Green's theorem

Google Classroom

Green's theorem relates the double integral curl to a certain line integral. It's actually really beautiful.

Background

- Double integrals
- Line integrals in a vector field
- Two-dimensional curl

Not strictly required, but helpful for a fuller understanding:

- Formal definition of curl in two dimensions

Other resources

You can find examples of how Green's theorem is used to solve problems in the next article. Here, I will walk through what I find to be a beautiful line of reasoning for why it is true. You can find a different perspective in Sal's video on the topic.

One lesson, four times the payoff

Green's theorem is one of four major theorems at the culmination of multivariable calculus:

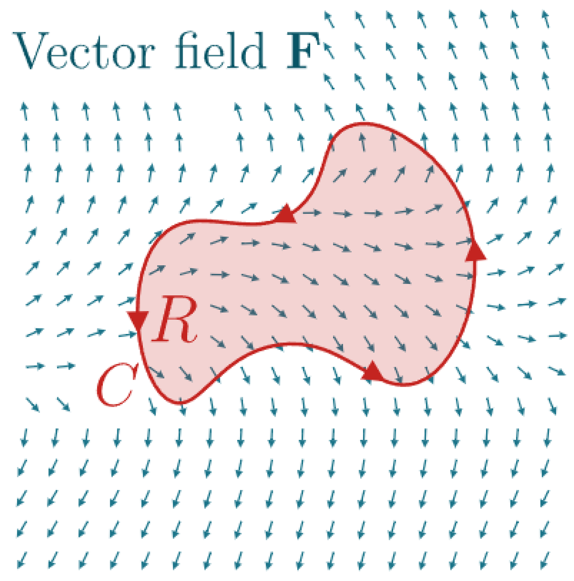
- **Green's theorem**
- 2D divergence theorem
- Stokes' theorem
- 3D Divergence theorem

Here's the good news: All four of these have very similar intuitions. So if you really get to the point where you feel Green's theorem in your bones, you're already most of the way there to understanding these other three!

What we're building to

- Setup:

- $\mathbf{F}$ is a two-dimensional vector field.
    - R is some region in the xy-plane.
    - C is the boundary of that region, oriented **counterclockwise**.



- Green's theorem states that the line integral of $\mathbf{F}$ around the boundary of R is the same as the double integral of the *curl* of $\mathbf{F}$ within R:

$$\iint_R \text{2d-curl}\, \mathbf{F}\, dA = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

- You think of the left-hand side as adding up all the little bits of rotation at every point within a region R, and the right-hand side as measuring the total fluid rotation around the boundary C of R.

- Often times $\mathbf{F}$ is written component-wise as follows:

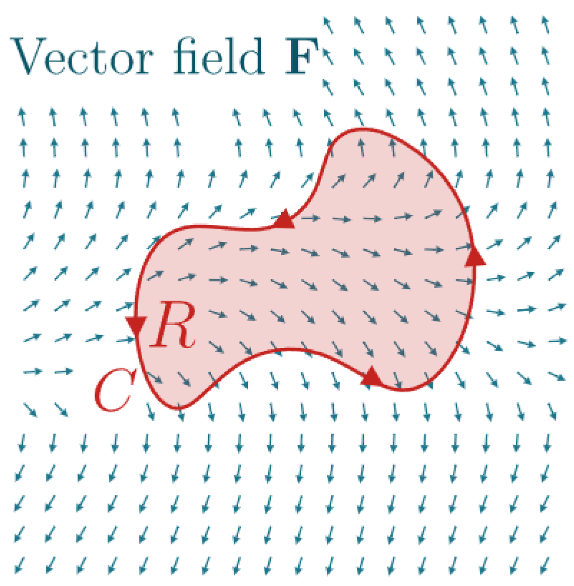
$$\mathbf{F}(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}}$$

In terms of P and Q, here's what Green's theorem looks like:

$$\oint_C P\, dx + Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Fluid rotation around a boundary

As you read on, the picture to have in your head is a blob in a vector field.



- $\mathbf{F}(x, y)$ is the function for the vector field. And, as you're probably getting used to if you've read other articles like this involving vector fields, imagine that $\mathbf{F}$ represents a fluid flow.

- R is some region in the xy-plane. In practice, and in problems, it will be some well-defined shape like a circle or the boundary between two graphs, but while thinking abstractly I like to just draw it as a blob.

- $\color{red}{C}$ is the boundary of $\color{red}{R}$, oriented **counterclockwise**. Remember that orientation, because it actually matters when you solve problems. Counterclockwise. Are you remembering it? Counterclockwise.

Concept check: How can you interpret the following line integral in terms of a fluid flow?

$$\oint_{C} \mathbf{F} \cdot d\mathbf{r}$$

(Remember, in a line integral through a vector field, the term $d\mathbf{r}$ represents a tiny step along the curve, as a vector, which in this case will always point in the counterclockwise direction.)

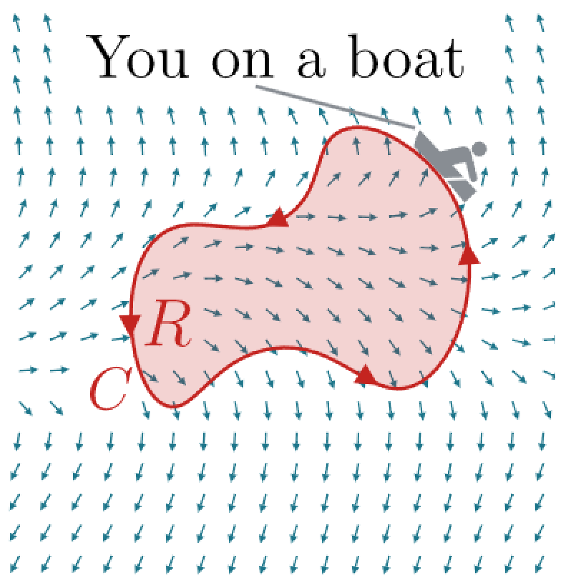
Choose 1 answer:

(A) It will be positive if the fluid has an overall counterclockwise rotation around the boundary of R, and negative if that overall rotation is clockwise.

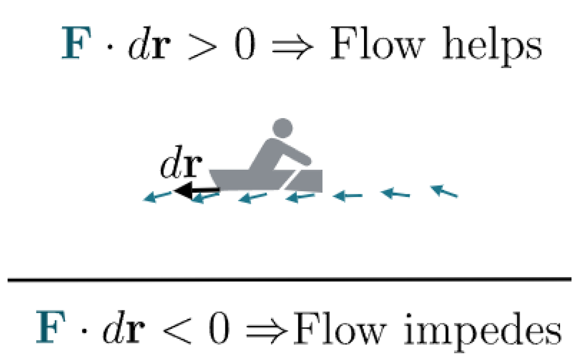
(B) It will be positive if the fluid tends to flow out of the region R, and positive if the fluid tends to flow into R.

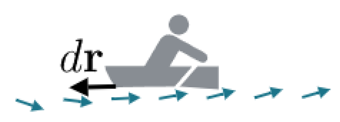
Check

Here's one way to think about the line integral $\oint_{C} \mathbf{F} \cdot d\mathbf{r}$: Image rowing a boat around the line C, counterclockwise.



At each point of your journey, the vector $d\mathbf{r}$ gives the direction of your motion. The dot product $\mathbf{F} \cdot d\mathbf{r}$ will be positive at points where the fluid flow is with you, and negative at points where it's against you.





On the whole, the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ adds up all these dot products to tell you if the flow was generally helpful or burdensome.

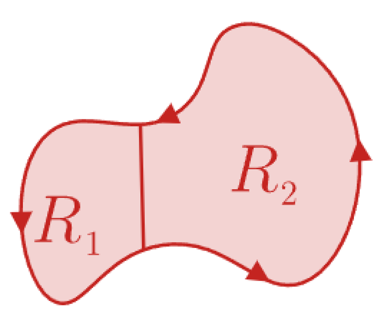
So this line integral is positive when the fluid flow has a general counterclockwise tendency around the boundary C (meaning it was generally helpful), and it will be negative if it has a clockwise tendency (generally burdensome).

Bringing the boundary to the interior

Green's theorem is all about taking this idea of fluid rotation around the boundary of R, and relating it to what goes on inside R. Conceptually, this will involve chopping up R into many small pieces. In formulas, the end result will be taking the double integral of $2\text{d-curl}\,\mathbf{F}$.

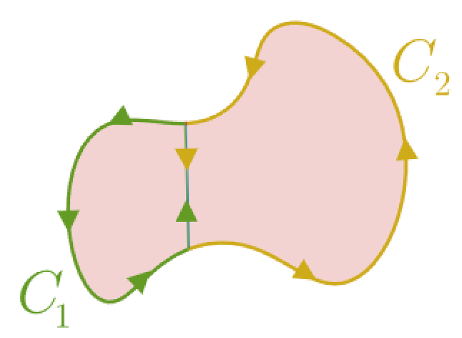
Cut the region

Imagine chopping up the region R with a line straight down the middle, giving two subregions R_1 and R_2:



Name the boundaries of these two regions C_1 and C_2. What happens if we take the line integral of $\mathbf{F}$ around these two boundaries, and add them up?

$$\oint_{C_1} F \cdot d\mathbf{r} + \oint_{C_2} F \cdot d\mathbf{r}$$



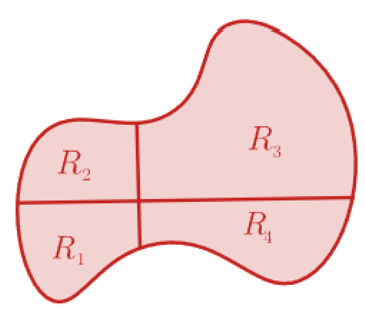
Notice, these line integrals will cancel out along the vertical line cut that you made. Namely, the integral around C_1 goes "up" this line, while the integral around C_2 integrates "down" this line. (Remember, when performing a line integral in a vector field, changing the direction along a curve multiplies your result by -1).

This means the sum of our two integrals is the same as just going around the full boundary C.

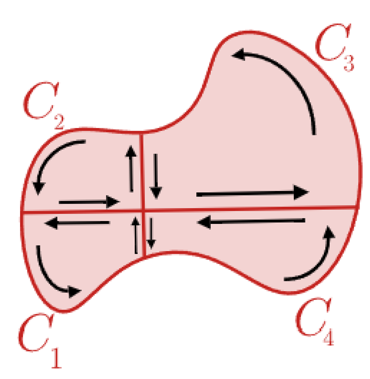
$$\oint_{C_1} F \cdot d\mathbf{r} + \oint_{C_2} F \cdot d\mathbf{r} = \oint_{C} F \cdot d\mathbf{r}$$

Cut it again

You could do this one more time, maybe with a horizontal cut this time:



If you integrate around the boundaries of the resulting four subregions, the integrals will all cancel out along the cuts you made in the interior of R:



In a formula, this means the sum of the line integrals around all four subregions end up just equalling the line integral around the full region:

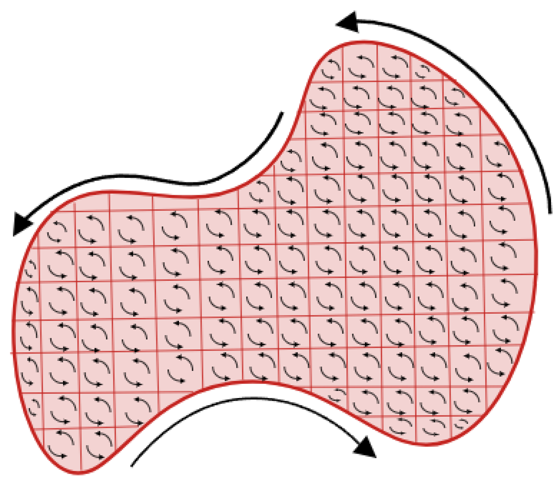
$$\oint_{C_1} F \cdot d\mathbf{r} + \oint_{C_2} F \cdot d\mathbf{r} + \oint_{C_3} F \cdot d\mathbf{r} + \oint_{C_4} F \cdot d\mathbf{r} = \oint_{C} F \cdot d\mathbf{r}$$

I should emphasize that this only works if we make sure that all the boundaries $C_1, \dots, C_4$ are oriented the same way. Otherwise, they might not cancel each other out along the cuts. It's common to think of counterclockwise as being the positive direction, so think about everything as being oriented counterclockwise.

Cut it many, many times

You might be able to see where I'm going with this. Imagine chopping of the region R into many many tiny pieces, $R_1, \dots, R_n$. Orient all of their boundaries

$C_1, \ldots, C_n$ counterclockwise, and integrate the function $\mathbf{F}$ over each one.



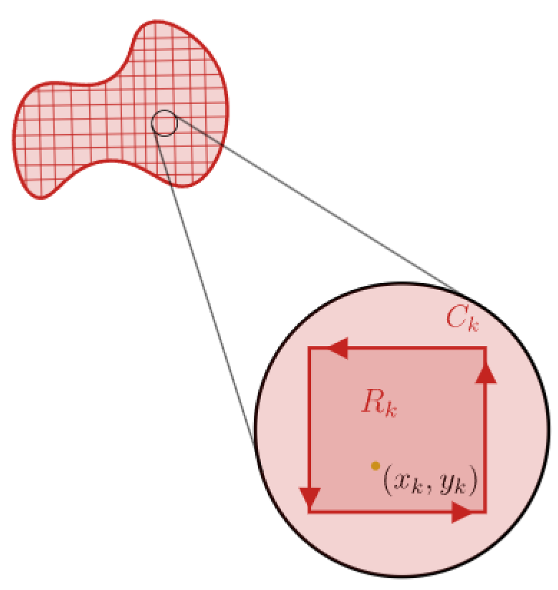
The integrals will cancel out along all the cuts inside R itself. This is because for any cut, one of the integrals will go along it in one direction, while another goes along it in the other direction. In the end, the only parts where these integrals don't cancel are the pieces of the boundary C.

This means adding up the line integrals along the tiny boundaries of the pieces will give the same result as just integrating across the full region:

$$\sum_{k=1}^{n} \left(\oint_{C_k} F \cdot d\mathbf{r} \right) = \oint_{C} F \cdot d\mathbf{r}$$

Integrating curl

So... why am I doing this? It's because there is another way to interpret each of these line integrals around a tiny piece using two-dimensional curl. Pick one of those pieces and zoom in on it.



- Let R_k be the piece you chose, with boundary C_k.
- Let $|R_k|$ represent the area of R_k, which we are thinking of as being some very small number.
- Let (x_k, y_k) be some point sitting inside this piece, any point really.

The fluid rotation around this piece due to $\mathbf{F}$ can be measured with the line integral $\oint_{C_k} \mathbf{F} \cdot d\mathbf{r}$. Think tiny row boat. But since this is a really small piece, there is another multivariable calculus concept that measures fluid rotation: Curl.

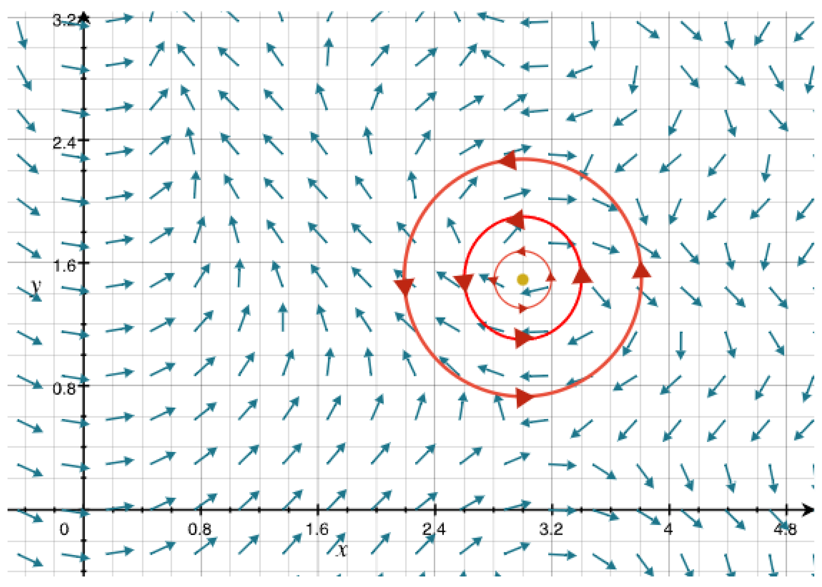
This line integral can be approximated by taking the 2d-curl of $\mathbf{F}$ at any point within R_k, and multiplying it by the (tiny) area $|R_k|$:

$$\underbrace{\oint_{C_k} \mathbf{F} \cdot d\mathbf{r}}_{\substack{\text{Integral around a} \\ \text{tiny piece } R_k}} \approx \left(\text{2d-curl}\, \mathbf{F} \underbrace{(x_k, y_k)}_{\text{Point in } R_k} \right) \underbrace{|R_k|}_{\text{Area of } R_k}$$

Also, and this is important, the smaller R_k is, the better this approximation is.

[Hide explanation]

For those of you who read the article on the formal definition of curl in two dimensions, this approximation might feel very familiar. In fact, it's very close to the formal definition of curl itself, where you imagine shrinking a region around a point:



$$\text{2d-curl}\, \mathbf{F}(x, y) = \lim_{|R_{(x,y)}| \to 0} \left(\frac{1}{|R_{(x,y)}|} \oint_{C_{(x,y)}} \mathbf{F} \cdot d\mathbf{r} \right)$$

Here, I'm using $R_{(x,y)}$ to represent any region containing the point (x, y).

The meaning of the limit as $|R_{(x,y)}| \to 0$ is that the smaller the area of the region, the closer the following approximation will be:

$$\text{2d-curl}\, \mathbf{F}(x, y) \approx \frac{1}{|R_{(x,y)}|} \oint_{C_{(x,y)}} \mathbf{F} \cdot d\mathbf{r}$$

This is essentially what the approximation written above says, except rather than starting with a point (x, y) and considering a region $R_{(x,y)}$ around that point, I started with a tiny region R_k and took some point (x_k, y_k) inside of it:

$$\left(\text{2d-curl}\, \mathbf{F} \underbrace{(x_k, y_k)}_{\text{Point in } R_k} \right) \approx \frac{1}{|R_k|} \underbrace{\oint_{C_k} \mathbf{F} \cdot d\mathbf{r}}_{\substack{\text{Integral around a} \\ \text{tiny piece } R_k}}$$

Now just multiply both sides by $|R_k|$, and you get the approximation I originally claimed:

$$\underbrace{\oint_{C_k} \mathbf{F} \cdot d\mathbf{r}}_{\substack{\text{Integral around a} \\ \text{tiny piece } R_k}} \approx \left(\text{2d-curl}\, \mathbf{F} \underbrace{(x_k, y_k)}_{\text{Point in } R_k} \right) \underbrace{|R_k|}_{\text{Area of } R_k}$$

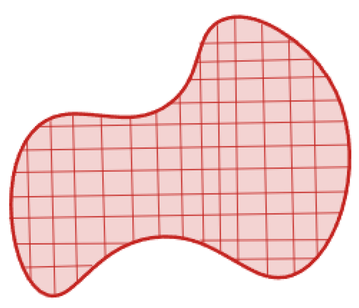
Moreover, if you *really* want to wrap you mind around the technical details here, you can make the following observation: We started with a certain approximation, whose error goes to zero for smaller pieces. Since our final approximation came from multiplying both sides by the area $|R_k|$, its error not only goes to zero, but it must remain significantly smaller than $|R_k|$ as it does so. It's okay if you want to ignore this fact, but it becomes important for a technical detail down the road.

Adding up these approximations over all of the tiny pieces R_k, here's what you get:

$$\sum_{k=1}^{n} \left(\oint_{C_k} \mathbf{F} \cdot d\mathbf{r} \right) \approx \sum_{k=1}^{n} \left(\text{2d-curl}\, \mathbf{F} \underbrace{(x_k, y_k)}_{\text{Point in } R_k} |R_k| \right)$$

Taking the conclusion from the previous section, the left-hand side above is the same as a single line integral around the full boundary of R, so we can rewrite this approximation as follows:

$$\oint_{C} \mathbf{F} \cdot d\mathbf{r} \approx \sum_{k=1}^{n} \left(\text{2d-curl}\, \mathbf{F} \underbrace{(x_k, y_k)}_{\text{Point in } R_k} |R_k| \right)$$



Now take a close look at the sum on the right-hand side.

- It includes a scalar-valued function, $\text{2d-curl}\, \mathbf{F}$
- The sum is taken over many tiny pieces R_k of a two-dimensional region, R.

- For each piece within the sum, the function is evaluated on a point inside that piece, then multiplied by its area.

Sound familiar? This is all the recipe for a double integral! (If this does not sound familiar, consider taking a look at this article on double integrals).

In particular, if you imagine chopping up the region R more and more finely, you can replace the sum above with a double integral of $\text{2d-curl}\,\mathbf{F}$ over R:

$$\sum_{k=1}^{n}\left(\text{2d-curl}\,\mathbf{F}\underbrace{(x_k, y_k)}_{\text{Point in } R_k}|R_k|\right) \rightarrow \iint_R \text{2d-curl}\,\mathbf{F}\,dA$$

Putting everything together, here's what we have:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \sum_{k=1}^{n}\left(\oint_{C_k}\mathbf{F}\cdot d\mathbf{r}\right)$$

$$\approx \sum_{k=1}^{n}\left(\text{2d-curl}\,\mathbf{F}\underbrace{(x_k, y_k)}_{\text{Point in } R_k}|R_k|\right)$$

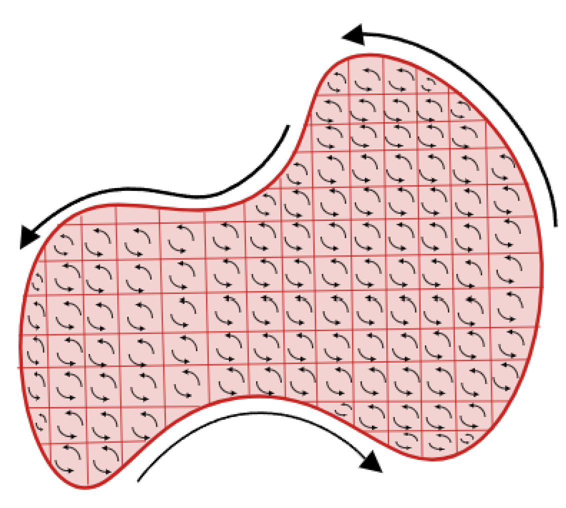
$$\rightarrow \iint_R \text{2d-curl}\,\mathbf{F}\,dA$$

This is actually more than a mere approximation, the line integral around the boundary *equals* the double integral of two-dimensional curl:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \text{2d-curl}\,\mathbf{F}\,dA$$

[Wait, how did that approximation turn into an equality?]

This marvelous fact is called **Green's theorem**. When you look at it, you can read it as saying that the rotation of a fluid around the full boundary of a region (the left-hand side) is the same as looking at all the little "bits of rotation" inside the region and adding them up (the right-hand side).



Alternative notation

It is very common to see Green's theorem written like this:

$$\oint_{C} P\,dx + Q\,dy = \iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

This is just spelling out the dot product in the left-hand side line integral, as well as the curl in the right-hand side double integral. For whatever reason, it is common to use the letters P and Q to denote the components of the vector-valued function $\mathbf{F}(x, y)$:

$$\mathbf{F}(x, y) = P(x, y)\hat{\mathbf{i}} + Q(x, y)\hat{\mathbf{j}} = \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$$

- [Hide explanation]

> In the line integral $\oint_{C} \mathbf{F} \cdot d\mathbf{r}$, the differential term $d\mathbf{r}$ is a vector representing a small step along the curve C. You can break it down as the x-component of that small step, dx, and the y-component of that step, dy:
>
> $$d\mathbf{r} = \begin{bmatrix} dx \\ dy \end{bmatrix}$$
>
> This means the dot product within the integral expands as follows:
>
> $$\oint_{C} \mathbf{F} \cdot d\mathbf{r} = \oint_{C} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix} = \oint_{C} P\,dx + Q\,dy$$

- [Hide explanation]

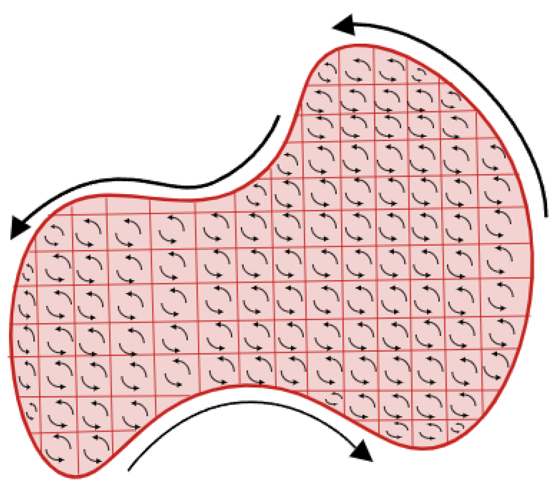
> The formula for $\text{2d-curl}\,\mathbf{F}$ is as follows:
>
> $$\text{2d-curl}\left(\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}\right) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$
>
> A memory trick for this is to think about the following determinant:
>
> $$\det\left(\begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ \\ P(x,y) & Q(x,y) \end{bmatrix}\right)$$
>
> If you are rusty on this, or wish to see a bit of intuition for why this formula relates to fluid rotation, see [this article on two-dimensional curl](#).

In the [next article](#), you will find examples of how this formula can be used to make either line integrals or double integrals simpler.

Summary

- You can think about the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ as measuring the rotation of the fluid flow represented by the vector field $\mathbf{F}(x, y)$ around the curve C. It is conventional to think of counterclockwise rotation as being positive, in which case C should be oriented counterclockwise.

- Imagine chopping up the two-dimensional region R enclosed by C into many tiny pieces. Name the boundaries of these pieces $C_1, \dots, C_n$, and orient them all counterclockwise. Then adding up the line integrals of $\mathbf{F}$ around each piece-boundary C_k boils down to the same thing as the line integral around the full boundary C.



$$\sum_{k=1}^{n}\left(\oint_{C_k} \mathbf{F} \cdot d\mathbf{r}\right) = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

That is to say, the little line integrals cancel out along all the cuts within R

- As you consider smaller and smaller pieces, the line integral around each tiny piece can be approximated using two-dimensional curl:

$$\underbrace{\oint_{C_k} \mathbf{F} \cdot d\mathbf{r}}_{\substack{\text{Integral around a} \\ \text{tiny piece } R_k}} \approx \left(\text{2d-curl}\, \mathbf{F} \underbrace{(x_k, y_k)}_{\text{Point in } R_k} \right) \underbrace{|R_k|}_{\text{Area of } R_k}$$

- Adding up these little "bits of curl" using a double integral over R, and applying the fact that the sum of the line integrals cancels out along interior cuts, you get **Green's theorem**:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{2d-curl}\, \mathbf{F} \, dA$$

Okay, there are some technicalities I should mention for the detail-oriented among you. Remember the step of adding up all the individual line-integral-to-curl approximations? That's what gave us this approximation:

$$\sum_{k=1}^{n} \left(\oint_{C_k} \mathbf{F} \cdot d\mathbf{r} \right) \approx \sum_{k=1}^{n} \left(\text{2d-curl}\, \mathbf{F} \underbrace{(x_k, y_k)}_{\text{Point in } R_k} |R_k| \right)$$

For all we know, without further investigation, the errors for each of these approximations add up to something substantial. This might mean that when we chop up R more and more finely, and the sum gets bigger and bigger, the accumulation of errors gets out of hand, even though each individual error-per-piece approaches 0.

Luckily, this does not happen. For those of you who read the end of the little technical note above justifying the original approximation using the formal definition of curl, do you remember what I said at the end? The error of that approximation is significantly smaller than the area of R_k. Since the sum of the areas of each piece R_k is the area of the full region R, a constant, this means the sum of the error terms will be significantly smaller than a constant. That is to say, they will be significantly small, period.

This means the overall approximation really does approach equality when we pass to the double integral, so we can feel justified in writing this equation:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{2d-curl}\, \mathbf{F}\, dA$$