

Line integrals in a scalar field

 Google Classroom

Learn how to compute and interpret line integrals, also known as path integrals or curve integrals.

Background

- [Notation for integrating along a curve](#)
- [Multivariable functions](#)
- [Parametric function](#)

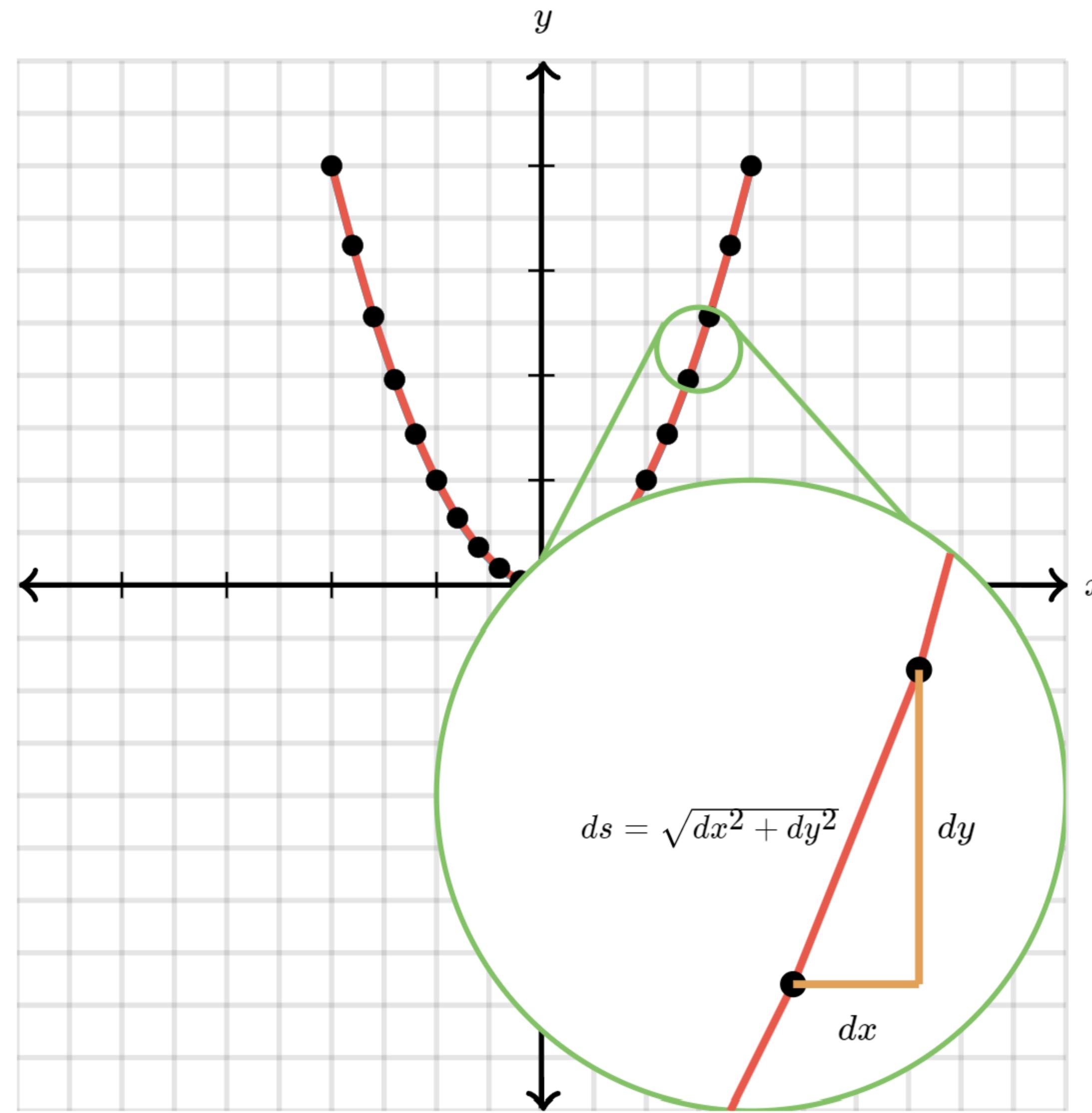
What we're building to

- In the same way that an ordinary integral $\int_a^b f(x)dx$ has you walking along the x axis and adding up certain tiny quantities as you go, a line integral $\int_C f(x, y)ds$ has you walking on a curve the xy -plane, adding up certain quantities depending on the multivariable function $f(x, y)$.
- If a curve C is parameterized by a vector value function $\vec{r}(t)$ between the values $t = a$ and $t = b$, the line integral is written as follows:

$$\int_C f ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt$$

- In this case, f is a scalar valued function, so we call this process "line integration in a scalar field", to distinguish from a related idea we'll cover next: [line integration in a vector field](#).

What is a line integral



In the [last article](#), I introduced some compact notation for an arc length integral:

$$\int_C ds$$

- The term ds represents a tiny change in length along the curve.
[\[Hide explanation\]](#)

Let's say you are given the parametric equations defining the curve:

$$x(t) = \text{yada yada yada}$$

$$y(t) = \text{blah blah blah}$$

Then ds , the tiny step along C , is given explicitly in terms of x and y using the pythagorean theorem.

$$ds = \sqrt{dx^2 + dy^2}$$

You should think of dx and dy as being the x and y components of the tiny step ds . Computationally they are the derivatives of $x(t)$ and $y(t)$:

$$dx = x'(t)dt$$

$$dy = y'(t)dt$$

- C is just a name we give to the curve. Putting it at the base of the integral, like this \int_C , is a way to postpone the need to place actual bounds on the integral.

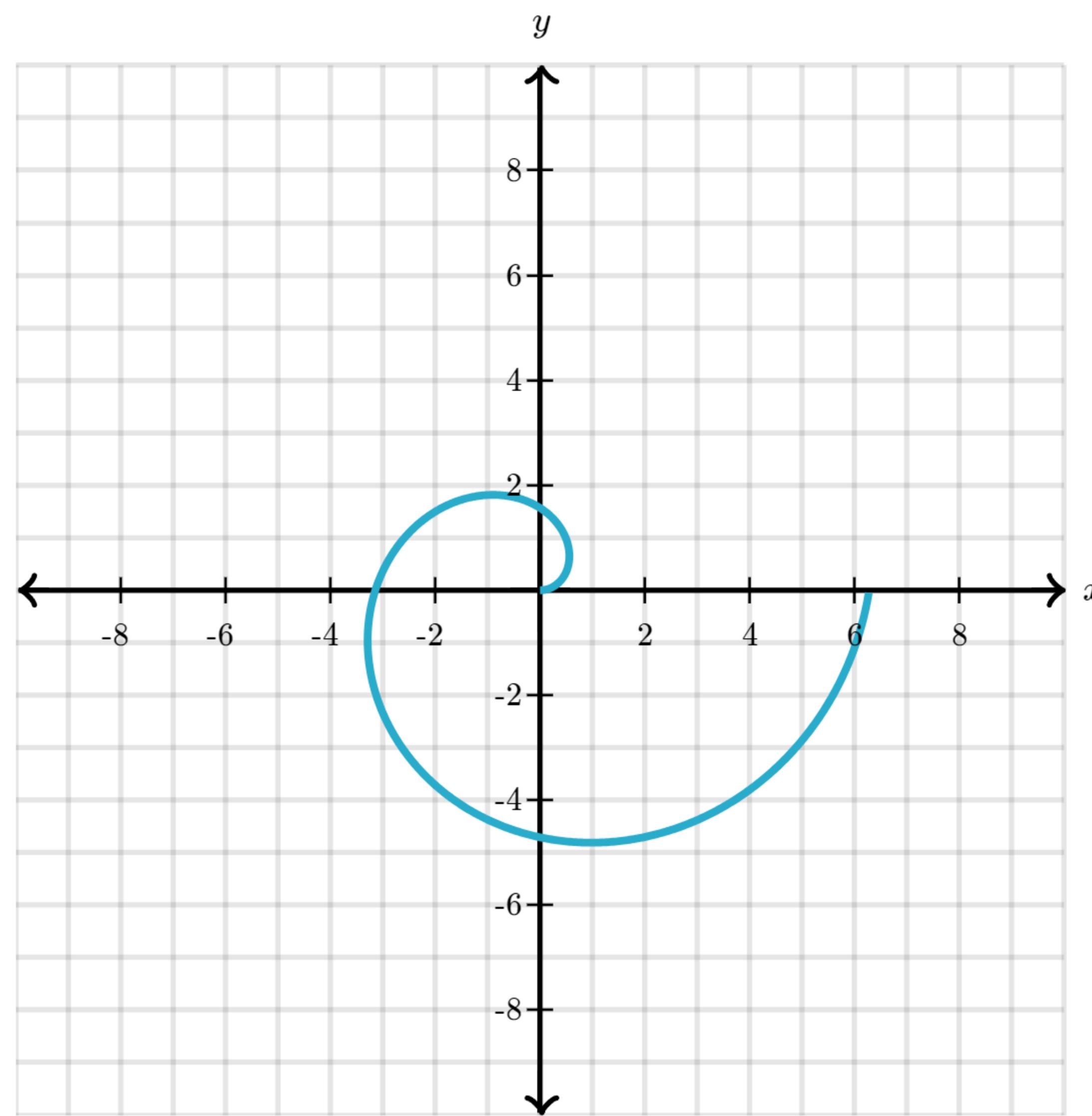
[\[Hide example\]](#)

Consider the curve defined by these parametric equations:

$$x(t) = t \cos(t)$$

$$y(t) = t \sin(t)$$

Let's say we are looking at the portion of this curve between $t = 0$ and $t = 2\pi$, which looks like this:



We start by writing down the integral

$$\int_C ds = \int_C \sqrt{dx^2 + dy^2}$$

Both dx and dy need to be written in terms of dt . To do this, we take the derivative of each parametric equation,

$$\begin{aligned} dx &= d(t \cos(t)) \\ &= t(-\sin(t))dt + \cos(t)dt \quad \leftarrow \text{Product rule} \\ &= (-t \sin(t) + \cos(t))dt \end{aligned}$$

$$\begin{aligned} dy &= d(t \sin(t)) \\ &= t(\cos(t))dt + \sin(t)dt \quad \leftarrow \text{Product rule} \end{aligned}$$

$$= (t \cos(t) + \sin(t))dt$$

Both of these values are then plugged into the integral, and we factor the dt^2 out from under the radical:

$$\begin{aligned} & \int_C \sqrt{dx^2 + dy^2} \\ &= \int_C \sqrt{((-t \sin(t) + \cos(t))dt)^2 + ((t \cos(t) + \sin(t))dt)^2} \\ &= \int_C \sqrt{(-t \sin(t) + \cos(t))^2 + (t \cos(t) + \sin(t))^2} dt \end{aligned}$$

Now that everything inside the integral is written in terms of t , we place the bounding values of t on the integral:

$$\int_0^{2\pi} \sqrt{(-t \sin(t) + \cos(t))^2 + (t \cos(t) + \sin(t))^2} dt$$

Super nasty integral, right? Welcome to the world of line integrals.

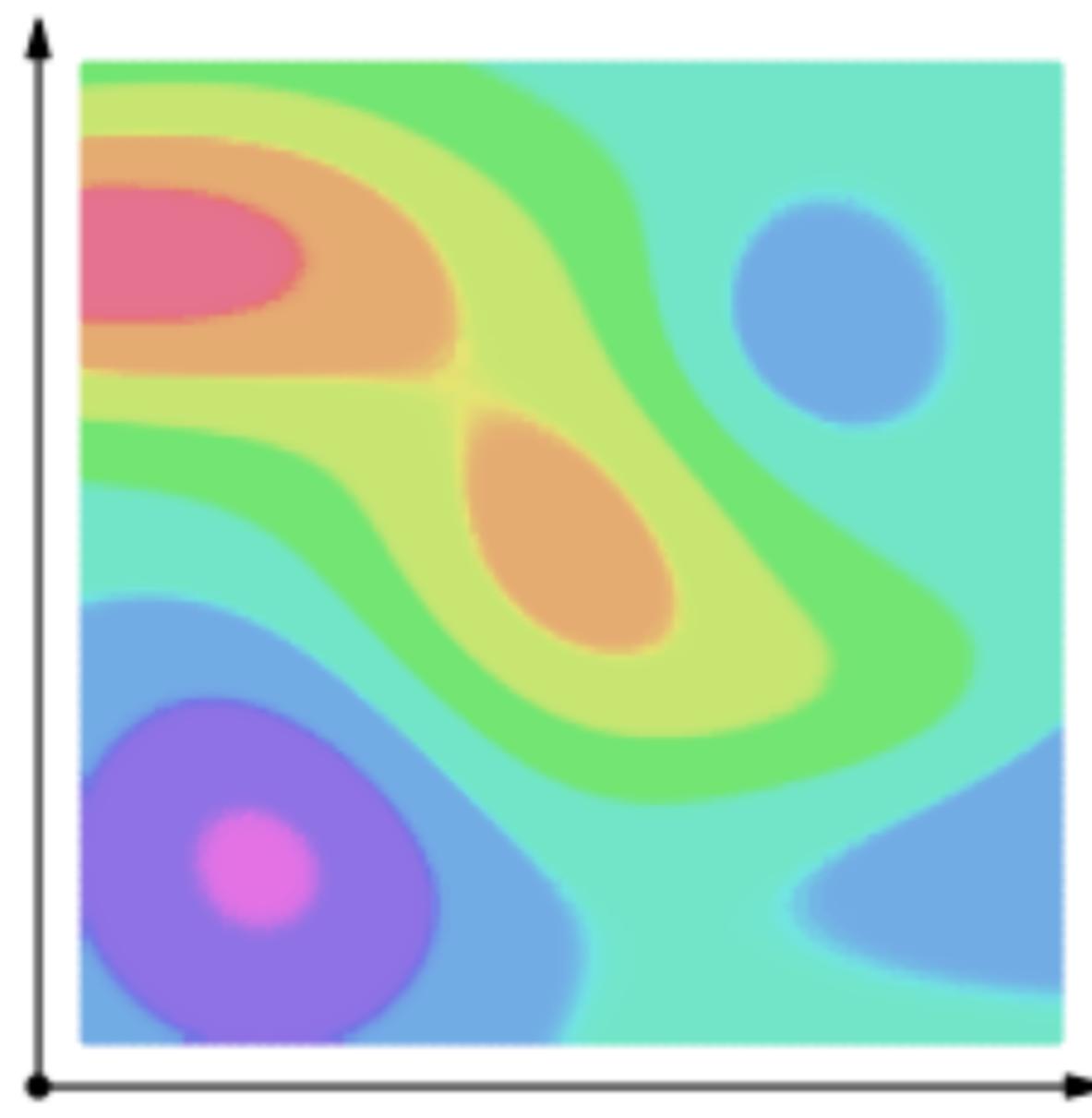
Line integrals extend this idea by placing a multivariable function inside the integral,

$$\int_C f(x, y) ds$$

You can think of this integral as saying

"Hey, while you're walking along the curve taking tiny steps of size ds , rather than just adding up the sizes of those steps, multiply each step size by the value of the function $f(x, y)$ at the point where you are standing."

The following animation relates this to the more familiar idea of finding the area under a curve. Imagine a curtain draped under the three-dimensional graph of $f(x, y)$ along the curve C in the xy -plane. The line integral gives the area of that curtain. (The initial image is a colored [contour plot](#) of the function f).



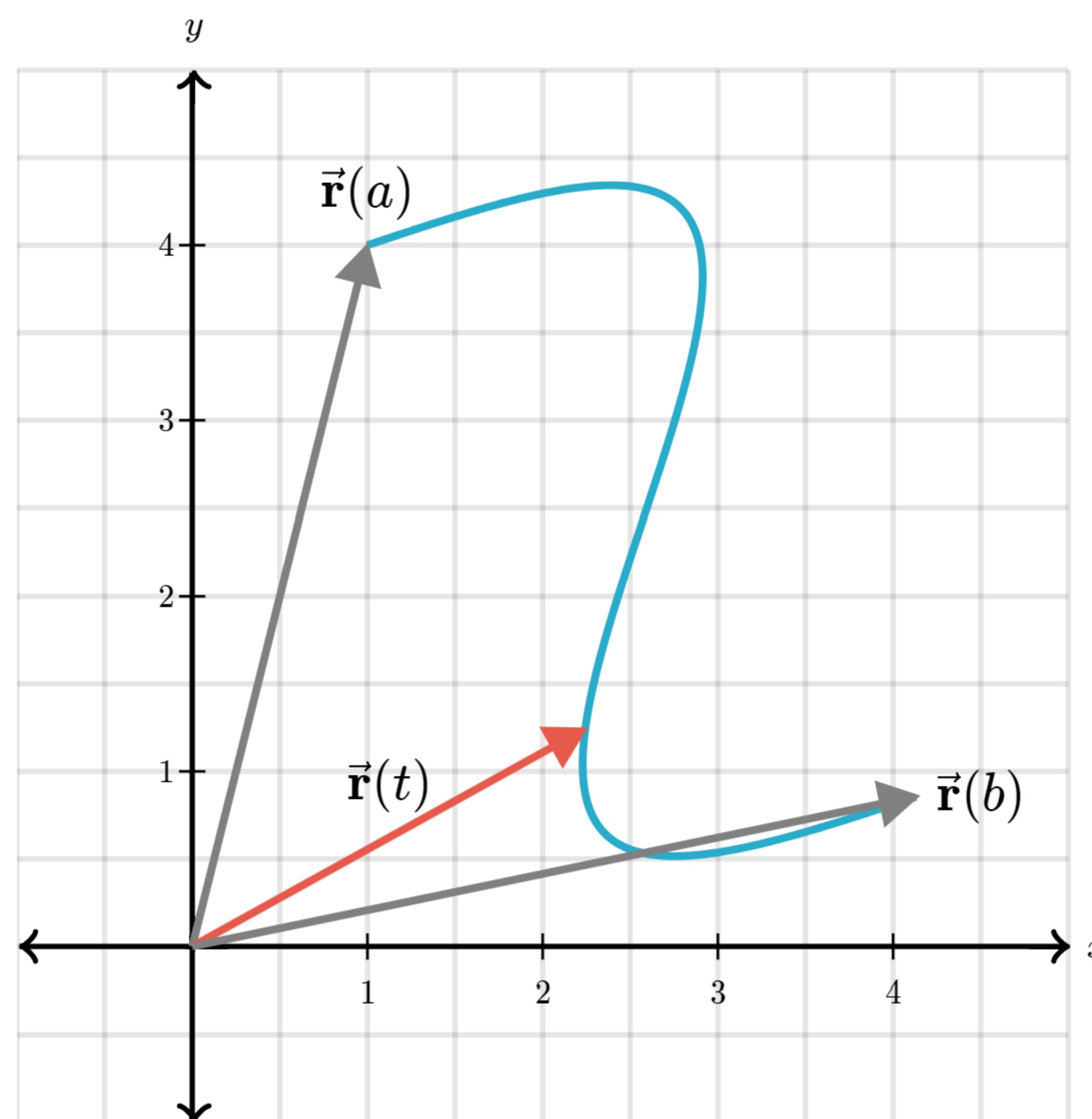
You imagine the area of that curtain being broken up into infinitely many infinitely thin rectangles. The infinitesimal base of each rectangle is $ds = |\mathbf{r}'(t)|$, the size of a tiny step along the curve. The height of each rectangle is $f(x, y)$, the height of the graph of f at that point.

Vector notation for line integrals

At the end of the animation above, the line integral is written like this:

$$\int_C f \, ds = \int_a^b f(\vec{\mathbf{r}}(t)) |\vec{\mathbf{r}}'(t)| dt$$

Let's break down what each part of this means.



The parameterization of C

$\vec{\mathbf{r}}(t)$ is some vector-valued function which parameterizes the curve C . In two dimensions, it might look something like this:

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

The bounds of the integral, a and b , are values for t which determine where the curve starts and ends.

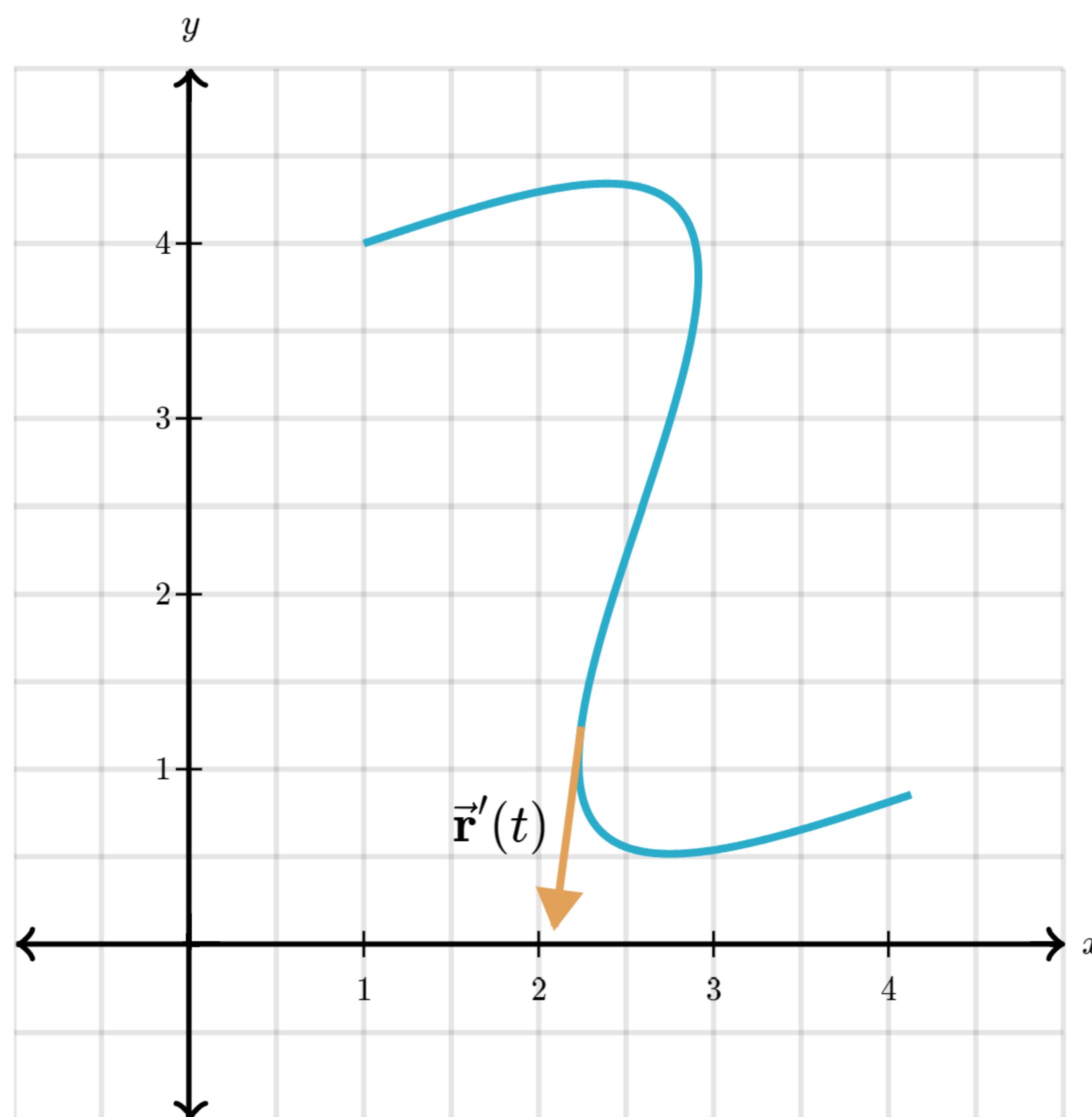
Essentially this means that as the value of t ranges between a and b , the tip of the vector $\vec{r}(t)$ traces over the curve C .

Composing f with the parameterization

Evaluating $f(\vec{r}(t))$ means taking the components $x(t)$ and $y(t)$ of $\vec{r}(t)$, and plugging them in as the inputs of $f(x, y)$:

$$f(\vec{r}(t)) = f(x(t), y(t))$$

You can think of it this way: A given value of t puts us at some point on the xy -plane, expressed as the tip of the vector $\vec{r}(t)$. Then $f(\vec{r}(t))$ gives the value of the function f at that point on the plane.



ds is the magnitude of the derivative (times dt)

The term ds , representing a tiny step along the curve, is expanded as $|\vec{r}'(t)|dt$, the magnitude of the derivative of $\vec{r}(t)$ times dt .

Intuitively, this is because the derivative answers the question "what happens if you nudge the input slightly by a value dt ?" The answer is that the output

gets nudged along the vector $\vec{r}'(t)dt$ in the xy -plane. The magnitude of this nudge in the output space gives you the size of a step ds along the curve, as a function of the size of the step in the parameter space dt .

You can also see this as the compact vector-notation way of saying $\sqrt{dx^2 + dy^2}$:

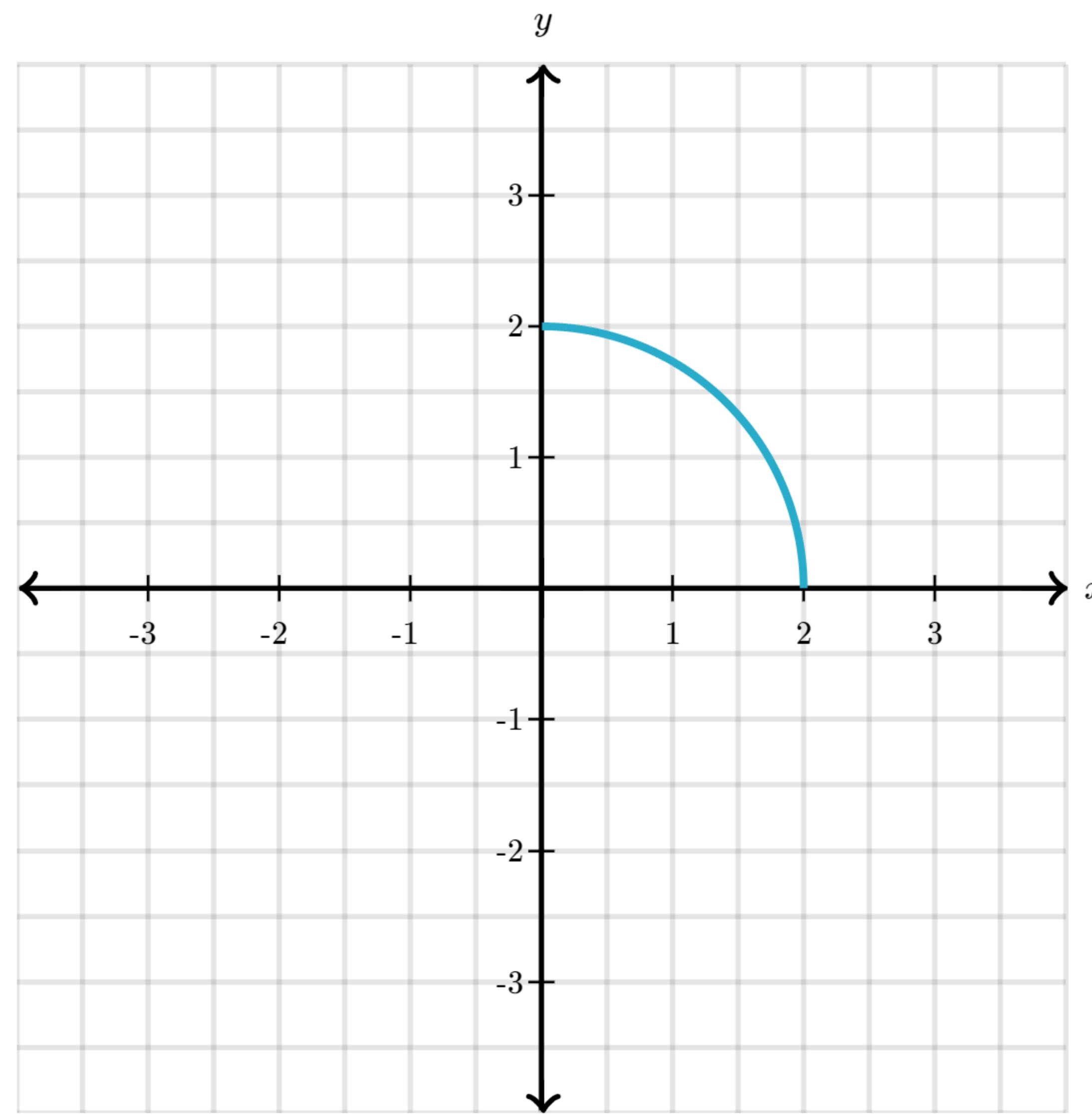
$$\begin{aligned} |\vec{r}'(t)|dt &= \left| \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \right| dt \\ &= \sqrt{(x'(t))^2 + (y'(t))^2} dt \\ &= \sqrt{(x'(t)dt)^2 + (y'(t)dt)^2} \\ &= \sqrt{(dx)^2 + (dy)^2} \end{aligned}$$

Putting all this together, we get the vector representation of a line integral:

$$\int_C \textcolor{teal}{f} \, \textcolor{red}{ds} = \int_a^b \textcolor{teal}{f}(\vec{r}(t)) |\vec{r}'(t)| dt$$

Example 1: Compute a simple line integral

Let C be one quarter of a circle with radius 2 centered at the origin. The quarter in the first quadrant to be specific.



We can describe this quarter circle parametrically with the following function:

$$\vec{r}(t) = \begin{bmatrix} 2 \cos(t) \\ 2 \sin(t) \end{bmatrix}$$

If we let t range from 0 to $\pi/2$, this traces around C .

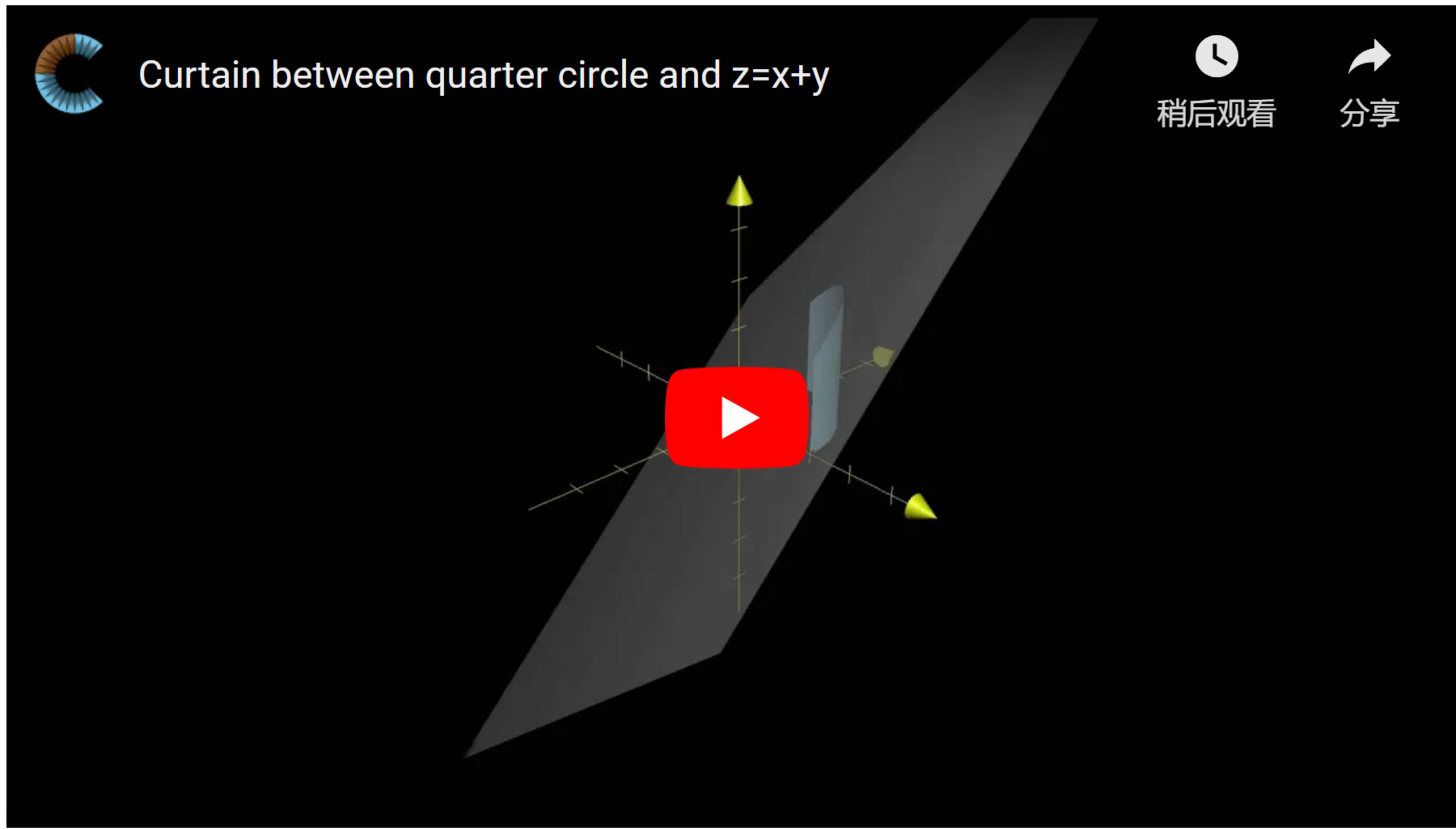
Now defined the multivariable function f as

$$f(x, y) = x + y$$

Our goal is to compute the line integral

$$\int_C f(x, y) ds$$

The following video shows what the "curtain" beneath the graph of f along the curve C looks like. The translucent white plane is the graph of $f(x, y) = x + y$, and the blue surface is the curtain whose area we are computing.

[See video transcript](#)

Step 1: Write ds in terms of dt

Given that the size of a tiny step ds along the curve is given by the magnitude of the derivative of $\vec{r}(t)$ times dt ,

$$ds = |\vec{r}'(t)|dt$$

solve for ds in our example.

For reference, the parameterization of our curve is $\vec{r}(t) = \begin{bmatrix} 2\cos(t) \\ 2\sin(t) \end{bmatrix}$

$$ds = \boxed{} dt$$

[Check](#)

[\[Hide explanation\]](#)

$$ds = |\vec{r}'(t)| dt$$

$$\begin{aligned} &= \left| \begin{bmatrix} \frac{d}{dt} 2 \cos(t) \\ \frac{d}{dt} 2 \sin(t) \end{bmatrix} \right| dt \\ &= \left| \begin{bmatrix} -2 \sin(t) \\ 2 \cos(t) \end{bmatrix} \right| dt \\ &= \sqrt{(-2 \sin(t))^2 + (2 \cos(t))^2} dt \\ &= \sqrt{2^2(\sin^2(t) + \cos^2(t))} dt \\ &= \sqrt{2^2(1)} dt \\ &= 2 dt \end{aligned}$$

In this case, it turns out to be a simple constant multiple of dt . This corresponds with the fact that the given parameterization of C has us walking along the curve at a constant rate.

Step 2: Write $f(x, y)$ in terms of t

What is $f(\vec{r}(t))$ in this case?

$$f(\vec{r}(t)) = \boxed{\quad}$$

[Check](#)

[\[Hide explanation\]](#)

Every point on the curve C is of the form

$$(2 \cos(t), 2 \sin(t))$$

Evaluating $f(x, y) = x + y$ at such a point, we get

$$f(2 \cos(t), 2 \sin(t)) = 2 \cos(t) + 2 \sin(t)$$

Step 3: Write the integral completely in terms of t and solve

From the previous two steps, our integral becomes

$$\int_C f(x, y) \, ds = \int_C (2 \cos(t) + 2 \sin(t)) \, 2dt$$

Since our parameterization of C has t running from 0 to $\frac{\pi}{2}$, these are the bounds of the integral. Now solve the integral.

$$\int_0^{\pi/2} (2 \cos(t) + 2 \sin(t)) \, 2dt = \boxed{\quad}$$

[Check](#)

[\[Hide explanation\]](#)

$$\begin{aligned}\int_0^{\pi/2} (2 \cos(t) + 2 \sin(t)) \, 2dt &= 4 \int_0^{\pi/2} (\cos(t) + \sin(t)) dt \\ &= 4 [\sin(t) - \cos(t)]_0^{\pi/2} \\ &= 4 \left(\left(\sin\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right) \right) - (\sin(0) - \cos(0)) \right) \\ &= 4((1 - 0) - (0 - 1)) \\ &= 8\end{aligned}$$

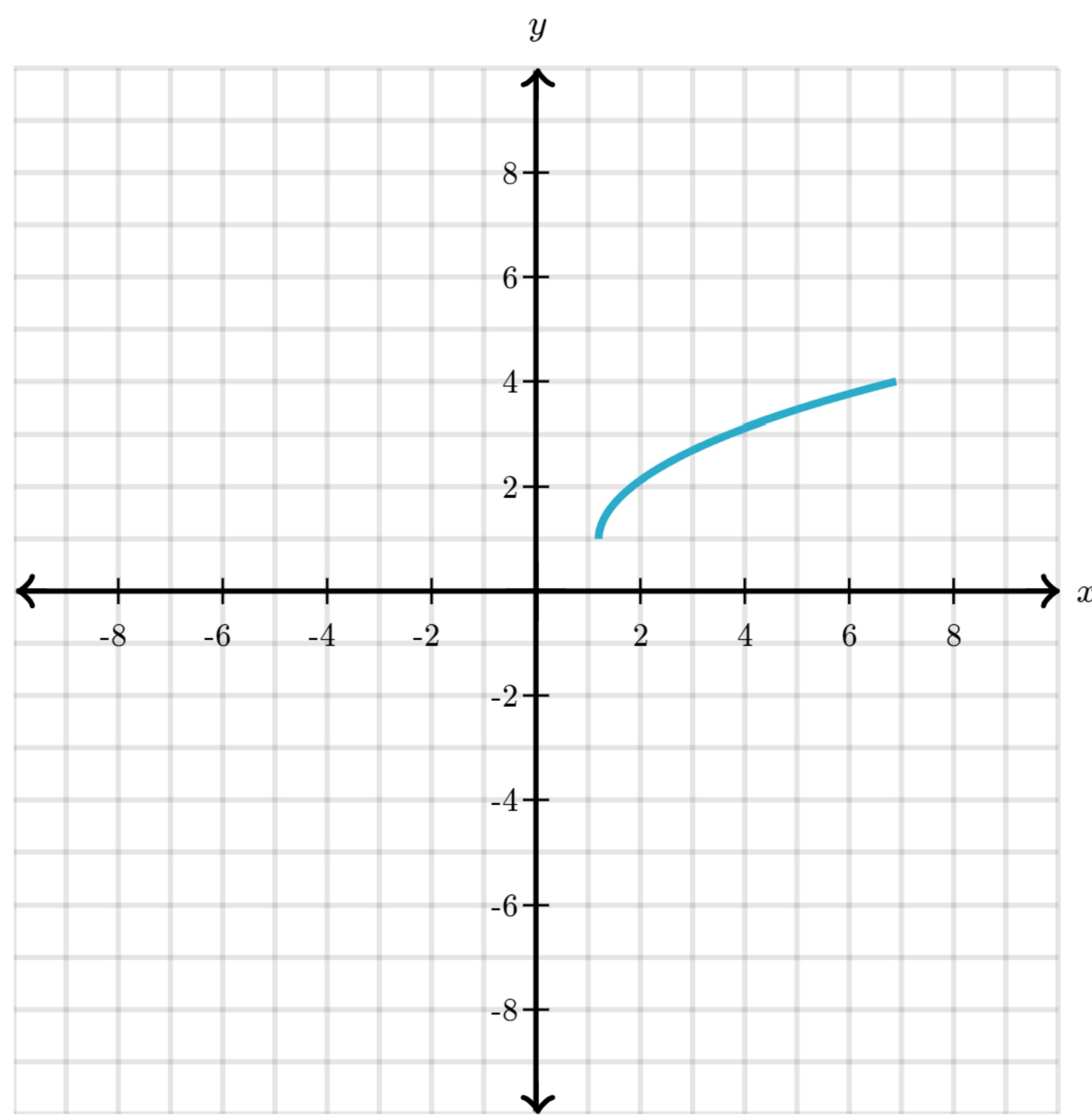


Example 2: More intricate practice

In principle, line integration is not too bad once you get the hang of it. You just have to know how to expand the term ds , and to rewrite the inputs of the function $f(x, y)$ in terms of the parameterization. Knowing how to set this up just takes a little practice, which is what we're doing here.

That said, line integrals can be a real pain to actually compute. Most of them end up in a state where you need to plug the integral into a computer to get an answer, but even when the integral is solvable the numbers involved can quickly grow cumbersome.

The following example shows how even for relatively simple functions, computing a line integral to its completion can be a very involved problem. You will definitely want to have a pencil and scratch paper ready if you choose to go through this example.



Let C be the curve defined by the function

$$\vec{s}(t) = \begin{bmatrix} \frac{1}{t} + \frac{1}{5}t^5 \\ t^2 \end{bmatrix}$$

on the interval $1 \leq t \leq 2$.

Let f be the function

$$f(x, y) = xy^2$$

Compute the line integral

$$\int_C f(x, y) ds$$

Step 1: Write ds in terms of dt

$$ds = \boxed{} dt$$

[Check](#)

[\[Hide explanation\]](#)

$$ds = |\vec{s}'(t)|dt$$

$$= \left| \begin{bmatrix} \frac{d}{dt} \left(\frac{1}{t} + \frac{1}{5}t^5 \right) \\ \frac{d}{dt}(t^2) \end{bmatrix} \right| dt$$

$$= \left| \begin{bmatrix} -\frac{1}{t^2} + t^4 \\ 2t \end{bmatrix} \right| dt$$

$$= \sqrt{\left(-\frac{1}{t^2} + t^4 \right)^2 + (2t)^2} dt$$

$$= \sqrt{\frac{1}{t^4} - 2\frac{1}{t^2}t^4 + t^8 + 4t^2} dt$$

$$= \sqrt{\frac{1}{t^4} - 2t^2 + t^8 + 4t^2} dt$$

$$= \sqrt{\frac{1}{t^4} + 2t^2 + t^8} dt$$

$$= \sqrt{\left(\frac{1}{t^2} + t^4 \right)^2} dt$$

$$= \left(\frac{1}{t^2} + t^4 \right) dt$$

By the way, the function under the radical almost never factors so nicely into a square. For practice problems like this, those of us writing the practice problems tend to purposefully engineer them to come out simply.

More often than not, a random line integral you come up with will not give an easily solvable integral.

Step 2: Replace $f(x, y)$ with $f(\vec{s}(t))$

What do you get when you plug the components of $\vec{s}(t)$ into $f(x, y) = xy^2$?

$$f(\vec{s}(t)) = \boxed{\quad}$$

[Check](#)

[\[Hide explanation\]](#)

$$f(\vec{s}(t)) = f\left(\frac{1}{t} + \frac{1}{5}t^5, t^2\right)$$

$$= \left(\frac{1}{t} + \frac{1}{5}t^5\right)(t^2)^2$$

$$= t^3 + \frac{1}{5}t^9$$

Step 3: Solve the integral

Plug the answers from the previous two steps into the integral, then solve. Since the curve is defined for $1 \leq t \leq 2$, and our integral is now given with respect to t , the bounds of the integral are 1 and 2.

(Warning, this gets super ugly to compute, so feel free to put it in a calculator when the time comes.)

$$\int_1^2 f(\vec{s}(t)) |\vec{s}'(t)| dt = \boxed{\quad}$$

[Check](#)

[\[Hide explanation\]](#)

$$\begin{aligned}\int_1^2 f(\vec{s}(t)) |\vec{s}'(t)| dt &= \int_1^2 \left(t^3 + \frac{1}{5}t^9\right) \left(\frac{1}{t^2} + t^4\right) dt \\ &= \int_1^2 \left(t + \frac{1}{5}t^7 + t^7 + \frac{1}{5}t^{13}\right) dt \\ &= \frac{1}{5} \int_1^2 (5t + 6t^7 + t^{13}) dt \\ &= \frac{1}{5} \left(\frac{5}{2}t^2 + \frac{6}{8}t^8 + \frac{1}{14}t^{14}\right)_1 \\ &= \frac{1}{5} \left(\frac{5}{2}(2)^2 + \frac{6}{8}(2)^8 + \frac{1}{14}(2)^{14}\right) - \frac{1}{5} \left(\frac{5}{2}(1)^2 + \frac{6}{8}(1)^8\right) \\ &= \text{Ugh! Just throw this in a calculator}\end{aligned}$$

$$= \frac{38,331}{140}$$

Notice, the hard part of this problem is not the new principles of line integration, knowing how to expand ds and all that. What makes it hard is that the complexity of the terms involved quickly blows up.

(Also, if you think this is bad, just wait until we get to surface integrals.)

Line integrals in a scalar field

In everything written above, the function f is a scalar-valued function, meaning it outputs a number (as opposed to a vector). There is a slight variation on line integrals, where you can integrate a vector-valued function along a curve, which we will cover in the [next article](#).

To distinguish these ideas, everything we just covered is typically called **line integration in a scalar field** and the alternative is referred to as **line integration in a vector field**. The term "scalar field" is just another way of thinking about what a multivariable function does: It associates each point in the xy -plane with some scalar (i.e. number), so that the entire plane is like a field of numbers just waiting for someone to wander along a path through that field and integrate those values.

[\[Hide explanation\]](#)

Line integral animations from [wikipedia](#), originally by Lucas V. Barbosa.