Surface area integrals

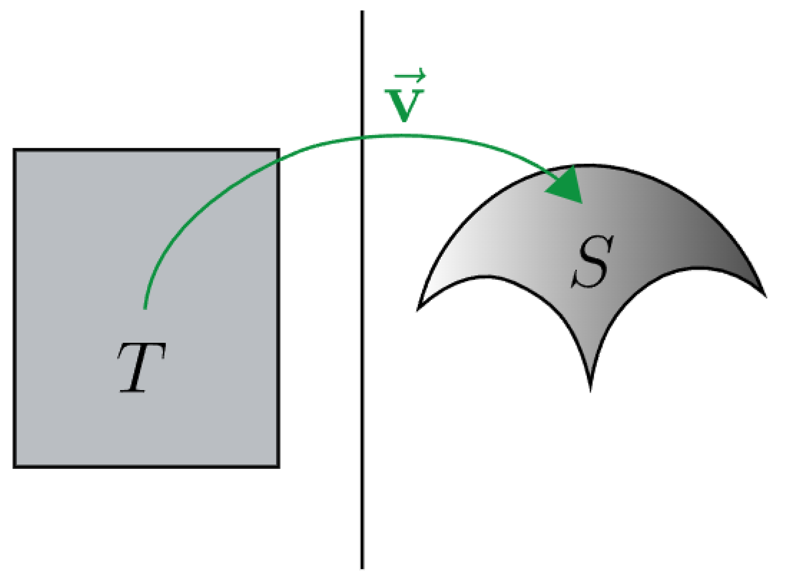
Google Classroom

How do you find the surface area of a parametric surface? This will lead to the more general idea of a surface integral.

Background

- Partial derivatives of parametric surfaces
  In particular, make sure you have a strong intuition for the partial derivatives of a function parameterizing a surface, and what they represent.
- Double integrals
- Cross product (video)

What we are building to



Setup:

- S is some surface in three-dimensional space.
- $\vec{\mathbf{v}}(t, s)$ is a vector-valued function parameterizing S. [Hide explanation]

> If you are unfamiliar with the idea of parameterization, see the article on parametric functions with two parameters.

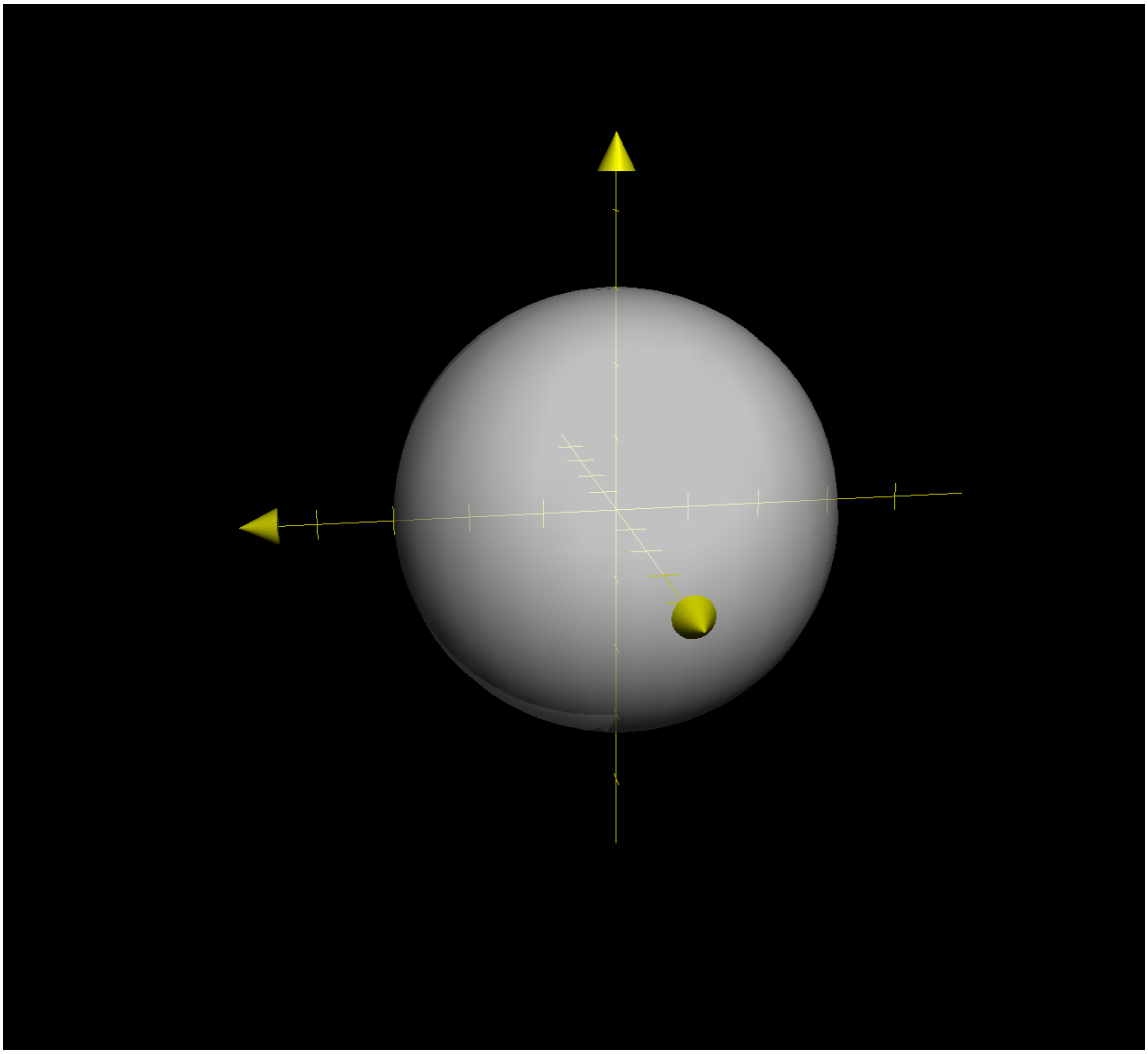
- T is the region of the ts-plane (also known as the parameter space) that corresponds with S.

The surface area of S can be computed with the following double integral:

$$\iint_T \left| \frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s} \right| dt\, ds$$

These integrals can be very labor intensive to compute.

Surface area



From geometry, you might be familiar with the surface areas of a few specific shapes. For example, the surface area of a sphere with radius r is $4\pi r^2$.

But what if someone gives you an arbitrary surface, defined using some parametric function that maps a region of two-dimensional parameter space into three-dimensional space? How do you find its surface area?

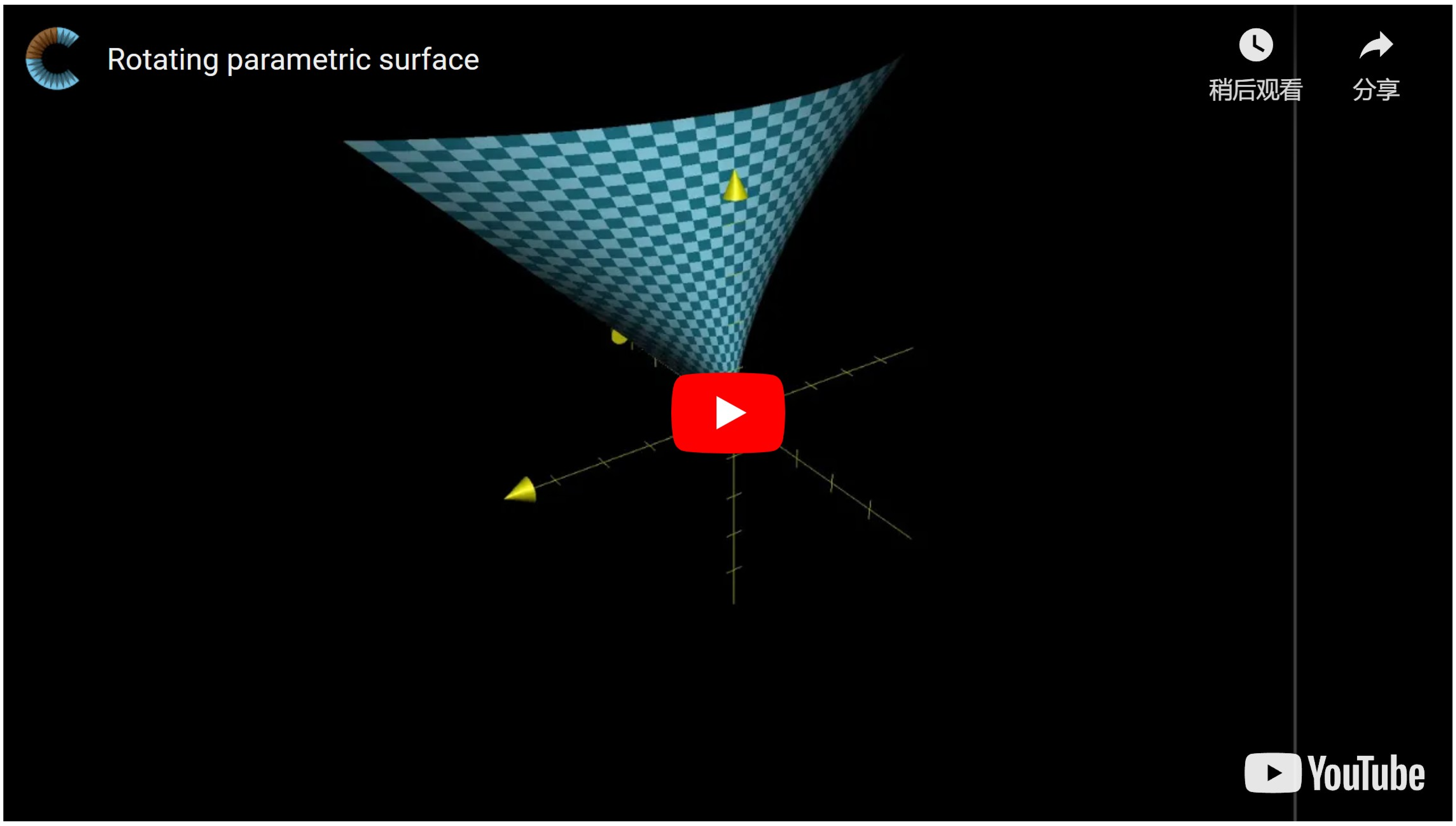
The answer is to use a certain integral, or rather a certain *double* integral, which you are about to learn. This is analogous to how you can find the arc length of an arbitrary curve using a certain single integral, or the volume of a strangely shaped solid using the appropriate triple integral.

Example: Breaking down surface area

Define a parametric surface with the following function:

$$\vec{\mathbf{v}}(t, s) = \begin{bmatrix} t^2 \\ st \\ s \end{bmatrix}$$

Let's name this surface S.

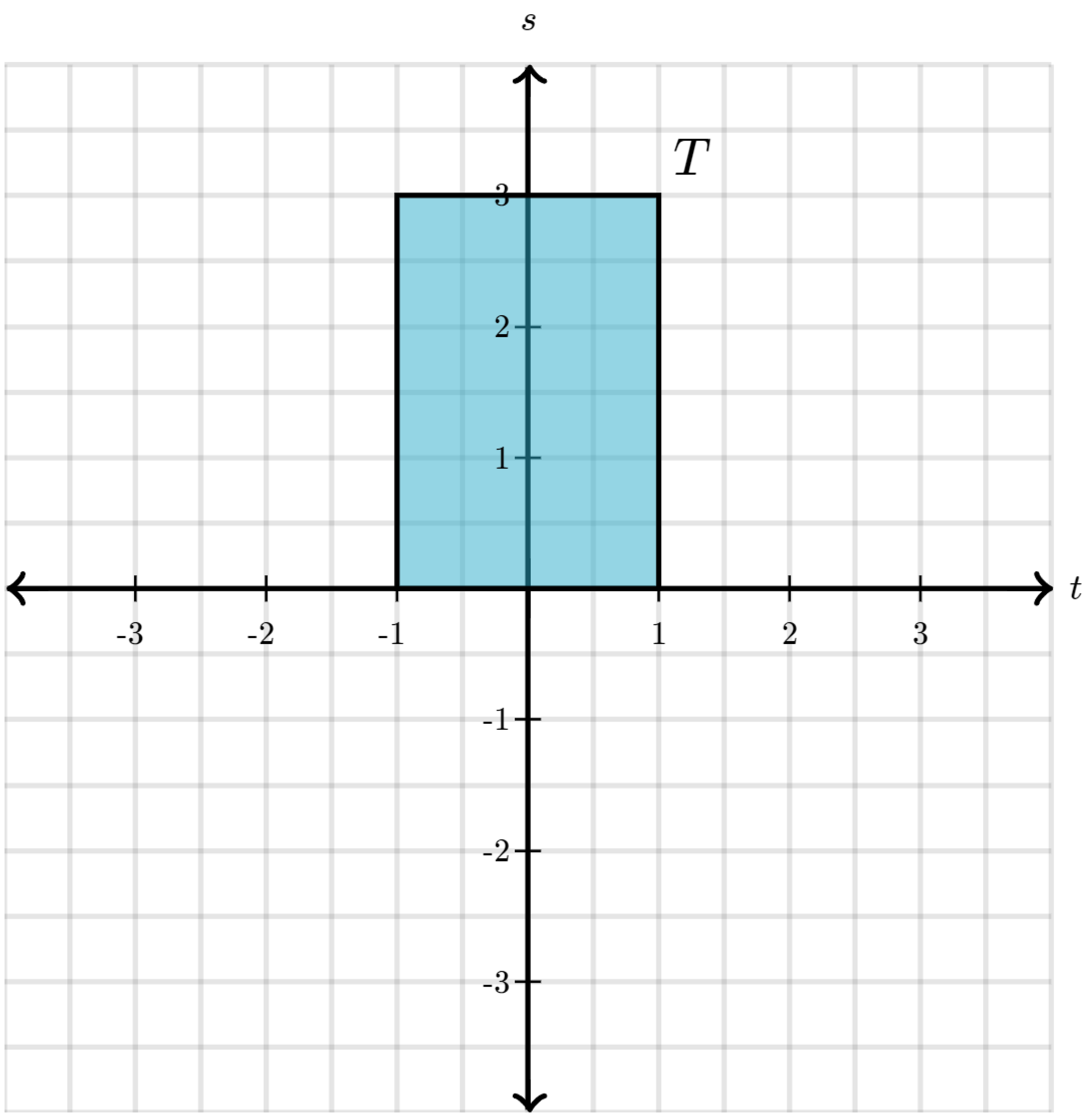


See video transcript

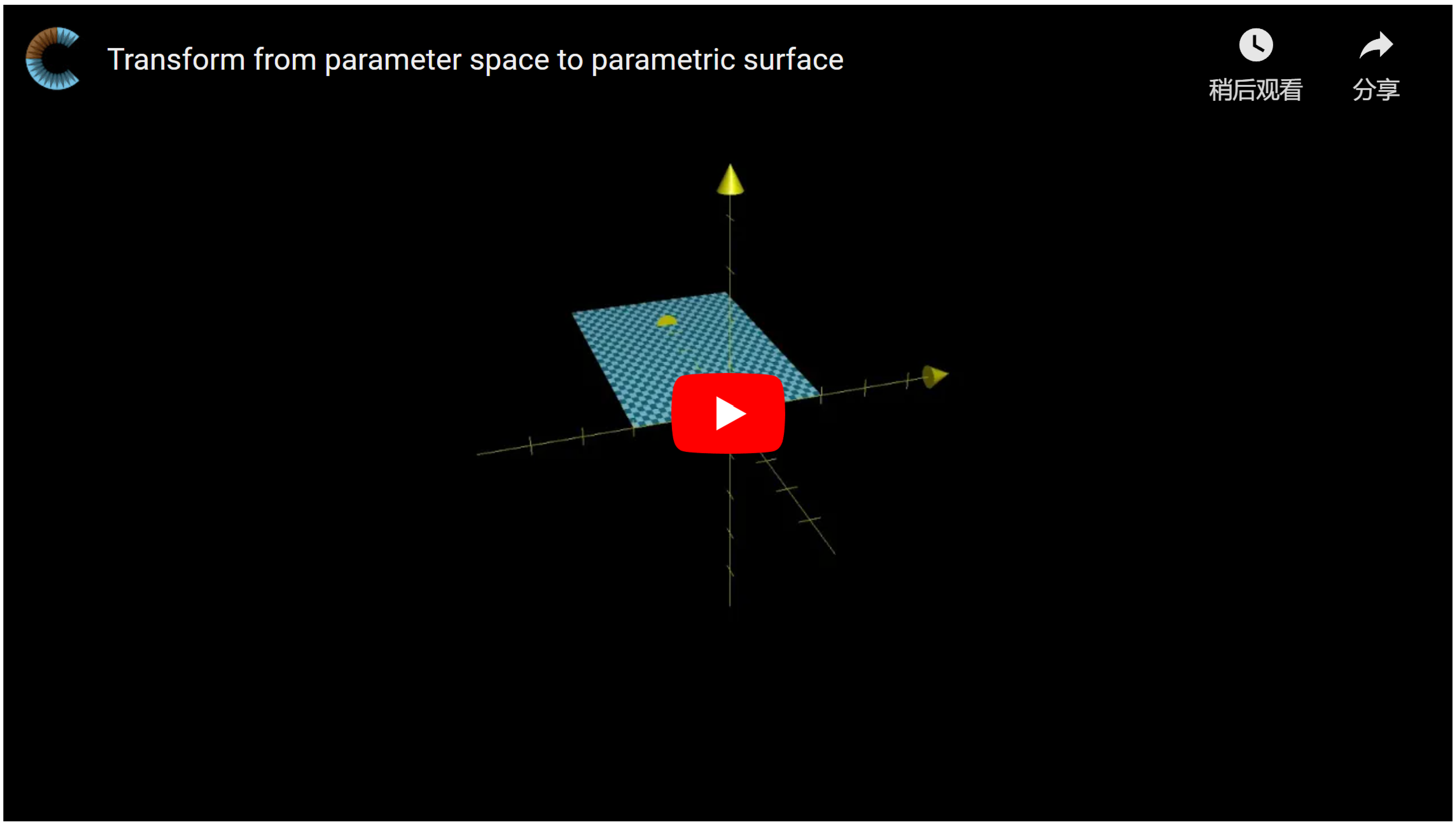
Of course, with parametric surfaces, it is not enough just to specify the function which parameterizes it. We also need to know the region of the parameter space that gets mapped onto the surface. "Parameter space" is a fancy word for where the point (t, s) lives, also known as the "domain". In this case, let's say it is the rectangle defined by

$$-1 \leq t \leq 1$$

$$0 \leq s \leq 3$$



Let's call this rectangle T. Here is what it looks like for $\vec{\mathbf{v}}$ to transform the rectangle T in the parameter space into the surface S in three-dimensional space.



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Our strategy for computing this surface area involves three broad steps:

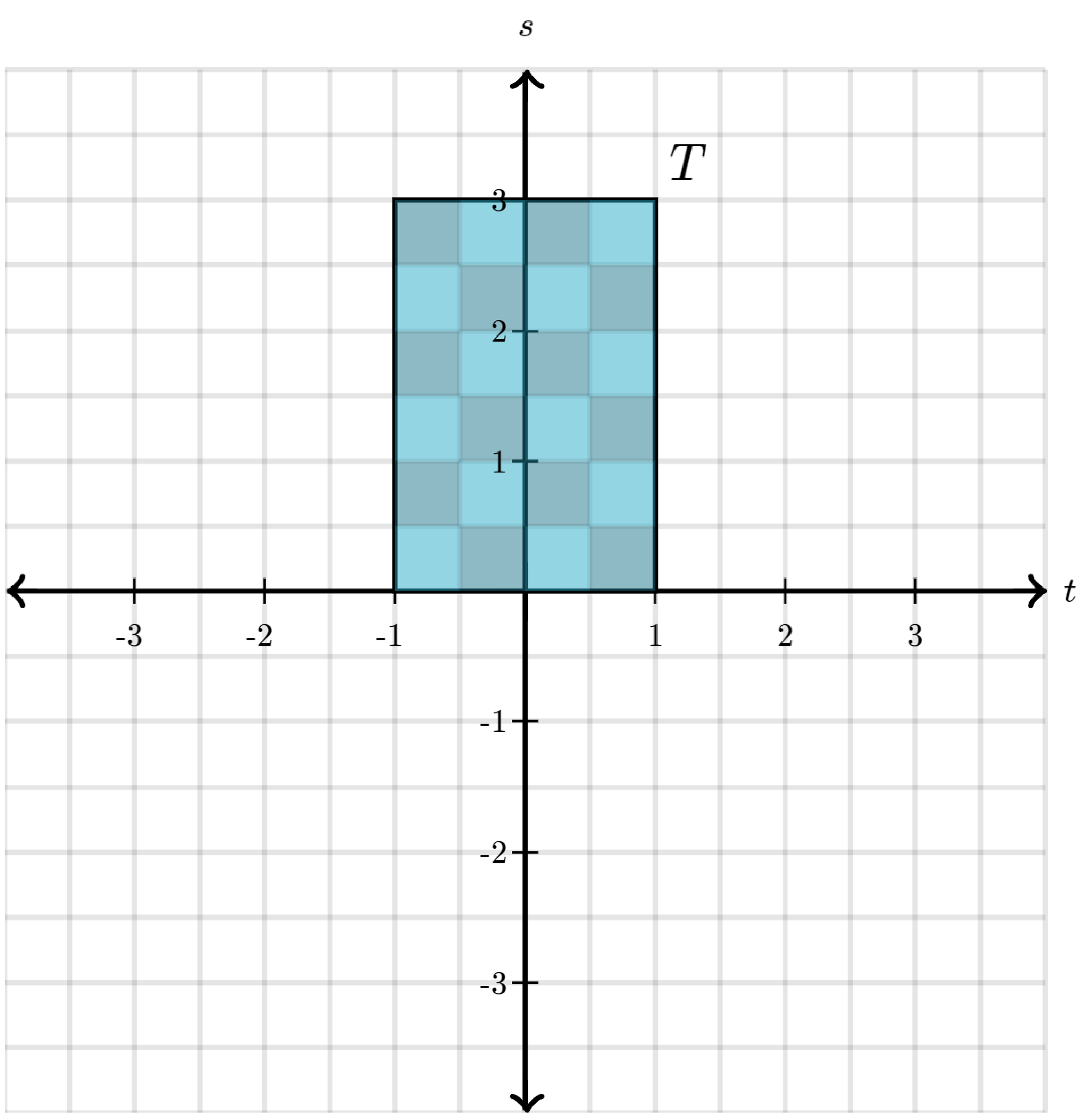
- **Step 1**: Chop up the surface into little pieces.
- **Step 2**: Compute the area of each piece.
- **Step 3**: Add up these areas.

After studying line integrals, double integrals and triple integrals, you may recognize this idea of chopping something up and adding all its pieces as a more general pattern in how integration can be used to solve problems. As with those examples, our final computation will not *actually* involve chopping up the surface into a specific number of pieces and adding them up; we let an integral take care of that for us. [Hide explanation]

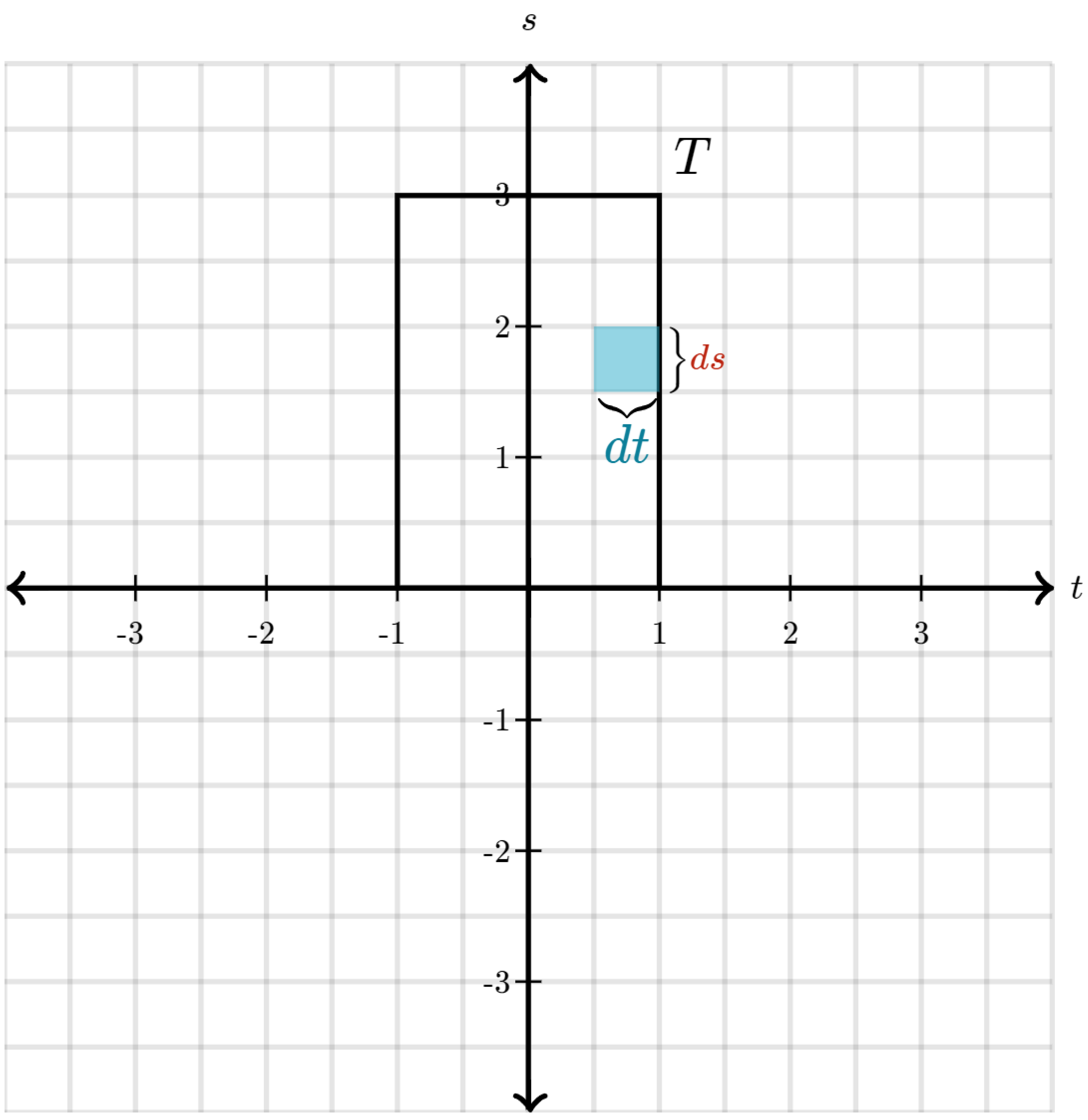
> To be more precise, the integral we are shooting for will do something better than chopping up a surface and adding up the areas of each piece. It considers what happens in the limit as you chop up the surface more and more finely into smaller and smaller pieces. This is why it will give the *exact* surface area, not a mere approximation (as literal chopping would).

Step 1: Chopping up the surface

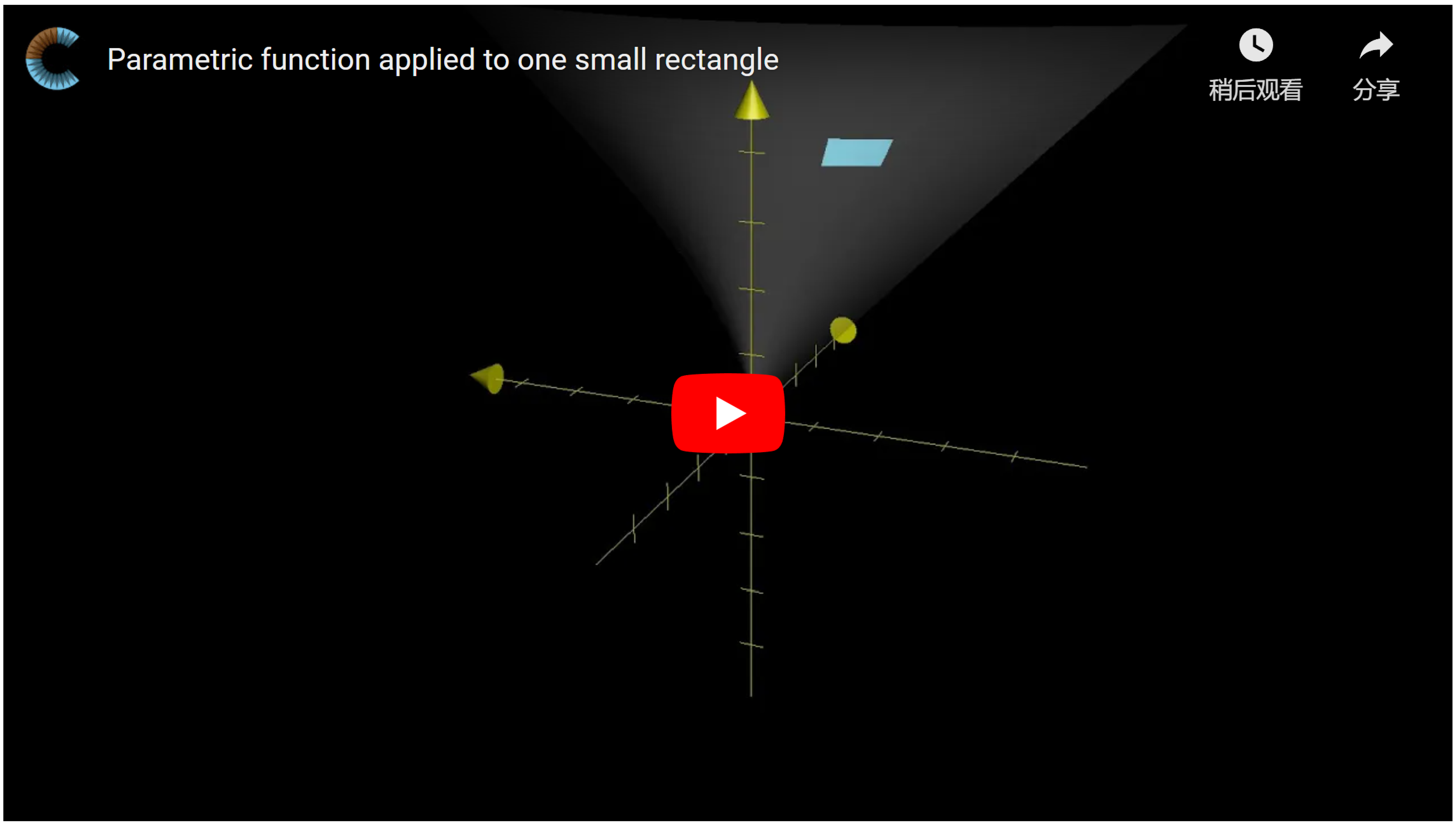
To start, think of chopping up the rectangle T in the parameter space into many tiny little rectangles. In the drawing, I'll only chop it into a few rectangles so that we can see and reference each one, but in principle you should think of very many, really small rectangles.



For one of these tiny rectangles, you can think its width as being dt, a tiny change to the parameter t. Similarly, think of its height as being ds, a tiny change to the parameter s.



Now consider how the function $\vec{\mathbf{v}}(t, s)$ maps one of these tiny rectangles onto the surface S. In the following animation, I'll make most of the surface a faded grey, and leave just one of the tiny rectangles colored as we watch T transform into S.



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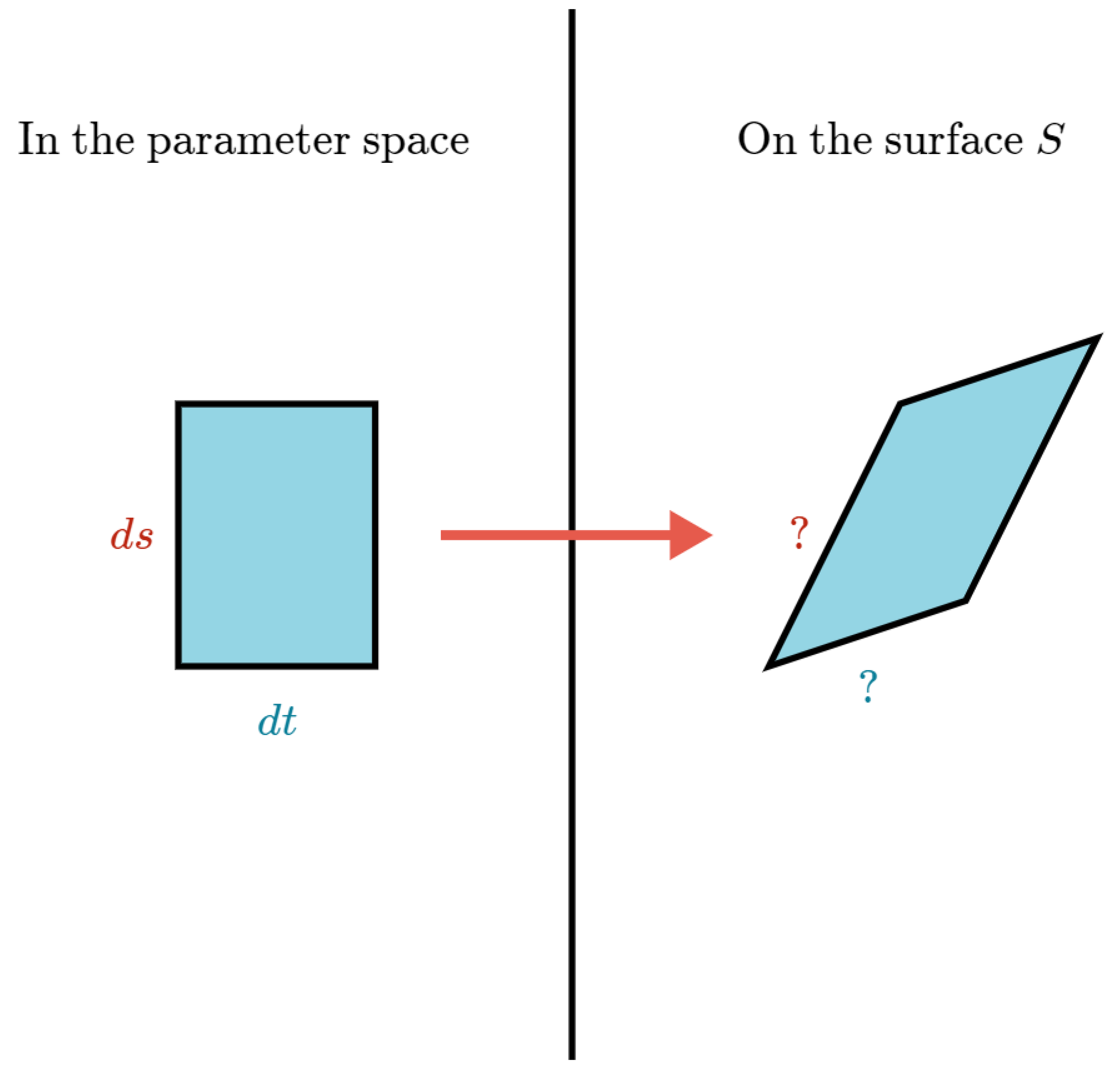
Strictly speaking, the rectangle will become slightly curved as it is pasted onto S. However, as you consider smaller and smaller rectangles, that curvature becomes more and more negligible, and we can basically treat this tiny piece as if it was flat.

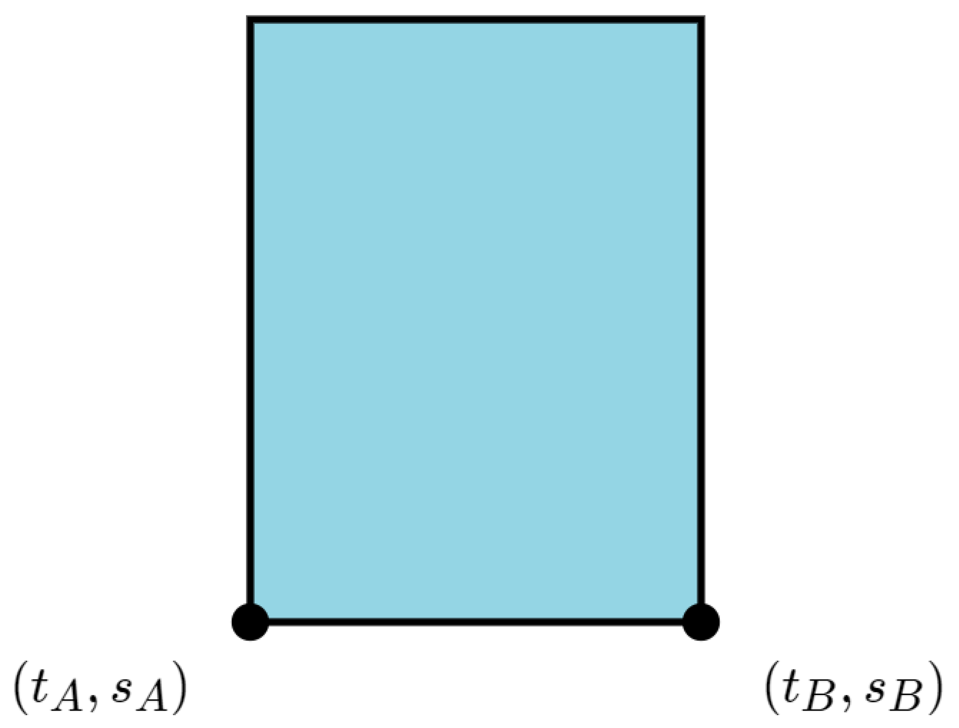
In fact, as we consider smaller and smaller rectangles in the parameter space, the portions of the surface S that these rectangles map to will look more and more like parallelograms.



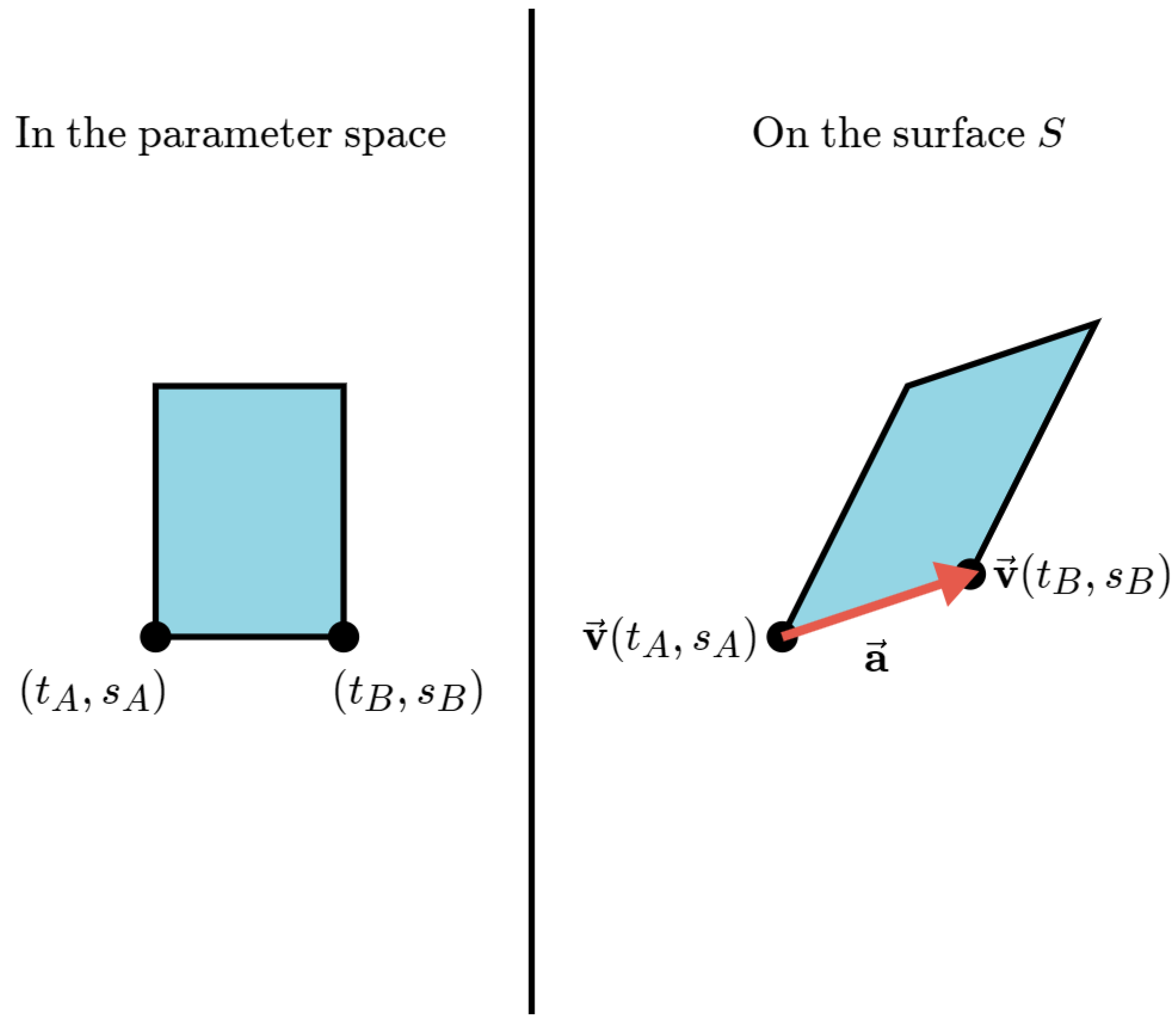
Our first task, then, will be to find a formula giving the area of these parallelograms.

Step 2: Seeking the area of a parallelogram piece

For one of these tiny rectangles that we chopped T into, let (t_A, s_A) represent its lower left corner, and (t_B, s_B) represent its lower right corner.



Now consider the vector pointing from $\vec{\mathbf{v}}(t_A, s_A)$ to $\vec{\mathbf{v}}(t_B, s_B)$ on the surface. Let's name that vector $\vec{\mathbf{a}}$.



Concept check: If we describe the distance between (t_A, s_A) and (t_B, s_B) as being dt, which of the following expressions represents a good approximation of $\vec{\mathbf{a}}$?

Choose 1 answer:

(A) $\dfrac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)$

(B) $\dfrac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)$

(C) $\dfrac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)\, dt$

(D) $\dfrac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)\, dt$

Check

Hide explanation

The third answer choice is correct:

$$\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)\, dt$$

If this is confusing, or if it seems unfamiliar, consider reviewing the article on partial derivatives of parametric surfaces.

One way to think about this is to imagine traveling in the positive t direction of the parameter space with a speed of 1. If you map your motion onto the surface S, the function $\dfrac{\partial \vec{\mathbf{v}}}{\partial t}$ describes your velocity vector on that surface as a function of time.

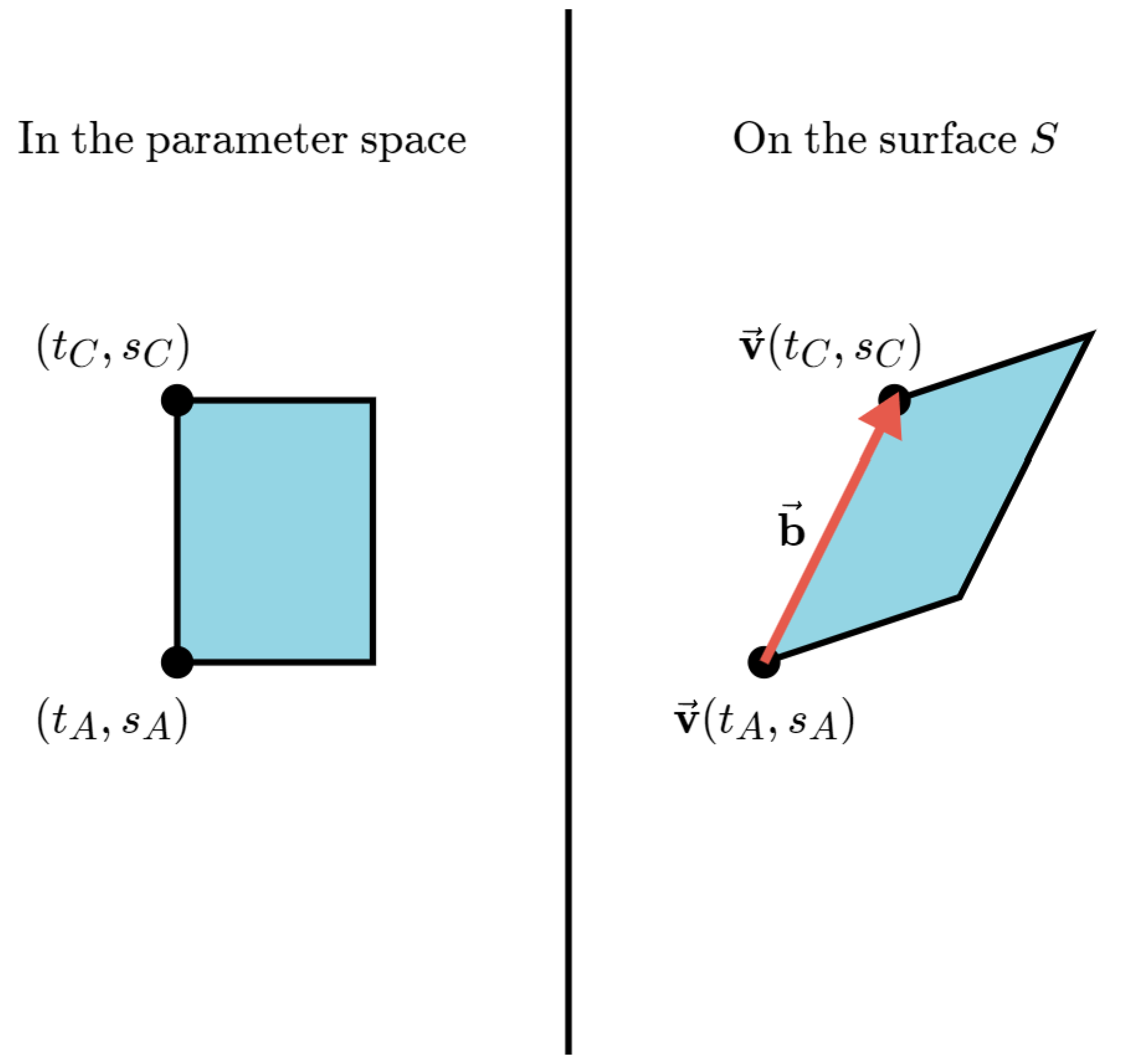
In particular, if we consider the instant when you are passing the point (t_A, s_A) in the parameter space, the corresponding velocity vector for your mapped-motion on S is the partial derivative of $\vec{\mathbf{v}}$ with respect to t:

$$\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A) \quad \leftarrow \text{This is a three-dimensional vector}$$

Multiplying it by dt, which can be thought of as a tiny change in time since your speed in the positive t-direction is 1, you will get a tiny change in displacement. This change in displacement is the vector in question.

It's worth noting that the partial derivative $\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)$ is a vector which doesn't really care about the location of (t_B, s_B). However, scaling it down by dt is what gives it just the right length to reach (t_B, s_B).

Concept check: Take the same setup as the previous problem, but let (t_C, s_C) be the upper left corner of the tiny rectangle. Let's give the vector pointing from $\vec{\mathbf{v}}(t_A, s_A)$ to $\vec{\mathbf{v}}(t_C, s_C)$ a name, $\vec{\mathbf{b}}$.



If we describe the distance between (t_A, s_A) and (t_C, s_C) as being ds, which of the following best approximates $\vec{\mathbf{b}}$?

Choose 1 answer:

(A) $\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)$

(B) $\frac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)$

(C) $\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)\, ds$

(D) $\frac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)\, ds$

Check

Hide explanation

This time the fourth answer choice is correct:

$$\frac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)\, ds$$

The reasoning is essentially identical to that of the previous question.

Okay, here's where we are so far: We are thinking about a tiny rectangle in the parameter space with the following properties

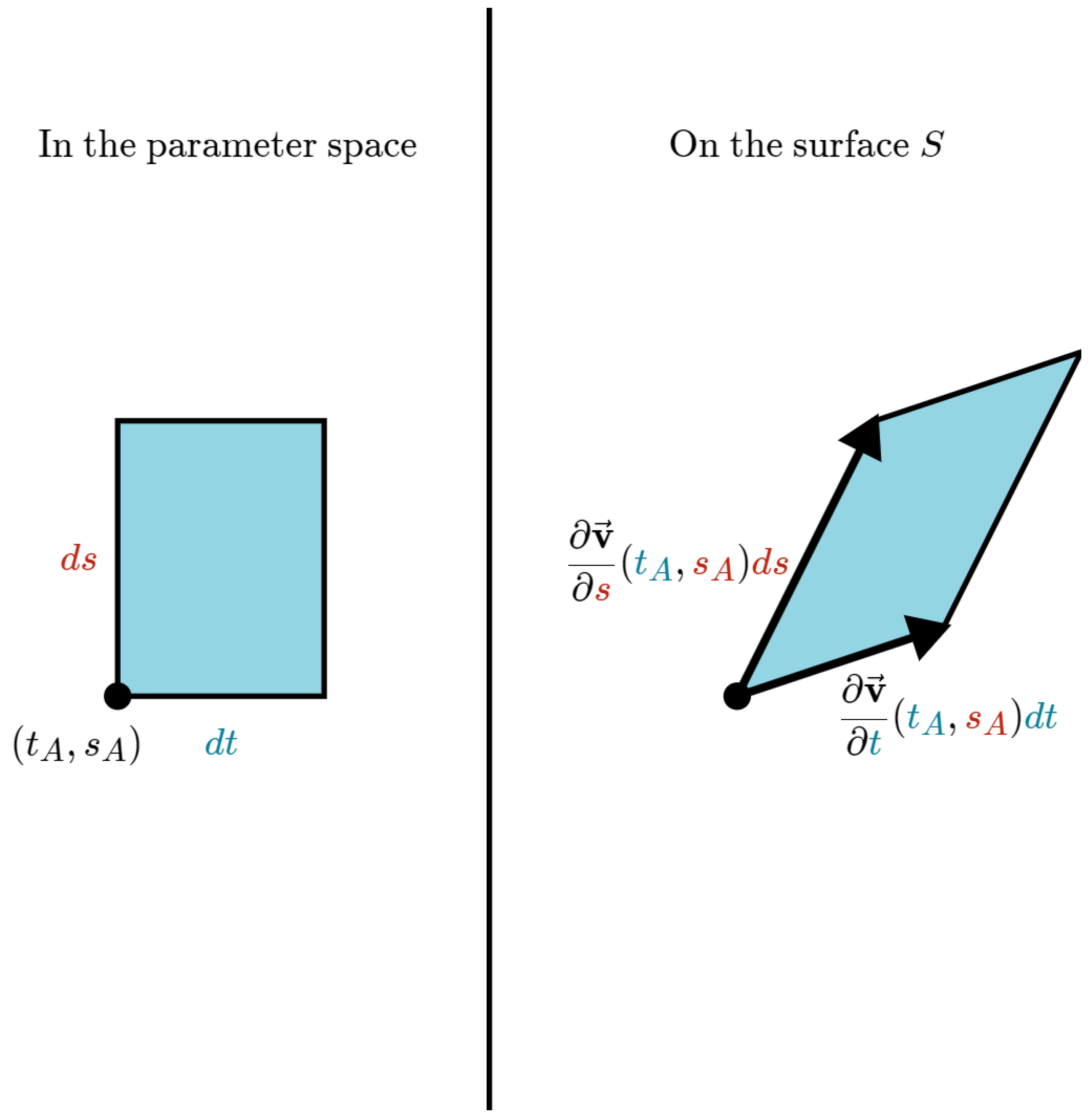
- Bottom left corner: (t_A, s_A)
- Width: dt
- Height: ds

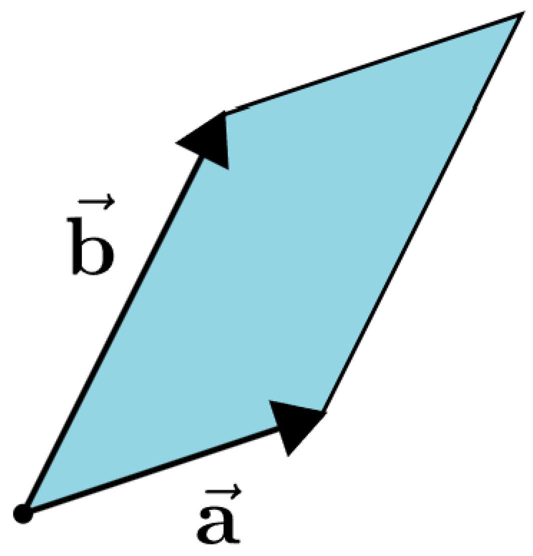
When you apply the function $\vec{\mathbf{v}}$ to this rectangle, you end up with what is basically a parallelogram on the surface S. Based on the previous two questions, the sides of this parallelogram are determined by the vectors

$$\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)\, dt$$

and

$$\frac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)\, ds$$





Concept check: If the side lengths of a parallelogram in three-dimensional space are described with the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, as pictured to the right, which of the following represents the area of that parallelogram?

Choose 1 answer:

(A) $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$

(B) $|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$

(C) $|\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}|$

We couldn't grade your answer. It looks like you left something blank or entered in an invalid answer.

Check

Hide explanation

The second answer choice is correct:

$|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$

Recall how to interpret the cross product. It returns a vector which is perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, and whose magnitude equals the area of a parallelogram spanned by those two vectors.

If this is unfamiliar, consider reviewing the video about cross product intuition.

Concept check: Putting all this together, when $\vec{\mathbf{v}}(t, s)$ maps the little dt-by-ds rectangle with lower-left corner (t_A, s_A) onto some parallelagram on the surafce S, what is the area of that parallelogram?

Choose 1 answer:

(A) $\left|\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)\right)\right|$

(B) $\left|\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)\right) \cdot \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)\right)\right|$

(C) $\left|\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)\, dt\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)\, ds\right)\right|$

(D) $\left|\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)\, dt\right) \cdot \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)\, ds\right)\right|$

Check

Hide explanation

The third answer choice is correct:

$$\left|\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)\, dt\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)\, ds\right)\right|$$

Where this gets labor intensive

Boy is this a complicated expression. It involves two partial derivatives of a vector-valued function, taking their cross product, then taking the magnitude. It's as if someone was *trying* to create the most complicated expression they could imagine.

Right now we have a purely theoretical expression for the area of one of these little parallelograms:

$$\left|\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)\, dt\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)\, ds\right)\right|$$

However, if you want to get a feel for what this actually entails, I encourage you to work through it.

Work it out: Given the definition of $\vec{\mathbf{v}}(t, s)$ that we started with,

$$\vec{\mathbf{v}}(t, s) = \begin{bmatrix} t^2 \\ st \\ s \end{bmatrix}$$

evaluate the expression found in the previous question to get a function in terms of t, s, dt and ds.

Area of parallelogram:

[ ] $dt\, ds$

Check

Hide explanation

Before diving into anything, notice that we can factor out the dt and ds terms from our expression. This is because they are each there to represent some length, which is constant, and both cross products and absolute values each allow us to factor out constants.

$$\left|\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A) \underbrace{dt}_{\text{Factor out}}\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A) \underbrace{ds}_{\text{Factor out}}\right)\right|$$

$$= \left|\left(\frac{\partial \vec{\mathbf{v}}}{\partial t}(t_A, s_A)\right) \times \left(\frac{\partial \vec{\mathbf{v}}}{\partial s}(t_A, s_A)\right)\right| dt\, ds$$

Next, let's compute each partial derivative.

$$\frac{\partial}{\partial t}\vec{\mathbf{v}}(t, s) = \begin{bmatrix} \frac{\partial}{\partial t}t^2 \\ \frac{\partial}{\partial t}(st) \\ \frac{\partial}{\partial t}s \end{bmatrix} = \begin{bmatrix} 2t \\ s \\ 0 \end{bmatrix}$$

and

$$\frac{\partial}{\partial s}\vec{\mathbf{v}}(t, s) = \begin{bmatrix} \frac{\partial}{\partial s}t^2 \\ \frac{\partial}{\partial s}(st) \\ \frac{\partial}{\partial s}s \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}$$

Moving right along, now compute the cross product of these two vectors using the usual determinant trick:

$$\begin{bmatrix} 2t \\ s \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ t \\ 1 \end{bmatrix}$$

$$= \det\left(\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2t & s & 0 \\ 0 & t & 1 \end{bmatrix}\right)$$

$$= ((s)(1) - 0)\hat{\mathbf{i}} + (0 - (2t)(1))\hat{\mathbf{j}} + ((2t)(t) - 0)\hat{\mathbf{k}}$$

$$= \begin{bmatrix} s \\ -2t \\ 2t^2 \end{bmatrix}$$

Finally, we compute the norm of this vector.

$$\left\| \begin{bmatrix} s \\ -2t \\ 2t^2 \end{bmatrix} \right\| = \sqrt{s^2 + (-2t)^2 + (2t^2)^2}$$

$$= \sqrt{s^2 + 4t^2 + 4t^4}$$

That is to say, when you map a tiny dt-by-ds rectangle from the parameter space onto the surface S using the function $\vec{\mathbf{v}}(t, s)$, it turns into a parallelogram with the following area:

$$\boxed{\left(\sqrt{s^2 + 4t^2 + 4t^4}\right)\, dt\, ds}$$

Step 3: Integrating everything together

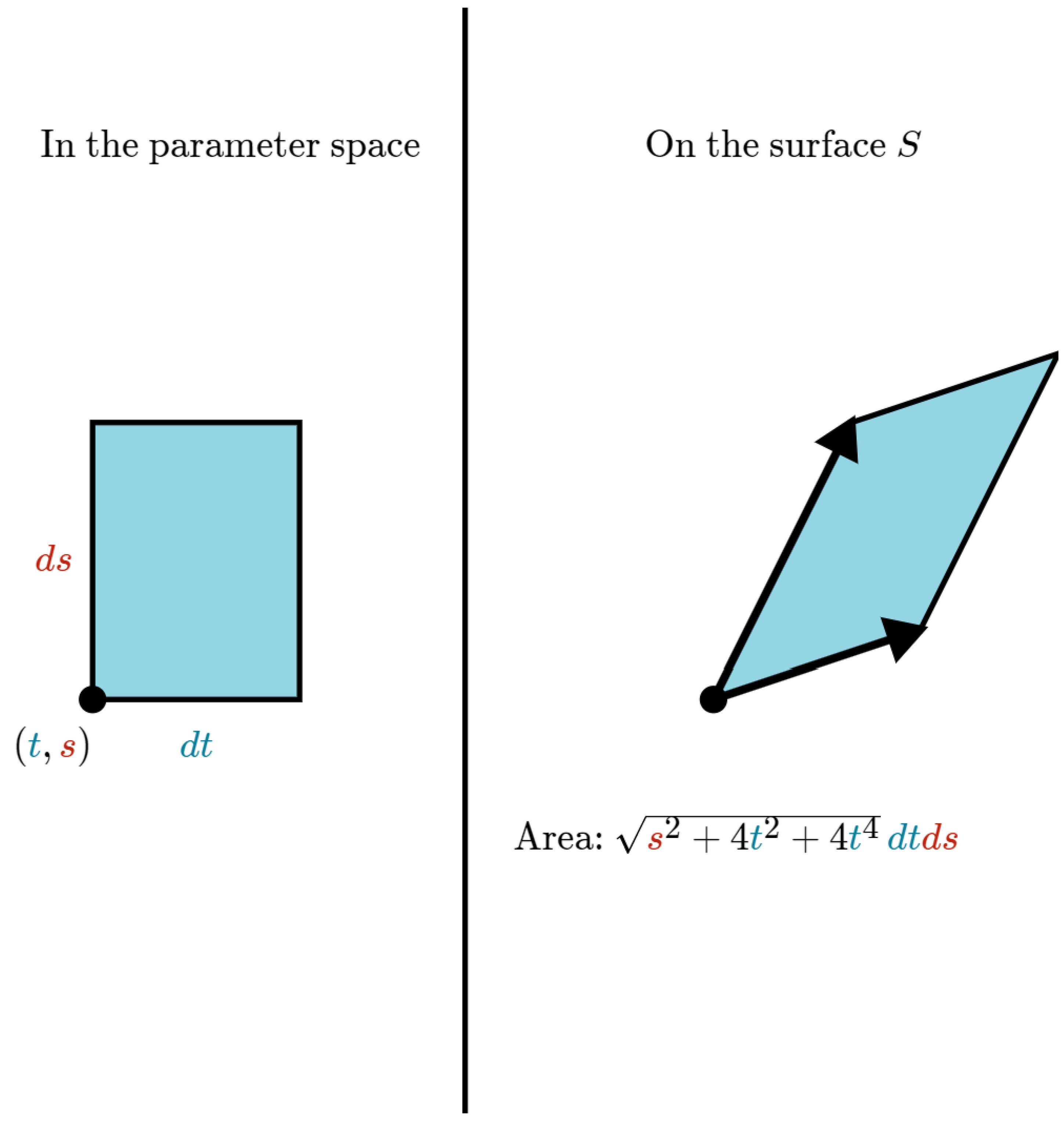
Here's where we are so far. After breaking up the rectangle T of the parameter space into many tiny little rectangles, I told you that those rectangles get turned into parallelograms on the surface S. Well, more accurately, they each get turned into some slightly curved piece of S, which can be well-approximated by a parallelogram. The smaller your initial rectangle, the more accurate the approximation.

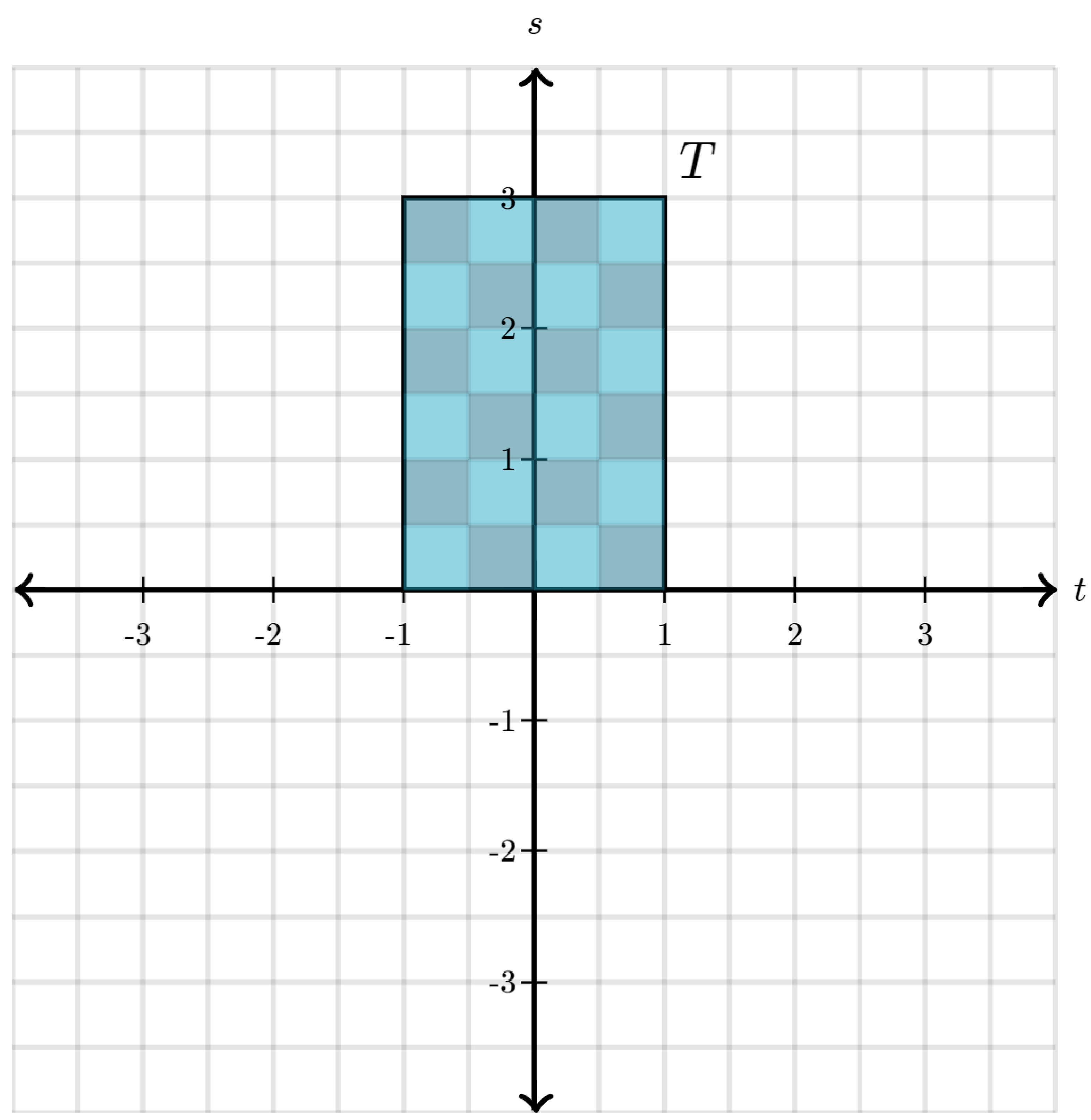
Then, through many computations, you found an expression for the area of one of these parallelograms:

$$\left(\sqrt{s^2 + 4t^2 + 4t^4}\right)\, dt\, ds$$

Where

- (t, s) describes the position of the initial little rectangle.
- dt is its width.
- ds is its height.





To add up the areas of all these little parallelograms, we take a double integral of this quantity over the region T. As a reminder, T was defined as the region where

$$-1 \leq t \leq 1$$

$$0 \leq s \leq 3$$

Using those bounds, here is the double integral representing the surface area of S:

$$\int_0^3 \int_{-1}^1 \left(\sqrt{s^2 + 4t^2 + 4t^4}\right) dt\, ds$$

Working this out by hand seems tricky, given that finding the antiderivative of $\sqrt{s^2 + 4t^2 + 4t^4}$ will be difficult. But using a calculator (or Wolfram Alpha), we can find the answer:

$$\int_0^3 \int_{-1}^1 \left(\sqrt{s^2 + 4t^2 + 4t^4}\right) dt\, ds \approx \boxed{12.6153}$$

The important thing to remember here is how to construct the appropriate double integral, and to think about adding up many tiny pieces of area on the surface itself.

Summary: This ain't easy

Generalizing everything we did in the previous example, the surface area of our parametric surface S is expressed using the integral

$$\iint_T \left| \frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s} \right| dt\, ds$$

where S is described using a parametric function $\vec{\mathbf{v}}(t, s)$ applied to a region T of the ts-plane.

You've already had a glimpse of this, but it's worth pointing out that this can be a really complicated thing to compute.

- First you have to take two partial derivatives of vector-valued functions, which if you count each component includes 6 partial derivatives in total.
- You then have to take the cross product of these two partial derivative vectors, which itself requires taking a determinant whose components are vectors and functions.
- Then you have to compute the norm of that cross product.
- After all that, there is still a double integral ahead of you. And remember, just setting up a double integral isn't always easy, especially if the region you are integrating over is not rectangular.
- And all this is assuming you already know the function $\vec{\mathbf{v}}(t, s)$ and the region T. Sometimes you are just given a surface which is implicitly defined, like a sphere defined by $x^2 + y^2 + z^2 = 1$. In that case you need to find a function which parameterizes this surface, as well as which specific

region of the parameter space corresponds to the surface.

The key when going through all of this is to stay organized, and be patient. One way to think about it is that setting up and computing just one of these surface area integrals is akin to doing 10 practice problems in single-variable calculus.

The thought process that goes into all of this is actually very useful for thinking about surfaces and three-dimensional geometry in general, beyond the specific case of computing surface area. For example, how do you think computer graphics works? Quite often, displaying a three-dimensional figure involves subdividing a surface into polygons, and getting the computer to display those polygons. Even if this never involves performing a surface area integral, per se, the reasoning associated with how to do this is remarkably similar, using cross products of partial derivatives, etc.

If you want to practice this more, the next article walks through another full example. If you do choose to work through it, prepare to mark up a lot of paper.