

## Examples: Second partial derivative test

 Google Classroom

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*Practice using the second partial derivative test*

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## Background

- [Second partial derivative test](#)

## Prepare for the slog

I have a challenge for you.

In this article, you can walk through two examples of finding maxima and minima in multivariable functions. In modern applications, most of the steps involved in solving these sorts of problems would be performed by a computer. However, the only way to test that you *really* understand how the second partial derivative test is used is to walk through it yourself, at least once.

After all, you may one day need to write the program to tell a computer how to do this, which requires somewhat of an intimate knowledge of all the steps involved. Moreover, it is a good way to become more fluent with partial derivatives.

**So my challenge to you is this:** try entering the answer to each step as you move through the article to test your own understanding.

## The statement of the second partial derivative test (for reference)

Start by finding a point  $(x_0, y_0)$  where both partial derivatives of  $f$  are 0.

$$f_x(x_0, y_0) = 0$$

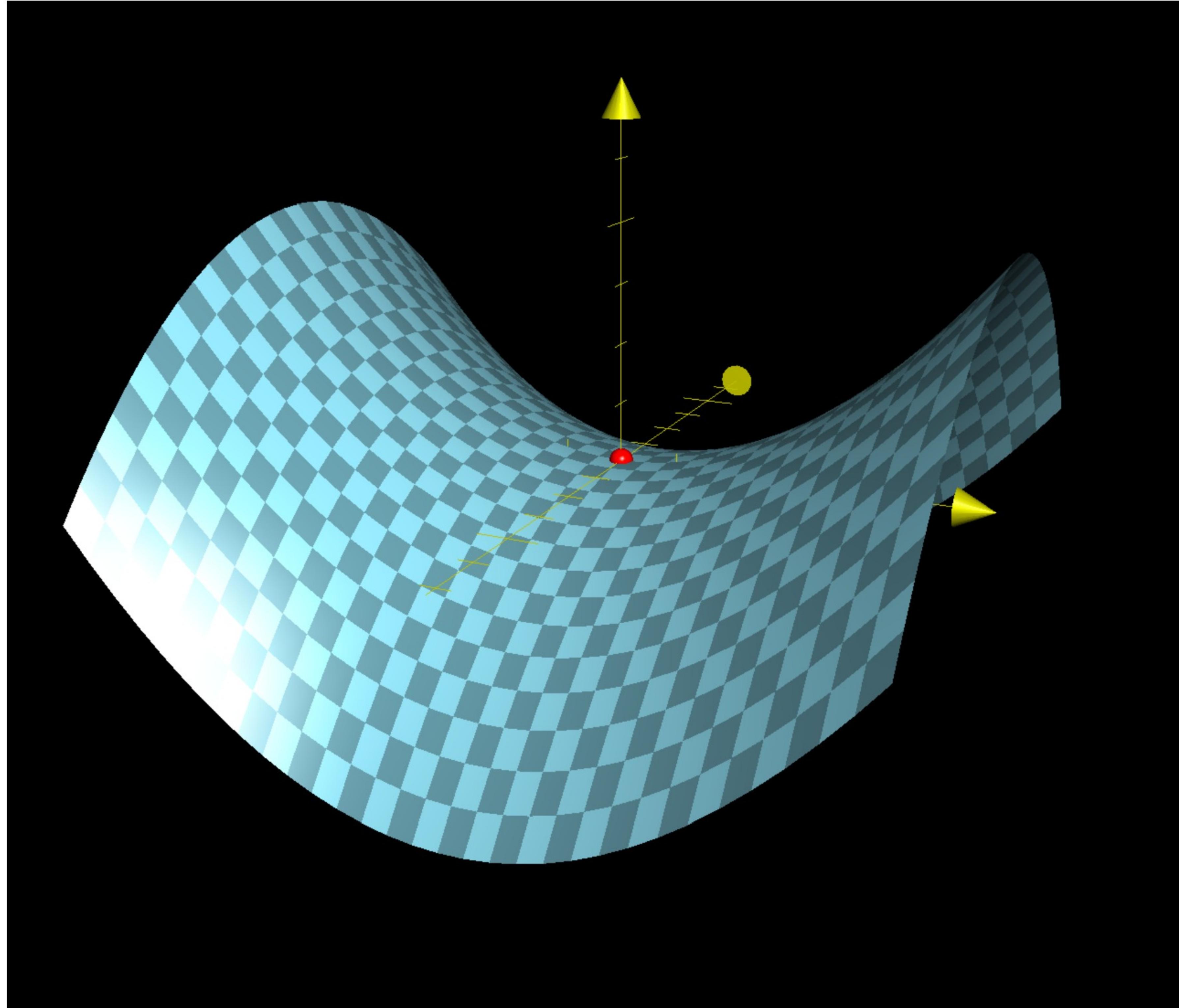
$$f_y(x_0, y_0) = 0$$

The [second partial derivative test](#) tells us how to determine if  $(x_0, y_0)$  is a local maximum, local minimum, or saddle point. Start by computing this term:

$$H = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

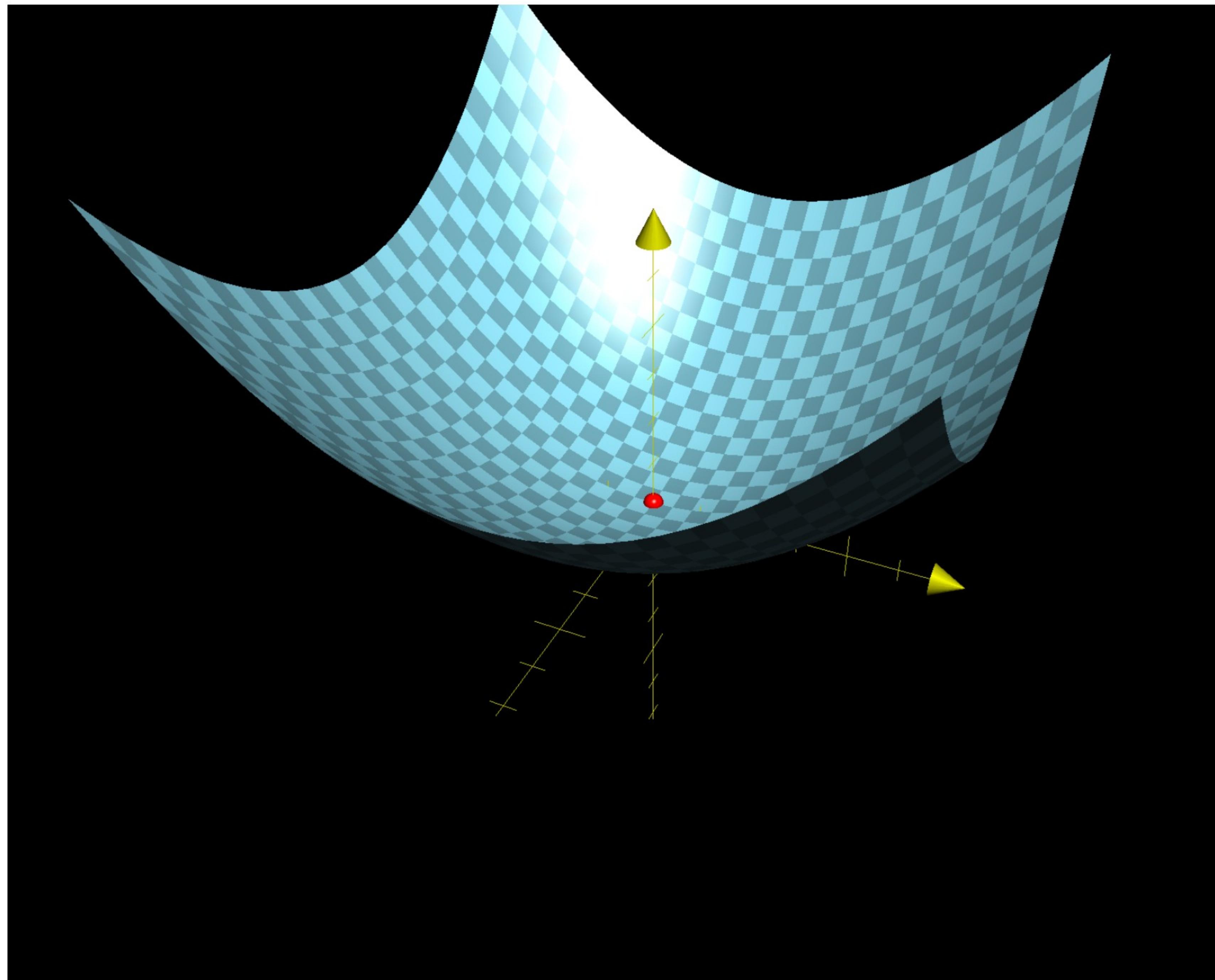
where  $f_{xx}$ ,  $f_{yy}$  and  $f_{xy}$  are the second partial derivatives of  $f$ .

If  $H < 0$ , then  $f$  has a neither minimum nor maximum at  $(x_0, y_0)$ , but instead has a saddle point. [\[Hide explanation\]](#)

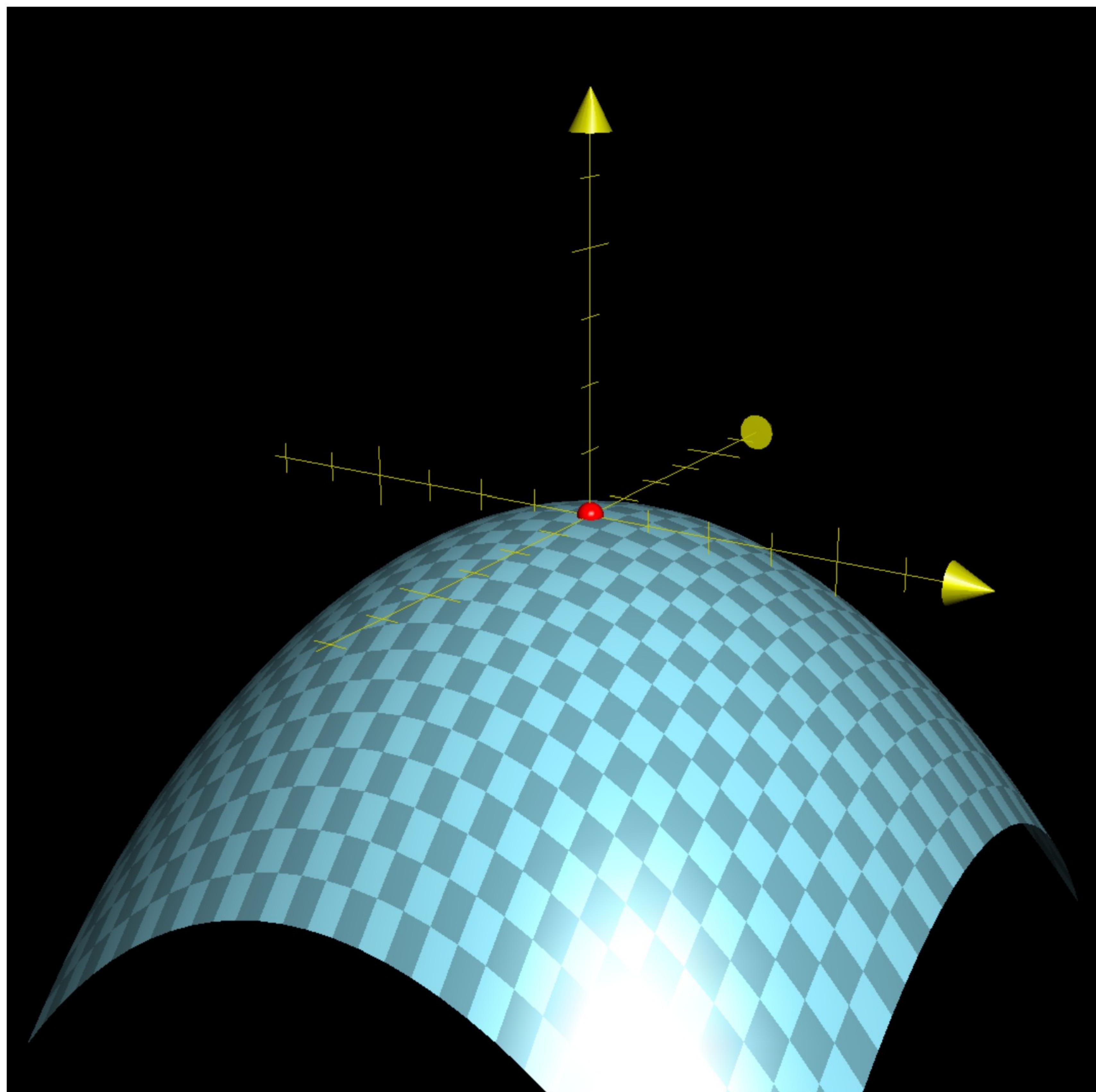


If  $H > 0$ , then  $f$  definitely has either a maximum or minimum at  $(x_0, y_0)$ , and we must look at the sign of  $f_{xx}(x_0, y_0)$  to figure out which one it is.

- If  $f_{xx}(x_0, y_0) > 0$ , then  $f$  has a local minimum. [\[Hide explanation\]](#)



- If  $f_{xx}(x_0, y_0) < 0$ , then  $f$  has a local maximum. [\[Hide explanation\]](#)



If  $H = 0$ , the second derivatives alone cannot tell us whether  $f$  has a local minimum or maximum.

## Example 1: All of the stable points!

**Problem:** Find all the stable points (also called critical points) of the function

$$x^4 - 4x^2 + y^2$$

And determine whether each one gives a local maximum, local minimum, or a saddle point.

### Step 1: Find all stable points

The stable points are all the pairs  $(x_0, y_0)$  where both partial derivatives equal 0. First, compute each partial derivative

$$f_x(x, y) = \boxed{\phantom{00}}$$

$$f_y(x, y) = \boxed{\phantom{00}}$$

[Check](#)

[Hide explanation](#)

$$\begin{aligned} f_x(x, y) &= \frac{\partial f}{\partial x}(x^4 - 4x^2 + y^2) \\ &= 4x^3 - 8x \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= \frac{\partial f}{\partial y}(x^4 - 4x^2 + y^2) \\ &= 2y \end{aligned}$$

Next, find all the points  $(x_0, y_0)$  where both partial derivatives are 0, which is to say, solve the system of equations

$$f_x(x_0, y_0) = 0$$

$$f_y(x_0, y_0) = 0$$

Which of the following pairs satisfying the system of equations?

Choose all answers that apply:

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A  $(2, 0)$

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B  $(-\sqrt{2}, 0)$

---

C  $(0, 0)$

---

D  $(-2, 0)$

---

E  $(-\sqrt{2}, \frac{1}{2})$

---

F  $(\sqrt{2}, \frac{1}{2})$

---

G  $(\sqrt{2}, 0)$

---

[Check](#)

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Putting in our partial derivatives, the system of equations looks like

$$4x^3 - 8x = 0$$

$$2y = 0$$

We are lucky, because the first equation has only an  $x$ , and the second equation has only  $y$ , so solving the system in this case just means solving each problem separately.

The first equation factors as

$$4x^3 - 8x = 0$$

$$4x(x^2 - 2) = 0$$

so its roots are  $x = 0$ ,  $x = \sqrt{2}$ , and  $x = -\sqrt{2}$ .

The second equation has only the solution  $y = 0$ .

Therefore our stable points are

(0, 0)
( $\sqrt{2}$ , 0)
( $-\sqrt{2}$ , 0)

## Step 2: Apply second derivative test

To start, find all three second partial derivatives of  $f(x, y) = x^4 - 4x^2 + y^2$

$$f_{xx}(x, y) = \boxed{\phantom{00}}$$

$$f_{yy}(x, y) = \boxed{\phantom{00}}$$

$$f_{xy}(x, y) = \boxed{\phantom{00}}$$

[Check](#)

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$$f_{xx}(x, y) = \frac{\partial^2}{\partial x^2}(x^4 - 4x^2 + y^2)$$

$$= \frac{\partial}{\partial x}(4x^3 - 8x)$$

$$= 12x^2 - 8$$

$$f_{yy}(x, y) = \frac{\partial^2}{\partial y^2}(x^4 - 4x^2 + y^2)$$

$$= \frac{\partial}{\partial y}(2y)$$

$$= 2$$

$$f_{xy}(x, y) = \frac{\partial^2}{\partial x \partial y}(x^4 - 4x^2 + y^2)$$

$$= \frac{\partial}{\partial x}(2y)$$

$$= 0$$

The expression we care about for the second partial derivative test is

$$f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2$$

If we apply the second derivatives we just found, what does this expression become (as a function of  $x$  and  $y$ )?

$$f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2 = \boxed{\phantom{00}}$$

**Check**

[Hide explanation](#)

$$(12x^2 - 8)(2) - 0^2 = 24x^2 - 16$$

To apply the second derivative test, we plug in each of our stable points to this expression and see if it becomes positive or negative.

- **Stable point 1:**

At  $(x, y) = (0, 0)$ , the expression evaluates as

$$24x^2 - 16 = 24(0)^2 - 16 = -16$$

This is negative, so according to the second partial derivative test, the point  $(0, 0)$  is a

Choose 1 answer:

- 
- A Saddle point

---

  - B Local maximum

---

  - C Local minimum
- 

**Check**

- **Stable point 2:** At  $(x_0, y_0) = (\sqrt{2}, 0)$ , the expression becomes

$$24x^2 - 16 = 24(\sqrt{2})^2 - 16$$

$$= 48 - 16$$

$$= 32$$

This is positive. Also,

$$f_{xx}(\sqrt{2}, 0) = 12(\sqrt{2})^2 - 8$$

$$= 24 - 8$$

$$= 16$$

Therefore, the point  $(\sqrt{2}, 0)$  must be a

Choose 1 answer:

- A Saddle point
- B Local maximum
- C Local minimum

We couldn't grade your answer. It looks like you left something blank or entered in an invalid answer.

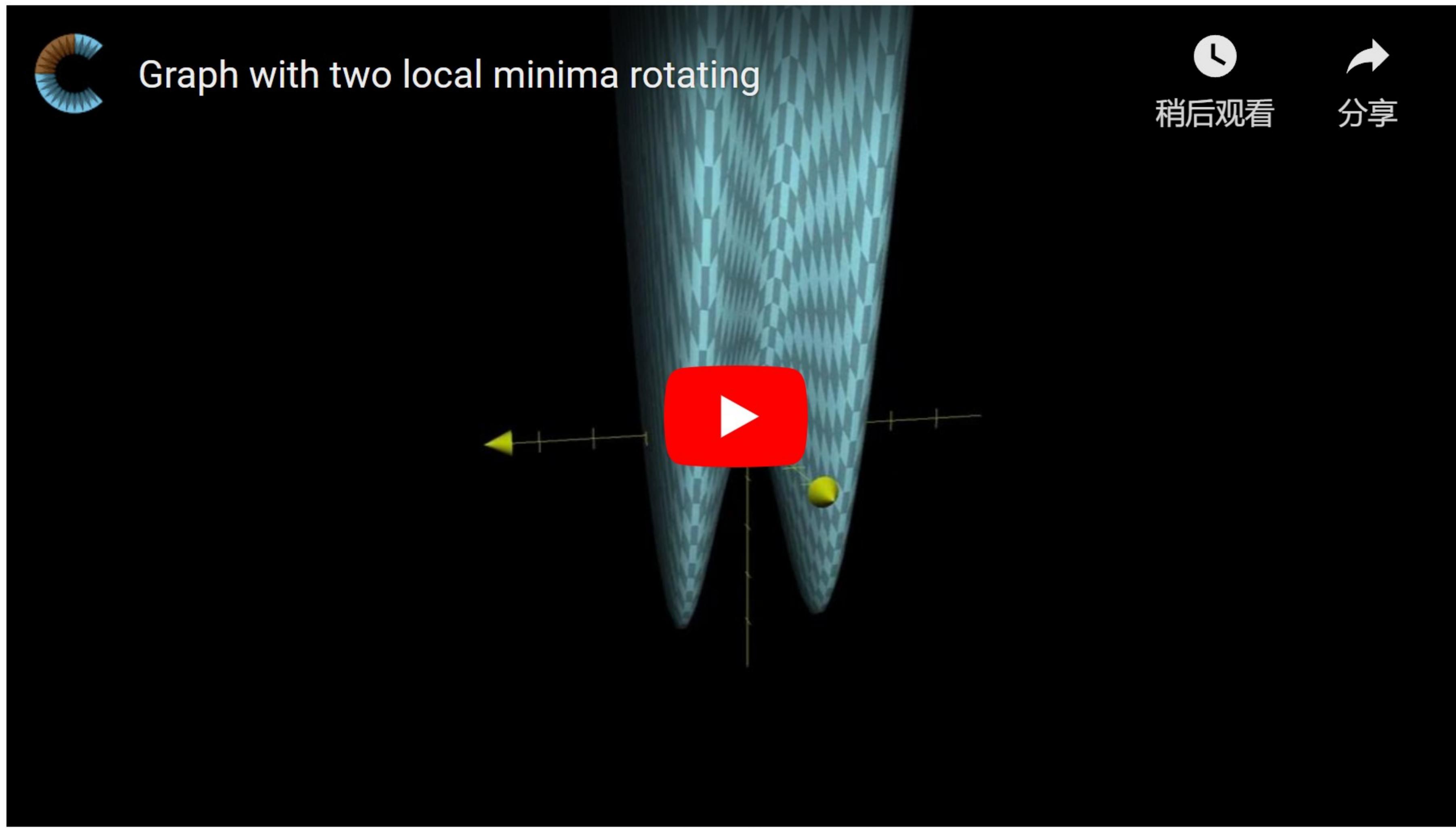
**Check**

- **Stable point 3:** We could plug in the point  $(-\sqrt{2}, 0)$  just as we have with the other stable points, but we could also notice that the function  $f(x, y) = x^4 - 4x^2 + y^2$  is symmetric, in the sense that replacing  $x$  with  $-x$  will yield the same expression:

$$(-x)^4 - 4(-x)^2 + y^2 = x^4 - 4x^2 + y^2$$

Therefore the point  $(-\sqrt{2}, 0)$  will have precisely the same behavior as  $(\sqrt{2}, 0)$

Here is a clip of the graph of  $f(x, y)$  rotating, where the two local minima are clear, and we can see that the point at the origin is indeed a saddle point.



[See video transcript](#)

## Example 2: Getting more intricate

Let's not sugarcoat things; optimization problems can get long. Very long.

**Problem:** Find all the stable points (also called critical points) of the function.

$$f(x, y) = x^2y - y^2x - x^2 - y^2$$

And determine whether each one gives a local maximum, local minimum, or a saddle point.

### Step 1: Find stable points

We need to find where both partial derivatives are zero, so start by finding both partial derivatives of  $f(x, y) = x^2y - y^2x - x^2 - y^2$

$$f_x(x, y) = \boxed{\phantom{00}}$$

$$f_y(x, y) = \boxed{\phantom{00}}$$

[Check](#)

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$$f_x(x, y) = \frac{\partial}{\partial x} x^2y - y^2x - x^2 - y^2$$

$$= 2xy - y^2 - 2x$$

$$f_y(x, y) = \frac{\partial}{\partial y} x^2 y - y^2 x - x^2 - y^2$$

$$= x^2 - 2yx - 2y$$

So we must solve the system of equations

$$2xy - y^2 - 2x = 0$$

$$x^2 - 2xy - 2y = 0$$

In the real world, when you come across a system of equations, you should almost certainly use a computer to solve it. For the sake of practice, though, and to see that optimization problems are not always that simple, let's do something crazy and actually work it out ourselves.

In general, the way you might go about this would go something like this:

- Solve one equation to get  $y$  in terms of  $x$ .
- Plug that into the other expression to get an equation with only  $x$ .
- Solve for  $x$ .
- Plug each solution for  $x$  into both equations and solve for  $y$ .
- Check which resulting  $(x, y)$  pairs actually solve the expression.

This can be a real mess since you might use the quadratic formula to solve for  $y$  treating  $x$  as a constant, and plug that nasty expression in elsewhere. Otherwise, you might find yourself solving a degree 4 equation, which aside from being a pain gives quite a few solutions to plug in.

In this particular system, the equations feel very symmetric, which is an indication that adding/subtracting them might make things simpler. Indeed, if we add them together, we get

$$2xy - y^2 - 2x = 0$$

$$+ \quad x^2 - 2xy - 2y = 0$$

---

$$x^2 - y^2 - 2(x + y) = 0$$

$$(x + y)(x - y) - 2(x + y) = 0$$

$$(x + y)(x - y - 2) = 0$$

What does this equation imply about the relationship between  $x$  and  $y$ ?  
(Express each answer as an **equation** involving the variables  $x$  and  $y$ )

Either

 or 

**Check**

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For  $(x + y)(x - y - 2) = 0$  to be true, one of the factors  $(x + y)$  or  $(x - y - 2)$  must be zero, which implies

$$x + y = 0 \Rightarrow x = -y$$

or

$$x - y - 2 = 0 \Rightarrow x = y + 2$$

Each of these possibilities lets us write  $x$  in terms of  $y$ , which in turn lets us write one of our equations purely in terms of  $y$ .

For example, if you plug in the relation  $x = -y$  to the first expression  $2xy - y^2 - 2x$ , you can get a quadratic expression purely in terms of  $y$ . What are the roots of this expression?

 and 

**Check**

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$$2xy - y^2 - 2x = 2(-y)y - y^2 - 2(-y)$$

$$= -3y^2 + 2y$$

$$= -y(3y - 2)$$

The roots of this expression are  $y = 0$  and  $y = \frac{2}{3}$ .

Since this arose from assuming  $x = -y$ , the corresponding  $x$  values are  $x = -0$  and  $x = -\frac{2}{3}$  respectively. This gives us **our first two solution pairs**:

$$(x, y) = \boxed{(0, 0)}$$

$$(x, y) = \boxed{\left(-\frac{2}{3}, \frac{2}{3}\right)}$$

Alternatively, if we consider the case where  $x = y + 2$ . Again, when we plug this relation into the expression  $2xy - y^2 - 2x$ , we have a quadratic expression purely in terms of  $y$ . What are the roots of this expression?

and

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$$\begin{aligned}2xy - y^2 - 2x &= 2(y + 2)y - y^2 - 2(y + 2) \\&= 2y^2 + 4y - y^2 - 2y - 4 \\&= y^2 + 2y - 4\end{aligned}$$

Applying the quadratic formula, we have

$$y = \frac{-2 \pm \sqrt{2^2 - 4(1)(-4)}}{2}$$

$$= \frac{-2 \pm \sqrt{4(1+4)}}{2}$$

$$= \frac{-2 \pm 2\sqrt{1+4}}{2}$$

$$= -1 \pm \sqrt{5}$$

This gives two values of  $y$ ,

$$y = -1 + \sqrt{5}$$

$$y = -1 - \sqrt{5}$$

Because we found these under the assumption that  $x = y + 2$ , the corresponding values of  $x$  are

$$x = 2 - 1 + \sqrt{5} = 1 + \sqrt{5}$$

$$x = 2 - 1 - \sqrt{5} = 1 - \sqrt{5}$$

This gives **two more solution pairs**:

$$(x, y) = \boxed{(1 + \sqrt{5}, -1 + \sqrt{5})}$$

$$(x, y) = \boxed{(1 - \sqrt{5}, -1 - \sqrt{5})}$$

We've now exhausted all possibilities since we initially found that either  $x = -y$  or  $x = y + 2$ , and we completely solved the equations resulting from each assumption.

## Step 2: Apply second derivative test

Man, that was already a lot of work for one example, and we're not even halfway done! Now we have to apply the second derivative test to each one of these. First, find all of the second derivatives of our function

$$f(x, y) = x^2y - y^2x - x^2 - y^2$$

$$f_{xx}(x, y) = \boxed{\phantom{00}}$$

$$f_{yy}(x, y) = \boxed{\phantom{00}}$$

$$f_{xy}(x, y) = \boxed{\phantom{00}}$$

[Check](#)

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$$\begin{aligned} f_{xx}(x, y) &= \frac{\partial^2}{\partial x^2} (x^2y - y^2x - x^2 - y^2) \\ &= \frac{\partial}{\partial x} (2xy - y^2 - 2x) \\ &= 2y - 2 \end{aligned}$$

$$\begin{aligned}
f_{yy}(x, y) &= \frac{\partial^2}{\partial y^2} (x^2 y - y^2 x - x^2 - y^2) \\
&= \frac{\partial}{\partial y} (x^2 - 2yx - 2y) \\
&= -2x - 2
\end{aligned}$$

$$\begin{aligned}
f_{yx}(x, y) &= \frac{\partial^2}{\partial x \partial y} (\textcolor{teal}{x}^2 y - y^2 \textcolor{red}{x} - \textcolor{teal}{x}^2 - y^2) \\
&= \frac{\partial}{\partial x} (\textcolor{teal}{x}^2 - 2yx - 2y) \\
&= 2\textcolor{teal}{x} - 2y
\end{aligned}$$

According to the second derivative test, to analyze whether each of our stable points is a local maximum or minimum, we plug them into the expression

$$f_{xx}(x, y) f_{yy}(x, y) - (f_{xy}(x, y))^2$$

**What does this expression become when we apply the second derivatives you just found?**

**Check**

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$$(2y - 2)(-2x - 2) - (2x - 2y)^2$$

Since we only care about whether this expression is positive or negative, we can divide everything by 4 to make things a bit simpler.

$$(y - 1)(-x - 1) - (x - y)^2 \quad \leftarrow \text{Key expression}$$

Now we see what the sign of this expression is for each of our stable points.

- **Stable point  $(0, 0)$ :**

At the point  $(x, y) = (0, 0)$ , the key expression above evaluates to .

**Check**

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Plugging  $(0, 0)$  into the expression above, we get

$$(0 - 1)(0 - 1) - (0 - 0)^2 = 1$$

Because this is positive,  $(0, 0)$  is either a local minimum point or a local maximum point. To find out which, we look at the sign of  $f_{xx}(0, 0)$ :

$$2y - 2 \rightarrow 2(0) - 2 = -2 < 0$$

From this we conclude that  $(0, 0)$  is

Choose 1 answer:

- 
- A a saddle point.
  - B either a local maximum point or a local minimum point.
- 

Now,

Choose 1 answer:

- 
- A We are done.
  - B We evaluate  $f_{xx}(0, 0) = 2(0) - 2 = -2$ , and conclude that  $f$  has a local **maximum** at  $(0, 0)$
  - C We evaluate  $f_{xx}(0, 0) = 2(0) - 2 = -2$ , and conclude that  $f$  has a local **minimum** at  $(0, 0)$
- 

[Check](#)

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Because  $-2$  is negative,  $f$  has negative concavity at  $(0, 0)$ , so **the stable point  $(0, 0)$  must be a local maximum point**.

- **Stable point**  $\left(-\frac{2}{3}, \frac{2}{3}\right)$ :

At the point  $(x, y) = \left(-\frac{2}{3}, \frac{2}{3}\right)$ , the key expression above evaluates to

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[Hide explanation](#)

Plugging  $\left(-\frac{2}{3}, \frac{2}{3}\right)$  into the expression above, we get

$$(y - 1)(-x - 1) - (x - y)^2$$

$$\begin{aligned}
&= \left( \frac{2}{3} - 1 \right) \left( -\left( -\frac{2}{3} \right) - 1 \right) - \left( -\frac{2}{3} - \frac{2}{3} \right)^2 \\
&= \frac{-1}{3} \cdot \frac{-1}{3} - \left( \frac{-4}{3} \right)^2 \\
&= \frac{1}{9} - \frac{16}{9} \\
&= \frac{-15}{9}
\end{aligned}$$

From this we conclude that  $\left( -\frac{2}{3}, \frac{2}{3} \right)$  is

Choose 1 answer:

- 
- (A) a saddle point.
- 
- (B) either a local maximum point or a local minimum point.
- 

Now,

Choose 1 answer:

- 
- (A) We are done.
- 
- (B) We evaluate  $f_{xx} \left( -\frac{2}{3}, \frac{2}{3} \right) = 2 \left( \frac{2}{3} \right) - 2 = -\frac{2}{3}$  and conclude that  $f$  has a local **maximum** at  $(0, 0)$ .
- 
- (C) We evaluate  $f_{xx} \left( -\frac{2}{3}, \frac{2}{3} \right) = 2 \left( \frac{2}{3} \right) - 2 = -\frac{2}{3}$  and conclude that  $f$  has a local **minimum** at  $(0, 0)$ .
- 

**Check**

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Because  $-\frac{15}{9}$  is negative,  $\left( -\frac{2}{3}, \frac{2}{3} \right)$  is a saddle point. Looking at  $f_{xx}$  is unnecessary.

- **Stable point**  $(1 + \sqrt{5}, -1 + \sqrt{5})$ :

At the point  $(x, y) = (1 + \sqrt{5}, -1 + \sqrt{5})$ , the key expression above evaluates to .

**Check**

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Plugging  $(1 + \sqrt{5}, -1 + \sqrt{5})$  into the expression above, we get

$$\begin{aligned} & (y - 1)(-x - 1) - (x - y)^2 \\ &= (-1 + \sqrt{5} - 1)(-1 - \sqrt{5} - 1) - (1 + \sqrt{5} - (-1 + \sqrt{5}))^2 \\ &= (-2 + \sqrt{5})(-2 - \sqrt{5}) - (2)^2 \\ &= 4 - 5 - 4 \\ &= -5 \end{aligned}$$

From this we conclude that  $(1 + \sqrt{5}, -1 + \sqrt{5})$  is

Choose 1 answer:

- 
- A a saddle point.
  - B either a local maximum point or a local minimum point.
- 

Now,

Choose 1 answer:

- 
- A We are done.
  - B We evaluate  $f_{xx}\left(-\frac{2}{3}, \frac{2}{3}\right) = 2\left(\frac{2}{3}\right) - 2 = -\frac{2}{3}$  and conclude that  $f$  has a local **maximum** at  $(0, 0)$ .
  - C We evaluate  $f_{xx}\left(-\frac{2}{3}, \frac{2}{3}\right) = 2\left(\frac{2}{3}\right) - 2 = -\frac{2}{3}$  and conclude that  $f$  has a local **minimum** at  $(0, 0)$ .
- 

**Check**

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Because  $-5$  is negative,  $(1 + \sqrt{5}, -1 + \sqrt{5})$  is a saddle point.

Looking at  $f_{xx}$  is unnecessary.

- **Stable point**  $(1 - \sqrt{5}, -1 - \sqrt{5})$ :

The arithmetic here is almost identical to the previous case.

[\[Hide explanation\]](#)

In a deep sense, this is not just similar to the previous case, but *equivalent*. This is because our function  $f$  has a certain symmetry, namely

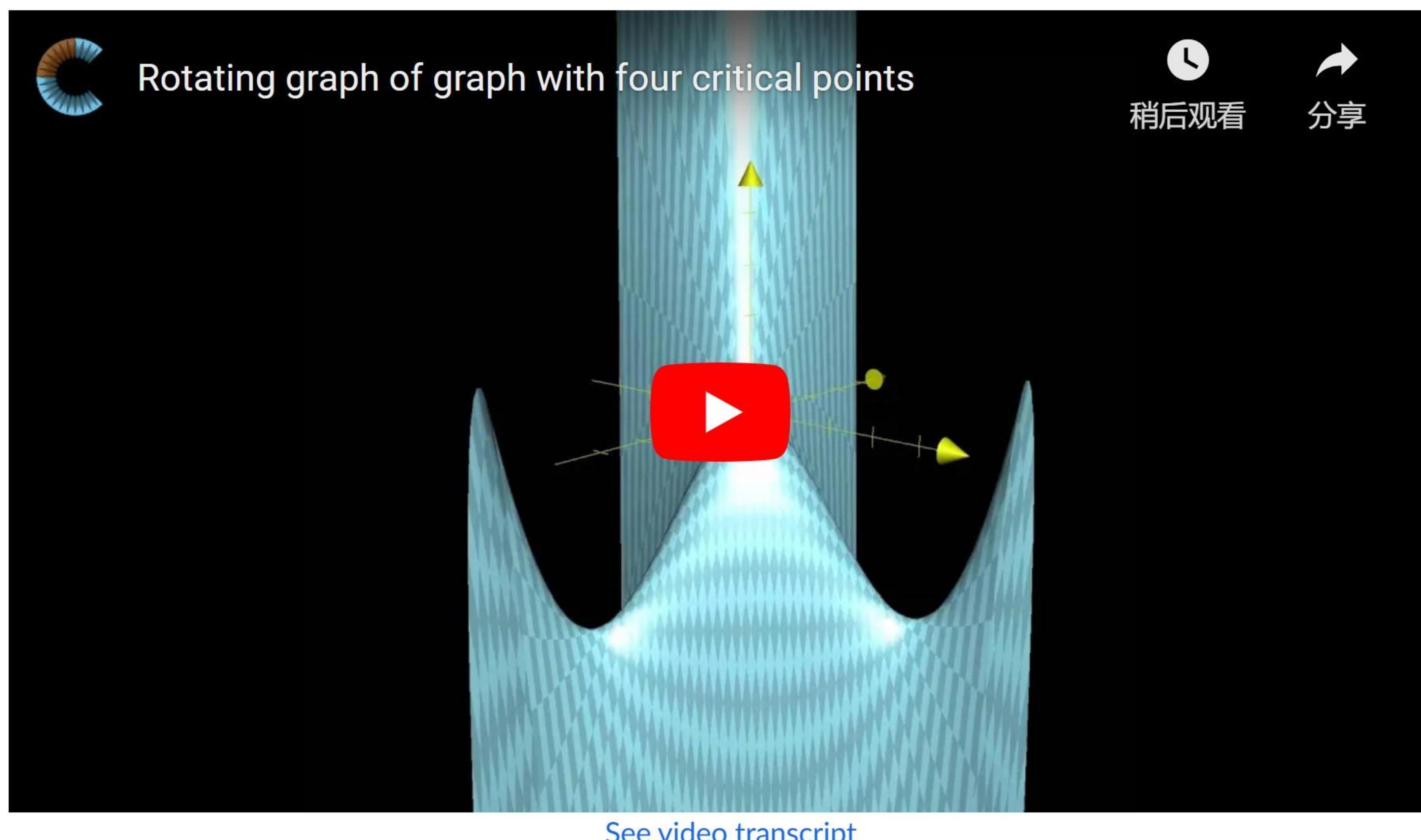
$$f(x, y) = f(-y, -x)$$

**Try it yourself to see why!**

Applying the operation  $(x, y) \rightarrow (-y, -x)$  to the third stable point  $(1 + \sqrt{5}, -1 + \sqrt{5})$  gives the fourth stable point  $(1 - \sqrt{5}, -1 - \sqrt{5})$ , so the behavior of the function at both of these points should be identical.

The symmetry of this function is very visible in the graph pictured below.

Here is a short clip of the graph of  $f(x, y) = x^2y - y^2x - x^2 - y^2$  rotating, where you can see the three saddle points and the one local maximum at the origin.



## Pat yourself on the back

These are long problems, so if you actually worked through them, give yourself some hearty congratulations!