Surface integrals

Google Classroom

How do you add up infinitely many infinitely small quantities associated with points on a surface?

Background

- Surface area
- Double integrals

Not strictly required, but useful for intuition and analogy:

- Line integrals

What we're building to

- In principle, the idea of a surface integral is the same as that of a double integral, except that instead of "adding up" points in a flat two-dimensional region, you are adding up points on a surface in space, which is potentially curved. The abstract notation for surface integrals looks very similar to that of a double integral:

$$\underbrace{\iint_S}_{S \text{ represents a surface}} f(x, y, z) \overbrace{d\Sigma}^{\text{Tiny piece of area in } S}$$

- Computing a surface integral is almost identical to computing surface area using a double integral, except that you stick a function inside the integral:

$$\iint_T f(\vec{\mathbf{v}}(t, s)) \underbrace{\left| \frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s} \right| dt\, ds}_{\text{Tiny piece of area}}$$

Here, $\vec{\mathbf{v}}(t, s)$ is a function parameterizing the surface S from the region T of the ts-plane.

(This is analogous to how computing line integrals is basically the same as computing arc length integrals, except that you throw a function inside the integral itself.)

- You can find an example of working through one of these integrals in the next article.

The idea of surface integrals

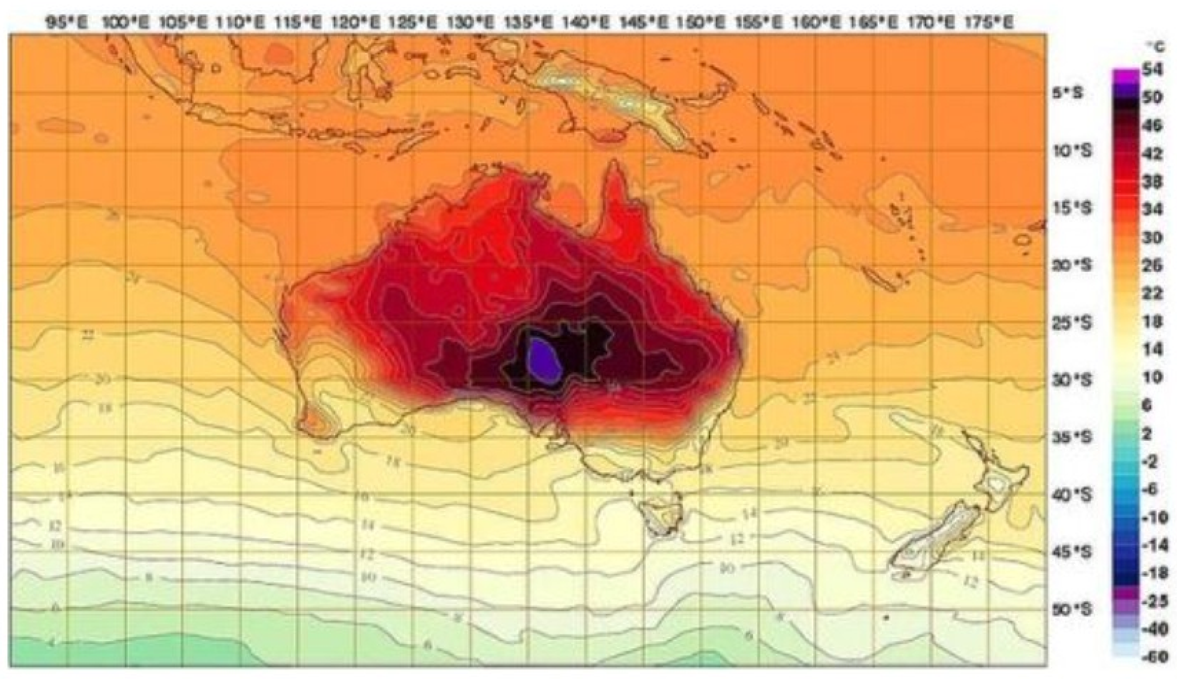
If you understand double integrals, and you understand how to compute the surface area of a parametric surface, you basically already understand surface integrals. It's just a matter of smooshing the two intuitions together. I'll go over the computation of a surface integral with an example in just a bit, but first, I think it's important for you to have a good grasp on what exactly a surface integral *does*.

Refresher of double integrals

Recall what a double integral does:

$$\iint_R f(x, y)\, dA$$

Here, R represents some region of the xy-plane, and $f(x, y)$ is a way to associate each point of R with a number.



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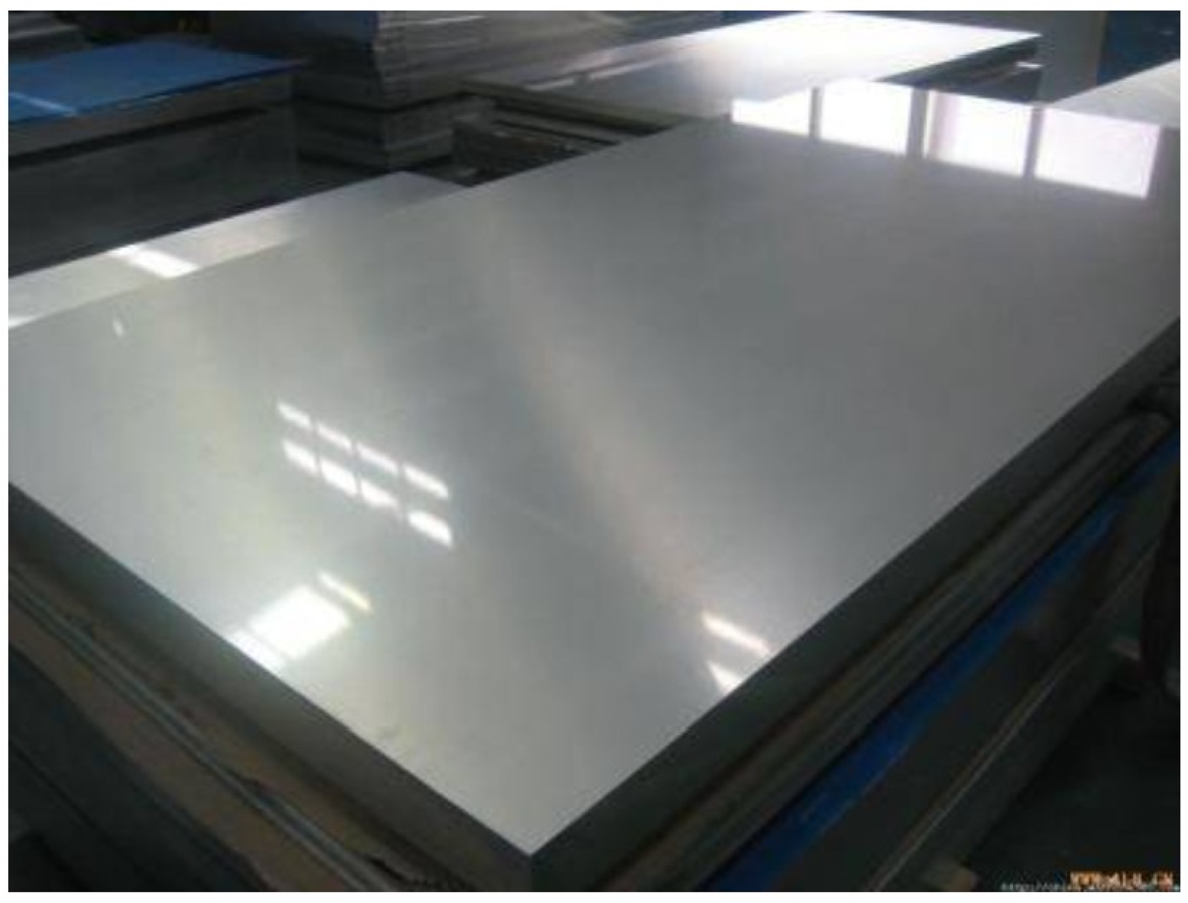

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- Maybe R represents a metal sheet, and $f(x, y)$ represents the density at each point.

- Or perhaps R represents a geographic region, and $f(x, y)$ represents the temperature at each point.

The double integral provides a way to "add up" the values of f on this region. However, the idea of "adding up" points in a continuous region is vague, so I like to imagine the following process:

- Chop up the region R into many tiny pieces.

- Multiply the area of each piece, thought of as dA, by the value of f at one of the points inside that piece.

- Add up the resulting values.

For example,

- If R represents a metal sheet, and $f(x, y)$ is a density function, the double integral will give you the mass of the sheet. (Why?)

- If R represents a geographic region, and $f(x, y)$ give the temperature at each location, taking this double integral then dividing by the area of R will give you the average temperature in that region. (Why?)

Double integrals over curved regions

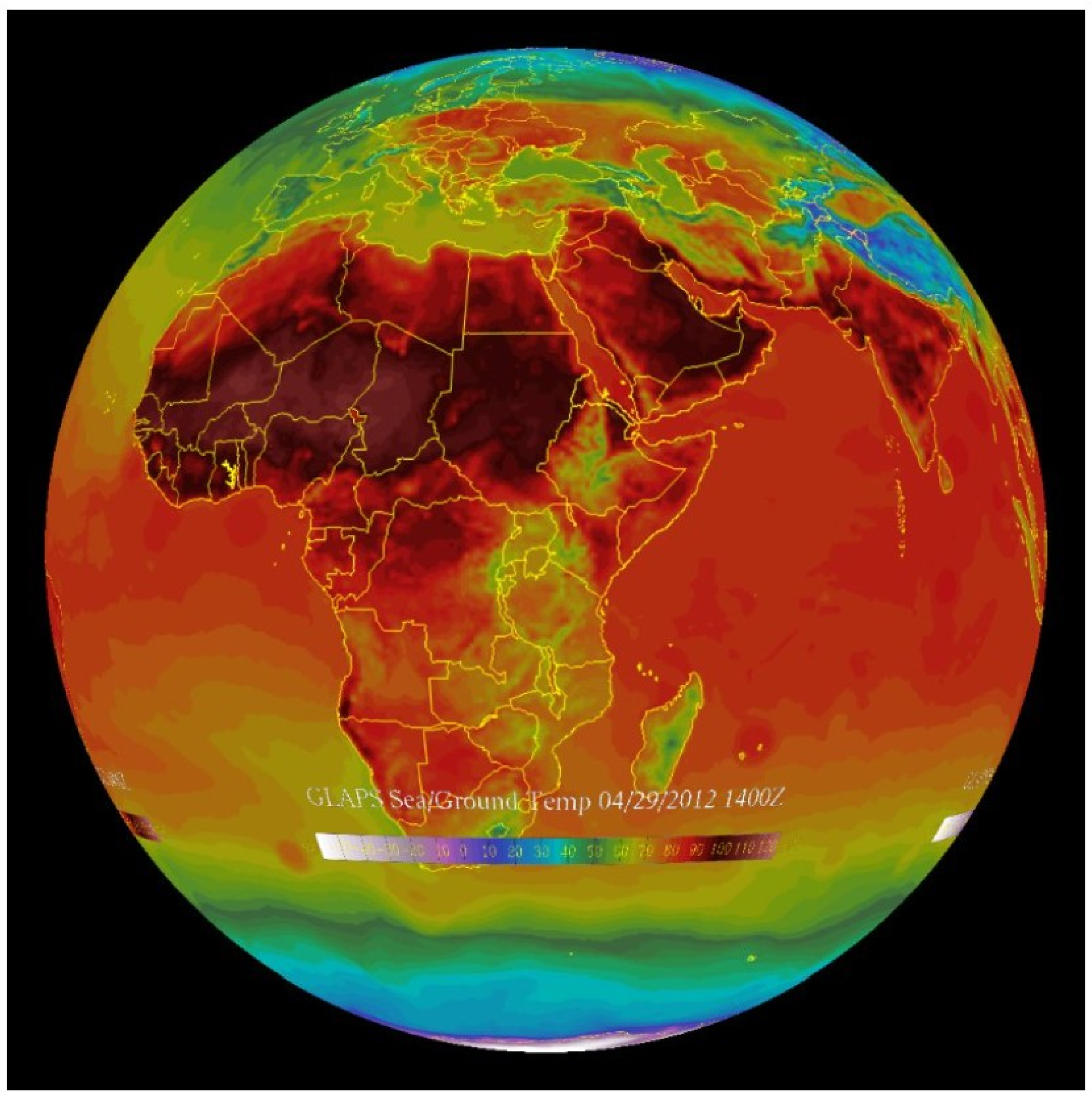

Image credit: "GLAPS Model: Sea Surface and Ground Temperature", by the National Oceanic and Atmospheric Administration.

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However, why stay so flat? This idea of adding up values over a continuous two-dimensional region can be useful for curved surfaces as well.

- What if you are considering the surface of a curved airplane wing with variable density, and you want to find its total mass?

- What if you have the temperature for every point on the curved surface of the earth, and you want to figure out the average temperature?

This time, the function f, which represents density, temperature, etc., must take in point of three dimensions since points on the surface live in three dimensions. The abstract notation for integrating a three-variable function $f(x, y, z)$ over a surface is pretty much the same as the abstract notation for double integrals:

$$\underbrace{\iint_S}_{S \text{ represents some surface}} f(x, y, z) \overbrace{d\Sigma}^{\text{Tiny piece of area on } S}$$

(Different authors might use different notation).

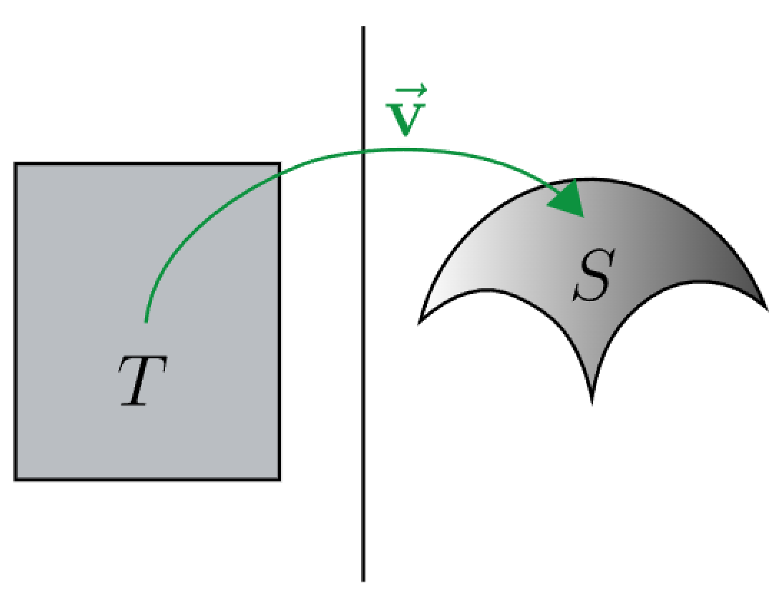
This is called a **surface integral**. The little S under the double integral sign represents the surface itself, and the term $d\Sigma$ represents a tiny bit of area piece of this surface. You can think about surface integrals the same way you think about double integrals:

- Chop up the surface S into many small pieces.
- Multiply the area of each tiny piece by the value of the function f on one of the points in that piece.
- Add up those values.

Why write $d\Sigma$ instead of dA? There's no real difference; each one represents a tiny bit of area of the thing you are integrating over. However, when it comes time to compute things, the way to handle a tiny bit of area on a curved surface is fundamentally different from doing it on a flat surface, so it's worth emphasizing this difference by using a different variable.

How to compute a surface integral

Abstract notation and visions of chopping up airplane wings are all well and good, but how do you actually *compute* one of these surface integrals? The trick is to sneakily turn it into an ordinary, *flat*, double integral.

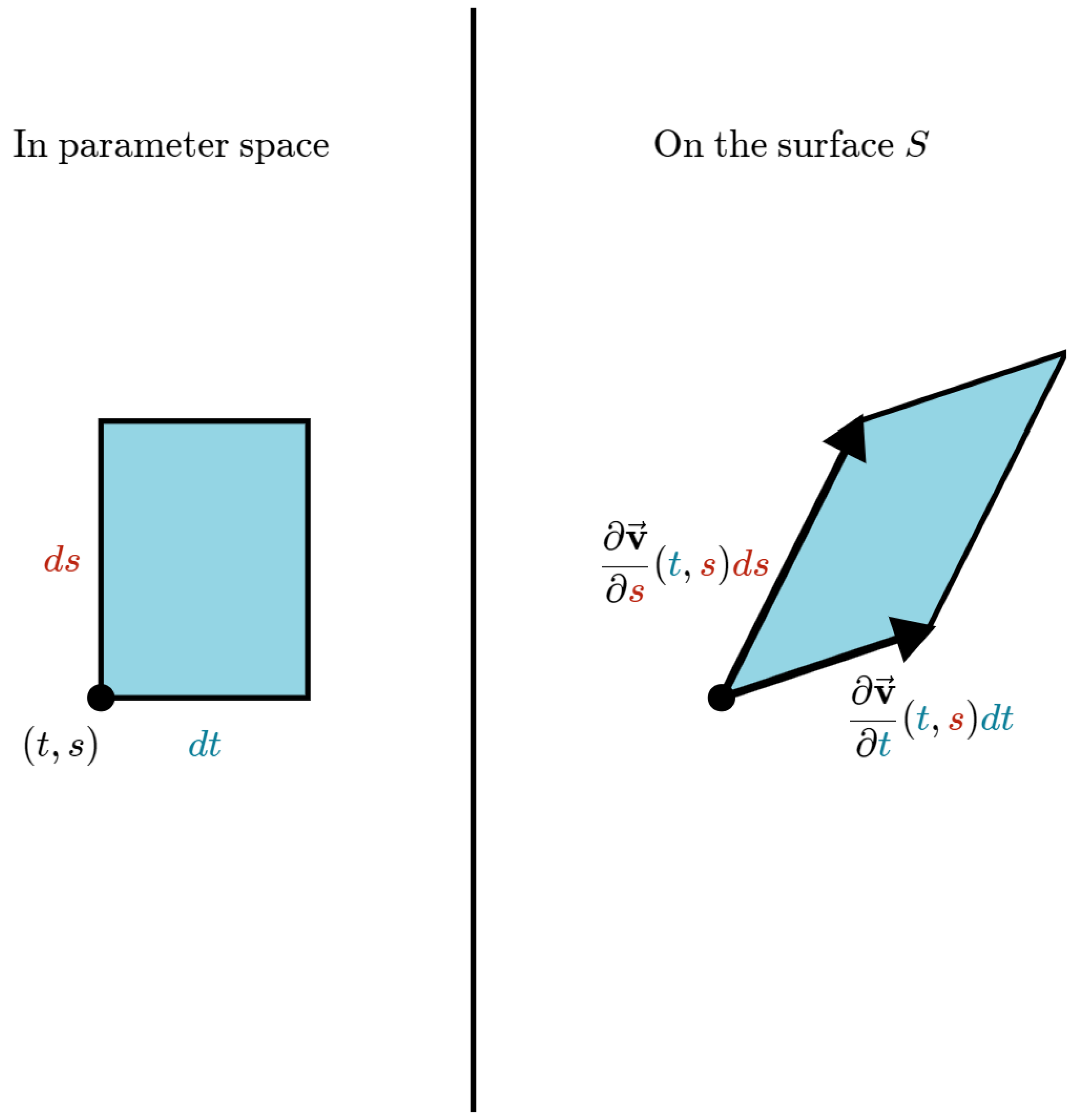


Specifically, the way you tend to represent a surface mathematically is with a parametric function. You'll have some vector-valued function $\vec{\mathbf{v}}(t, s)$, which takes in points on the two-dimensional ts-plane (lovely and flat), and outputs points in three-dimensional space. You also need to specify the region T of the ts-plane which maps onto the surface S.

The trick for surface integrals, then, is to find a way of integrating over the flat region T that gives the same effect as integrating over the curved surface S.

This requires describing "tiny piece of area" of S in terms of something inside the parameter.

Almost all of the work for this was done in the article on [surface area](). There, we saw how a tiny rectangle inside T with area $dt\, ds$ gets transformed into a parallelogram on S with area $\left|\frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s}\right| dt\, ds$



For our surface integral desires, this means you expand $d\Sigma$ as follows:

$$d\Sigma = \left|\frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s}\right| dt\, ds$$

Specifically, here's how to write a surface integral with respect to the parameter space:

$$\iint_S f(x, y, z)\, d\Sigma = \iint_T f(\vec{\mathbf{v}}(t, s)) \left|\frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s}\right| dt\, ds$$

Let's break that down a bit:

$$\underbrace{\iint_S}_{\text{Integral over surface}} f(x, y, z) \overbrace{d\Sigma}^{\text{Area of a tiny piece of } S} = \underbrace{\iint_T}_{\text{Integral in parameter space}} \overbrace{f(\vec{\mathbf{v}}(t, s))}^{\text{See where each point } (t,s) \text{ lands on } S\text{, then evalute } f} \underbrace{\left|\frac{\partial \vec{\mathbf{v}}}{\partial t}\right.}_{\text{How much } T \text{ is sca mapped}}$$

The main thing to focus on here, and what makes computations particularly labor intensive, is the way to express $d\Sigma$.

In the next article, you can go through a full example of one of these surface integrals.

Summary

- Surface integrals are used anytime you get the sensation of wanting to add a bunch of values associated with points on a surface. This is the two-dimensional analog of line integrals. Alternatively, you can view it as a way of generalizing double integrals to curved surfaces.

$$\underbrace{\iint_S}_{S \text{ represents a surface}} f(x, y, z) \overbrace{d\Sigma}^{\text{Tiny piece of area in } S}$$

- Computing a surface integral is almost identical to computing surface area using a double integral, except that you stick a function inside the integral:

$$\iint_T f(\vec{\mathbf{v}}(t, s)) \left| \frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s} \right| dt\, ds$$

Like so many things in multivariable calculus, while the theory behind surface integrals is beautiful, actually computing one can be painfully labor intensive.