Triple integrals in spherical coordinates

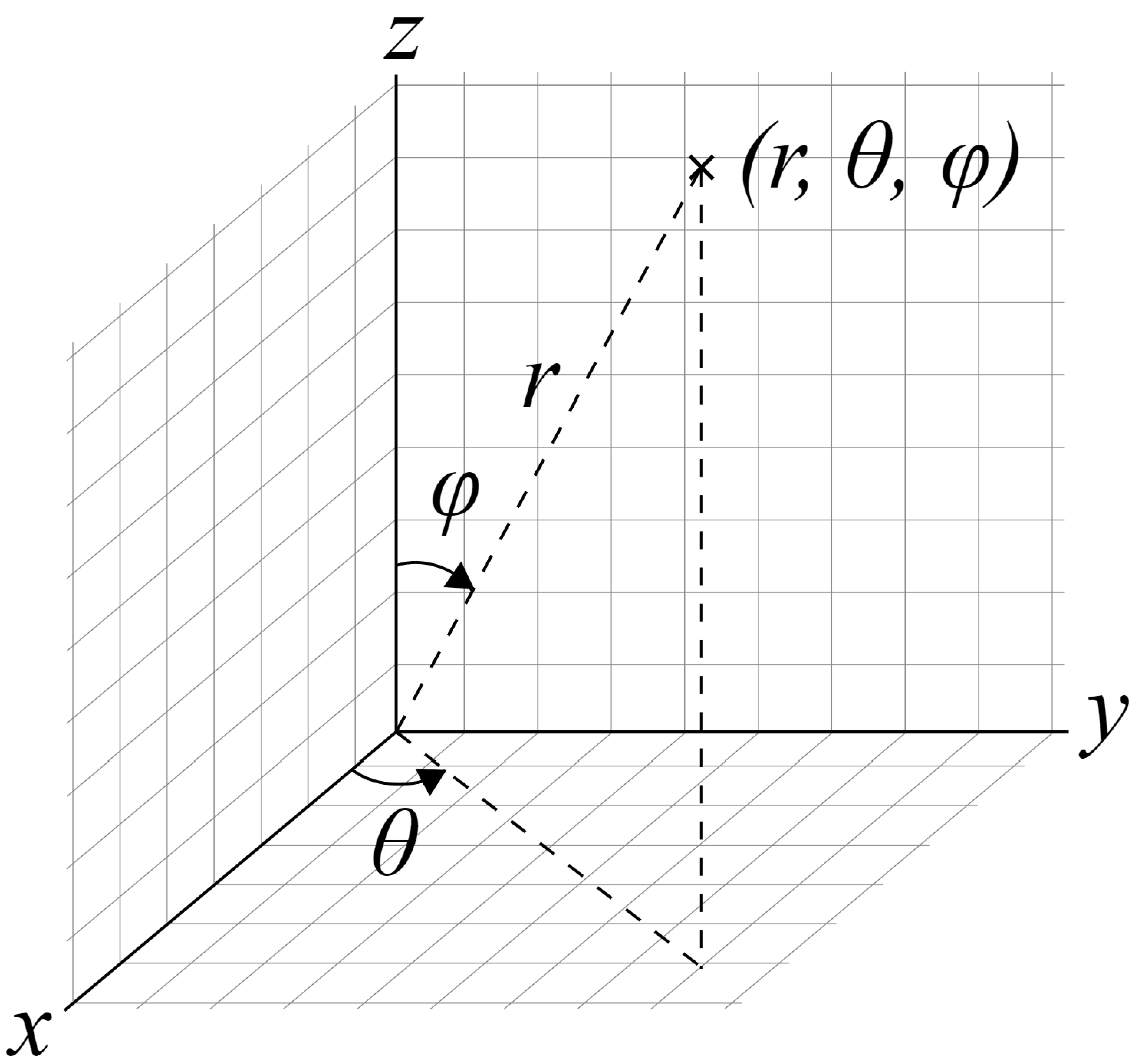
Google Classroom

How to perform a triple integral when your function and bounds are expressed in spherical coordinates.

Background

- Triple integrals

- Spherical coordinates:



Different authors have different conventions on variable names for spherical coordinates. For this article, I will use the following convention. (In each description the "radial line" is the line between the point we are giving coordinates to and the origin).

- r indicates the length of the radial line.

- θ the angle around the z-axis. Specifically, if you project the radial line onto the xy-plane, θ is the angle that line makes with the x-axis.

- ϕ the angle between the radial line and the z-axis.

The following two are not strictly required, but they might help as warm up and practice for this topic.

- Double integrals in polar coordinates
- Triple integrals in cylindrical coordinates

What we're building to

- When you are performing a triple integral, if you choose to describe the function and the bounds of your region using spherical coordinates, (r, ϕ, θ), the tiny volume dV should be expanded as follows:

$$\iiint_R f(r, \phi, \theta)\, dV$$

$$= \iiint_R f(r, \phi, \theta)\, (dr)(r\, d\phi)(r \sin(\phi)\, d\theta)$$

$$= \iiint_R f(r, \phi, \theta)\, r^2 \sin(\phi)\, d\theta\, d\phi\, dr$$

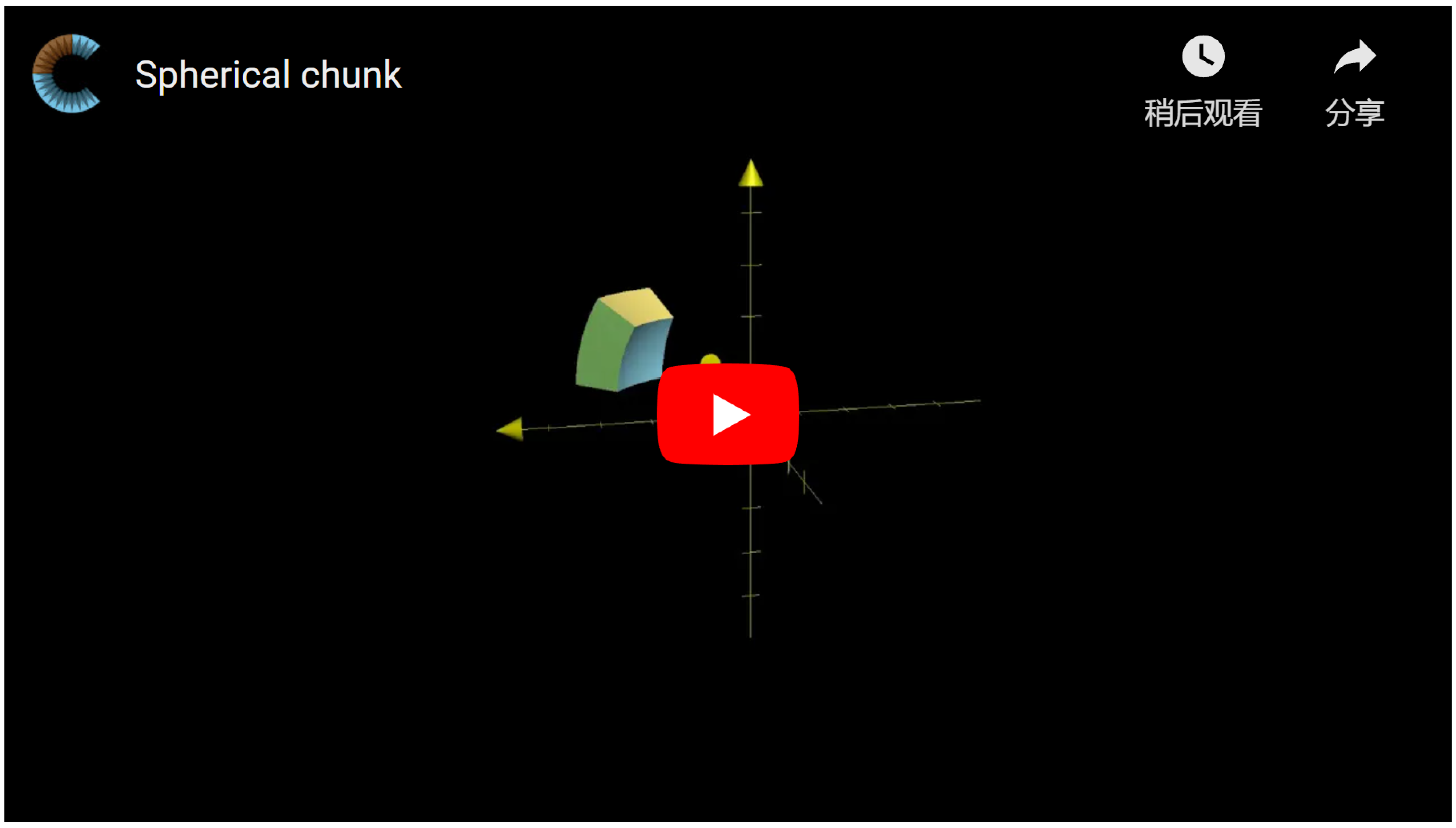
The key term to remember (or re-derive) is $r^2 \sin(\phi)$

- Converting to spherical coordinates can make triple integrals much easier to work out when the region you are integrating over has some spherical symmetry.

Dissecting tiny volumes in spherical coordinates

As discussed in the introduction to triple integrals, when you are integrating over a three-dimensional region R, it helps to imagine breaking it up into infinitely many infinitely small pieces, each with volume dV.

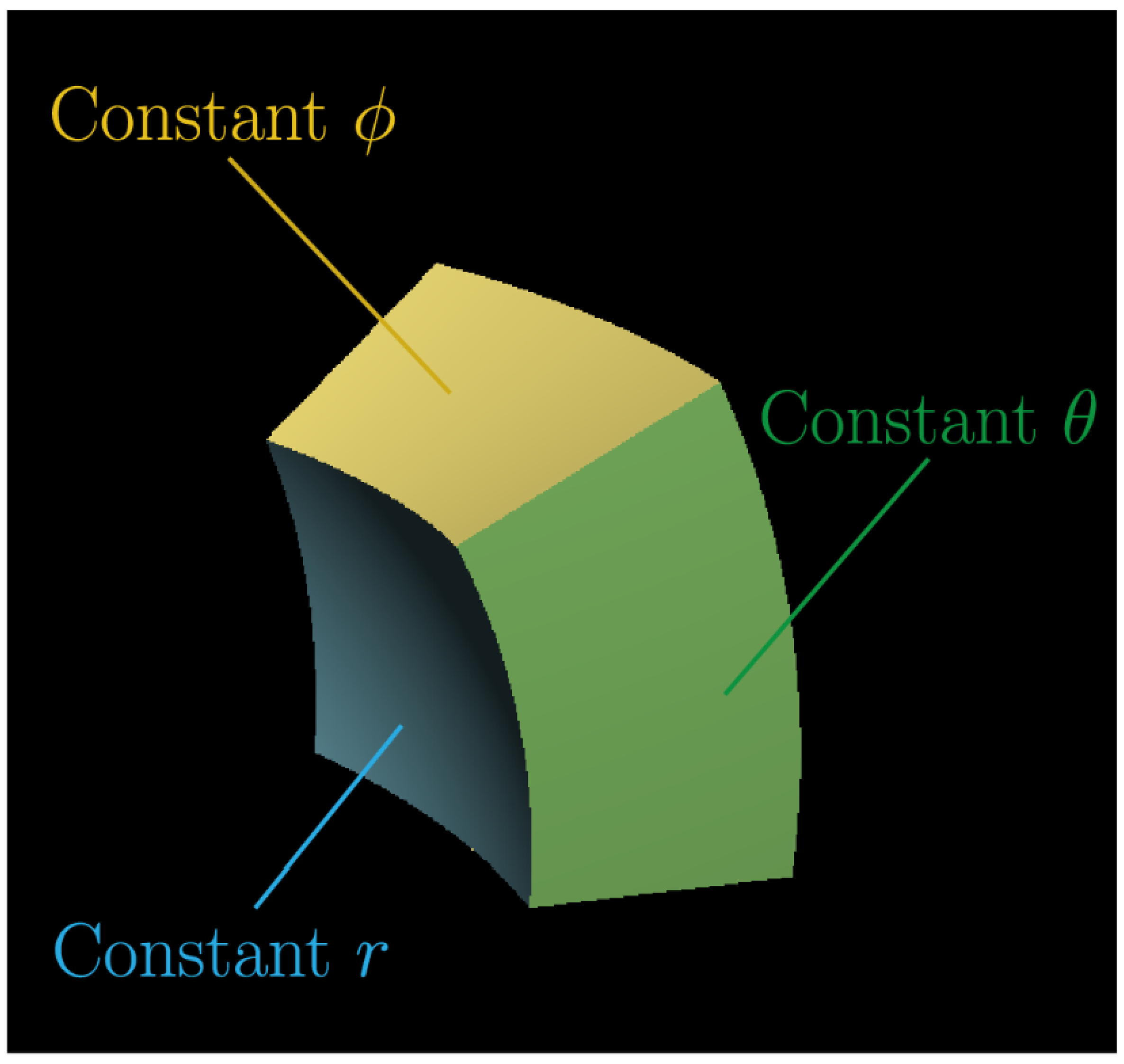
When you were working in cartesian coordinates, these tiny pieces were thought of as rectangular blocks. In spherical coordinates, on the other hand, it helps to think of your tiny pieces as being slightly curved blocks "hugging" a sphere. I'll be drawing a fairly large version of one of these chunks, partly to exaggerate its curvature, and partly just so we can see it. For example, here's what one looks like in three dimensions:



See video transcript

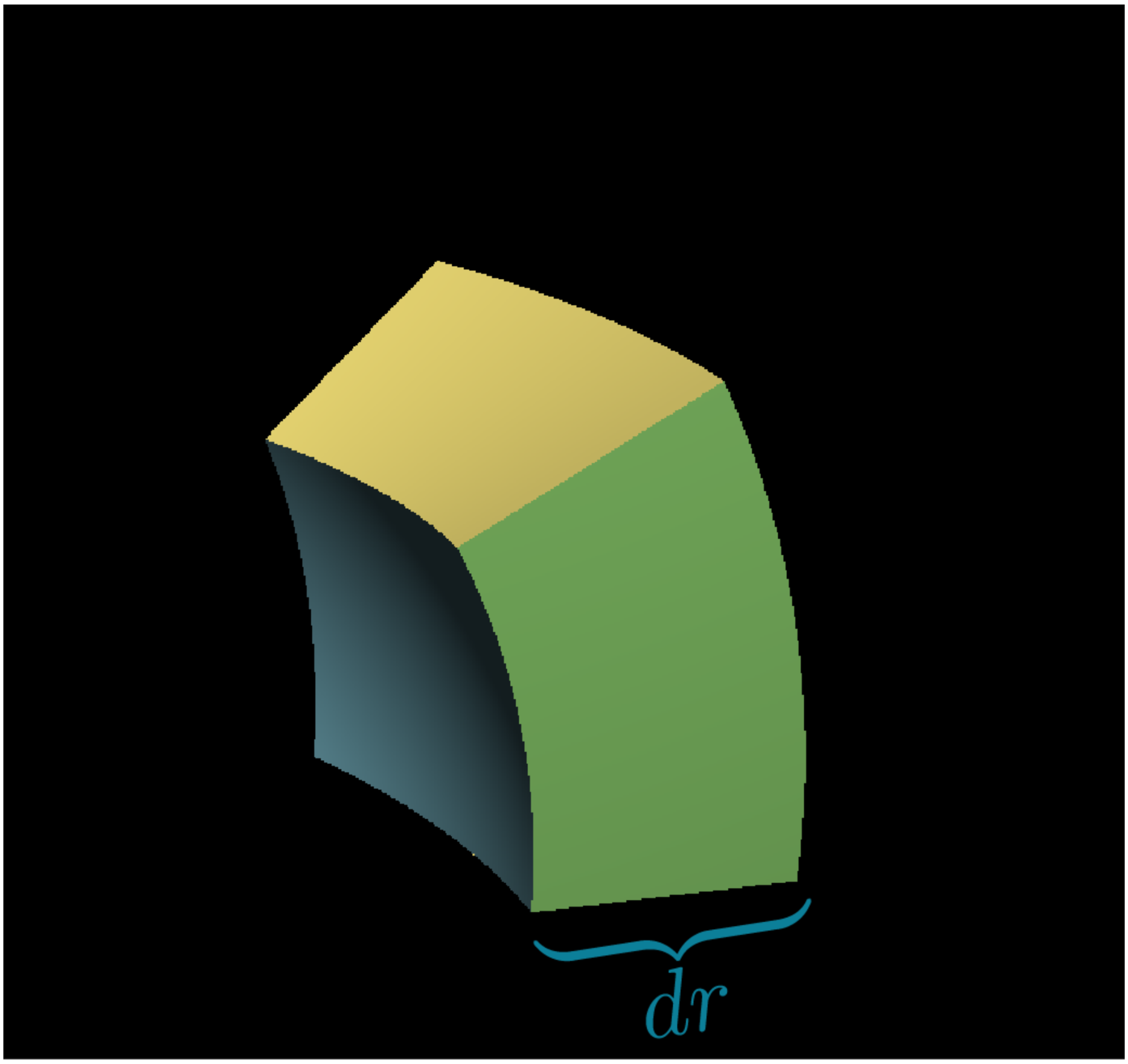
The reason for this shape is that each face represents a constant value for one of the spherical coordinates:

- One pair of faces represents constant values of r (these will be slightly curved, as if hugging a sphere).
- One pair of faces represents constant values of ϕ.
- One pair of faces represents constant values of θ.



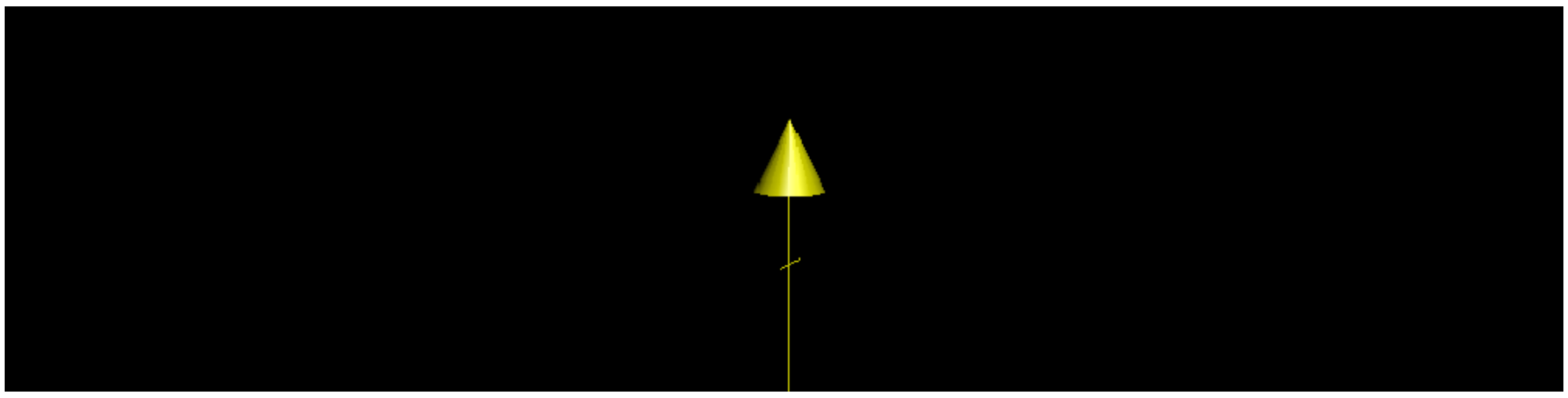
Why is that significant? Because the way multiple integrals work is that each individual integral treats all coordinate as constants, except for one. Therefore, as we consider how the multiple integral as a whole assembles these tiny pieces together, it is more natural to think about pieces whose volume can be expressed in terms of changes to individual coordinates. This will become clearer as you read further.

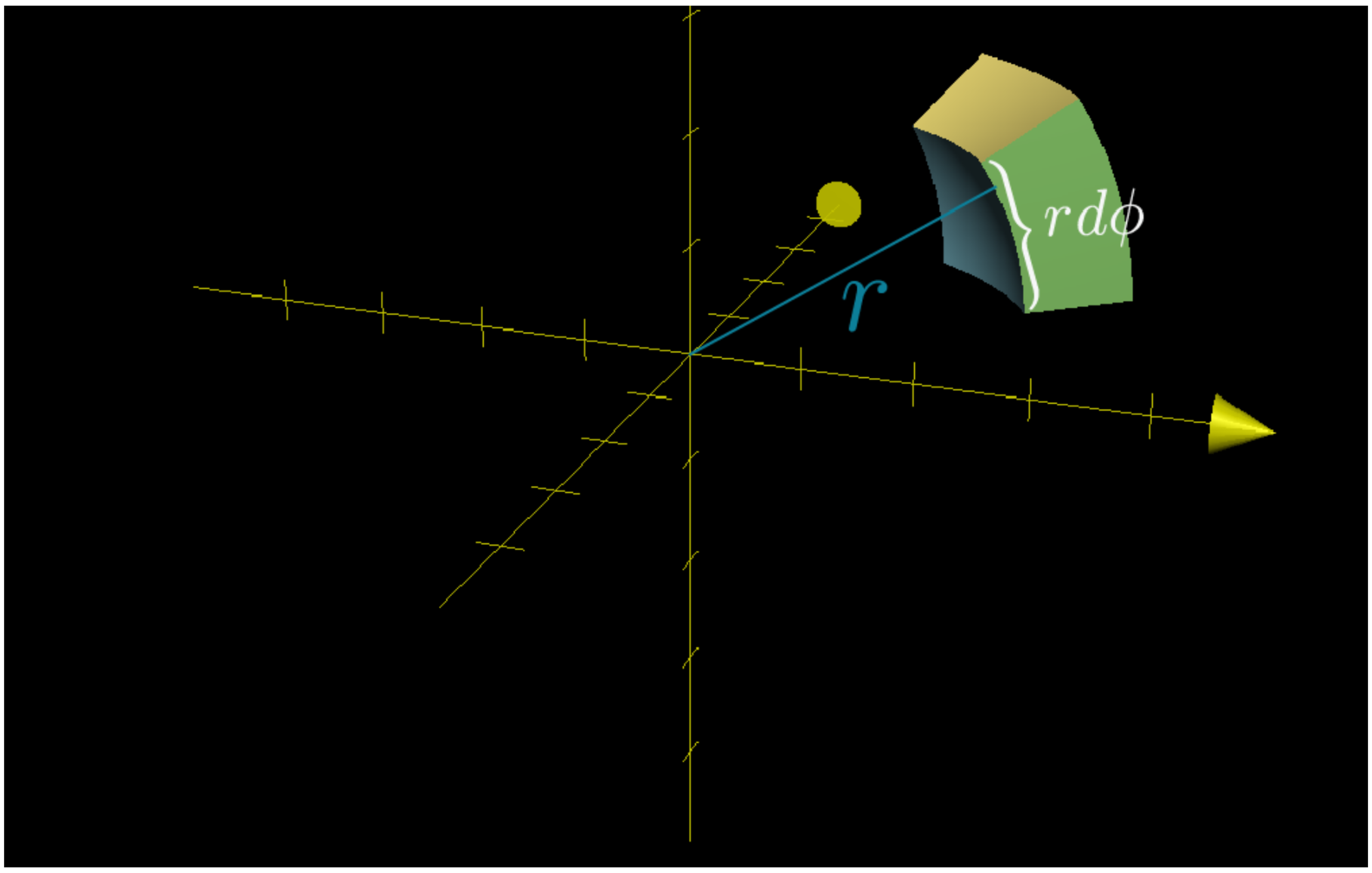
As the size of these blocks approaches zero, the curve will become so negligible that we can treat them as rectangular prisms. One edge represents a tiny change in the length in the distance from the origin, dr:



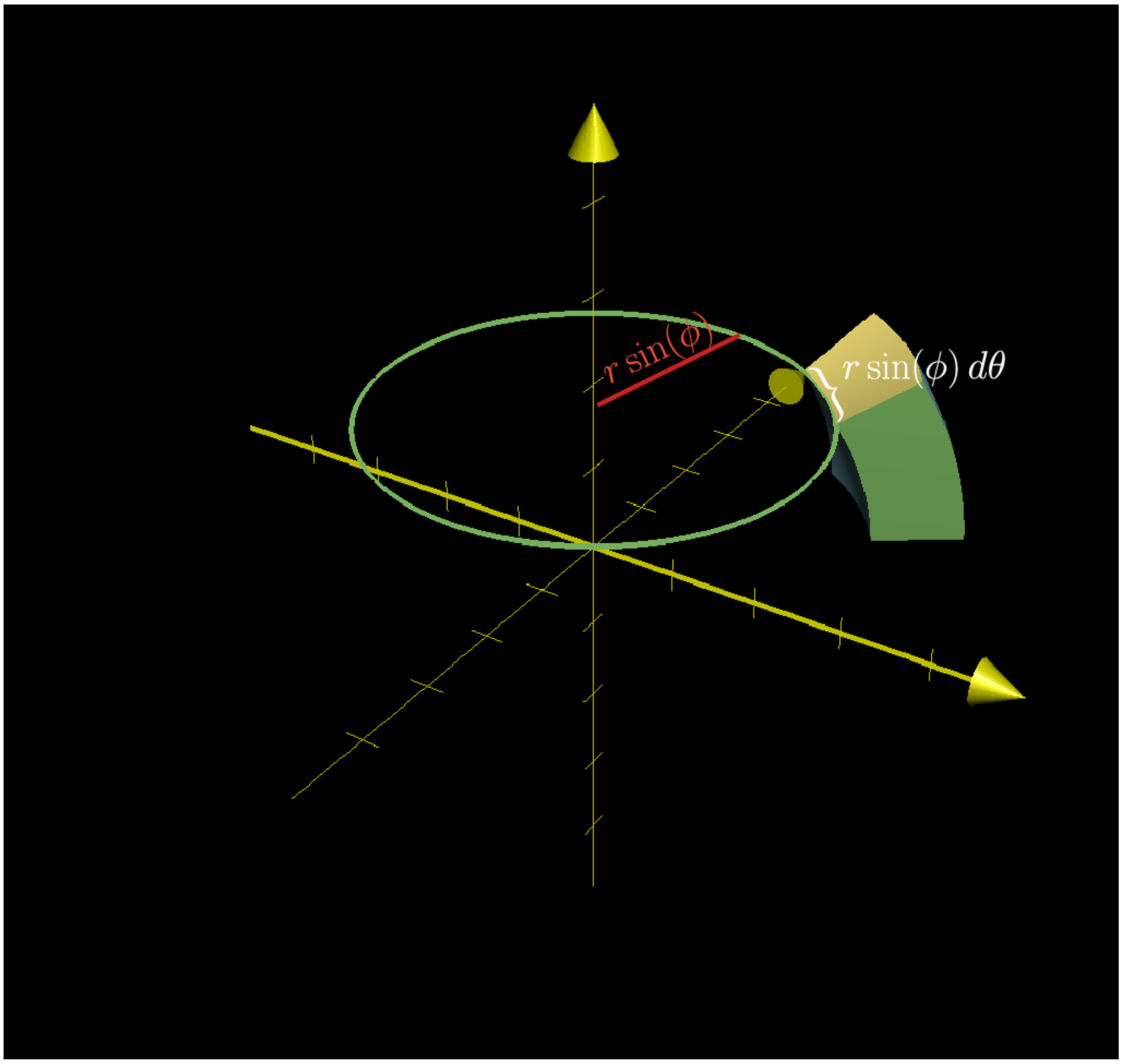
The other two edges are related to the tiny changes in the other two coordinates, $d\theta$ and $d\phi$. However, **since θ and ϕ measure radians, not a unit of length, these values must be multiplied by a unit of length in order to properly reflect the lengths of the edges in our rectangular prism**.

For example, the edge representing a change in ϕ has length $r\,d\phi$:





The edge representing a change in θ is a little trickier. This edge is part of some circle wrapping around the z-axis, and the radius of that circle is not r, but $r\sin(\phi)$. This means the arc length due to a small change in θ is $r\sin(\phi)\,d\theta$.



That can be confusing at first, so it might be worth a moment of contemplation to ensure you understand how that works.

Putting all this together, we can express the volume of our "rectangular" block in terms of dr, $d\phi$ and $d\theta$ by taking the product of all its side lengths.

$$dV = (dr)(r\,d\phi)(r\sin(\phi)\,d\theta) = r^2\sin(\phi)\,dr\,d\phi\,d\theta$$

In other words, when you have some triple integral,

$$\iiint_R f\,dV$$

and you choose to express the bounds and the function using spherical coordinates, **you cannot just replace dV with $dr\,d\phi\,d\theta$**. You must also remember the $r^2\sin(\phi)$ term:

$$\iiint_R f(r,\theta,\phi)\,r^2\sin(\phi)\,dr\,d\phi\,d\theta$$

Personally, I can never quite remember exactly how to expand the dV term off the top of my head

> "Was it $\sin(\phi)$ or $\sin(\theta)$... and is it r or r^2...?"

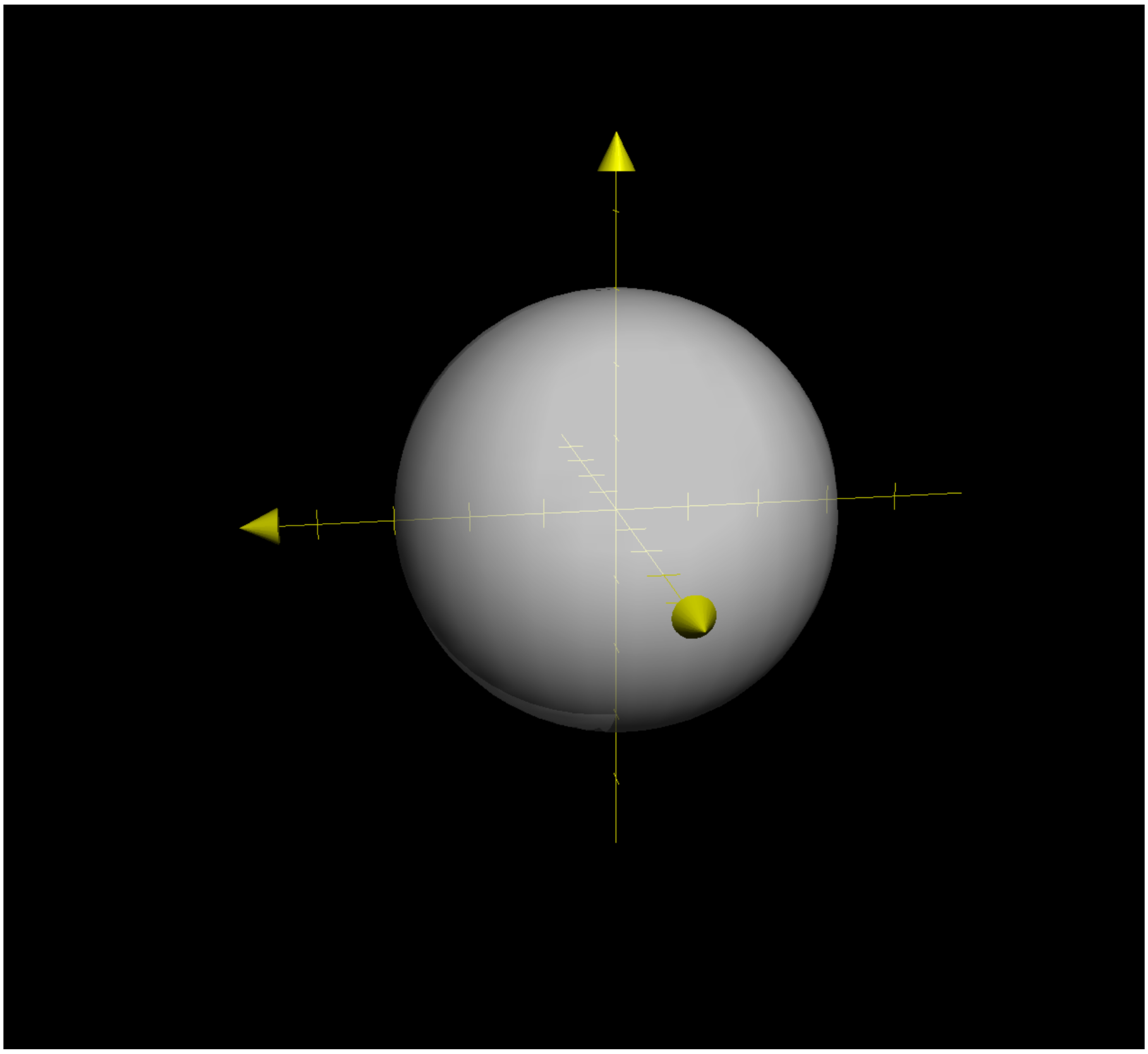
Instead, I think through the process I just illustrated above, asking what the arc lengths resulting from changes to ϕ and θ are.

Example 1: Volume of a sphere revisited

This might be the simplest possible starting example for triple integration in spherical coordinates, but it let's us compute an interesting non-trivial fact: The volume of a sphere.

Question: What is the volume of a sphere with radius R?

Situate the sphere such that its center is on the origin.



If we were doing this integral in cartesian coordinates, we would have that ugly-but-common situation where the bounds of inner integrals are functions of the outer variables. However, because spherical coordinates are so well suited to describing, well, *actual spheres*, our bounds are all constants.

Concept check: Which of the following sets of bounds on the coordinates r, ϕ and θ accurately describes all the points inside a sphere of radius R (without running over the entire sphere multiple times)

Choose 1 answer:

(A)
- $0 \leq r \leq R$
- $0 \leq \phi \leq 2\pi$
- $0 \leq \theta \leq 2\pi$

(B)
- $0 \leq r \leq R$
- $0 \leq \phi \leq \pi$
- $0 \leq \theta \leq 2\pi$

(C)
- $0 \leq r \leq R$
- $0 \leq \phi \leq \pi$
- $0 \leq \theta \leq \pi$

Check

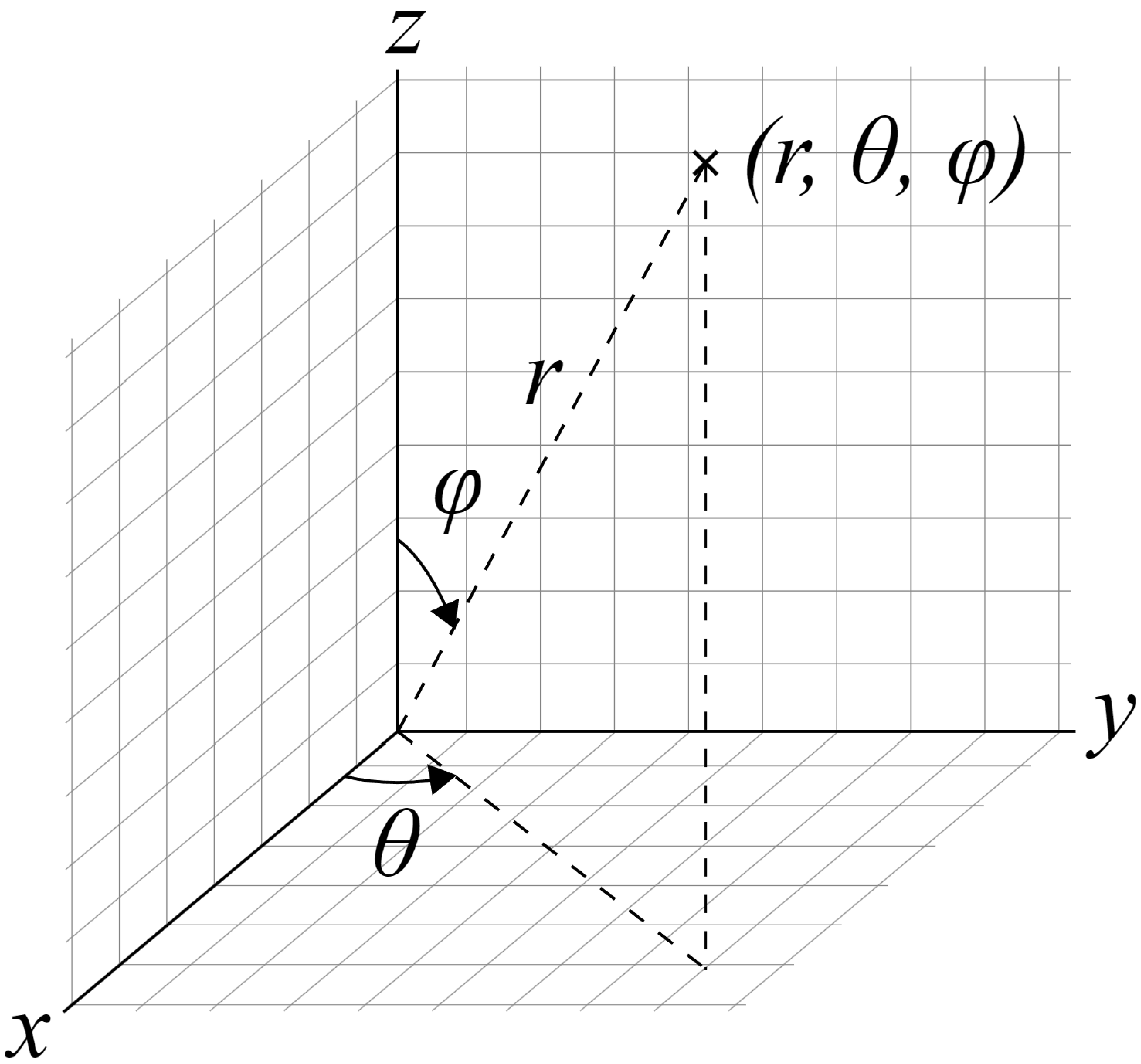
[Hide explanation]

The second answer choice is correct.

- $0 \leq r \leq R$

- $0 \leq \phi \leq \pi$

- $0 \leq \theta \leq 2\pi$



Although θ runs around a full circle, which you can think of as lines of latitude), ϕ only needs to run from 0 to π, which you can think of as running from the north pole to the south pole.

If you accidentally let ϕ run from 0 to 2π, your integral will run over every point in the sphere twice, and hence evaluate to twice the value that you want.

If you only let θ run from 0 to π, you will miss half the sphere.

Using these bounds, together with the fact that

$$dV = r^2 \sin(\phi)\, dr\, d\phi\, d\theta$$

we can start setting up our integral like this:

$$\iiint_{\text{Ball}} dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} r^2 \sin(\phi)\, dr\, d\phi\, d\theta$$

Concept check: Work through this integral, and appreciate just how lovely it is compared with the other nasty triple integrals you may have encountered.

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{R} r^2 \sin(\phi)\, dr\, d\phi\, d\theta = \boxed{}$$

Check

[Hide explanation]

$$\int_0^{2\pi} \int_0^{\pi} \underbrace{\int_0^{R} r^2 \sin(\phi)\, dr}_{\text{Start with the inner } r \text{ integral}} d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left[\frac{r^3}{3} \sin(\phi) \right]_{r=0}^{r=R} d\phi\, d\theta$$

$$= \int_0^{2\pi} \underbrace{\int_0^{\pi} \frac{R^3}{3} \sin(\phi)\, d\phi}_{\text{Next, tackle the } \phi\text{-integral}} d\theta$$

$$= \int_0^{2\pi} \left[\frac{R^3}{3}(-\cos(\phi)) \right]_{\phi=0}^{\phi=\pi} d\theta$$

$$= \int_0^{2\pi} \left(\frac{R^3}{3}(-\cos(\pi) - (-\cos(0))) \right) d\theta$$

$$= \underbrace{\int_0^{2\pi} \frac{R^3}{3}(2)\, d\theta}_{\text{Finally, solve the } \theta \text{ integral}}$$

$$= \left[\frac{R^3}{3}(2)\theta \right]_{\theta=0}^{\theta=2\pi}$$

$$= \frac{R^3}{3}(2)(2\pi)$$

$$= \frac{4}{3}\pi R^3$$

If you dare, imagine trying to do this integral in cartesian coordinates. It's a nightmare! This gives us an important takeaway:

Key takeaway If you are integrating over a region with some spherical symmetry, passing to spherical coordinates can make the bounds *much* nicer to deal with.

Example 2: Integrating a function

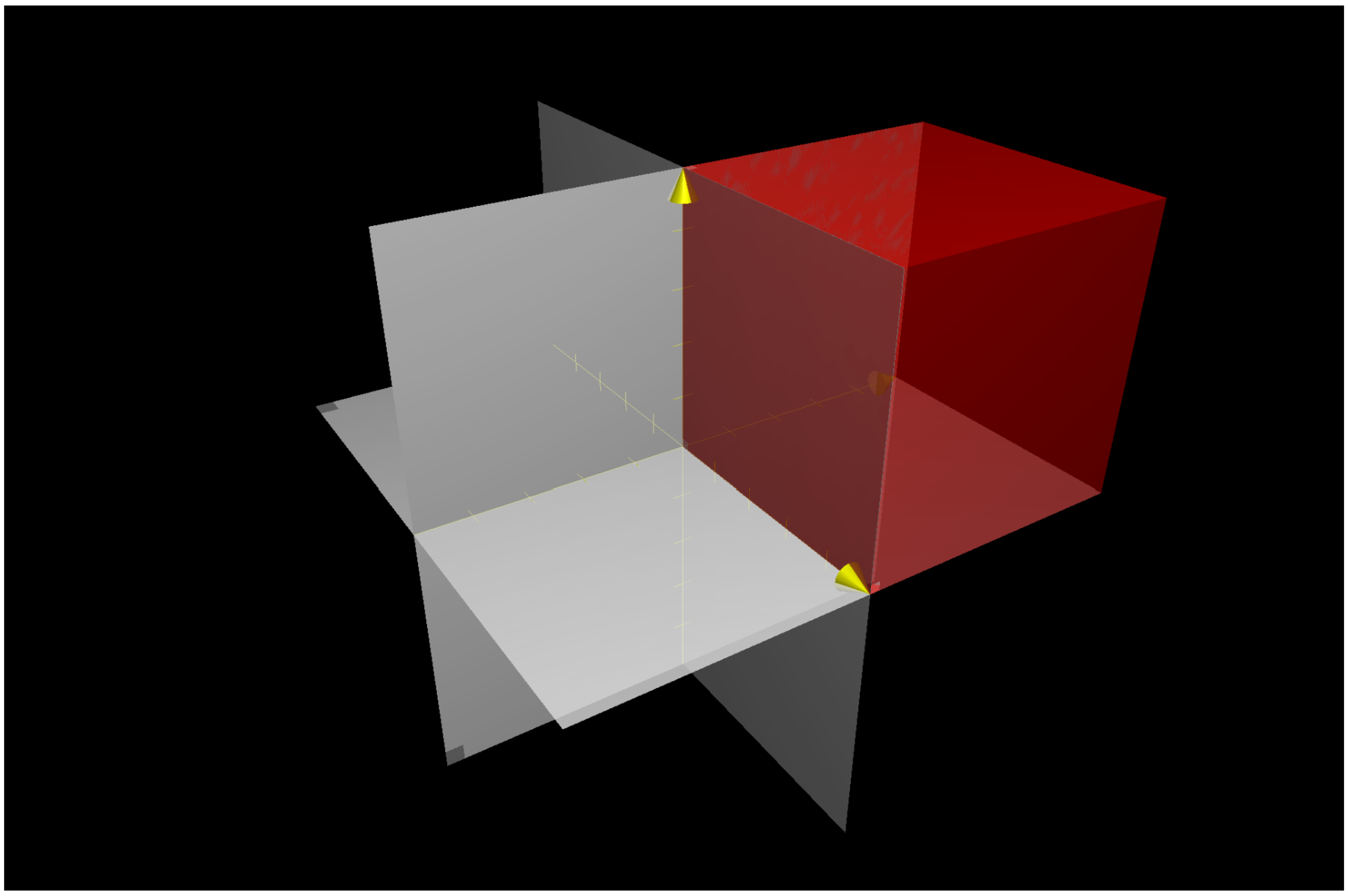
Integrate the function

$$f(x, y, z) = x + 2y + 3z$$

in the region of the **first octant** where

$$x^2 + y^2 + z^2 \leq 3$$

[Hide explanation]



When referring to three-dimensional space, especially in terms of the coordinates x, y and z, the "octants" are the 8 regions that space is divided into when you slice it using the xy-plane, the xz-plane, and the yz-plane.

The "first" octant is the one where x, y and z are all positive.

> (This is analogous to the "first quadrant" of the xy-plane)

Step 1: Express the region in spherical coordinates.

How could you know that we should pass to spherical coordinates? We could do this whole integral in cartesian coordinates, couldn't we? Cylindrical coordinates would work too.

The fact that our boundary includes the condition $x^2 + y^2 + z^2 \leq 3$ is a description of the distance between points of our region and the origin. Since the spherical coordinate r expresses precisely this idea, we can feel confident that describing the boundary of our region using r will make the bounds of our three integrals simpler than if we did so in terms of x, y and z.

Specifically, this condition becomes

$$x^2 + y^2 + z^2 \leq 3$$

$$r^2 \leq 3$$

$$r \leq \sqrt{3}$$

Concept check: What about θ and ϕ? What bounds should we place on these two coordinates to keep our integral within the first octant?

[ ] $\leq \theta \leq$ [ ]

[ ] $\leq \phi \leq$ [ ]

Check

[Hide explanation]

The correct answer is

$$0 \leq \theta \leq \frac{\pi}{2}$$

$$0 \leq \phi \leq \frac{\pi}{2}$$

You can think of θ as sweeping out the region $x \geq 0$ and $y \geq 0$ as it ranges from 0 to $\frac{\pi}{2}$.

Similarly, ϕ sweeps out the "top half" of space, the region where $z \geq 0$, as it ranges from 0 to $\frac{\pi}{2}$.

Step 2: Express the function in spherical coordinates

Next, we convert the function

$$f(x, y, z) = x + 2y + 3z$$

into spherical coordinates. To do this, we use the conversions for each individual cartesian coordinate.

- $x = r\sin(\phi)\cos(\theta)$
- $y = r\sin(\phi)\sin(\theta)$
- $z = r\cos(\phi)$

Plugging each of these in, we get

$$\begin{aligned} f(x, y, z) &= x + 2y + 3z \\ &= r\sin(\phi)\cos(\theta) + 2r\sin(\phi)\sin(\theta) + 3r\cos(\phi) \\ &= r\Big(\sin(\phi)\cos(\theta) + 2\sin(\phi)\sin(\theta) + 3\cos(\phi)\Big) \end{aligned}$$

You might say that this makes things more complicated than they were in cartesian coordinates. And you'd be right! But when it comes to triple integrals, a more complicated function is a relatively small price to pay for getting our bounds to be constants.

Step 3: Compute the triple integral

Concept check: Putting the previous two steps together, what is the integral that we need to solve?

Choose 1 answer:

(A) $\int_0^{\pi}\int_0^{2\pi}\int_0^{3} r\Big(\sin(\phi)\cos(\theta) + 2\sin(\phi)\sin(\theta) + 3\cos(\phi)\Big)r^2\sin(\phi)\,dr\,d\theta\,d$

(B) $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\sqrt{3}} r\Big(\sin(\phi)\cos(\theta) + 2\sin(\phi)\sin(\theta) + 3\cos(\phi)\Big)r^2\sin(\phi)\,dr$

(C) $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\sqrt{3}} r\Big(\sin(\phi)\cos(\theta) + 2\sin(\phi)\sin(\theta) + 3\cos(\phi)\Big)\,dr\,d\theta\,d\phi$

Check

Hide explanation

The second answer choice is correct:

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\sqrt{3}} r\Big(\sin(\phi)\cos(\theta) + 2\sin(\phi)\sin(\theta) + 3\cos(\phi)\Big)r^2\sin(\phi)\,dr\,d\theta\,d\phi$$

To start, this accurately reflects the bounds discussed in step 1:

- $0 \le r \le \sqrt{3}$
- $0 \le \theta \le \frac{\pi}{2}$
- $0 \le \phi \le \frac{\pi}{2}$

Also, unlike the third choice, this one includes the crucial $r^2\sin(\phi)$ term.

Bring it on home: Solve that integral!

Integral from previous question:

Check

[Hide explanation]

Remember, for triple integrals that come up in practice, once you know how to express the bounds, it will typically be a computer which takes care of the computation itself.

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\sqrt{3}} \underbrace{r\Big(\sin(\phi)\cos(\theta)+2\sin(\phi)\sin(\theta)+3\cos(\phi)\Big)r^2 \text{si}}_{\text{Factor out all non-}r\text{ terms}}$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\Big(\sin(\phi)\cos(\theta)+2\sin(\phi)\sin(\theta)+3\cos(\phi)\Big)\sin(\phi)\int_0^{\sqrt{}}$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\Big(\sin(\phi)\cos(\theta)+2\sin(\phi)\sin(\theta)+3\cos(\phi)\Big)\sin(\phi)\Bigg[\frac{r^4}{4}$$

$$=\underbrace{\int_0^{\pi/2}\int_0^{\pi/2}\Big(\sin(\phi)\cos(\theta)+2\sin(\phi)\sin(\theta)+3\cos(\phi)\Big)\sin(\phi)\Bigg(\frac{\sqrt{}}{}}_{\text{Factor out non-}\theta\text{ terms appropriately}}$$

$$=\frac{9}{4}\int_0^{\pi/2}\sin(\phi)\int_0^{\pi/2}\Big(\sin(\phi)\cos(\theta)+2\sin(\phi)\sin(\theta)+3\cos(\phi)\Big)\,d$$

$$=\frac{9}{4}\int_0^{\pi/2}\sin(\phi)\Big[\sin(\phi)\sin(\theta)+2\sin(\phi)(-\cos(\theta))+3\cos(\phi)\theta\Big]_{\theta=0}^{\theta=\pi/}$$

$$=\frac{9}{4}\int_0^{\pi/2}\sin(\phi)\Big(\sin(\phi)(1-0)+2\sin(\phi)(-(0-1))+3\cos(\phi)\Big(\frac{\pi}{2}$$

$$=\frac{9}{4}\int_0^{\pi/2}\sin(\phi)\Big(\sin(\phi)+2\sin(\phi)+\frac{3\pi}{2}\cos(\phi)\Big)\,d\phi$$

$$=\frac{9}{4}\int_0^{\pi/2}\Big(3\sin^2(\phi)+\frac{3\pi}{2}\cos(\phi)\sin(\phi)\Big)\,d\phi$$

(Look up these antiderivatives if necessary)

- Antiderivative of $\sin^2(\phi)$ is $\frac{\phi}{2}-\frac{1}{2}\cos(\phi)\sin(\phi)$
- Antiderivative of $\cos(\phi)\sin(\phi)$ is $-\frac{1}{2}\cos^2(\phi)$

Jumping back in where we were and plugging these in, we get:

$$=\frac{9}{4}\int_0^{\pi/2}\Big(3\sin^2(\phi)+\frac{3\pi}{2}\cos(\phi)\sin(\phi)\Big)\,d\phi$$

$$= \frac{9}{4}\left[3\left(\frac{\phi}{2} - \frac{1}{2}\cos(\phi)\sin(\phi)\right) + \frac{3\pi}{2}\left(-\frac{1}{2}\cos^2(\phi)\right)\right]_{\phi=0}^{\phi=\pi/2}$$

$$= \frac{9}{4}\left(3\left(\frac{\pi}{4} - 0\right) + \frac{3\pi}{2}\left(\frac{1}{2}\right)\right)$$

$$= \frac{9}{4}\left(\frac{3\pi}{4} + \frac{3\pi}{4}\right)$$

$$= \frac{9}{4}\left(\frac{3\pi}{2}\right)$$

$$= \frac{27\pi}{8}$$

Phew! Working through triple integrals is the worst. Just remember, the main skill to aquire here is setting up these integrals. And the point here is that setting up this integral was much easier using spherical coordinates than it would have been using cartesian coordinates.

Summary

- When you are performing a triple integral, if you choose to describe the function and the bounds of your region using spherical coordinates, (r, ϕ, θ), the tiny volume dV should be expanded as follows:

$$\iiint_R f(r, \phi, \theta)\, dV$$

$$= \iiint_R f(r, \phi, \theta)\, (dr)(r\, d\phi)(r \sin(\phi)\, d\theta)$$

$$= \iiint_R f(r, \phi, \theta)\, r^2 \sin(\phi)\, d\theta\, d\phi\, dr$$

The key term to remember (or re-derive) is $r^2 \sin(\phi)$

- Converting to spherical coordinates can make triple integrals much easier to work out when the region you are integrating over has some spherical symmetry.