# Formal definition of curl in three dimensions

Google Classroom

*After learning how two-dimensional curl is defined, you are ready to learn about the formal definition of three-dimensional curl. This is advanced, so be prepared to take things slowly.*

## Background

- Curl in three dimensions
- Formal definition of curl in two dimensions

Understanding this article *really does* require these two prerequisites. The definition of curl in three dimensions has so many moving parts that having a solid mental grasp of the two-dimensional analogy, as well as the three-dimensional concept we are trying to capture, is crucial.

In particular, if you did not just come from reading the article giving the formal definition of curl in two dimensions, I would highly recommend taking a quick look at it right now, even if you've seen it before, and even if it's just the summary.

## What we're building to

- We define three-dimensional curl one component at a time, looking at the components of fluid rotation which are parallel to the $yz$-plane, the $xz$-plane, and the $xy$-plane.

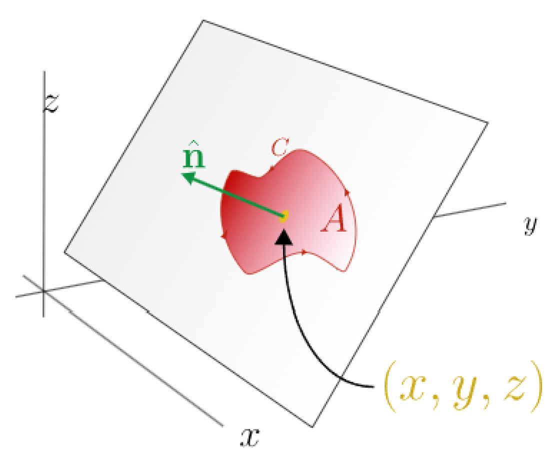
- You can capture all three of these coordinate-by-coordinate definitions of $\operatorname{curl} \mathbf{F}$ by defining what the dot product between $\operatorname{curl} \mathbf{F}$ and any arbitrary unit vector $\hat{\mathbf{n}}$ should be.

$$(\operatorname{curl} \mathbf{F}(x, y, z)) \cdot \hat{\mathbf{n}} = \lim_{|A_{((x,y,z),\hat{\mathbf{n}})}| \to 0} \left( \frac{1}{|A_{((x,y,z),\hat{\mathbf{n}})}|} \oint_C \mathbf{F} \cdot d\mathbf{r} \right)$$

[Hide explanation]

- $\mathbf{F}$ is a three-dimensional vector field.

(Vector field $\mathbf{F}$ not pictured)




- $(x, y, z)$ is some specific point in 3d space.

- $\operatorname{curl} \mathbf{F}(x, y, z)$ returns a three-dimensional vector.

- $\hat{\mathbf{n}}$ is an arbitrary unit vector in three-dimensions.

- $A_{((x,y,z),\hat{\mathbf{n}})}$ represents some two-dimensional region around the point $(x, y, z)$ on a plane perpendicular to the vector $\hat{\mathbf{n}}$.

- $|A_{((x,y,z),\hat{\mathbf{n}})}|$ indicates the area of $A_{((x,y,z),\hat{\mathbf{n}})}$.

- $|A_{((x,y,z),\hat{\mathbf{n}})}| \to 0$ indicates we are considering the limit as the area of $A_{((x,y,z),\hat{\mathbf{n}})}$ goes to zero, meaning that region shrinks around $(x, y, z)$.

- $C$ is the boundary of $A_{((x,y,z),\hat{\mathbf{n}})}$.

- The orientation of $C$ is determined based on the right-hand rule: Stick the thumb of your right hand in the direction of $\hat{\mathbf{n}}$, and the direction your fingers point as they wrap around $C$ is the direction of integration.

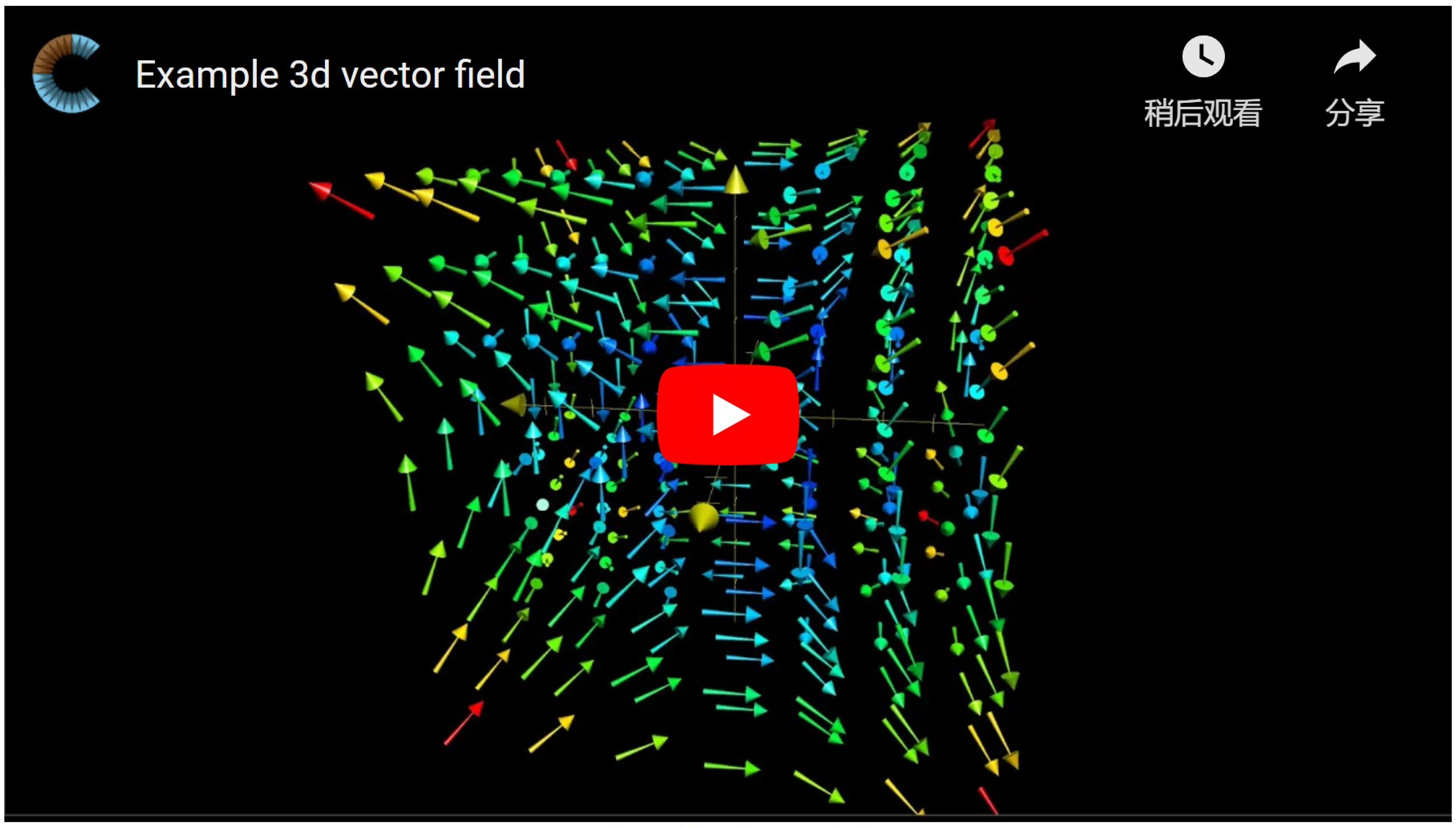
- $\oint_C$ is the line integral around $C$, written as $\oint$ instead of $\int$ to emphasize that $C$ is a closed curve.

## Limit our view to one plane

Curl in three-dimensions is a rather complicated thing to think about. For example, let $\mathbf{F}(x, y, z)$ be a three-dimensional vector field:

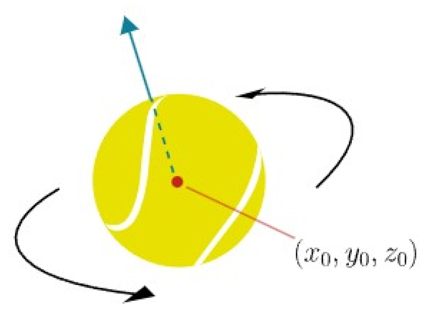
$$\mathbf{F}(x, y, z) = \begin{bmatrix} F_1(x, y, z) \\ F_2(x, y, z) \\ F_3(x, y, z) \end{bmatrix}$$

An example of what this could look like is shown in the following video.




See video transcript

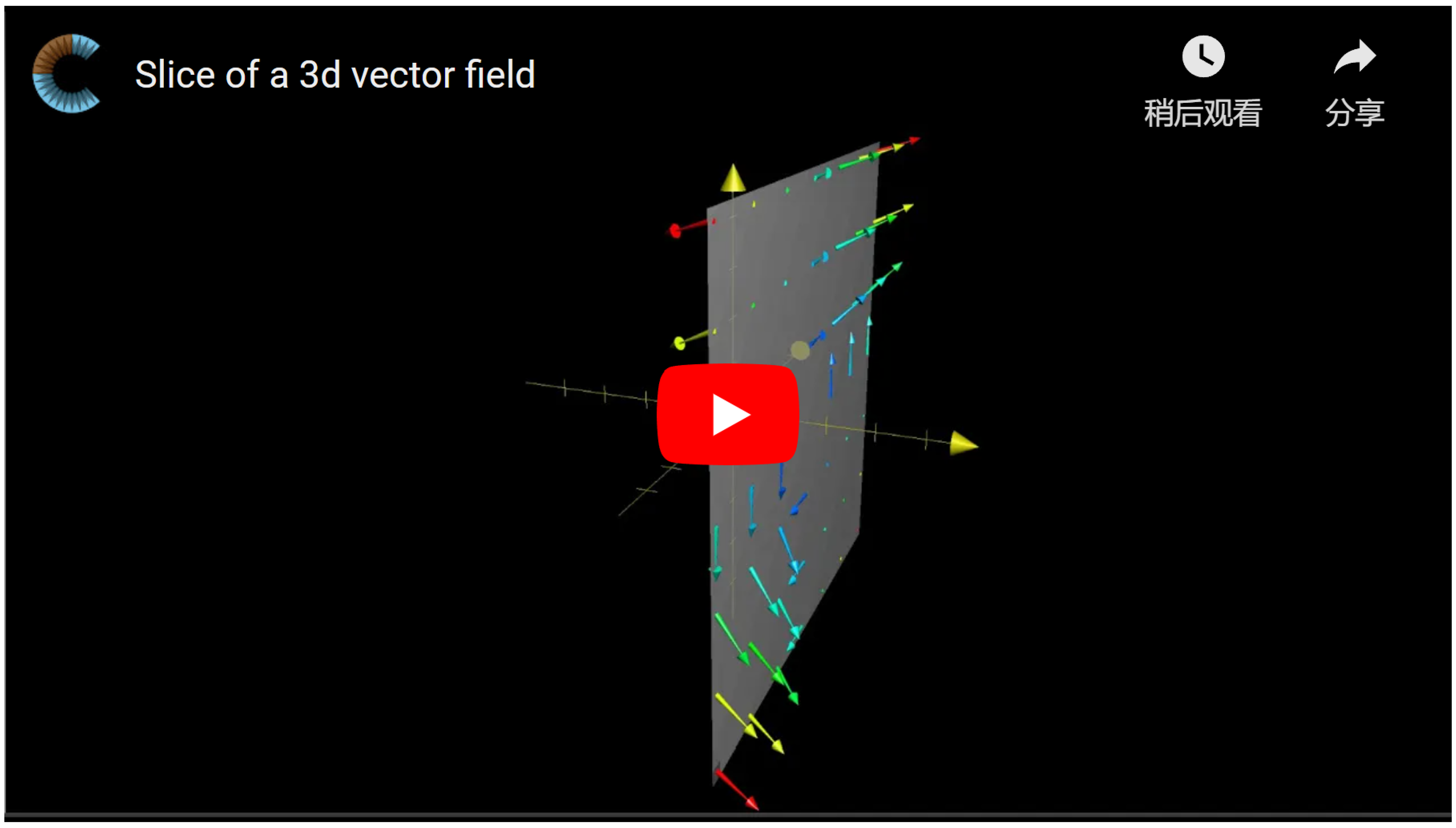
Now imagine the three-dimensional fluid flow that $\mathbf{F}$ could represent. As you know, $\operatorname{curl} \mathbf{F}(x_0, y_0, z_0)$ is a way to measure rotation in that fluid flow near the point $(x_0, y_0, z_0)$, but this is a tricky concept to quantify rigorously.




There are some good analogies out there to gain an intuition for curl. One of my favorites is to think of a tiny tennis ball centered at the point $(x_0, y_0, z_0)$, and how the surrounding fluid flow would cause it to rotate. In this analogy, $\operatorname{curl} \mathbf{F}(x_0, y_0, z_0)$ gives the vector of the tennis ball's resulting rotation.

However, these descriptions can only go so far when the goal is to formally define what curl is; to capture this intuition with mathematical rigour.

Our basic strategy moving forward will be to **limit our view to the rotation in a specific plane**. For example, the following video shows a plane representing a constant $x$ value, $x = 1.6$ to be specific, as well as the vectors from $\mathbf{F}$ which stem from this plane.

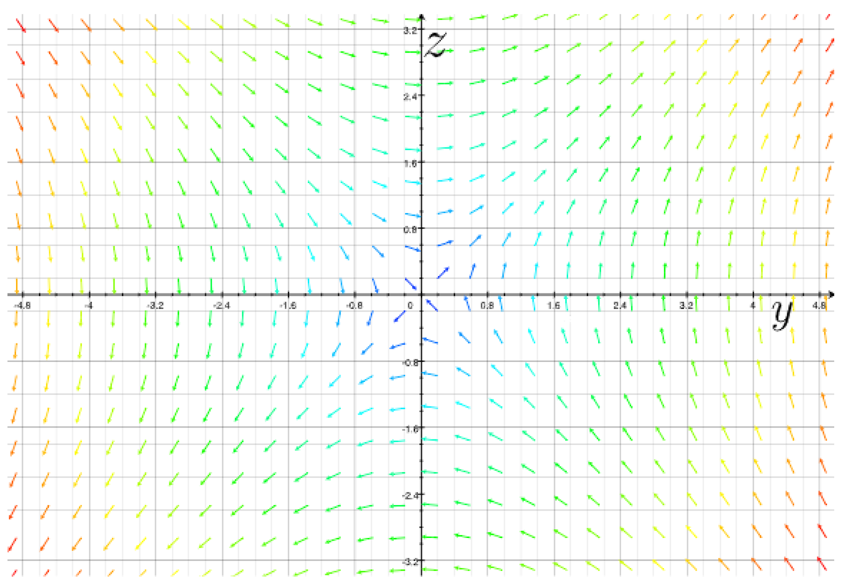



See video transcript

In formulas, you might describe this as all the vectors of the form

$$\mathbf{F}(1.6, y, z)$$

Here, $y$ and $z$ range freely. When we project those vectors onto the plane and lay it out flat as a picture, we'd get something like this:




Note, the axes are labeled "$y$" and "$z$" because this plane was originally parallel to the $yz$-plane in three-dimensional space. We could describe this two-dimensional vector field with a new two-dimensional function $\mathbf{F}_{1.6}(y, z)$ defined as follows:

$$\mathbf{F}_{1.6}(y, z) = \left[ \begin{array}{c} F_2(1.6, y, z) \\ F_3(1.6, y, z) \end{array} \right]$$

More generally, if we slice the vector field with any plane of the form $x = x_0$ for some constant $x_0$, then project the vectors stemming from that plane onto the plane itself, we will get a two-dimensional vector field described by a function that looks like this:

$$\mathbf{F}_{x_0}(y, z) = \left[ \begin{array}{c} F_2(x_0, y, z) \\ F_3(x_0, y, z) \end{array} \right]$$

**Concept check**: Why does the definition of $\mathbf{F}_{x0}(y, z)$ not include $F_1$, the $x$-component of $\mathbf{F}(x, y, z)$?

**Choose 1 answer:**

(A) Because it represents the projection of vectors $\mathbf{F}(x_0, y, z)$ onto a plane parallel to the $yz$-plane, and that projection involves ignoring the $F_1$ component.

(B) Because only $y$ and $z$ vary in its input.

Check

[Hide explanation]

> This first answer is correct.
>
> Leaving off the first coordinate of a three-dimensional vector corresponds to projecting it onto the $yz$-plane.
>
> The fact that only $y$ and $z$ vary indicates that $\mathbf{F}_{x0}$ is a function on the $yz$-plane, but says nothing about how the output of that function should look.

**Concept check**: For a given point $(y_0, z_0)$ in the plane above, what does $\text{2d-curl}\,\mathbf{F}_{x0}(y_0, z_0)$ represent?

**Choose all answers that apply:**

[A] How much the fluid flow defined by $\mathbf{F}$ rotates parallel to the $yz$-plane near the point $(x_0, y_0, z_0)$

[B] The $x$-component of $\text{curl}\,\mathbf{F}(x_0, y_0, z_0)$

Check

[Hide explanation]

> Both answers are correct. In fact, they are two ways of saying the same thing.
>
> $\mathbf{F}_{x0}$ takes vectors from $\mathbf{F}$ stemming from points in the plane $x = x_0$, and projects them onto that same plane. In terms of fluid flow, this means if you follow the three-dimensional motion of particles starting in the plane $x = x_0$ according to the velocity vectors given by $\mathbf{F}$, then project those trajectories onto the plane $x = x_0$, you get the two-dimensional fluid motion described by $\mathbf{F}_{x0}$.
>
> So what does this mean for $\text{2d-curl}\,\mathbf{F}_{x0}$, which measures the rotation in the two-dimensional fluid flow defined by $\mathbf{F}_{x0}$? It measures the extent to which the three-dimensional fluid flow described by $\mathbf{F}$ rotates in the plane $x = x_0$. You could aptly describe this as the component of three-dimensional rotation which is parallel to the $yz$-plane.
>
> On the other hand, since the vector $\operatorname{curl} \mathbf{F}(x_0, y_0, z_0)$ represents the total three-dimensional rotation near $(x_0, y_0, z_0)$, the component of this vector pointing in the $x$-direction also represents the component of fluid rotation which is parallel to the $yz$-plane.

## A component-wise definition

So why am I talking about projecting vectors and trajectories in three dimensions onto a two-dimensional plane? Basically, it's hard to think about three dimensions, so it's worth doing everything you can to frame things two-dimensions at a time.

The importance of this last concept check is that we can describe the $x$-component of the three-dimensional curl of $\mathbf{F}$ purely in terms of the two-dimensional curl of the function $\mathbf{F}_x$:

$$x\text{-component of } \underbrace{\operatorname{curl} \mathbf{F}(x, y, z)}_{\text{3d vector}} = \underbrace{\text{2d-curl}\,\mathbf{F}_x(y, z)}_{\text{Scalar value}}$$

We can also more elegantly pull out the $x$-component of $\operatorname{curl} \mathbf{F}$ by dotting it with the unit vector in the $x$-direction,

$$\hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This means our expression looks like this:

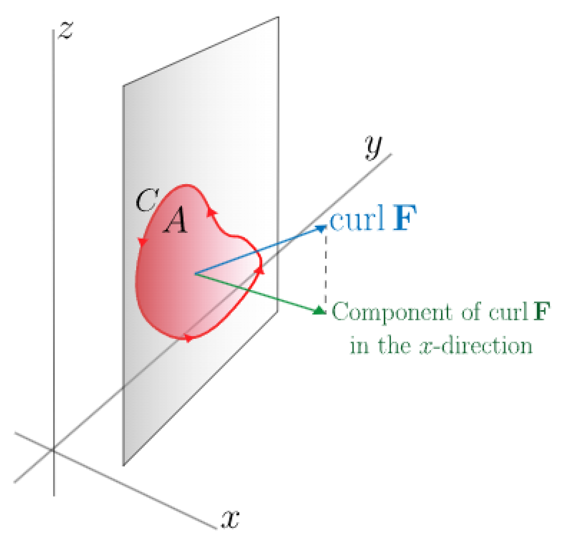
$$\left(\operatorname{curl} \mathbf{F}(x, y, z)\right) \cdot \hat{\mathbf{i}} = \text{2d-curl}\, \mathbf{F}_x(y, z)$$

In terms of the formula you already know, this explains why the $x$-component of $\operatorname{curl} \mathbf{F}$ has the form that it does,

$$\operatorname{curl} \mathbf{F}(x, y, z) = \overbrace{\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)}^{\text{2d-curl}\, \mathbf{F}_x(y,z)} \hat{\mathbf{i}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \hat{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} \right.$$

But remember, the whole point of this article is that curl is one of those funny operations where the formula we use to compute it is not its definition. Our goal is to find a definition of curl by directly representing fluid rotation. With that in mind, the significance of representing the $x$-component of $\operatorname{curl} \mathbf{F}$ using a two-dimensional curl is that we can take the line-integral-limit definition of $\text{2d-curl}$ found in the last article, and use it to *define* the $x$-component of $\operatorname{curl} \mathbf{F}$.

$$\left(\operatorname{curl} \mathbf{F}(x, y, z)\right) \cdot \hat{\mathbf{i}} \overbrace{=}^{\text{definition}} \lim_{A \to 0} \left(\frac{1}{|A|} \oint_C \mathbf{F} \cdot d\mathbf{r}\right)$$




- $A$ is some two-dimensional region in the plane perpendicular to $\hat{\mathbf{i}}$, and passing through the point $(x, y, z)$.
- $C$ is the boundary of $A$.
- The orientation of $C$ is determined based on the right-hand rule: Stick the thumb of your right hand in the direction of $\hat{\mathbf{i}}$, and curl your fingers. The direction your fingers point as they wrap around $C$ is the direction of integration.
- $|A|$ represents the area of $A$.
- $\lim_{|A| \to 0}$ indicates that we are considering the limit as $A$ shrinks to the point $(x, y, z)$ on the plane where $x$ is constant.

Although it will clutter things up, for clarity's sake it will help if our formula expresses the fact that the region $A$ must always include the point $(x, y, z)$, and that it is perpendicular to $\hat{\mathbf{i}}$. To do this, I'll write $A$ with subscripts, $A_{(x,y,z),\hat{\mathbf{i}}}$

This means our full definition looks like this:

$$(\text{curl } \mathbf{F}(x, y, z)) \cdot \hat{\mathbf{i}} \overbrace{=}^{\text{definition}} \lim_{|A_{((x,y,z),\hat{\mathbf{i}})}| \to 0} \left( \frac{1}{\left|A_{((x,y,z),\hat{\mathbf{i}})}\right|} \oint_C \mathbf{F} \cdot d\mathbf{r} \right)$$

This is a *very* heavy definition, which assumes a lot of prior knowledge from the reader. And that's just for one component! The key to understanding this is to:

- Make sure you have a full grasp of the definition of curl in two-dimensions.
- Understand how this definition is applying that same concept to a plane sitting in three-dimensional space.
- Make sure you understand why two-dimensional curl of $\mathbf{F}_{x0}$ should represent the $x$-component of the curl of $\mathbf{F}$.

## Full definition

There is of course nothing special about the $x$-direction, we can also define the other two coordinates of $\text{curl}\,\mathbf{F}$ similarly:

$$(\text{curl } \mathbf{F}(x, y, z)) \cdot \hat{\mathbf{j}} = \lim_{|A_{((x,y,z),\hat{\mathbf{j}})}| \to 0} \left( \frac{1}{\left|A_{((x,y,z),\hat{\mathbf{j}})}\right|} \oint_C \mathbf{F} \cdot d\mathbf{r} \right)$$

$$(\text{curl } \mathbf{F}(x, y, z)) \cdot \hat{\mathbf{k}} = \lim_{|A_{((x,y,z),\hat{\mathbf{k}})}| \to 0} \left( \frac{1}{\left|A_{((x,y,z),\hat{\mathbf{k}})}\right|} \oint_C \mathbf{F} \cdot d\mathbf{r} \right)$$

**Concept check**: What does $A_{((x,y,z),\hat{\mathbf{j}})}$ represent?

**Choose 1 answer:**

(A) A region in the plane passing through the point $(x, y, z)$ which is parallel to the $xz$-plane.

(B) A region in the $yz$-plane.

Check

[Hide explanation]

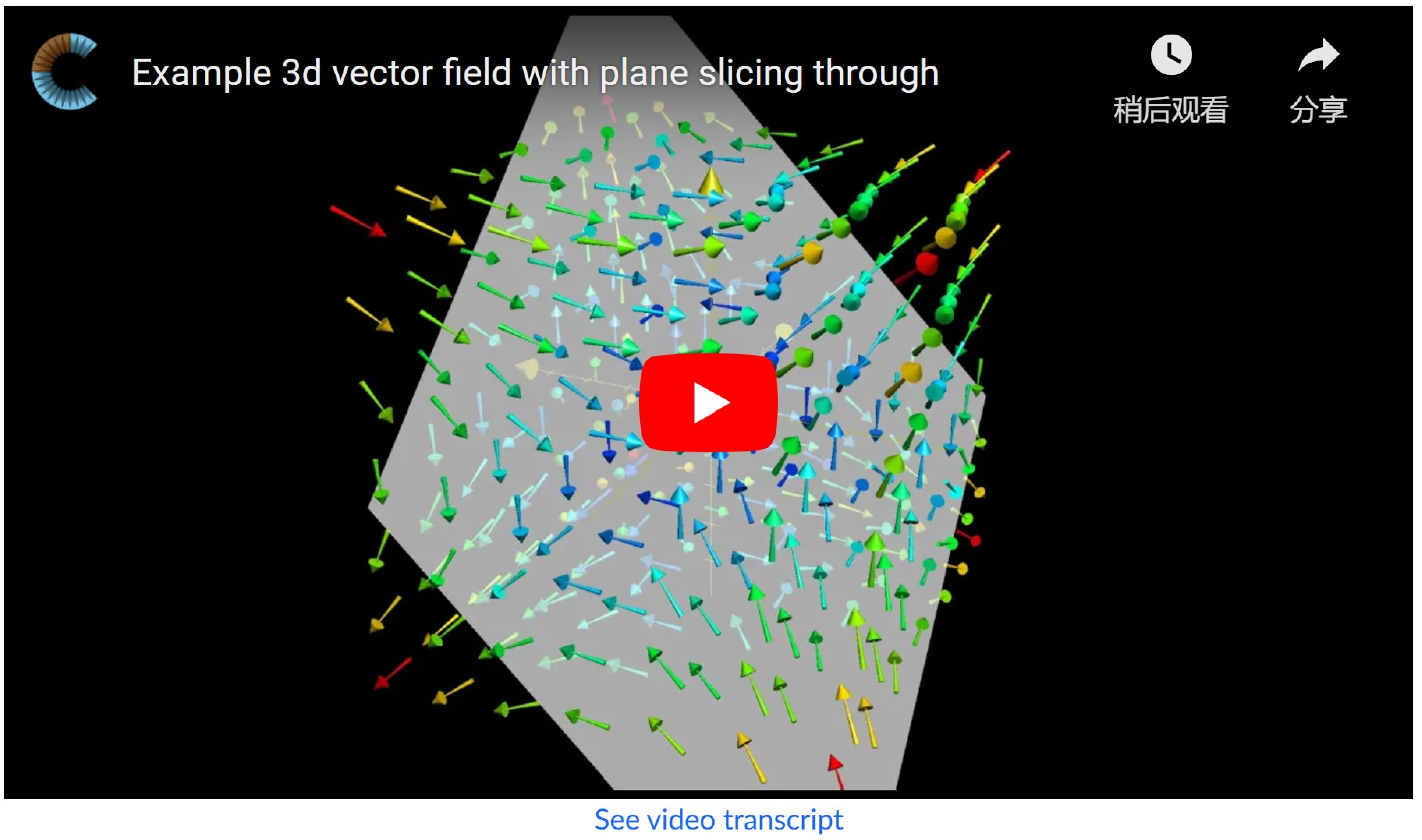
> $A_{((x,y,z),\hat{\mathbf{j}})}$ represents a region in the plane passing through the point $(x, y, z)$ which is perpendicular to $\hat{\mathbf{j}}$. The plane perpedicular to $\hat{\mathbf{j}}$ is parallel to the $xz$-plane.

This gives a full definition, since each component of $\operatorname{curl} \mathbf{F}$ is accounted for.

## Arbitrary unit normal vectors

However, it is a little inelegant to define curl with three separate formulas. Also, when curl is used in practice, it is common to find yourself taking the dot product between the vector $\operatorname{curl} \mathbf{F}$ and some other vector, so it is handy to have a definition suited to interpreting the dot product between $\operatorname{curl} \mathbf{F}$ and *any* vector, not just $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$.

Think about an arbitrary plane cutting through the vector field $\mathbf{F}(x, y, z)$:

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Suppose that this plane is defined to be perpendicular to some unit vector $\hat{\mathbf{n}}$, such as

$$\hat{\mathbf{n}} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Now imagine mimicking everything we did before with the plane $x = x_0$.

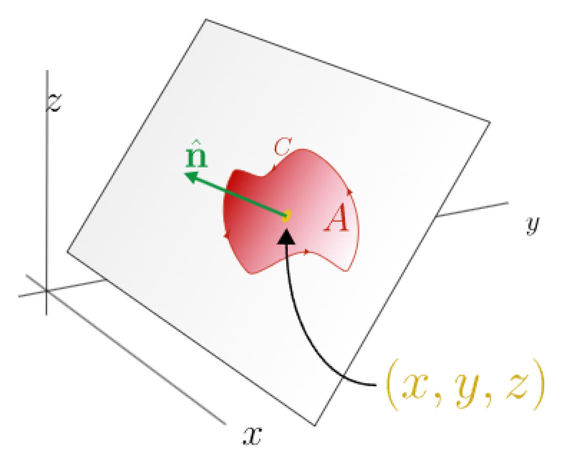
- Considering the vectors which stem from points on this plane.
- Project them onto the plane
- Measure the resulting two-dimensional curl on that plane.

This will let us define the component of three-dimensional curl in the $\hat{\mathbf{n}}$-direction:

$$(\text{curl } \mathbf{F}(x, y, z)) \cdot \hat{\mathbf{n}} = \lim_{|A_{((x,y,z),\hat{\mathbf{n}})}| \to 0} \left( \frac{1}{|A_{((x,y,z),\hat{\mathbf{n}})}|} \oint_C \mathbf{F} \cdot d\mathbf{r} \right)$$

[Hide explanation]

(Vector field $\mathbf{F}$ not pictured)




- $\mathbf{F}$ is a three-dimensional vector field.
- $(x, y, z)$ is some specific point in 3d space.
- $\text{curl } \mathbf{F}(x, y, z)$ returns a three-dimensional vector.
- $\hat{\mathbf{n}}$ is an arbitrary unit vector in three-dimensions.
- $A_{((x,y,z),\hat{\mathbf{n}})}$ represents some two-dimensional region around the point $(x, y, z)$ on a plane perpendicular to the vector $\hat{\mathbf{n}}$.
- $|A_{((x,y,z),\hat{\mathbf{n}})}|$ indicates the area of $A_{((x,y,z),\hat{\mathbf{n}})}$.
- $|A_{((x,y,z),\hat{\mathbf{n}})}| \to 0$ indicates we are considering the limit as the area of $A_{((x,y,z),\hat{\mathbf{n}})}$ goes to zero, meaning that region shrinks around $(x, y, z)$.
- $C$ is the boundary of $A_{((x,y,z),\hat{\mathbf{n}})}$.
- The orientation of $C$ is determined based on the right-hand rule: Stick the thumb of your right hand in the direction of $\hat{\mathbf{n}}$, and the direction your fingers point as they wrap around $C$ is the direction of integration.
- $\oint_C$ is the line integral around $C$, written as $\oint$ instead of $\int$ to emphasize that $C$ is a closed curve.

This is the definition of curl that you might come across in other texts. It's a weird definition, since instead of defining the vector $\operatorname{curl}\mathbf{F}$ itself, it only defines what the dot product between this vector and anything else would be.

But here's why it kind of makes sense to do things this way, even if it feels convoluted: Rotation is an inherently two-dimensional idea, and when we try to talk about rotation in three dimensions (e.g. rotation of the earth) we are somewhat awkwardly forced to use vectors. A given rotation vector is saying "the rotation is really happening in some two-dimensional plane, and I'm just telling you what plane that is."

When it comes to fluid rotation, what we really want is a way of taking any possible rotation vector (which is the same as saying any possible plane in which rotation occurs), and asking "how much does the fluid rotation near a given point look like *this* vector?" The curl gives us a way to answer this question. For a given vector, representing some rotation, when you dot that vector against the curl of a fluid flow, it tells you how much the fluid rotation resembles the rotation represented by that vector.

## Summary

- To define curl in three dimensions, we take it two dimensions at a time. Project the fluid flow onto a single plane and measure the two-dimensional curl in that plane.

- Using the formal definition of curl in two dimensions, this gives us a way to define each component of three-dimensional curl. For example, the $x$-component is defined like this:

$$(\operatorname{curl}\mathbf{F}(x,y,z))\cdot\hat{\mathbf{i}} \overbrace{=}^{\text{definition}} \lim_{|A_{((x,y,z),\hat{\mathbf{i}})}|\to 0}\left(\frac{1}{\left|A_{((x,y,z),\hat{\mathbf{i}})}\right|}\oint_C \mathbf{F}\cdot d\mathbf{r}\right)$$

- You can replace $\hat{\mathbf{i}}$ with any unit vector, thus defining what the component of $\operatorname{curl}\mathbf{F}$ should be in any direction.

## Congrats!

Understanding this complicated definition fully is a sign that you have fully grasped both curl and line integrals, each of which are formidable concepts in their own right.

Also, this definition will prepare you very well for understanding Stokes' theorem, a topic which stands at the very pinnacle of multivariable calculus.