Unit normal vector of a surface

Google Classroom

Learn how to find the vector that is perpendicular, or "normal", to a surface. You will need this skill for computing flux in three dimensions.

Background

- Partial derivatives of parametric surfaces
    - In particular, make sure you have a strong intuition for the partial derivatives of a function parameterizing a surface, and what they represent.
- Cross product (video)

What we're building to

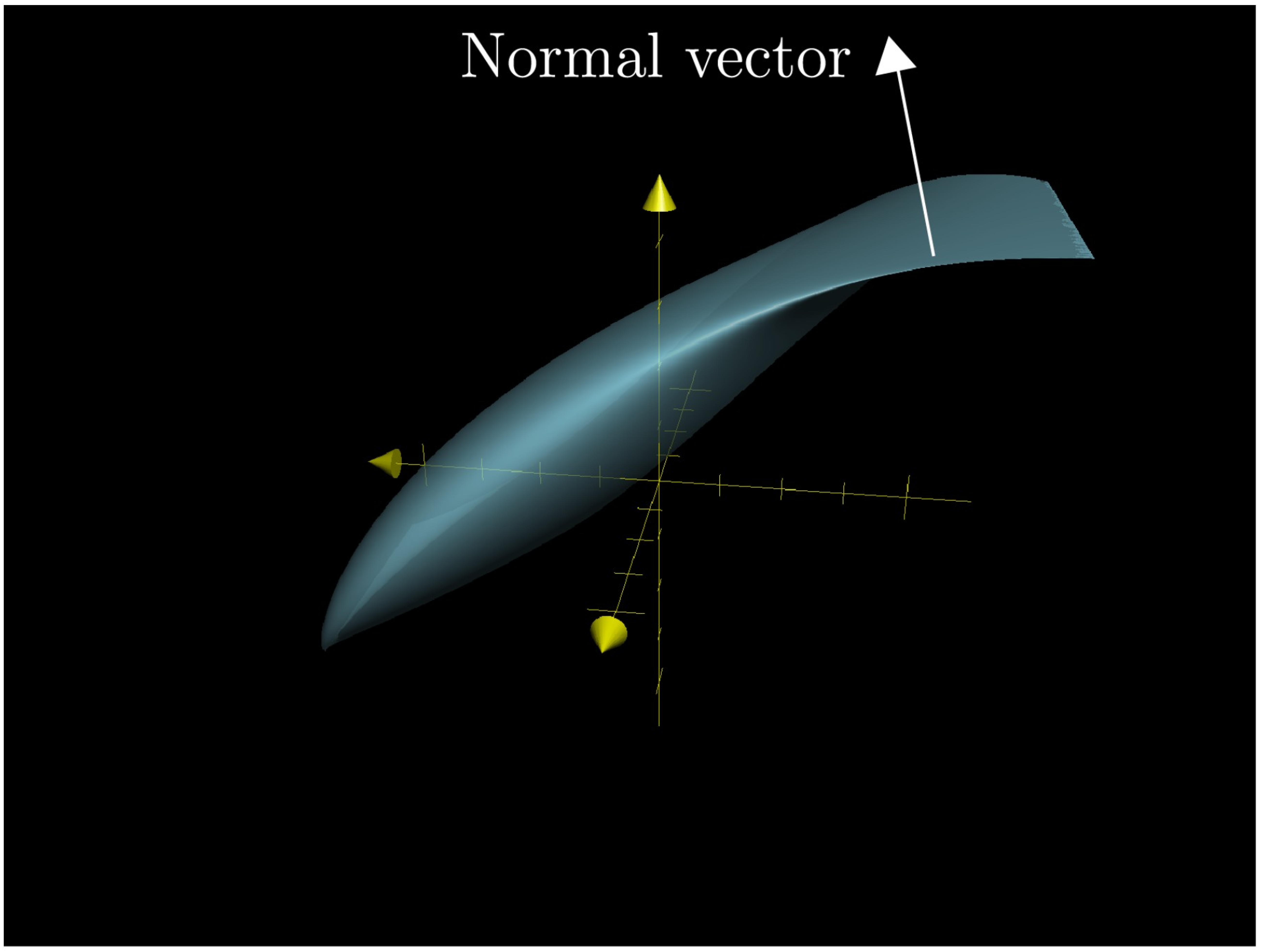
- If a surface is parameterized by a function $\vec{\mathbf{v}}(t, s)$, the unit normal vector to this surface is given by the expression

$$\pm \frac{\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t, s)\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t, s)\right)}{\left|\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t, s)\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t, s)\right)\right|}$$

- You always have two choices for a unit vector function. If a surface is closed, like a sphere or a torus, those choices can be interpreted as outward-facing and inward-facing vectors.

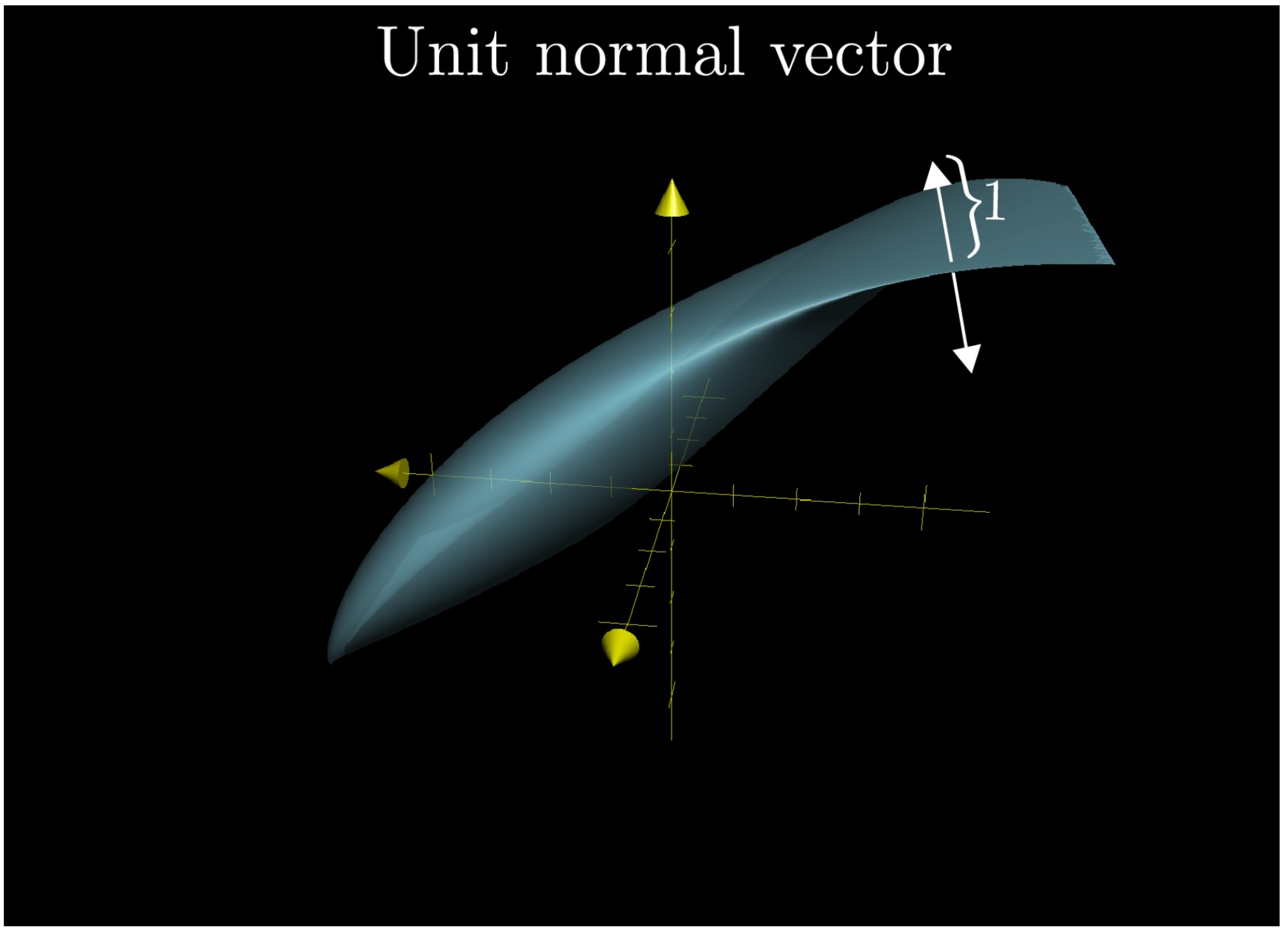
- This is useful for the idea of flux in three-dimensions, covered in the next article.

Unit normal vector



Let's say you have some surface, S. If a vector at some point on S is perpendicular to S at that point, it is called a **normal vector** (of S at that point). More precisely, you might say it is perpendicular to the *tangent plane* of S at that point, or that it is perpendicular to all possible tangent *vectors* of S at that point.

When a normal vector has magnitude 1, it is called a **unit normal vector**. Notice, there will always be two unit normal vectors, each pointing in opposite directions:



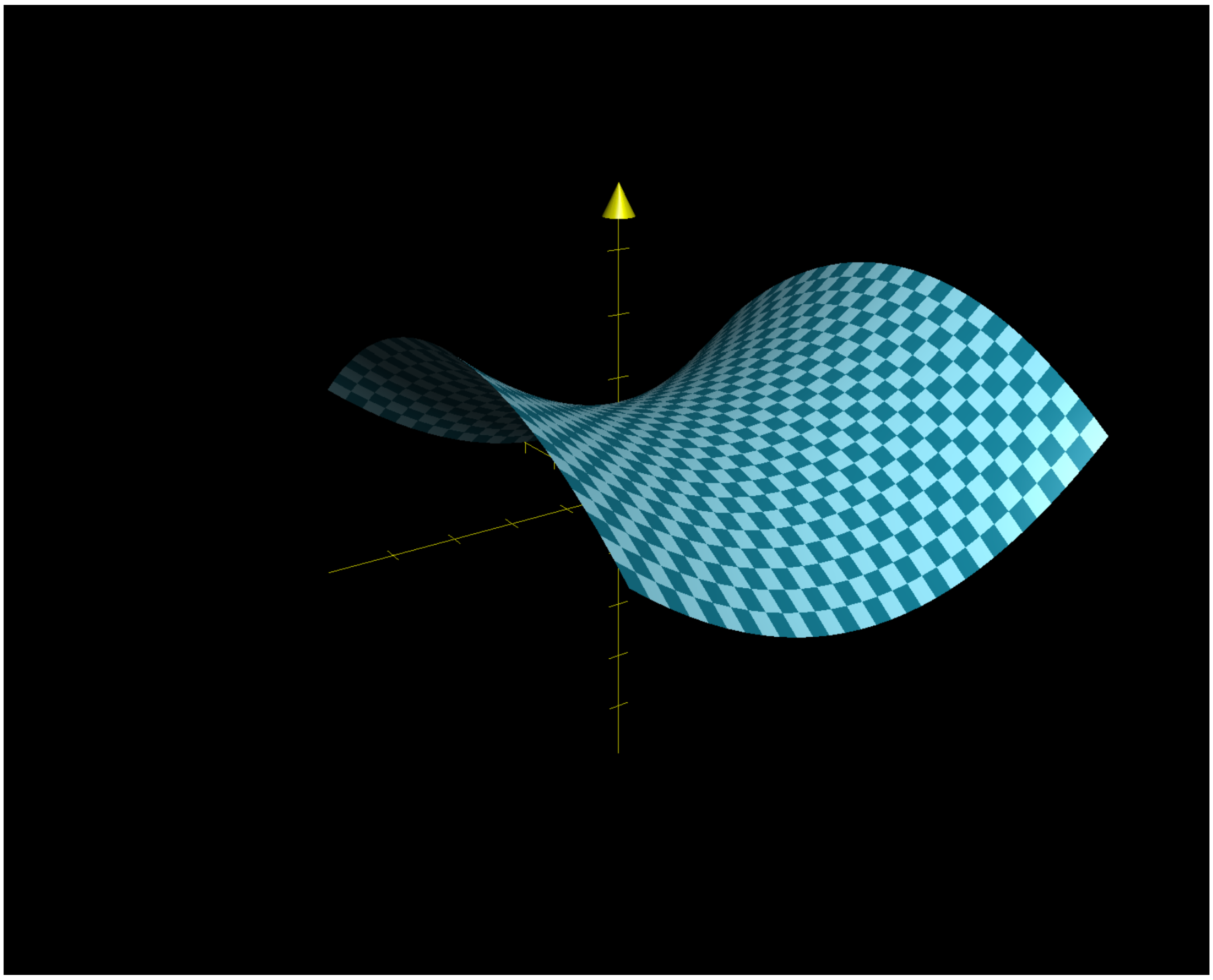
Why do we care? To compute surface integrals in a vector field, also known as three-dimensional flux, you will need to find an expression for the unit normal vectors on a given surface. This will take the form of a multivariable, vector-valued function, whose inputs live in three dimensions (where the surface lives), and whose outputs are three-dimensional vectors.

Example: How to compute a unit normal vector

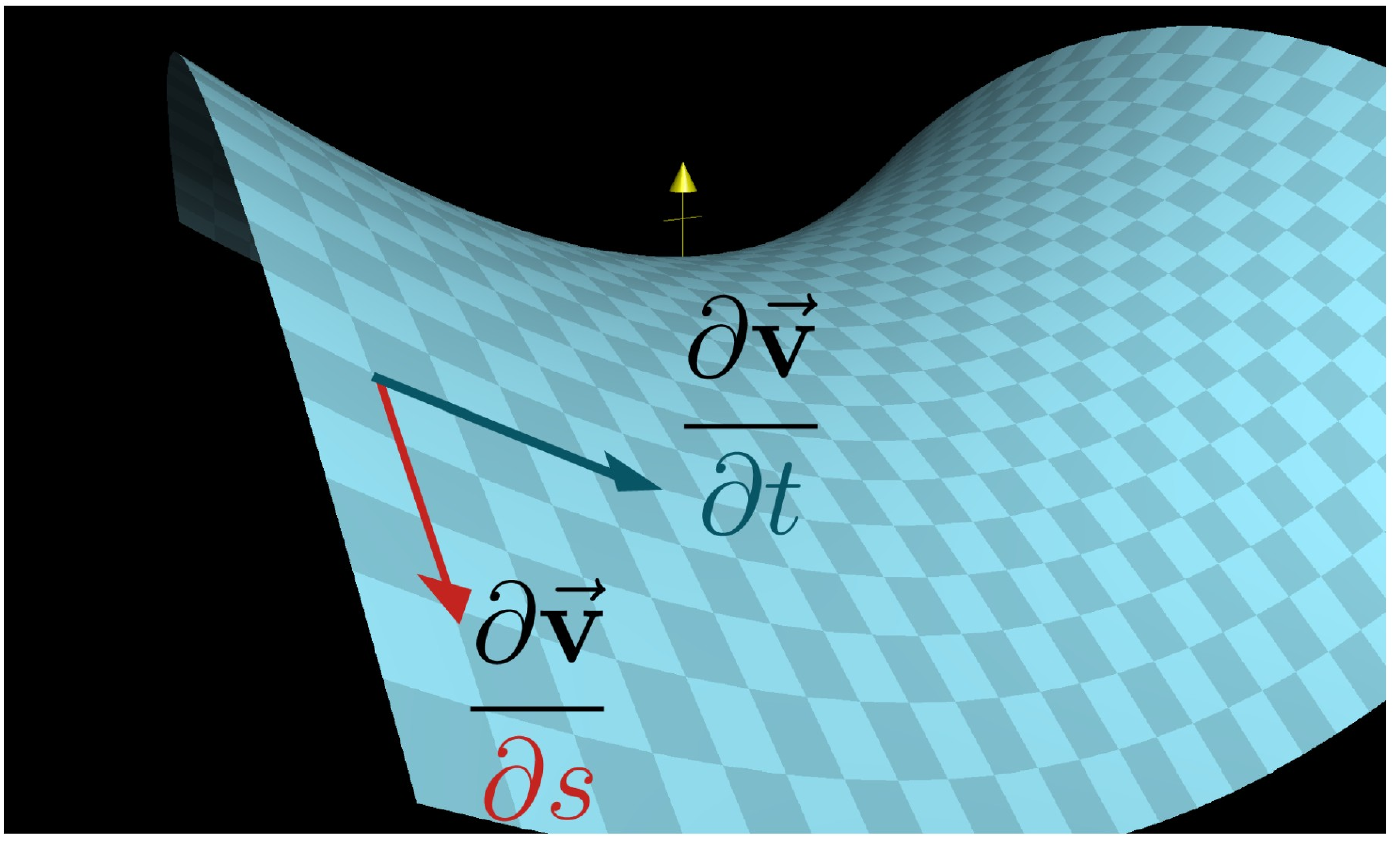
Consider the surface described by the following parametric function:

$$\vec{\mathbf{v}}(t, s) = \begin{bmatrix} t+1 \\ s \\ s^2 - t^2 + 1 \end{bmatrix}$$

In the range where $-2 \leq t \leq 2$ and $-2 \leq s \leq 2$, here's what that surface looks like:



For what follows, I am assuming you know that the two partial derivatives of a parametric surface give vectors which are each tangent to the surface, but in different directions.



Step 1: Find a (not necessarily unit) normal vector

Concept check: Which of the following will give a vector which is perpendicular to the surface parameterized by $\vec{\mathbf{v}}$ at the point $\vec{\mathbf{v}}(1, -2)$?

Choose 1 answer:

Ⓐ $\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(1, -2)\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(1, -2)\right)$

Ⓑ $\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(1, -2)\right) \cdot \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(1, -2)\right)$

Ⓒ $\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(1, -2)\right) + \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(1, -2)\right)$

Check

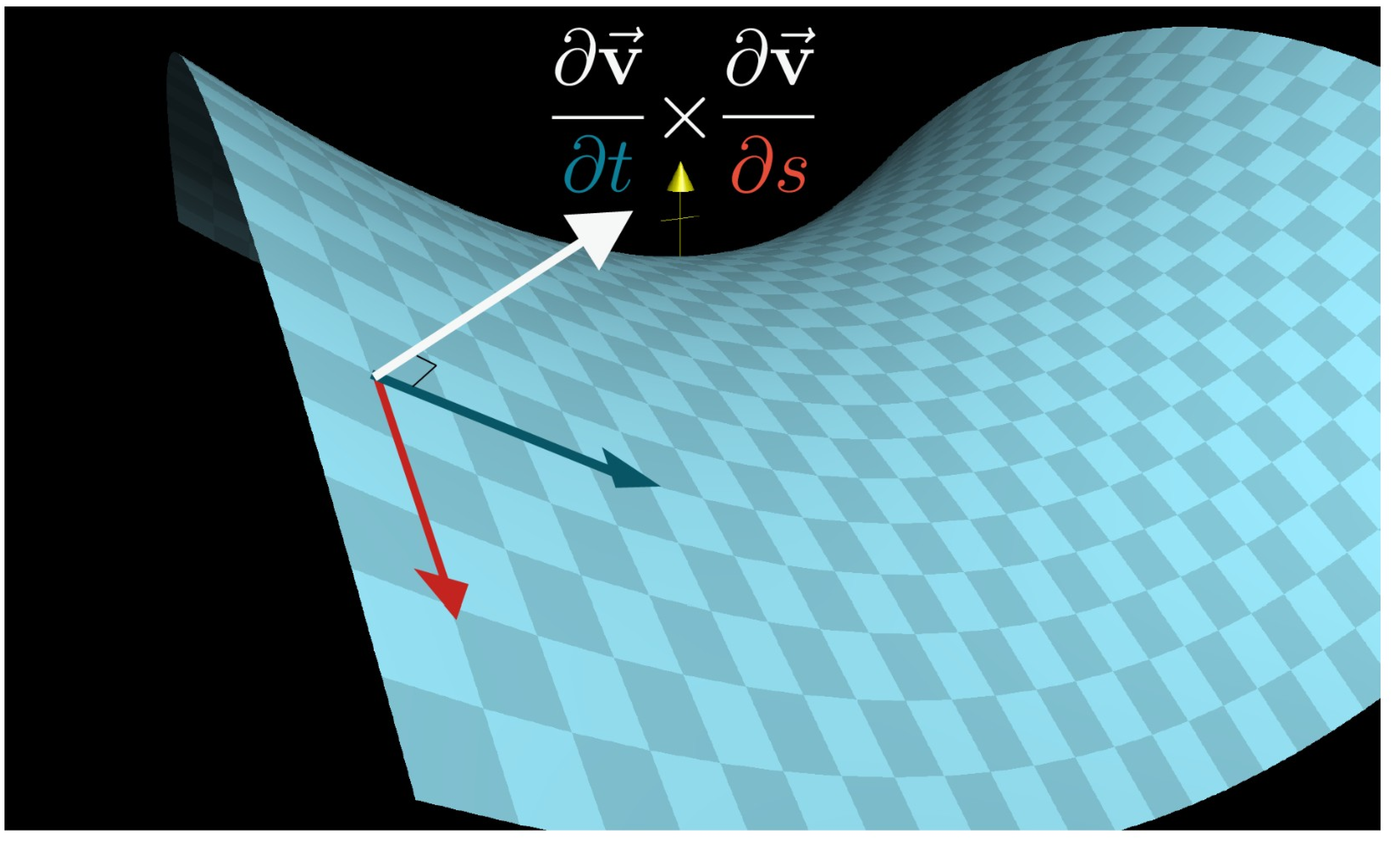
[Hide explanation]

The first answer choice is correct:

$$\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(1, -2)\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(1, -2)\right)$$

If this seems unclear, consider reviewing the video on cross products.

Both partial derivative vectors, $\frac{\partial \vec{\mathbf{v}}}{\partial t}(1, -2)$ and $\frac{\partial \vec{\mathbf{v}}}{\partial s}(1, -2)$, are tangent to the surface at the point $\vec{\mathbf{v}}(1, -2)$. Their cross product gives a vector that is perpendicular to both of them, and which is therefore normal to the surface at that point.



This is a pretty complicated expression, with two vector-valued partial derivatives and a cross product. If you have computed some surface integrals before, you are all-too familiar with the expression and how ugly it can be to compute.

Once again, here's how $\vec{\mathbf{v}}(t, s)$ is defined:

$$\vec{\mathbf{v}}(t, s) = \begin{bmatrix} t+1 \\ s \\ s^2 - t^2 + 1 \end{bmatrix}$$

Concept check: Now compute the cross product of the partial derivatives of $\vec{\mathbf{v}}$. Do this for a general point (t, s), meaning each component of your answer will be a function of t and s. As described in the previous problem, this will give you a function for normal vectors of the surface.

$$\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t,s)\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t,s)\right) =$$

[ ] $\hat{\mathbf{i}}+$ [ ] $\hat{\mathbf{j}}+$ [ ] $\hat{\mathbf{k}}$

Check

[Hide explanation]

Start off by computing each partial derivative. First, let's do it with respect to t:

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} = \begin{bmatrix} \frac{\partial}{\partial t}(t+1) \\ \frac{\partial}{\partial t}(s) \\ \frac{\partial}{\partial t}(s^2 - t^2 + 1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2t \end{bmatrix}$$

Next up, with respect to s:

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} = \begin{bmatrix} \frac{\partial}{\partial s}(t+1) \\ \frac{\partial}{\partial s}(s) \\ \frac{\partial}{\partial s}(s^2 - t^2 + 1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2s \end{bmatrix}$$

Now, take the cross product:

$$\begin{bmatrix} 1 \\ 0 \\ -2t \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 2s \end{bmatrix} = \det\begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -2t \\ 0 & 1 & 2s \end{pmatrix}$$

$$= \Big(0 - (-2t)\Big)\hat{\mathbf{i}} + \Big(0 - 2s\Big)\hat{\mathbf{j}} + \Big(1 - 0\Big)\hat{\mathbf{k}}$$

$$= 2t\hat{\mathbf{i}} + -2s\hat{\mathbf{j}} + 1\hat{\mathbf{k}}$$

Or, written as an upright vector, here's what we get:

$$\begin{bmatrix} 2t \\ -2s \\ 1 \end{bmatrix}$$

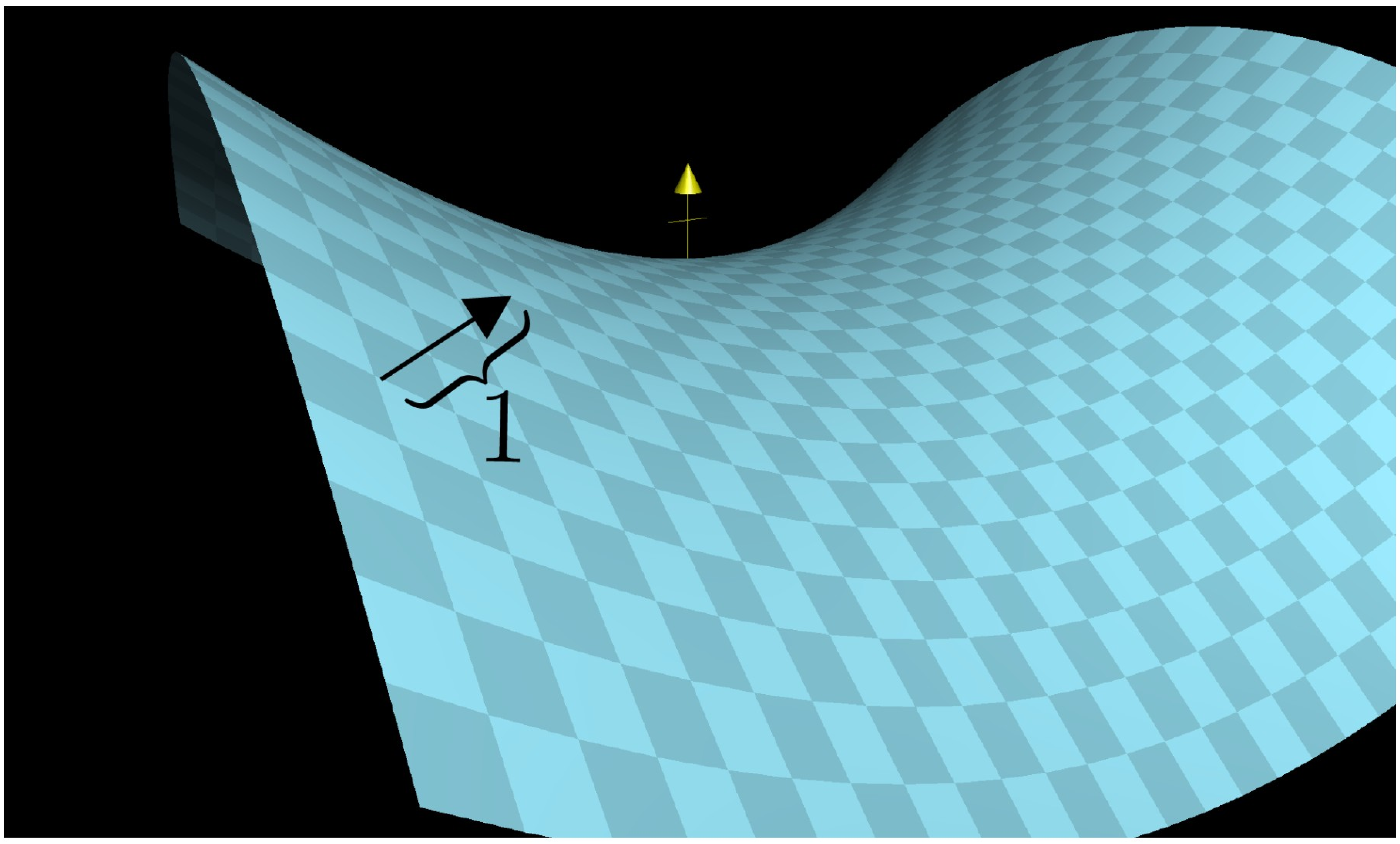
For example, if we plugged in $(t, s) = (1, -2)$, here's what we'd get:

$$\begin{bmatrix} 2(1) \\ -2(-2) \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

This is a vector which is perpendicular to the surface at the point $\vec{\mathbf{v}}(1, -2)$. However, it is not a unit vector, as you can see by computing its magnitude:

$$\sqrt{2^2 + 4^2 + 1^2} = \sqrt{4 + 16 + 1} = \sqrt{21}$$

Step 2: Make that a unit normal vector



So we have this expression $\begin{bmatrix} 2t \\ -2s \\ 1 \end{bmatrix}$ that gives us a normal vector for each point $\vec{\mathbf{v}}(t, s)$. The next step is to massage this a bit to get an expression for a **unit normal vector**.

Concept check: What is the unit normal vector to our surface at the point $\vec{\mathbf{v}}(1, -2)$?

[ ] $\hat{\mathbf{i}} +$ [ ] $\hat{\mathbf{j}} +$ [ ] $\hat{\mathbf{k}}$

Check

[Hide explanation]

As I said in the previous section, when we plug in $(t, s) = (1, -2)$ to our general expression for a normal vector, we get:

$$\begin{bmatrix} 2(1) \\ -2(-2) \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

And this vector has the following magnitude:

$$\sqrt{2^2 + 4^2 + 1^2} = \sqrt{4 + 16 + 1} = \sqrt{21}$$

To make this a unit vector, then, we will scale it from a vector with length $\sqrt{21}$ to one with length 1. To do this, divide each component by $\sqrt{21}$:

$$\begin{bmatrix} 2/\sqrt{21} \\ 4/\sqrt{21} \\ 1/\sqrt{21} \end{bmatrix} \quad \leftarrow \text{Unit vector}$$

Concept check: More generally, what is the unit normal vector to our surface at an arbitrary point $\vec{\mathbf{v}}(t, s)$, as a function of t and s?

$\square\ \hat{\mathbf{i}} + \square\ \hat{\mathbf{j}} + \square\ \hat{\mathbf{k}}$

Check

[Hide explanation]

Our general expression for a (not necessarily unit) normal vector is

$$\begin{bmatrix} 2t \\ -2s \\ 1 \end{bmatrix}$$

As a function of t and s, this vector has magnitude

$$\sqrt{(2t)^2 + (-2s)^2 + 1} = \sqrt{4t^2 + 4s^2 + 1}$$

To turn our normal vector expression into a *unit* normal vector expression, divide each term by this magnitude:

$$\begin{bmatrix} 2t/\sqrt{4t^2 + 4s^2 + 1} \\ -2s/\sqrt{4t^2 + 4s^2 + 1} \\ 1/\sqrt{4t^2 + 4s^2 + 1} \end{bmatrix}$$

Bada boom bada bang, you've got yourself a unit normal vector.

If you plug in any value (t_0, s_0) to this expression, you will get a vector which has magnitude 1, and is normal to the surface parameterized by the function $\vec{\mathbf{v}}$ at the point $\vec{\mathbf{v}}(t_0, s_0)$.

Choosing orientation

Notice, if you multiply your function for a unit normal vector by -1, it will still produce unit normal vectors. They will all just point in the opposite directions. The choice of direction for the unit normal vectors of your surface is what's called an **orientation of that surface**.

You will see the significance of this in the next article on three-dimensional flux. In short, orienting your surface is analogous to giving a one-dimensional curve a direction.

When your surface is closed, like a sphere or a torus, the two choices for unit normal vectors are often called outward-facing and inward-facing unit normal vectors.

Summary

- Given a surface parameterized by a function $\vec{\mathbf{v}}(t, s)$, to find an expression for the unit normal vector to this surface, take the following steps:

- **Step 1**: Get a (non necessarily unit) normal vector by taking the cross product of both partial derivatives of $\vec{\mathbf{v}}(t, s)$:

$$\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t, s)\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t, s)\right)$$

- **Step 2**: Turn this vector-expression into a unit vector by dividing it by its own magnitude:

$$\frac{\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t, s)\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t, s)\right)}{\left|\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t, s)\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t, s)\right)\right|}$$

- You can also multiply this expression by -1, and it will still give unit normal vectors.

- The main reason for learning this skill is to compute three-dimensional flux.