Notation for integrating along a curve

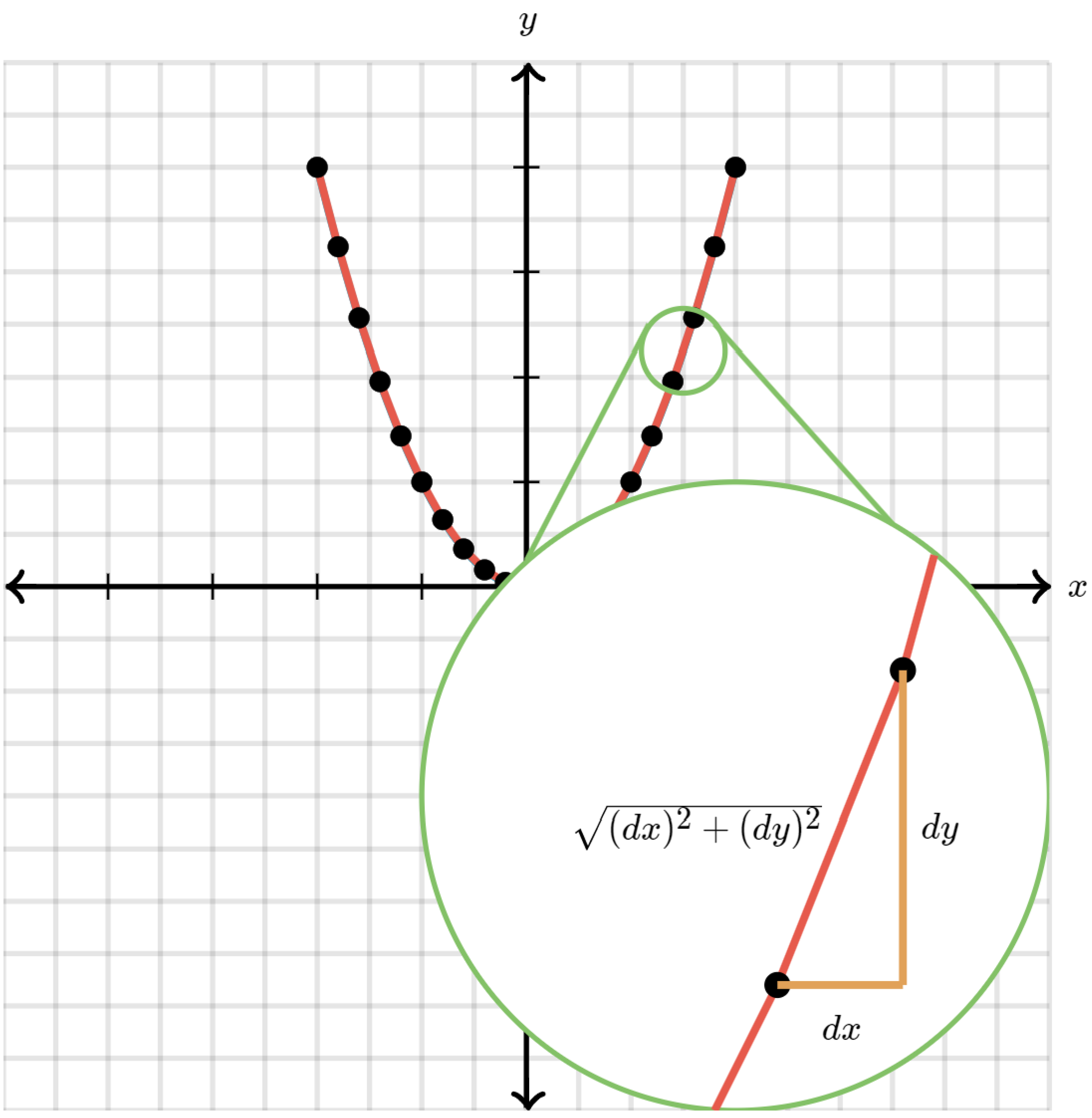
Google Classroom

There is a very compact way to express arc length integrals, which lays a foundation for writing line integrals.

Background:

- [Arc length of parametric curves]
- [Derivatives of vector valued function]

What we're building to



- The arc length integral

$$\int \sqrt{dx^2 + dy^2}$$

may alternatively be written as

$$\int_C ds$$

where C represents the curve, and ds is shorthand for $\sqrt{dx^2 + dy^2}$, representing the length of a tiny step along the curve.

- When the parametric curve is given by a vector-valued function $\vec{\mathbf{r}}(t)$ in the range $a \leq t \leq b$, the arc length integral looks like

$$\int_a^b |\vec{\mathbf{r}}'(t)|dt$$

In other words, the small step ds along the curve is the magnitude of the derivative of $\vec{\mathbf{r}}(t)$

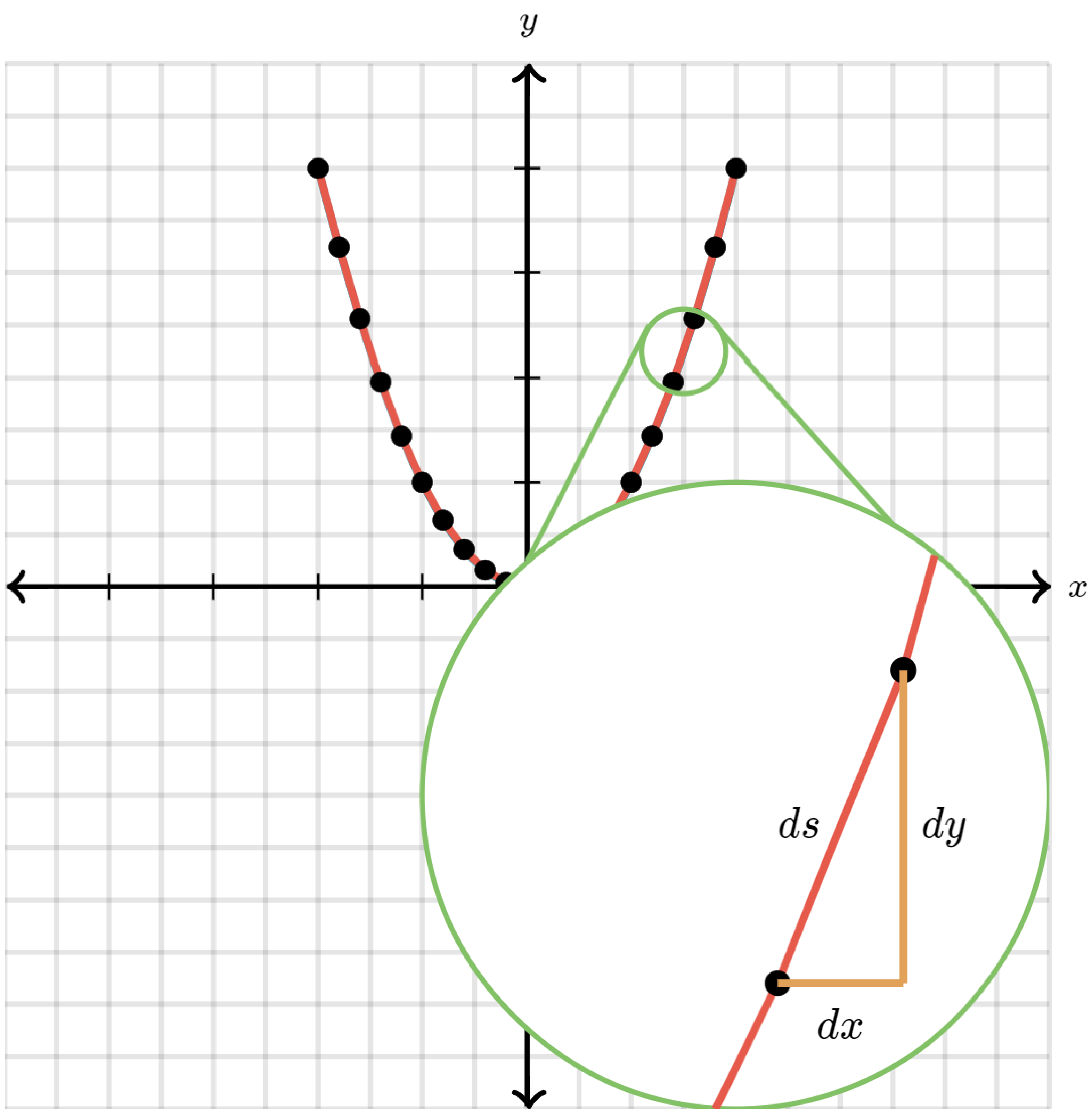
- This is the standard notation for line integrals, introduced in the next article.

Writing arc length compactly

When we talked about finding the arc length of function graphs and the arc length of parametric curves, we started by setting up an integral of the form

$$\int \sqrt{dx^2 + dy^2}$$

Instead of always writing $\sqrt{dx^2 + dy^2}$ to represent a tiny change in arc length, a common convention is to express this tiny change as ds.



You think of ds as a tiny step along whatever curve we're talking about, in the same way that dx is a tiny step in the x-direction or dy is a tiny step in the y-direction.

Settling bound awkwardness

Throughout the last few articles, we procrastinated putting bounds on the integral

$$\int \sqrt{(dx)^2 + (dy)^2}$$

(which we now know could be written simply as $\int ds$.)

If everything inside the integral was written in terms of x, the bounds will reflect x values. If it is all in terms of t, the bounds reflect t values, etc.

If you are uncomfortable with your integral looking so naked but you don't want to make a commitment about which variable owns the bounds, here's what you do. You say,

> "Let C be the curve defined by . . ."

and you go on defining your curve. Then you just write your integral with a little C at the bottom:

$$\int_C ds$$

This basically tells the person reading it to go find where the curve C is defined, then plug in the relevant boundary values when it comes time to compute.

On the one hand, this notation is so simple as to be nearly meaningless. You might read it out loud by saying

> "The arc length of C is the integral over C of tiny steps along C"

Silly, right? This entirely sweeps under the rug the details of what solving the arc-length problem entails, expanding ds and encoding the definition of C into the integral.

But, that's actually the point. Part of the reason for talking about arc length integrals is to set the stage for the broader idea of line integrals. When we get to line integrals, you don't always want the full details of the curve and the tiny change in arc length ds to spill out into your notation. There will be other things to deal with. In that context, abstracting the arc length away to something as simple as $\int_C ds$ will be a more-than-welcome simplification.

In the language of vector calculus

In vector calculus, we move away from thinking about a parametric curve as a set of parametric equations like

$$x(t) = t\cos(t)$$

$$y(t) = t\sin(t)$$

Instead, we think of these curves as the output of a single vector-valued function,

$$\vec{\mathbf{r}}(t) = \left[\begin{array}{c} x(t) \\ y(t) \end{array}\right]$$

The derivative of a vector-valued function like this gives another vector valued function,

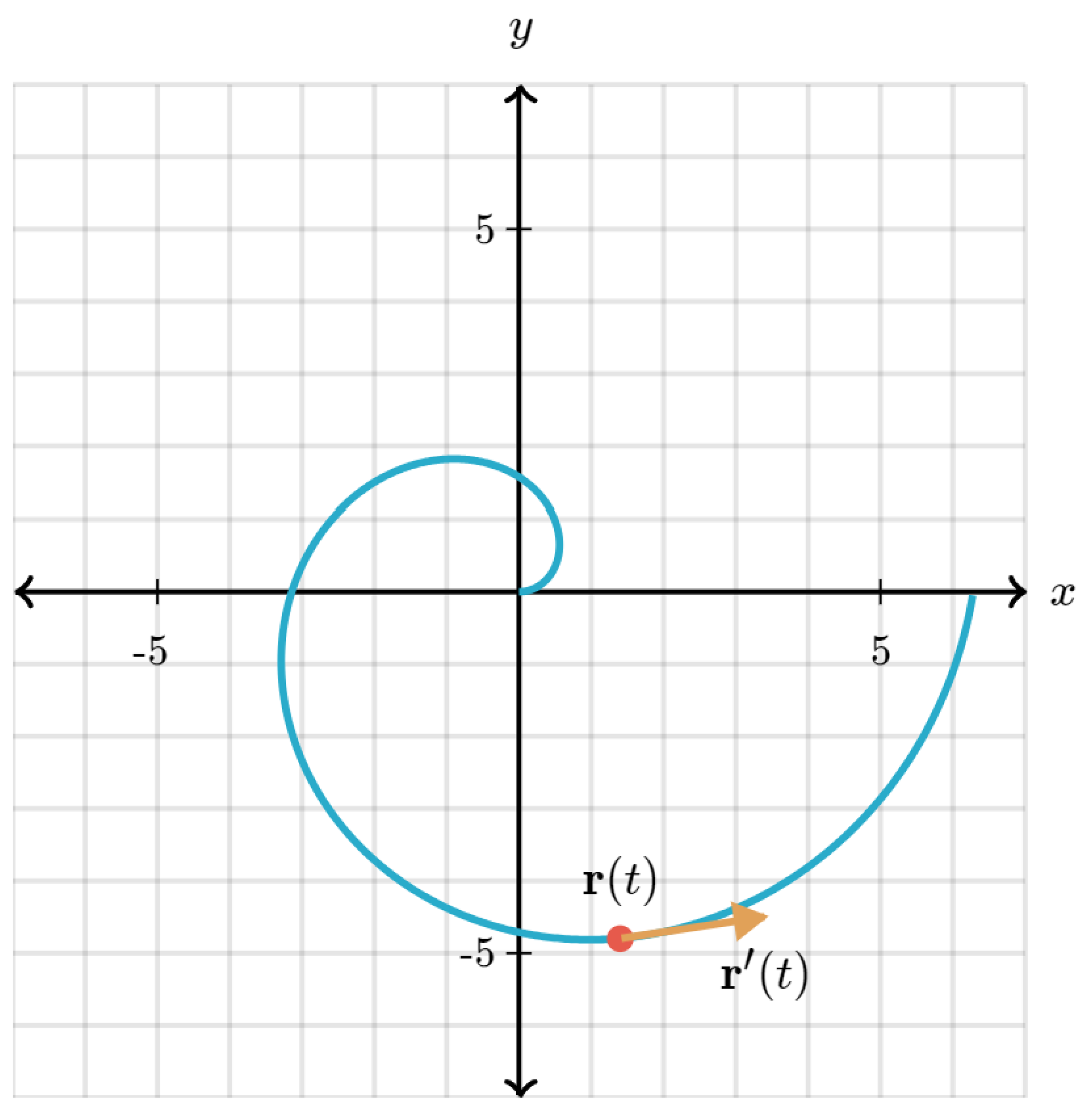
$$\vec{\mathbf{r}}'(t) = \left[\begin{array}{c} x'(t) \\ y'(t) \end{array}\right]$$

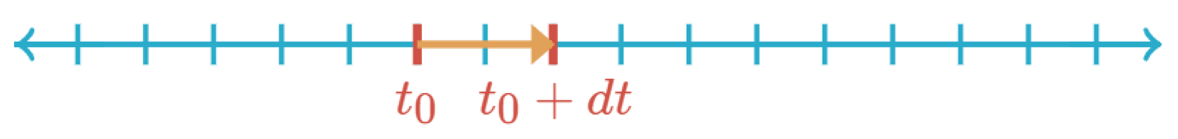
This gives us a very nice way to express ds, the length of a tiny step along the curve:

$$ds = |\vec{\mathbf{r}}'(t)|dt$$

Why is this true? One way is to expand out the expression $|\vec{\mathbf{r}}'(t)|dt$ and simplify. Try it! *[Hide explanation]*

$$\begin{aligned} |\vec{\mathbf{r}}'(t)|dt &= \left|\left[\begin{array}{c} x'(t) \\ y'(t) \end{array}\right]\right| dt \\ &= \sqrt{(x'(t))^2 + (y'(t))^2} dt \\ &= \sqrt{(x'(t))^2 dt^2 + (y'(t))^2 dt^2} \\ &= \sqrt{(x'(t)dt)^2 + (y'(t)dt)^2} \\ &= \sqrt{(dx)^2 + (dy)^2} \\ &= ds \end{aligned}$$





Alternatively, think about how we interpret vector-derivatives. Imagine standing on a value t_0 in the input space, also known as the parameter space, and getting a slight nudge of size dt, bringing you up to the point $t_0 + dt$.

The derivative vector $\vec{\mathbf{r}}'(t)$ is the resulting "nudge" in the output space along the curve. When we multiply that derivative by the tiny amount dt to get

$$\vec{\mathbf{r}}'(t)dt,$$

it's helpful to think about this as a tiny step along the curve.

Technically it's a tiny step *in the tangent direction*, which might be slightly off from the curve. However, as dt approaches 0, a step in the tangent direction and a step along the curve can be treated as the same thing.

The magnitude of this vector is the size of our small step along the curve, ds.

$$ds = |\vec{\mathbf{r}}'(t)dt| = |\vec{\mathbf{r}}'(t)|dt,$$

This means the arc length integral for a parametric curve defined by a function $\vec{\mathbf{r}}(t)$ between $t = a$ and $t = b$ could look like

$$\int_a^b |\vec{\mathbf{r}}'(t)|dt$$

Actually computing this will look no different from when we thought of these curves as a set of equations, since $|\vec{\mathbf{r}}'(t)|dt$ will always expand to look like $\sqrt{dx^2 + dy^2}$. However, people generally favor this notation. For one thing, it is compact, and for another, it generalizes well to higher dimensions.

Onward to line integrals!

Armed with this notation, and an understanding of how portrays tiny steps along a curve, you are now ready to learn about line integrals.