Derivatives of vector-valued functions

Google Classroom

How to compute, and more importantly how to interpret, the derivative of a function with a vector output.

Background

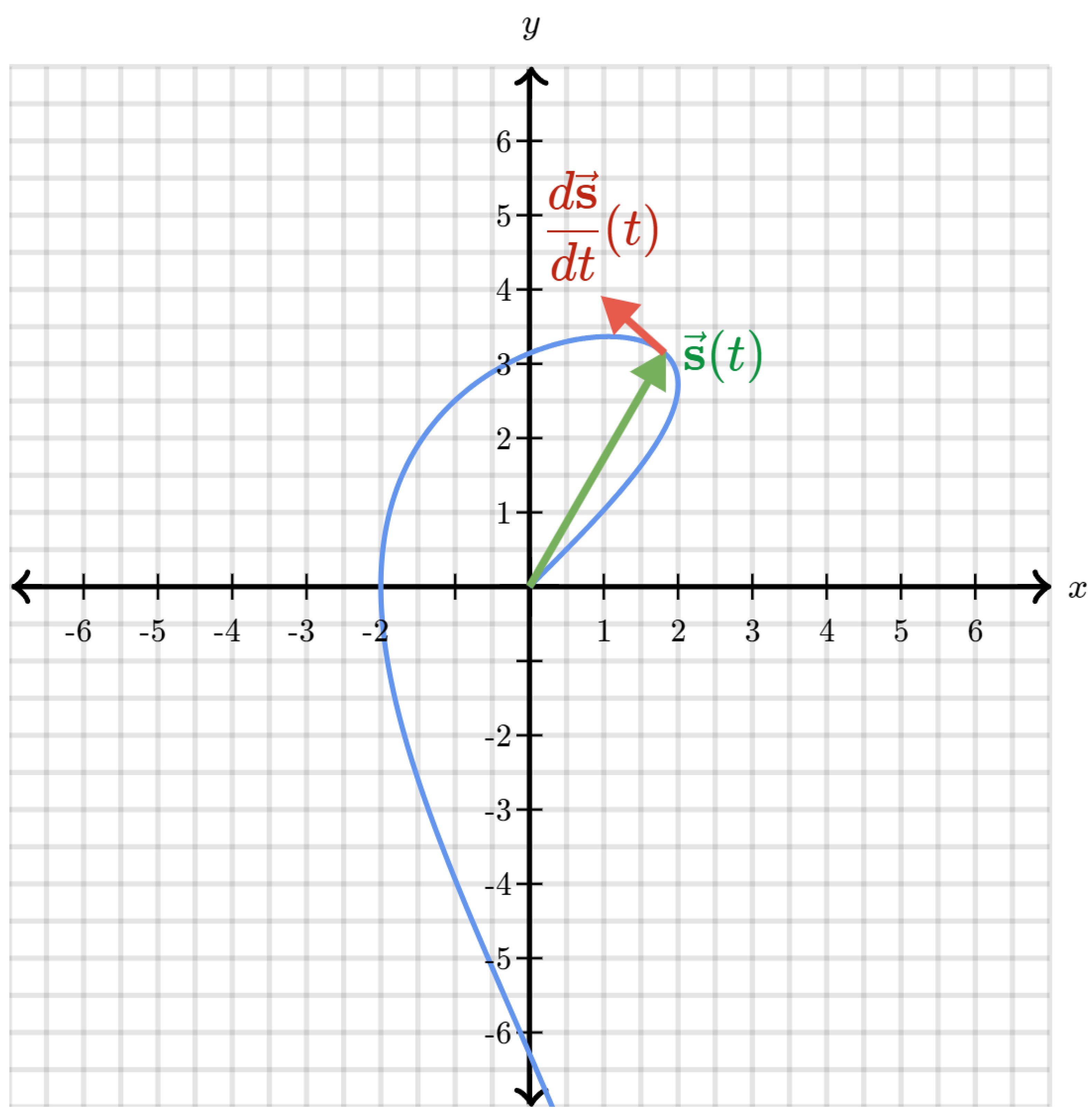
- Ordinary derivatives
- Parametric function

What we're building to

- To take the derivative of a vector-valued function, take the derivative of each component:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$$

- If you interpret the initial function as giving the position of a particle as a function of time, the derivative gives the velocity vector of that particle as a function of time.



The derivative of a vector-valued function

Good news! Computing the derivative of a vector-valued function is nothing really new. As such, I'll keep this article pretty short. The main new takeaway is interpreting the vector derivative.

Dive in with an example

Let's start with a relatively simple vector-valued function $\vec{\mathbf{s}}(t)$, with only two components,

$$\vec{\mathbf{s}}(t) = \left[\begin{array}{c} 2\sin(t) \\ \\ 2\cos(t/3)t \end{array}\right]$$

To take the derivative of $\vec{\mathbf{s}}$, just take the derivative of each component:

$$\frac{d\vec{\mathbf{s}}}{dt}(t) = \left[\begin{array}{c} \dfrac{d}{dt}(2\sin(t)) \\ \\ \dfrac{d}{dt}(2\cos(t/3))t \end{array}\right]$$

$$= \left[\begin{array}{c} 2\cos(t) \\ \\ 2\cos(t/3) - \dfrac{2}{3}\sin(t/3)t \end{array}\right]$$

You might also write this derivative as $\vec{\mathbf{s}}'(t)$. This derivative is a new vector-valued function, with the same input t that $\vec{\mathbf{s}}$ has, and whose output has the same number of dimensions.

More generally, if we write the components of $\vec{\mathbf{s}}$ as $x(t)$ and $y(t)$, we write its derivative like this:

$$\vec{\mathbf{s}}'(t) = \left[\begin{array}{c} x'(t) \\ \\ y'(t) \end{array}\right]$$

Derivative gives a velocity vector.

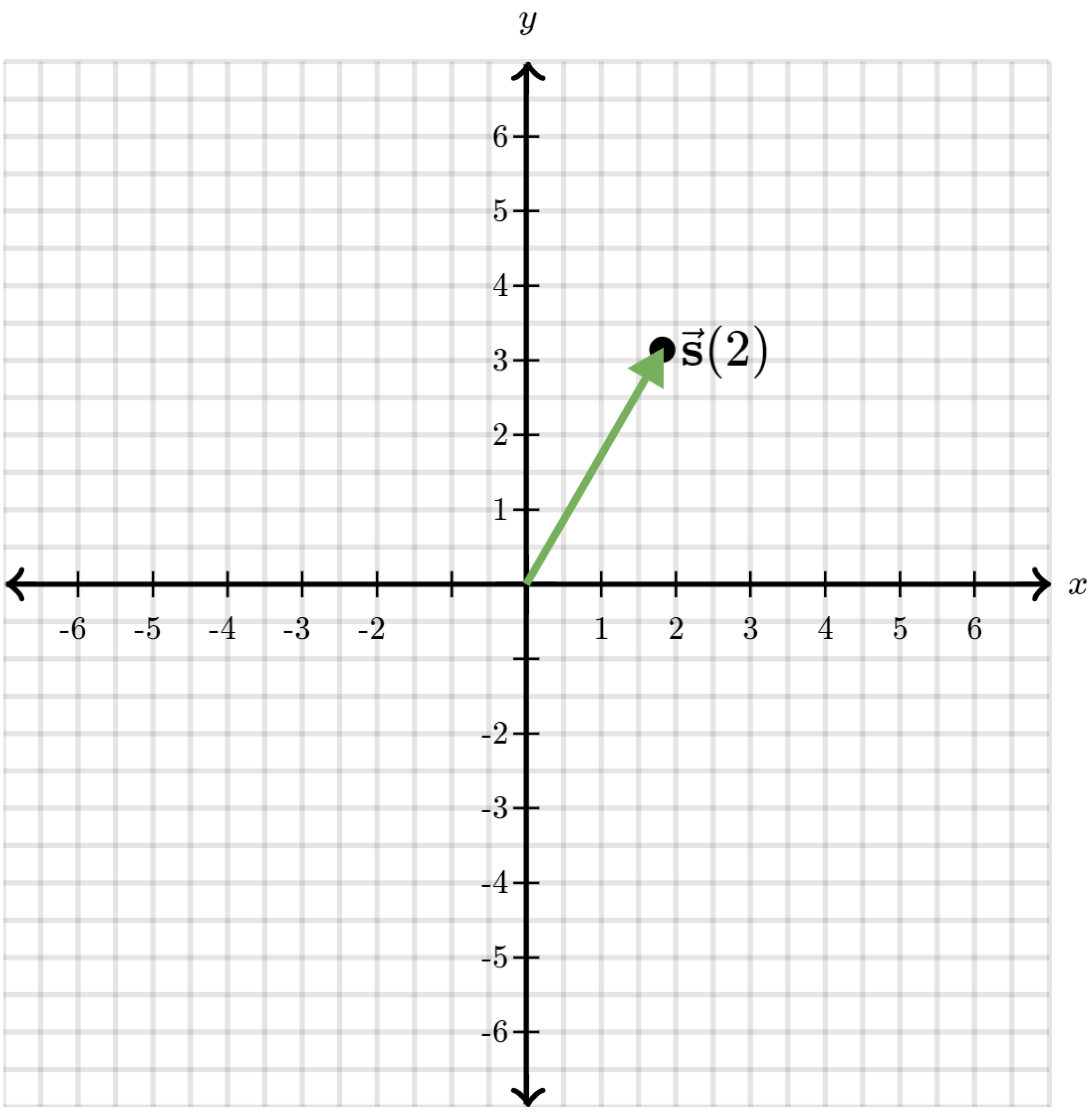
For the example above, how can we visualize what the derivative means? First, to visualize

$$\vec{\mathbf{s}}(t) = \begin{bmatrix} 2\sin(t) \\ \\ 2\cos(t/3)t \end{bmatrix}$$

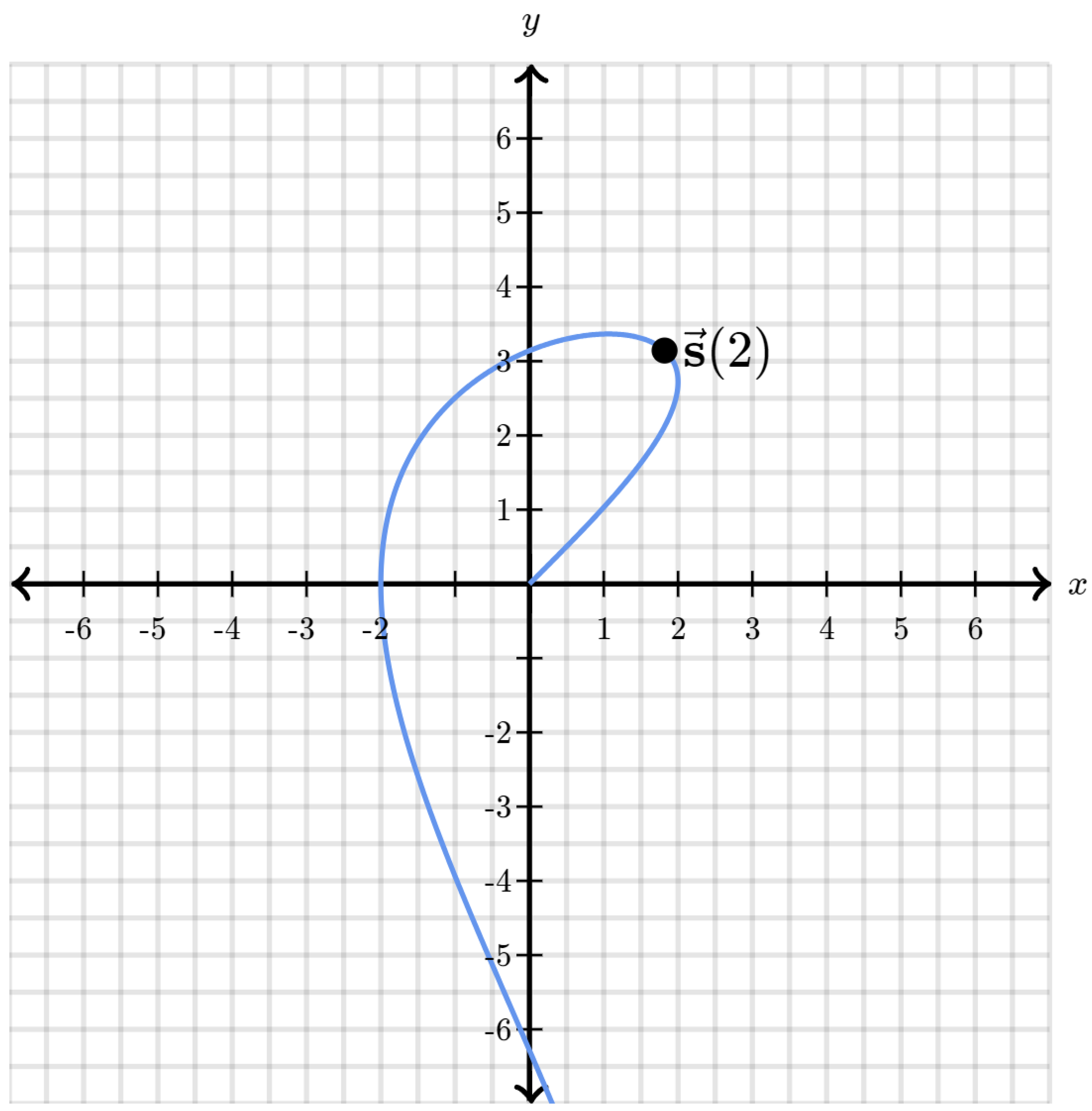
we note that the output has more dimensions than the input, so it is well-suited to be viewed as a parametric function.

Each point on the curve represents the tip of a vector $\begin{bmatrix} 2\sin(t_0) \\ 2\cos(t_0/3)t_0 \end{bmatrix}$ for some specific number t_0. For instance, when $t_0 = 2$ we draw a vector to the point

$$\vec{\mathbf{s}}(2) = \begin{bmatrix} 2\sin(2) \\ \\ 2\cos(2/3)\cdot 2 \end{bmatrix} \approx \begin{bmatrix} 1.819 \\ \\ 3.144 \end{bmatrix}$$



When we do this for all possible inputs t, the tips of the vectors $\vec{\mathbf{s}}(t)$ will trace out a certain curve:

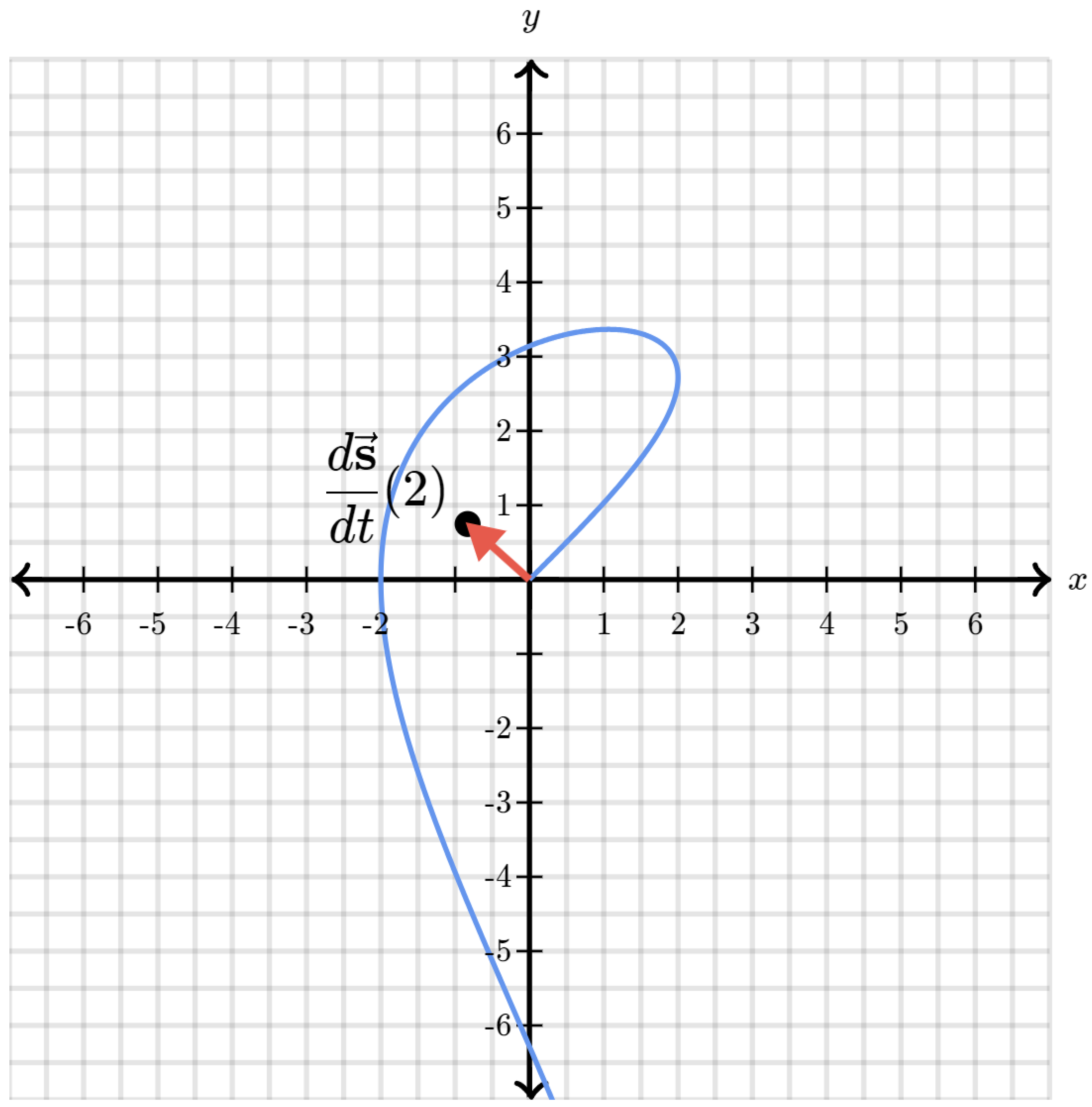


What do we get when we plug in some value of t, perhaps 2 again, to the derivative?

$$\frac{d\vec{\mathbf{s}}}{dt}(2) = \begin{bmatrix} 2\cos(2) \\ 2\cos(2/3) - \frac{2}{3}\sin(2/3)\cdot 2 \end{bmatrix}$$

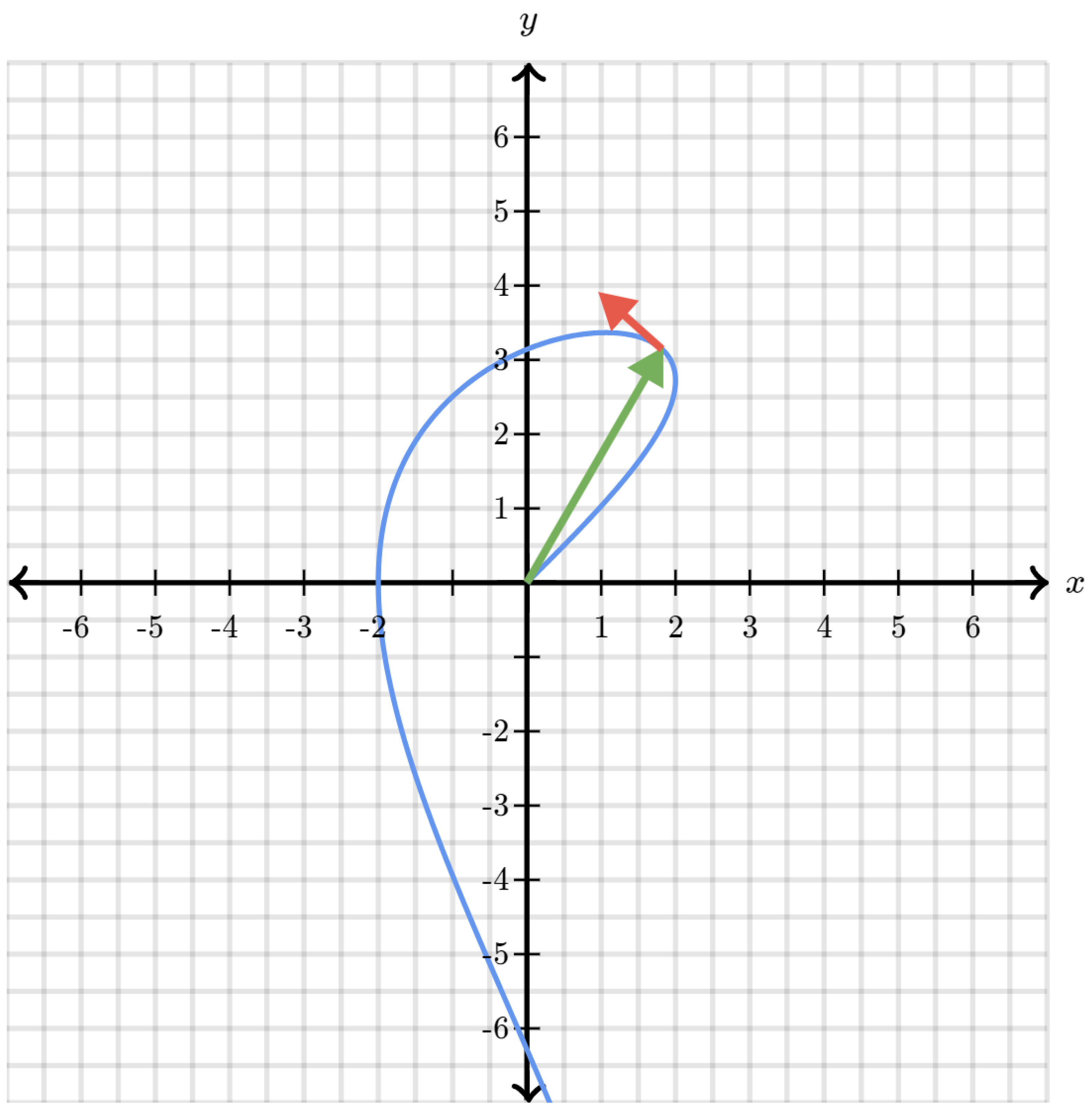
$$\approx \begin{bmatrix} -0.832 \\ 0.747 \end{bmatrix}$$

This is also some two-dimensional vector.



It's hard to see what this derivative vector represents when it just sits at the origin, but if we shift it so that its tail sits on the tip of the vector $\vec{s}(2)$, it has a wonderful interpretation:

- **If $\vec{s}(t)$ represents the position of a traveling particle as a function of time, $\frac{d\vec{s}}{dt}(t_0)$ is the velocity vector of that particle at time t_0.**



In particular, this means the direction of the vector is tangent to the curve, and its magnitude indicates the speed at which one travels along this curve as t increases at a constant rate (as time tends to do).

Concept Check: Suppose the position in two-dimensional space of a particle, as a function of time, is given by the function

$$\vec{s}(t) = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$$

PROBLEM 1

What is $\frac{d\vec{s}}{dt}$?

Choose 1 answer:

(A) $\begin{bmatrix} 3t^2 \\ 2t \end{bmatrix}$

(B) $\begin{bmatrix} 2t \\ 3t^2 \end{bmatrix}$

(C) $\begin{bmatrix} t \\ t^2 \end{bmatrix}$

Check

Hide explanation

Take the derivative of each component:

$$\frac{d}{dt}\begin{bmatrix} t^2 \\ t^3 \end{bmatrix} = \begin{bmatrix} \frac{d}{dt}(t^2) \\ \frac{d}{dt}(t^3) \end{bmatrix} = \begin{bmatrix} 2t \\ 3t^2 \end{bmatrix}$$

PROBLEM 2

What is the speed of the particle at time $t = 3$?

Choose 1 answer:

(A) $\sqrt{6^2 + 27^2} \approx 27.659$

(B) $6^2 + 27^2 = 765$

Check

Hide explanation

Plug in the value $t = 3$ to the derivative:

$$\frac{d\vec{s}}{dt}(3) = \begin{bmatrix} 2(3) \\ 3(3)^2 \end{bmatrix} = \begin{bmatrix} 6 \\ 27 \end{bmatrix}$$

The speed of the particle at time 3 is then given by the magnitude of this vector:

$$\sqrt{6^2 + 27^2} \approx 27.659$$

Summary

- To take the derivative of a vector-valued function, take the derivative of each component.
- If you interpret the initial function as giving the position of a particle as a function of time, the derivative gives the velocity vector of that particle as a function of time.