Assignment - 1

Problem 1 :

→ A and B are two sets and $pow(S)$ is the power set of set S.

a) **To prove:** $pow(A \cap B) = pow(A) \cap pow(B)$

Proof : Let $x \in pow(A \cap B)$

$\Rightarrow x \subseteq A \cap B$

$\Rightarrow x \subseteq A$ and $x \subseteq B$

$\Rightarrow x \in pow(A)$ and $x \in pow(B)$

$\Rightarrow x \in pow(A) \cap pow(B)$

As this is true for all $x \in pow(A \cap B)$,

~~$pow(A) \cap pow(B) \subseteq pow(A \cap B)$~~

$pow(A \cap B) \subseteq pow(A) \cap pow(B)$ — (1)

Let $x \in pow(A) \cap pow(B)$

$\Rightarrow x \in pow(A)$ and $x \in pow(B)$

$\Rightarrow x \subseteq A$ and $x \subseteq B$

$\Rightarrow x \subseteq A \cap B \Rightarrow x \in pow(A \cap B)$

As this is true for all $x \in pow(A) \cap pow(B)$,

$pow(A) \cap pow(B) \subseteq pow(A \cap B)$. — (2)

∴ From (1) and (2), we get that

$pow(A) \cap pow(B) = pow(A \cap B)$.

b) <u>To prove</u>: $pow(A) \cup pow(B) \subseteq pow(A \cup B)$

<u>Proof</u>: Let $x \in pow(A) \cup pow(B)$

$\Rightarrow x \in pow(A)$ or $x \in pow(B)$

$\Rightarrow x \subseteq A$ or $x \subseteq B$

$\Rightarrow x \subseteq A \cup B \Rightarrow x \in pow(A \cup B)$

$\therefore$ ~~pow~~ As this is true for all x in $pow(A) \cup pow(B)$

$pow(A) \cup pow(B) \subseteq pow(A \cup B)$

If $A \subseteq B$, then $A \cup B = B \Rightarrow pow(A \cup B) = pow(B)$

$\Rightarrow pow(A) \subseteq pow(B) \Rightarrow pow(A) \cup pow(B)$

$= pow(B)$

$\therefore$ LHS = RHS

Now, if $a \in A$ and $a \notin B$ and
$b \in B$ and $b \notin A$,

$\Rightarrow a \in pow(A)$ and $a \notin pow(B)$

$b \notin pow(A)$ and $b \in pow(B)$

<u>RHS</u>: $pow(A \cup B)$. $\Rightarrow \{a, b\} \in pow(A \cup B)$

<u>LHS</u>: $pow(A) \cup pow(B)$

$\Rightarrow \{a\} \in pow(A)$ and $b \in pow(B)$ but

$pow(A) \cup pow(B) = \{\{a\}, \{b\}\} \neq \{a, b\}$

since $\{a, b\} \notin pow(A)$ and $\{a, b\} \notin pow(B)$,

$\therefore pow(A) \cup pow(B) \neq pow(A \cup B)$ if $A \nsubseteq B$ or $B \nsubseteq A$

Equality only holds when $A \subseteq B$ or $B \subseteq A$

Problem 2:

A and B are finite sets such that $A \subseteq B$.

To calculate: $\sum_{\forall C: A \subseteq C \subseteq B} (-1)^{|C \setminus A|}$

where $C \setminus A = \{x \in C : x \notin A\}$, $|C \setminus A| \Rightarrow$ cardinality

As $A \subseteq C$, set C can be written as $A + D$, where $D \subseteq B \setminus A$

$\therefore C \setminus A = D$

If $|B| = b$ and $|A| = a$,

$|B \setminus A| = b - a \Rightarrow 0 \leq |D| \leq b - a$

~~All~~ possible no. of Cs essentially means finding all possibilities of Ds.

possibilities of Ds are $b-a+1$ where no. of Ds per k is ${}^{b-a}C_k$ where k is no. of elements in D.

$\therefore |C \setminus A| = k \Rightarrow 0 \leq k \leq b-a$

$$\therefore \sum_{\forall C: A \subseteq C \subseteq B} (-1)^{|C \setminus A|} = \sum_{k=0}^{b-a} {}^{b-a}C_k (-1)^k = S$$

$$\left[(1-x)^n = \sum_{k=0}^{n} {}^{n}C_k (-x)^k \right]$$

$$\therefore S = (1-1)^{b-a} = \underline{\underline{0}}$$

Problem 3 :

A and B are independent events (given)

$\Rightarrow P(A \cap B) = P(A) . P(B)$

$\Rightarrow P(A^c \cap B^c) = [1-P(A)][1-P(B)]$

$= 1 - (P(A)+P(B)) + P(A) . P(B)$

$= 1 - [P(A)+P(B) - P(A \cap B)]$

$= 1 - P(A \cup B) = P[(A \cup B)^c] = P(A^c \cap B^c)$

$\therefore A^c$ and B^c are independent.

$P(A^c \cap B) = (1-P(A)) P(B) = P(B) - P(A). P(B)$

$= P(B) - P(A \cap B) = P(A^c). P(B)$

$\therefore A^c$ and B are also independent

Problem 4 :

N turtles in total

m are initially picked and tagged.

Later, n are picked again

No. of ways of picking m out of N turtles $= {}^N C_m$

Picking k out of m tagged turtles $= {}^m C_k$

Picking remaining n-k turtles out of N-m turtles $= {}^{N-m} C_{n-k}$

$\therefore$ Probability $= \dfrac{\cancel{{}^N C_m} \times {}^m C_k \times {}^{N-m} C_{n-k}}{\cancel{{}^N C_m} \times {}^N C_n} = \dfrac{{}^m C_k \cdot {}^{N-m} C_{n-k}}{{}^N C_n}$

Problem 5 :

(a) K people are present in a room.

$\Rightarrow P\left[\begin{array}{l}\text{two or more with} \\ \text{same birthday}\end{array}\right] = 1 - P\left[\begin{array}{l}\text{no common} \\ \text{birthday}\end{array}\right]$

No. of possibilities of K people ~~with~~ without same birthday $= {}^{365}P_K$

$\therefore P\left(\begin{array}{l}\text{no common} \\ \text{birthday}\end{array}\right) = \dfrac{{}^{365}P_K}{365^K}$

$\therefore P\left[\begin{array}{l}\text{two or more with} \\ \text{same birthday}\end{array}\right] = 1 - \left(\dfrac{{}^{365}P_K}{365^K}\right)$

$= 1 - \dfrac{\lfloor 365}{365^K \lfloor 365-K}$

For this value to be $\frac{1}{2}$,

$\dfrac{\lfloor 365}{\lfloor 365-K} \times \dfrac{1}{365^K} = \dfrac{1}{2} \Rightarrow K = 23$

$\therefore$ Minimum of 23 people must be present for the probability of two or more people ~~to~~ having the same birthday to be ≥ 0.5.

(b) i) P(atleast one birthday match)

$= 1 - P($ no birthday match$)$

$P(E) = 1 - P(E^c)$

$P(E^c) = \sum_{p_{g_1}+p_{g_2}+\dots+p_{g_K}} \prod_{j=1}^{K} p_{ij}$

Here, P_{ij} denotes birthday on ith day for jth person.

$$\therefore P(E) = 1 - \sum_{p_{i1} + p_{i2} + \dots + p_{ik}} \left(\prod_{j=1}^{k} P_{ij} \right)$$

ii) $P(E)$ is minimized ~~maximized~~ when ~~$P(E) = 0$~~ $P(E^c)$ = is maximized

This happens when all p_i are equal and $p_i = \frac{1}{365}$

Problem 6:

50 seats are uniquely assigned to 50 people.

1st person randomly chooses a seat.

Further, people sit in the assigned seat if empty, else, ~~or~~ sit randomly

Here, no. of people $N = 50$

Let's take the case where $N = 2$, $\{P_1, P_2\}$

$P(P_2$ sitting in their seat$) = P(P_1$ picking their own seat$)$

$= \frac{1}{2}$

Now, let $N = 3$, $\{P_1, P_2, P_3\}$

$P(P_3$ sitting in their seat$) = P(P_1$ sitting in their place$)$

$+ P\left[\begin{matrix} P_1 \text{ sitting in} \\ P_2\text{'s place} \end{matrix}\right] \times P\left[\begin{matrix} P_2 \text{ sitting in} \\ P_1\text{'s place} \end{matrix}\right]$

$= \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

By induction, we can assume this to be true for any ~~all~~ N.

$\Rightarrow P(P_N$ sitting in their place$) = \frac{1}{2}$