Sample LaTeX work for Mathspace by Alex Li. Questions taken from CLRS 3rd Ed.

6.1-1. What are the minimum and maximum number of elements in a heap of height h?

Solution: For a binary heap of height h, the last row contains at most 2^h nodes. The row above the last row contains exactly 2^{h-1} nodes and the row before that contains exactly 2^{h-2} nodes and so on until we reach the first row which contains exactly 1 node, the root. So the maximum number of elements in a binary heap of height h is given by the sum, $1 + 2^1 + 2^2 + \ldots + 2^h = 2^{h+1} - 1$ by the Geometric Series.

Similarly, for a binary heap of height h, the last row must contain at least 1 element. The row above the last contains exactly 2^{h-1} elements and so on until we reach the root. So the minimum number of elements in a binary heap of height h is given by the sum, $1 + (1 + 2^1 + 2^2 + \ldots + 2^{h-1}) = 2^h$.

6.1-2. Show that an n-element heap has height $\lfloor \log(n) \rfloor$

Solution: Let h be the height of a binary heap of n elements. Then by the previous question, $2^h \leq n \leq 2^{h+1} - 1$. Now we do a sneaky little trick to the upper bound of n, $2^{h+1} - 1 < 2^{h+1}$. Since log is an increasing function,

$2^h \leq n \leq 2^{h+1} \iff \log(2^h) \leq \log(n) < \log(2^{h+1}) \iff h \leq \log(n) \leq h + 1$. Hence by the definition of the floor function, $h = \lfloor \log(n) \rfloor$.

End of Sample