Triple integrals in cylindrical coordinates

Google Classroom

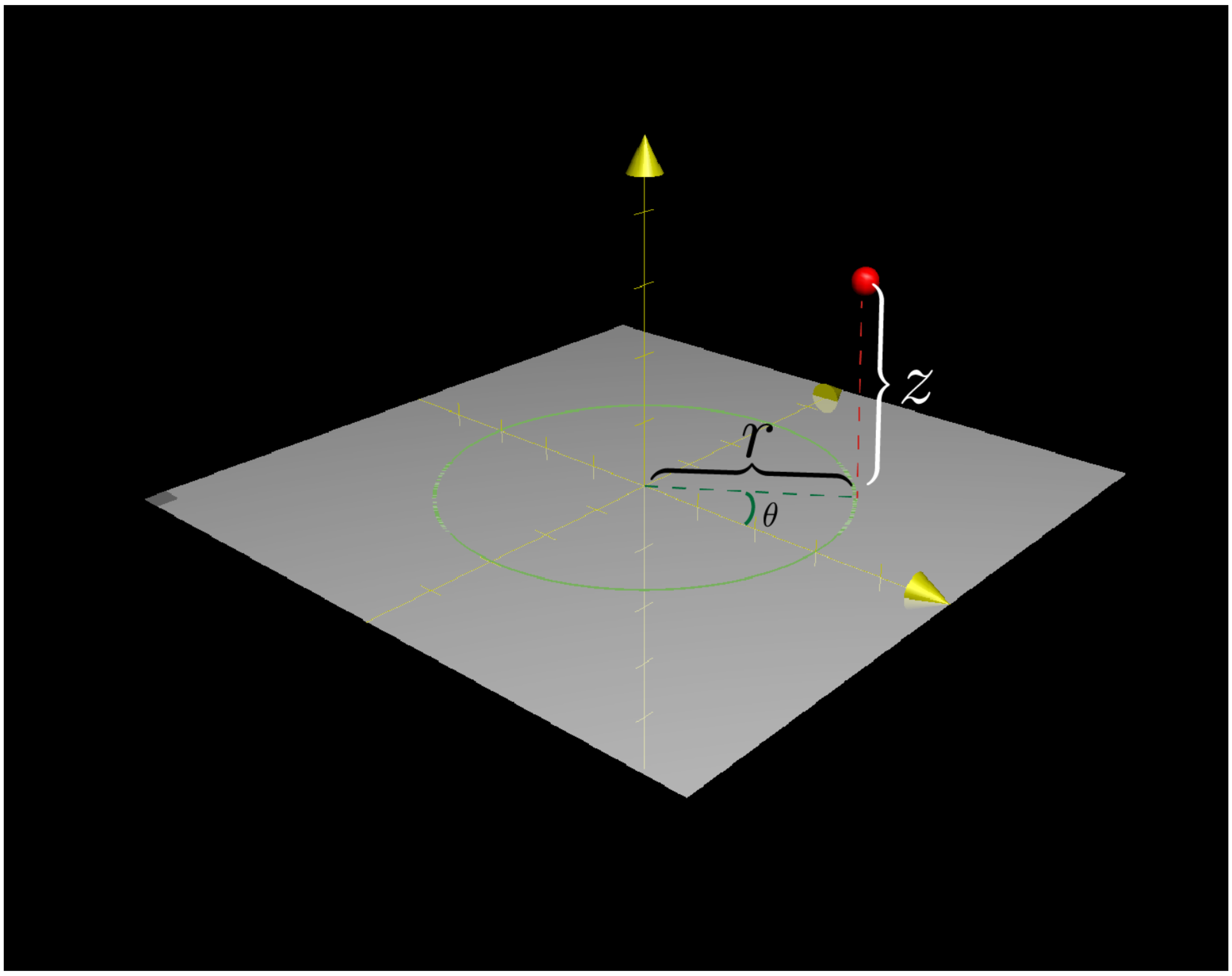
How to perform a triple integral when your function and bounds are expressed in cylindrical coordinates.

Background

- Triple integrals
- Double integrals in polar coordinates

[Hide explanation]

Cylindrical coordinates are essentially the same as polar coordinates in two-dimensions, just with a z-coordinate thrown in to make it three-dimensional.



Each point in space is described with three coordinates:

- r
- θ
- z

As with any coordinate system, these three numbers give instructions for how to get to a given point starting at the origin.

- Start by drawing a line of length r away from the origin, along the x-axis.

- Rotate that line by θ radians on the xy-plane, counterclockwise away from the x-axis, fixing one end at the origin.
- From the tip of the line, move a distance z in the third dimension, perpendicular to the xy-plane.

Converting between cylindrical coordinates and cartesian coordinates is the same as converting to polar coordinates in two dimensions.

- $x^2 + y^2 = r^2$
- $\tan(\theta) = \frac{y}{x}$

The only new coordinate to three dimensions, z, remains unchanged as you convert back and forth.

Note, many authors will use the coordinates (ρ, ϕ, z) instead of (r, θ, z), but I have opted to use (r, θ, z) to emphasize the connection with polar coordinates.

What we're building to

- The main thing to remember about triple integrals in cylindrical coordinates is that dV, representing a tiny bit of volume, is expanded as

  $$dV = r\,d\theta\,dr\,dz$$

  (Don't forget to include the r)

- Using cylindrical coordinates can greatly simplify a triple integral when the region you are integrating over has some kind of rotational symmetry about the z-axis.

The one rule

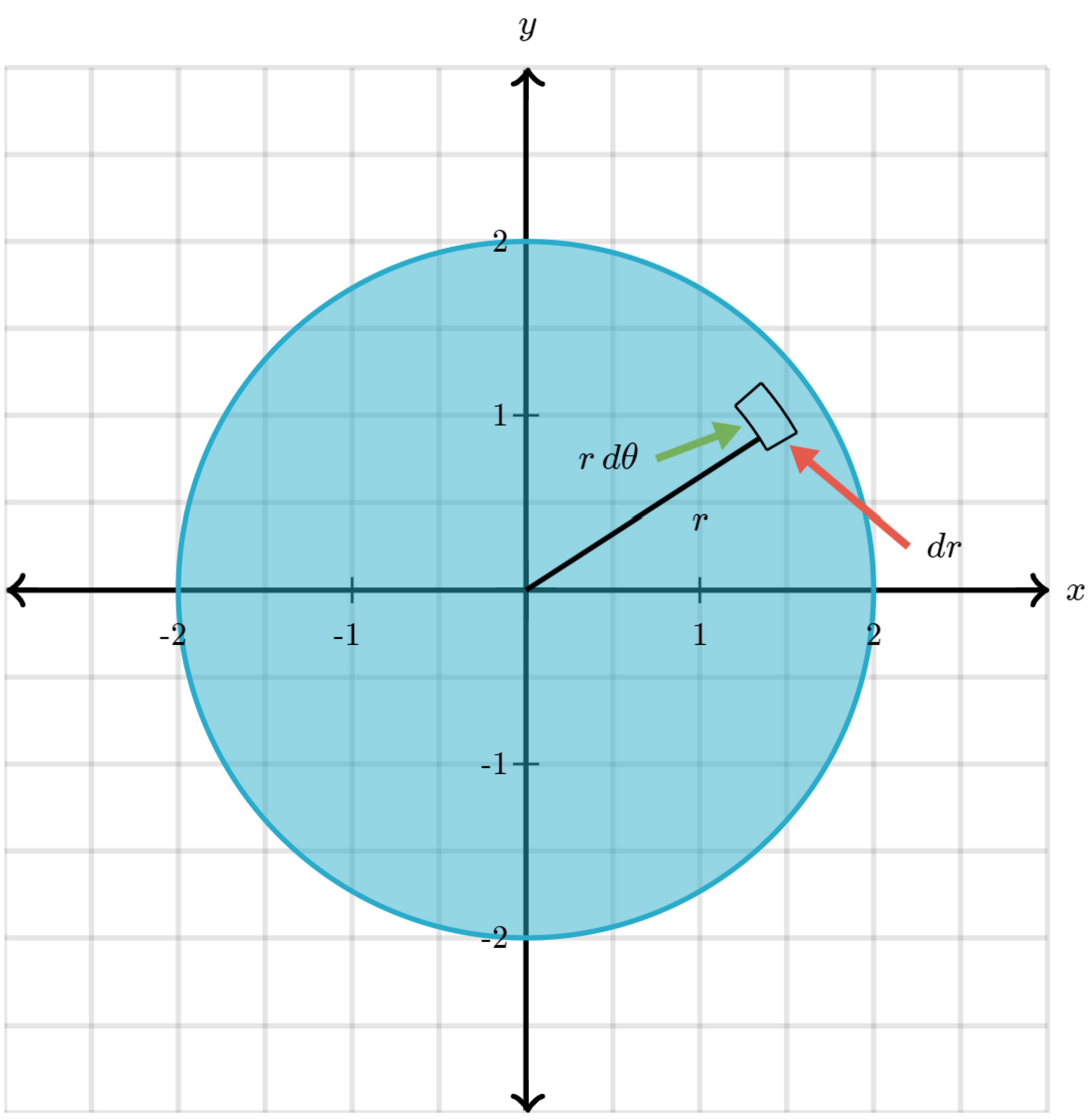
When performing double integrals in polar coordinates, the one key thing to remember is how to expand the tiny unit of area dA in terms of dr and $d\theta$

$$\iint_R f(r,\theta)\,dA = \iint_R f(r,\theta)\,r\,d\theta\,dr$$

Note that the variable r is part of this expansion. Expanding the tiny unit of volume dV in a triple integral over cylindrical coordinates is basically the same, except that now we have a dz term:

$$\iiint_R f(r, \theta, z)\, dV = \iiint_R f(r, \theta, z)\, r\, d\theta\, dr\, dz$$

Remember, the reason this little r shows up for polar coordinates is that a tiny "rectangle" cut by radial and circular lines has side lengths $r\, d\theta$ and dr.

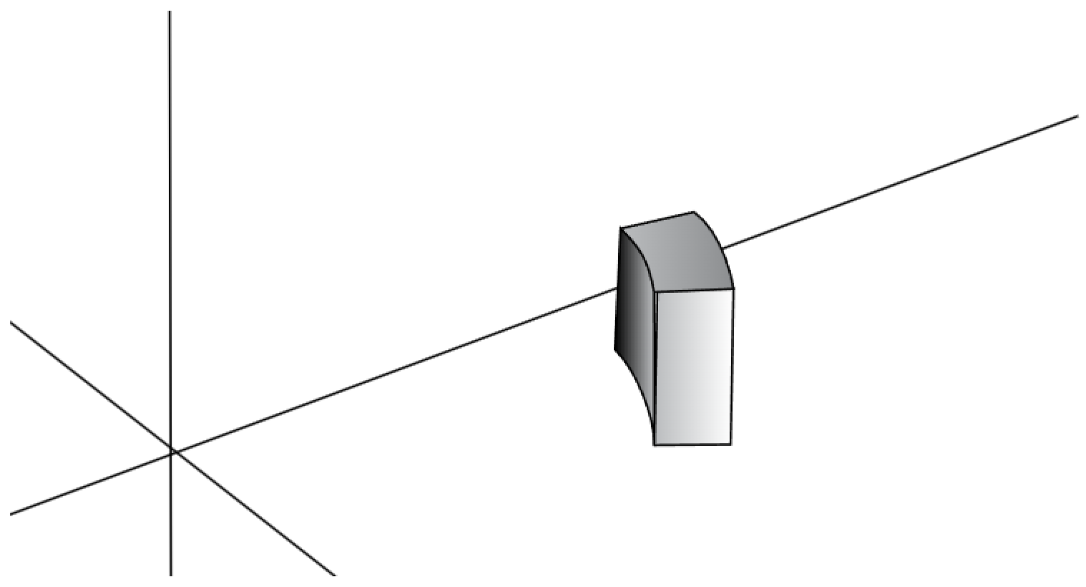


The key thing to remember here is that θ **is not a unit of length**, so $d\theta$ does not represent a tiny length in the same way that dr and dz do. It measures radians, which need to be multiplied by the distance r from the origin to become a length.

[Hide explanation]

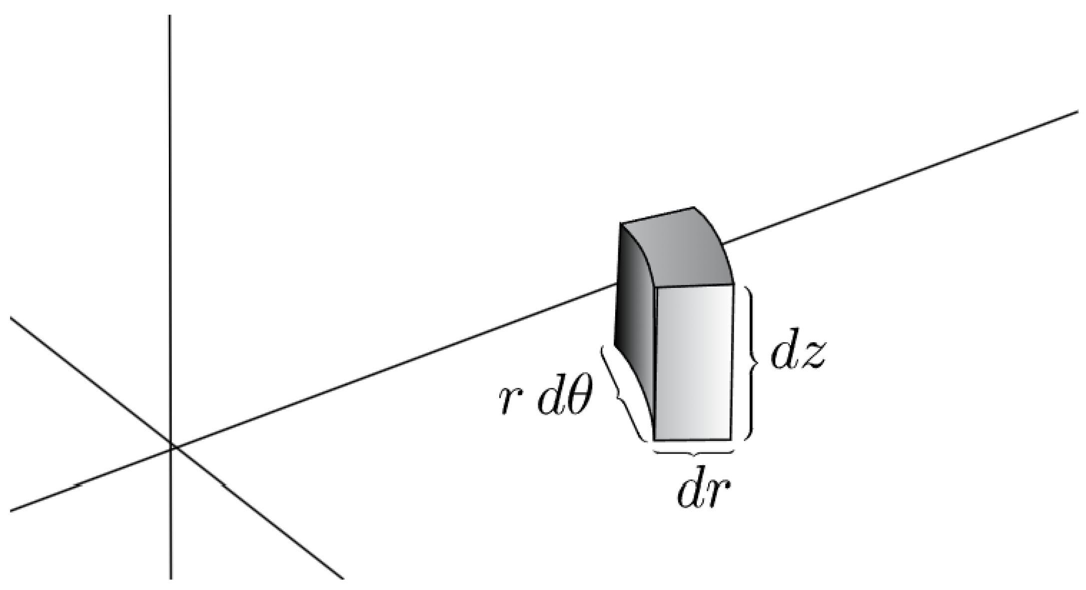
Think of your little unit of volume dV as a block with the following properties.

- One pair of faces represent constant values of θ.
- One pair of faces represent constant values of r (these will be slightly curved, as if hugging a cylinder).
- One pair of faces represent constant values of z (these are the flat top and bottom).



When you are integrating over a three-dimensional region R, you can think of it as being composed of infinitely many infinitely small blocks like this. When they are small enough, the curved edge will basically be a straight line, so you can treat it as a rectangular prism.

As the size of the block approaches zero, each edge will represent an infinitesimal change in one variable.



- One edge will have length dr, representing a small change in r while θ and z stay constant.

- One edge will have length dz, representing a small change in z while r and θ stay constant.

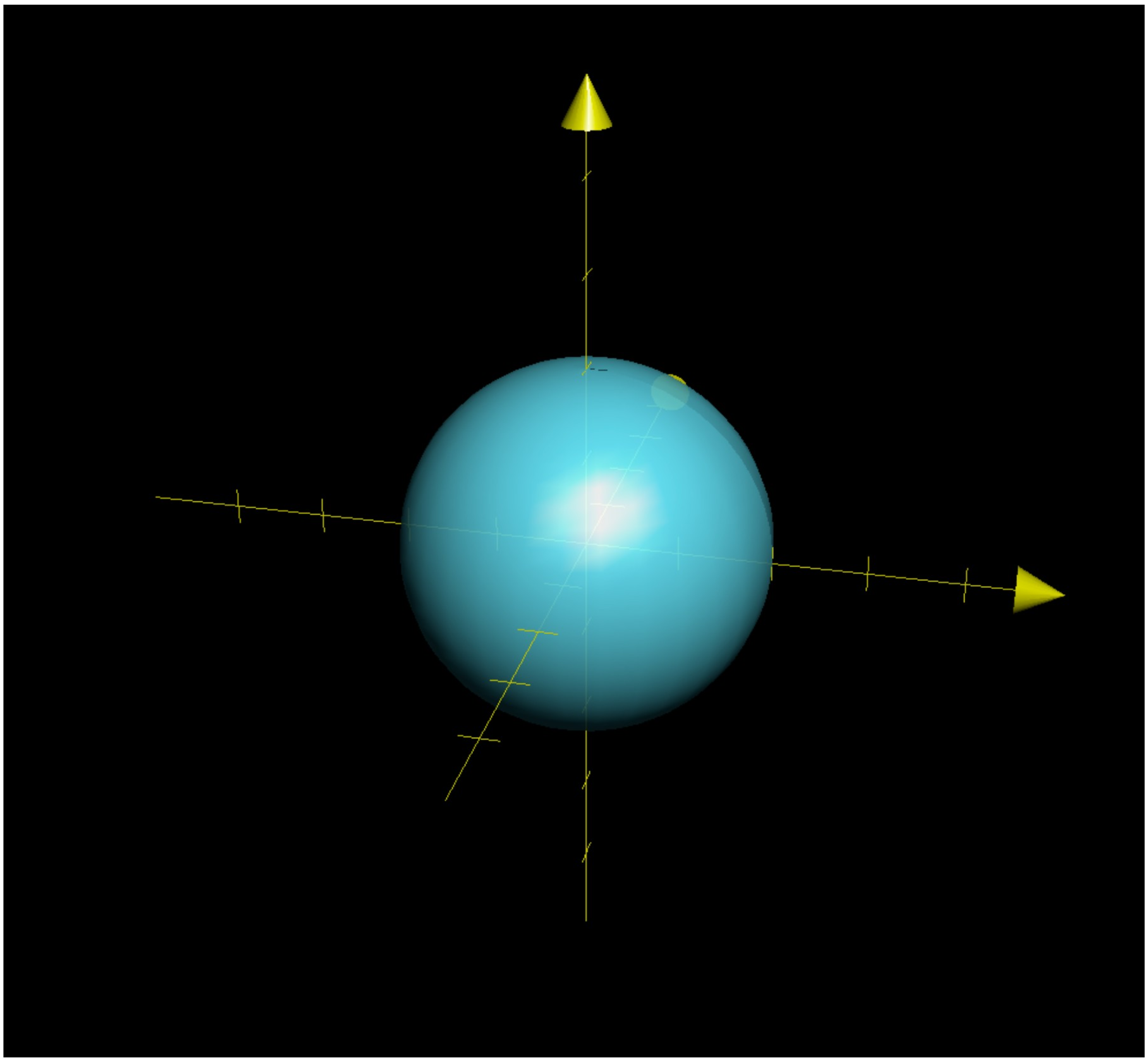
- Another edge will represent a tiny change in θ while r and z remain constant. The length of this edge is not $d\theta$, since θ is not a unit of length. Its length is $r\,d\theta$ where r is the distance from the box to the origin.

In total, the volume of this box is $dV = (r\,d\theta)(dr)(dz)$.

Example 1: Volume of a sphere

Problem: Find the volume of a sphere with radius 1 using a triple integral in cylindrical coordinates.

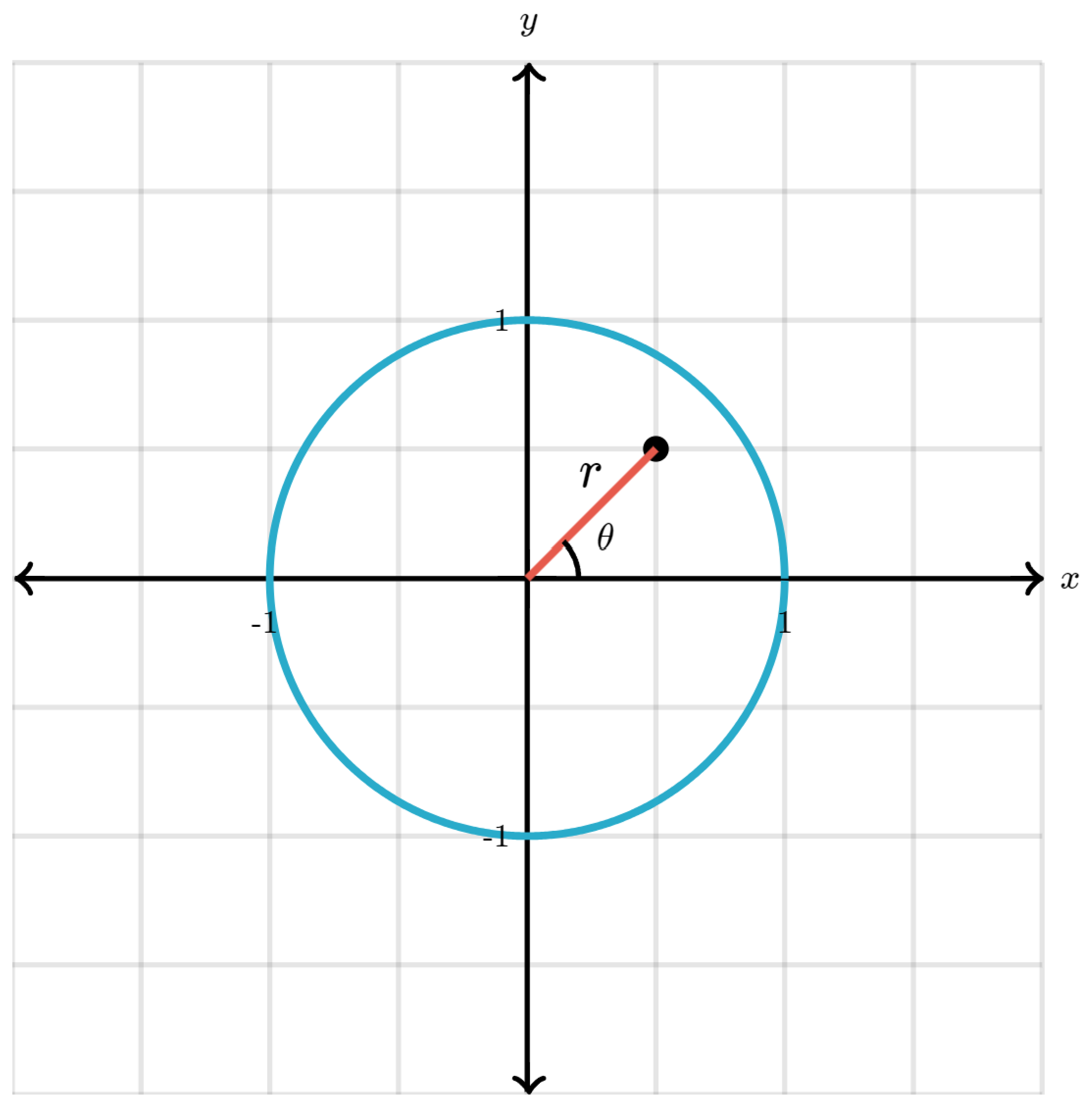
First of all, to make our lives easy, let's place the center of the sphere on the origin.



Next, I'll give the sphere a name, S, and write the abstract triple integral to find its volume.

$$\iiint_S dV = \iiint_S r \, d\theta \, dr \, dz$$

As always, the hard part is putting bounds on the integral. However, doing this with cylindrical coordinates is much easier than it would be for cartesian coordinates. In particular, r and θ will just live within the unit disc, which is very natural to describe in polar coordinates:



Concept check: Which of the following sets of bounds for r and θ should we use to integrate over the unit disc?

Choose 1 answer:

(A)
- $0 \leq r \leq 1$
- $0 \leq \theta \leq 2\pi$

(B)
- $-1 \leq r \leq 1$
- $0 \leq \theta \leq 2\pi$

(C)
- $0 \leq r \leq 1$
- $0 \leq \theta r \leq 2\pi$

Check

[Hide explanation]

The first choice is correct:

- $0 \leq r \leq 1$

- $0 \leq \theta \leq 2\pi$

Within the unit disc, the radius varies between 0 and 1, and the angle θ runs along its full range from 0 to 2π.

Since the bounds of z will depend on the value of r, we let the innermost integral handle z, while the outer two integrals take care of r and θ. Writing down what we have so far, we get

$$\int_0^{2\pi}\int_0^1\int_?^? r\,dz\,dr\,d\theta$$

Remember, it's important to make sure the order of the differential terms dz, dr and $d\theta$ matches up with the appropriate integral.

This next question is a little trickier.

Concept check: For a given value of r, which of the following shows the right range of values for z?

Choose 1 answer:

(A) $-\sqrt{1-r^2} \le z \le \sqrt{1-r^2}$

(B) $-\sqrt{1+r^2} \le z \le \sqrt{1+r^2}$

(C) $-r^2 \le z \le r^2$

(D) $-r \le z \le r$

Check

[Hide explanation]

The first choice is correct.

$$-\sqrt{1-r^2} \le z \le \sqrt{1-r^2}$$

In cylindrical coordinates, the distance from a point (r, θ, z) to the origin is

$$\sqrt{r^2+z^2}$$

To see why, relate this to cartesian coordinates using the fact that $r^2 = x^2 + y^2$

$$\sqrt{r^2+z^2} = \sqrt{x^2+y^2+z^2}$$

Therefore, the unit sphere could be defined as all points such that

$$r^2 + z^2 = 1^2.$$

Solving for z, we get its upper and lower bounds if it is to remain inside the sphere.

$$r^2 + z^2 = 1$$

$$z^2 = 1 - r^2$$

$$z = \pm\sqrt{1-r^2}$$

Applying this bound to our innermost integral, we get something that can be worked out.

Concept check: Solve this triple integral.

$$\int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta = \square$$

Check

[Hide explanation]

$$\int_0^{2\pi} \int_0^1 \underbrace{\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r\,dz}_{\text{Integrate with respect to } z} \quad dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^1 \Big[rz\Big]_{z=-\sqrt{1-r^2}}^{z=\sqrt{1-r^2}} dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^1 r\left(\sqrt{1-r^2} - (-\sqrt{1-r^2})\right)\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^1 2r\sqrt{1-r^2}\,dr\,d\theta$$

Using the inverse chain rule, we can quickly scribble down some notes to see what this antiderivative is:

$$\int 2r\sqrt{1-r^2}\,dr = 2r \underbrace{\left(\frac{2}{3}\left(1-r^2\right)^{3/2}\right)}_{\text{Antiderivative of } \sqrt{\text{something}}} \overbrace{\frac{1}{-2r}}^{\text{Divide by } \frac{d}{dr}(1-r^2)}$$

$$= -\frac{2}{3}\left(1-r^2\right)^{3/2}$$

Continuing on with our integral then,

$$= \int_0^{2\pi}\int_0^1 2r\sqrt{1-r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{-2}{3}(1-r^2)^{3/2}\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi}\left(\frac{-2}{3}(1-(1)^2)^{3/2} - \left(\frac{-2}{3}(1-(0)^2)^{3/2}\right)\right) d\theta$$

$$= \int_0^{2\pi}\left(\frac{-2}{3}(0) - \frac{-2}{3}(1)\right) d\theta$$

$$= \int_0^{2\pi}\frac{2}{3}\,d\theta$$

$$= \frac{4}{3}\pi$$

And with that, you just found the volume of a unit sphere!

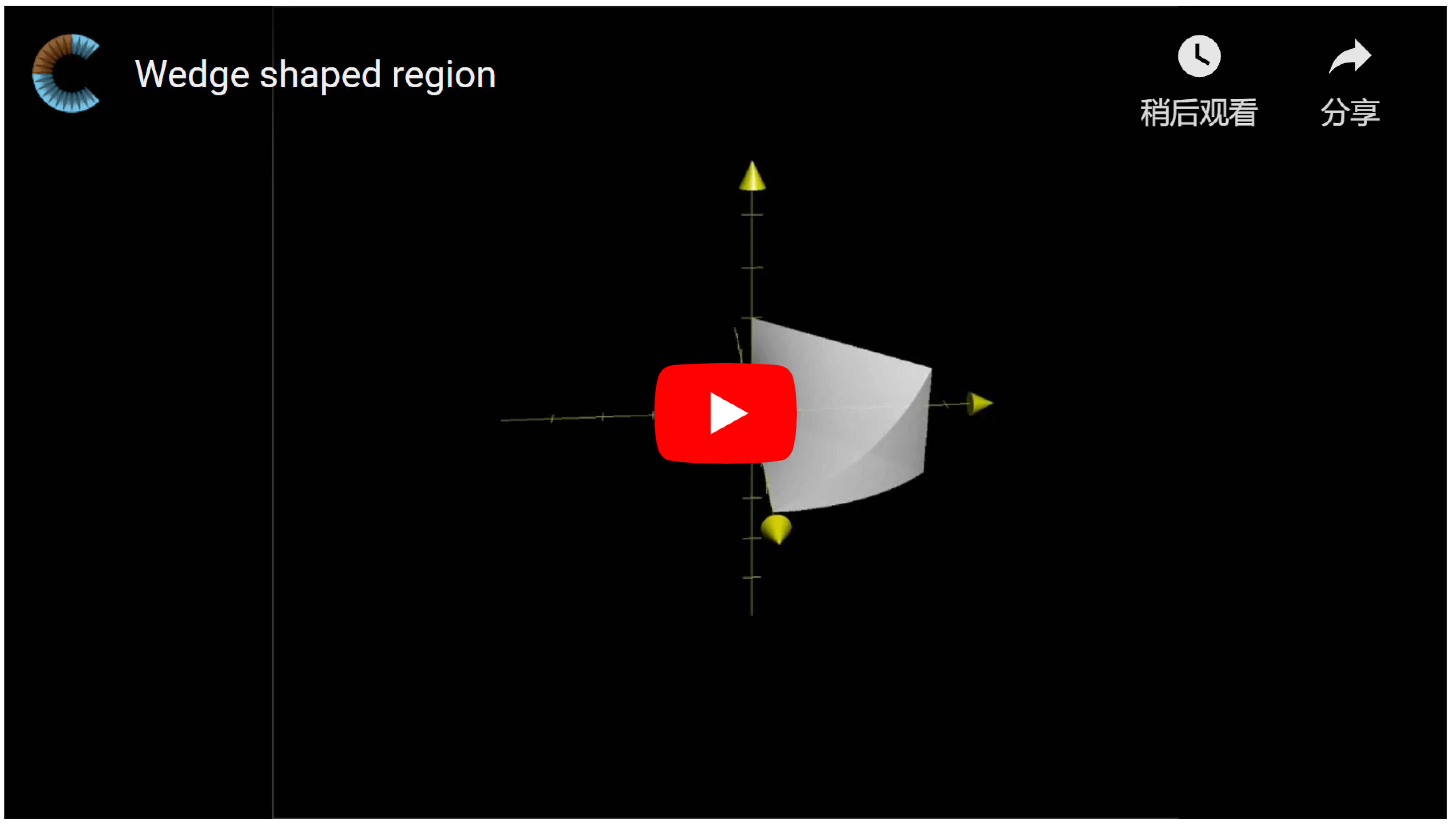
Moreover, this tool is powerful enough to do more than just find the volume of the sphere. For example, you could integrate a three-variable function $f(r, \theta, z)$ inside the sphere,

$$\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} f(r,\theta,z)r\,dz\,dr\,d\theta$$

The hard part of finding the bounds is no different, but the computation of the integrals (done by either you or a computer) will change.

Example 2: Integrating over a pie slice

For this example, we will integrate over a region which looks kind of like a slanted pie slice:



See video transcript

In a problem, this region might be described to you using the following list of properties:

- $x \geq 0$
- $y \geq 0$
- $z \geq 0$
- $y \leq x$
- $x^2 + y^2 \leq 4$
- $z \leq \dfrac{y}{x}$

This time, we will not just be finding the volume of this region. Instead, our task is to integrate the following three-variable function:

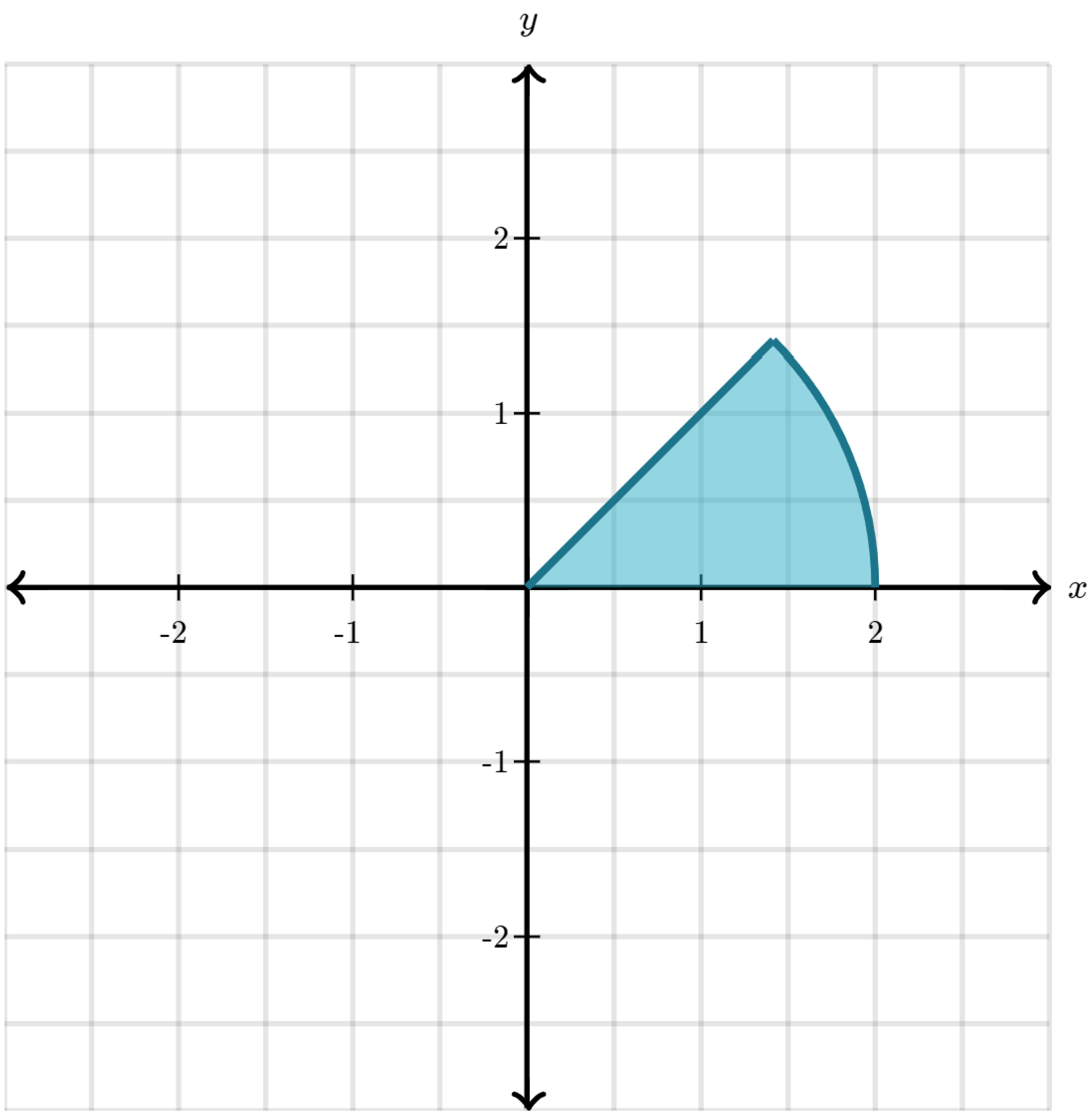
$$f(x, y, z) = z - x^2 - y^2$$

This might seem out of place in an article about integrating in cylindrical coordinates, since everything here is given in cartesian coordinates. Indeed, you could setup the triple integral using cartesian coordinates if you wanted. However, there's one key fact suggesting that our lives can be made dramatically easier by converting to cylindrical coordinates first:

- The expression $x^2 + y^2$ shows up in the function f, as well as in the description of the bounds. This suggests some rotational symmetry around the z-axis, which cylindrical coordinates are well-suited for.

For example, look at the range for our x and y values:

- $x \geq 0$
- $y \geq 0$
- $y \leq x$
- $x^2 + y^2 \leq 4$



Describing this with a pair of integrals over dx and dy is a real pain. However, in polar coordinates, this becomes very simple:

- $0 \leq \theta \leq \frac{\pi}{4}$
- $0 \leq r \leq 2$

This means the bounds on the integrals handling $d\theta$ and dr will be constants. You can't do better than that!

What about the other criteria, such as

- $z \leq \frac{y}{x}$

Since converting to polar coordinates involves the property

$$\tan(\theta) = \frac{y}{x}$$

The bounds on z can be translated to

- $0 \leq z \leq \tan(\theta)$

Putting this together, our triple integral looks like this:

$$\int_0^{\pi/4} \int_0^2 \int_0^{\tan(\theta)} f\, dV$$

Notice how simple the bounds are. If you are up for a little pain, you can try finding the appropriate triple integral bounds in cartesian coordinates to see just how much uglier they are.

We now write the function f using polar coordinates.

$$\begin{gathered} f(x, y, z) = z - x^2 - y^2 \\ \Downarrow \\ f(r, \theta, z) = z - r^2 \end{gathered}$$

And of course, we incorporate the main takeaway of this article, which is how to write dV in polar coordinates:

$$dV = r\, d\theta\, dr\, dz$$

Putting this all together, we get our triple integral in its final solvable state.

More practice: Solve this integral

$$\int_0^{\pi/4} \int_0^2 \int_0^{\tan(\theta)} (z - r^2) r\, dz\, dr\, d\theta =$$

Check

[Hide explanation]

Again, in practice, the computation of integrals like this one are taken care of by programs performing numerical integration, or even computer algebra systems. By hand, here's what it looks like:

$$\int_0^{\pi/4}\int_0^2\int_0^{\tan(\theta)}(z-r^2)r\,dz\,dr\,d\theta$$

$$=\int_0^{\pi/4}\int_0^2\int_0^{\tan(\theta)}(zr-r^3)\,dz\,dr\,d\theta$$

$$=\int_0^{\pi/4}\int_0^2\left[\frac{z^2}{2}r-r^3z\right]_0^{\tan(\theta)}dr\,d\theta$$

$$=\int_0^{\pi/4}\int_0^2\left(\frac{\tan^2(\theta)}{2}r-r^3\tan(\theta)\right)dr\,d\theta$$

$$=\int_0^{\pi/4}\left[\frac{r^2\tan^2(\theta)}{4}-\frac{r^4}{4}\tan(\theta)\right]_{r=0}^{r=2}d\theta$$

$$=\int_0^{\pi/4}\left(\frac{(2)^2\tan^2(\theta)}{4}-\frac{(2)^4}{4}\tan(\theta)\right)d\theta$$

$$=\int_0^{\pi/4}\left(\tan^2(\theta)-4\tan(\theta)\right)\,d\theta$$

Here, I had to look things up.

- The antiderivative of $\tan^2\theta$ is $\tan(\theta)-\theta$
- The antiderivative of $\tan(\theta)$ is $-\ln(|\cos(\theta)|)$

Continuing on our merry way then:

$$=\int_0^{\pi/4}\left(\tan^2(\theta)-4\tan(\theta)\right)\,d\theta$$

$$=\Big[\tan(\theta)-\theta-4(-\ln(\cos(\theta)))\Big]_0^{\pi/4}$$

$$=\Big[\tan(\theta)-\theta+4\ln(\cos(\theta))\Big]_0^{\pi/4}$$

$$=(\tan(\pi/4)-\pi/4+4\ln(\cos(\pi/4)))-(\tan(0)-0+4\ln(\cos(0)))$$

$$= \left(1 - \frac{\pi}{4} + 4\ln\left(\frac{\sqrt{2}}{2}\right)\right) - (0 - 0 + 4\ln(1))$$

$$= 1 - \frac{\pi}{4} + 4\ln\left(\frac{\sqrt{2}}{2}\right)$$

Summary

- The main thing to remember about triple integrals in cylindrical coordinates is that dV, representing a tiny bit of volume, is expanded as

$$dV = r\, d\theta\, dr\, dz$$

(Don't forget to include the r)

- Using cylindrical coordinates can greatly simplify a triple integral when the region you are integrating over has some kind of rotational symmetry about the z-axis.