Formal definition of divergence in three dimensions

Google Classroom

Learn how surface integrals and 3D flux are used to formalize the idea of divergence in 3D.

Background

- [Formal definition of divergence in two-dimensions]
- [Flux in three-dimensions]

It is a short step between these two prerequisites, and understanding the formal definition of divergence in three dimensions. For that reason, I'm going to keep this article relatively short, assuming that you have the intuition behind both of those pieces of background knowledge.

What we're building to

- The goal is to capture the intuition of **outward fluid flow at a point** in a mathematical formula.

- In three-dimensions, divergence is defined using the following limit:

$$\operatorname{div} \mathbf{F}(x, y, z) = \lim_{|R_{(x,y,z)}| \to 0} \overbrace{\frac{1}{|R_{(x,y,z)}|} \underbrace{\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, d\Sigma}_{\text{Flux through the surface of } R}}^{\text{Average outward flow from } R \text{ per unit volume}}$$

[Hide explanation]

- $F(x, y, z)$ is a three-dimensional vector field, thought of as defining a three-dimensional fluid flow.

- (x, y, z) is some specific point in space.

- $R_{(x,y,z)}$ is a three-dimensional region which contains the point (x, y, z).

- $|R_{(x,y,z)}|$ is the volume of $R_{(x,y,z)}$

- $|R_{(x,y,z)}| \to 0$ indicates that we are considering the limit as the volume of the region goes to 0. Since $R_{(x,y,z)}$ must contain (x, y, z) by definition, you can think of the region as shrinking around the specific point.

- S is the boundary of $R_{(x,y,z)}$, which is a surface.

- $\hat{\mathbf{n}}(x, y, z)$ is a vector-valued function which returns an outward facing unit normal vector at each point on S.

- The surface integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, d\Sigma$ gives the flux of F through the surface S.

There is quite a lot going on in this definition, but most of the complexity lies in that flux integral. If you understand that part, the rest comes from taking the limit with respect to a region shrinking around a point.

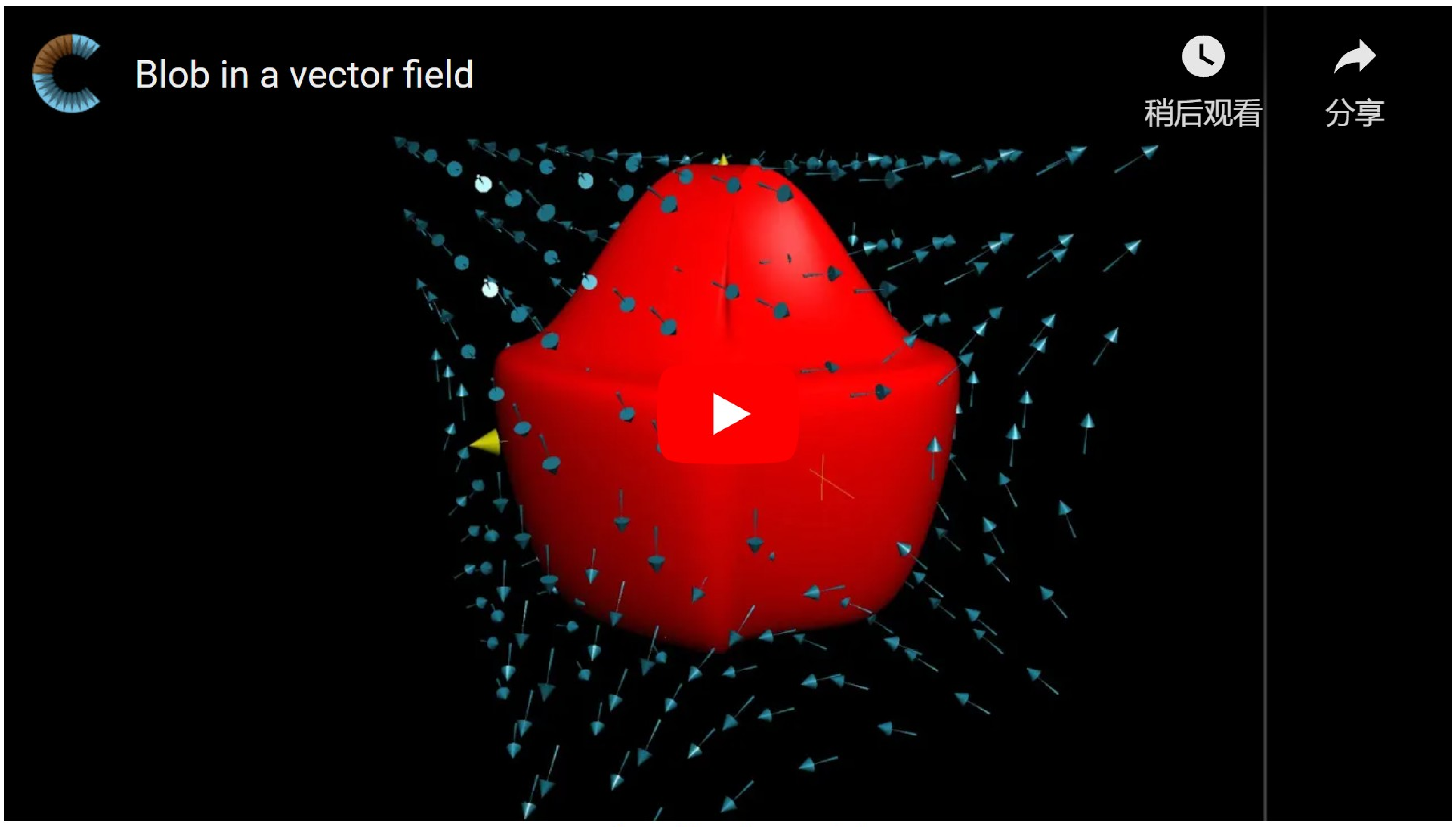
From a region to a point

Let's say you have a three-dimensional vector field.

$$F(x, y, z) \quad \leftarrow \text{Three-dimensional vector field}$$

As always, think of this vector field as representing a fluid flow. The divergence $\text{div} F$ tries to measure the "outward flow" of this fluid at each point. However, it doesn't quite make sense to talk about what it means for fluid to flow out of a *point*.

What *does* make sense is the idea of fluid flowing out of region. Specifically, picture some region R in the vector field.



See video transcript

Let's name the surface of this region "S". In the article on flux in three dimensions, I showed how you can measure the rate at which fluid is leaving this region by taking the flux of $\mathbf{F}$ over the surface S:

$$\underbrace{-\frac{d(\text{fluid mass in } R)}{dt}}_{\text{Rate at which fluid exits } R} = \underbrace{\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, d\Sigma}_{\text{Flux surface integral}}$$

Here, $\hat{\mathbf{n}}(x, y, z)$ is a vector-valued function which returns the outward facing unit normal vector at each point on S.

Divergence itself is concerned with the change in fluid *density* around each point, as opposed mass. We can get the change in fluid density of R by dividing the flux integral by the volume of R. To denote the volume of R, put bars around it:

$$|R| \quad \leftarrow \text{Volume of } R$$

So here's what rate at which fluid density changes inside R looks like:

$$-\frac{d(\text{fluid density in } R)}{dt} = \frac{1}{|R|} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, d\Sigma$$

The divergence of $\mathbf{F}$ at a point (x, y, z) is defined as the limit of this change-in-fluid-density expression as the region shrinks around the point (x, y, z).

$$\operatorname{div} \mathbf{F}(x, y, z) = \underbrace{\lim_{R \to (x,y,z)}}_{R \text{ shrinks around } (x,y,z)} \frac{1}{|R|} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, d\Sigma$$

In that equation, I wrote $R \to (x, y, z)$ to communicate the idea of R shrinking around the point (x, y, z). At the end of the day, all this notation is just a desperate attempt to communicate a heavily visual idea with symbols. You will see different authors use different notation. If you prefer, you could alternatively start by saying $R_{(x,y,z)}$ is a region which contains the point (x, y, z), then write the following:

$$\operatorname{div} \mathbf{F}(x, y, z) = \lim_{|R_{(x,y,z)}| \to 0} \frac{1}{|R_{(x,y,z)}|} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, d\Sigma$$

I have a slight preference for this last notation, just because it makes it a bit easier to see the connection between (x, y, z) on the left hand side and the right hand side without relying so heavily on the context in which all the terms are defined.

Congratulations!

If you are at the point where you can understand this (rather complicated) definition, it is a good sign that you have a solid mental grasp of both divergence and surface integrals. It also means you are in a strong position to understand the divergence theorem, which connects this idea to that of triple integrals.