Curl warmup, fluid rotation in two dimensions

Google Classroom

Curl measures the rotation in a fluid flowing along a vector field. Formally, curl only applies to three dimensions, but here we cover the concept in two dimensions to warmup.

Background

- Partial derivatives
- Vector fields

Note: Throughout this article I will use the following convention:

- $\hat{\mathbf{i}}$ represents the unit vector in the x-direction.
- $\hat{\mathbf{j}}$ represents the unit vector in the y-direction.

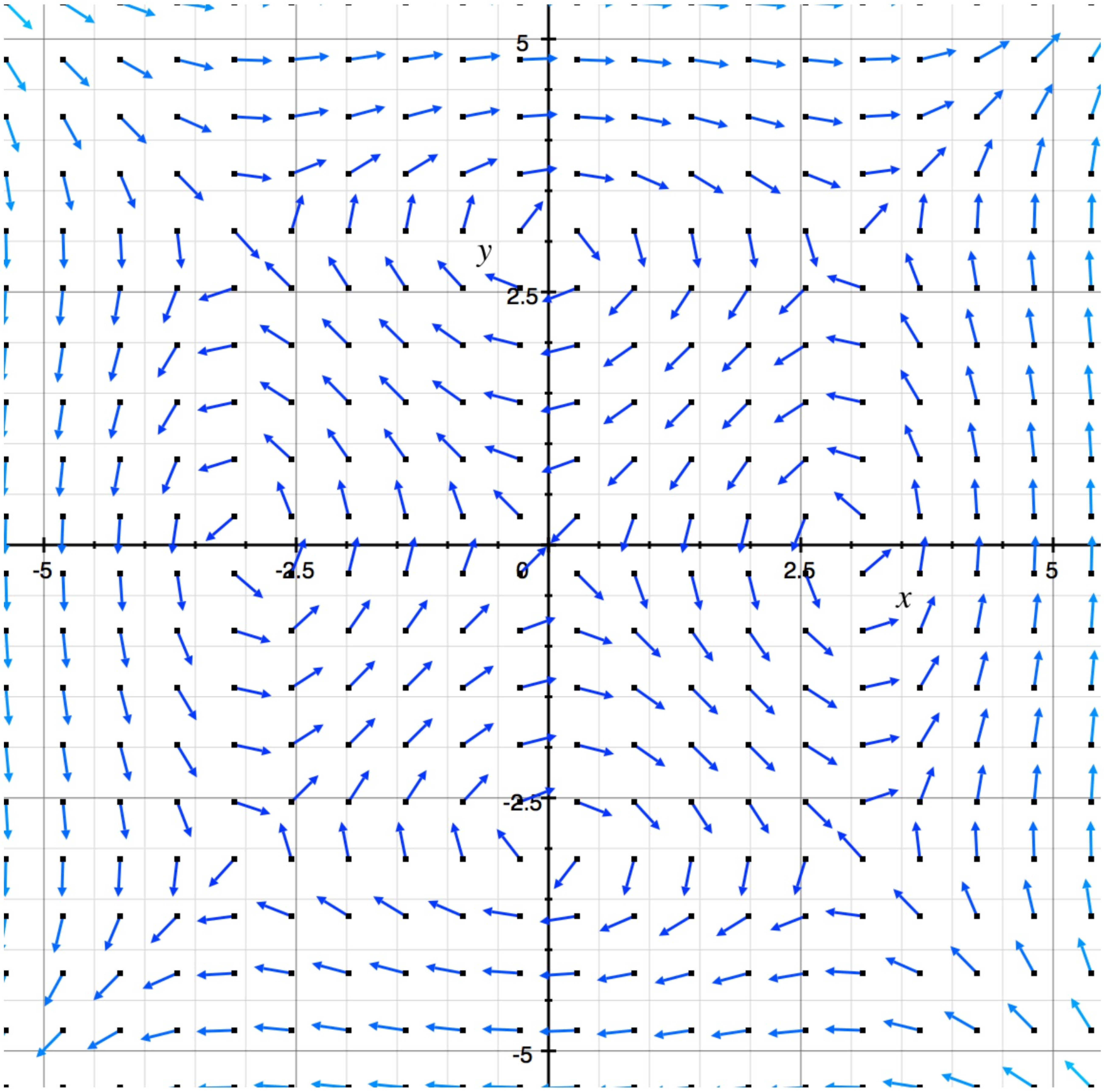
What we're building to

- Curl measures the "rotation" in a vector field.
- In two dimensions, if a vector field is given by a function $\vec{\mathbf{v}}(x, y) = v_1(x, y)\hat{\mathbf{i}} + v_2(x, y)\hat{\mathbf{j}}$, this rotation is given by the formula

$$\text{2d-curl } \vec{\mathbf{v}} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$$

Rotation in fluid flow

Have yourself a nice swirly vector field:

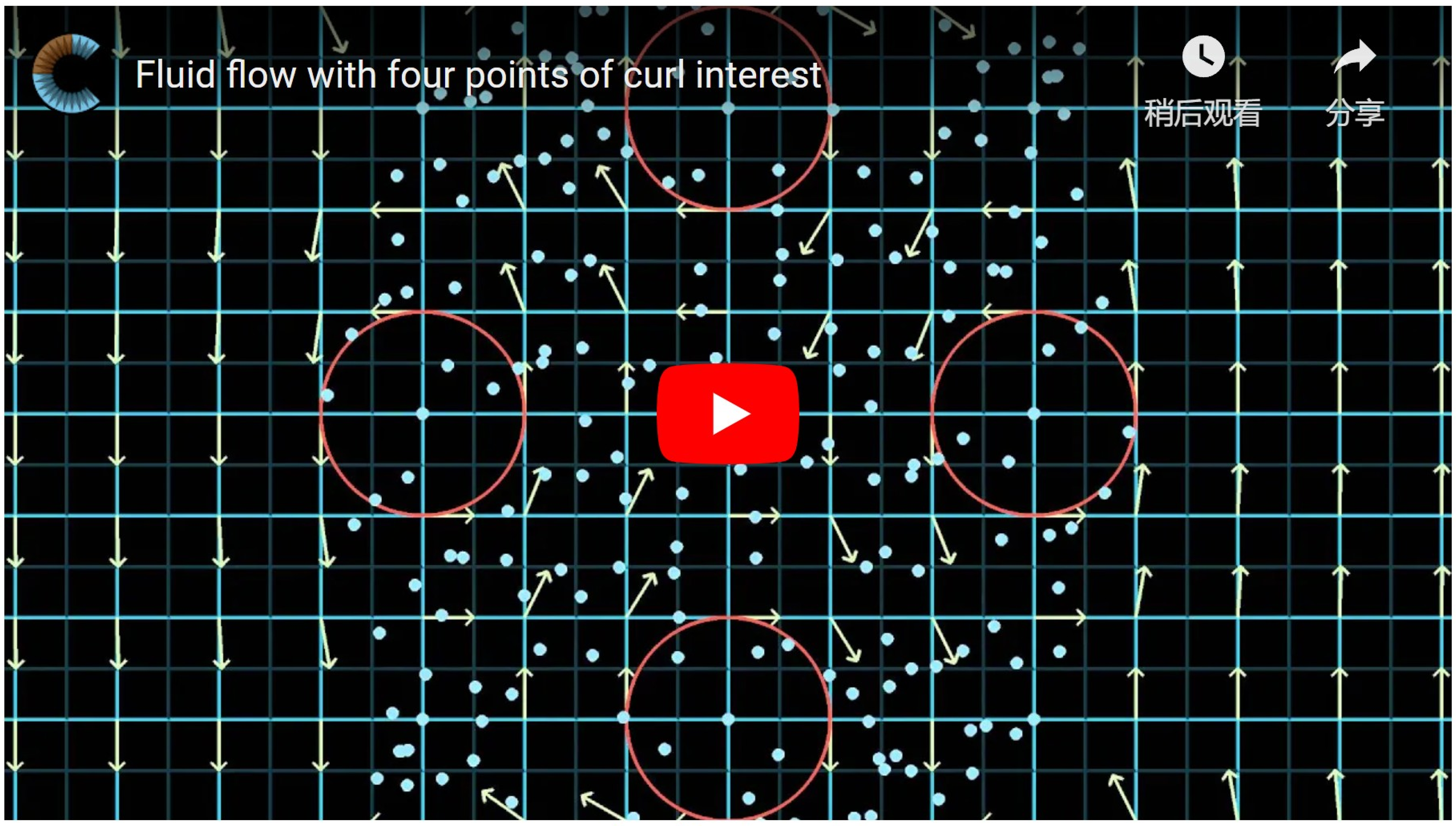


This particular vector field is defined with the following function:

$$\vec{\mathbf{v}}(x, y) = \left[\begin{array}{c} y^3 - 9y \\ x^3 - 9x \end{array} \right]$$

$$= (y^3 - 9y)\hat{\mathbf{i}} + (x^3 - 9x)\hat{\mathbf{j}}$$

Now I want you to imagine that this vector field describes a fluid flow, perhaps in a chaotic part of a river. The following video shows a simulation of what this might look like. A sample of fluid particles, shown as blue dots, will flow along the vector field. This means that at any given moment, each dot moves along the arrow it is closest to. Focus in particular on what happens in the four circled regions.



See video transcript

Amidst all the chaos, you might notice that the fluid is rotating within the circled regions. In the left and right circles, the rotation is counterclockwise, and in the top and bottom circles, the rotation is clockwise.

- **Key Question**: If we are given a function $\vec{\mathbf{v}}(x, y)$ that defines a vector field, along with some specific point in space, (x_0, y_0), how much does a fluid flowing along the vector field rotate at the point (x_0, y_0)?

The vector calculus operation **curl** answer this question by turning this idea of fluid rotation into a formula. It is an operator which takes in a function defining a vector field and spits out a function that describes the fluid rotation given by that vector field at each point.

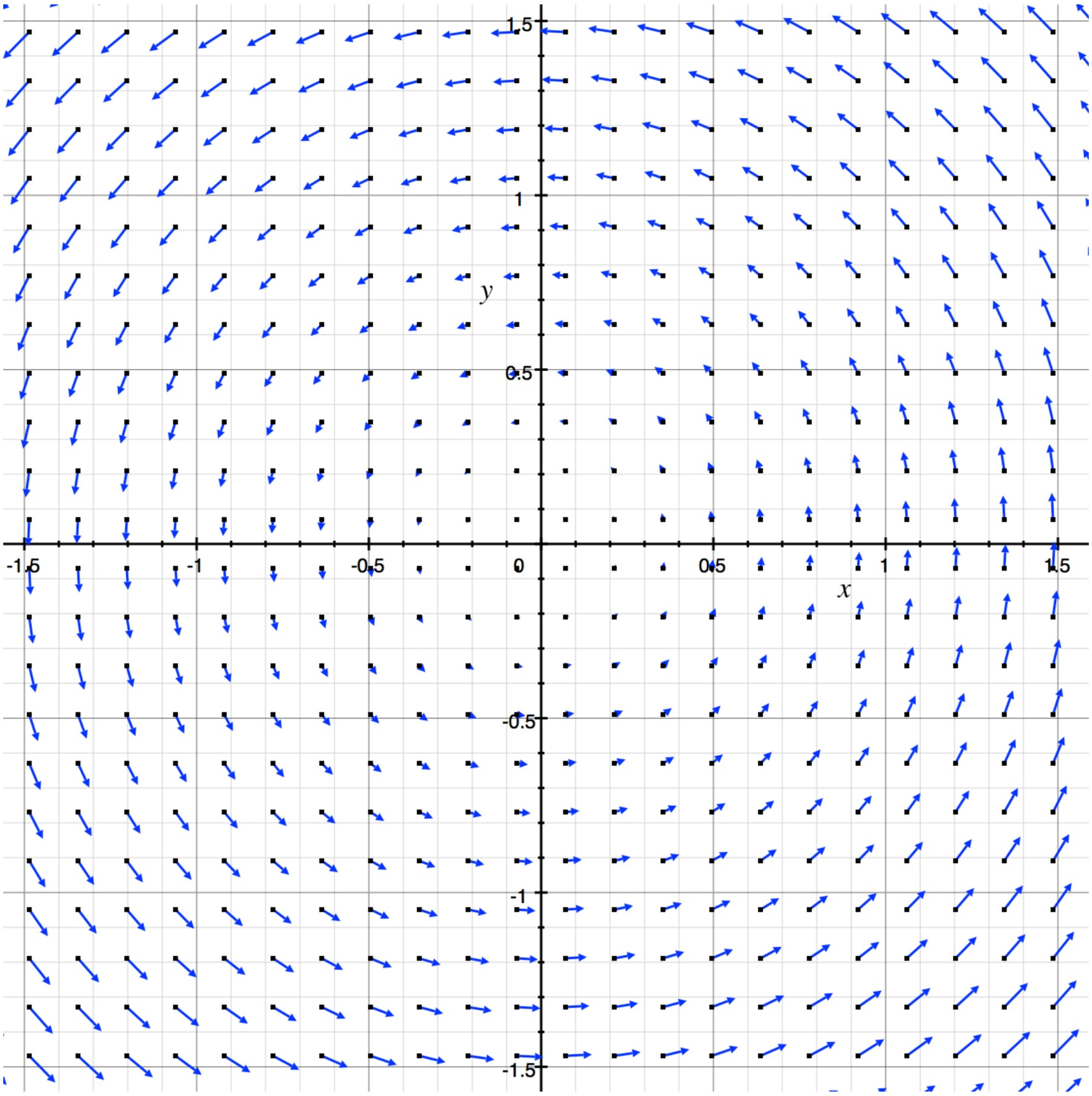
Technically, the curl operation only applies to three dimensions. You can see what that means and how it is computed in the next article, but in this article, we warm up by describing fluid rotation in two dimensions with a formula.

Capturing two-dimensional rotation with a formula

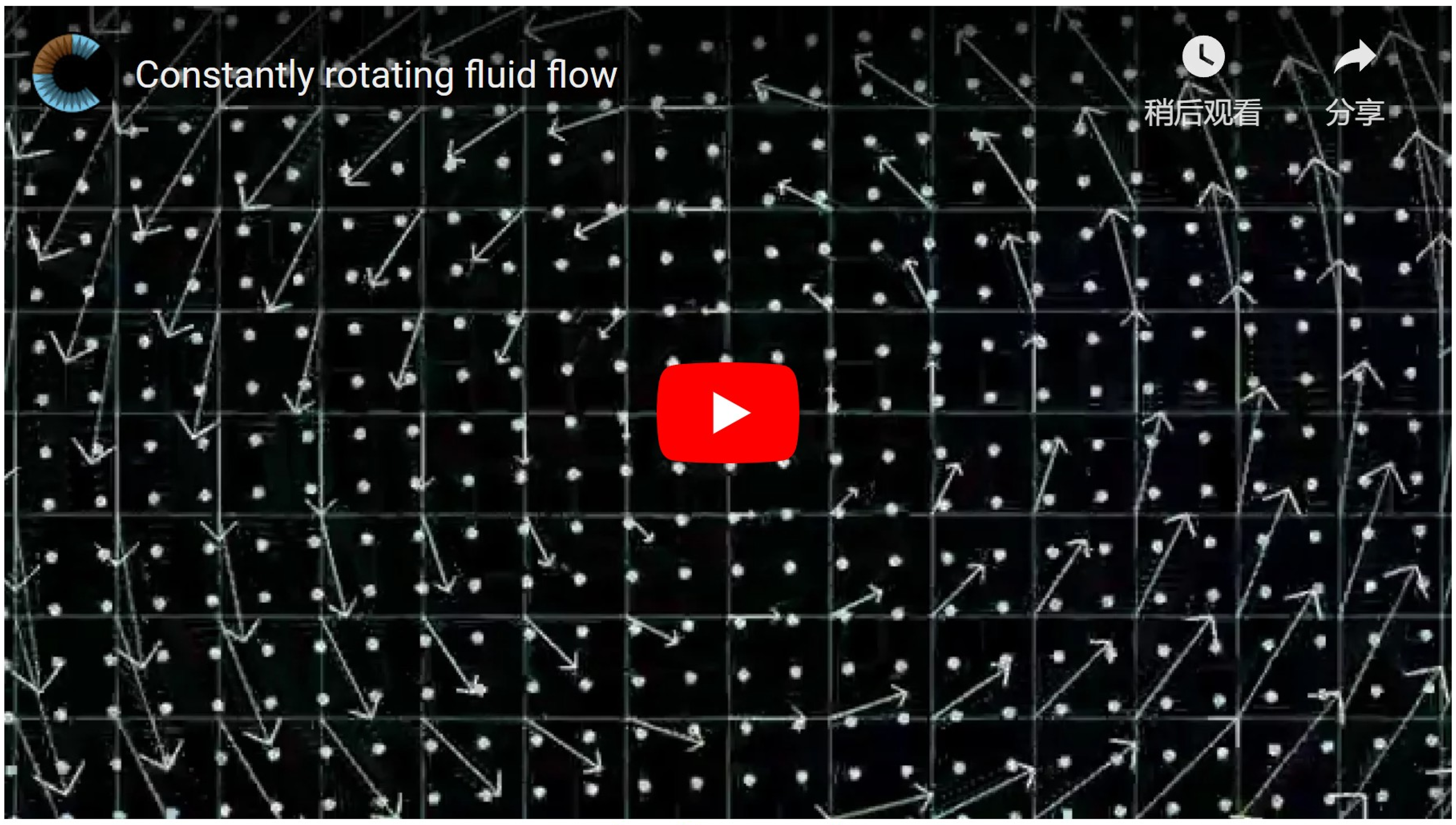
One of the simplest examples of a vector field which describes a rotating fluid is

$$\vec{\mathbf{v}}(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}.$$

Here's what it looks like.



Animated, all the fluid particles just go in circles.

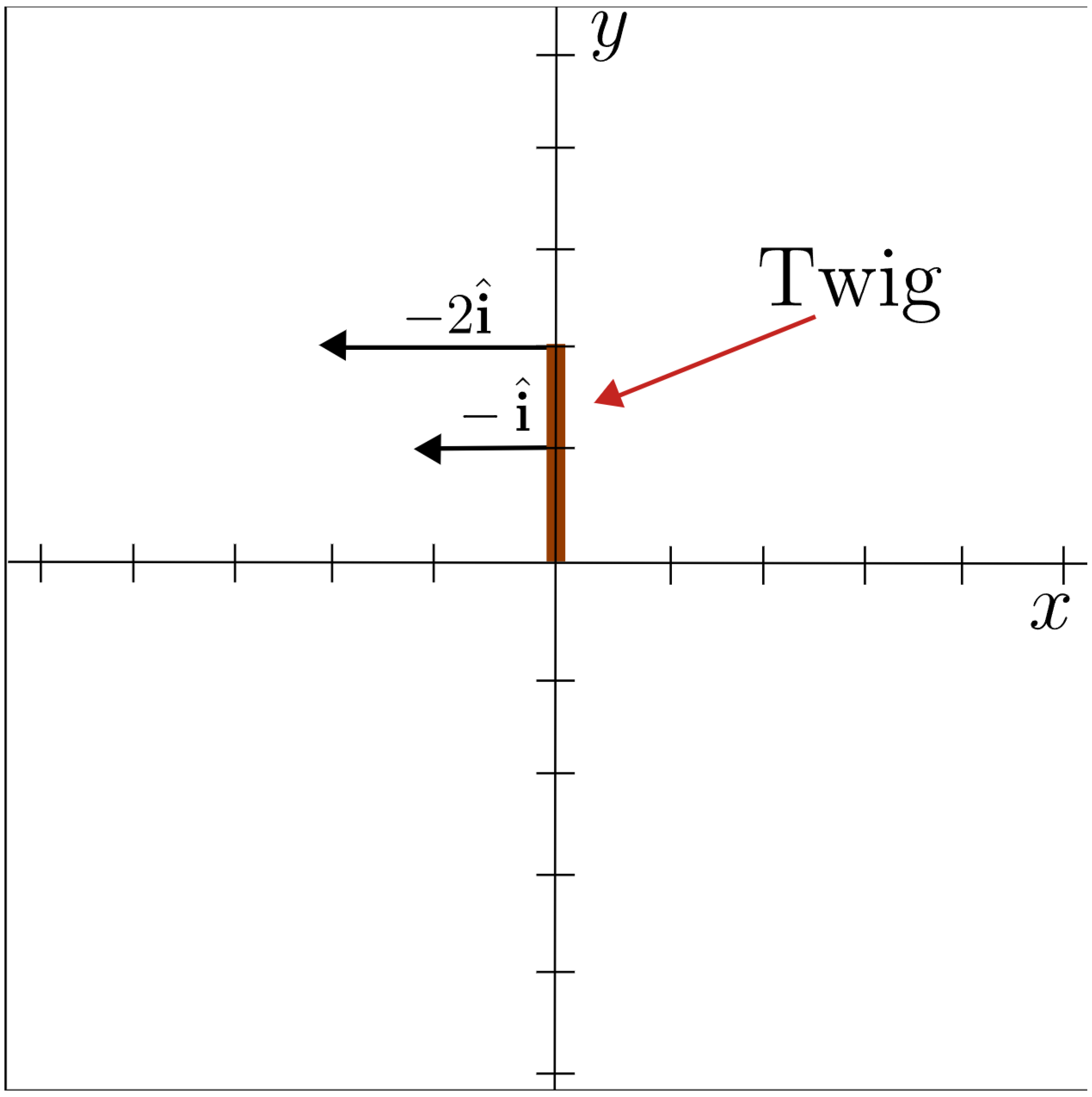


See video transcript

In some sense, this is the most perfect example of counterclockwise rotation, and you can understand the general formula for rotation in a two-dimensional vector field just by understanding why the function $\vec{\mathbf{v}}(x, y) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ gives counterclockwise rotation.

The $\hat{\mathbf{i}}$-component

First, let's understand why the $-y\hat{\mathbf{i}}$ component suggests counterclockwise rotation. Imagine a small twig sitting in our fluid, oriented parallel to the y-axis. More specifically, let's say one end is at the origin $(0, 0)$, and the other is at the point $(0, 2)$. What does the $-y\hat{\mathbf{i}}$ component of the vector field imply for the fluid velocity at points on this twig?



This means the velocity at the top of the twig is $-2\hat{\mathbf{i}}$, a leftward vector, while the velocity at the bottom of the twig is 0.

For the twig, this means the important factor for counterclockwise rotation is that **vectors point more to the left as we move up the vector field**. Said with a few more symbols, the important point here is that the $\hat{\mathbf{i}}$-component of a vector attached to a point (x_0, y_0) decreases as y_0 increases.

Said with even more symbols,

$$\frac{\partial}{\partial y}(-y) = -1 < 0$$

Let's generalize this idea a bit.

- **Question**: Consider a more general vector field.

$$\vec{\mathbf{v}}(x, y) = v_1(x, y)\hat{\mathbf{i}} + v_2(x, y)\hat{\mathbf{j}}$$

The components v_1 and v_2 are any scalar-valued functions. If you place a small twig at some point (x_0, y_0), oriented parallel to the y-axis, how can you tell if the twig will rotate just by looking at v_1, v_2 and (x_0, y_0)?

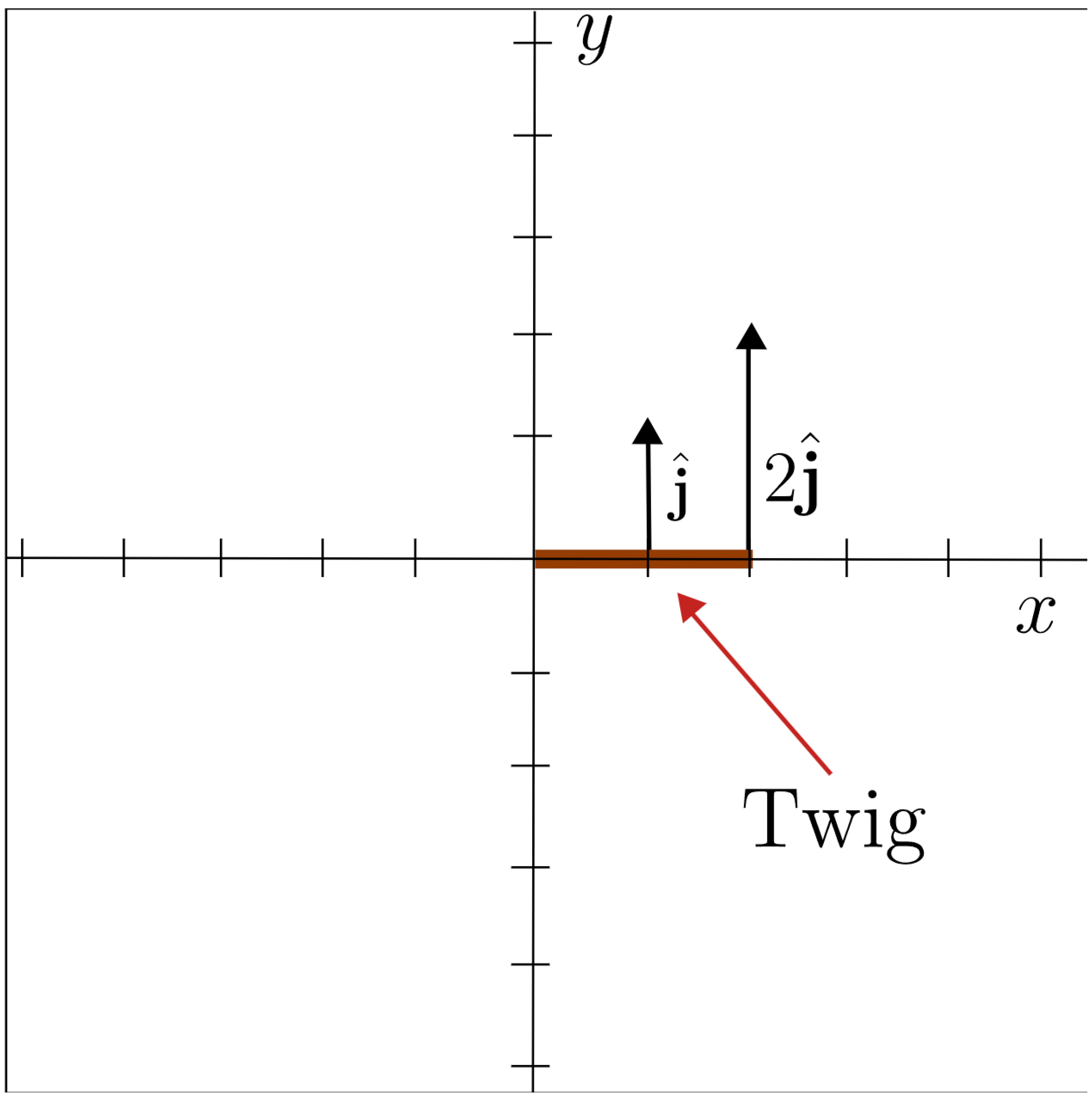
- **Answer**: Look at the rate of change of v_1 as y varies near the point of interest, (x_0, y_0):

  $$\frac{\partial v_1}{\partial y}(x_0, y_0) \quad \leftarrow \text{ Suggests counterclockwise rotation if negati}$$

  If this is negative, it indicates that vectors point more to the left as y_0 increases, so rotation would be counterclockwise. If it is positive, vectors point more to the right as y_0 increases, indicating a clockwise rotation.

The $\hat{\mathbf{j}}$-component

Next, let's see why the $x\hat{\mathbf{j}}$ component of the original vector field suggests counterclockwise rotation as well. This time, imagine a twig which is parallel to the x-axis. Specifically, put one end of the twig at the origin $(0, 0)$, and put the other at the point $(2, 0)$.



The vector attached to the origin is 0, but the vector attached to the other end at $(2, 0)$ is $2\hat{\mathbf{j}}$, an upward vector. Therefore, the fluid pushes the right end of the stick upwards, and the left end experiences no force, so there will be a counterclockwise rotation.

For this second twig, **the vertical component of vectors increases as we move right, suggesting counterclockwise rotation**. That is to say, the y component of a vector attached to a point (x_0, y_0) increases as x_0 increases.

In the case of a more general vector field function,

$$\vec{\mathbf{v}}(x, y) = v_1(x, y)\hat{\mathbf{i}} + v_2(x, y)\hat{\mathbf{j}}$$

we can measure this effect near a point (x_0, y_0) by looking at the change in v_2 as x changes.

$$\frac{\partial v_2}{\partial x} \quad \leftarrow \text{Suggests counterclockwise rotation if positive}$$

Combining both components

Putting these two components together, the rotation of a fluid flowing along a vector field $\vec{\mathbf{v}}$ near a point (x_0, y_0) can be measured using the following quantity:

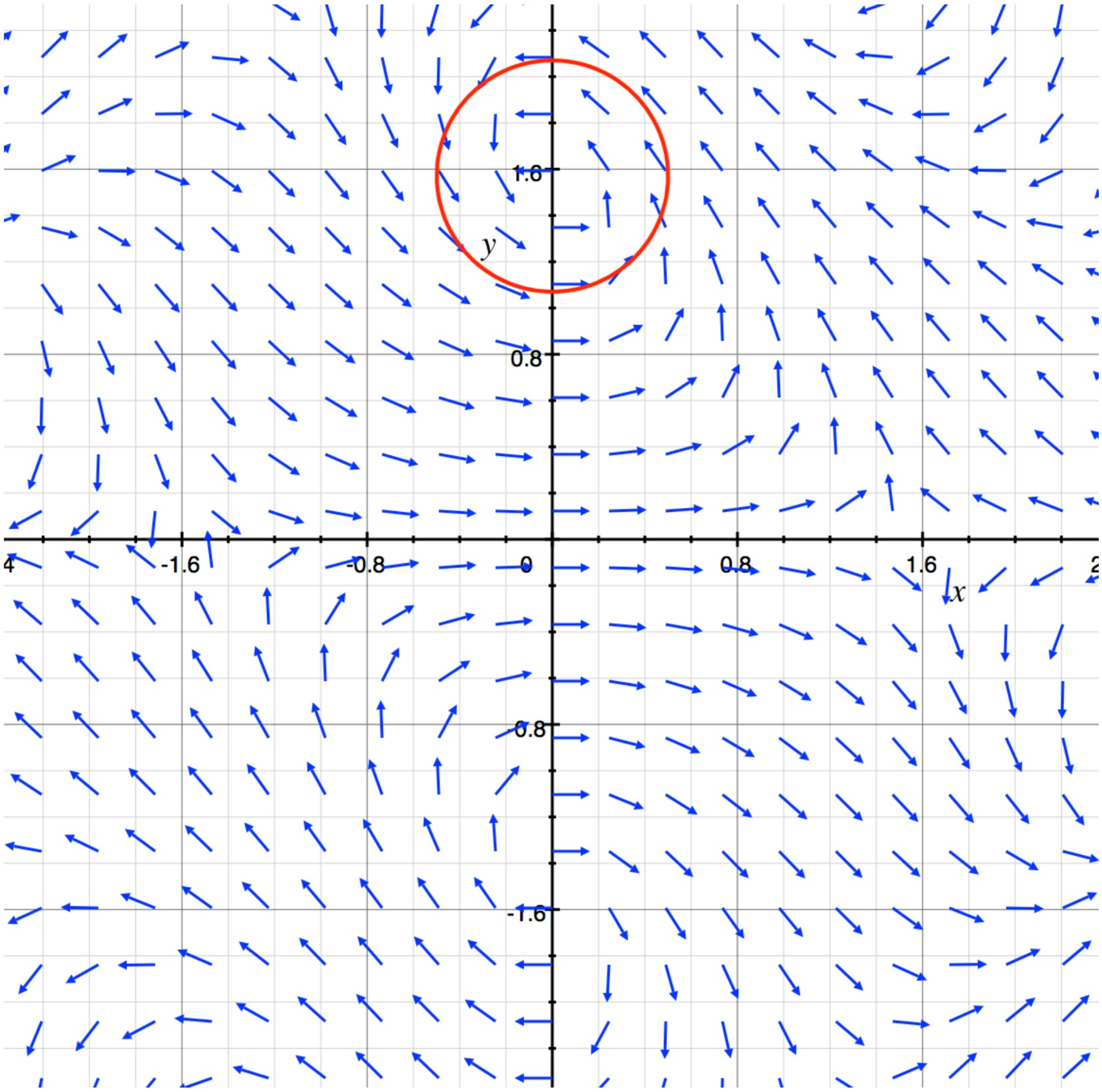
$$\frac{\partial v_2}{\partial x}(x_0, y_0) - \frac{\partial v_1}{\partial y}(x_0, y_0)$$

When you evaluate this, a **positive** number will indicate a general tendency to rotate **counterclockwise** around (x_0, y_0), a negative quantity indicates the opposite, clockwise rotation. If it equals 0, there is no rotation in the fluid around (x_0, y_0). If you are curious about the specifics, this formula gives precisely *twice* the angular velocity of the fluid near (x_0, y_0).

Some authors will call this the "**two-dimensional curl**" of $\vec{\mathbf{v}}$. This isn't standard, but let's write this formula as if "**2d-curl**" was an operator.

$$\text{2d-curl } \vec{\mathbf{v}} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$$

Example: Analyzing rotation in a 2d vector field using curl



Problem: Consider the vector field defined by the function

$$\vec{\mathbf{v}}(x, y) = \begin{bmatrix} \cos(x+y) \\ \sin(xy) \end{bmatrix}$$

Determine whether a fluid flowing according to this vector field has clockwise or counterclockwise rotation at the point

$$p = \left(0, \frac{\pi}{2}\right)$$

Step 1: Compute the 2d-curl of this function.

$\text{2d-curl } \vec{\mathbf{v}} =$

Check

[Hide explanation]

We apply the **2d-curl** formula we just found to this function. This will give us a new, scalar-valued function that indicates the rotation near each point. Then, we'll plug in the point $(0, \pi/2)$ to see whether the rotation there is positive (counterclockwise), zero, or negative (clockwise).

$$\begin{aligned}
\text{2d-curl } \vec{\mathbf{v}} &= \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \\
&= \frac{\partial}{\partial x}(\sin(xy)) - \frac{\partial}{\partial y}(\cos(x+y)) \\
&= \cos(xy)y - (-\sin(x+y)) \\
&= \boxed{\cos(xy)y + \sin(x+y)}
\end{aligned}$$

Step 2: Plug in the point $(0, \pi/2)$.

$\text{2d-curl } \vec{\mathbf{v}}(0, \pi/2) =$

Check

[Hide explanation]

$$\begin{aligned}
p = \left(0, \frac{\pi}{2}\right) \rightarrow \cos\left(0 \cdot \frac{\pi}{2}\right)\frac{\pi}{2} + \sin\left(0 + \frac{\pi}{2}\right) \\
= 1 \cdot \frac{\pi}{2} + 1 \\
= \frac{\pi + 2}{2}
\end{aligned}$$

Step 3: Interpret. How does the fluid tend to rotate near this point?

Choose 1 answer:

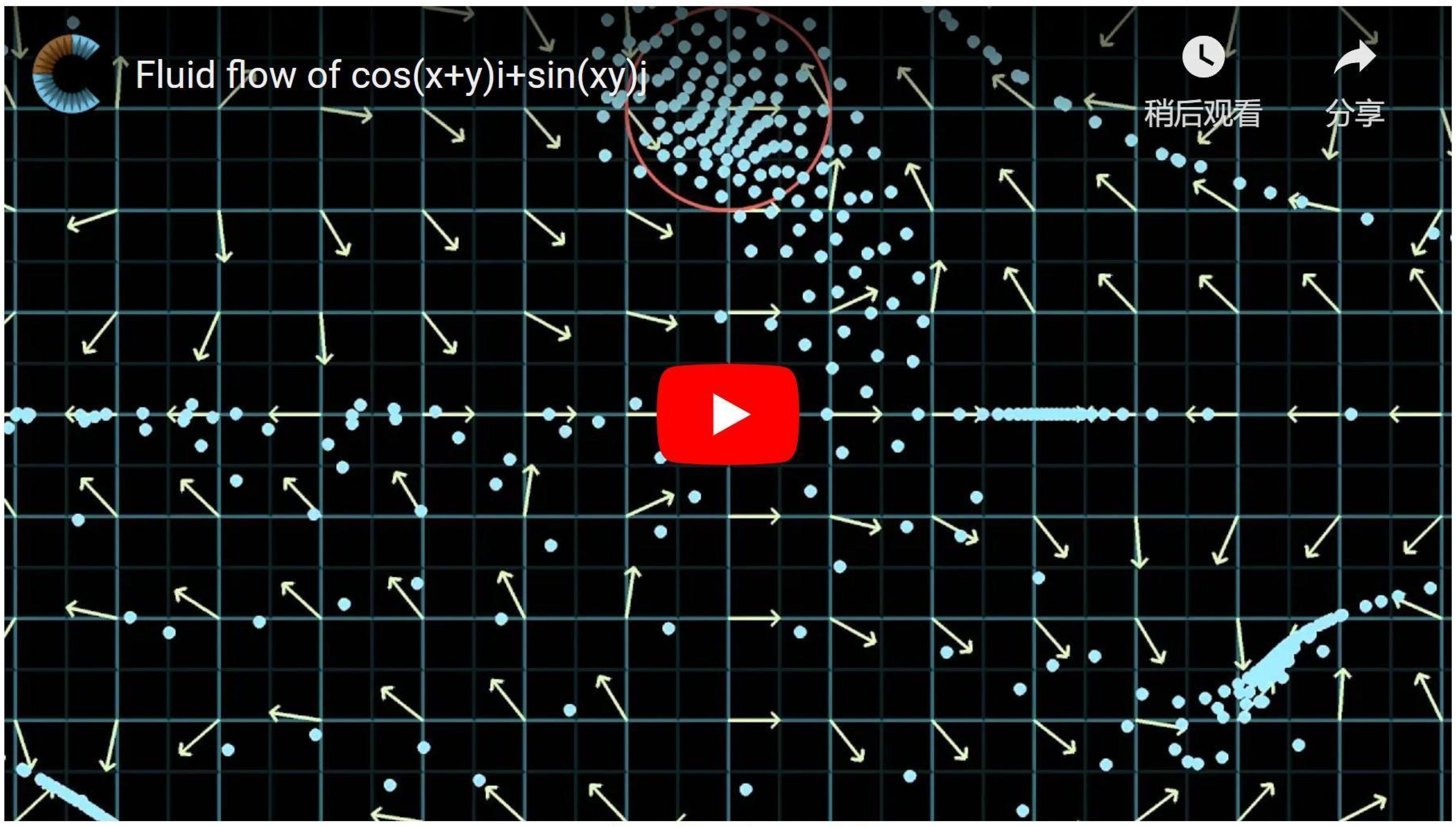
- (A) Clockwise
- (B) Counterclockwise
- (C) No rotation

Check

[Hide explanation]

Because this is a positive value, **the fluid will tend to flow counterclockwise at the point** p.

Let's watch a sample of particles in this fluid flow:



See video transcript

The point towards the top where all the particles congregate corresponds with $p = \left(0, \frac{\pi}{2}\right)$. Particles rotate counterclockwise in this region, which should be consistent with your 2d-curl calculations.

Summary

- Curl measures the "rotation" in a vector field.

- In two dimensions, if a vector field is given by a function $\vec{\mathbf{v}}(x, y) = v_1(x, y)\hat{\mathbf{i}} + v_2(x, y)\hat{\mathbf{j}}$, this rotation is given by the formula

$$2\text{d-curl}\,\vec{\mathbf{v}} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$$

On to the third dimension!

The true curl operation, covered in the next article, extends this idea and this formula to three dimensions.