6.3 c) **Defn:-** **Totally Bounded Sets:-**

let $\langle M, \rho \rangle$ be a Metric space. we say that the subset A of M is totally bounded if given $\epsilon > 0$, there exists a finite number of subsets $A_1, A_2, A_3, \ldots A_n$ of M such that diam $A_K < \epsilon$ $(K = 1, 2, \ldots n)$ and such that $A \subset \bigcup_{K=1}^{n} A_K$

(1)

6.3 D **Thm:-**

If the subset A of the metric space $\langle M, \rho \rangle$ is totally bounded then it is bounded.

Pf:-

let A be a totally bounded subset of M.

By defn.

there exists a finite number of non-empty subsets $A_1, A_2, A_3 \ldots A_n$ of M such that diam $A_K < 1$ $(K = 1, 2, \ldots n)$ and $A \subset \bigcup_{K=1}^{n} A_K$

$x, y \in A \subset A_1 \cup A_2 \cup \cdots \cup A_n$

$x \in A_i$ $y \in A_j$

for each $K = 1, 2, \ldots n$.

let a_K be a point in A_K

(i) $a_1 \in A_1$, $a_2 \in A_2$, ... $a_n \in A_n$

let x & y be any two points in A.

(ii) $x, y \in A$

$\therefore A \subset \bigcup_{K=1}^{n} A_K \Rightarrow x \in A_i$ & $y \in A_j$, for some i & j

We assume that $i \leq j$

let $D = \rho(a_1, a_2) + \rho(a_2, a_3) + \cdots + \rho(a_{n-1}, a_n)$

since diam $A_i < 1$.

(iii) diam $A_i = \underset{u, v \in A_i}{\text{l.u.b}}\, \rho(u, v)$

$\therefore \rho(u, v) \leq$ diam $A_i < 1$.

$\therefore \rho(u, v) < 1$.

$\rho(x, a_i) < 1$

Similarly, diam $A_j < 1$

$\therefore \rho(a_j, y) < 1$

By Δ^{le} inequality

$\rho(x, y) \leq \rho(x, a_i) + \rho(a_i, a_{i+1}) + \rho(a_{i+1}, a_{i+2}) + \cdots$

$+ \rho(a_{j-1}, a_j) + \rho(a_j, y)$

$< 1 + D + 1 = D + 2$

$\boxed{\rho(x, y) < L}$, $\forall x, y \in A$

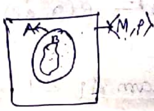
A is bdd

6.3.E Corollary

6.3 F) **Defn:**

Let A be a subset of the metric space M. The subset B of A is said to be ϵ-dense in A (where $\epsilon > 0$) if for every $x \in A$ there exists $y \in B$ such that $\rho(x, y) < \epsilon$

(ie) B is ϵ-dense in A if each point of A is within distance ϵ from some point of B



6.3.G) **Theorem:** (6)

The subset A of the metric space $\langle M, \rho \rangle$ is totally bounded if and only if for every $\epsilon > 0$, "A" contains a finite subset $\{a_1, a_2, \dots a_n\}$ which is ϵ-dense in A.

Pf:-

Fix $\epsilon > 0$

If A is totally bounded

By defn.

there exists finite subsets $A_1, A_2, \dots, A_n$ such that $\operatorname{diam} A_i < \epsilon$ $(i = 1, 2, \dots n)$ and

$A \subset \bigcup_{i=1}^{n} A_i \longrightarrow (1)$

Always $\bigcup_{i=1}^{n} A_i \subset A \longrightarrow (2)$

From (1) & (2),

$A = \bigcup_{i=1}^{n} A_i$

we may assume $A_i \neq \phi$.

If $a_i \in A_i$ $(i = 1, 2, \dots n)$

let $B = \{a_1, a_2, \dots, a_n\}$ be finite subset of A.

Let $x \in A = \bigcup_{i=1}^{n} A_i$

$\Rightarrow x \in A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$

$\Rightarrow x \in A_i$, for some i.

Now, $\operatorname{diam} A_i = \text{l.u.b}\{\rho(x, y) : x, y \in A_i\}$

$\therefore \rho(x, y) \leq \operatorname{diam} A_i < \epsilon$, $\forall x, y \in A_i$

$\Rightarrow \rho(x, y) < \epsilon$, $\forall x, y \in A_i$

$\therefore \rho(x, a_i) < \epsilon$, x; $y = a_i \in A_i$

$\therefore B = \{a_1, a_2, \dots a_n\}$ is ϵ-dense in A

Scanned by CamScanner

Conversely,

If $B = \{x_1, x_2, \ldots x_n\}$ is $\frac{\epsilon}{3}$ dense in A.

let $A_K = B[x_K : \epsilon/3]$, (for each $K = 1, 2 \ldots n$)

$\therefore$ diam A_K = 2 × radius of open ball A_K

$= 2(\frac{\epsilon}{3})$

$= \frac{2\epsilon}{3}$

$\boxed{\text{diam } A_K < \epsilon}$

Since

$B = \{x_1, x_2, \ldots x_n\}$ is $\frac{\epsilon}{3}$ dense in A.

$\Rightarrow$ For each $x \in A$, there exists some point x_K in B s.t $\rho(x, x_K) < \frac{\epsilon}{3}$

$\Rightarrow x \in B[x_K : \epsilon/3]$

$\Rightarrow x \in A_K$

$\therefore$ Each point of A lies in some A_K $(K = 1, 2, \ldots n)$

$\therefore x \in A \Rightarrow x \in A_1 \cup A_2 \cup \ldots \cup A_n$

$\Rightarrow A \subset \bigcup_{K=1}^{n} A_K$

$\therefore$ "A" is totally bounded

6.3 H) Thm (9)

let $\langle M, \rho \rangle$ be a metric space. The subset A of M is totally bounded iff every sequence of points of A contains a Cauchy subsequence.

Pf:-

let A be a totally bounded subset of M

let $\{x_n\}_{n=1}^{\infty}$ be a sequence of pts in A

claim:-

sequence $\{x_n\}_{n=1}^{\infty}$ has a Cauchy subsequence

Since A is totally bdd.

By defn.

There exists a finite $A_1, A_2 \ldots A_n$ s.t diam $A_K < 1$ $(K = 1, 2, \ldots n)$ and $A \subset \bigcup_{K=1}^{n} A_K$

then, One of these subsets (say) A_1 must contain x_n, for infinitely many "n"

Choose $n_1 \in I$ such that $x_{n_1} \in A_1$

Now diam $A_1 < 1$.

$\therefore A_1$ is also totally bounded.

$\therefore A_1$ is covered by a finite no. of subsets of A_1 each of whose diam $< \frac{1}{2}$

Let A_2 be the one of the subsets of A_1 which contain 'x_n' for infinitely many 'n'.

choose $n_2 > n_1$ such that $x_{n_2} \in A_2$ and $A_2 \subset A_1$

$\therefore$ diam $A_2 < \frac{1}{2}$.

Further A_2 is also totally bounded

Continuing this process,

for each $k = 1, 2, \ldots$ we obtain the subset A_k such that $n_k > n_{k-1} > n_{k-2} > \cdots > n_2 > n_1$

and $A_k \subset A_{k-1}$ and diam $A_k < \frac{1}{k}$

$A_k \subset A_{k-1}$
$x_{n_1} \in A_1$
$x_{n_2} \in A_2$
$x_{n_k} \in A_k$
$x_{n_{k+1}} \in A_{k+1} \subset A_k$

for this k,

the points $x_{n_k}, x_{n_{k+1}}, \ldots$ are in A_k.

$\because$ diam A_k = l.u.b $\rho(x_{n_k}, x_{n_{k+1}})$

$\therefore \rho(x_{n_k}, x_{n_{k+1}}) \leq$ diam A_k.

$< \frac{1}{k}$ $\quad (\because \text{diam} A_k < 1/k)$

$\therefore \rho(x_{n_k}, x_{n_{k+1}}) < 1/k$.

$\Rightarrow \{x_{n_k}\}_{k=1}^{\infty}$ is a Cauchy subsequence of $\{x_n\}$.

$\therefore \{x_n\}$ Contains a cauchy subsequence

Conversely

Let $\{x_n\}_{n=1}^{\infty}$ be any sequence of points of a subset A of the Metric Space M.

Assume that $\{x_n\}_{n=1}^{\infty}$ Contains a Cauchy subsequence

Claim:-

A is totally bounded.

Assume that A is not totally bounded

By known thm

there is $\epsilon > 0$ such that A contains no finite subset ϵ-dense subset in A.

If $x_1 \in A$.

$\therefore \{x_1\}$ is not a ϵ-dense in A.

$\therefore$ There is some $x_2 \in A$ such that

$\rho(x_1, x_2) \geq \epsilon$

Now, $\{x_1, x_2\}$ is not again ϵ-dense in A.

Scanned by CamScanner

$\therefore$ There is some $x_3 \in A$ such that

$$\rho(x_1, x_3) \geq \epsilon \text{ and } \rho(x_2, x_3) \geq \epsilon.$$

Continuing this process,

we obtain a sequence $\{x_n\}_{n\geq 1}^{\infty}$ of pts. in A

such that $\rho(x_i, x_j) \geq \epsilon$, $\forall i \neq j \in I$

This implies,

$\{x_n\}_{n\geq 1}^{\infty}$ has no cauchy subsequence

It is a contradiction to fact that

$\{x_n\}$ has a Cauchy subsequence.

So A is totally bounded

Hence the theorem

———— x ————