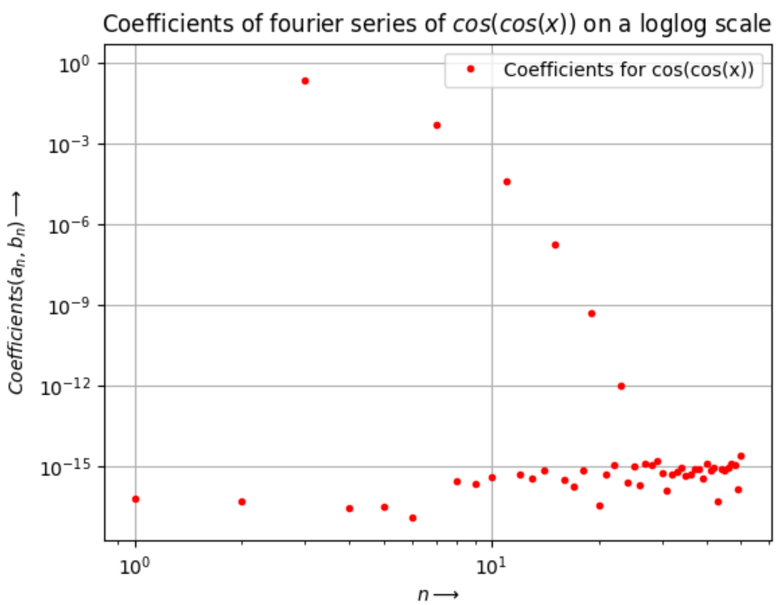




Coefficients of fourier series of e^x on a loglog scale
Coefficients for exp(x)
$Coefficients(a_n, b_n) \longrightarrow$
$n \longrightarrow$
10^2
10^1
10^0
10^0
10^1



Coefficients of fourier series of $cos(cos(x))$ on a semilogy scale
Coefficients for cos(cos(x))
$Coefficients(a_n, b_n) \longrightarrow$
$n \longrightarrow$
10^0
10^{-3}
10^{-6}
10^{-9}
10^{-12}
10^{-15}
0
10
20
30
40
50


Coefficients of fourier series of $cos(cos(x))$ on a loglog scale
Coefficients for cos(cos(x))
10^{0}
10^{-3}
10^{-6}
10^{-9}
10^{-12}
10^{-15}
$Coefficients(a_n, b_n) \longrightarrow$
10^{0}
10^{1}
$n \longrightarrow$

3.4 Least Squares Model

Based on our previous week's work we now try the Least Squares Approach to this problem. We linearly choose 400 values of x in the range [0,2π).It should be noted that far better approximations were achieved when a larger value(10000) was used instead of 400. We try to solve Equation (1) By using regression on these 400 values

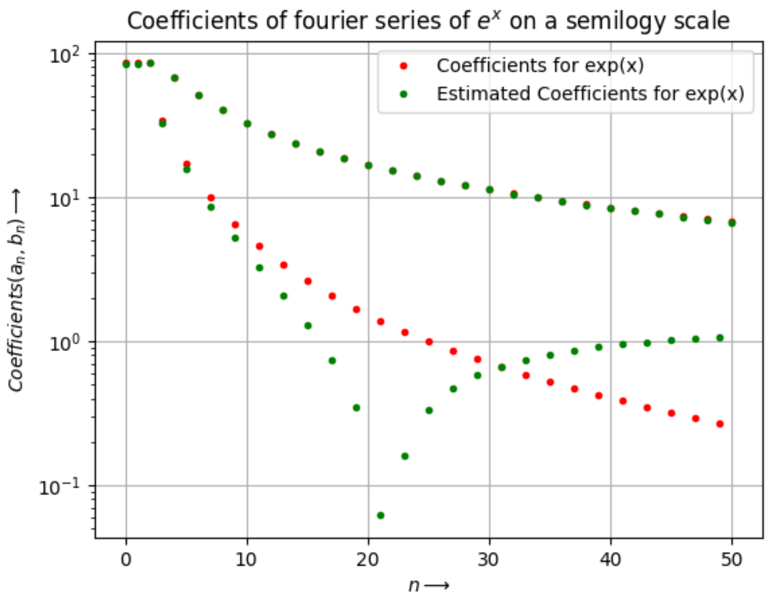
$$\begin{pmatrix} 1 & \cos(x_1) & \sin(x_1) & .... & \cos(25x_1) & \sin(25x_1) \\ 1 & \cos(x_2) & \sin(x_2) & .... & \cos(25x_2) & \sin(25x_2) \\ ... & ... & ... & .... & ... & ... \\ 1 & \cos(x_{400}) & \sin(x_{400}) & .... & \cos(25x_{400}) & \sin(25x_{400}) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ b_1 \\ ... \\ a_{25} \\ b_{25} \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ ... \\ f(x_{400}) \end{pmatrix}$$

We call the matrix on the left side as A . We want to solve $Ac = b$ where c are the fourier coefficients.

```
#This block estimates the Fourier series coefficients using the Least Square method

x=np.linspace(0,2*math.pi,401)
x=x[:-1]
b1= exp_func(x)
b2= cos_cos_func(x)

#This block creates the matrix A
A=np.zeros((400,51))
A[:,0]=1
for k in range(1,26):
    A[:,2*k-1]=np.cos(k*x)
    A[:,2*k]=np.sin(k*x)

c1= lstsq(A,b1)[0]
c2= lstsq(A,b2)[0]
```

3.5 Analysing the resulting plots of the Least Squares model

Below are the plots of the estimated coefficients using the Least squares model along with the actual coefficients for comparison.


Coefficients of fourier series of e^x on a semilogy scale
Coefficients for exp(x)
Estimated Coefficients for exp(x)
$Coefficients(a_n, b_n) \longrightarrow$
$n \longrightarrow$
10^2
10^1
10^0
10^{-1}
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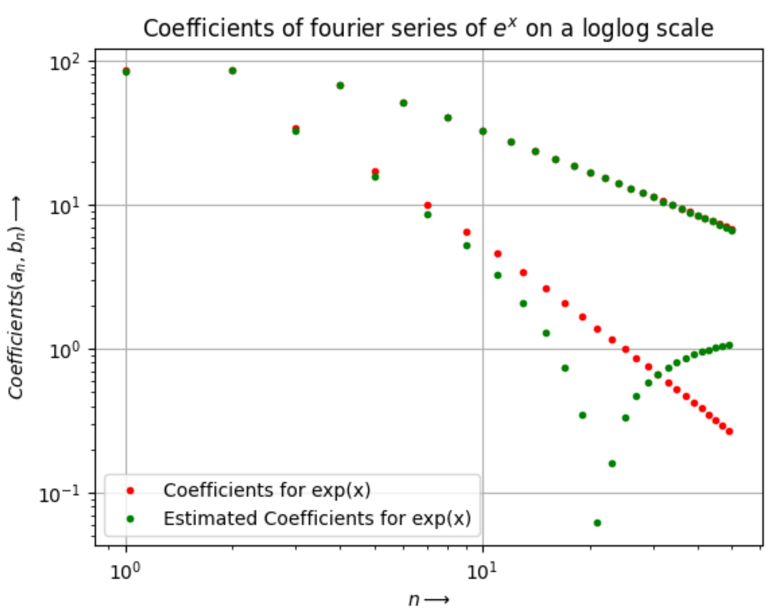
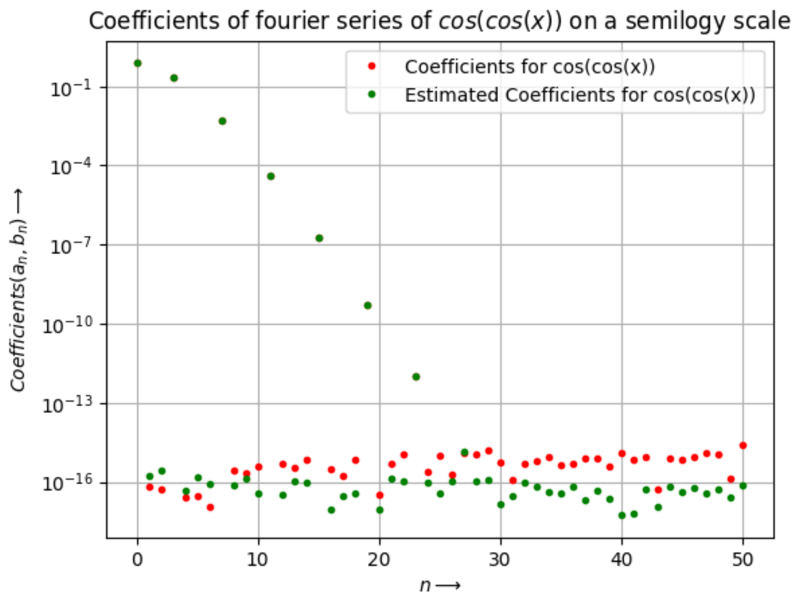
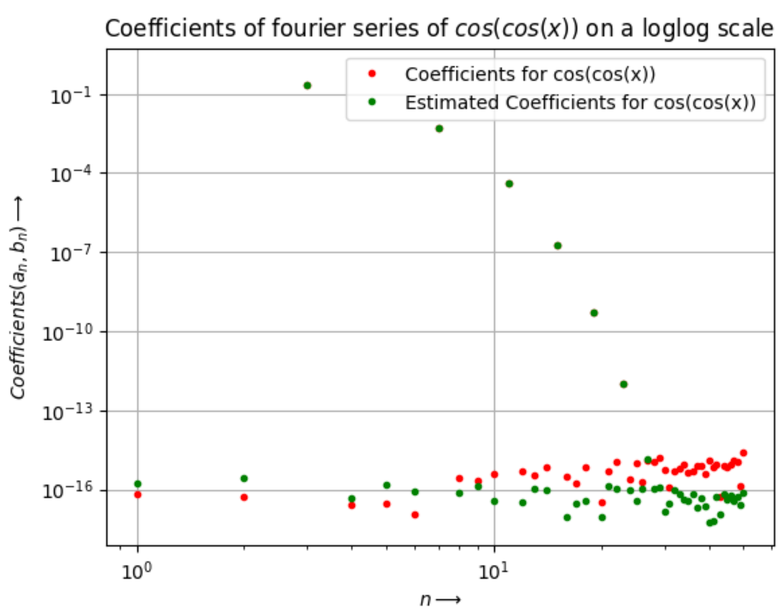
20
30
40
50


Coefficients of fourier series of e^x on a loglog scale
Coefficients for exp(x)
Estimated Coefficients for exp(x)
$Coefficients(a_n, b_n) \longrightarrow$
$n \longrightarrow$
10^2
10^1
10^0
10^{-1}
10^0
10^1



Coefficients of fourier series of $cos(cos(x))$ on a semilogy scale
Coefficients for cos(cos(x))
Estimated Coefficients for cos(cos(x))
$Coefficients(a_n, b_n) \longrightarrow$
$n \longrightarrow$



Coefficients of fourier series of $cos(cos(x))$ on a loglog scale
Coefficients for cos(cos(x))
Estimated Coefficients for cos(cos(x))
$Coefficients(a_n, b_n) \longrightarrow$
$n \longrightarrow$

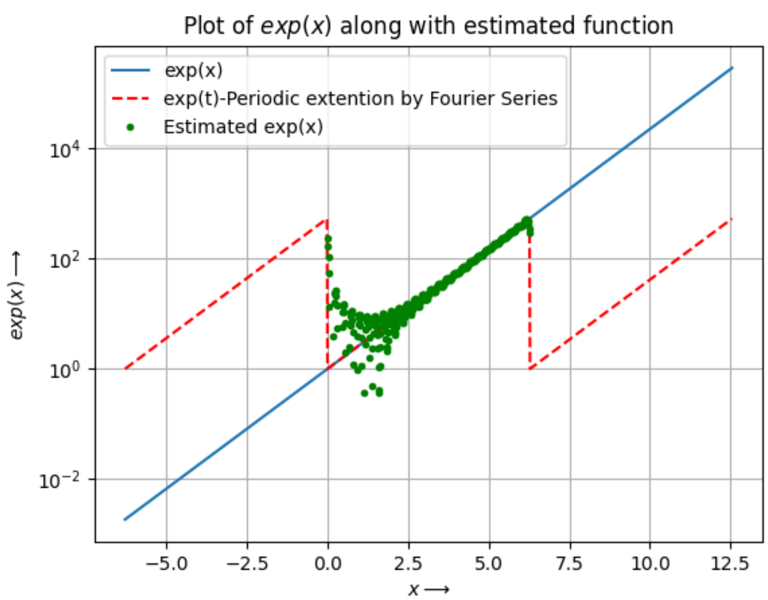
3.6 Comparing Least squares with direct integration

* The maximum absolute deviation for e^x =1.3327308703353538
* The maximum absolute deviation for cos(cos(x)) = 2.519109152022848e-15

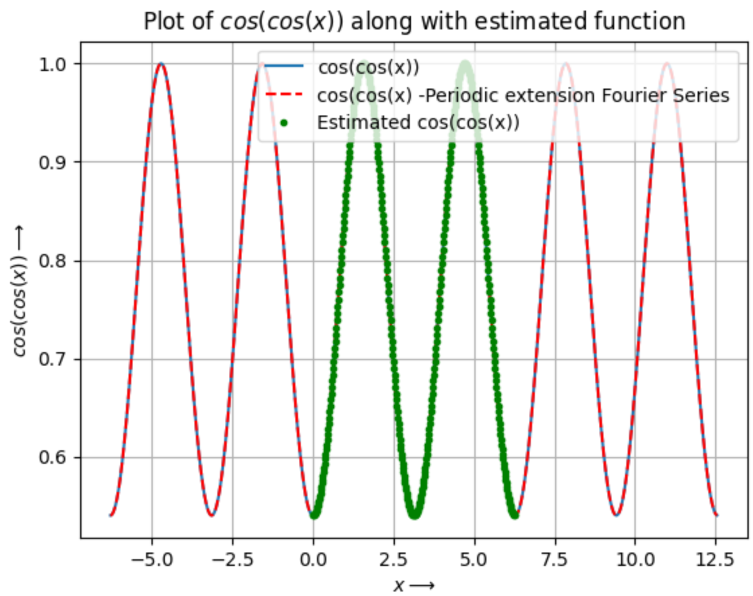
We can note that our Predictions for e^x are very poor compared to that of cos(cos(x)). This can be fixed by sampling at a larger number of points.(say 10^5 points)
But this would cost more computational load on the system.

3.7 Regenerating the functions using the estimated coefficients

Below are the plots for the functions generated using the estimated coefficients along with the actual functions for comparison.



It should be noted that e^x is a non periodic function and Fourier series' exists only for periodic functions. Hence we have considered a variation of e^x with period 2pi that has the actual value of e^x only in the range $[0,2\pi)$. Hence it is acceptable that there is a large discrepancy in the predicted value of e^x at these boundaries



4 Conclusion

We have examined the case of approximating functions using their Fourier coefficients upto a threshold. Whilst doing so, we perform the same for two cases, one a continuous function, and the other a function with finite discontinuities.

We adopted 2 methods in finding the respective Fourier coefficients,
(i) direct evaluation of the Fourier series formula,
(ii) Least Square model .

We notice close matching of the two methods in case of $\cos(\cos(x))$ while, there is a larger discrepancy in $\exp(x)$.