3D divergence theorem examples

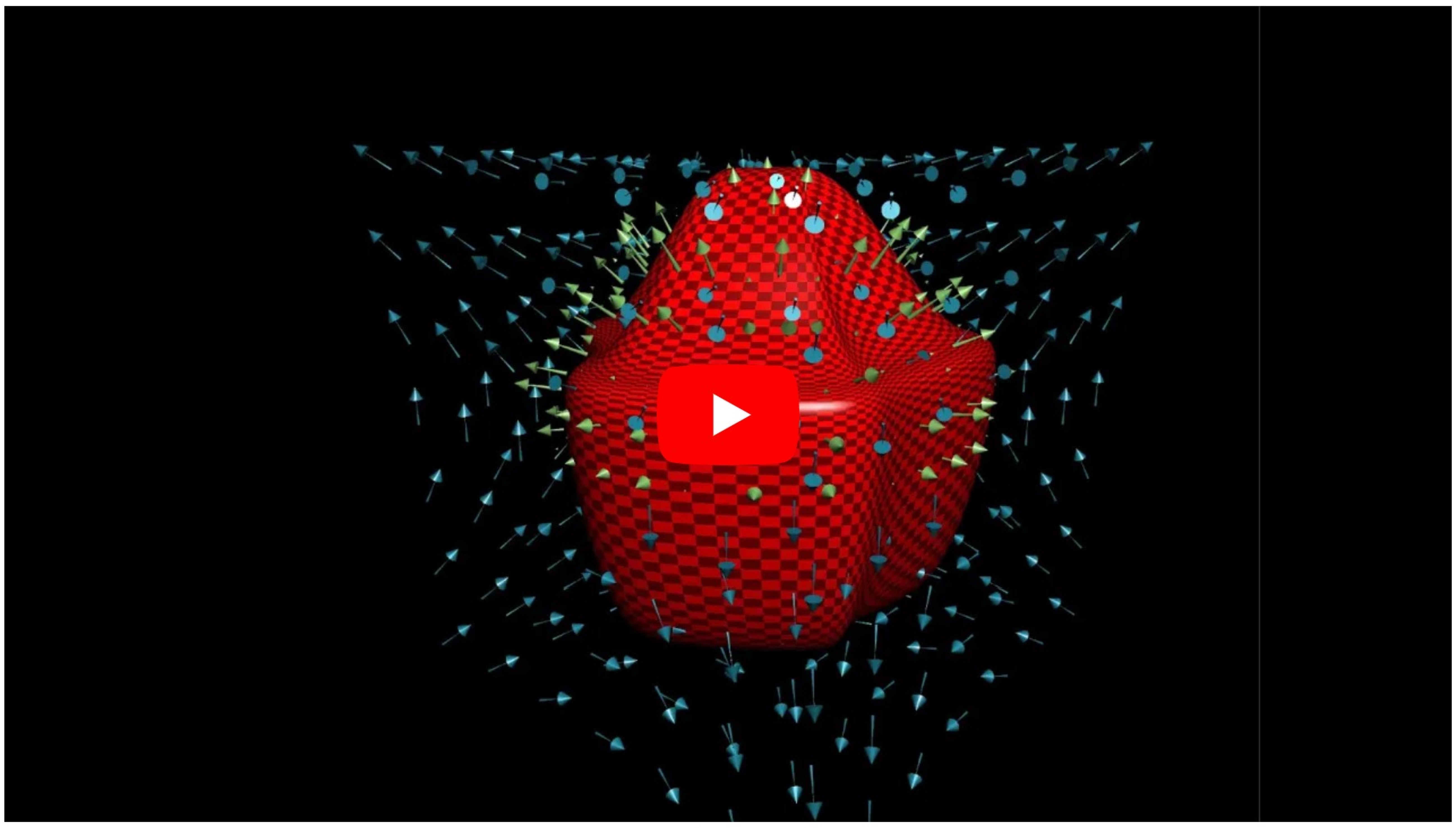
Google Classroom

See how to use the 3d divergence theorem to make surface integral problems simpler.

Background

- 3D divergence theorem
    - Flux in three dimensions
        - Divergence
        - Triple integrals

The divergence theorem (quick recap)



See video transcript

Setup:

- $\mathbf{F}(x, y, z)$ is some three-dimensional vector field.
- V is a three-dimensional volume (think of a blob in space).
- S is the surface of V.
- $\hat{\mathbf{n}}$ is a function which gives unit normal vectors on the surface of S.

Here's what the divergence theorem states:

$$\underbrace{\iiint_V \operatorname{div} \mathbf{F}\, dV}_{\substack{\text{Add up little bits} \\ \text{of outward flow in } V}} = \overbrace{\underbrace{\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, d\Sigma}_{\substack{\text{Measures total outward} \\ \text{flow through } V\text{'s boundary}}}}^{\text{Flux integral}}$$

The intuition here is that both integrals measure the rate at which a fluid flowing along the vector field $\mathbf{F}$ is exiting the region V (or entering V, if the values of both integrals are negative). Triply integrating divergence does this by counting up all the little bits of outward flow of the fluid inside V, while taking the flux integral measures this by checking how much is leaving/entering along the boundary of V.

Strategizing

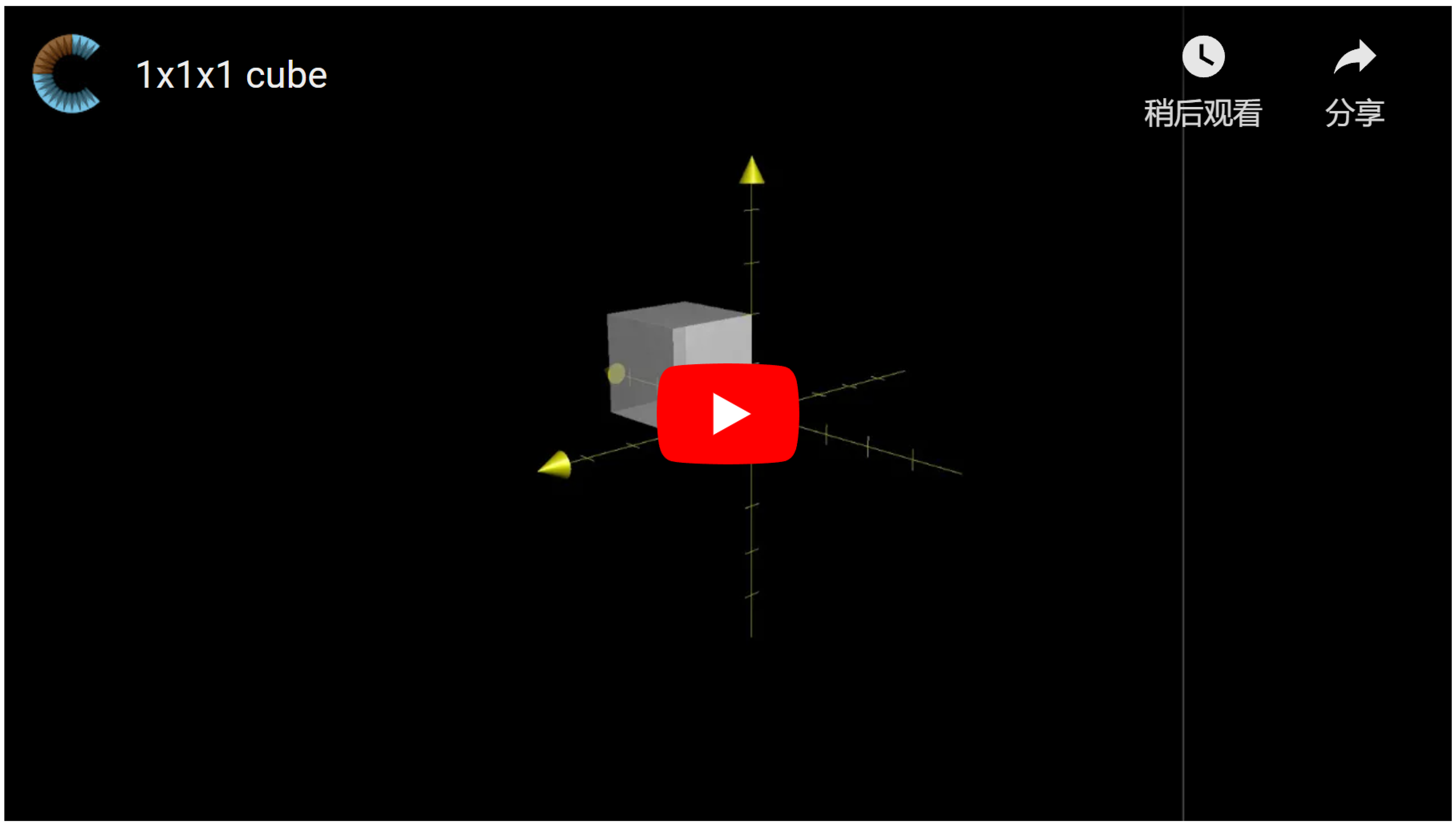
The divergence theorem lets you translate between surface integrals and triple integrals, but this is only useful if one of them is simpler than the other. In each of the following examples, take note of the fact that the volume of the relevant region is simpler to describe than the surface of that region.

In general, when you are faced with a surface integral over a closed surface, consider if it would be easier to integrate over the volume enclosed by that surface. If it is, it's a strong signal that the divergence theorem will come in handy.

Example 1: Surface integral through a cube.

Problem

Let's say C is a $1 \times 1 \times 1$ cube, situated in space such that one corner is on the origin, one corner is at $(1, 1, 1)$, and so that all its edges are parallel to one of the coordinate axes.



See video transcript

Let S represents the surface of this cube, which consists of 6 square faces, oriented using outward facing normal vectors. Compute the following surface integral:

$$\iint_S (2y\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}} + 4z\hat{\mathbf{k}}) \cdot d\Sigma$$

[Hide explanation]

> Whenever people talk about taking a surface integral of a *vector field*, this always means taking the dot product with the unit normal vector to that surface:
>
> $$\iint_S \left(\vec{\mathbf{v}} \cdot \hat{\mathbf{n}}\right) d\Sigma$$
>
> So really, this is a surface integral of a scalar-valued function, it's just that the scalar value arises as the dot product between two vector-valued functions: $\left(\vec{\mathbf{v}} \cdot \hat{\mathbf{n}}\right)$.
>
> This is so universal that it often goes without saying that the unit normal vector is involved, so people instead drop the $\hat{\mathbf{n}}$ and write this:
>
> $$\iint_S \vec{\mathbf{v}} \cdot d\Sigma$$

Solution

Concept check: According to the divergence theorem, which of the following is equal to the surface integral we are asked to compute?

Choose 1 answer:

(A) $\int_0^1 \int_0^1 \int_0^1 \nabla \times (2y\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}} + 4z\hat{\mathbf{k}})\, dx\, dy\, dz$

(B) $\int_0^1 \int_0^1 \int_0^1 \nabla \cdot (2y\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}} + 4z\hat{\mathbf{k}})\, dx\, dy\, dz$

(C) $6\int_0^1 \int_0^1 \nabla \times (2y\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}} + 4z\hat{\mathbf{k}})\, dx\, dy$

(D) $6\int_0^1 \int_0^1 \nabla \cdot (2y\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}} + 4z\hat{\mathbf{k}})\, dx\, dy$

Check

[Hide explanation]

$$\int_0^1 \int_0^1 \int_0^1 \nabla \cdot (2y\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}} + 4z\hat{\mathbf{k}})\, dx\, dy\, dz$$

The divergence theorem says that the surface integral of our vector field $(2y\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}} + 4z\hat{\mathbf{k}})$ through the surface of the cube is the same as the triple integral of the *divergence* of that vector field throughout the cube.

The divergence is written like this:

$$\nabla \cdot (2y\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}} + 4z\hat{\mathbf{k}})$$

Describing the cube with a triple integral is relatively straightforward, since each variable x, y and z ranges from 0 to 1, so the appropriate triple integral structure just looks like this:

$$\int_0^1 \int_0^1 \int_0^1 \ldots dx\, dy\, dz$$

The cube is a great example of an object whose volume is simpler than its surface. To do this surface integral directly, you would have to address each of the 6 square faces separately. Furthermore, the vector-valued function we are integrating becomes simpler when we take the divergence, as you are about to see. So using the divergence theorem will be doubly helpful!

Concept check: Compute the divergence of the vector-valued function in the surface integral above.

$$\nabla \cdot (2y\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}} + 4z\hat{\mathbf{k}}) = \boxed{}$$

Check

[Hide explanation]

If this feels unfamiliar, consider reviewing the article on divergence.

To compute divergence, think of the symbol ∇ as a vector of partial derivative operators:

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Then take the "dot product" between this vector-ish thing and the vector-valued function whose divergence you are computing:

$$\begin{aligned} &\nabla \cdot (2y\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}} + 4z\hat{\mathbf{k}}) \\ =&\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} 2y \\ 3y^2 \\ 4z \end{bmatrix} \\ =&\frac{\partial}{\partial x}(2y) + \frac{\partial}{\partial y}(3y^2) + \frac{\partial}{\partial z}(4z) \\ =&0 + 6y + 4 \end{aligned}$$

Concept check: Use the divergence theorem to finish the problem by plugging the divergence you just computed to the triple integral you chose in the question before that:

$$\int_0^1 \int_0^1 \int_0^1 \nabla \cdot (2y\hat{\mathbf{i}} + 3y^2\hat{\mathbf{j}} + 4z\hat{\mathbf{k}})\, dx\, dy\, dz = \boxed{}$$

Check

[Hide explanation]

$$\int_0^1\int_0^1\int_0^1 \underbrace{\nabla\cdot(2y\hat{\mathbf{i}}+3y^2\hat{\mathbf{j}}+4z\hat{\mathbf{k}})}_{\text{Plug in answer from previous question}}\;dx\,dy\,dz$$

$$=\underbrace{\int_0^1\int_0^1\int_0^1 (6y+4)\,dx\,dy\,dz}_{\text{Break up integral}}$$

$$=\underbrace{\int_0^1\int_0^1\int_0^1 6y\;dx\,dy\,dz}_{\substack{\text{Two of these integrals}\\ \text{treat } 6y \text{ as constant}}}+\underbrace{\int_0^1\int_0^1\int_0^1 4\;dx\,dy\,dz}_{\text{Pull out constant } 4}$$

$$=\int_0^1 6y\underbrace{\left(\int_0^1\int_0^1 dx\,dz\right)}_{\text{Equals } 1}dy+4\underbrace{\int_0^1\int_0^1\int_0^1 dx\,dy\,dz}_{\text{Equals } 1\text{, volume of the cube}}$$

$$=\left(\int_0^1 6y\,dy\right)+4$$

$$=\left[3y^2\right]_{y=0}^{y=1}+4$$

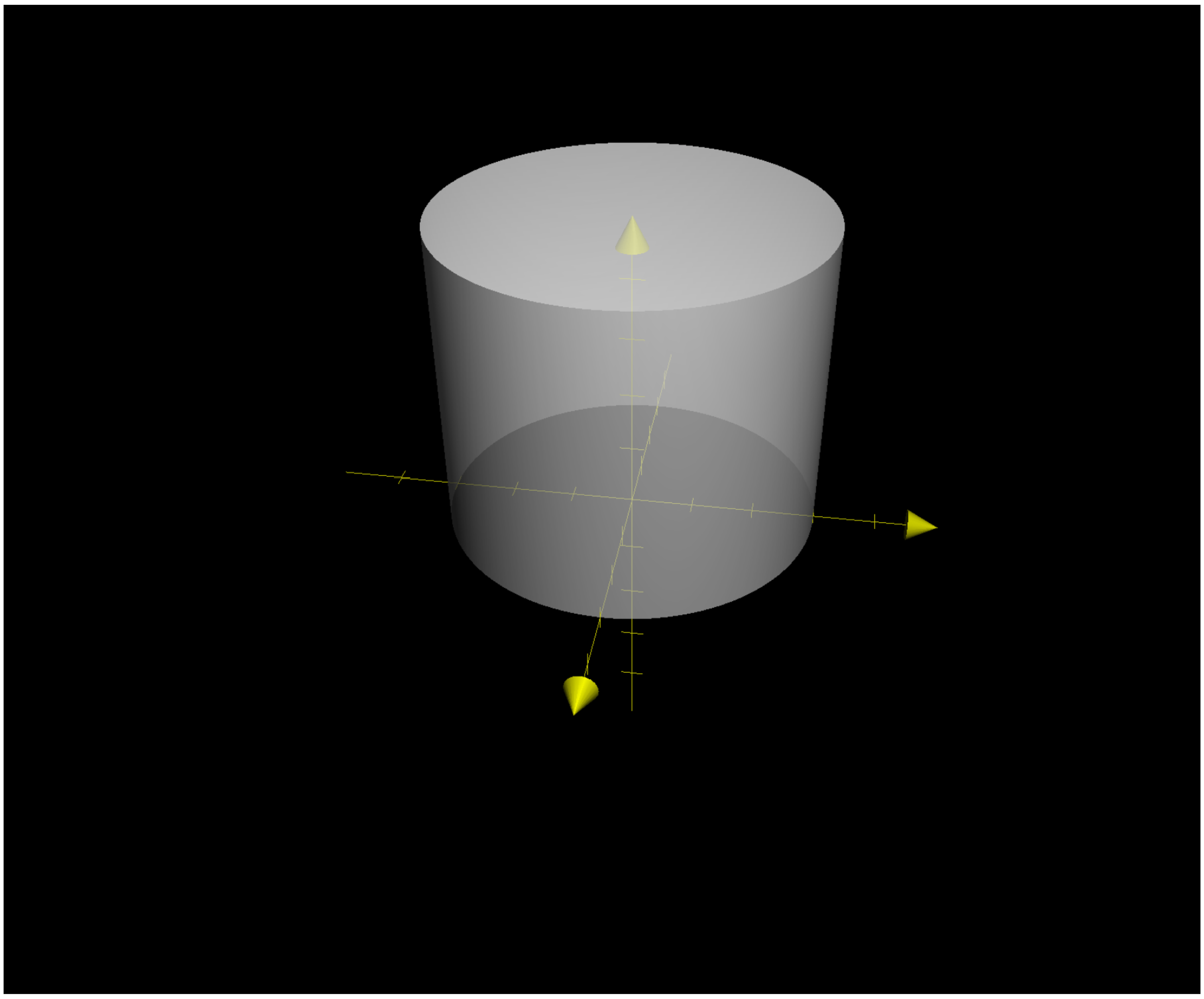
$$=3+4$$

$$=7$$

Example 2: Surface integral through a cylinder

Problem

Let C be a cylinder, whose base is a circle with radius 3 sitting on the xy-plane centered at the origin, and whose height is 5.



Letting S represent the surface of this cylinder, oriented with outward facing unit normal vectors compute the following surface integral:

$$\iint_S \begin{bmatrix} x^3 \\ y^3 \\ x^3 + y^3 \end{bmatrix} \cdot d\Sigma$$

Solution

As with the previous example, what signals that the divergence theorem might be useful is that the volume of our region is easier to describe than its surface. This is especially true if we anticipate integrating using cylindrical coordinates. And again, the divergence of the relevant function will make it simpler.

Concept check: Compute the divergence of the vector-valued function in the integral above.

$$\nabla \cdot \begin{bmatrix} x^3 \\ y^3 \\ x^3 + y^3 \end{bmatrix} = \boxed{}$$

Check

[Hide explanation]

$$\nabla \cdot \begin{bmatrix} x^3 \\ y^3 \\ x^3 + y^3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} x^3 \\ y^3 \\ x^3 + y^3 \end{bmatrix}$$

$$= \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(x^3 + y^3)$$

$$= 3x^2 + 3y^2 + 0$$

Concept check: Compute the triple integral of this divergence inside the cylinder C described in the problem. As a reminder, its base is a circle of radius 3 on the xy-plane centered at the origin, and it has height 5.

$$\iiint_C \left(\nabla \cdot \begin{bmatrix} x^3 \\ y^3 \\ x^3 + y^3 \end{bmatrix} \right) dV = \square$$

Check

[Hide explanation]

First, just plug in the value you found from the last problem:

$$\iiint_C \nabla \cdot \begin{bmatrix} x^3 \\ y^3 \\ x^3 + y^3 \end{bmatrix} dV$$

$$= \iiint_C 3(x^2 + y^2)\, dV$$

Next, because we are integrating over a cylinder, it is very natural to integrate using cylindrical coordinates. For this problem, that involves three steps:

- Replace dV with $r\, d\theta\, dr\, dz$ (don't forget the little r in front, this is what turns $d\theta$ into a unit of length).
- Replace $x^2 + y^2$ with r^2.

- Set bounds which describe our cylinder. Luckily, all of these are constant:
  - r ranges from 0 to 3.
    - θ ranges from 0 to 2π.
    - z ranges from 0 to 5.

Applying all this to the integral, you get something that can be worked out:

$$\underbrace{\iiint_C}_{\text{Plug in bounds}} \underbrace{3(x^2+y^2)}_{r^2} \underbrace{dV}_{r\,d\theta\,dr\,dz}$$

$$= \int_{z=0}^{z=5} \int_{r=0}^{r=3} \int_{\theta=0}^{\theta=2\pi} (3r^2) r\,d\theta\,dr\,dz$$

$$= \underbrace{3 \int_{z=0}^{z=5} \int_{r=0}^{r=3} \int_{\theta=0}^{\theta=2\pi} r^3\,d\theta\,dr\,dz}_{\substack{\text{Two of these integrals} \\ \text{treat } r \text{ as a constant.}}}$$

$$= 3 \int_{r=0}^{r=3} r^3 \int_{z=0}^{z=5} \underbrace{\int_{\theta=0}^{\theta=2\pi}}_{=2\pi} d\theta\,dz\,dr$$

$$= 3(2\pi) \int_{r=0}^{r=3} r^3 \underbrace{\int_{z=0}^{z=5}}_{=5} dz\,dr$$

$$= 3(2\pi)(5) \int_{r=0}^{r=3} r^3\,dr$$

$$= 3(2\pi)(5) \left[\frac{1}{4}r^4\right]_{r=0}^{r=3}$$

$$= 3(2\pi)(5)\left(\frac{1}{4}3^4 - \frac{1}{4}0^4\right)$$

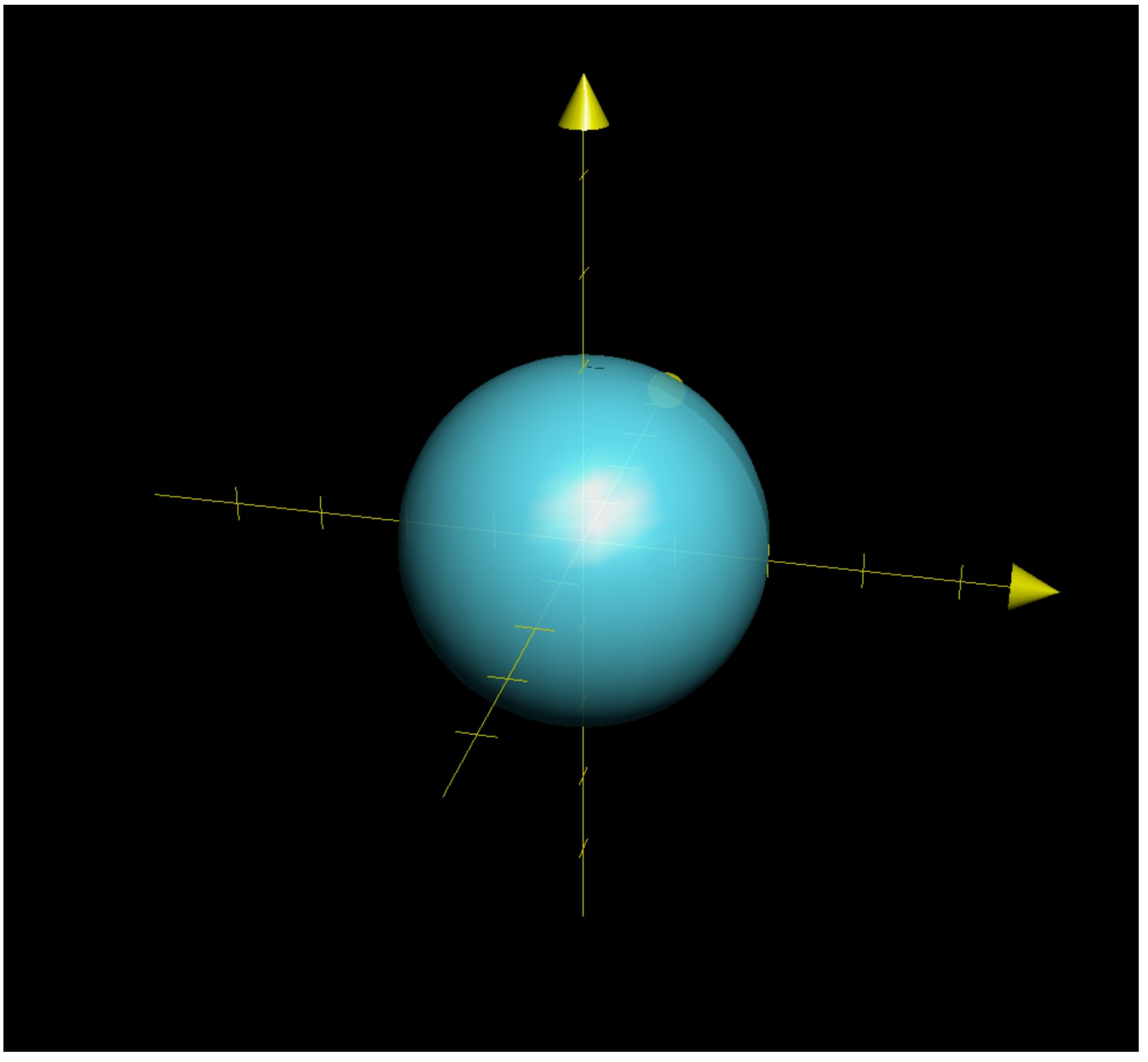
$$= \frac{3(2\pi)(5)(3^4)}{4}$$

$$= \frac{1{,}215\pi}{2}$$

Example 3: Surface area from a volume integral

Problem:

Use the divergence theorem to compute the surface area of a sphere with radius 1, given the fact that the volume of that sphere is $\frac{4}{3}\pi$.



Solution

This feels a bit different from the previous two examples, doesn't it? To start, there is no vector field in the problem, even though the divergence theorem is all about vector fields!

If you let S describe the surface of the sphere, its surface area will be given by the following as-simple-as-they-come surface integral:

$$\iint_S 1 \, d\Sigma$$

However, this is a surface integral of a scalar-valued function, namely the constant function $f(x, y, z) = 1$, but the divergence theorem applies to surface integrals of a *vector field*. In other words, the divergence theorem applies to surface integrals that look like this:

$$\iint_S (\mathbf{F} \cdot \hat{\mathbf{n}})\, d\Sigma$$

Here, $\hat{\mathbf{n}}$ is some function which gives unit normal vectors to S, the surface of the sphere. You can turn this into the desired surface area integral by finding a vector-valued function $\mathbf{F}$ such that $\mathbf{F} \cdot \hat{\mathbf{n}}$ always equals 1.

Concept check: Which of the following definitions of a vector field $\mathbf{F}$ will ensure the property $\mathbf{F} \cdot \hat{\mathbf{n}} = 1$ at all points on the surface of the sphere?

Choose 1 answer:

(A) $\mathbf{F} = \hat{\mathbf{n}}$

(B) $\mathbf{F} = \hat{\mathbf{n}} \times (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$

Check

[Hide explanation]

> $\mathbf{F} = \hat{\mathbf{n}}$
>
> With this choice, $\mathbf{F} \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}$, and any unit vector dotted against itself equals 1.

Concept check: Using this choice for $\mathbf{F}$, together with the divergence theorem, which of the following integrals will give the surface area of the unit sphere? Let B represent the volume enclosed by the sphere, also known as the "unit ball".

Choose 1 answer:

(A) $\iiint_B \nabla \cdot \hat{\mathbf{n}}\, dV$

(B) $\iiint_B \nabla \cdot (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\, dV$

Check

[Hide explanation]

The first choice is correct. In fact, the other choice doesn't even make sense, since $(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})$ gives a scalar valued function, and you can only take the divergence of vector-valued functions.

The surface integral we want to compute looks like this:

$$\iint_S 1\, d\Sigma = \iint_S (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\, d\Sigma$$

This is the flux of the unit normal vector function $\hat{\mathbf{n}}$ through the surface. According to the divergence theorem, this is the same as integrating the divergence of that vector field within the volume of the sphere:

$$\iiint_B \nabla \cdot \hat{\mathbf{n}}\, dV$$

Concept check: Which of the following functions gives unit normal vectors to the surface of the unit sphere?

Choose 1 answer:

(A) $\hat{\mathbf{n}}(x, y, z) = (x + y + z)(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

(B) $\hat{\mathbf{n}}(x, y, z) = \frac{1}{\sqrt{3}} x\hat{\mathbf{i}} + \frac{1}{\sqrt{3}} y\hat{\mathbf{j}} + \frac{1}{\sqrt{3}} z\hat{\mathbf{k}}$

(C) $\hat{\mathbf{n}}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Check

[Answer]

Concept check: Compute the divergence of this function.

$\nabla \cdot \hat{\mathbf{n}} =$

Check

[Hide explanation]

Plugging in $\hat{\mathbf{n}}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, here's what we get:

$$\nabla \cdot \hat{\mathbf{n}}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$= 3$$

Concept check: Finally, given the fact that the volume within the unit sphere is $\frac{4}{3}\pi$, compute the following integral:

$$\iiint_B \nabla \cdot \hat{\mathbf{n}} \, dV = \boxed{}$$

Check

[Hide explanation]

Luckily, the expression $\nabla \cdot \hat{\mathbf{n}}$ turned out to be a constant, 3. This means we are left with a volume integral:

$$\iiint_B \underbrace{\nabla \cdot \hat{\mathbf{n}}}_{=3} \, dV$$

$$= \iiint_B (3) \, dV$$

$$= 3 \underbrace{\iiint_B dV}_{\text{Volume of unit ball}}$$

$$= 3\frac{4}{3}\pi$$

$$= 4\pi$$

> And wouldn't you know it, that's the surface area of a unit sphere!

[Hide explanation]

> In this example, when you computed $\nabla \cdot \hat{\mathbf{n}}$, the result was just a constant, 3. This was underlying reason behind the fact that the surface area of a unit sphere is 3 times the volume of that sphere.
>
> If you did this problem in two dimensions, the analogous two-dimensional divergence $\nabla \cdot \hat{\mathbf{n}}$ would have been 2. And in fact, you could use the two-dimensional divergence theorem to justify why the circumference of a unit circle is 2 times the area of that circle.
>
> In general, an analogous argument using the higher-dimensional analog of the divergence theorem can show that the $(n-1)$-dimensional volume of the boundary of an n-dimensional unit sphere must be n times the n-dimensional volume of that sphere.

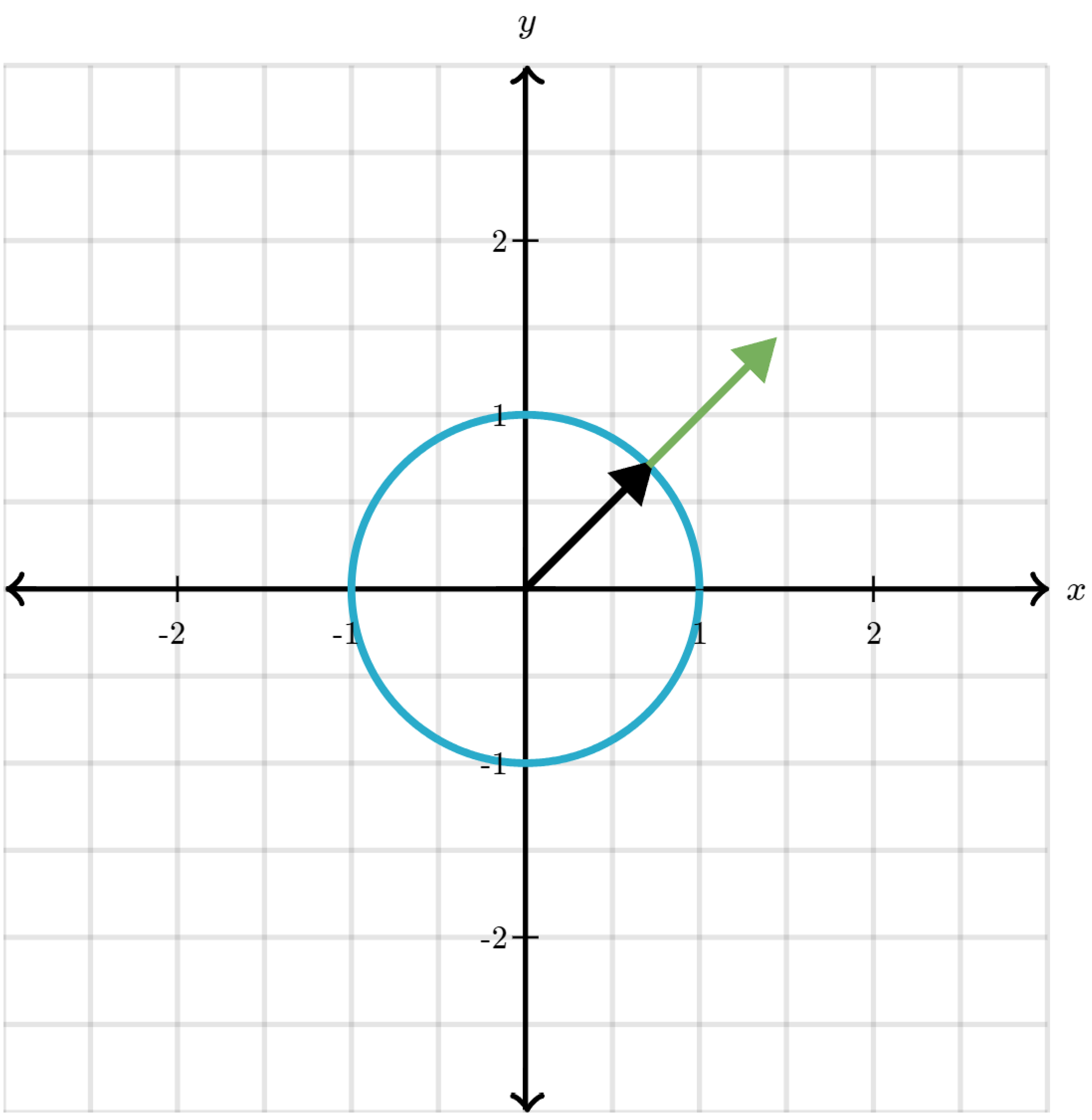
Summary

- The divergence theorem is useful whenever the interior volume of a region is easier to describe than its surface.

- It also helps if the divergence of the relevant vector field turns it into a simpler function.

$$\hat{\mathbf{n}}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

The neat thing about the unit sphere centered at the origin is that if you take a vector pointing from the origin to a point on the sphere, then move it so that its tail is on that point, it will be normal to the sphere. This is easiest to draw and see on a circle, but the same principle applies for a sphere:



What's more, since the problem asks about a unit sphere, these vectors are all automatically unit vectors, so there's no need to normalize!