Tangent planes

Google Classroom

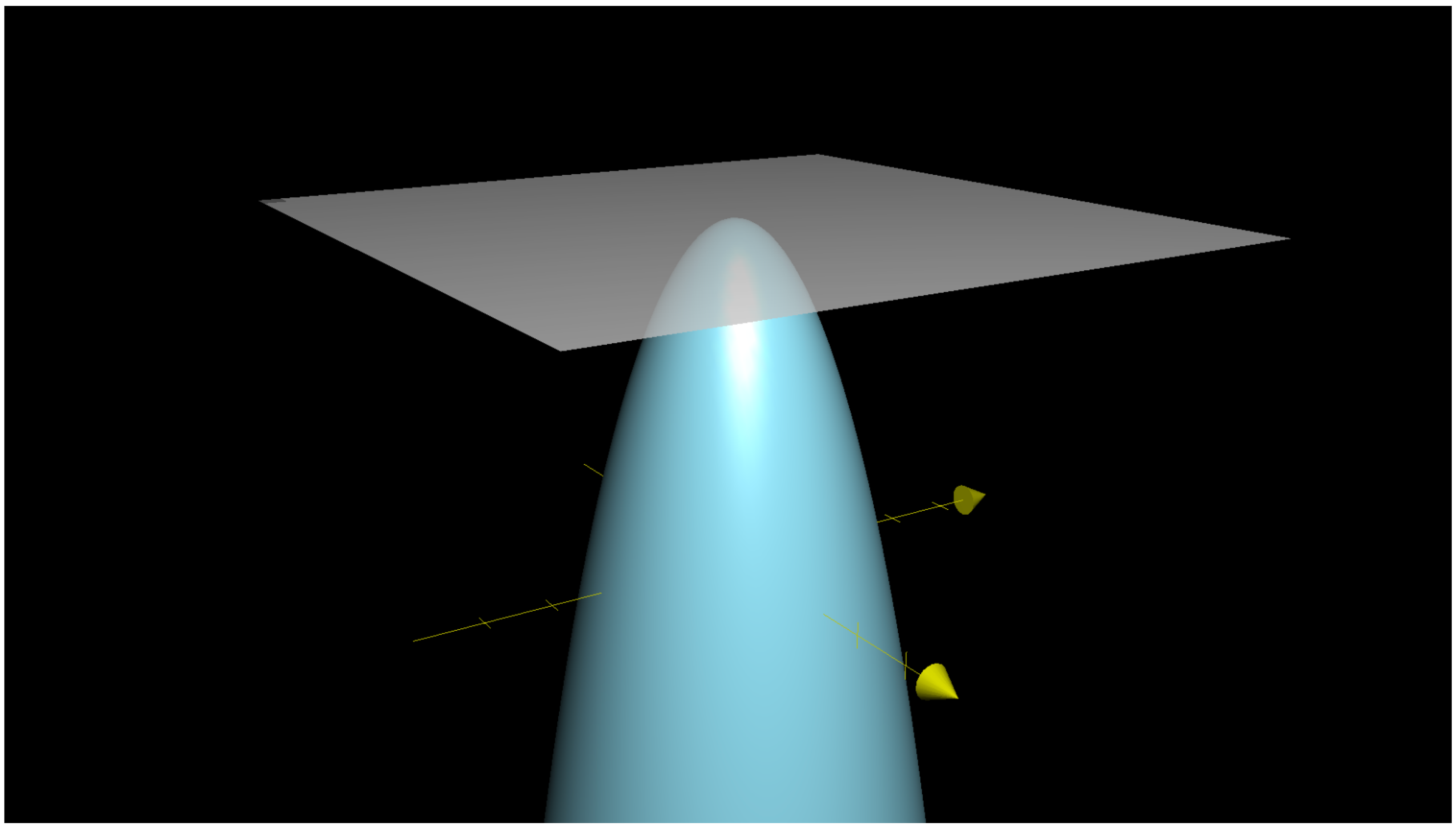
Just as the single variable derivative can be used to find tangent lines to a curve, partial derivatives can be used to find the tangent plane to a surface.

Background

- Partial derivatives

What we're building to

- A **tangent plane** to a two-variable function $f(x, y)$ is, well, a plane that's tangent to its graph.



- The equation for the tangent plane of the graph of a two-variable function $f(x, y)$ at a particular point (x_0, y_0) looks like this:

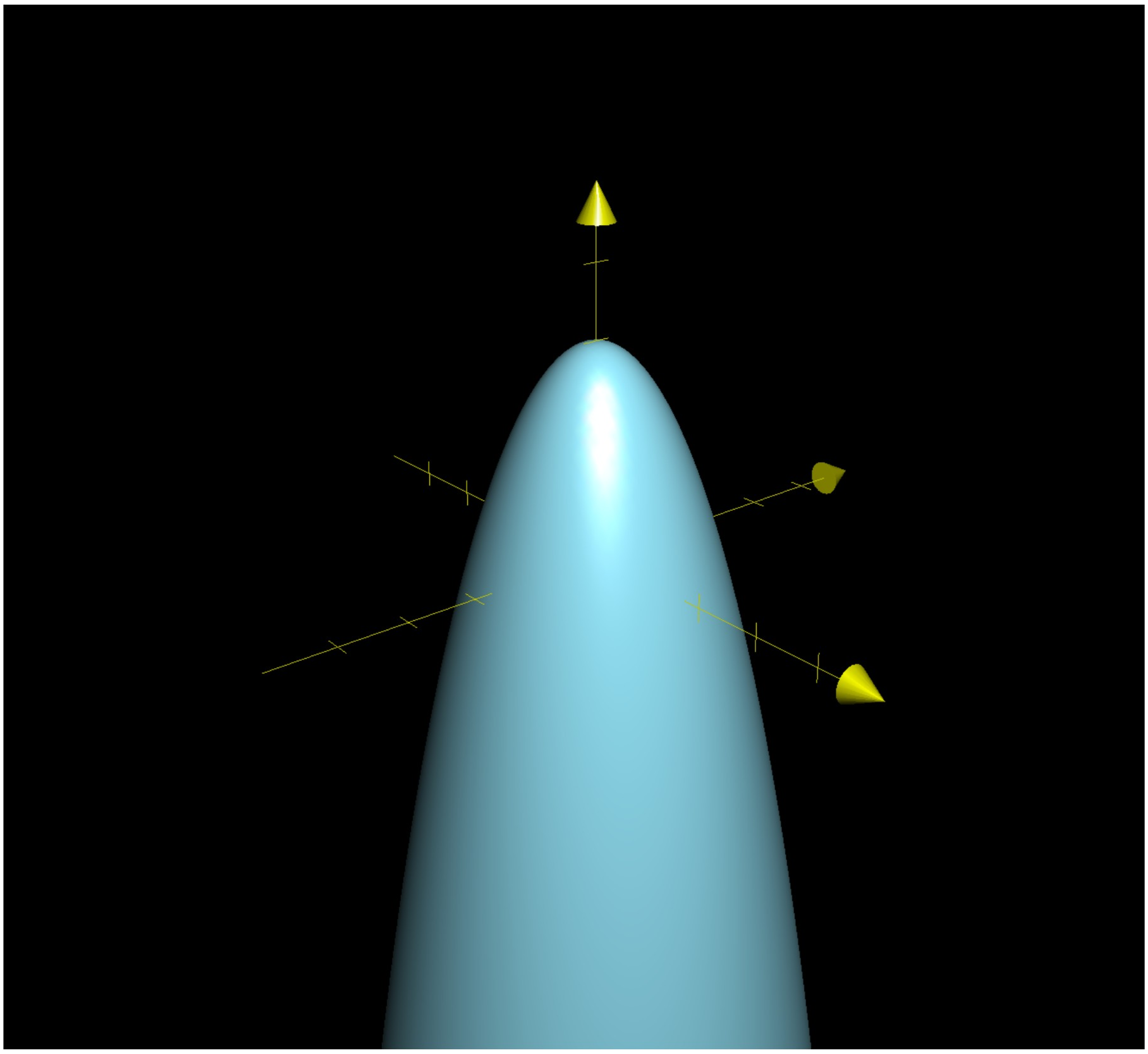
$$T(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The task at hand

Think of a scalar-valued function with a two-coordinate input, like this one:

$$f(x, y) = -x^2 - y^2 + 3$$

Intuitively, it's common to visualize a function like this with its three-dimensional graph.

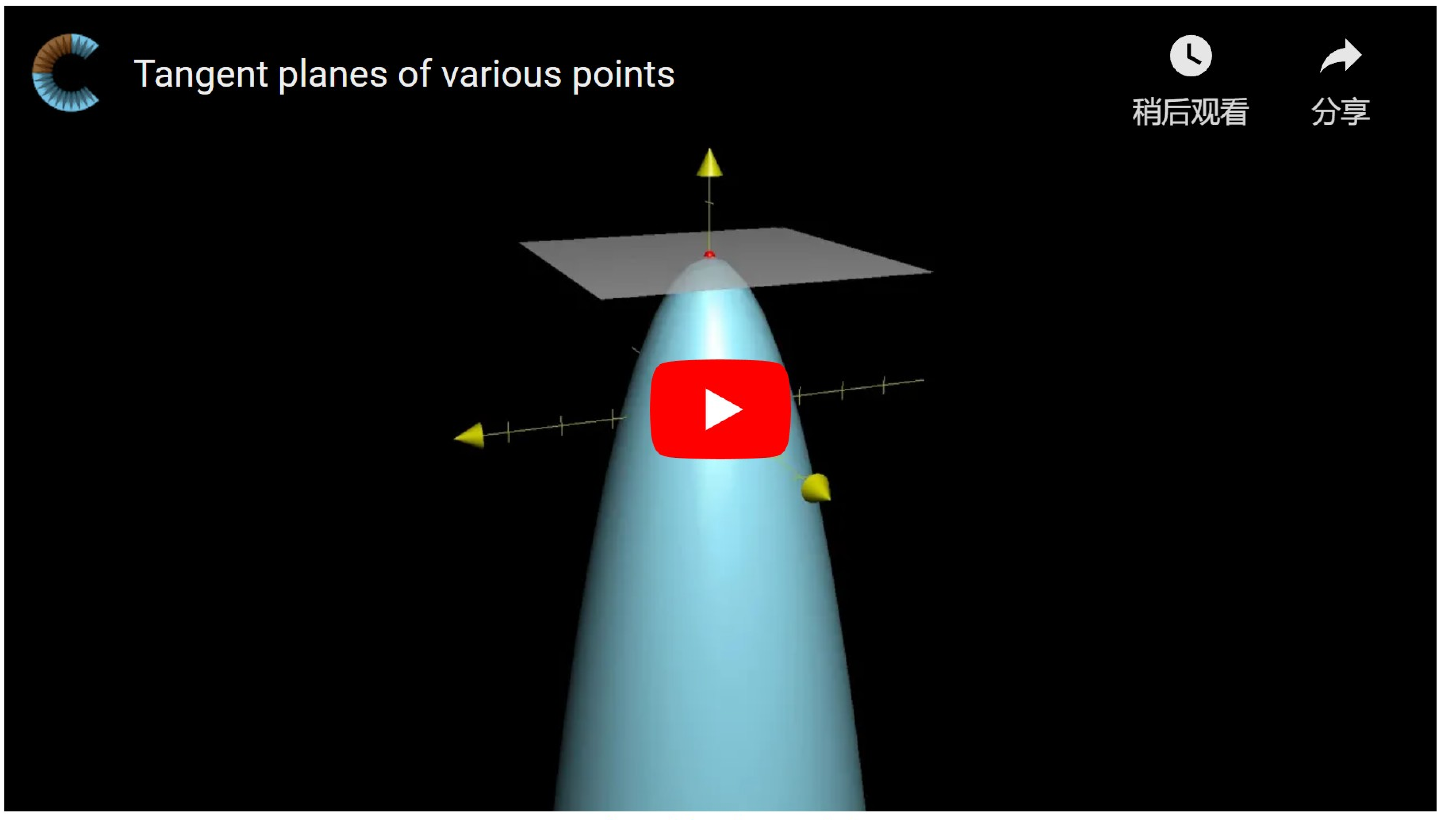


Remember, you can describe this graph more technically by describing it as a certain set of points in three-dimensional space. Specifically, it is all the points that look like this:

$$(x, y, f(x, y)) = (x, y, -x^2 - y^2 + 3)$$

Here, x and y can range over all possible real numbers.

A **tangent plane** to this graph is a plane which is tangent to the graph. Hmmm, that's not a good definition. This is hard to describe with words, so I'll just show a video with various different tangent planes.

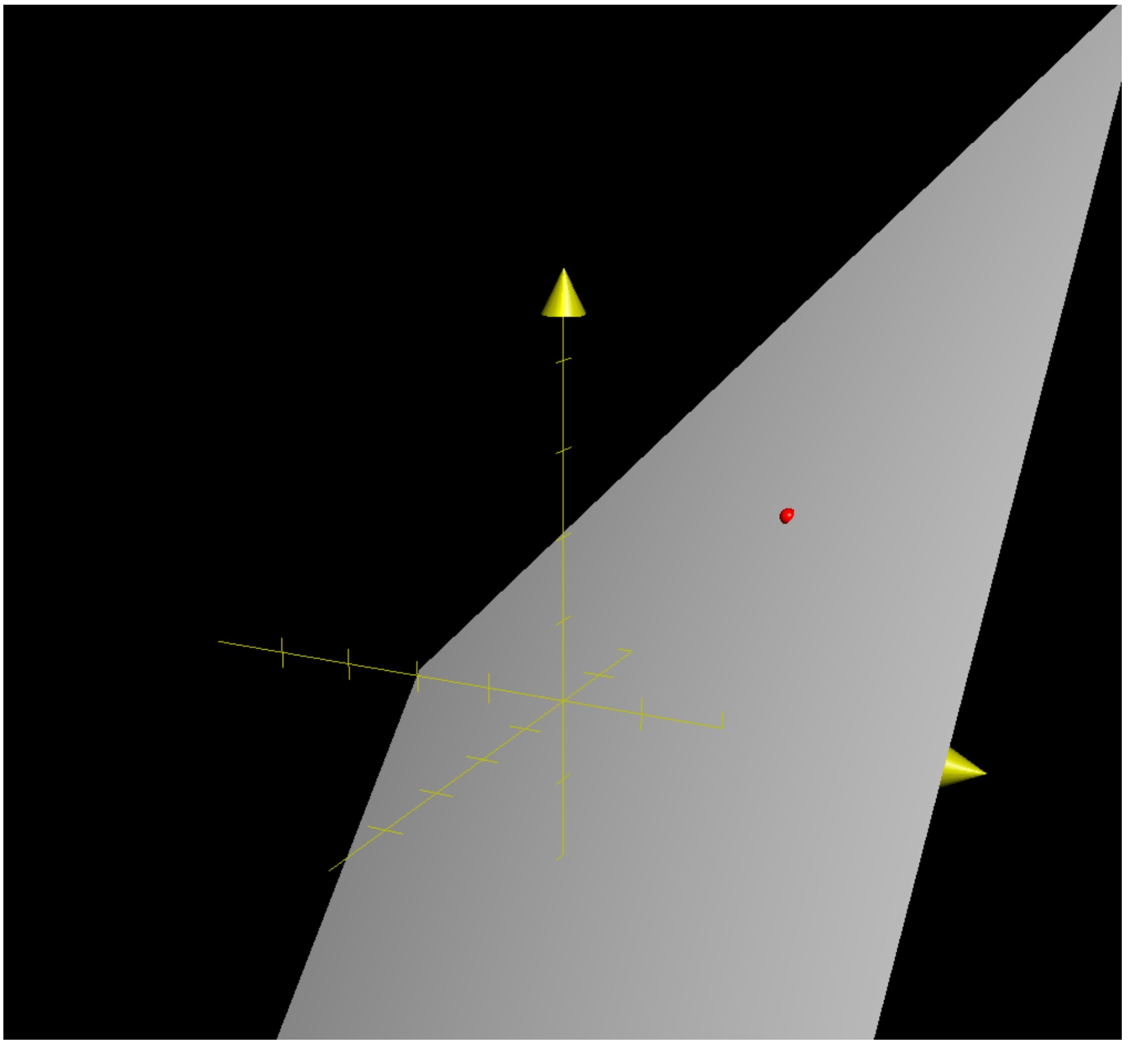


See video transcript

Key question: How do you find an equation representing the plane tangent to the graph of the function at some specific point $(x_0, y_0, f(x_0, y_0))$ in three-dimensional space?

Representing planes as graphs

Well, first of all, which functions $g(x, y)$ have graphs that look like planes?



A plane passing through (2, 2, 2)

The slope of a plane in any direction is constant over all input values, so both partial derivatives g_x and g_y would have to be constants. The functions with constant partial derivatives look like this:

$$g(x, y) = ax + by + c$$

Here, a, b, and c are each some constant. These are called **linear functions**. Well, technically speaking they are **affine functions** since linear functions must pass through the origin, but it's common to call them linear functions anyway.

Question: How can you guarantee that the graph of a linear function passes through a particular point (x_0, y_0, z_0) in space?

One clean way to do this is to write our linear function as

$$g(x, y) = {\color{teal}a}(x - x_0) + {\color{red}b}(y - y_0) + z_0$$

[Hide explanation]

> In effect, we have fixed the constant c in the expression $ax + by + c$ to be
>
> $$c = z_0 - ax_0 - by_0,$$

Concept check: With g defined this way, compute $g(x_0, y_0)$.

Choose 1 answer:

- (A) (x_0, y_0, z_0)
- (B) z_0

Check

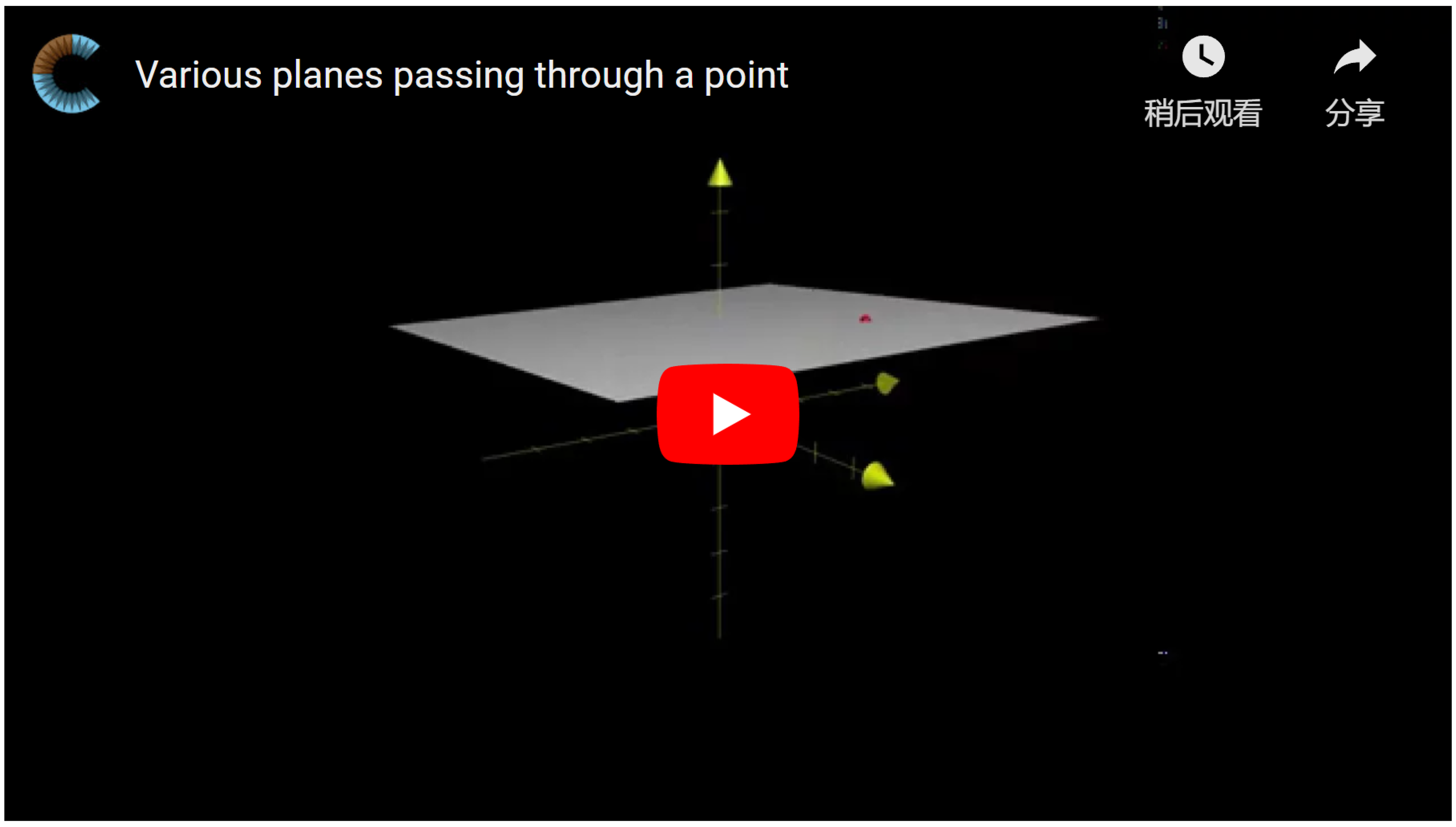
[Hide explanation]

> $$\begin{aligned} g(x_0, y_0) &= a(x_0 - x_0) + b(y_0 - y_0) + z_0 \\ &= a(0) + b(0) + z_0 \\ &= z_0 \end{aligned}$$

Writing $g(x, y)$ like this makes it clear that $g(x_0, y_0) = z_0$. This guarantees that the graph of g must pass through (x_0, y_0, z_0):

$$(x_0, y_0, g(x_0, y_0)) = (x_0, y_0, z_0)$$

The other constants a and b are free to be whatever we want. Different choices for a and b result in different planes passing through the point (x_0, y_0, z_0). The video below shows how those planes change as we tweak a and b:



See video transcript

Equation for a tangent plane

Back to the task at hand. We want a function $T(x, y)$ that represents a plane tangent to the graph of some function $f(x, y)$ at a point $(x_0, y_0, f(x_0, y_0))$, so we substitute $f(x_0, y_0)$ for z_0 in the general equation for a plane.

$$T(x, y) = f(x_0, y_0) + a(x - x_0) + b(y - y_0)$$

As you tweak the values of a and b, this equation will give various planes passing through the graph of f at the desired point, but only one of them will be a *tangent* plane.

Of all the planes passing through $(x_0, y_0, f(x_0, y_0))$, the one tangent to the graph of f will **have the same partial derivatives as** f. Pleasingly, the partial derivatives of our linear function are given by the constants a and b.

- **Try it!** Take the partial derivatives of the equation for $T(x, y)$ above.

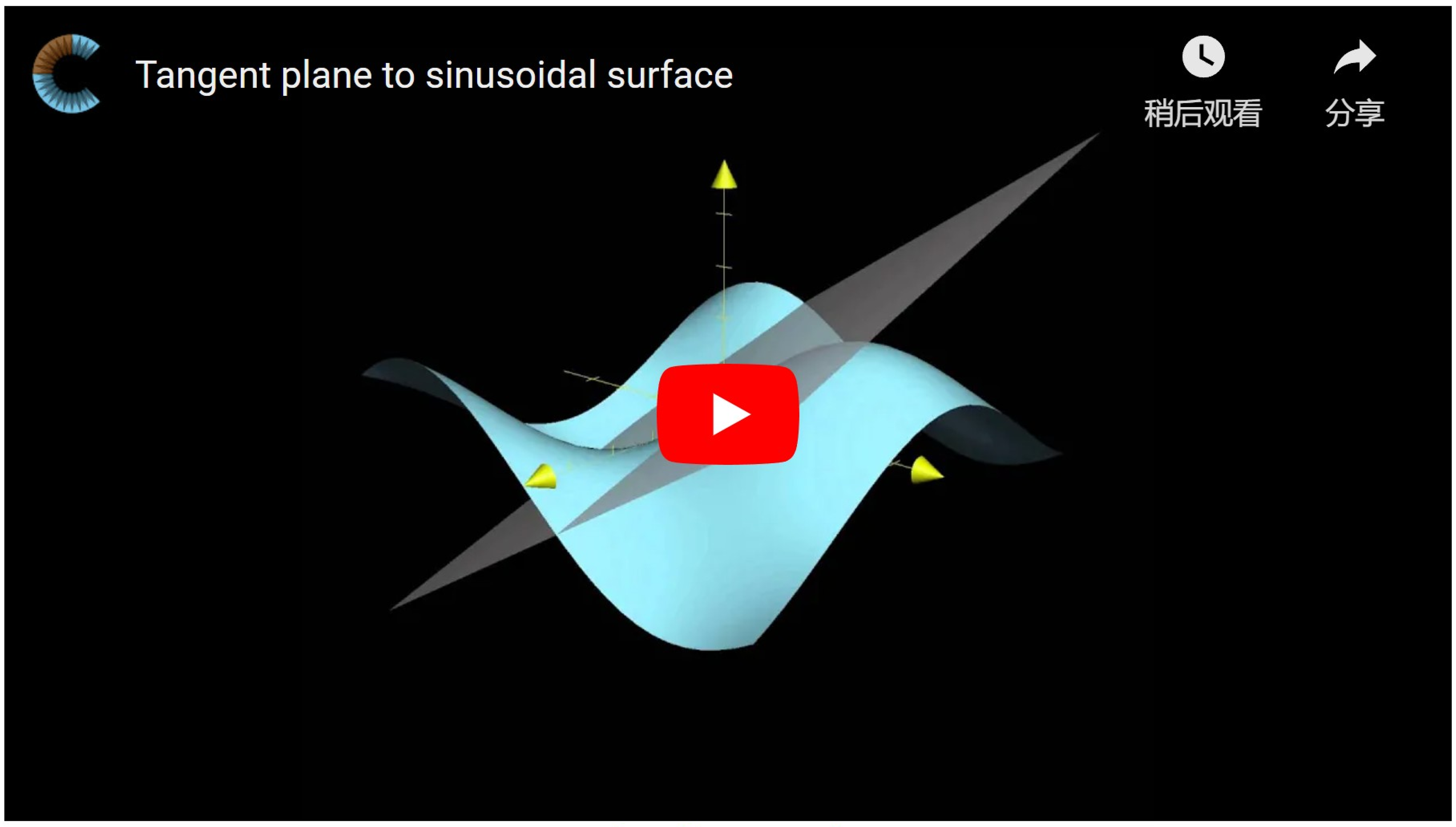
[Hide explanation]

> $$T_x(x,y) = \frac{\partial}{\partial x}(f(x_0,y_0) + \underbrace{a(x - x_0)}_{\text{Only part with } x} + b(y - y_0)) = a$$
>
> $$T_y(x,y) = \frac{\partial}{\partial y}(f(x_0,y_0) + a(x - x_0) + \underbrace{b(y - y_0)}_{\text{Only part with } y}) = b$$

Therefore setting $a = f_x(x_0, y_0)$ and $b = f_y(x_0, y_0)$ will guarantee that the partial derivatives of our linear function T match the partial derivatives of f. Well, at least they will match for the input (x_0, y_0), but that's the only point we care about. Putting this together, we get a usable formula for the tangent plane.

$$T(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y -$$

Example: Finding a tangent plane



See video transcript

Problem:

Given the function

$$f(x, y) = \sin(x)\cos(y),$$

find the equation for a plane tangent to the graph of f above the point $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

The tangent plane will have the form

$$T(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

Step 1: Find both partial derivatives of f.

$f_x(x, y) =$ [ ]

$f_y(x, y) =$ [ ]

Check

[Hide explanation]

$$f_x = \frac{\partial}{\partial x}(\sin(x)\cos(y)) = \cos(x)\cos(y)$$

$$f_y = \frac{\partial}{\partial y}(\sin(x)\cos(y)) = -\sin(x)\sin(y)$$

Step 2: Evaluate the function f as well as both these partial derivatives at the point $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$:

$f(\pi/6, \pi/4) =$ [ ]

$f_x(\pi/6, \pi/4) =$ [ ]

$f_y(\pi/6, \pi/4) =$ [ ]

Check

[Hide explanation]

$$f\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}$$
$$f_x\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4}$$
$$f_y\left(\frac{\pi}{6}, \frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) = -\frac{1}{2} \cdot \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{4}$$

Putting these three numbers into the general equation for a tangent plane, you can get the final answer

$T(x, y) =$

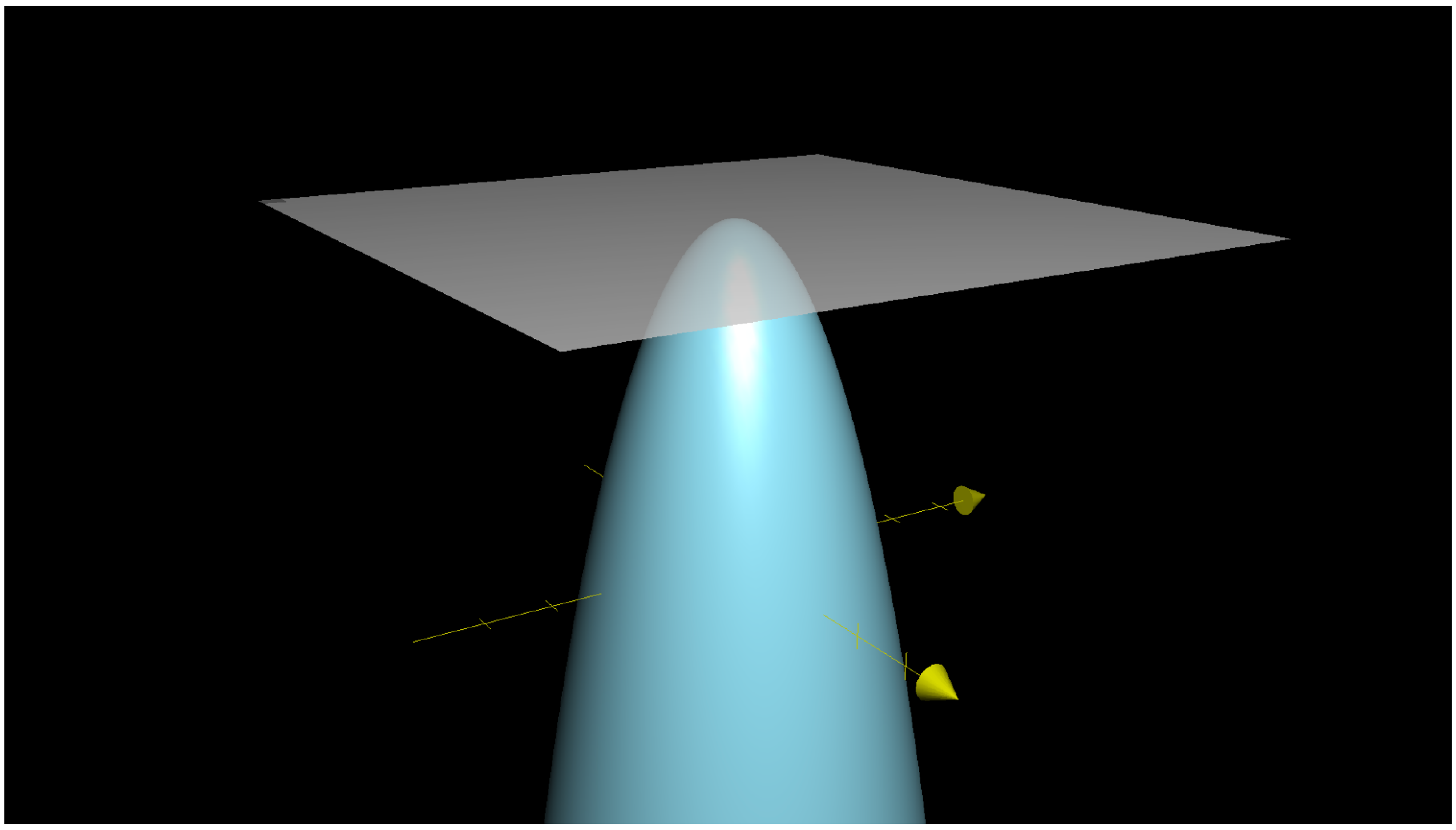
Check

[Hide explanation]

$$\boxed{T(x, y) = \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\left(x - \frac{\pi}{6}\right) - \frac{\sqrt{2}}{4}\left(y - \frac{\pi}{4}\right)}$$

Summary

- A **tangent plane** to a two-variable function $f(x, y)$ is, well, a plane that's tangent to its graph.



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