Arc length of parametric curves

Google Classroom

How to find the length of a parametric curve? This will lead to the idea of a line integral.

Background:

- Arc length of function graphs
- Parametric curves
- Derivatives of vector valued function

What we're building to

- To find the arc length of a curve, set up an integral of the form

$$\int \sqrt{(dx)^2 + (dy)^2}$$

- We now care about the case when the curve is defined parametrically, meaning x and y are defined as functions of some new variable t. To apply the arc length integral, first take the derivative of both these functions to get dx and dy in terms of dt.

$$dx = \frac{dx}{dt}dt$$

$$dy = \frac{dy}{dt}dt$$

Plug these expressions into the integral and factor the dt^2 term out of the radical.

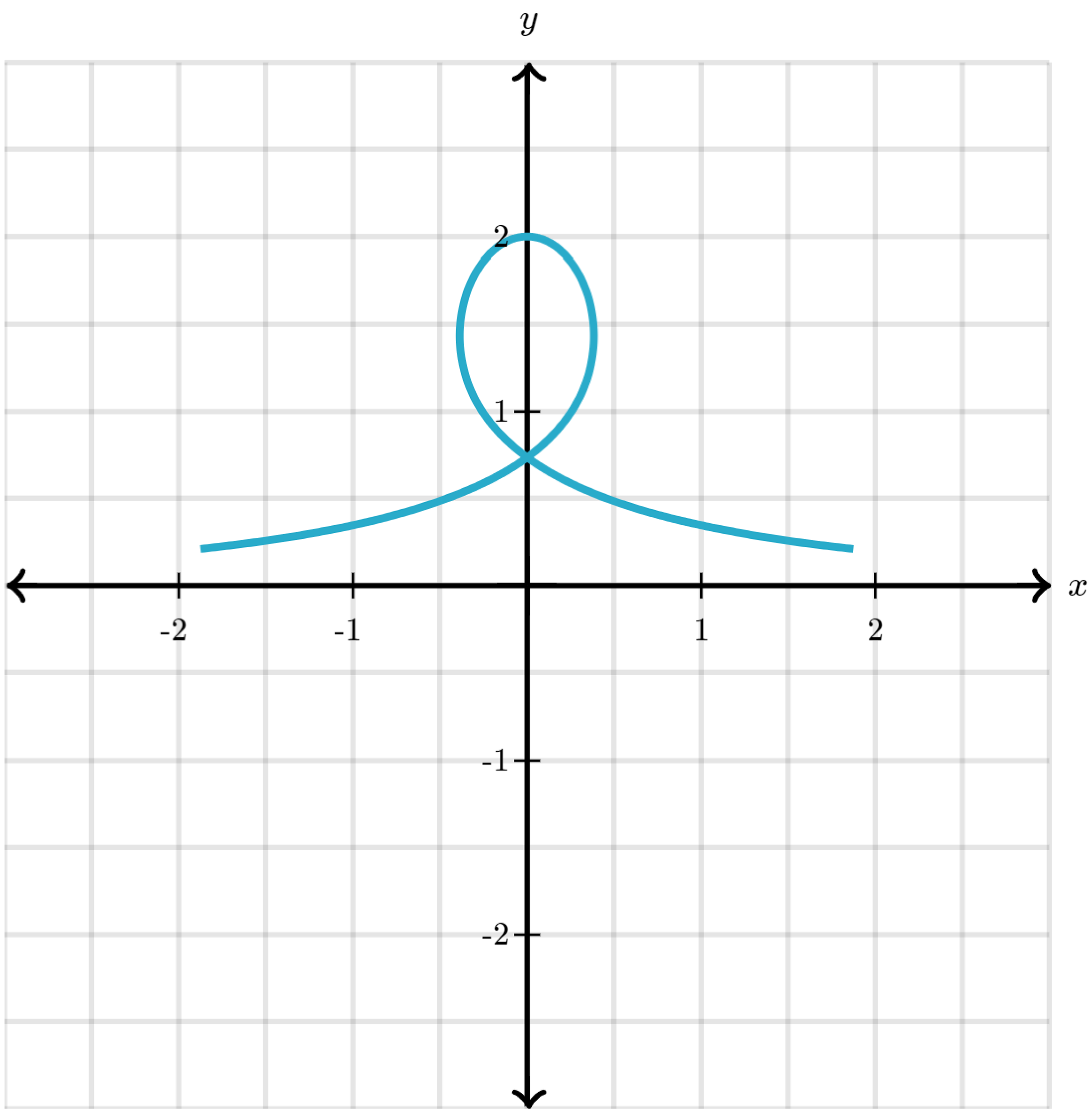
The length of a parametric curve

Consider the parametric curve defined by the following set of equations:

$$x(t) = t^3 - t$$

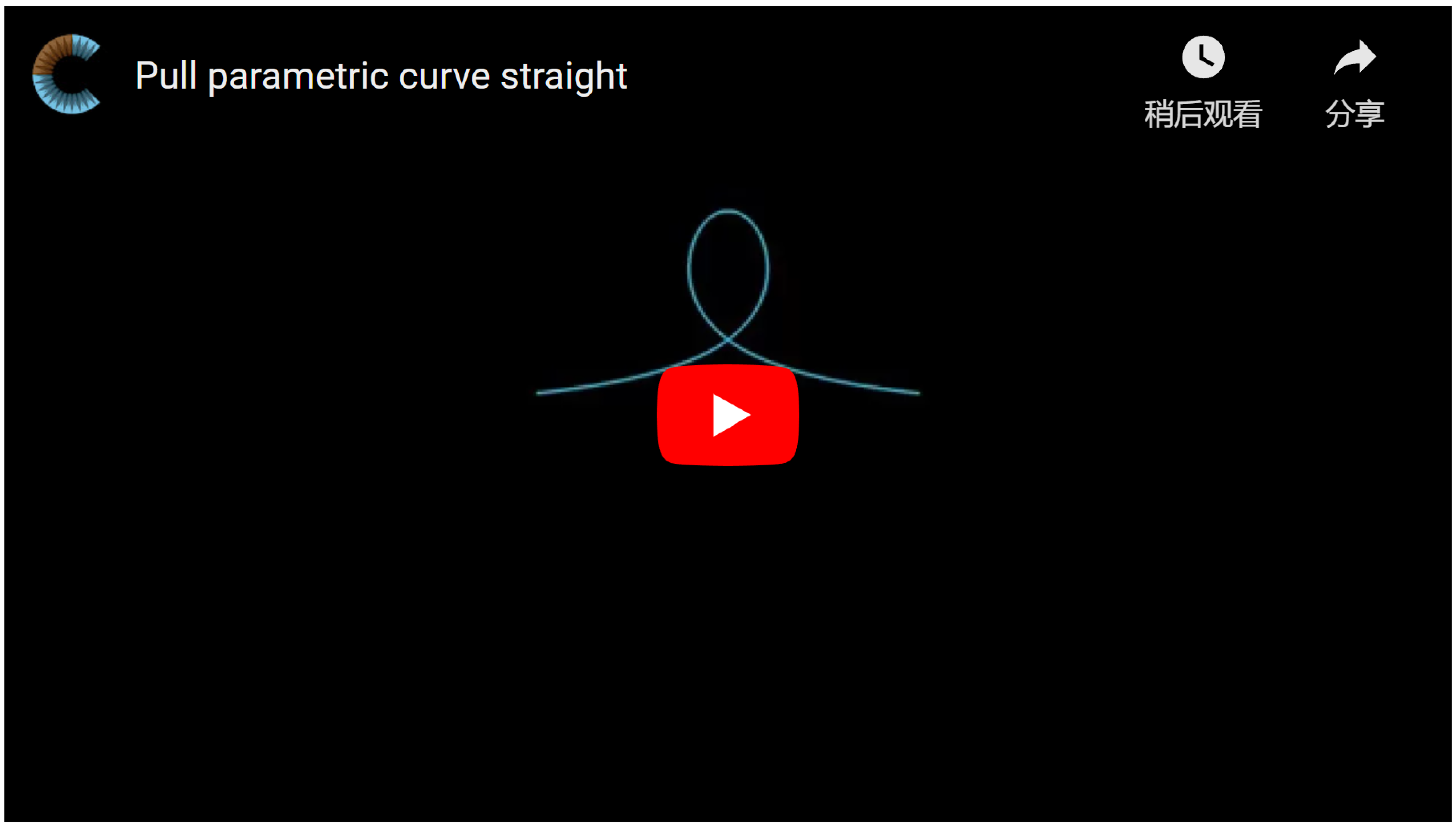
$$y(t) = 2e^{-t^2}$$

If we let t range from -1.5 to 1.5, the resulting curve looks like this:



Key question: What is the length of this curve?

That is, imagine pulling the line straight, as if you were tightening a loose piece of string, then measuring it with a ruler. What value would you get?

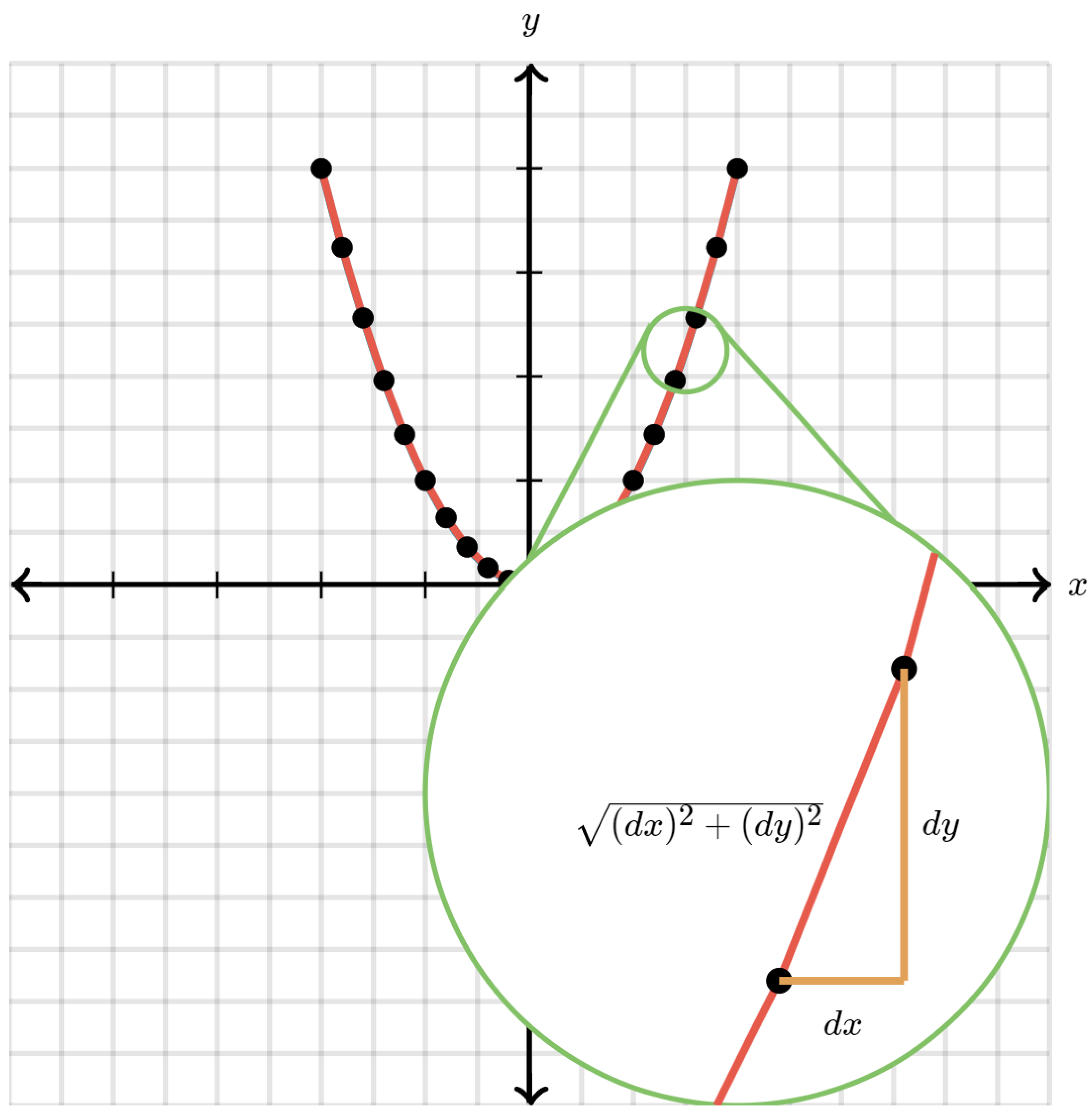


See video transcript

In the last article, we saw how to find the arc length of *function graphs*, not parametric curves. We started by writing down the following integral:

$$\int \sqrt{(dx)^2 + (dy)^2}$$

Let's quickly recap the meaning behind this integral.



- Imagine approximating the curve with a bunch of tiny straight lines.

- The length of each such tiny line is given using the Pythagorean theorem,

  $$\sqrt{dx^2 + dy^2}$$

  dx and dy represent the tiny change in x and y values from the start to the end of the line.

This same integral can apply to *parametric* curves as well as function graphs. This time, since x and y are given as functions of t, we write dx and dy in terms of dt by taking the derivative of these two functions.

For example, differentiating the function defining x, we get

$$x = t^3 - t$$

$$d(x) = d(t^3 - t)$$

$$dx = (3t^2 - 1)dt$$

And similarly with y:

$$y = 2e^{-t^2}$$

$$d(y) = d(2e^{-t^2})$$

$$dy = (2(-2t)e^{-t^2})\,dt$$

$$dy = -4te^{-t^2}\,dt$$

You can think of these expressions as answering the question "when you take some value t, and increase it slightly by some tiny amount dt, how much does it change x and y?" The answer is expressed in terms of t an dt.

Putting these into the integral, we get

$$\int \sqrt{(dx)^2 + (dy)^2} = \int \sqrt{((3t^2-1)dt)^2 + ((-4te^{-t^2})dt)^2}$$

$$= \int \sqrt{((3t^2-1)^2 + (-4te^{-t^2})^2)dt^2}$$

$$= \int \sqrt{9t^4 - 6t^2 + 1 + 16t^2e^{-2t^2}}\,dt$$

Now everything inside the integral is written in terms of t, so the bounds we place on the integral correspond with the starting and ending values of the parameter t. In this case, we are letting t range from -1.5 to 1.5, so we have

$$\int_{-1.5}^{1.5} \sqrt{9t^4 - 6t^2 + 1 + 16t^2e^{-2t^2}}\,dt$$

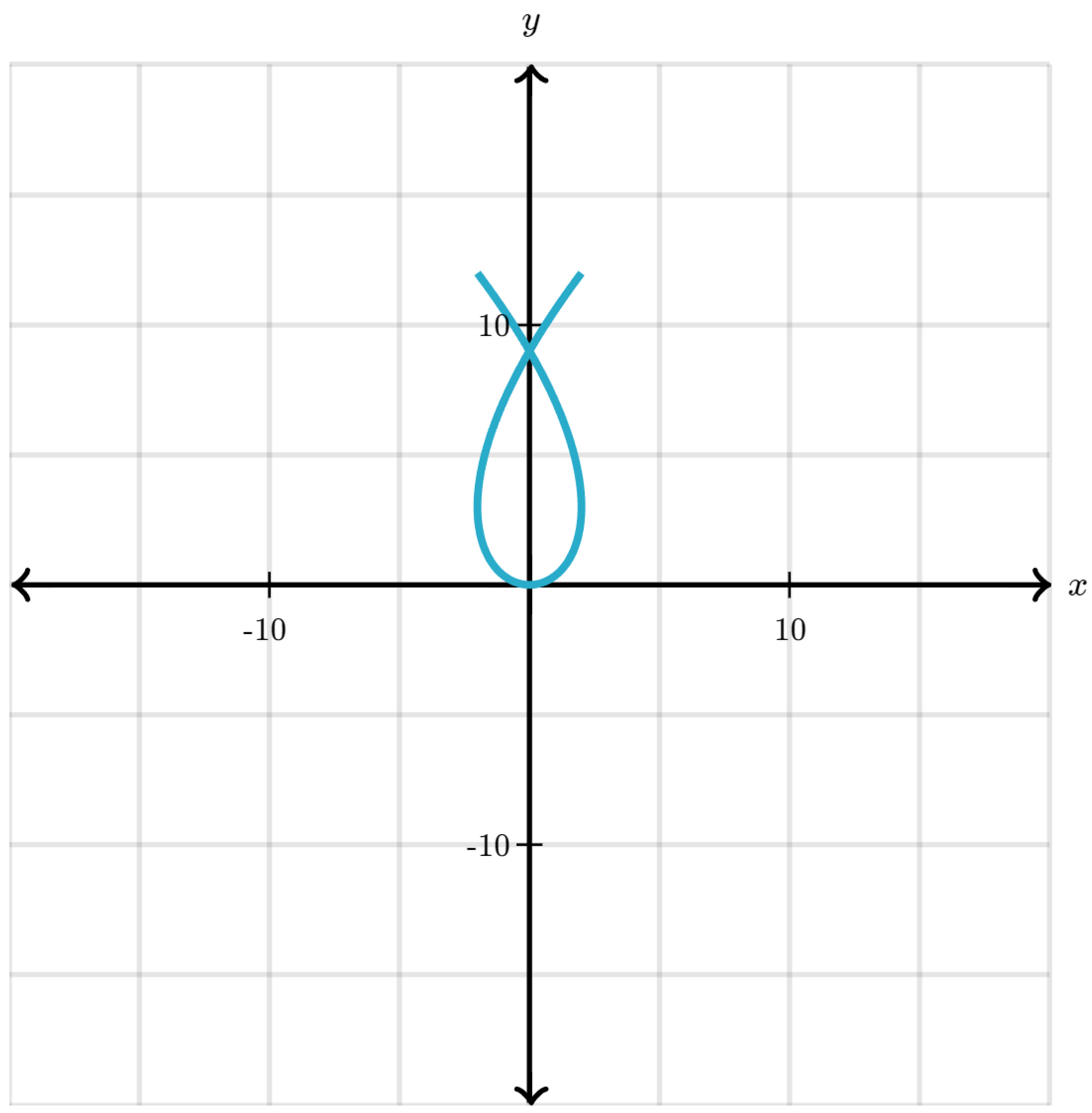
This is a very nasty integral to compute. I'm not even sure that an antiderivative exists. However, we've at least reduced the arc length problem down to a state where you can plug it into a computer.

Practice a parametric arc length integral

Let's look at the parametric curve defined by

$$x(t) = t^3 - 3t$$

$$y(t) = 3t^2$$



Consider the segment of this curve between the points where $t = -2$ and $t = 2$.

What is the length of this segment?

Since our curve is expressed in terms of x and y, our arc length integrals begin life looking like

$$\int \sqrt{dx^2 + dy^2}$$

To get this integral in terms of t, we must write dx and dy each as some expression of t

Step 1: Write dx and dy in terms of t

What is dx in terms of t?

$dx =$ [ ] dt

Check

[Hide explanation]

Take the derivative of each side of the expression defining x:

$$\begin{aligned} x &= t^3 - 3t \\ d(x) &= d(t^3 - 3t) \\ dx &= (3t^2 - 3)dt \end{aligned}$$

What is dy in terms of t?

$dy =$ [ ] dt

Check

[Hide explanation]

Take the derivative of each side of the expression defining y:

$$\begin{aligned} y &= 3t^2 \\ d(y) &= d(3t^2) \\ dy &= 6t\,dt \end{aligned}$$

Step 2: Put these expressions in the integral

What does our integral look like after we plug in these expressions for dx and dy? Simplify it down to the point where there is no radical.

$\int$ [ ] dt

Check

[Hide explanation]

$$\begin{aligned} \int \sqrt{dx^2 + dy^2} &= \int \sqrt{((3t^2 - 3)\,dt)^2 + (6t\,dt)^2} \\ &= \int \sqrt{3^2\left((t^2-1)^2 + (2t)^2\right)dt^2} \\ &= \int 3\sqrt{(t^2-1)^2 + (2t)^2}\;dt \\ &= \int 3\sqrt{t^4 - 2t^2 + 1 + 4t^2}\;dt \\ &= \int 3\sqrt{t^4 + 2t^2 + 1}\;dt \\ &= \int 3\sqrt{(t^2+1)^2}\;dt \\ &= \int 3(t^2+1)dt \end{aligned}$$

Step 3: Place the appropriate bounds on the integral and solve

The problem states that the curve runs from -2 to 2. Solve the integral with these bounds.

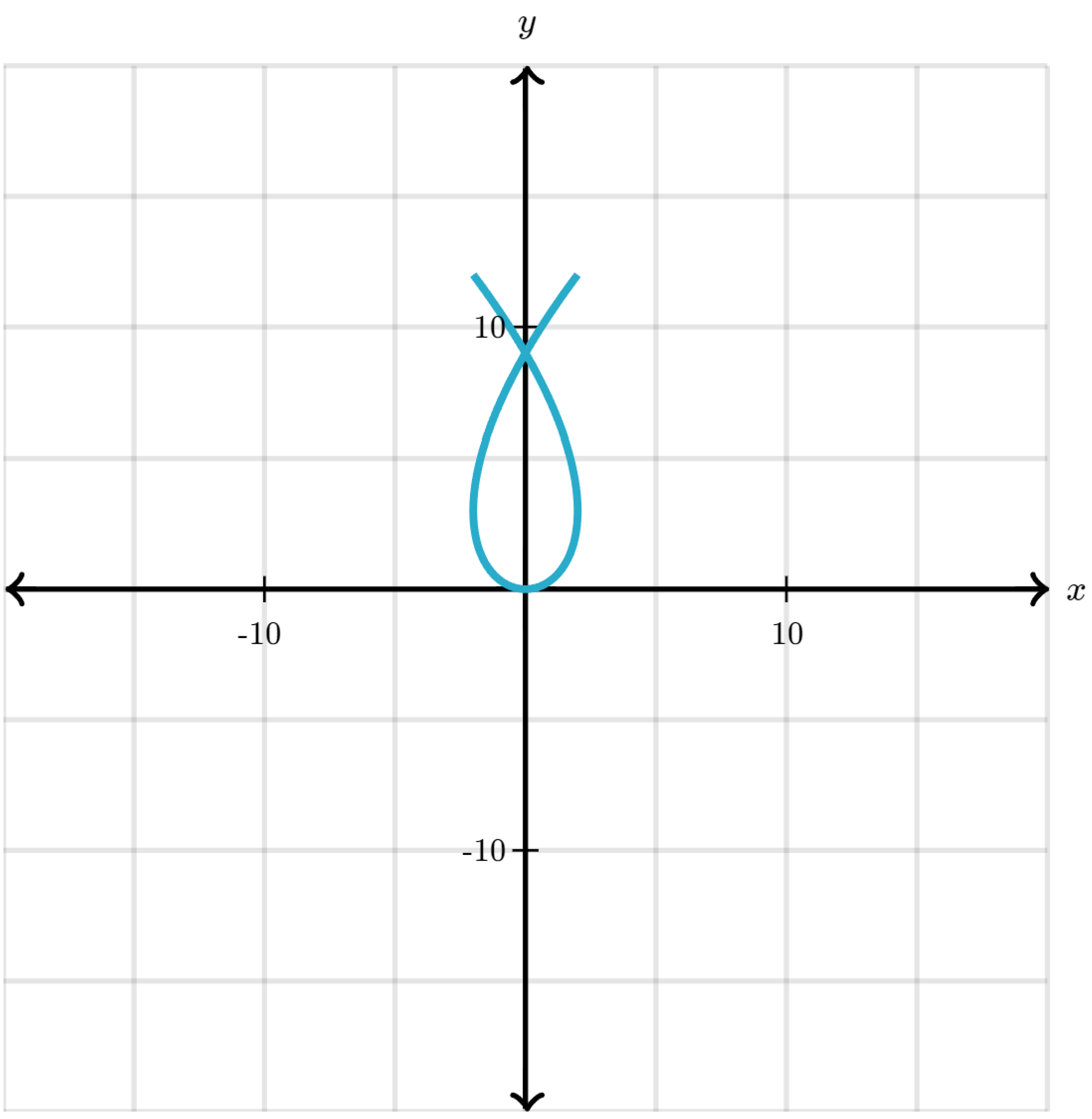
[ ]

Check

[Hide explanation]

$$
\begin{aligned}
\int_{-2}^{2}(3t^2+3)dt &= \left[t^3+3t\right]_{-2}^{2} \\
&= (2^3+3(2)) - ((-2)^3+3(-2)) \\
&= 14-(-14) \\
&= 28
\end{aligned}
$$

So evidently the length of this curve is 28. Looking at the picture, this seems about right. The curve starts from about $y \approx 12$, goes down to the x-axis and back, which takes at least 24 units of length. Since it has some curvature, wandering left and right as it goes down and up, the true length is a bit more than 24.



What's next?

Arc length of parametric curves is a natural starting place for learning about line integrals, a central notion in multivariable calculus. To keep things from getting too messy as we do so, I first need to go over some more compact notation for these arc length integrals, which you can find in the next article.

Summary

- To find the arc length of a curve, set up an integral of the form

$$\int \sqrt{(dx)^2 + (dy)^2}$$

- When the curve is defined parametrically, with x and y given as functions of t, take the derivative of both these functions to get dx and dy in terms of dt.

$$dx = \frac{dx}{dt}dt$$

$$dy = \frac{dy}{dt}dt$$

plug these expressions into the integral and factor the dt^2 term out of the radical.