Green's theorem examples

Google Classroom

Green's theorem is beautiful and all, but here you can learn about how it is actually used.

Background

- Green's theorem

Remembering the formula

Green's theorem is most commonly presented like this:

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

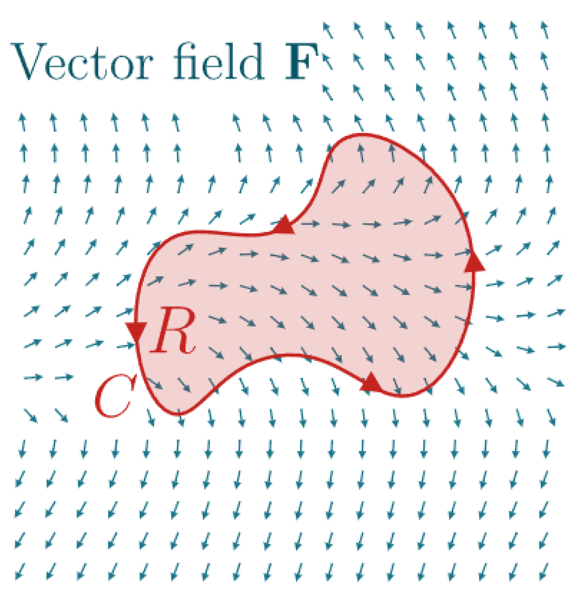
This is also most similar to how practice problems and test questions tend to look. But personally, I can never quite remember it just in this P and Q form.

> "Was it $\frac{\partial Q}{\partial x}$ or $\frac{\partial Q}{\partial y}$?"
>
> "Which term is subtracted again?"

I always start by thinking about this form:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{2d-curl}\,\mathbf{F}\,dA$$

I find this easier to remember because it actually has a physical meaning (see the last article for more details):



- The line integral of a vector field $\mathbf{F}(x, y)$ around a closed curve C measures the fluid rotation around that boundary C.
- The double integral of the curl of $\mathbf{F}$ adds up all the tiny little bits of fluid rotation within the region R enclosed by C.
- Intuitively, it makes sense that these should be related. And in fact, they are equal.

To get to the PQ version of the theorem, write the components of $\mathbf{F}$ as $P(x, y)$

and $Q(x, y)$:

$$\mathbf{F}(x, y) = \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$$

(To remember that P is the x-component and Q is the y-component, think about the fact that P comes before Q in the alphabet).

And from here, expand each bit of the line integral, curl, etc. After doing this a couple times, it's natural enough to do in your head.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{2d-curl}\, \mathbf{F}\, dA$$

$$\Downarrow$$

$$\oint_C \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix} = \iint_R \text{2d-curl} \left(\begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix} \right) dA$$

$$\Downarrow$$

$$\oint_C P\, dx + Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

Of course, this requires remembering how to compute two-dimensional curl, but this is something which ought to be remembered outside the context of Green's theorem anyway.

> **Warning**: Green's theorem only applies to curves that are oriented **counterclockwise**. If you are integrating clockwise around a curve and wish to apply Green's theorem, you must flip the sign of your result at some point.

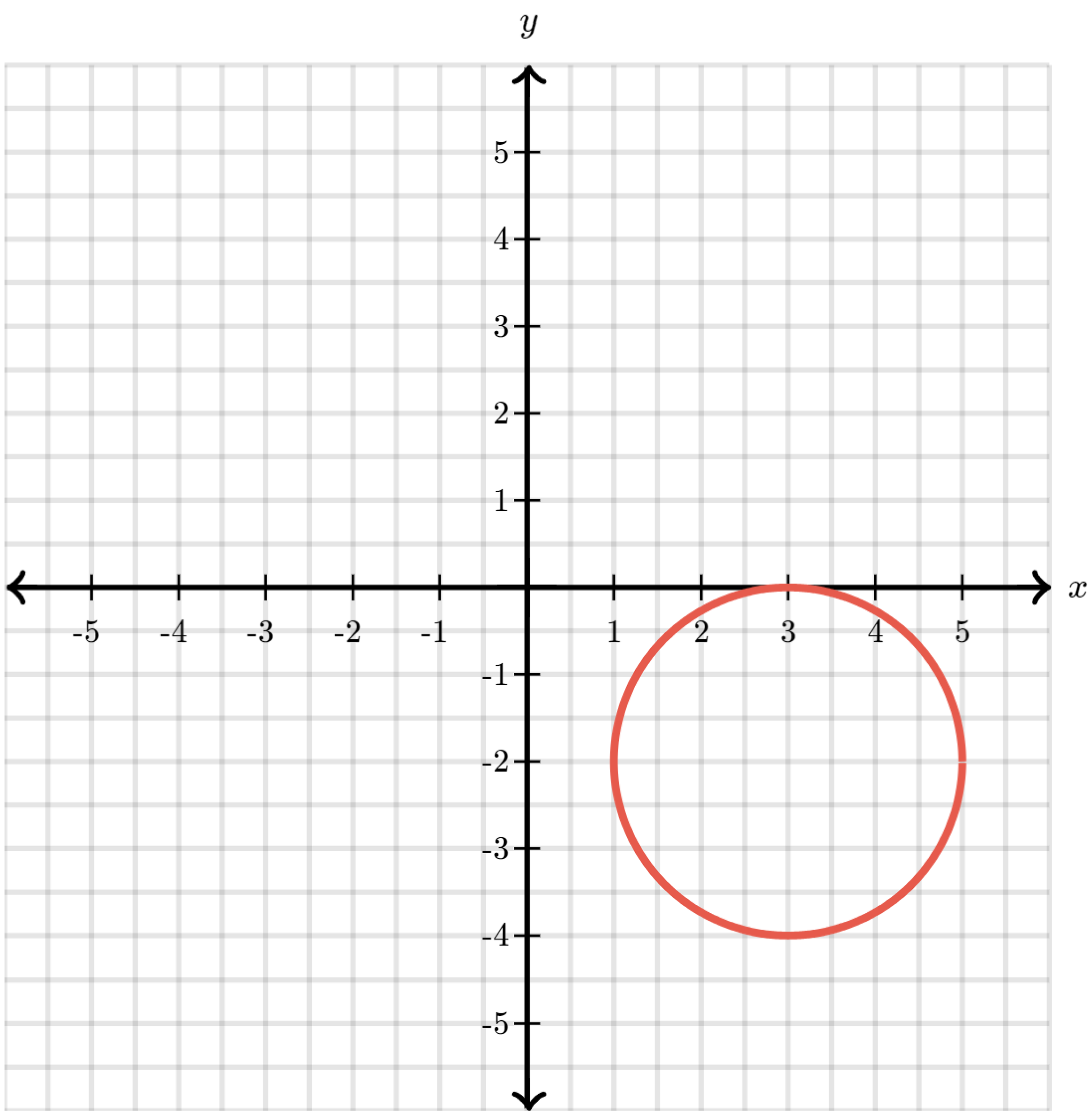
How do you know when to use Green's theorem?

> "Mathematics is not a spectator sport" - George Polya

The best way to get a feel for its usefulness is to simply jump into some examples to get a feel for it. I'll debrief after each example to help extract the intuition for each one.

Example 1: Line integral $\rightarrow$ Area

Problem: Let C represent a circle with radius 2 centered at $(3, -2)$:



If you orient C counterclockwise, compute the following line integral:

$$\oint_C 3y\,dx + 4x\,dy$$

Solution

Step 1: Is the curve in question oriented clockwise or counterclockwise?

Choose 1 answer:

- (A) Clockwise
- (B) Counterclockwise

Check

I know that might feel silly to ask, given that it was just stated explicitly in the problem. But it's important to remember that you must always ask this when using Green's theorem.

Step 2: As we apply Green's theorem to this integral $\oint_C 3y\,dx + 4x\,dy$, what should we substitute for $P(x, y)$ and $Q(x, y)$?

$P(x, y) =$ [ ]

$Q(x, y) =$ [ ]

Check

[Hide explanation]

The line integral in Green's theorem looks like this:

$$\oint_C P\,dx + Q\,dy$$

And here's the integral we need to compute:

$$\oint_C 3y\,dx + 4x\,dy$$

Just matching terms, we should apply

$$P(x, y) = 3y$$

and

$$Q(x, y) = 4x$$

Step 3: Now compute the appropriate partial derivatives of $P(x, y)$ and $Q(x, y)$.

$\frac{\partial Q}{\partial x} =$ [ ]

$\frac{\partial P}{\partial y} =$ [ ]

Check

[Hide explanation]

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(4x) = 4$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(3y) = 3$$

Step 4: Finally, compute the double integral from Green's theorem. In this case, R represents the region enclosed by the circle with radius 2 centered at $(3, -2)$. (Hint, don't work too hard on this one).

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \square$$

Check

[Hide explanation]

Start by plugging in the expressions for $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$ that you found in the previous problem.

$$\begin{aligned}\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA &= \iint_R (4-3)\ dA \\ &= \iint_R dA\end{aligned}$$

Now before you go crazy describing the region R with the appropriate bounds for x and y, notice what this integral represents. Since there is no function inside it is just adding up tiny bits of area of R, so it will equal the area of R itself!

Since R is a circle with radius 2, that area is $\pi r^2 = \pi(2)^2 = 4\pi$. Therefore our line integral evaluates to 4π.

Example 1 debrief

Why did the line integral in the last example become simpler as a double integral when we applied Green's theorem? It's because the curl of the relevant function was a constant:

$$\text{2d-curl}\left(\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} \right) = \text{2d-curl}\left(\begin{bmatrix} 3y \\ 4x \end{bmatrix} \right)$$

$$= \frac{\partial}{\partial x}(4x) - \frac{\partial}{\partial y}(3y)$$

$$= 4 - 3$$

$$= 1$$

More generally, if it looks like the partial derivative of Q with respect to x is simple, and/or that the partial derivative of P with respect to y is simple, think Green's theorem.

$$\oint_C \overbrace{P(x,y)}^{\text{Is } \frac{\partial}{\partial y} \text{ simple?}} \, dx + \overbrace{Q(x,y)}^{\text{Is } \frac{\partial}{\partial x} \text{ simple?}} \, dy$$

It was also important that we could easily compute the area of the region in question. If that were not true, the double integral might not have been simpler at all.

Example 2: Two function graphs

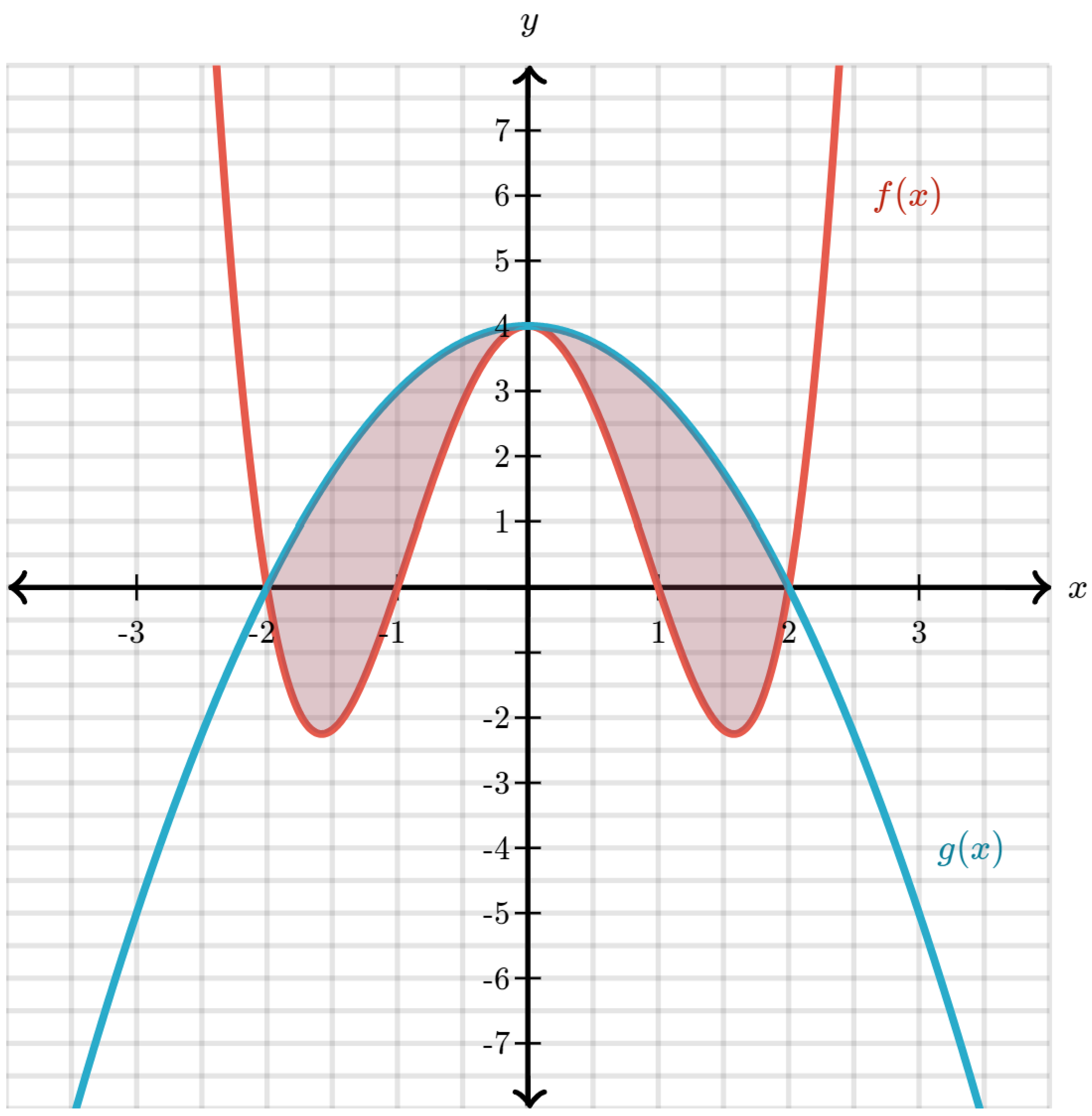
Problem

Consider the following two functions:

$$f(x) = (x^2 - 4)(x^2 - 1)$$

$$g(x) = 4 - x^2$$

Now consider the region between the graphs of these functions.



Let D be the clockwise-oriented boundary of this region (D for droopy). Compute the following line integral:

$$\oint_D x^2 y\,dx - y^2 dy$$

Solution

Step 1: Is the curve in question oriented clockwise or counterclockwise?

Choose 1 answer:

- (A) Clockwise
- (B) Counterclockwise

Check

Since Green's theorem applies to counterclockwise curves, this means we will need to take the negative of our final answer.

Step 2: What should we substitute for $P(x, y)$ and $Q(x, y)$ in the integral $\oint_D x^2 y\,dx - y^2 dy$?

$P(x, y) =$

$Q(x, y) =$

Check

[Hide explanation]

The line integral in Green's theorem looks like this:

$$\oint_D P\,dx + Q\,dy$$

And here's the integral we need to compute:

$$\oint_D x^2 y\,dx - y^2 dy$$

Just matching terms, we should apply

$$P(x,y) = x^2 y$$

and

$$Q(x,y) = -y^2$$

Step 3: Now compute the appropriate partial derivatives of $P(x,y)$ and $Q(x,y)$.

$$\frac{\partial Q}{\partial x} = \square$$

$$\frac{\partial P}{\partial y} = \square$$

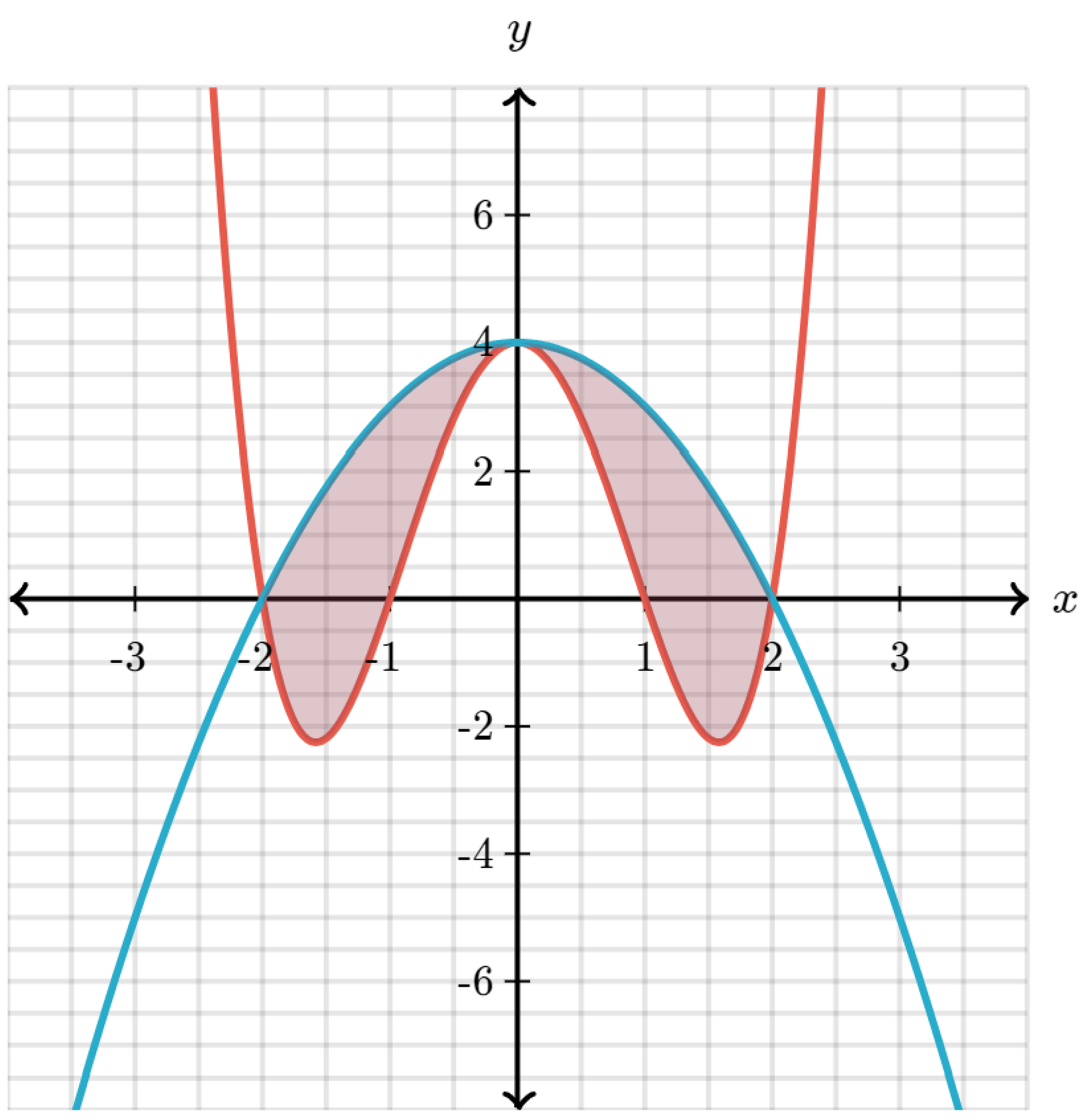
Check

[Hide explanation]

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(-y^2) = 0$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x^2 y) = x^2$$

Step 4: To apply Green's theorem, we will perform a double integral over the droopy region D, which was defined as the region above the graph $y = (x^2 - 4)(x^2 - 1)$ and below the graph $y = 4 - x^2$. This double integral will be something of the following form:



$$\int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} \dots \, dy \, dx$$

Fill in all of those bounds:

$x_1 =$

$x_2 =$

$y_1(x) =$

$y_2(x) =$

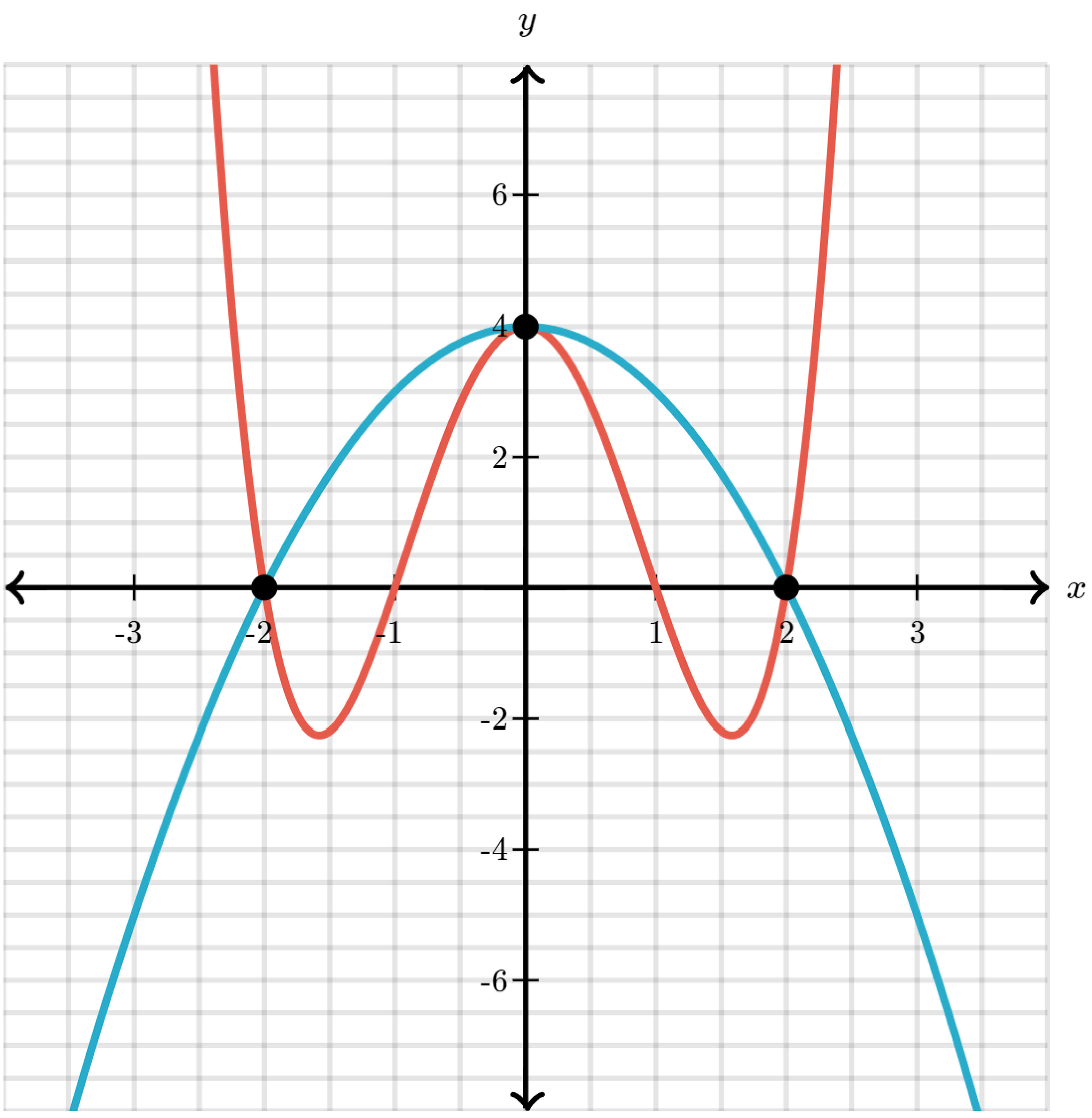
Check

[Hide explanation]

Let's start with the easier ones. The bounds for y are explicitly given in the problem: The region is defined to be above the graph $y = (x^2 - 4)(x^2 - 1)$ and below the graph $y = 4 - x^2$. This means the inner integral will look like this:

$$\int_{(x^2-4)(x^2-1)}^{4-x^2} \langle \text{yadda yadda yadda} \rangle \, dy$$

The trickier part is knowing the bounds of x. The definition of our region doesn't even mention x. Let's take a look at the graphs of the y-bounds:



The bounds on the x-values are determined by where these two graphs intersect on the left and right. Just looking at the graph, you can probably guess that they intersect at $x = -2$ and $x = 2$. You can also check this more precisely by plugging in 2 and -2 to each equation.

If you did not have the graph to look at to guess-and-check, you would solve the equation $(x^2 - 4)(x^2 - 1) = 4 - x^2$.

This means the double integral as a whole will have the following setup:

$$\int_{-2}^{2} \int_{(x^2-4)(x^2-1)}^{4-x^2} \overbrace{\text{<yadda yadda yadda>}}^{\text{We'll put something here in just a sec}} \, dy \, dx$$

Step 5: Finally, to apply Green's theorem, we plug in the appropriate value to this integral. If our original line integral was oriented counterclockwise, we would plug in

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

However, since the curve is oriented clockwise, we make this negative:

$$-\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$$

Using the answers to the previous two questions, plugging in this value to the double integral you set up, find the answer to the original line integral problem:

$$\oint_{D} x^2 y\,dx - y^2 dy = \square$$

Check

[Hide explanation]

$$\int_{-2}^{2}\left(\int_{(x^2-4)(x^2-1)}^{4-x^2} \underbrace{\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}_{\text{Replace with values from step } 3} dy\right) dx$$

$$\int_{-2}^{2}\left(\int_{(x^2-4)(x^2-1)}^{4-x^2} \underbrace{(x^2 - 0)}_{\text{Factor out from } y \text{ integral}} dy\right) dx$$

$$\int_{-2}^{2}\left(x^2 \underbrace{\int_{(x^2-4)(x^2-1)}^{4-x^2} dy}_{\text{Simply the difference of bounds}}\right) dx$$

$$\int_{-2}^{2} x^2\left(\underbrace{(4-x^2) - (x^2-4)(x^2-1)}_{\text{Factor out } x^2 - 4}\right) dx$$

$$\int_{-2}^{2} x^2(x^2-4)\underbrace{(-1-(x^2-1))}_{-x^2}\,dx$$

$$\int_{-2}^{2} -x^4(x^2-4)\,dx$$

$$\int_{-2}^{2} -(x^6-4x^4)\,dx$$

$$\int_{-2}^{2} 4x^4 - x^6\,dx$$

$$\left[4\frac{x^5}{5} - \frac{x^7}{7}\right]_{x=-2}^{x=2}$$

$$\left(4\frac{2^5}{5} - \frac{2^7}{7}\right) - \left(4\frac{(-2)^5}{5} - \frac{(-2)^7}{7}\right)$$

$$\left(4\frac{32}{5} - \frac{128}{7}\right) \quad \underbrace{-\left(4\frac{-32}{5} - \frac{-128}{7}\right)}_{\text{This is just adding the same amount again}}$$

$$2\left(4\frac{32}{5} - \frac{128}{7}\right)$$

$$2\left(\frac{128}{5} - \frac{128}{7}\right)$$

$$256\left(\frac{1}{5} - \frac{1}{7}\right)$$

$$256\left(\frac{7}{35} - \frac{5}{35}\right)$$

$$256\left(\frac{2}{35}\right)$$

$$\boxed{\frac{512}{35}}$$

Example 2 debrief

As in Example 1, part of the reason this line integral became simpler is that the terms simplified once we looked at the appropriate partial derivatives.

$$\oint_C \overbrace{x^2 y}^{\substack{\text{Is } \frac{\partial}{\partial y} \text{ simple?} \\ \text{A little bit, yes.}}} \, dx + \overbrace{(-y^2)}^{\substack{\text{Is } \frac{\partial}{\partial x} \text{ simple?} \\ \text{Very much so, yes.}}} \, dy$$

Also, the region in question was defined by two separate curves. Computing the line integral directly requires setting up two separate line integrals for each curve. But the double integral very naturally went over the full region in one fell swoop.

Another thing to note is that the ultimate double integral wasn't exactly *simple*. You still had to mark up a lot of paper during the computation. But this is okay. We can still feel confident that Green's theorem simplified things, since each individual term became simpler, since we avoided needing to parameterize our curves, and since what would have been two separate line integrals was just one double integral.

Sneaky area calculations

In the previous two examples, we used Green's theorem to turn a line integral into a double integral. Here, let's do things the other way around. Take a look at the double integral from Green's theorem:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Remember how in Example 1, we were lucky enough to have the following property:

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 1$$

This means our integral was just computing the area of R:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \to \iint_R dA = \text{Area of } R$$

Now imagine that we didn't already know the area of R, but we wished to compute it. One thing you could do is find some pair of functions $P(x, y)$ and $Q(x, y)$ satisfying this curl-equals-one property:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1,$$

According to Green's theorem, any pair of functions like this let's you compute the area of a region using a line integral:

$$\begin{aligned}\oint_C P\,dx + Q\,dy &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= \iint_R (1)\,dA \\ &= \text{Area of } R\end{aligned}$$

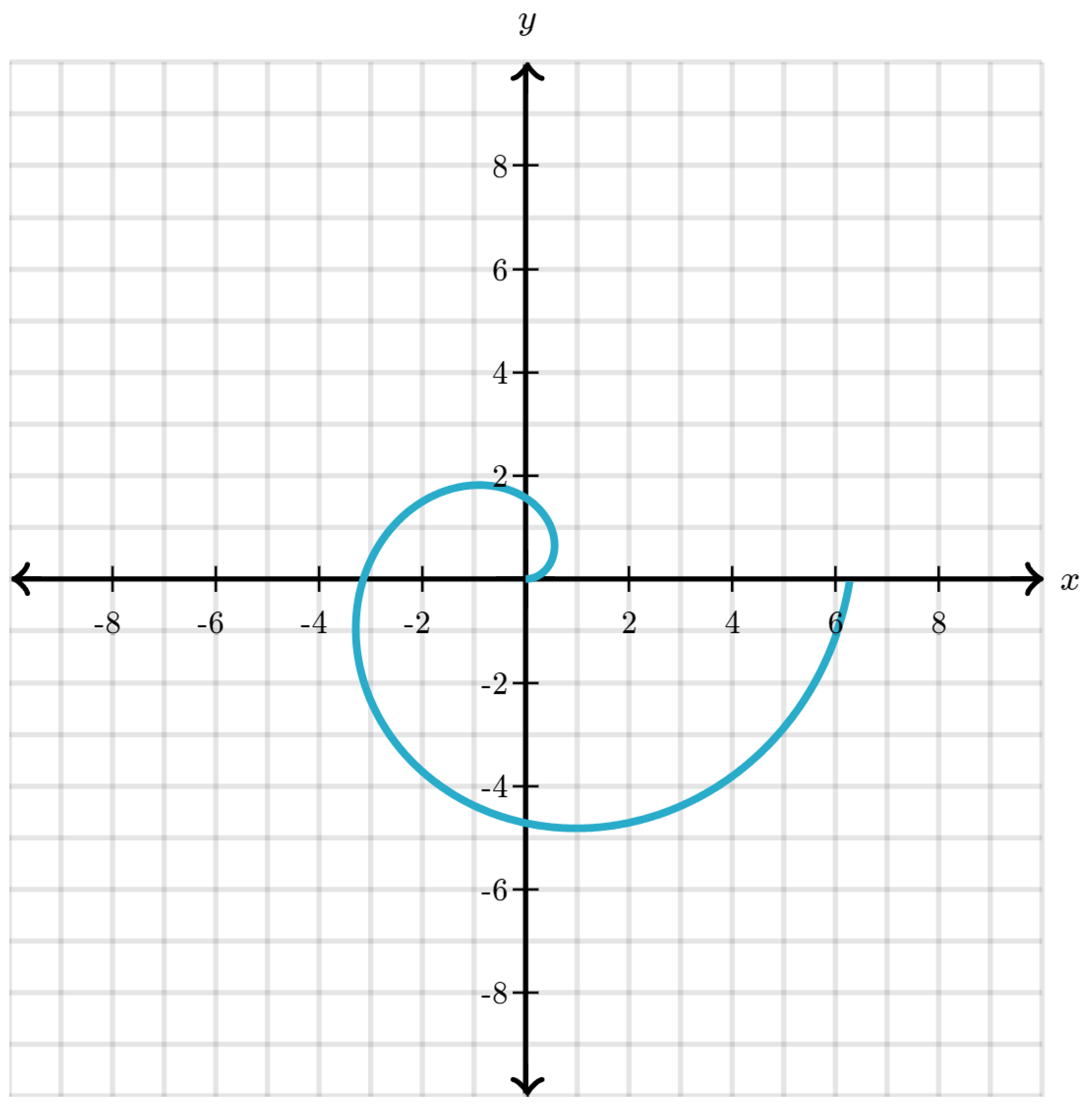
Doesn't that feel strange, computing the area of a region using a line integral around its boundary? Let' see what it looks like in action.

Example 3: Area of a seashell

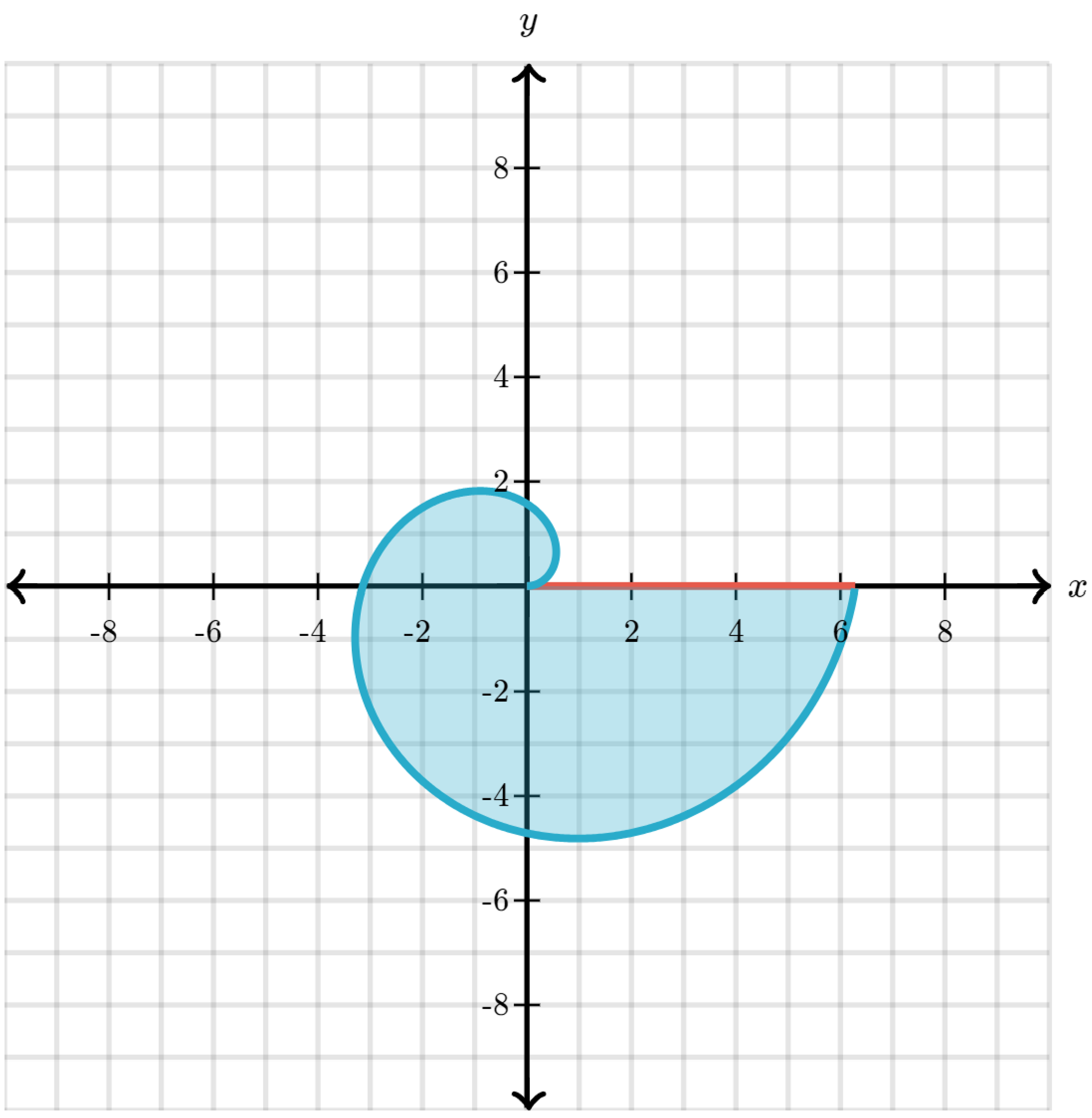
Problem

Consider the spiral defined by the following parametric equations in the range $0 \le t \le 2\pi$.

$$x(t) = t\cos(t)$$
$$y(t) = t\sin(t)$$



Now add the line from $(0,0)$ to $(2\pi, 0)$ to this spiral, and consider the seashell-shaped region it encloses.



What is the area of that region?

Solution

Step 1: How is the boundary of this seashell oriented?

Choose 1 answer:

- (A) Clockwise
- (B) Counterclockwise

Check

Step 2: Choose the appropriate $P(x, y)$ and $Q(x, y)$.

To apply the Green's theorem trick, we first need to find a pair of functions $P(x, y)$ and $Q(x, y)$ which satisfy the following property:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

Actually, quite a few pairs of functions satisfy this.

Concept check: Which of the following function pairs satisfies this property?

Choose all answers that apply:

A $P(x, y) = x$
$Q(x, y) = y$

B $P(x, y) = -y$
$Q(x, y) = 0$

C $P(x, y) = 0$
$Q(x, y) = x$

D $P(x, y) = -y/2$
$Q(x, y) = x/2$

Check

[Hide explanation]

> All but the first choice are correct. When you take the definitions of P and Q in the first choice, both partial derivatives are 0.

You might think the second or third choices above make things simplest. Interestingly, though, it's often the last choice that makes the line integral computation work out best. This means solving the following integral:

$$\oint_C \underbrace{-\frac{1}{2}y\,dx}_{P\,dx} + \underbrace{\frac{1}{2}x\,dy}_{Q\,dy}$$

Or, written more cleanly,

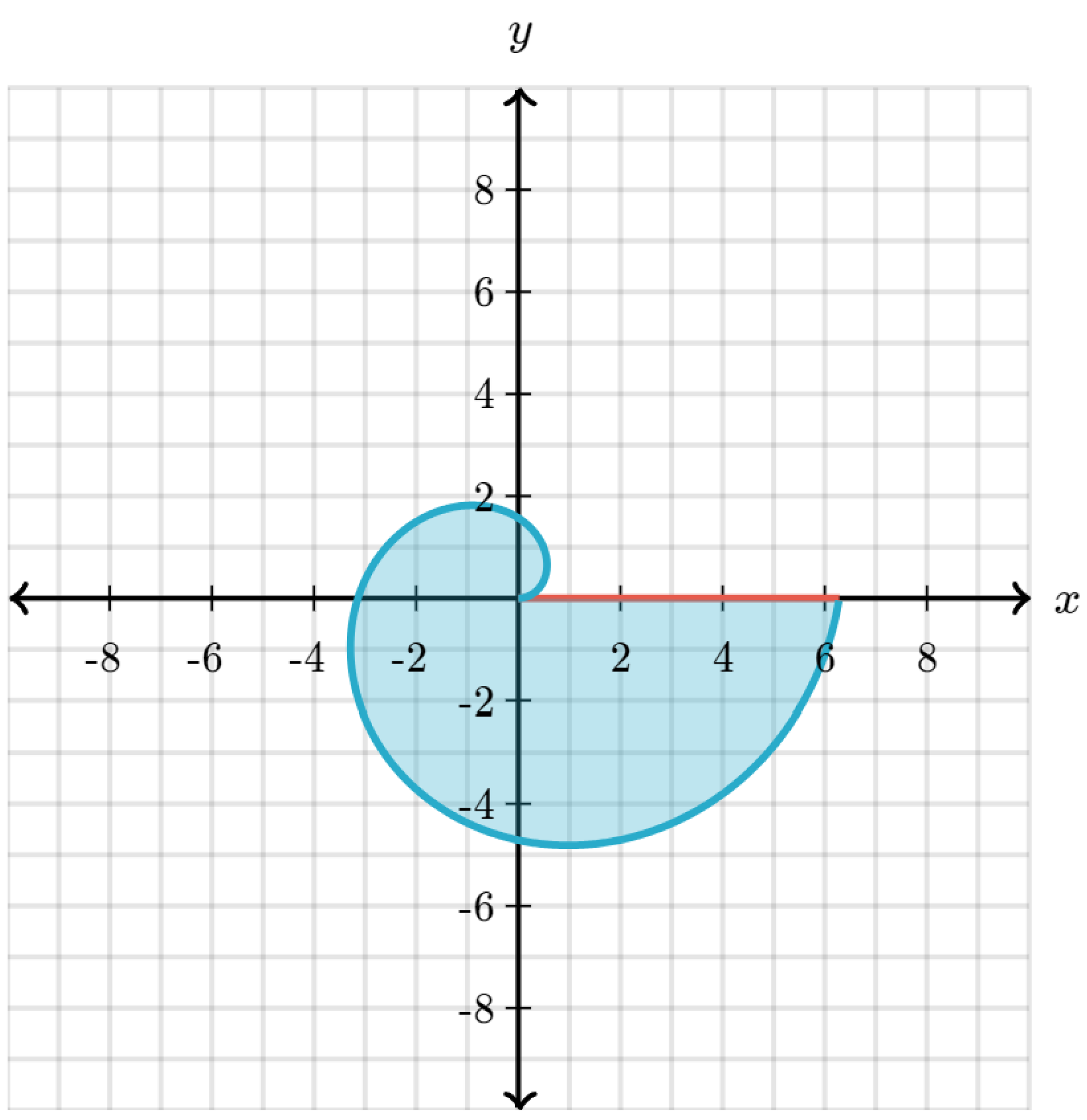
$$\oint_C \frac{1}{2}\left(x\,dy - y\,dx\right)$$

Why is this simpler? You will see in just a moment how things nicely cancel out, and it has to do with symmetrically including both x and y into the expression. Honestly, I'm not sure how you could have seen this ahead of time; it's just really clever.

Step 3: Compute the line integral.

The boundary of our region is defined with two curves. One is the spiral, defined by these two equations in the range $0 \le t \le 2\pi$:

$$x(t) = t\cos(t)$$
$$y(t) = t\sin(t)$$



The other is the line between $(0, 0)$ and $(2\pi, 0)$. Notice, this line is purely on the x-axis. Therefore y is always 0, and dy is also 0, since there's no change in y.
So consider the value of the line integral on this segment:

$$\int \frac{1}{2}\left(x\underbrace{dy}_{0} - \underbrace{y}_{0}\,dx\right)$$

Each part of the integrant is 0, so we can ignore it! Therefore, we can just take this line integral over our spiral and get the answer.

Concept check: Given that $x(t) = t\cos(t)$ and $y(t) = t\sin(t)$, what should we plug in for $x\,dy - y\,dx$ in the line integral? Try to work it out on paper and simplify.

$$x\,dy - y\,dx = \boxed{}\,dt$$

Check

[Hide explanation]

$$\underbrace{x\,dy - y\,dx}_{\text{Plug in functions } x(t),\, y(t)}$$

$$\underbrace{x(t)\left(\frac{dy}{dt}dt\right) - y(t)\left(\frac{dx}{dt}dt\right)}_{\text{Factor out } dt}$$

$$\underbrace{\left(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt}\right)}_{\text{Plug in definitions of } x(t) \text{ and } y(t)}\,dt$$

$$\left(t\cos(t)\underbrace{\frac{d}{dt}(t\sin(t))}_{\text{Product rule}} - t\sin(t)\underbrace{\frac{d}{dt}(t\cos(t))}_{\text{Product rule}}\right)dt$$

$$\Big(t\cos(t)(t\cos(t)+\sin(t)) - t\sin(t)(-t\sin(t)+\cos(t))\Big)\,dt$$

$$\Big(t^2\cos^2(t) + \cancel{t\cos(t)\sin(t)} + t^2\sin^2(t) - \cancel{t\sin(t)\cos(t)}\Big)\,dt$$

$$\Big(t^2\underbrace{(\cos^2(t)+\sin^2(t))}_{1}\Big)\,dt$$

$$t^2\,dt$$

Bring it on home: Use the last answer to compute the following line integral on the spiral, which will give the area of the seashell region as desired:

$$\int_{\text{Spiral}} \frac{1}{2}(x\,dy - y\,dx) = \boxed{}$$

Check

[Hide explanation]

Start by plugging in the value for $x\,dy - y\,dx$ that you found in the last problem, and put in the bounds $t = 0$ and $t = 2\pi$:

$$\int_{\text{Spiral}} \frac{1}{2}\underbrace{(x\,dy - y\,dx)}_{t^2\,dt} = \int_0^{2\pi} \frac{1}{2}t^2\,dt$$

$$= \left[\frac{1}{6}t^3\right]_{t=0}^{t=2\pi}$$

$$= \frac{(2\pi)^3}{6} - \frac{0^3}{6}$$

$$= \frac{8\pi^3}{6}$$

Summary

- Green's theorem can turn tricky line integrals into more straight-forward double integrals.
- To know if Green's theorem will actually make a line integral simpler, ask the following two questions:

$$\oint_C \overbrace{P(x,y)}^{\text{Is } \frac{\partial}{\partial y} \text{ simple?}} \, dx + \overbrace{Q(x,y)}^{\text{Is } \frac{\partial}{\partial x} \text{ simple?}} \, dy$$

- Also, consider if the region encompassed by the curve C will be easy to describe with a double integral, or if it has a known area.

- You can compute the area of a region with the following line integral around its counterclockwise-oriented boundary:

$$\oint_C \frac{1}{2}\left(x\,dy - y\,dx\right)$$