Why care about the formal definitions of divergence and curl?

Google Classroom

Before learning about how to formally define divergence and curl, read a little about why this is something worth doing.

About formal definitions

In some ways, the great art of mathematics is finding the right definitions. It involves taking a loose idea, an intuition, and turning it into something absolutely airtight.

For the next few articles, I am assuming you have learned about both divergence and curl. In particular, you should know how to compute them, and more importantly, you should feel comfortable with how to interpret each operation in terms of fluid flow.

The purposes of these articles, then, will be to turn these fluid-flow interpretations into mathematical definitions.

> "Wait, haven't I already seen the definition of divergence and curl? These are the formulas we use to compute them, right?"

Well, divergence and curl are two funny operations where the way they are defined is not the same as the way they are computed in practice. The formulas that we use for computations, i.e. the ones stemming from the notation $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$, are not the formal definitions. The formal definitions involve certain integrals which capture the appropriate fluid flow intuition.

Unfortunately, these definitions are not very practical to use for real computations, so it's more common to just introduce the formulas $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$. It's actually quite fortunate that these relatively computable formulas exist.

> "If these formal definitions are impractical for computations, why should I care? Let the mathematicians worry about the underlying theory, right?"

Well, yes and no. Yes, these definitions will not be something you need to memorize or to pull out of your pocket in a practical application. However, in my opinion, **there is no better way to solidify your understanding of how to interpret both divergence and curl than by understanding these definitions**.

They also serve as great practice in applying line integrals and surface integrals.

Moreover, and perhaps more importantly, some of the big topics you will soon learn in multivariable calculus include Green's theorem and Stokes' theorem, which relate curl to line integrals and surface integrals. I promise you, it will be much easier to see what these theorems are really saying if you have a solid grasp on how curl is actually defined.

And the same goes for divergence, Green's divergence theorem and the three-dimensional divergence theorem are two more big topics that are made easier to understand when you know what divergence really means.

That said, it's possible to understand all these theorems without learning the formal definitions of divergence and curl, so you can view these articles as optional reading. But you would be doing your future self a favor by front-loading the burden of conceptual understanding here.