Double integrals over non-rectangular regions

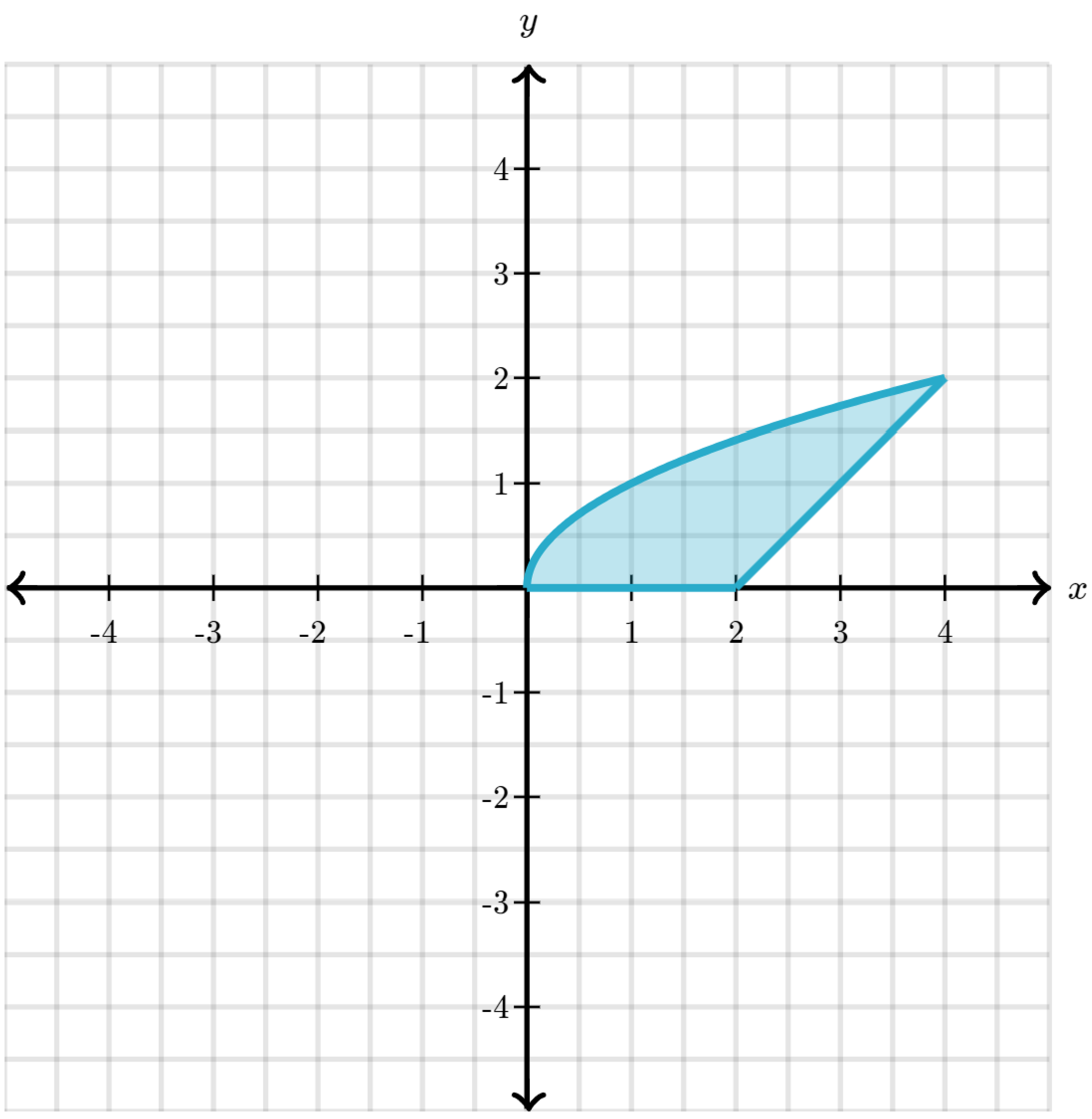
Google Classroom

What makes double integrals tricky is finding the bounds in non-rectangular regions. Here we go through what that means and practice a few examples.

Background

- Double integrals

What we're building to



Example of a non-rectangular region

- If you wish to perform an integral over a region of the xy-plane that is not rectangular, you have to express each of the bounds of the inner integral as a function of the outer variable.

$$\int_{y_1}^{y_2} \overbrace{\left(\int_{x_1(y)}^{x_2(y)} f(x, y)\, dx \right)}^{\text{Evaluates to some function of } y} dy$$

or alternatively,

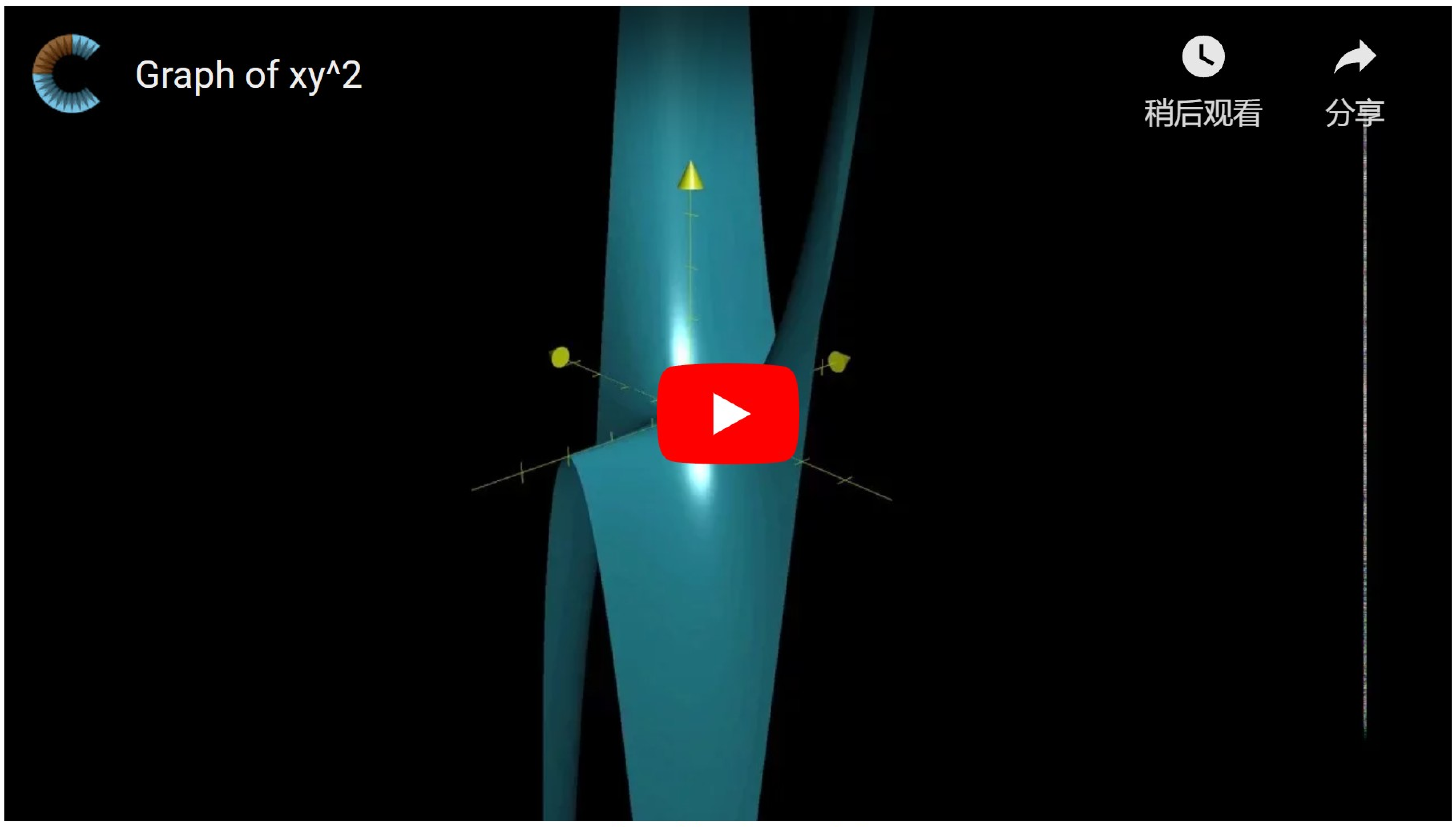
$$\int_{x_1}^{x_2} \overbrace{\left(\int_{y_1(x)}^{y_2(x)} f(x, y)\, dy \right)}^{\text{Evaluates to some function of } x} dx$$

The trouble with non-rectangular regions

Consider the function

$$f(x, y) = xy^2$$

Its graph looks like this:



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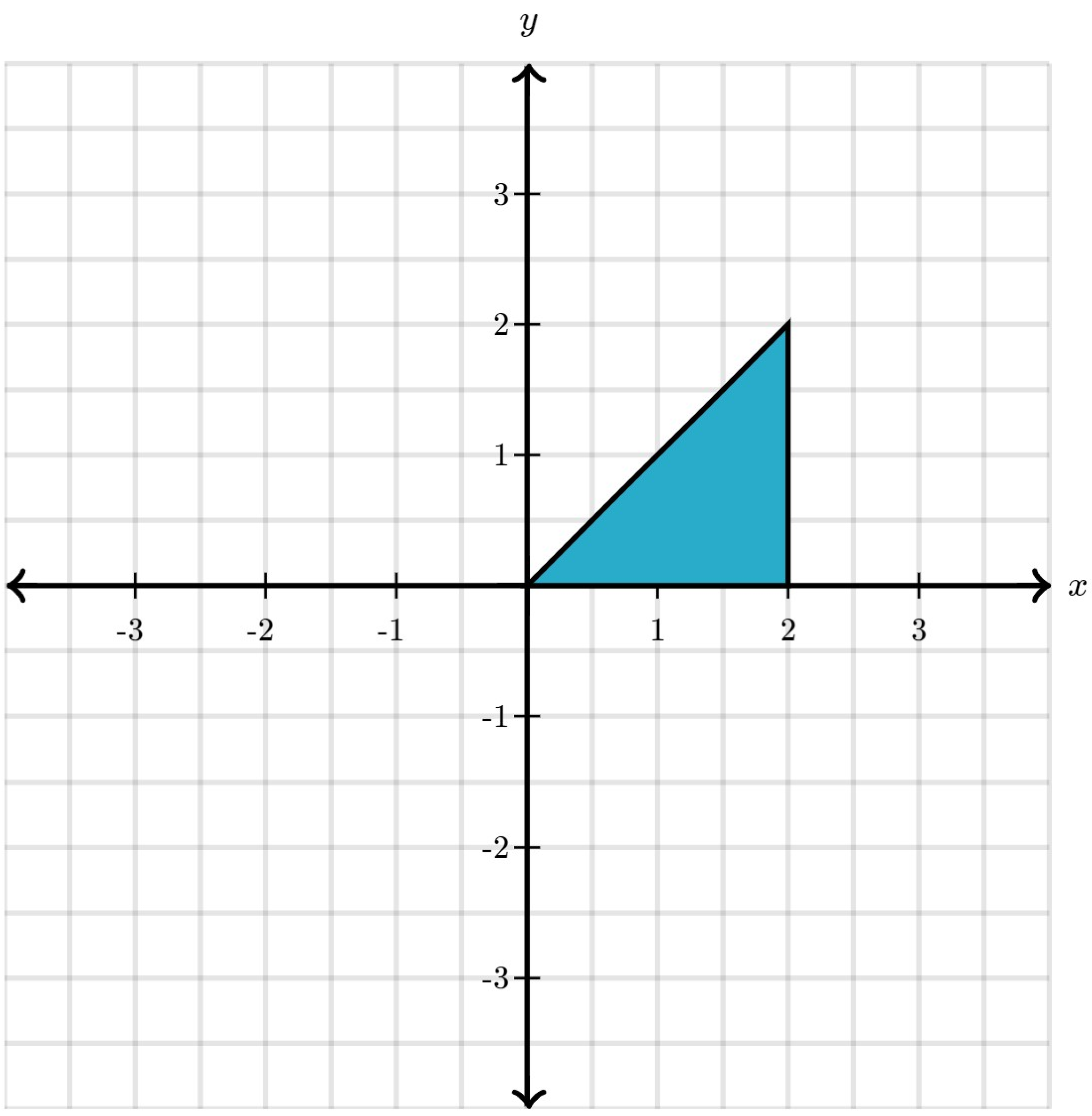
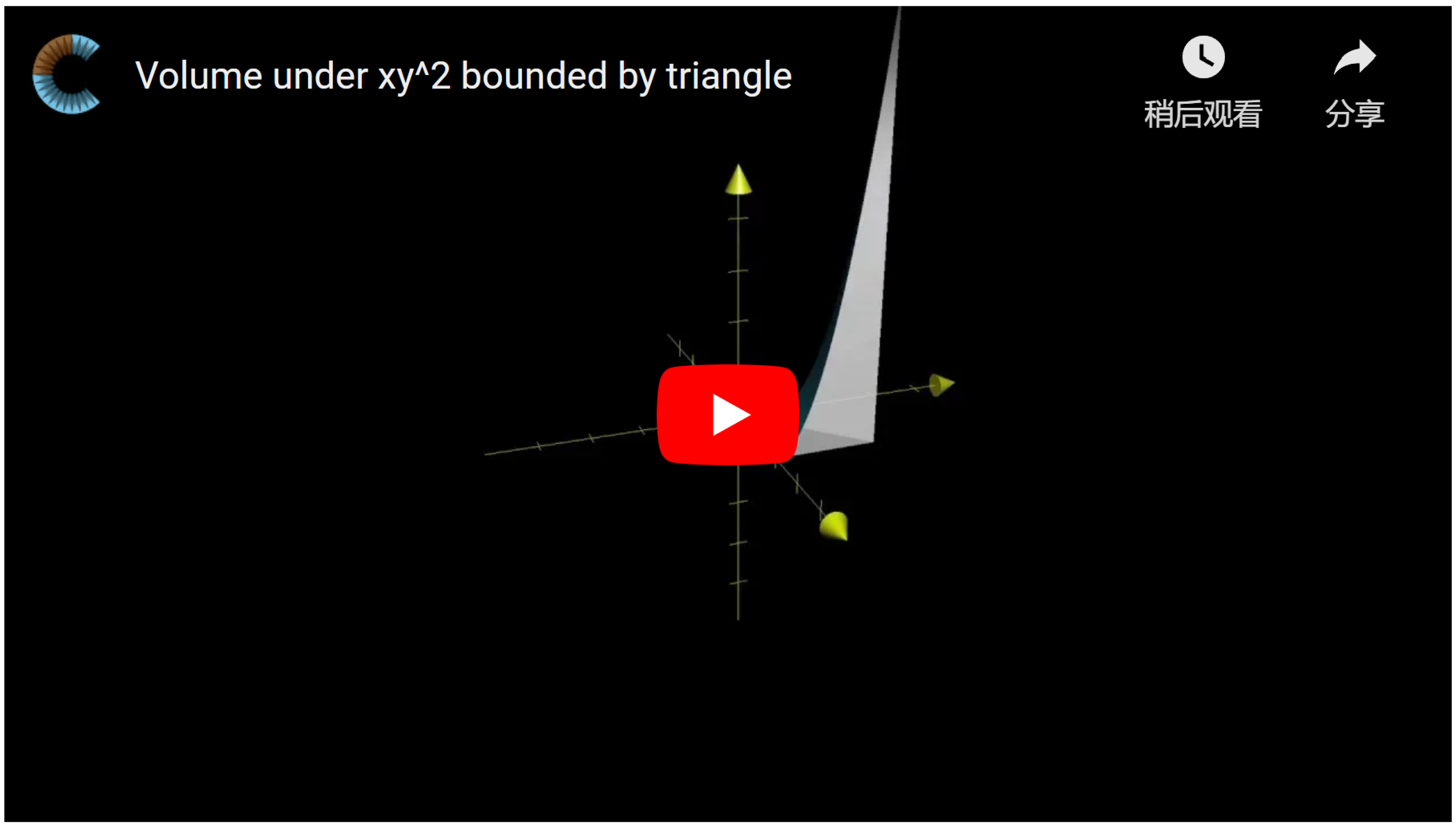
We will find the volume under a portion of this graph. Unlike the last article, this volume will not lie above a rectangular region on the xy-plane. Instead, we will look for a volume whose base is a triangle. The triangle pictured below, to be specific.



This is a right isosceles triangle, one of whose legs connects the points $(0,0)$ and $(2,0)$ on the x-axis, while the other leg connects the points $(2,0)$ and $(2,2)$. The volume above this triangle and below the graph of $f(x,y) = xy^2$ looks like this:



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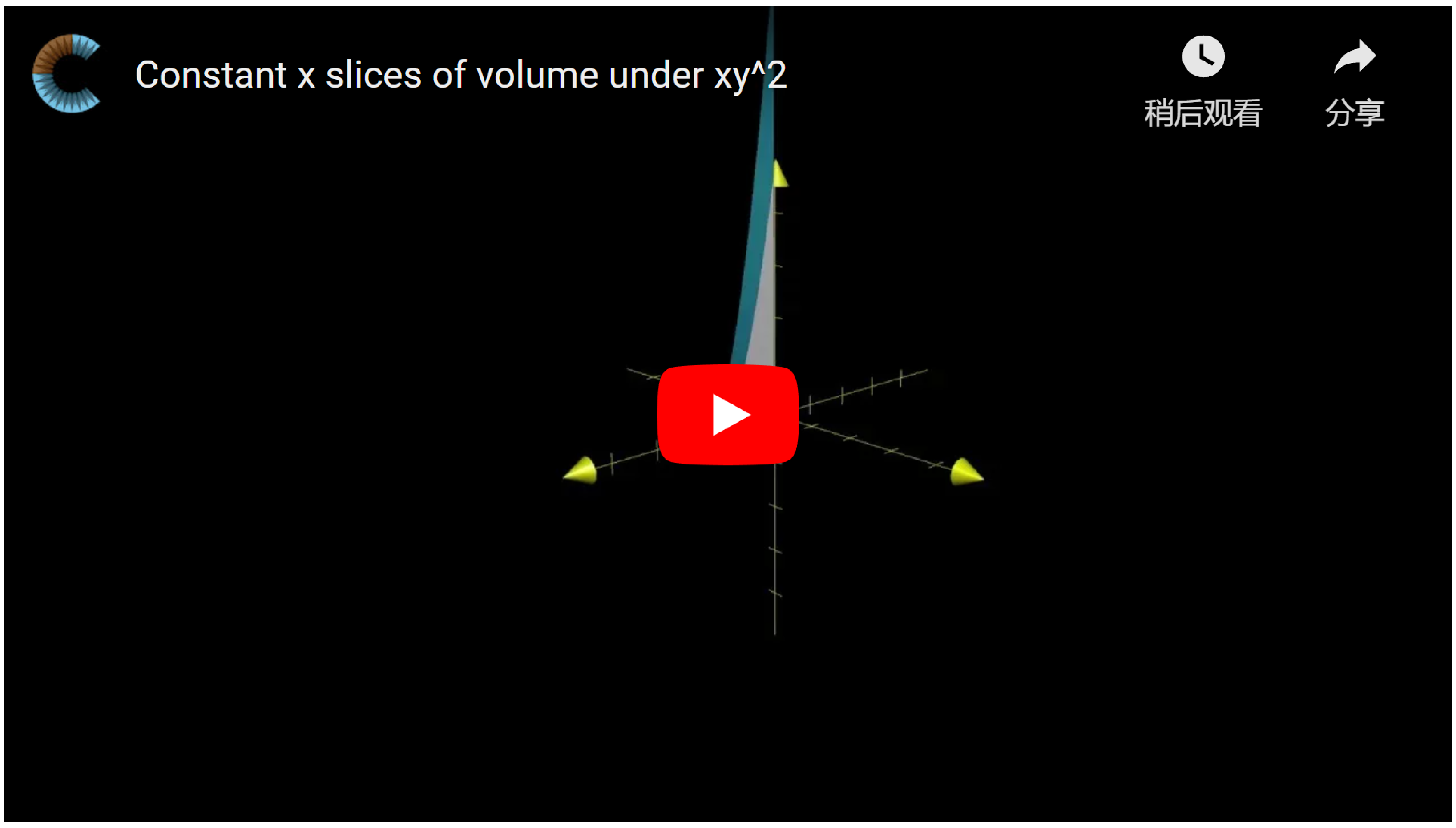
This is similar to the problem I showed in the last article, which introduced the double integral. And indeed, the way to solve it is similar.

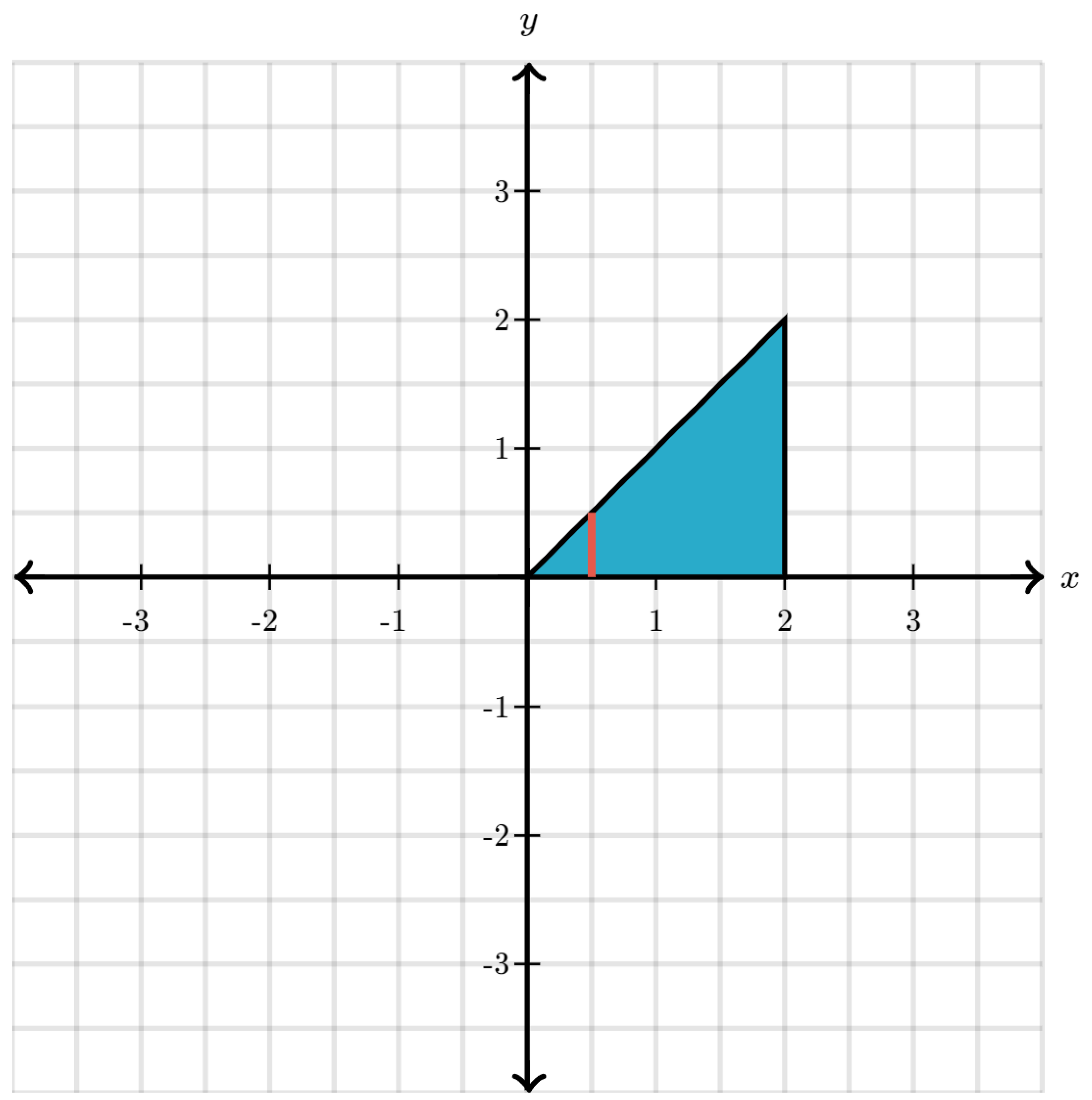
- Find a formula for slices of area using an integral.
- Use a second integral to add those infinitely many slices of area into a volume.

What gets tricky now is the bounds. For example, consider the slices of this volume which represent constant x values. The following animation shows what these slices look like, as the constant x-value varies back and forth between 0 and 2.

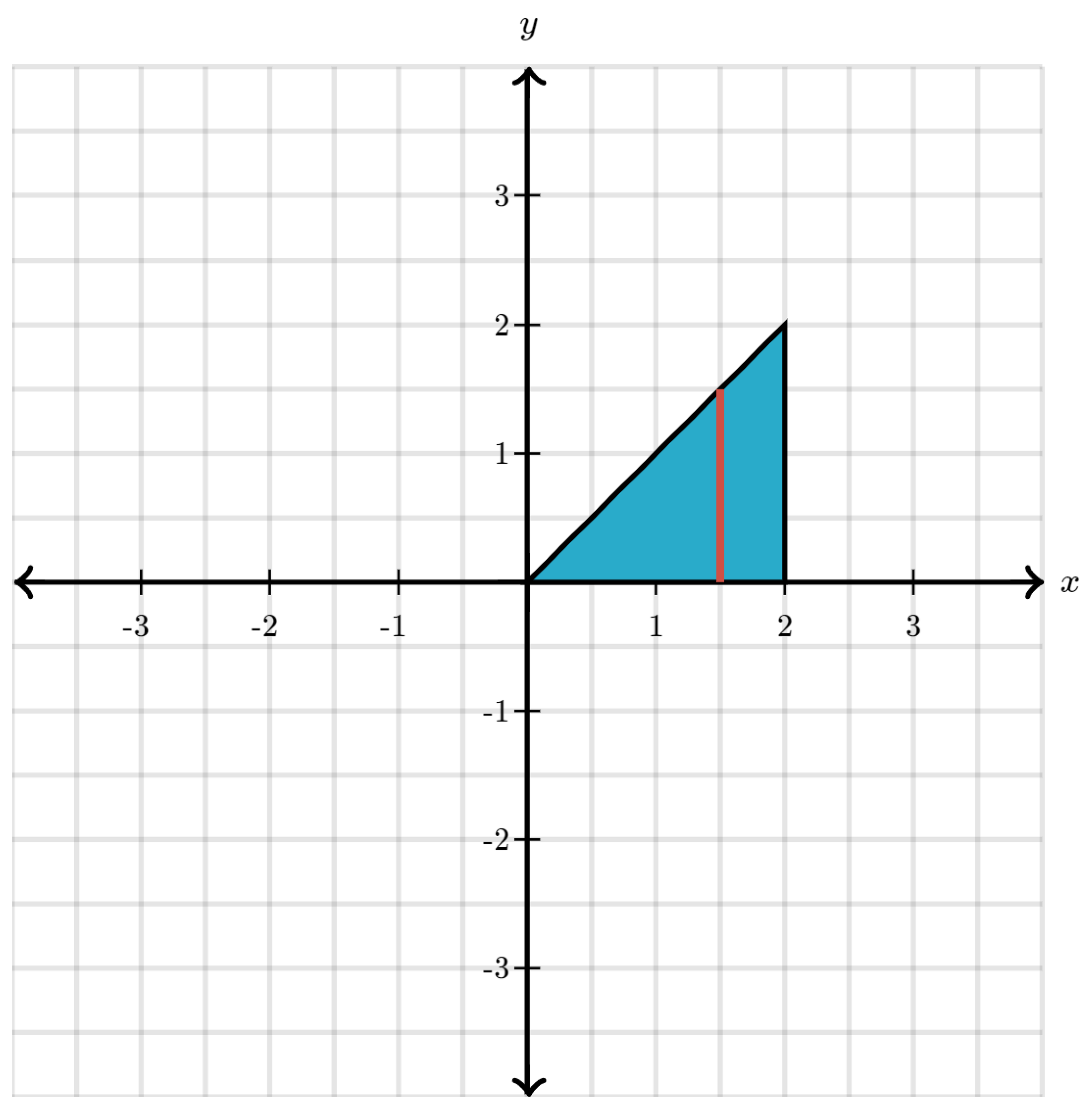


See video transcript

The height of one of these slices changes based the height of the graph of $f(x, y) = xy^2$ above its base. But the length of the base of the slice also changes. For example, when $x = 0.5$, the value of y at the base can range from 0 to 0.5, as in the vertical red stripe pictured below.



Alternatively, when $x = 1.5$, the value of y ranges from 0 to 1.5:



This means when we set up an integral to find the area of one of these constant-x-value slices, **the upper bound is written in terms of x.**

$$\int_0^{x} f(x, y)\, dy = \int_0^{x} xy^2\, dy$$

As far as our computations are concerned, it's perfectly fine to have one of the bounds written in terms of x. After all, we'll end up with an expression in terms of x anyway. Go ahead and work out the integral for yourself:

$$\int_0^x xy^2\,dy = \boxed{}$$

Check

[Hide explanation]

> Remember, in the eyes of the integral, x is a constant, since it is an integral with respect to y (as indicated by the "dy").
>
> $$\begin{aligned}\int_0^x xy^2\,dy &= \left(x\frac{y^3}{3}\right)\Bigg|_0^x \\ &= \left(x\frac{(x)^3}{3}\right) - \left(x\frac{(0)^3}{3}\right) \\ &= \frac{x^4}{3}\end{aligned}$$

From here, there is nothing new. Multiply this value by dx to give it a little depth, and hence make it an infinitesimal volume. Then when we integrate it with respect to x, the bounds are constants, $x = 0$ and $x = 2$, since this is where the base of our triangle sits on the x-axis.

$$\begin{aligned}\int_0^2 \frac{x^4}{3}\,dx &= \left(\frac{x^5}{(5)(3)}\right)\Bigg|_0^2 \\ &= \frac{2^5}{15} - \frac{0^5}{15} \\ &= \frac{32}{15}\end{aligned}$$

The total volume is therefore $\frac{32}{15} \approx 2.13$

Integrating over a disk

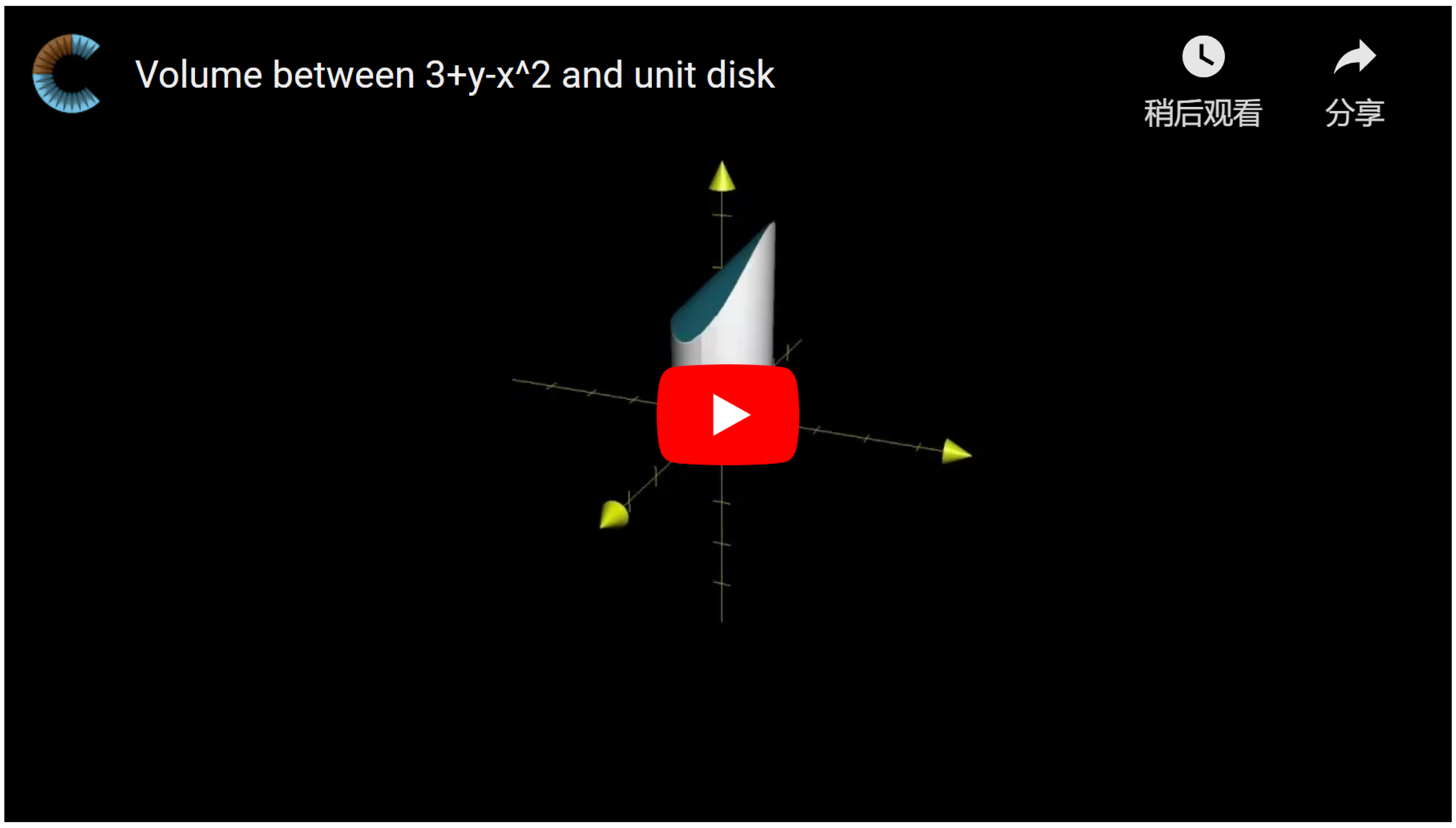
Now let's try something a little harder: finding the volume under a graph bounded by the **unit disk**. The unit disk on the xy-plane is all points (x, y) such that

$$x^2 + y^2 \leq 1$$

For example, the volume underneath the graph

$$f(x, y) = 3 + y - x^2$$

bound by the unit disk looks like this



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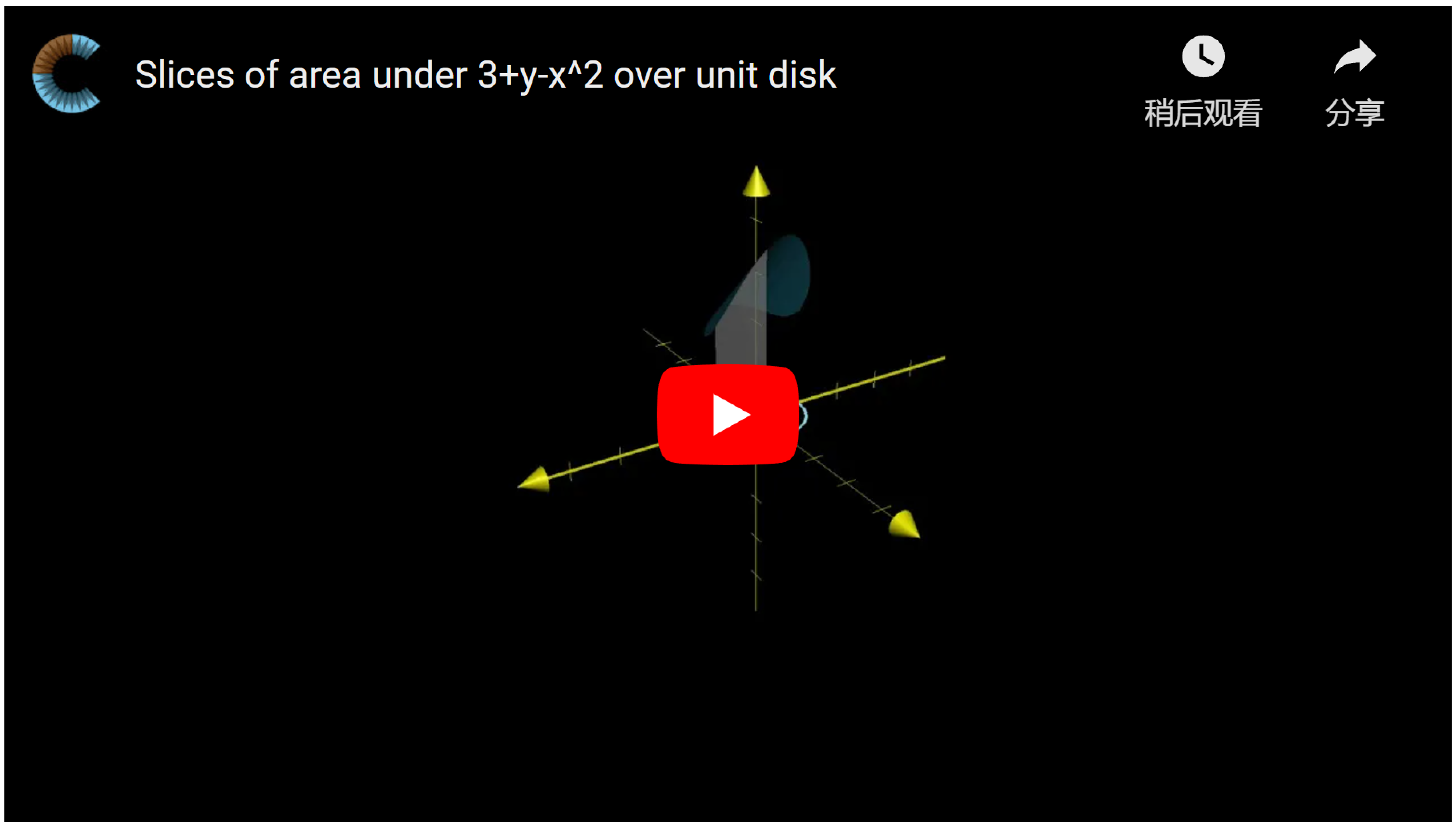
Once again, consider slices of this volume which correspond to constant x-values.



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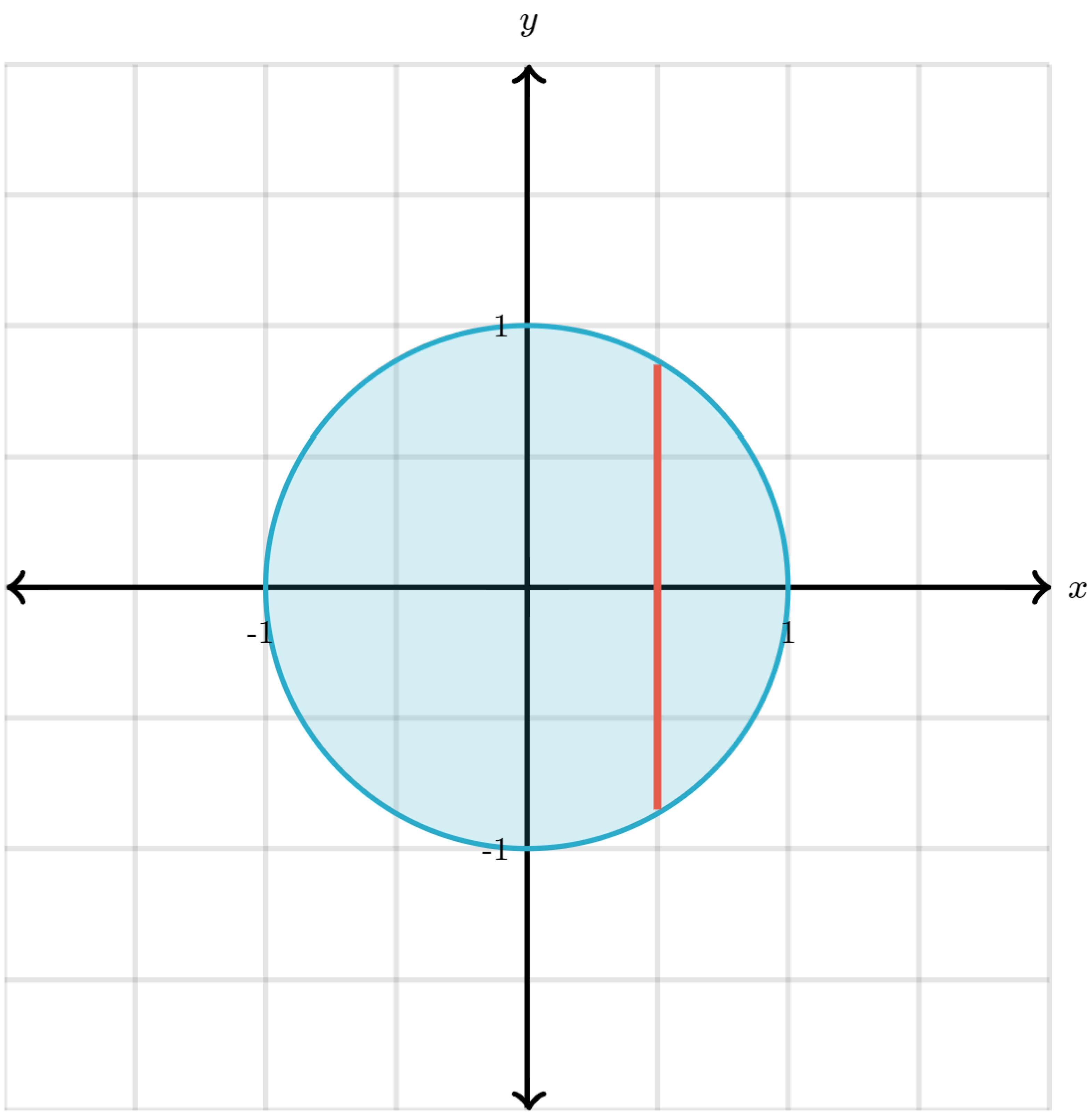
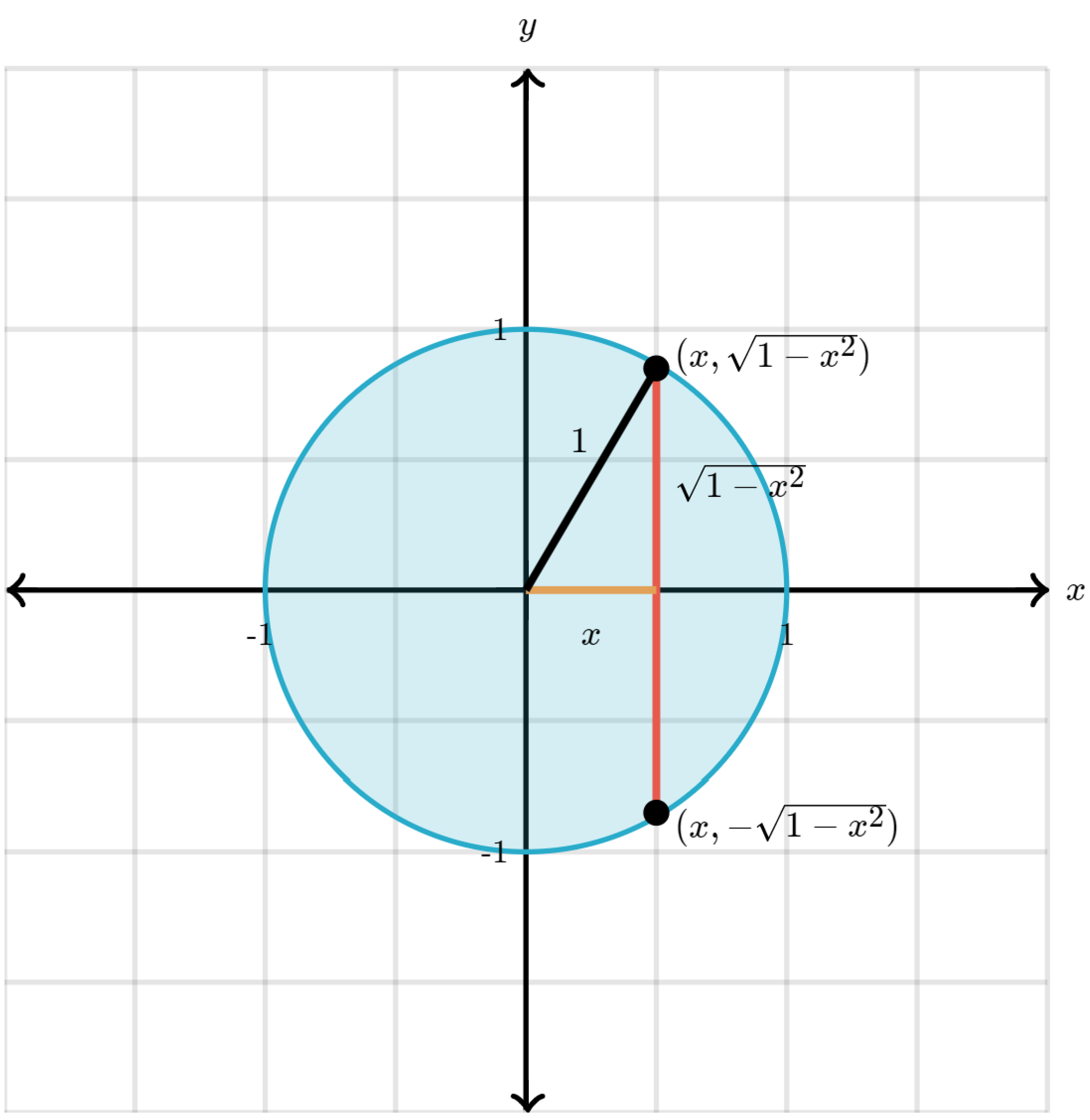
Think about what the base of each of these slices looks like on the xy-plane. Each slice corresponds with some vertical stripe in the unit disk.



Using the pythagorean theorem, we can find the y-values which determine the top and bottom of this stripe as a function of the x-value that the stripe represents.



We can now find the area of one of these constant-x-value slices by integrating $f(x, y)$ with respect to y. Again, where this is different from the case of rectangular regions is that the bounds are each a function of x.

Concept check: Which of the following integrals represents the area of a constant-x-value slice of the volume we are looking for?

Choose 1 answer:

(A) $\displaystyle\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3 + y - x^2)\, dy$

(B) $\displaystyle\int_{0}^{3+y-x^2} \left(\sqrt{1-x^2} - (-\sqrt{1-x^2})\right) dx$

Check

[Hide explanation]

The first choice is correct:

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3 + y - x^2)\, dy$$

In this integral, x should be thought of as a constant.

The height of the slice at each point (as a function of y) is $f(x, y) = 3 + y - x^2$. The bounds for the y-value at the base of the slice are what we just found geometrically:

$$y = -\sqrt{1 - x^2}$$

and

$$y = \sqrt{1 - x^2}$$

Work through it: This is a heavier computation than previous examples, but if you feel up to it, compute this integral to get a formula for the area of a constant-x-value slice, as a function of x.

Area of a constant-x-value slice:

Check

[Hide explanation]

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3 + y - x^2 \, dy$$

$$= \left(3y + \frac{y^2}{2} - x^2 y\right)\Bigg|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}$$

$$= \left(3\sqrt{1-x^2} + \frac{(\sqrt{1-x^2})^2}{2} - x^2\sqrt{1-x^2}\right)$$

$$- \left(3\left(-\sqrt{1-x^2}\right) + \frac{(-\sqrt{1-x^2})^2}{2} - x^2\left(-\sqrt{1-x^2}\right)\right)$$

$$= 6\sqrt{1-x^2} + \left(\frac{1-x^2}{2} - \frac{1-x^2}{2}\right) - 2x^2\sqrt{1-x^2}$$

$$= (6 - 2x^2)\sqrt{1-x^2}$$

The x-values in the unit disk range from $x = -1$ to $x = 1$, so to find the volume we are looking for, integrate the expression you just found with respect to x between the value -1 and 1. As before, you can imagine this as adding up many, many paper-thin volumes.

This turns out to be a tricky integral, but for pragmatism's sake we can solve it using any ol' computer algebra system or numerical integration tool, such as Wolfram Alpha.

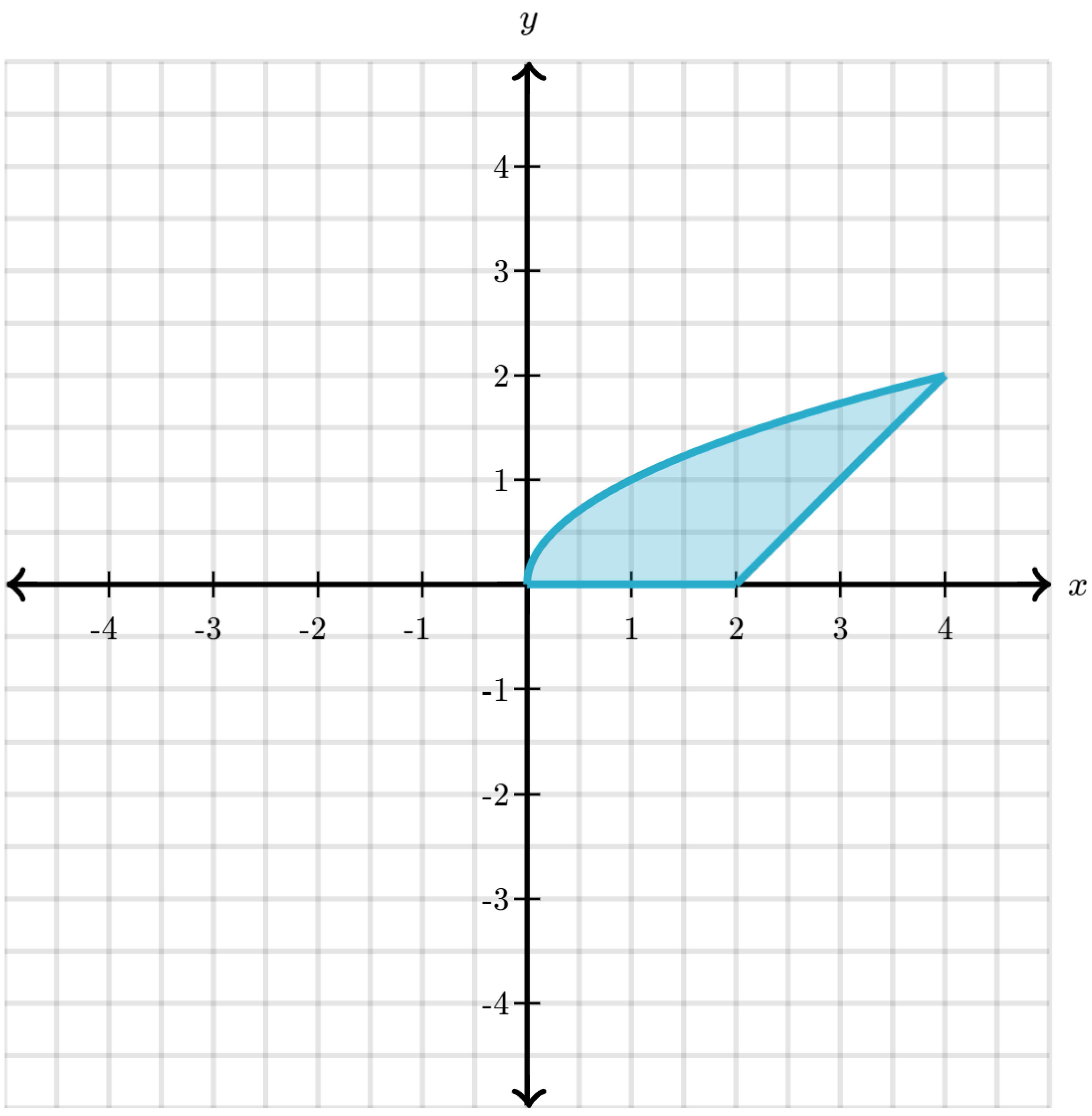
Total volume: $$\int_{-1}^{1} (6 - 2x^2)\sqrt{1-x^2} \, dx = \frac{11\pi}{4} \approx 8.6394$$

Slice the other way: Shark fin region

Sometimes it's easier to consider constant-y-value slices, which involves cutting your region in xy-plane along horizontal stripes. For example, consider region of the xy-plane satisfying the following properties:

- $x \geq y^2$
- $x \leq y + 2$
- $y \geq 0$

This region kind of looks like a shark's dorsal fin:

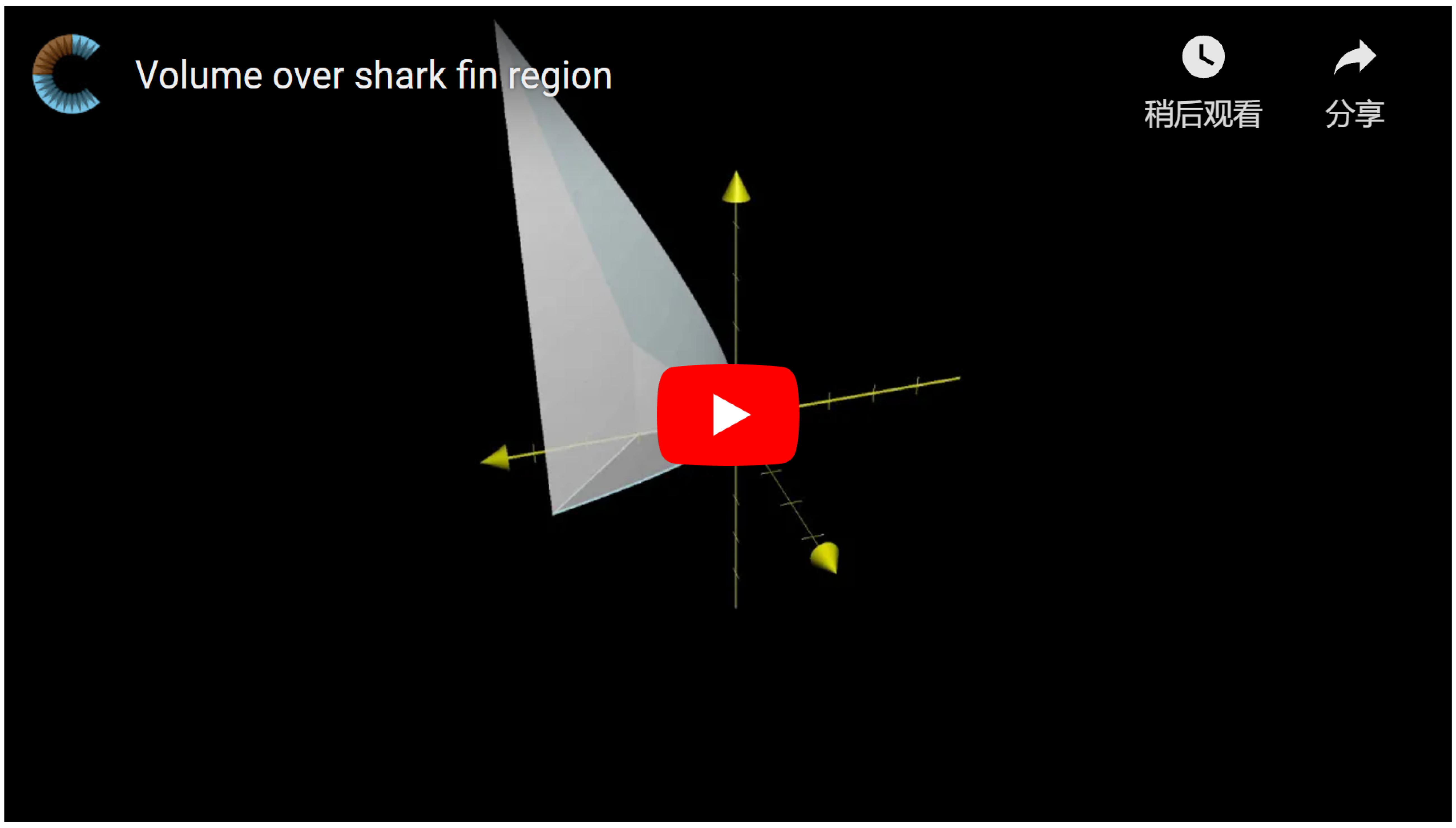


The upper right corner of the region is where the curve $x = y^2$ meets the line $x = y + 2$. That point is $(4, 2)$.

Let's find the volume of a solid that has this region as its footprint, and whose height is determined by a relatively simple multivariable function:

$$f(x, y) = x + 2y$$

Here's what the volume looks like:



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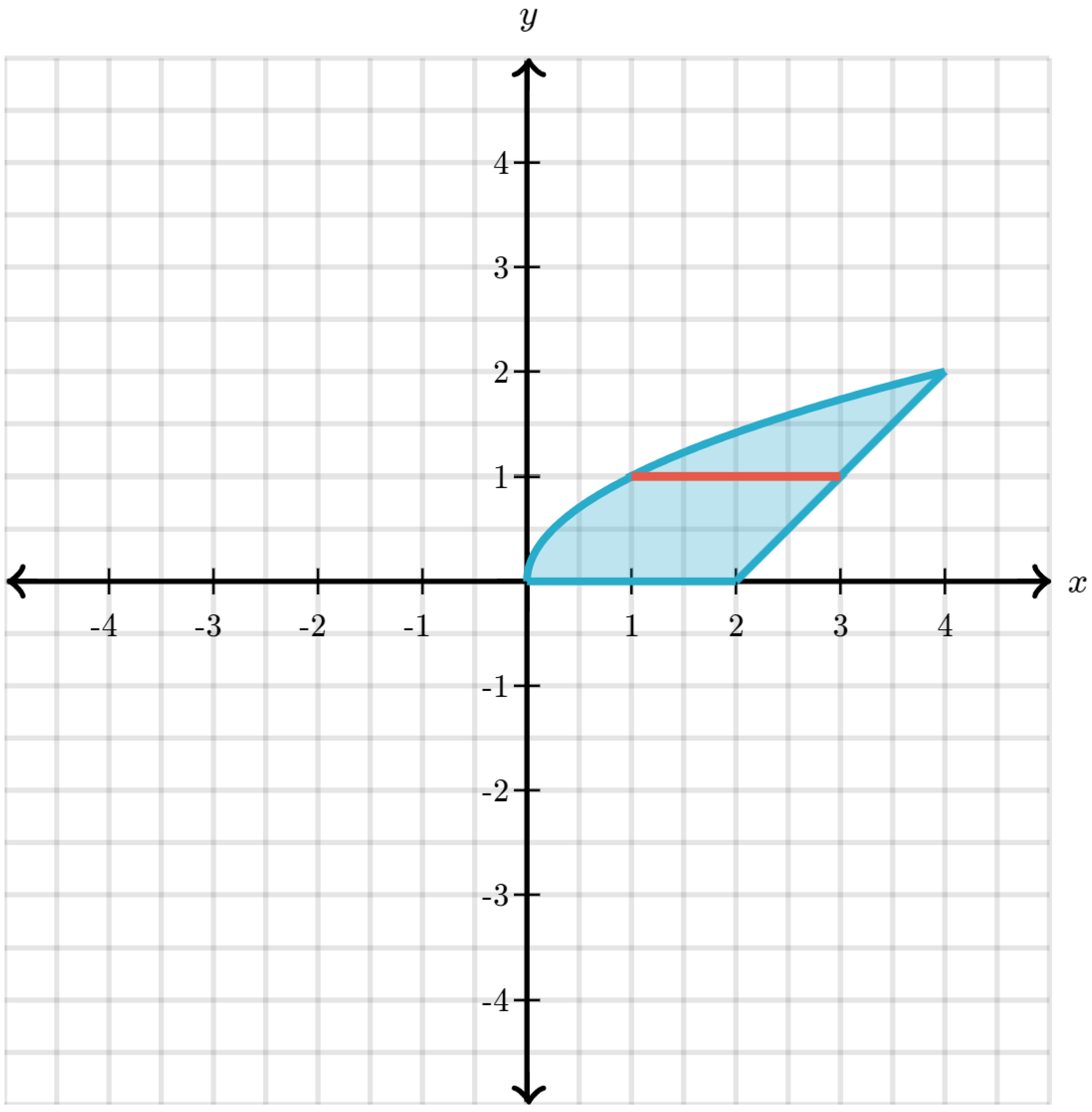
This time, imagine cutting constant-y-value slices of this volume. This will give the area above a horizontal strip of our shark fin region, such as the one pictured below in red.



Concept check: If one of these horizontal stripes corresponds to a value y, what are the bounds on the x-value of the stripe? That is, what are the x-coordinates of the left and right ends of this line as a function of y?

Lower bound: $x =$

Upper bound: $x =$

Check

[Hide explanation]

> **Lower bound**: $x = y^2$
>
> **Upper bound**: $x = y + 2$
>
> These come directly from the definition of the region given above.

Concept check: Which of the following integrals represents the slice of area above one of these stripes, and under the graph of $f(x, y) = x + 2y$, as a function of y?

Choose 1 answer:

(A) $\int_{y^2}^{y+2} (x + 2y)\, dx$

(B) $\int_0^2 (x+2y)\,dy$

Check

[Hide explanation]

The first choice is correct.

$$\int_{y^2}^{y+2} (x+2y)\,dx$$

The integration moves horizontally, as indicated by the "dx". It is also bounded by the values found in the last question, indicated that it stays within the shark fin region.

Concept check: Solve this integral to find the area of the constant y-value slices of our volume.

Area of constant-y-value slice:

Check

[Hide explanation]

$$\int_{y^2}^{y+2} (x+2y)\,dx = \left(\frac{x^2}{2} + 2yx\right)\Bigg|_{x=y^2}^{x=y+2}$$

$$= \left(\frac{(y+2)^2}{2} + 2y(y+2)\right) - \left(\frac{(y^2)^2}{2} + 2y(y^2)\right)$$

$$= \underbrace{\frac{y^2+4y+4}{2} + 2y^2 + 4y - \frac{y^4}{2} - 2y^3}_{\text{Factor out } 1/2}$$

$$= \frac{1}{2}\left(y^2 + 4y + 4 + 4y^2 + 8y - y^4 - 4y^3\right)$$

$$= \frac{1}{2}\left(-y^4 - 4y^3 + 5y^2 + 12y + 4\right)$$

Concept check: When we integrate this function of y to get the total volume, what bounds should we use?

Choose 1 answer:

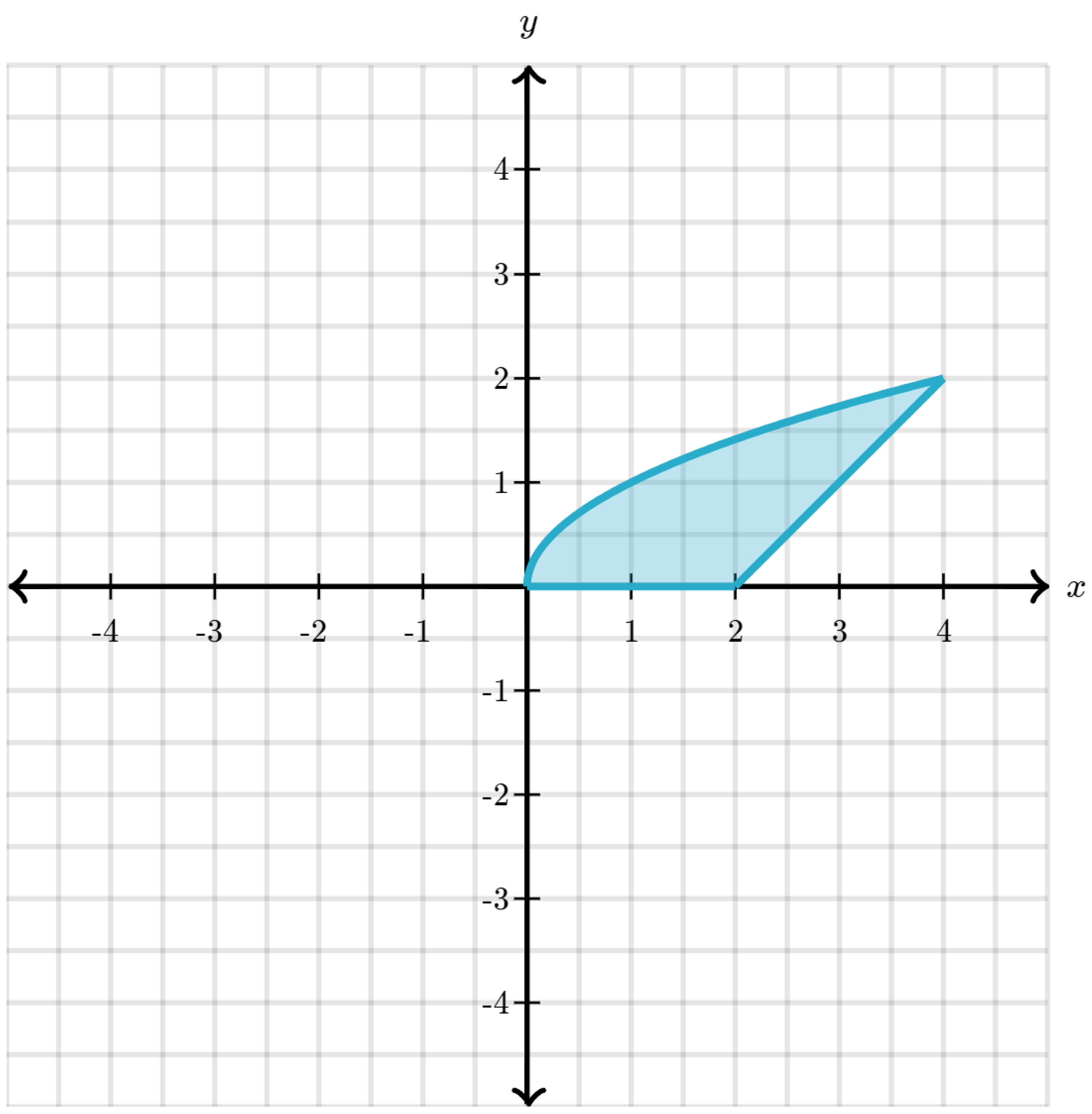
(A) $\displaystyle\int_0^4 \dots\, dy$

(B) $\displaystyle\int_0^2 \dots\, dy$

Check

[Hide explanation]

Looking at the picture of our shark fin region, y varies from 0 to 2



Therefore, the second choice is correct. The integral giving our desired volume looks like this:

$$\int_0^2 \frac{1}{2}\left(-y^4 - 4y^3 + 5y^2 + 12y + 4\right)\, dy$$

Bring it on home: Solve this integral to find the volume of the region defined at the start of this section. (Feel free to use a calculator).

Volume:

Check

[Hide explanation]

$$\int_0^2 \frac{1}{2}\left(-y^4 - 4y^3 + 5y^2 + 12y + 4\right)\, dy$$

$$= \frac{1}{2}\left(-\frac{y^5}{5} - y^4 + 5\frac{y^3}{3} + 6y^2 + 4y\right)\Bigg|_0^2$$

$$= \frac{1}{2}\left(-\frac{(2)^5}{5} - (2)^4 + 5\frac{(2)^3}{3} + 6(2)^2 + 4(2)\right) - \frac{1}{2}\left(-\frac{(0)^5}{5} - (0)^4 + 5\frac{(0)^3}{3} + 6(0)^2 + 4(0)\right)$$

$$= \frac{1}{2}\left(-\frac{32}{5} - 16 + 5\frac{8}{3} + 24 + 8\right)$$

$$= \ldots(\text{calculator})\ldots$$

$$= \frac{172}{15}$$

Summary

When you need to perform a double integral over a non-rectangular region, follow these steps.

- Start by cutting your region along slices that correspond with holding one of the variables constant. For example, holding x at some constant value will give a vertical stripe of your region.

- Find how to express the bounds of these stripes as a function of the other variable. For example, the top and bottom of a vertical stripe would be expressed as some function of x.

- When you set up your double integral, the inner integral will correspond to integrating along one of these stripes, and each of its bounds will be a function of the outer variable. If the inner integral corresponds to constant-x-values, the double integral as a whole might look like this:

$$\int_{x_1}^{x_2} \overbrace{\left(\int_{y_1(x)}^{y_2(x)} f(x,y)\, dy\right)}^{\text{Evaluates to some function of } x} dx$$

Alternatively, if you started with horizontal constant-y-value slices, the double integral might look like this:

$$\int_{y_1}^{y_2} \overbrace{\left(\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx \right)}^{\text{Evaluates to some function of } y} dy$$