Surface integral example

Google Classroom

Practice computing a surface integral over a sphere.

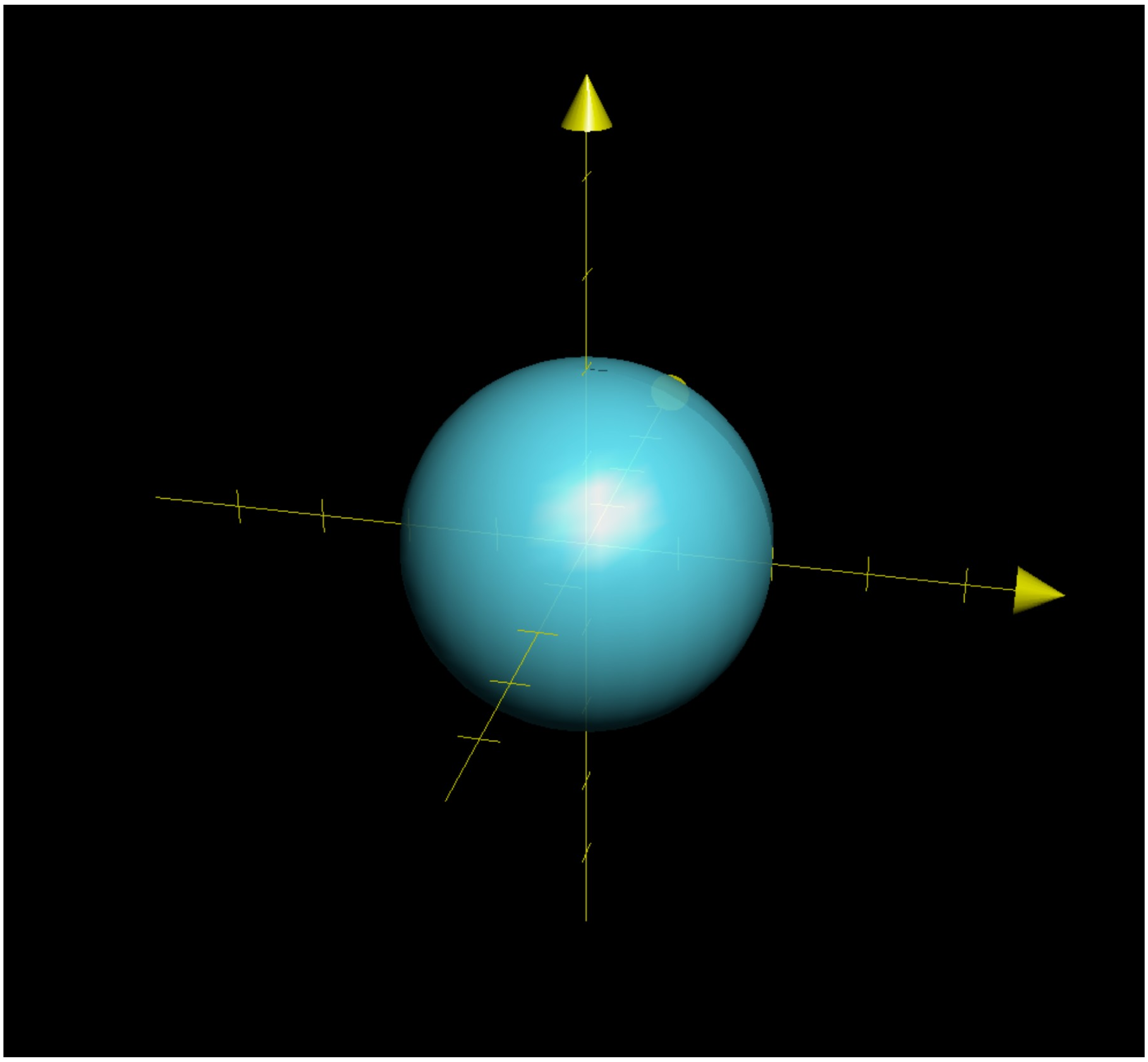
Background

- Surface integrals

The task at hand: Surface integral on a sphere.

In the last article, I talked about what surface integrals do and how you can interpret them. Here, you can walk through the full details of an example. If you prefer videos you can also watch Sal go through a different example.

Consider the sphere of radius 2, centered at the origin.



Your task will be to integrate the following function over the surface of this sphere:

$$f(x, y, z) = (x-1)^2 + y^2 + z^2$$

Step 1: Take advantage of the sphere's symmetry

The sphere with radius 2 is, by definition, all points in three-dimensional space satisfying the following property:

$$x^2 + y^2 + z^2 = 2^2$$

This expression is very similar to the function:

$$f(x, y, z) = (x-1)^2 + y^2 + z^2$$

In fact, we can use this to our advantage...

Concept check: When you evaluate $f(x, y, z) = (x-1)^2 + y^2 + z^2$ on points that happen to be on the sphere with radius 2, what simpler expression do you get?

Check

[Hide explanation]

$$\begin{aligned} f(x, y, z) &= (x-1)^2 + y^2 + z^2 \\ &= x^2 - 2x + 1 + y^2 + z^2 \\ &= -2x + 1 + \underbrace{x^2 + y^2 + z^2}_{\text{On the sphere, this is } 4} \\ &= -2x + 1 + 4 \\ &= -2x + 5 \end{aligned}$$

Keep in mind, $f(x, y, z)$ does not equal this simpler expression *everywhere*, but only on the points where $x^2 + y^2 + z^2 = 4$. Since we will only integrate over points on this sphere, though, we can justifiably replace the function f in the integral with this value.

$$\iint_{\text{Sphere}} \left((x-1)^2 + y^2 + z^2\right) d\Sigma = \iint_{\text{Sphere}} (-2x+5)\, d\Sigma$$

Of course, this is not something you can do for every surface integral, but it's a good lesson to take advantage of symmetry when you can to make these integrals easier.

Step 2: Parameterize the sphere

To relate this surface integral to a double integral on a flat plane, we need to first find a function which parameterizes the sphere.

Concept check: Which of the following functions parameterizes the sphere with radius 2?

Choose 1 answer:

(A) $\vec{\mathbf{v}}(t, s) = \begin{bmatrix} 2\cos(t)\sin(s) \\ 2\sin(t)\sin(s) \\ 2\cos(s) \end{bmatrix}$ in the region where $0 \le t \le 2\pi$ and $0 \le s \le \pi$.

(B) $\vec{\mathbf{v}}(t, s) = \begin{bmatrix} 2\cos(t)\cos(s) \\ 2\sin(t)\sin(s) \\ 2\sin(t)\cos(s) \end{bmatrix}$ in the region where $0 \le t \le 2\pi$ and $0 \le s \le 2\pi$.

Check

[Hide explanation]

The first choice is correct:

$$\vec{\mathbf{v}}(t, s) = \begin{bmatrix} 2\cos(t)\sin(s) \\ 2\sin(t)\sin(s) \\ 2\cos(s) \end{bmatrix}$$

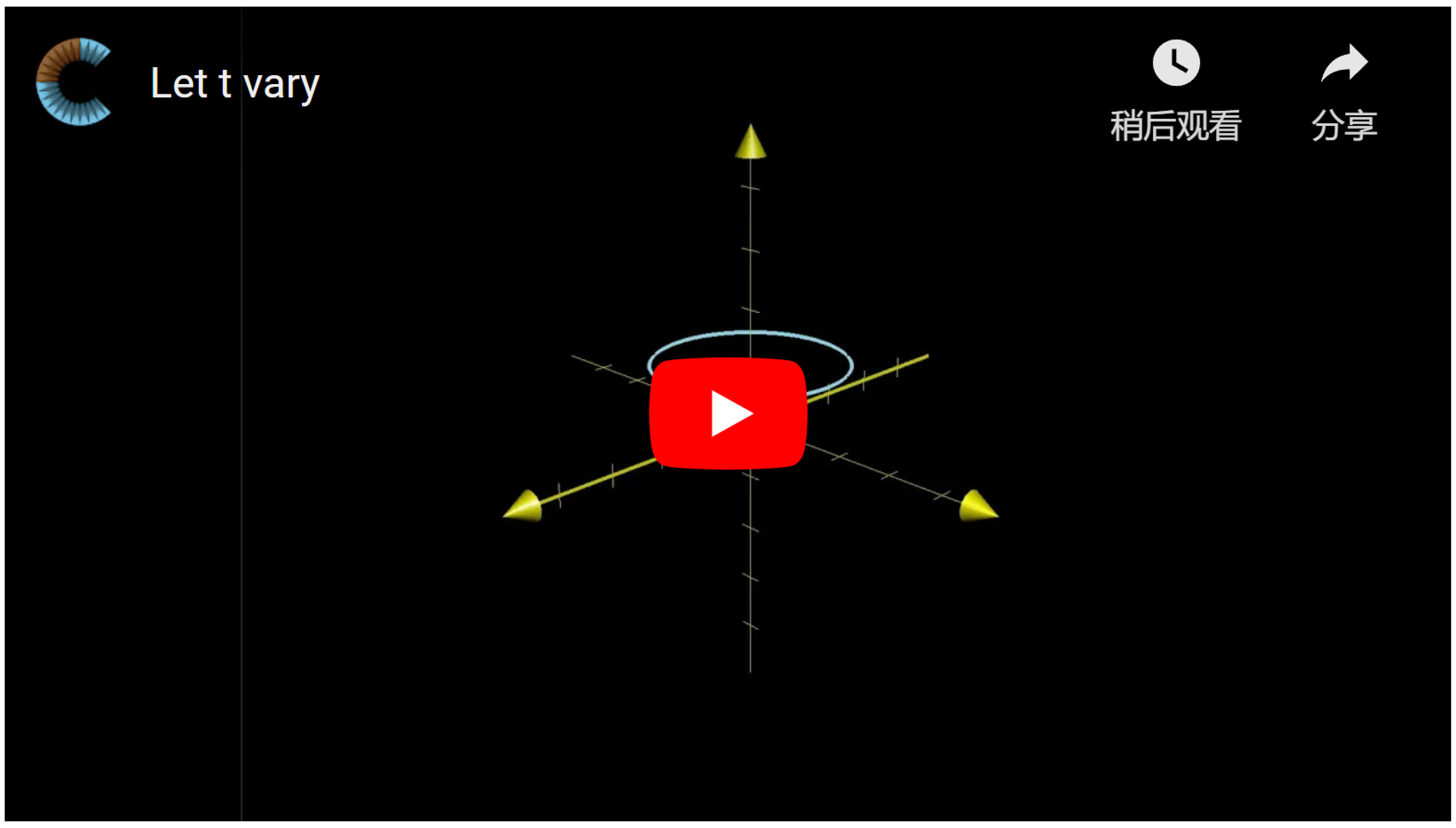
And you apply this to the region of the ts-plane where $0 \le t \le 2\pi$ and $0 \le s \le \pi$.

There are no two ways about it, parameterizing surfaces is hard. The trick to a problem like this, where you need to recognize what surface a given function will parameterize, is to think about what happens when you freeze one variable and let the other one vary.

For example, in the expression for $\vec{\mathbf{v}}(t, s)$ above, imagine freezing s and let t vary:

$$\begin{bmatrix} 2\cos(t)\sin(s) \\ 2\sin(t)\sin(s) \\ 2\cos(s) \end{bmatrix}$$

Since x is proportional to $\cos(t)$, and y is proportional to $\sin(t)$, letting t range from 0 to 2π will draw a circle around the z-axis:

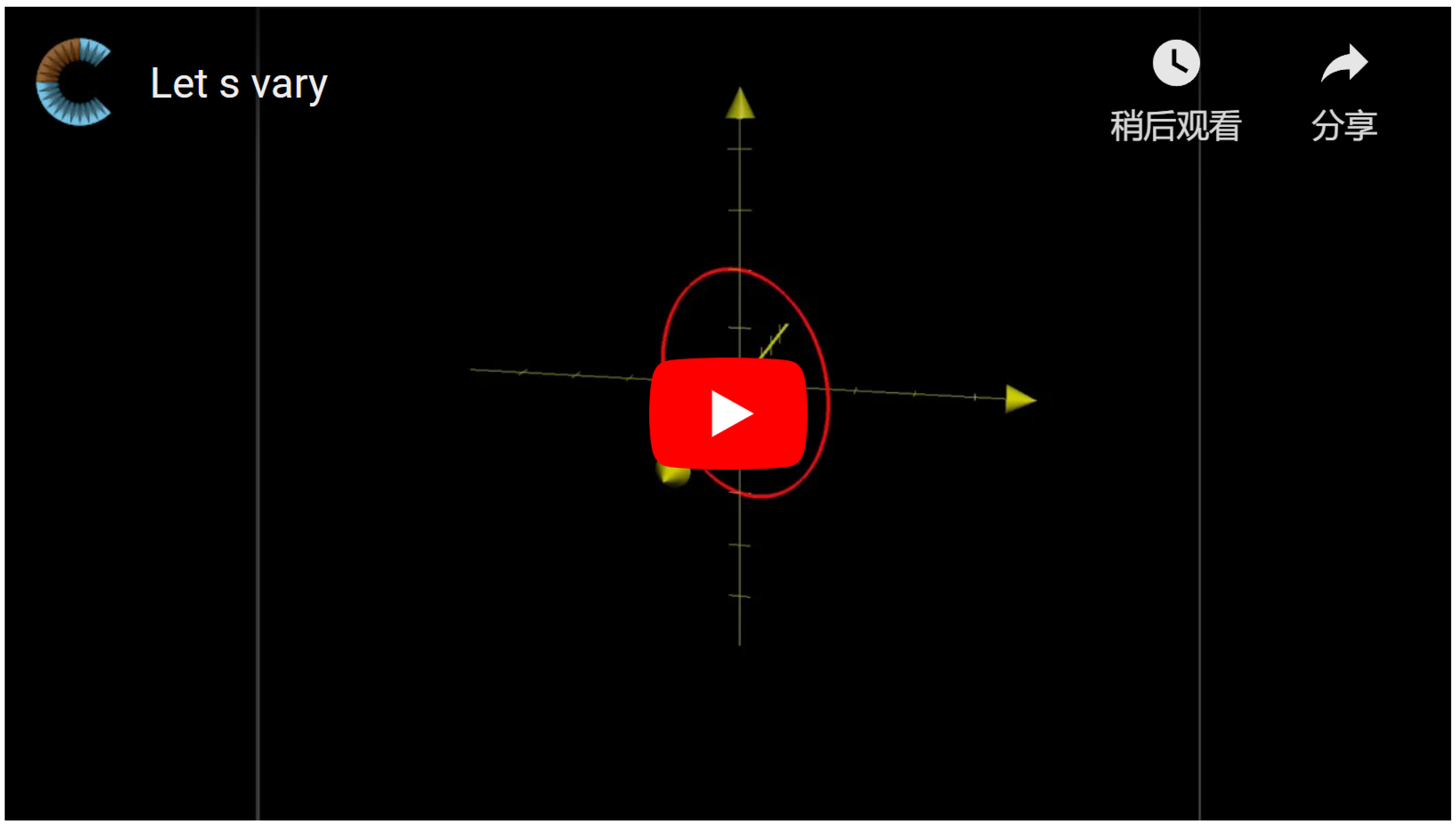


See video transcript

Alternatively, imagine fixing t and letting s vary:

$$\begin{bmatrix} 2\cos(t)\sin(s) \\ 2\sin(t)\sin(s) \\ 2\cos(s) \end{bmatrix}$$

This one also draws a circle (can you see why?)



See video transcript

> Now imagine letting s range from 0 to π, meaning only half of the red circle shown in that last animation is drawn. Rather than thinking of t as a fixed amount, picture the entire circle that t draws for each specific value of s. These circles will sweep over the sphere from top to bottom:
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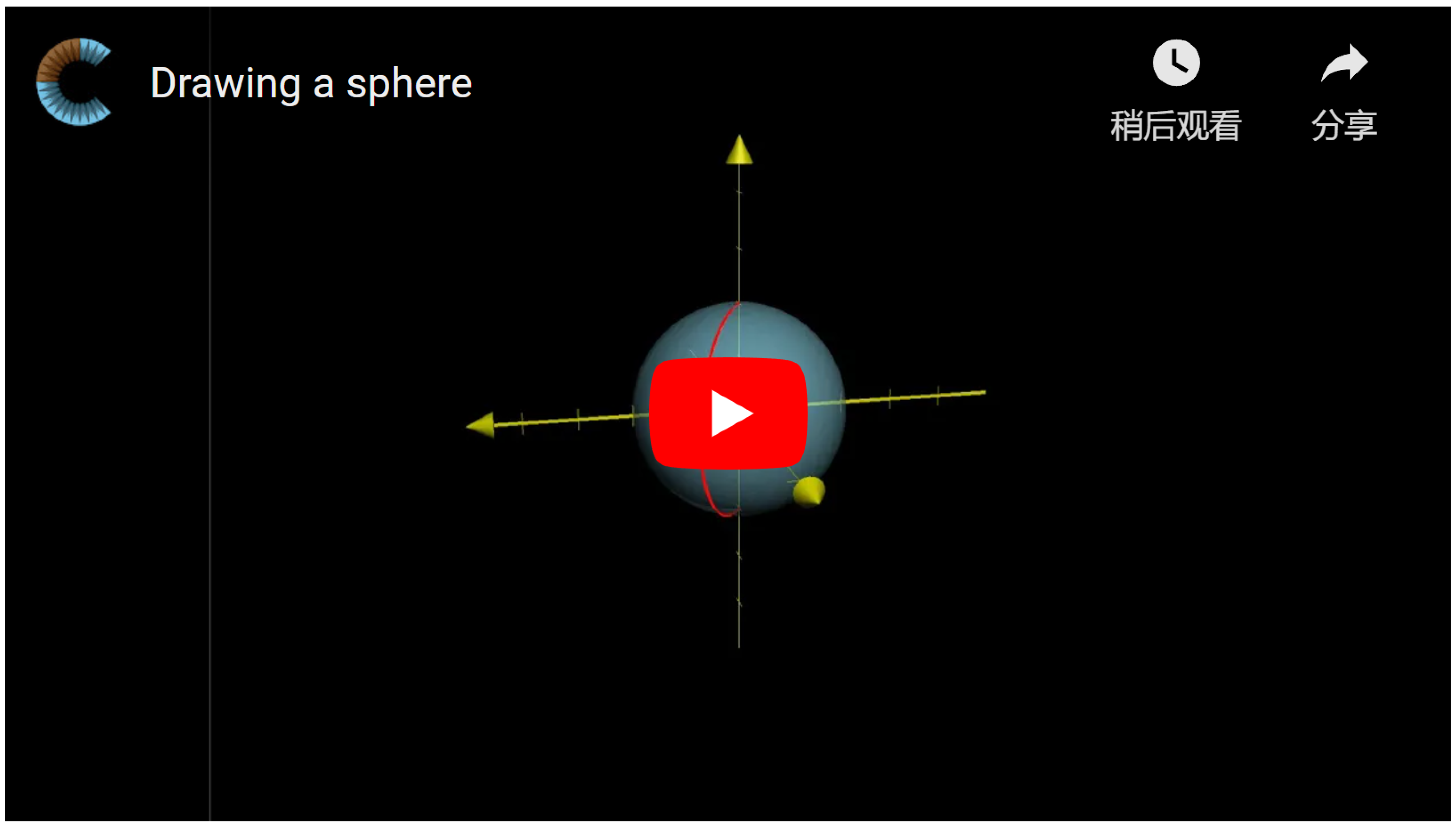
> 
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> See video transcript
>
> This is why we let t range from 0 to 2π, but we only let s range from 0 to π.

Great! Now we have a formula for the parameterization $\vec{\mathbf{v}}(t, s)$ of the sphere, along with a corresponding region on the ts-plane. We can start expanding out surface integral like this:

$$\iint_{\text{Sphere}} (-2x + 5)\, d\Sigma$$

$$= \int_0^{\pi} \int_0^{2\pi} \left(-2 \underbrace{(2\cos(t)\sin(s))}_{x\text{-value of parameterization}} + 5 \right) \underbrace{\left| \frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s} \right|}_{\text{We need work this out}} dt\, d$$

Step 3: Compute both partial derivatives

The main beast to wrangle with in any surface integral is this little guy:

$$\left| \frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s} \right|$$

Concept check: To start, compute both partial derivatives of our parametric function:

$$\vec{\mathbf{v}}(t, s) = \begin{bmatrix} 2\cos(t)\sin(s) \\ 2\sin(t)\sin(s) \\ 2\cos(s) \end{bmatrix}$$

$$\frac{\partial \vec{\mathbf{v}}}{\partial t}(t, s) = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}} + \square\,\hat{\mathbf{k}}$$

Check

[Hide explanation]

$$\frac{\partial \vec{\mathbf{v}}}{\partial t}(t, s) = \begin{bmatrix} \frac{\partial}{\partial t} 2\cos(t)\sin(s) \\ \frac{\partial}{\partial t} 2\sin(t)\sin(s) \\ \frac{\partial}{\partial t} 2\cos(s) \end{bmatrix} = \begin{bmatrix} -2\sin(t)\sin(s) \\ 2\cos(t)\sin(s) \\ 0 \end{bmatrix}$$

$$\frac{\partial \vec{\mathbf{v}}}{\partial s}(t, s) = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}} + \square\,\hat{\mathbf{k}}$$

Check

[Hide explanation]

$$\frac{\partial \vec{\mathbf{v}}}{\partial s}(t, s) = \begin{bmatrix} \frac{\partial}{\partial s} 2\cos(t)\sin(s) \\ \frac{\partial}{\partial s} 2\sin(t)\sin(s) \\ \frac{\partial}{\partial s} 2\cos(s) \end{bmatrix} = \begin{bmatrix} 2\cos(t)\cos(s) \\ 2\sin(t)\cos(s) \\ -2\sin(s) \end{bmatrix}$$

Step 4: Compute the cross product

Compute the cross product of the two partial derivative vectors that you just found.

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s} = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}} + \square\,\hat{\mathbf{k}}$$

Check

[Hide explanation]

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s}$$

$$= \begin{bmatrix} -2\sin(t)\sin(s) \\ 2\cos(t)\sin(s) \\ 0 \end{bmatrix} \times \begin{bmatrix} 2\cos(t)\cos(s) \\ 2\sin(t)\cos(s) \\ -2\sin(s) \end{bmatrix}$$

Now apply the usual determinant trick for cross products:

$$\det\left(\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2\sin(t)\sin(s) & 2\cos(t)\sin(s) & 0 \\ 2\cos(t)\cos(s) & 2\sin(t)\cos(s) & -2\sin(s) \end{bmatrix}\right)$$

We take this one component at a time.

$\hat{\mathbf{i}}$ component: Cross out the top row and left column, then take the determinant:

$$\det\left(\begin{bmatrix} 2\cos(t)\sin(s) & 0 \\ 2\sin(t)\cos(s) & -2\sin(s) \end{bmatrix}\right) = -4\cos(t)\sin^2(s)$$

$\hat{\mathbf{j}}$ component: Cross out the top row and middle column, then take the negative determinant:

$$-\det\left(\begin{bmatrix} -2\sin(t)\sin(s) & 0 \\ 2\cos(t)\cos(s) & -2\sin(s) \end{bmatrix}\right) = -4\sin(t)\sin^2(s)$$

$\hat{\mathbf{k}}$ component: Lastly, and most nastily, cross out the top row and last column, then take the determinant:

$$\det\left(\begin{bmatrix} -2\sin(t)\sin(s) & 2\cos(t)\sin(s) \\ 2\cos(t)\cos(s) & 2\sin(t)\cos(s) \end{bmatrix}\right)$$

$$= \underbrace{-4\sin^2(t)\sin(s)\cos(s) - 4\cos^2(t)\sin(s)\cos(s)}_{\text{factor out } -4\sin(s)\cos(s)}$$

$$= -4\sin(s)\cos(s)\left(\underbrace{\sin^2(t) + \cos^2(t)}_{1}\right)$$

$$= -4\sin(s)\cos(s)$$

Step 5: Find the magnitude of the cross product.

Find the magnitude of the cross product that you just found.

$$\left|\frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s}\right| = \boxed{}$$

Check

[Hide explanation]

$$\left|\frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s}\right|$$

$$= \left|\left[\begin{array}{c} -4\cos(t)\sin^2(s) \\ -4\sin(t)\sin^2(s) \\ -4\sin(s)\cos(s) \end{array}\right]\right|$$

$$= \sqrt{\left(-4\cos(t)\sin^2(s)\right)^2 + \left(-4\sin(t)\sin^2(s)\right)^2 + \left(-4\sin(s)\,\mathrm{co}\right.}$$

$$= \sqrt{\underbrace{4^2\cos^2(t)\sin^4(s) + 4^2\sin^2(t)\sin^4(s) + 4^2\sin^2(s)\cos^2(s)}_{\text{Factor out } 4^2 \text{ from the radical}}}$$

$$= 4\sqrt{\underbrace{\cos^2(t)\sin^4(s) + \sin^2(t)\sin^4(s)}_{\text{Factor out } \sin^4(s)} + \sin^2(s)\cos^2(s)}$$

$$= 4\sqrt{\sin^4(s)\left(\underbrace{\cos^2(t) + \sin^2(t)}_{1}\right) + \sin^2(s)\cos^2(s)}$$

$$= 4\sqrt{\underbrace{\sin^4(s) + \sin^2(s)\cos^2(s)}_{\text{Factor out } \sin^2(s)}}$$

$$= 4\sqrt{\sin^2(s)\left(\underbrace{\sin^2(s) + \cos^2(s)}_{1}\right)}$$

$$= 4\sin(s)$$

Notice, technically the answer should have an absolute value sign in it. However, because our parameterization only applies to the region where $0 \leq s \leq \pi$, the value of $\sin(s)$ will always be positive anyway, so we are free to leave that out.

Step 6: Compute the integral

Taking everything we've done so far, here's what the surface integral has turned into:

$$\iint_{\text{Sphere}} f(x, y, z)\, d\Sigma$$

$$= \iint_{\text{Sphere}} (-2x + 5)\, d\Sigma \quad \leftarrow \text{Step 1}$$

$$= \int_0^{\pi} \int_0^{2\pi} \Big(-2(2\cos(t)\sin(s)) + 5 \Big) \left| \frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s} \right| dt\, ds \quad \leftarrow$$

$$= \int_0^{\pi} \int_0^{2\pi} \Big(-2(2\cos(t)\sin(s)) + 5 \Big) (4\sin(s))\, dt\, ds \quad \leftarrow \text{St}$$

$$= \int_0^{\pi} \int_0^{2\pi} \Big(-16\cos(t)\sin^2(s) + 20\sin(s) \Big)\, dt\, ds$$

As a the final step, compute this double integral.

$$\int_0^{\pi} \int_0^{2\pi} \Big(-16\cos(t)\sin^2(s) + 20\sin(s) \Big)\, dt\, ds = \boxed{}$$

Check

[Hide explanation]

First, let's break this integral into two simpler ones:

$$\int_0^{\pi} \int_0^{2\pi} \Big(-16\cos(t)\sin^2(s) + 20\sin(s) \Big)\, dt\, ds$$

$$= -16 \int_0^{\pi} \int_0^{2\pi} \cos(t)\sin^2(s)\, dt\, ds + 20 \int_0^{\pi} \int_0^{2\pi} \sin(s)\, dt\, ds$$

Now let's go through each, one at a time:

$$\int_0^{\pi} \int_0^{2\pi} \cos(t)\sin^2(s)\, dt\, ds$$

$$= \int_0^{\pi} \Big[\sin(t)\sin^2(s) \Big]_{t=0}^{t=2\pi} ds$$

$$= \int_0^{\pi} \left(\sin(2\pi)\sin^2(s) - \sin(0)\sin^2(s)\right)\, ds$$

$$= \int_0^{\pi} (0 - 0)\, ds$$

$$= 0$$

Well, that makes things easier! What about the other integral:

$$\int_0^{\pi}\int_0^{2\pi} \underbrace{\sin(s)}_{\text{Constant with respect to } t}\, dt\, ds$$

$$= \int_0^{\pi} \sin(s)(2\pi)\, ds$$

$$= 2\pi\Big[-\cos(s)\Big]_{s=0}^{s=\pi}$$

$$= 2\pi\left(-\cos(\pi) - (-\cos(0))\right)$$

$$= 2\pi\,(2)$$

$$= 4\pi$$

Therefore, our final integral simplifies as

$$-16\underbrace{\int_0^{\pi}\int_0^{2\pi} \cos(t)\sin^2(s)\, dt\, ds}_{=0} + 20\underbrace{\int_0^{\pi}\int_0^{2\pi} \sin(s)\, dt\, ds}_{=4\pi}$$

$$= 80\pi$$