Curl, fluid rotation in three dimensions

Google Classroom

Curl is an operator which measures rotation in a fluid flow indicated by a three dimensional vector field.

Background

- Partial derivatives
- Vector fields
- Cross product
- Curl warmup

Note: Throughout this article I will use the convention that

- $\hat{\mathbf{i}}$ represents the unit vector in the x-direction.
- $\hat{\mathbf{j}}$ represents the unit vector in the y-direction.
- $\hat{\mathbf{k}}$ represents the unit vector in the z-direction.

What we're building to

- Curl is an operator which takes in a function representing a three-dimensional vector field and gives another function representing a different three-dimensional vector field.

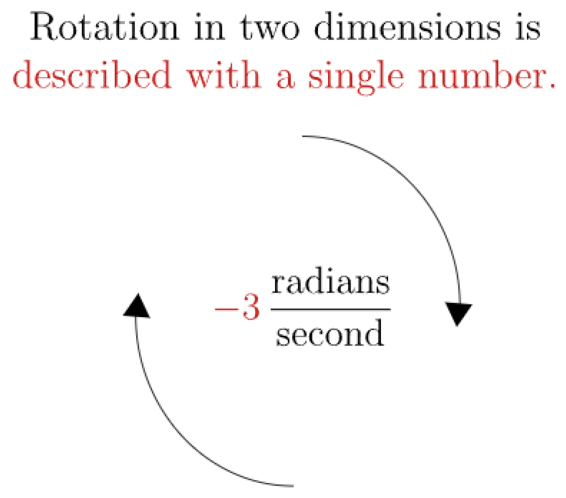
- If a fluid flows in three-dimensional space along a vector field, the rotation of that fluid around each point, represented as a vector, is given by the curl of the original vector field evaluated at that point. The curl vector field should be scaled by one-half if you want the magnitude of curl vectors to equal the rotational speed of the fluid.

- If a three-dimensional vector-valued function $\vec{\mathbf{v}}(x, y, z)$ has component functions $v_1(x, y, z)$, $v_2(x, y, z)$ and $v_3(x, y, z)$, the curl is computed as follows:

$$\underbrace{\nabla \times \vec{\mathbf{v}}}_{\text{Notation for curl}} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v}{\partial y}\right.$$

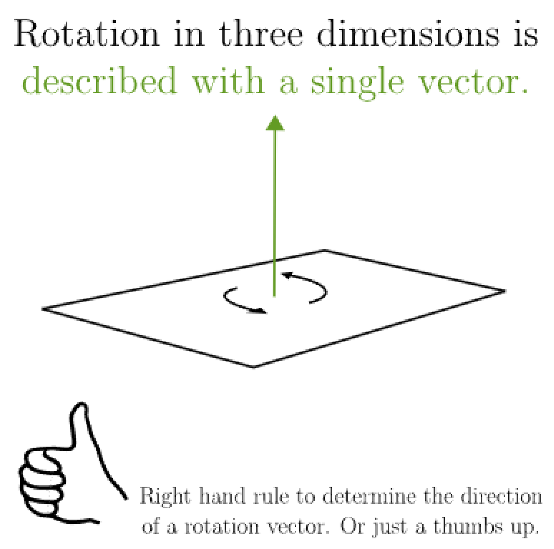
Describing rotation with a vector

If an object is rotating in two dimensions, you can describe the rotation completely with a single number: the angular velocity. A positive angular velocity indicates a counter-clockwise rotation while a negative number indicates a clockwise rotation. The absolute value of the angular velocity gives the speed of rotation, typically in radians per second.



For an object rotating in three dimensions, the situation is more complicated. We need to represent both angular velocity and the *direction* in three-dimensional space in which the object is rotating.

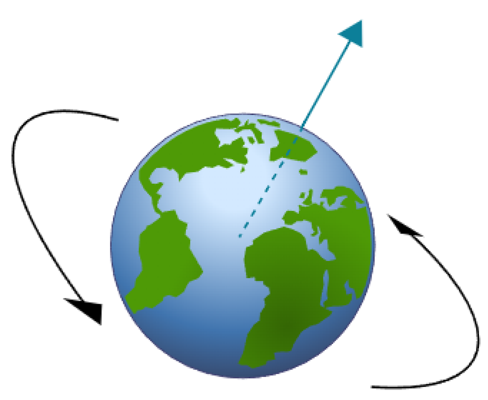
To do this, rotation in three dimensions is typically described using a single **vector**. The magnitude of the vector indicates the angular speed, and the direction is determined by a super-important convention called the "right-hand rule"



- **RIGHT-HAND RULE**: Curl the fingers of your right hand in the direction of rotation, and stick out your thumb. The vector representing this three-dimensional rotation is, by definition, oriented in the direction of your thumb.

Your thumb should point along the axis of rotation. Adopting the convention of using the right hand instead of the left lets us encode the difference between a certain three-dimensional rotation, and the reverse rotation. Basically, it extends the idea of clockwise vs. counterclockwise into three dimensions.

For example, the rotation of the earth in space would be described using a vector pointing from the center of the earth to its north pole, whose length is equal to the angular speed of the earth's rotation (which happens to be 0.0000729 radians/second).

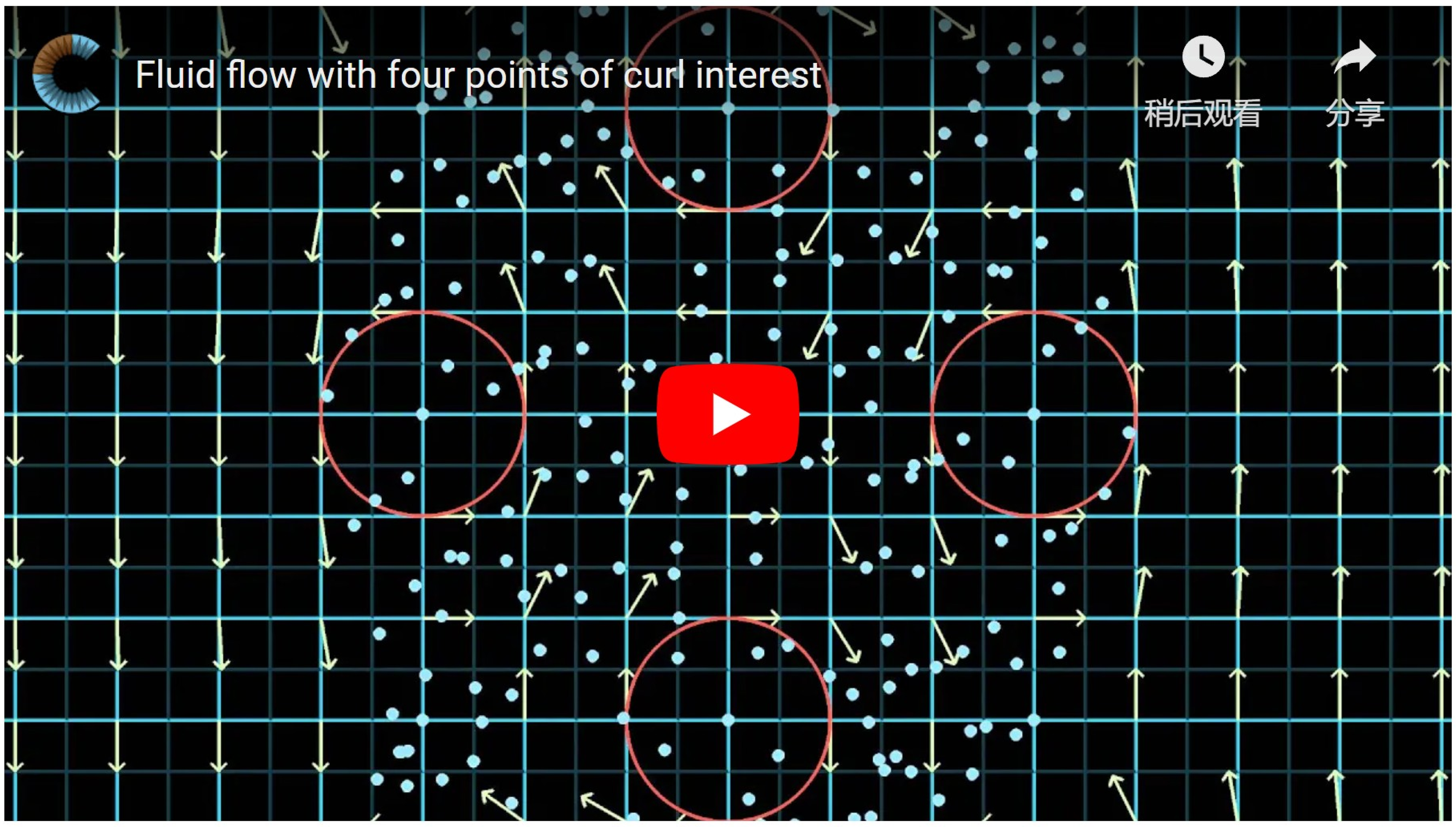


Two-dimensional fluid rotation revisited

In the curl warmup article, I introduce how fluid flows along a two-dimensional vector field defined by the function

$$\begin{aligned}\vec{\mathbf{v}}(x, y) &= \left[\begin{array}{c} y^3 - 9y \\ x^3 - 9x \end{array}\right] \\ &= (y^3 - 9y)\hat{\mathbf{i}} + (x^3 - 9x)\hat{\mathbf{j}}\end{aligned}$$

The following animation gives a simulation of this, where fluid particles (drawn as blue dots) always move in the direction of the vector they are closest to. For the purposes of studying curl, notice what happens in and around the circled regions.



See video transcript

The fluid rotates counterclockwise in the left and right circles, and clockwise in the top and bottom circles. In studying curl, the key question is this: **How much does the fluid rotate around each specific point (x_0, y_0) in the plane?**

In the last article, I gave an intuition for how the answer to this question is what you might call the **2d-curl** of $\vec{\mathbf{v}}$, which has the following formula:

$$\text{2d-curl } \vec{\mathbf{v}}(x_0, y_0) = \frac{\partial v_2}{\partial x}(x_0, y_0) - \frac{\partial v_1}{\partial y}(x_0, y_0)$$

Here, v_1 and v_2 are the components of the vector-valued function $\vec{\mathbf{v}}$. For example, with specific vector field given above, defined by $(y^3 - 9y)\hat{\mathbf{i}} + (x^3 - 9x)\hat{\mathbf{j}}$, this answer would be

$$\begin{aligned}\frac{\partial(x^3 - 9x)}{\partial x} - \frac{\partial(y^3 - 9y)}{\partial y} &= 3x^2 - 9 - (3y^2 - 9) \\ &= 3x^2 - 3y^2\end{aligned}$$

Notice, the result is a scalar-valued function. You plug in a point, like $(2, 1)$, and you get out a single number which indicates angular velocity of the fluid near your point, $3(2)^2 - 3(1)^2 = 12 - 3 = 9$. As it turns out this number represents *twice* the angular speed of the fluid near the point, so the speed of rotation is 4.5 radians/second (more on this later). The important point that you get a single scalar describing the rotation.

This should make sense because the rotation of a single object in two dimensions can be described with a single number (or scalar), so rotation around all possible points in a flowing fluid should be described with a **scalar**-valued function.

Reflection question: In the fluid flow animated above, does the fluid have a rotational component at the origin $(0, 0)$?

Choose 1 answer:

(A) Yes, clockwise

(B) Yes, counterclockwise

(C) No

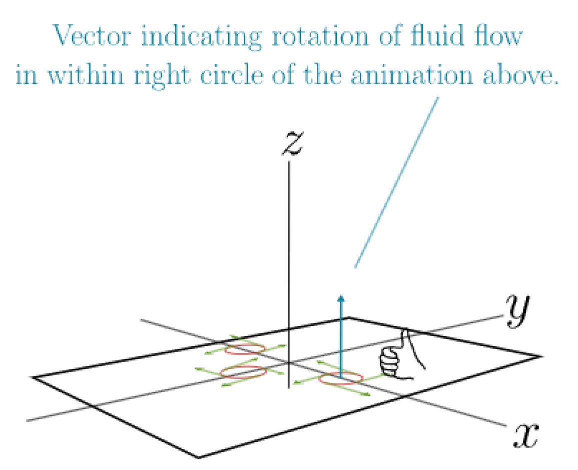
Check

[Hide explanation]

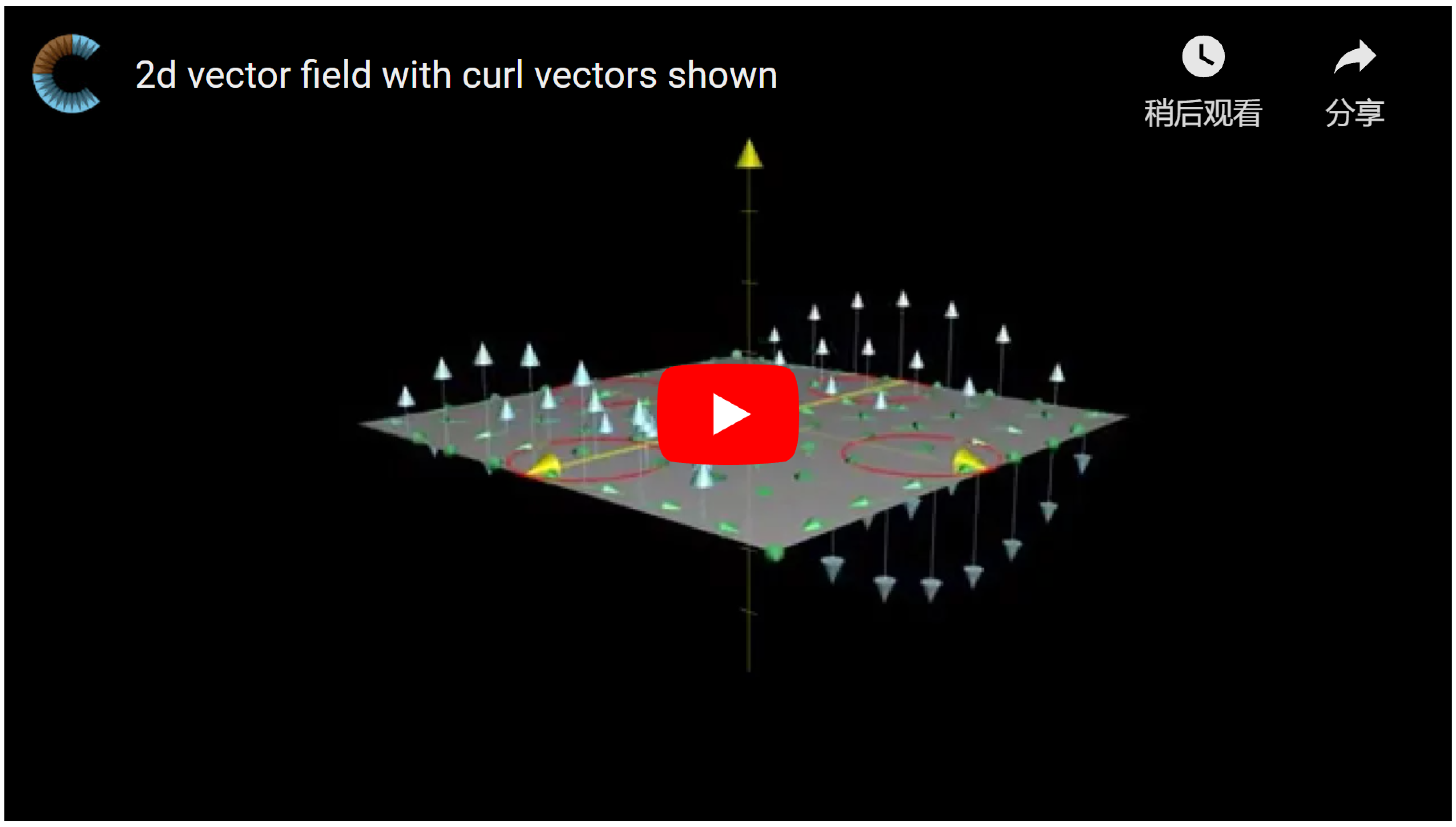
> Evaluating the formula $3x^2 - 3y^2$ found above at the point $(x, y) = (0, 0)$, we get $3(0)^2 - 3(0)^2 = 0$. This means there is no fluid rotation around this point.
>
> This should feel consistent with the animation above. Around the origin, fluid particles flow in from the upper right and lower left, and flow out towards the upper left and lower right, but there is no noticeable rotation in how they do this.

Moving to three dimensions

In preparation for moving to three dimensions, let's express the fluid rotation above using vectors. Focus on a region of counterclockwise rotation, such as the right-most circle in the animation above. Imagine wrapping the fingers of your right hand around this circle, so they point in the direction of the arrows (counterclockwise in this case), and stick out your thumb. Your thumb should be pointing out of the page, in the positive z-direction, parallel to the unit vector $\hat{\mathbf{k}}$.



If we did this at every point, assigning a vector to the rotation around each point on the xy-plane according to the formula $\text{2d-curl}\,\vec{\mathbf{v}}(x, y) = 3x^2 - 3y^2$, you would end up with something like this:



See video transcript

Vectors pointing in the positive z-direction indicate counterclockwise rotation near that point, and vectors pointing the other way indicate clockwise rotation, as viewed from above the xy-plane. The length of each vector indicates the speed of that rotation. You could describe this system of vectors with the expression

$$(3x^2 - 3y^2)\hat{\mathbf{k}}$$

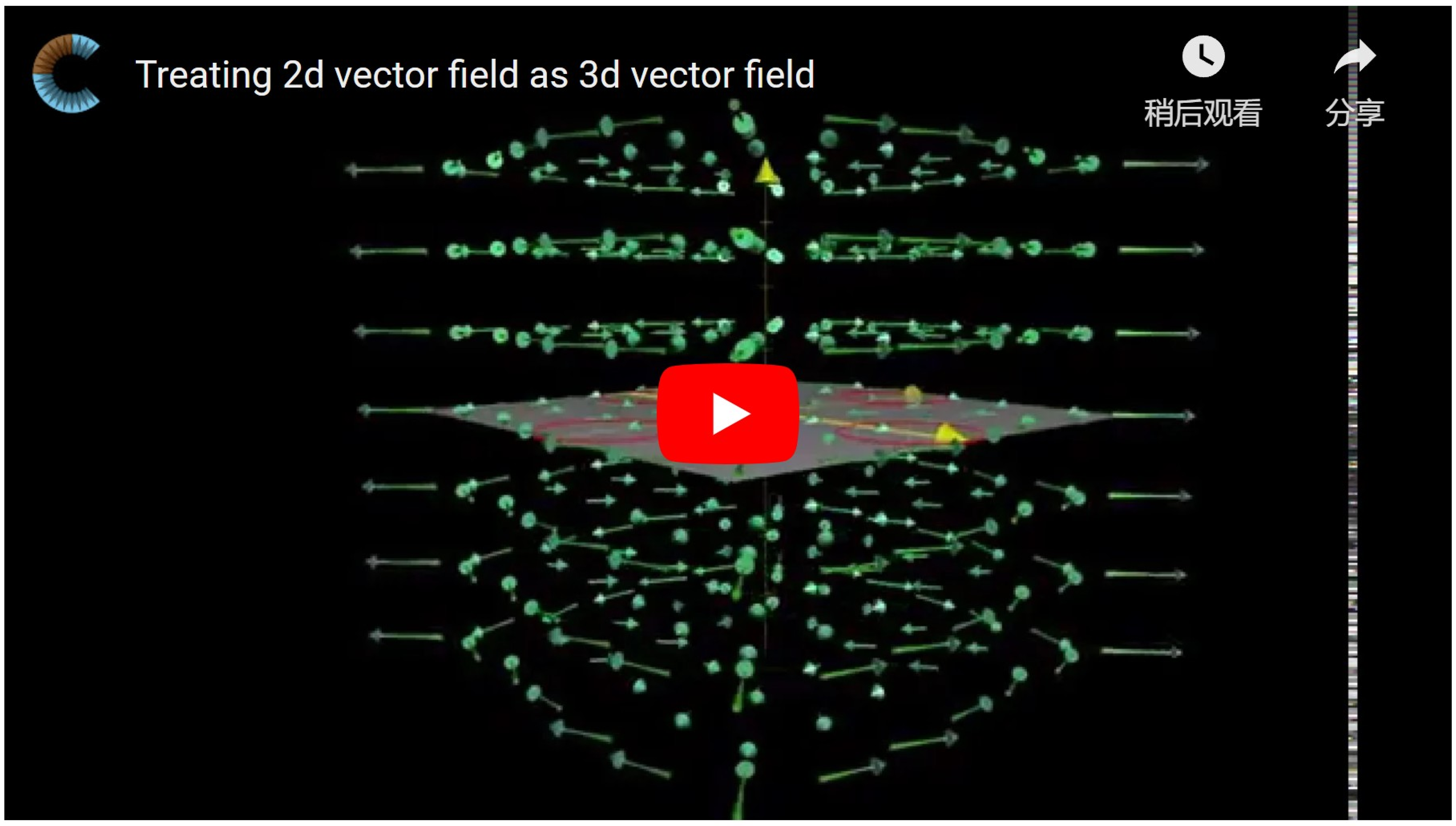
This is almost a three-dimensional vector field, except that we are only looking at points on the xy-plane, not in all of space. **Curl itself only applies to three-**

dimensional vector fields, so to properly set the stage for the material below, let's make this a fully three-dimensional example. To start, we extend our original vector-valued function $\vec{\mathbf{v}}$ to a similar three-dimensional function $\vec{\mathbf{v}}_{3d}$.

$$\vec{\mathbf{v}}_{3d}(x, y, z) = \begin{bmatrix} y^3 - 9y \\ x^3 - 9x \\ 0 \end{bmatrix} = (y^3 - 9y)\hat{\mathbf{i}} + (x^3 - 9x)\hat{\mathbf{j}} + (0)\hat{\mathbf{k}}$$

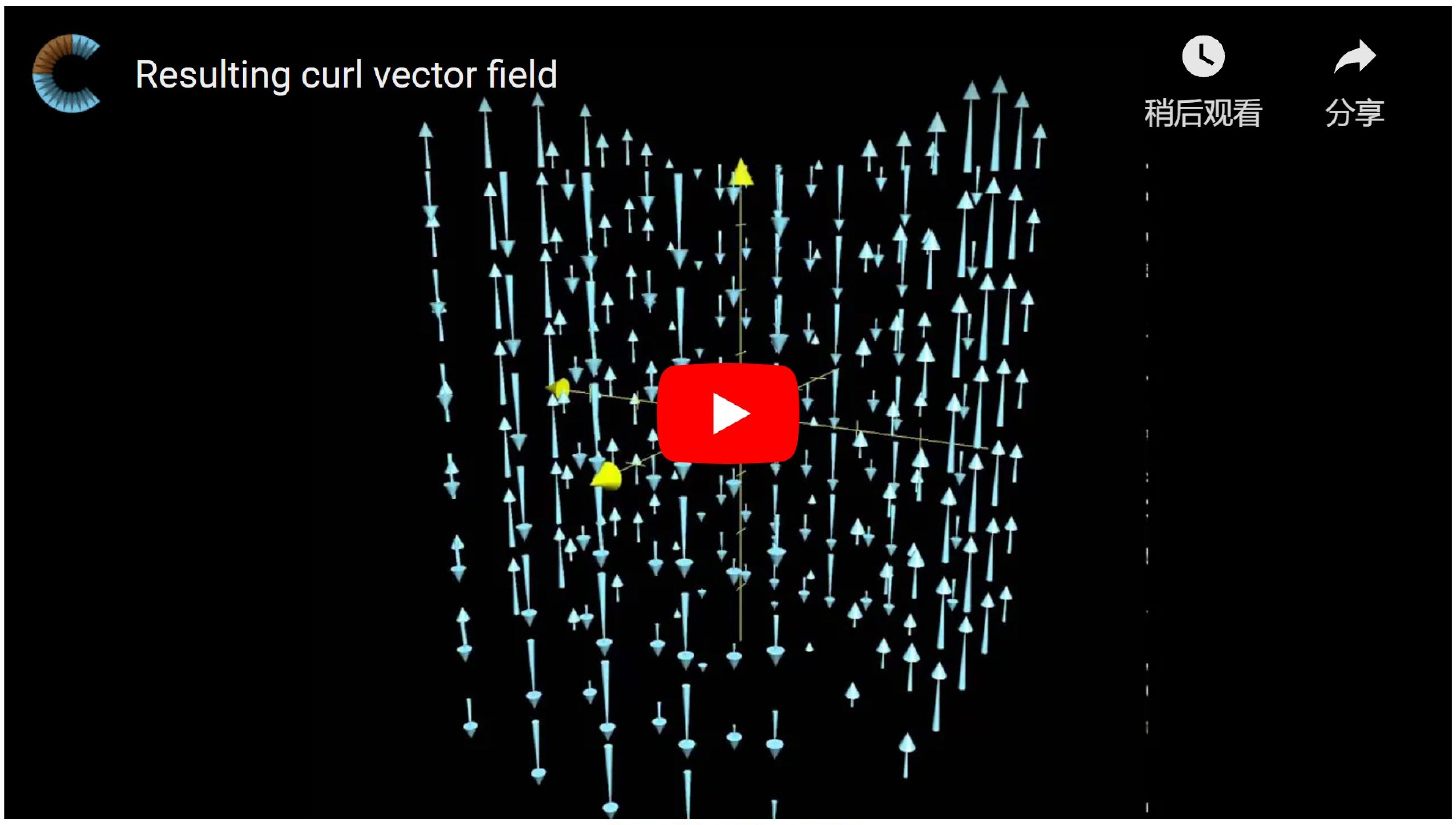
As three-dimensional vector fields go, this still feels very flat, doesn't it? The $\hat{\mathbf{k}}$ component is 0 everywhere, and none of the components depend on the z input variable at all. We have basically just copied the original two-dimensional vector field onto every slice of three-dimensional space parallel to the xy-plane.

The next video shows what that vector field $\vec{\mathbf{v}}_{3d}$ looks like, where we keep the flat xy-plane (drawn in grey) and red circles as reference points. Notice that at each layer parallel to the xy-plane, the vectors are identical to the original vectors we had sitting in the xy-plane from the purely 2d vector field $\vec{\mathbf{v}}$ in the previous section.



See video transcript

Again, imagine this vector field as representing a fluid flow, like air in a room or water in a pool. When we represent the rotation of this fluid around each point with a vector attached to that point, we get a new vector field, as shown in the next video:



See video transcript

This is given by the vector-valued function

$$\vec{\mathbf{w}}(x, y, z) = (0)\hat{\mathbf{i}} + (0)\hat{\mathbf{j}} + (3x^2 - 3y^2)\hat{\mathbf{k}}$$

This is the same formula that we had before, $(3x^2 - 3y^2)\hat{\mathbf{k}}$, but the important point is that now we apply it to all points (x, y, z) in space, not just the points (x, y) in the xy-plane.

- The fact that the z-input does not influence the output reflects the fact that our fluid motion is the same in all slices of space parallel to the xy-plane.
- The fact that the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components are 0 means all rotation vectors point purely in the z-direction, meaning all actual fluid rotation is parallel to the xy-plane.

This new (blue) vector field $\vec{\mathbf{w}}$ is called the "curl" of the initial (green) vector field $\vec{\mathbf{v}}_{3d}$. One way you might see this written is

$$\vec{\mathbf{w}} = \operatorname{curl} \vec{\mathbf{v}}_{3d}$$

This is our first example of honest-to-goodness three-dimensional curl: **Curl, as a mathematical operator, takes in a three-dimensional vector-valued function $\vec{\mathbf{v}}_{3d}$, thought of as representing a fluid flow, and outputs another three-dimensional vector-valued function "$\operatorname{curl} \vec{\mathbf{v}}_{3d}$" which represents the rotation near each point of that fluid.**

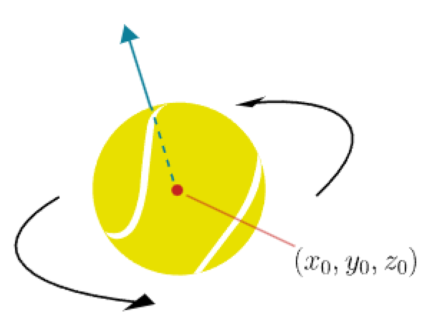
Visualizing fluid rotation in three dimensions

For a general fluid flow in three dimensions, the rotation may not always be purely parallel to the xy-plane. This can make it hard to picture what's going on. Really hard.

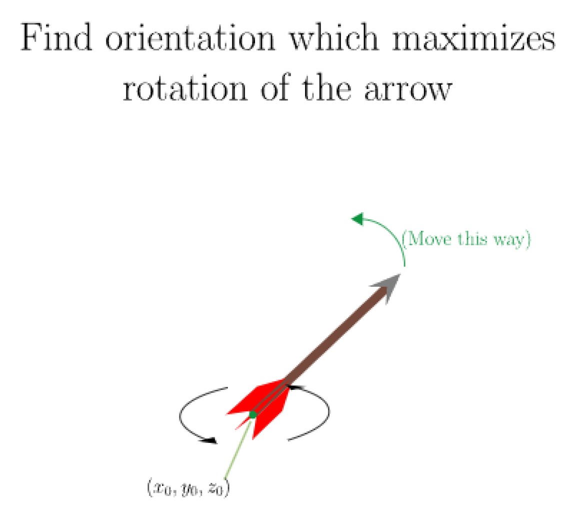
For instance, imagine that the air around you is blowing and swirling in some chaotic motion. Now pick some specific point (x_0, y_0, z_0) in space. How can you think about what "air rotation near that point" means?

Here are a couple of tactics:

- Imagine there is a tiny tennis ball whose center is fixed to the point (x_0, y_0, z_0), but which is free to rotate. Perhaps you have invented magic to hold it there, or otherwise have some sort of ingenious magnetic suspension device. The air blowing around it may cause it to spin in some way or another. The curl vector attached to that point will be the vector describing this tiny tennis ball's rotation, in the same way, we described the earth's rotation using a single vector above.



- Alternatively, take an archer's arrow with nice thick feathers. The kind you might imagine Robin Hood shooting. Situate the arrow in midair such that its feathers are at the point (x_0, y_0, z_0). Again, you've invented magic and finagle a way so that the base of the arrow is fixed to this point, but you are free to orient the arrow in any direction you want, and it freely rotates based on how the wind blows its feathers.



If you experiment with various orientations for the arrow and find the one direction in which the air currents cause the arrow to rotate the fastest, this is the direction of the curl vector at the point (x_0, y_0, z_0).

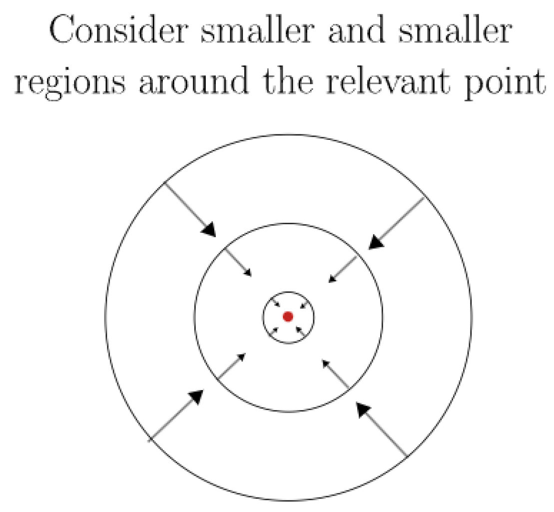
This is somewhat analogous to how the gradient points in the "direction of steepest ascent"; the curl points in the "direction of greatest rotation".

[Hide explanation]

Curl is a limiting operation

Technically, all the descriptions above describe fluid flow in a **region**: The circled regions of the animation, the region occupied by the tiny tennis ball, the region spanned by the feathers of the arrow, etc. None of these are literally rotation at just one point.

In fact, the idea of "rotation at a point" is kind of ridiculous, isn't it? It only ever makes sense to talk about the rotation of a fluid *around* a point. Nevertheless, the curl of a vector field, as a mathematical operation, takes in just one point as its input, not a region, so what does it *really* mean.



As with all differential operators, the formal definition of curl involves a limiting process. You can think of it as taking the limit of rotation in smaller and smaller regions which all surround a given point. For example, imagine the tennis ball above is shrinking. If you are hungry for a formal definition, I wrote about the formal definition of curl later on, once line integrals are covered

Just keep in mind that all descriptions using animations, images of small rotating balls, or anything of this sort, are necessarily approximations of what true curl is measuring.

Notation and formula for curl

Let's write $\vec{\mathbf{v}}$ as a general vector-valued function, with three inputs (x, y, z) and a three-coordinate output. We will write this three-coordinate output in terms of three scalar valued functions: $v_1(x, y, z)$, $v_2(x, y, z)$, and $v_3(x, y, z)$.

$$\vec{\mathbf{v}}(x, y, z) = \begin{bmatrix} v_1(x, y, z) \\ v_2(x, y, z) \\ v_3(x, y, z) \end{bmatrix}$$

$$= v_1(x, y, z)\hat{\mathbf{i}} + v_2(x, y, z)\hat{\mathbf{j}} + v_3(x, y, z)\hat{\mathbf{k}}$$

The notation for curl uses the same symbol "∇" used in the expressions for gradient and divergence, and once again we think of it as representing a vector of partial derivative operators:

$$\nabla = \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix}$$

The curl is thought of as the cross product of this "vector" and the function $\vec{\mathbf{v}}$, computed using the determinant as usual:

$$\begin{aligned}
\operatorname{curl} \vec{\mathbf{v}} &= \nabla \times \vec{\mathbf{v}} \\
&= \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} v_1(x, y, z) \\ v_2(x, y, z) \\ v_3(x, y, z) \end{bmatrix} \\
&= \det\left(\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{bmatrix} \right) \\
&= \boxed{\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \hat{\mathbf{k}}}
\end{aligned}$$

I know what you're thinking: "That's the funkiest determinant I've ever seen. None of the elements are even numbers! One row has vectors, one has operators, and one has functions. Can you even do that?" It's a bit weird, sure, but it works as a notational trick if nothing else. [Hide explanation]

> When you "multiply" an operator like $\frac{\partial}{\partial x}$ with a function like v_2, you're not really multiplying, but applying the operator to get a new function, $\frac{\partial v_2}{\partial x}$. Similarly, multiplying a vector like $\hat{\mathbf{k}}$ with a function like $\frac{\partial v_2}{\partial x}$ gives a vector-valued function $\frac{\partial v_2}{\partial x}\hat{\mathbf{k}}$.
>
> The final expression for the determinant is the sum of 6 vector-valued functions like this one.

Intuition for the formula

Let's take a close look at this final result:

$$\operatorname{curl} \vec{\mathbf{v}} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{\mathbf{k}}$$

Notice, each component is like its own version of 2d-curl operator we found in the curl warm up article. In fact, the $\vec{\mathbf{k}}$ component has precisely the same formula as the 2d-curl. This should make sense because the $\vec{\mathbf{k}}$-component of curl should measure the component of fluid rotation which is parallel to the xy -plane.

Likewise, the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ components measure the component of the fluid rotation parallel to the yz and xz planes respectively.

$$\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{\mathbf{i}} \quad \leftarrow \text{Rotational component parallel to the } yz\text{-plane}$$

$$\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{\mathbf{j}} \quad \leftarrow \text{Rotational component parallel to the } xz\text{-plane}$$

$$\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{\mathbf{k}} \quad \leftarrow \text{Rotational component parallel to the } xy\text{-plane}$$

One little nuance I should point out is that when you evaluate the curl near a point to get a vector (thought of as a rotation vector), the magnitude of that vector does **not** equal the angular speed of the imagined fluid near that point. Instead the magnitude is equal to **twice** the angular speed of the fluid.

[Hide explanation]

In short, this is because there are two terms in each component, which should be averaged and not summed.

Let's look at 2d-curl evaluated at some point (x_0, y_0), since each of the components of curl are of this form.

$$\frac{\partial v_2}{\partial x}(x_0, y_0) - \frac{\partial v_1}{\partial y}(x_0, y_0)$$

You may recall the intuition for this formula given in the 2d-curl article:

- The term $\frac{\partial v_2}{\partial x}(x_0, y_0)$ gives the angular speed of particles to the left and right (x_0, y_0).
- The term $-\frac{\partial v_1}{\partial y}(x_0, y_0)$ gives the angular speed of particles above and below (x_0, y_0).

What we really mean by "total rotation around (x_0, y_0)" is the average angular speed of all particles around (x_0, y_0), so we should really be averaging the influences of these two directions, not summing them.

For example, think of a wheel rotating at 3 radians per second. The points on the left and right sides of the rim could be described as having angular speeds of 3 radians per second each, relative to the center of the wheel. Likewise points on the top and bottom of the rim have angular speeds of 3 radians per second, relative to the center of the wheel. But this does not mean the total rotation of the wheel suddenly becomes 6 radians per second. The rotation as a whole is still 3 radians per second.

Rotating fluids are a bit more flimsy than a rigid wheel, so the angular speeds of particles above and below a point (x_0, y_0) may not equal the angular speeds of particles to the left and right of (x_0, y_0). Still, though, adding these two speeds is not the right way to measure "total rotation", averaging is.

Example: Finding rotation in a three-dimensional vector field using curl

Problem: Suppose a fluid flows in three dimensions according to the following vector field

$$\vec{\mathbf{v}}(x, y, z) = (x^3 + y^2 + z)\hat{\mathbf{i}} + (ze^x)\hat{\mathbf{j}} + (xyz - 9xz)\hat{\mathbf{k}}$$

Describe the rotation of the fluid near the point $(0, 1, 2)$

Step 1: Evaluate curl (you may want some paper for this one).

$$\nabla \times \vec{\mathbf{v}} = \boxed{}\,\hat{\mathbf{i}} + \boxed{}\,\hat{\mathbf{j}} + \boxed{}\,\hat{\mathbf{k}}$$

Check

[Hide explanation]

First, we find the curl of this function. Personally, I can never remember the final formula for curl off the top of my head. I just remember that curl is $\nabla \times \vec{\mathbf{v}}$, and write out the full cross product, determinant and all, whenever I need to compute curl by hand:

$$\nabla \times \vec{\mathbf{v}} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} x^3 + y^2 + z \\ ze^x \\ xyz - 9xz \end{bmatrix}$$

$$= \det\left(\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 + y^2 + z & ze^x & xyz - 9xz \end{bmatrix}\right)$$

Let's go through this determinant one component at a time. The $\hat{\mathbf{i}}$ component is

$$\begin{aligned} \hat{\mathbf{i}} \det\left(\begin{bmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^x & xyz - 9xz \end{bmatrix}\right) &= \left(\frac{\partial}{\partial y}(xyz - 9xz) - \frac{\partial}{\partial z}(ze^x)\right)\hat{\mathbf{i}} \\ &= (xz - 0 - e^x)\hat{\mathbf{i}} \\ &= (xz - e^x)\hat{\mathbf{i}} \end{aligned}$$

The $\hat{\mathbf{j}}$ component is

$$\begin{aligned} -\hat{\mathbf{j}} \det\left(\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x^3 + y^2 + z & xyz - 9xz \end{bmatrix}\right) &= -\left(\frac{\partial}{\partial x}(xyz - 9xz) - \frac{\partial}{\partial z}(x \right. \\ &= (-(yz - 9z) + 0 + 0 + 1)\hat{\mathbf{j}} \\ &= (1 - yz + 9z)\hat{\mathbf{j}} \end{aligned}$$

And the $\hat{\mathbf{k}}$ component is

$$\begin{aligned} \hat{\mathbf{k}} \det\left(\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^3 + y^2 + z & ze^x \end{bmatrix}\right) &= \left(\frac{\partial}{\partial x}(ze^x) - \frac{\partial}{\partial y}(x^3 + y^2 + z)\right)\hat{\mathbf{k}} \\ &= (ze^x - (0 + 2y + 0))\hat{\mathbf{k}} \\ &= (ze^x - 2y)\hat{\mathbf{k}} \end{aligned}$$

Putting this all together, the curl is

$$\nabla \times \vec{\mathbf{v}} = (xz - e^x)\hat{\mathbf{i}} + (1 - yz + 9z)\hat{\mathbf{j}} + (ze^x - 2y)\hat{\mathbf{k}}$$

Step 2: Plug in $(0, 1, 2)$

$$\nabla \times \vec{\mathbf{v}}(0,1,2) = \boxed{}\,\hat{\mathbf{i}} + \boxed{}\,\hat{\mathbf{j}} + \boxed{}\,\hat{\mathbf{k}}$$

Check

[Hide explanation]

$$\begin{aligned}\nabla \times \vec{\mathbf{v}}(0,1,2) &= ((0)(2) - e^{0})\hat{\mathbf{i}} + (1 - (1)(2) + 9(2))\hat{\mathbf{j}} + ((2)e^{0} - 2(1))\hat{\mathbf{k}} \\ &= (0-1)\hat{\mathbf{i}} + (1 - 2 + 18)\hat{\mathbf{j}} + (2-2)\hat{\mathbf{k}} \\ &= \boxed{-\hat{\mathbf{i}} + 17\hat{\mathbf{j}}}\end{aligned}$$

Step 3: Interpret

Near the point $(0, 1, 2)$, the fluid rotation is about [ ] radians per second, with rotation nearly parallel to the [ ]

Check

[Hide explanation]

From the previous question, the rotation of the fluid near $(0, 1, 2)$ is about the axis $-\hat{\mathbf{i}} + 17\hat{\mathbf{j}}$. Since the angular speed is given by one half the magnitude of curl, the angular speed of our fluid near the point $(0, 1, 2)$ is

$\frac{1}{2}\sqrt{1^2 + 17^2} \approx 8.51$ radians/second.

Since this vector points predominantly in the $\hat{\mathbf{j}}$ direction, the rotation of the fluid near this point is almost entirely parallel to the xz-plane.

Summary

- Curl is an operator which takes in a function representing a three-dimensional vector field, and gives another function representing a different three-dimensional vector field.
- If a fluid flows in three-dimensional space along a vector field, the rotation of that fluid around each point, represented as a vector, is given by the curl

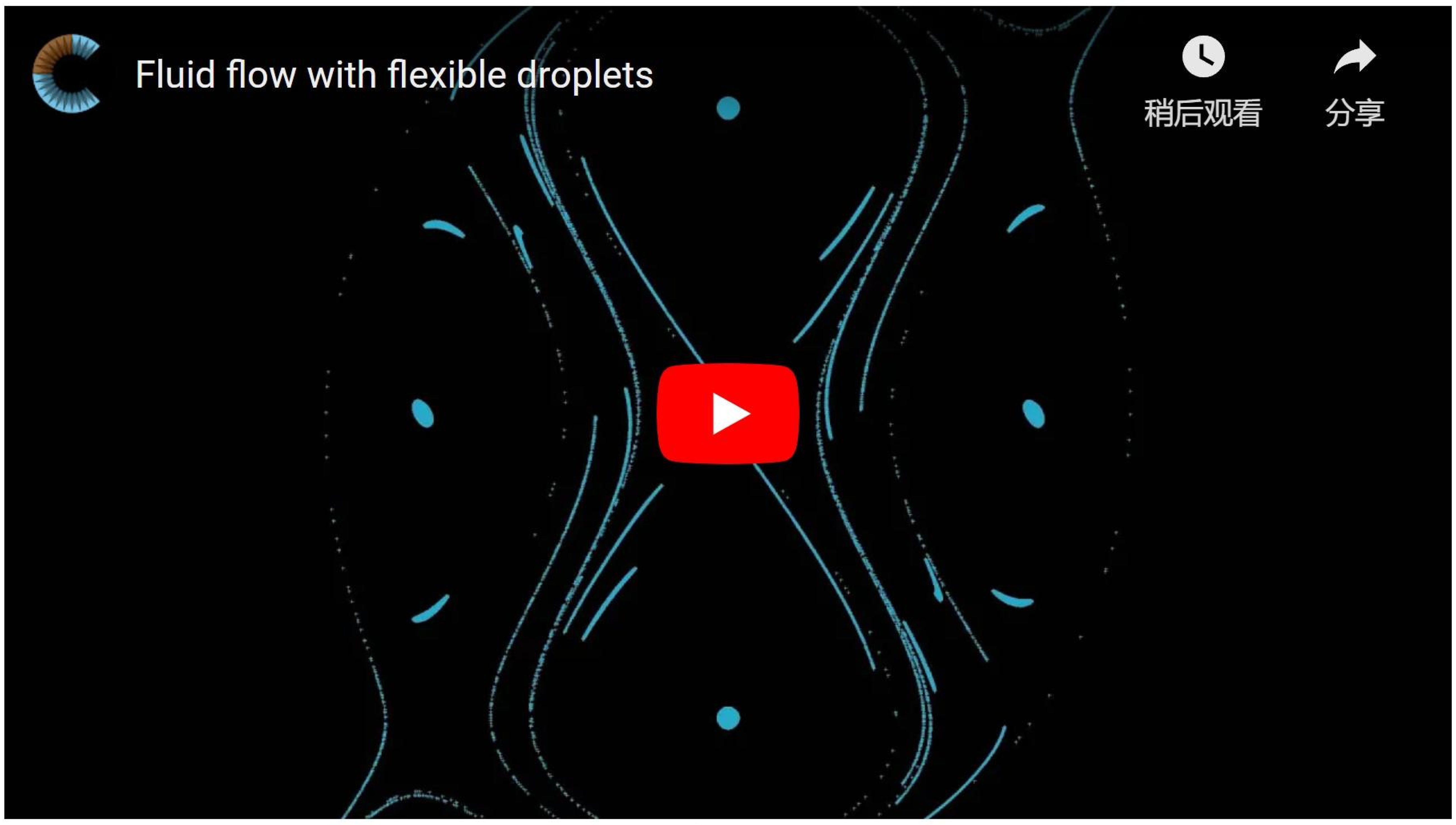
of the original vector field evaluated at that point. The curl vector field should be scaled by a half if you want the magnitude of curl vectors to equal the rotational speed of the fluid.

- If a three-dimensional vector-valued function $\vec{\mathbf{v}}(x, y, z)$ has component function $v_1(x, y, z)$, $v_2(x, y, z)$ and $v_3(x, y, z)$, the curl is computed as follows:

$$\nabla \times \vec{\mathbf{v}} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

Just for fun

Here's an animation of the fluid flow I showed at the very start of the article, but this time each dot is treated more accurately like a droplet of water, flexing and twisting based on how the vector field pulls on each individual particle in the droplet. I also took away the actual vectors from the vector field so that it's easier to see how the fluid moves. Hopefully this gives an impression for how complex yet beautiful the fluid-flow conception of vector fields can be.



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