Double integrals beyond volume

Google Classroom

Double integrals do more than find volume under three-dimensional graphs. Here we cover other uses, a more general notation for double integrals, and explain the "feel" of double integration.

Background

- [Double integrals over non-rectangular regions]

What we're building to

- Double integrals are used anytime you get that feeling where you want to chop up a two-dimensional region into infinitely many infinitely small areas, multiply each one by some value, then add them up.

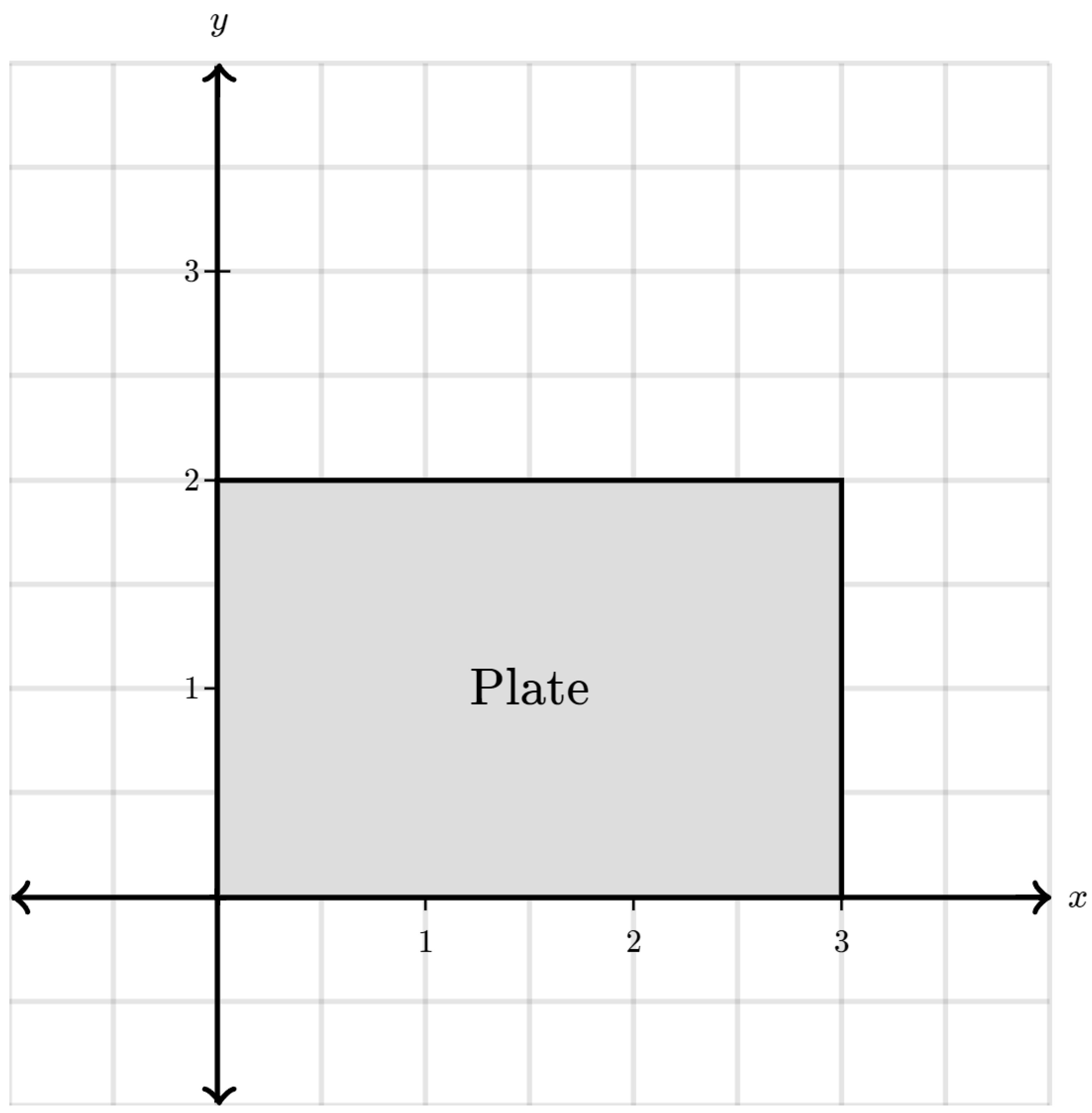
- The more general notation for a double integral is
$$\iint_{R} f\, dA$$
where

- R is the region that you are integrating over.
- dA signifies a "tiny chunk of area", which typically means $dx\,dy$ or $dy\,dx$, unless another coordinate system is being used.
- $f(x, y)$ is a two-variable function.

Example 1: Mass of a plate

Imagine a metal plate 3 meters wide and 2 meters tall. Our goal will be to find its mass based on its density, but the catch is that the density is not constant over the plate.

To be able to describe this variable density with a function, start by situating the plate on the xy-plane:



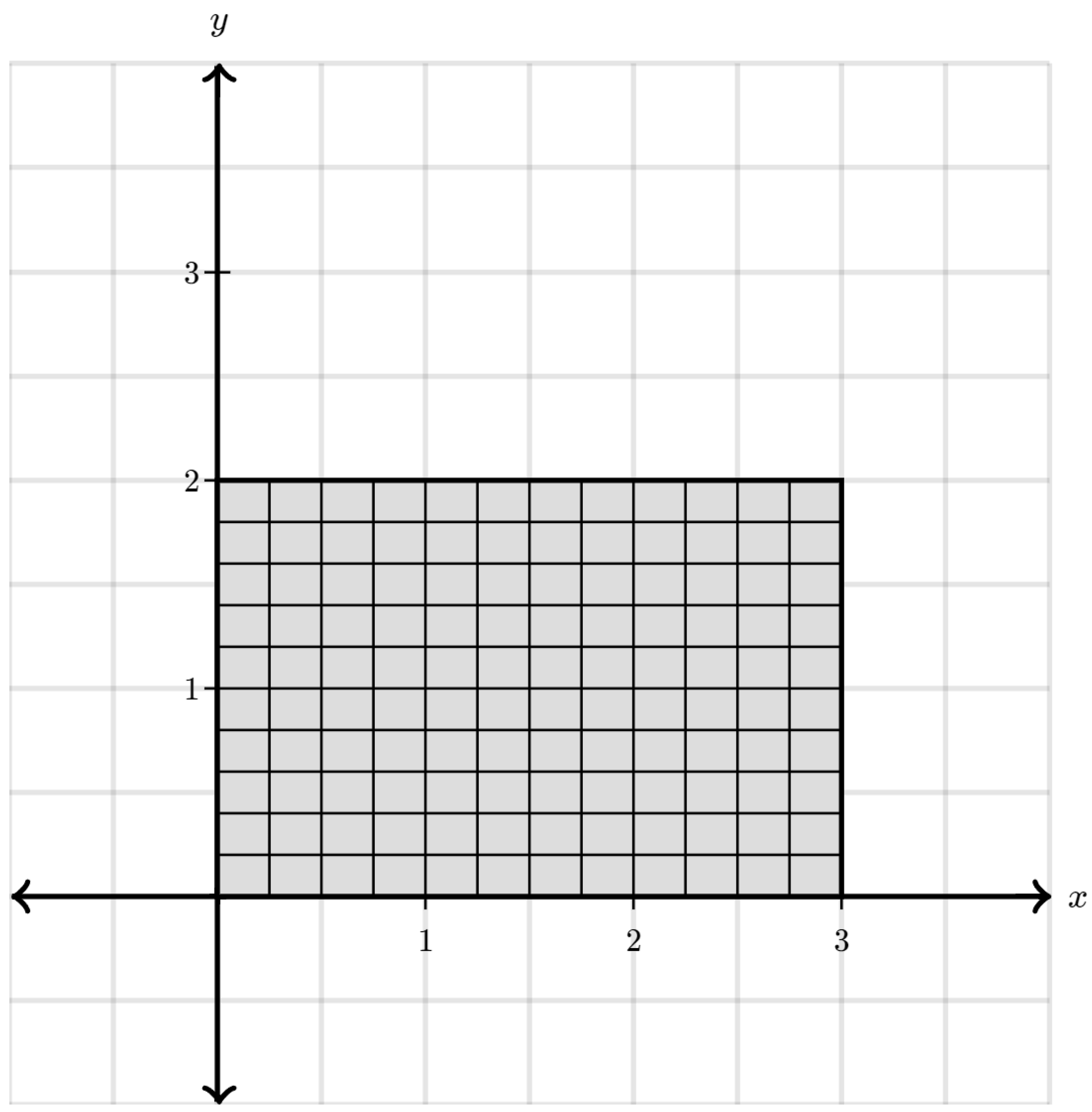
Its lower left corner at the origin, and its long side resting on the x-axis.

Let's say that the density of this plate, in kg/m^2, is expressed with the following function.

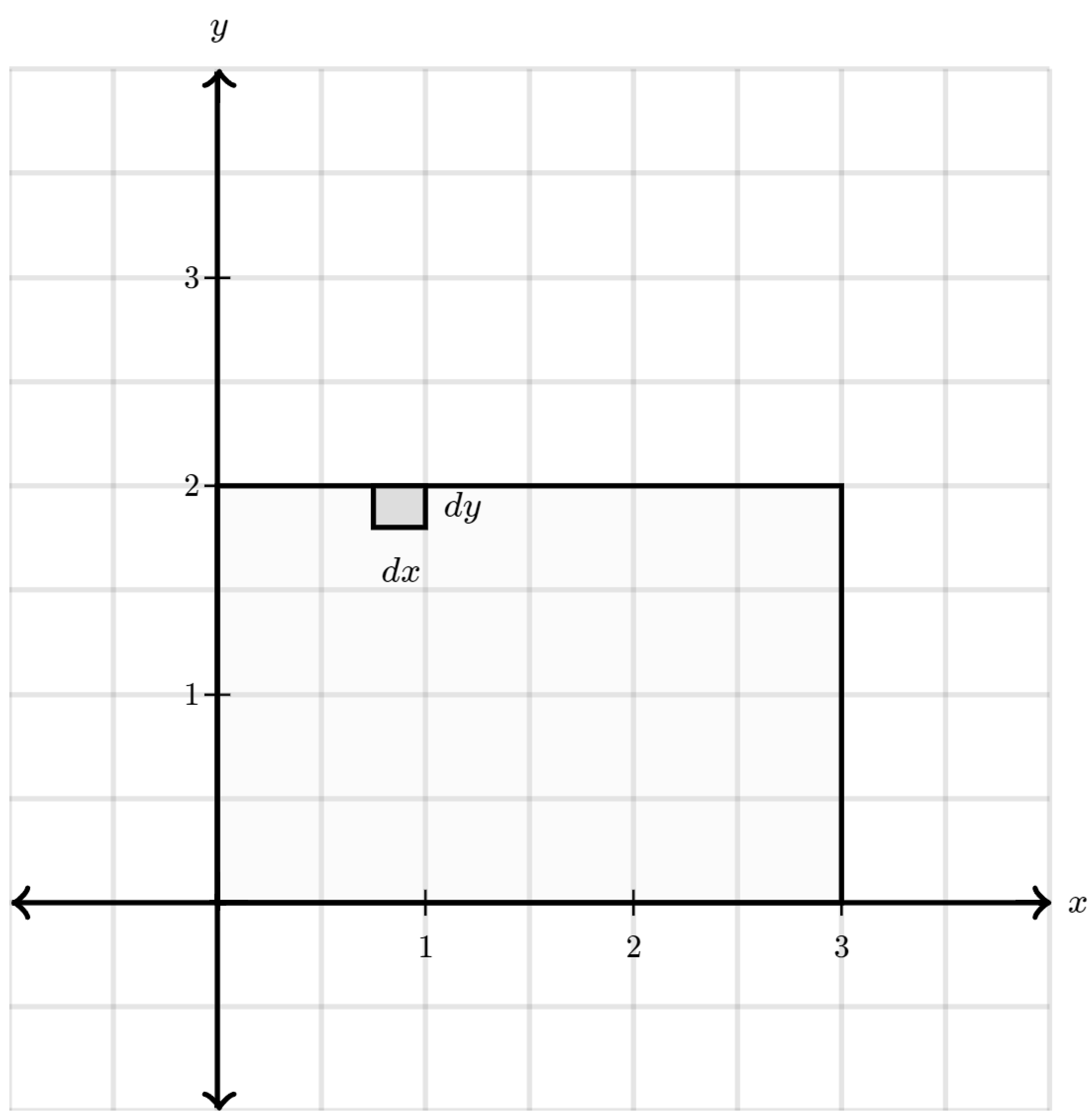
$$\sigma(x, y) = (\sin(\pi x) + 1)y$$

(σ is a typical variable name for two-dimensional density). Density is mass per unit area, so it might seem strange to define it using a function which takes in individual points. After all, what does it mean for a single point like $(1, 2)$ to have density $\sigma(1, 2)$? If you prefer, you can interpret this function as giving the density within tiny region *around* each point.

To find the mass of the plate, you can imagine chopping it up into many tiny pieces, each one a rectangle, then adding up their masses.



Think of each of these rectangles as having a tiny width, dx, and a tiny height, dy.



Think about a specific rectangle, perhaps the one containing the point $(1, 2)$. Since this rectangle is really small, the density within it will pretty much equal the constant $\sigma(1, 2)$. The more finely you cut things, and the smaller the rectangles, the closer it is to being true that the density of each rectangle is constant.

This means we can find the mass of each such rectangle. For example,

$$\underbrace{\sigma(1, 2)}_{\text{density}} \underbrace{dx\,dy}_{\text{tiny area}} = (\sin(\pi) + 1)(2)\,dx\,dy = 2\,dx\,dy$$

To get the total mass of the plate, we integrate all of these tiny masses together. Since we are integrating over a two-dimensional region, we use a double integral. **Caution: the order of your integrals depends on whether you express the tiny area of each rectangle as $dx\,dy$ or $dy\,dx$**

Concept check: Which of the following double integrals represents the mass of our metal plate, which is 3 meters wide and 2 meters tall:

Choose 1 answer:

(A) $\int_0^3 \int_0^2 \sigma(x, y)\,dx\,dy$

(B) $\int_0^2 \int_0^3 \sigma(x, y)\,dx\,dy$

Check

Hide explanation

The second choice is correct. Since dx is written first, the bounds and variable of integration within the inner integral must be in terms of x.

$$\int_0^2 \underbrace{\left(\int_0^3 \sigma(x, y)\,dx\right)}_{\substack{\text{This integral is} \\ \text{purely with respect to } x}} dy$$

Concept check: Using the function $\sigma(x, y) = (\sin(\pi x) + 1)y$, evaluate this double integral. (If you are unsure about how to do this, consider reviewing the article introducing double integrals)

$$\int_0^2 \int_0^3 (\sin(\pi x) + 1)y\,dx\,dy = =$$

Check

Hide explanation

$$\int_0^2 \underbrace{\left(\int_0^3 (\sin(\pi x) + 1)y\,dx\right)}_{\text{First do inner integral}} dy$$

$$= \int_0^2 \left(\left(-\frac{\cos(\pi x)}{\pi} + x\right)y\right)\Bigg|_{x=0}^{x=3} dy$$

$$= \int_0^2 \left(\left(-\frac{\overbrace{\cos(3\pi)}^{-1}}{\pi} + 3 \right) y - \left(-\frac{\overbrace{\cos(0)}^{1}}{\pi} + 0 \right) y \right) dy$$

$$= \int_0^2 \left(\frac{2}{\pi} + 3 \right) y \, dy$$

$$= \left(\frac{2}{\pi} + 3 \right) \int_0^2 y \, dy$$

$$= \left(\frac{2}{\pi} + 3 \right) \left(\frac{y^2}{2} \right) \Big|_0^2$$

$$= \left(\frac{2}{\pi} + 3 \right) \left(\frac{2^2}{2} - \frac{0^2}{2} \right)$$

$$= \left(\frac{2}{\pi} + 3 \right) (2)$$

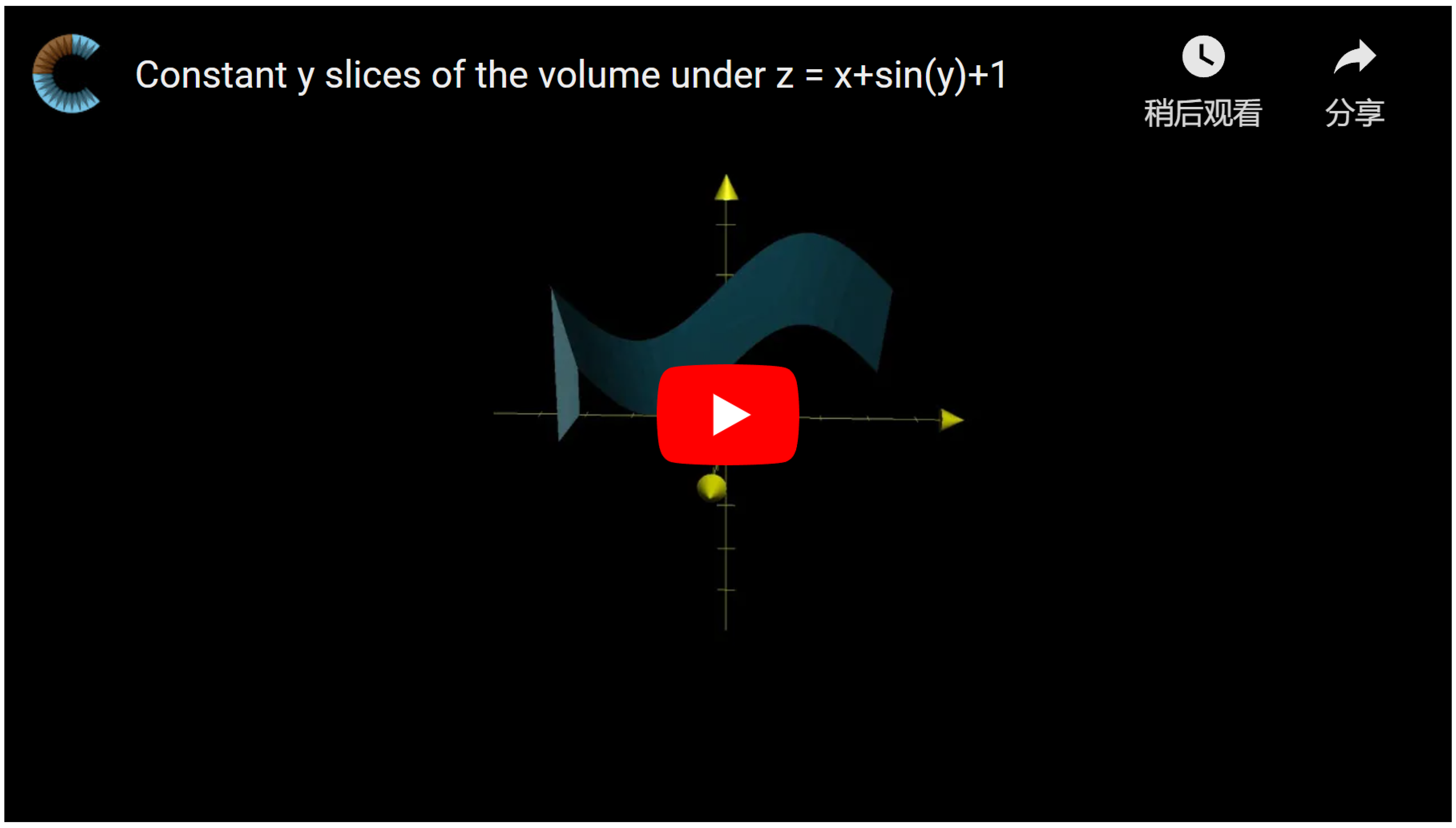
$$= \frac{4}{\pi} + 6$$

So evidently our metal plate with the density function

$\sigma(x, y) = (\sin(\pi x) + 1)y,$

has mass $\boxed{\frac{4}{\pi} + 6}$

Thinking about tiny areas



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When I first introduced double integrals, it was in the context of computing the volume under a graph. The thought process went something like this:

- First cut the volume into infinitely many slices. Each slice represents a constant value for one of the variables, for example $x = 0.78$.

- Find the area of each of those slices. (This is what the inner integral does).

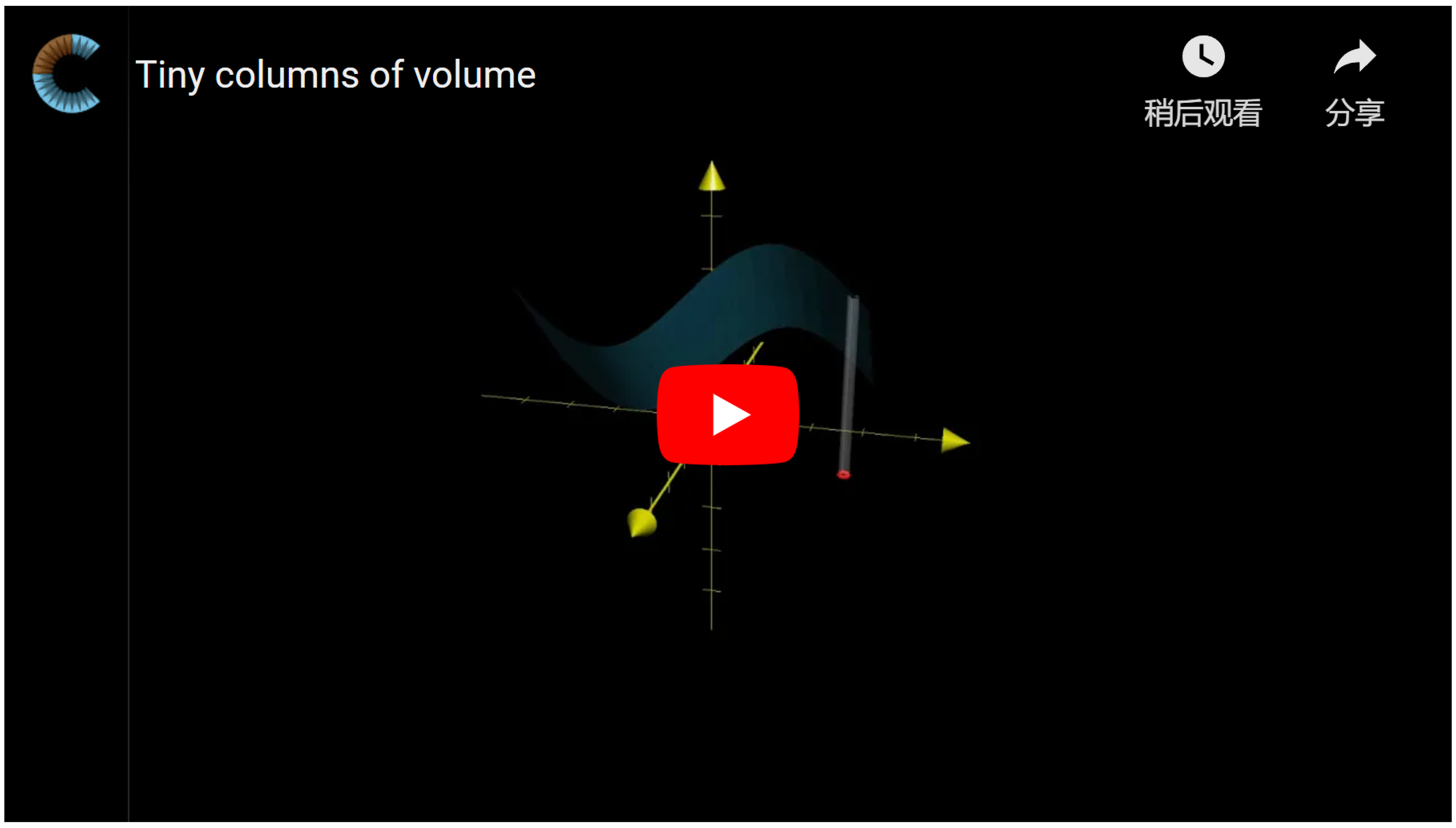
- Make each slice an infinitesimal volume by giving it a little depth. Mathematically, this means multiplying the area of each slice by either dx or dy, whichever one represents a tiny step perpendicular to the slice.

- Integrate those infinitesimal volumes together to get the volume of the solid as a whole. (This is what the outer integral does).

By contrast, the example from the previous section finding the mass of the plate has a different look and feel to it. We start by thinking about tiny areas, then we multiply each one by a constant (the density) and try to add all of them together at once.

Of course, both these perspectives are equivalent. And when it comes to the computation, nothing will look different. You will always set up one integral inside another, compute the inner integral, then compute the outer integral.

Nevertheless, in terms of visualization and conceptual understanding, framing a double integral in terms of tiny areas is distinct from framing it as one linear integral inside another. For example, if you thought about computing the volume under a graph by initially breaking your region of the xy-plane down into tiny areas, you might imagine adding together the volume of thin columns above those tiny areas.



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General notation for double integrals

When we think about a double integral with respect to tiny areas, it's common to write it abstractly like this:

$$\iint_R f\, dA$$

R

R represents the region that we are integrating over. The reason for writing it like this is that while you are setting things up, or reasoning about double integrals in general, you typically don't want to get bogged down with the specific (and potentially complicated) definition of your region while you scribble things down.

When it comes time to compute the integral, we replace this $\iint_R$ with an actual pair of single-integrals with bounds that can be computed. When R is a rectangle, those bounds will be constants:

$$\int_{y_1}^{y_2} \int_{x_1}^{x_2} \dots$$

More generally, when R is defined in terms of some curves in the xy-plane, the bounds of the inner integral are expressed as functions of the outer variable:

$$\int_{y_1}^{y_2} \int_{x_1(y)}^{x_2(y)} \dots$$

(See the last article for practice with this idea.)

dA

dA represents a tiny area, in the same way that dx represents a tiny length in an ordinary integral.

You will typically imagine chopping up the region R into many tiny pieces, and this term represents the area of one of those pieces. Once you get down to computing the double integral, you will replace this with $dx\,dy$, or $dy\,dx$. In other coordinate systems, there are different ways to break down dA, but I'll leave that for the next article.

$f(x, y)$

$f(x, y)$ is some two-variable function. When you chop up your region into many tiny pieces, each piece typically represents some value that you are hoping to add up. Perhaps this value is a tiny bit of mass, or the tiny volume of a slim column under a graph.

Hopefully, you are able to express this tiny amount as something times the area of your tiny piece. For example, the mass of a piece is its density times its area; and the volume of a column above a piece is the height of the column times the area.

In these examples, $f(x, y)$ represents density, or height. In general, it is the thing that needs to be multiplied by the area dA of a tiny piece, and it generally depends on the position of that tiny piece, expressed with (x, y)-coordinates.

"What if I cannot express the tiny value that I want to add up as something times dA?"

Well, in that case my friend, double integrals are not the tool for you. Although I cannot think of any examples where that comes up...

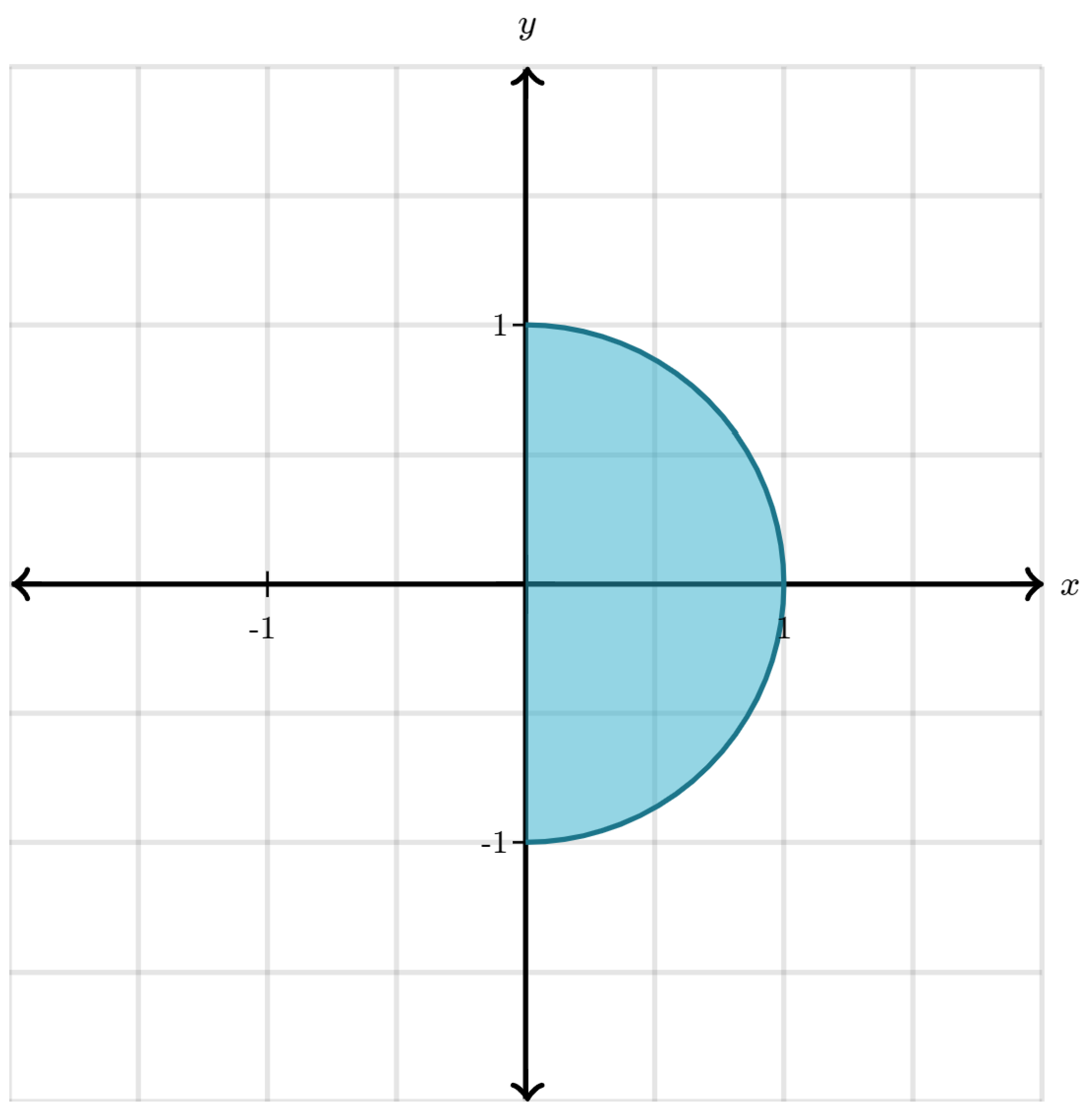
There are two benefits to this abstract notation:

- **Simplicity**: When you're starting to set something up, or if you want to quickly reference the idea of a certain double integral without getting into the implementation details, it's nice to be able to write something quickly. Also, many of the theorems and tools coming up in multivariable calculus are expressed abstractly in this notation.

- **Generality**: Writing your integral as $\iint_R f \, dA$ gives you options as you sit down to compute it. For example, in the next article, we will cover double integrals in polar coordinates, in which case the way you expand dA and the way you put bounds on the two integrals are different than they are for cartesian coordinates.

Example 2: Center of mass

What is the center of mass of a half-disk?



For simplicity, let's say the radius of the disk is 1, and let's orient it such that the diameter rests on the y-axis. Also, assume the disk has uniform density everywhere.

This is a pretty interesting problem, don't you think? You can guess that the answer is something slightly to the left of $(0.5, 0)$, but it's not obvious what the specific answer should be, is it?

By the vertical symmetry of this half-disk, you can know that the center of mass will lie on the x-axis. In a sense, what we're looking for is the "average x-value" of points in the disk.

Concept check: If we let H represent this half-disk, with $|H|$ representing its area, which of the following abstractly written integrals represents the x-value for the center of mass of H?

Choose 1 answer:

(A) $\frac{1}{|H|} \iint_H x \, dA$

(B) $\frac{1}{|H|} \iint_H \sqrt{1-x^2} \, dA$

Check

Hide explanation

The first choice is correct.

$$\frac{1}{|H|} \iint_H x \, dA$$

Since we are assuming uniform density, we can think of dA as representing a tiny piece of mass just as well as it represents a tiny area. Think of the density as being 1.

You can interpret this integral saying

> "Chop the disk into many many tiny pieces, multiply the mass (or area) of each one of those pieces by its x-value, then add up what you get."

Dividing this by the total mass of the disk, we get the "average x-value" of points on the disk, which is the x-coordinate of the center of mass.

Concept check: What is the area of the half-disk H?

$|H| =$ [ ]

Check

Hide explanation

Since the disk has radius 1, the area of the entire full disk would be

$$\pi R^2 = \pi(1)^2 = \pi$$

Chopping that in half, we get $\frac{\pi}{2}$.

Concept check: Which of the following represents the right way to expand the integral $\iint_H x\,dA$ into a computable form?

Choose all answers that apply:

A $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} x\,dx\,dy$

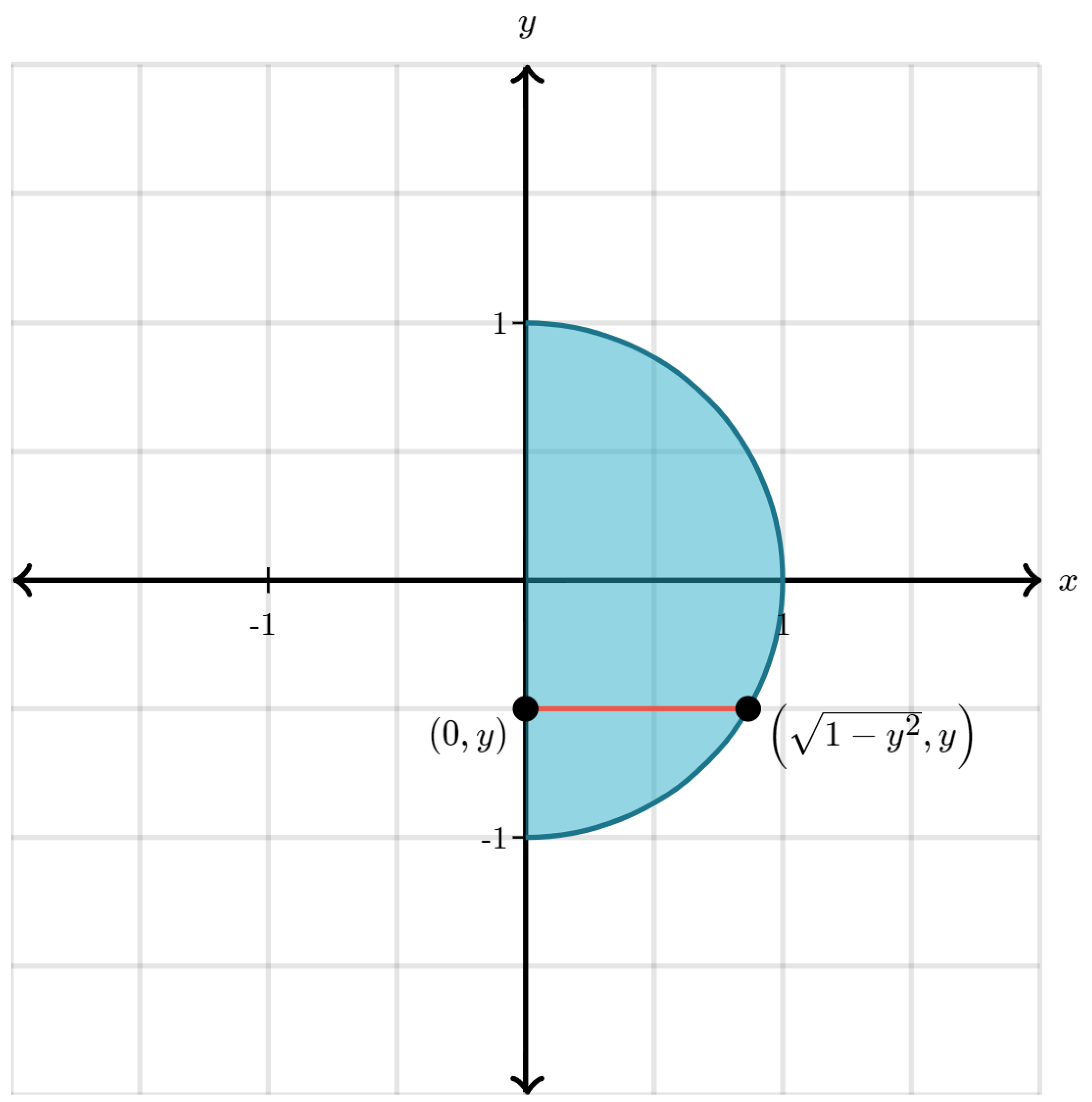
B $\displaystyle\int_{0}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dy\,dx$

Check

Hide explanation

This is kind of a trick question, both answers are correct.

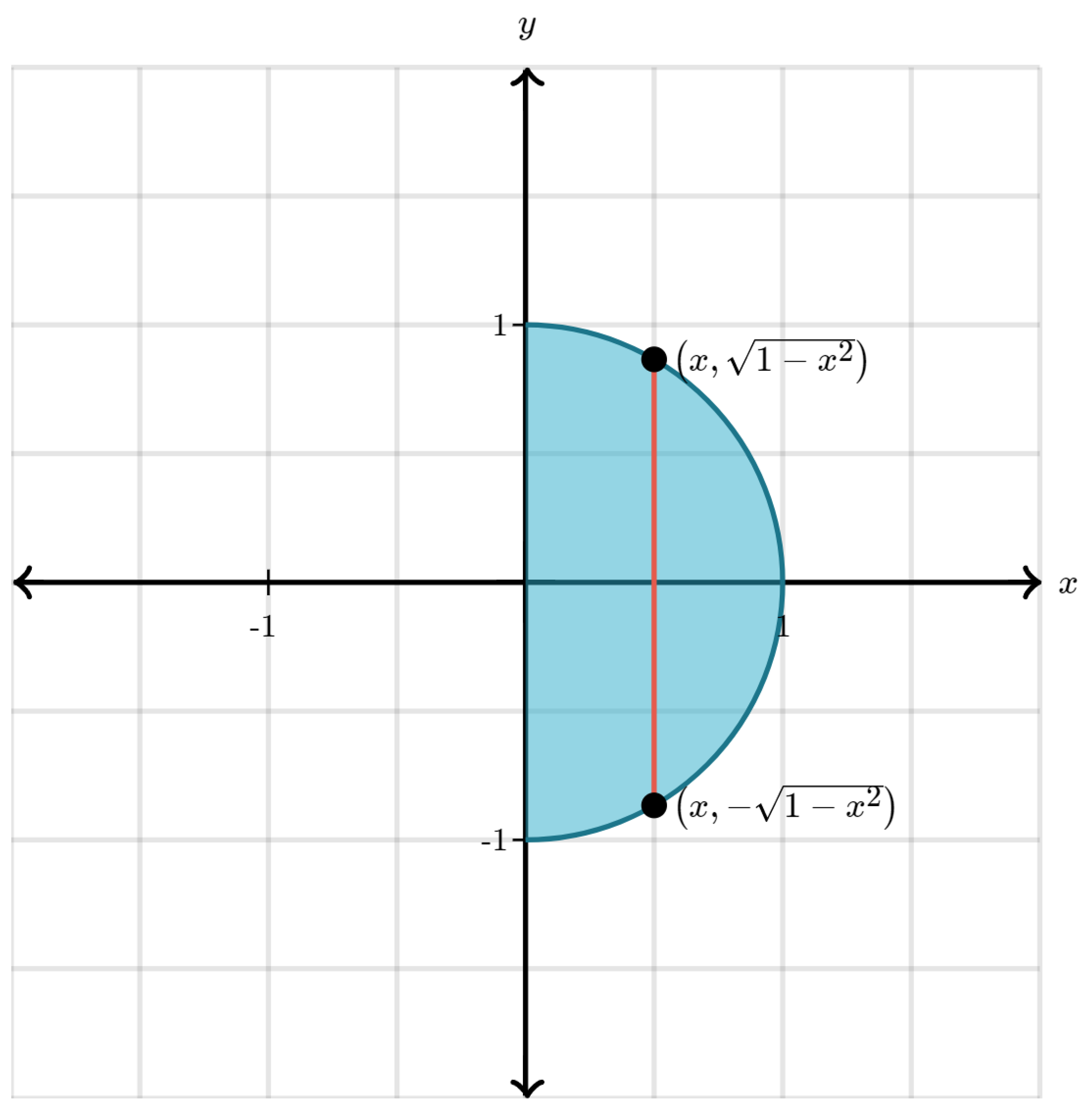
The first one shows what it looks like if you think about dividing H into horizontal stripes:



The right bound of this stripe, as a function of y, is found using the pythagorean theorem. The integral over H now starts by integrating along this stripe with respect to x (the inner integral), then integrating the result between -1 and 1 with respect to y, (the outer integral)

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}} x\,dx\,dy$$

The second integral shows what happens when you think of dividing H into vertical stripes.



The bounds are again found using the pythagorean theorem, but this time they must be expressed as functions of x. Now the inner integral is with respect to y, and the outer integral is with respect to x, going from 0 to 1.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x \, dy \, dx$$

Bring it on home: Solve this integral, and use it to find the center of mass of H.

x-coordinate of center of mass:

Check

Hide explanation

You can choose either one of the integrals shown in the previous question, but I think the one corresponding to horizontal stripes looks a bit easier.

$$\int_{-1}^{1} \underbrace{\int_0^{\sqrt{1-y^2}} x \, dx}_{\text{Start by evaluating the inner integral}} \, dy$$

$$= \int_{-1}^{1} \left(\frac{x^2}{2} \right)_0^{\sqrt{1-y^2}} dy$$

$$= \int_{-1}^{1} \left(\frac{(\sqrt{1-y^2})^2}{2} - \frac{0^2}{2} \right) dy$$

$$= \frac{1}{2} \int_{-1}^{1} (1 - y^2)\, dy$$

$$= \frac{1}{2} \left(y - \frac{y^3}{3} \right)_{-1}^{1}$$

$$= \frac{1}{2} \left(\left(1 - \frac{1^3}{3} \right) - \left(-1 - \frac{(-1)^3}{3} \right) \right)$$

$$= \frac{1}{2} \left(\frac{2}{3} + \frac{2}{3} \right)$$

$$= \frac{2}{3}$$

We are not done! To find the center of mass we need to divide this by the total area of the disk, which is $\frac{\pi}{2}$. Therefore, the x-coordinate of the center of mass of the half-disk H is

$$\frac{2/3}{\pi/2} = \boxed{\frac{4}{3\pi}} \approx 0.4244$$

It makes sense that this should be a little bit less than 0.5, since more of the mass is on the left portion of the half-disk.

Summary

Double integrals are used anytime you get that feeling where you want to chop up a two-dimensional region into infinitely many infinitely small areas, multiply each one by some value, then add them up.

The more general notation for a double integral is

$$\iint_{R} f\, dA$$

where

- R is the region that you are integrating over.
- dA signifies a "tiny chunk of area", which typically means $dx\, dy$ or $dy\, dx$, unless another coordinate system is being used.
- $f(x, y)$ is a two-variable function.

From this point forward, double integrals will be inextricably tied to most of the new topics in multivariable calculus. And in almost all cases, it helps to think about what's happening inside each "tiny area" of a given region, rather than thinking about integrating something along a line then integrating again in the perpendicular direction.