Second partial derivative test

Google Classroom

Learn how to test whether a function with two inputs has a local maximum or minimum.

Background

- Maximums, minimums, and saddle points
- Second partial derivatives

Not strictly necessary, but used in one section:

- The Hessian matrix

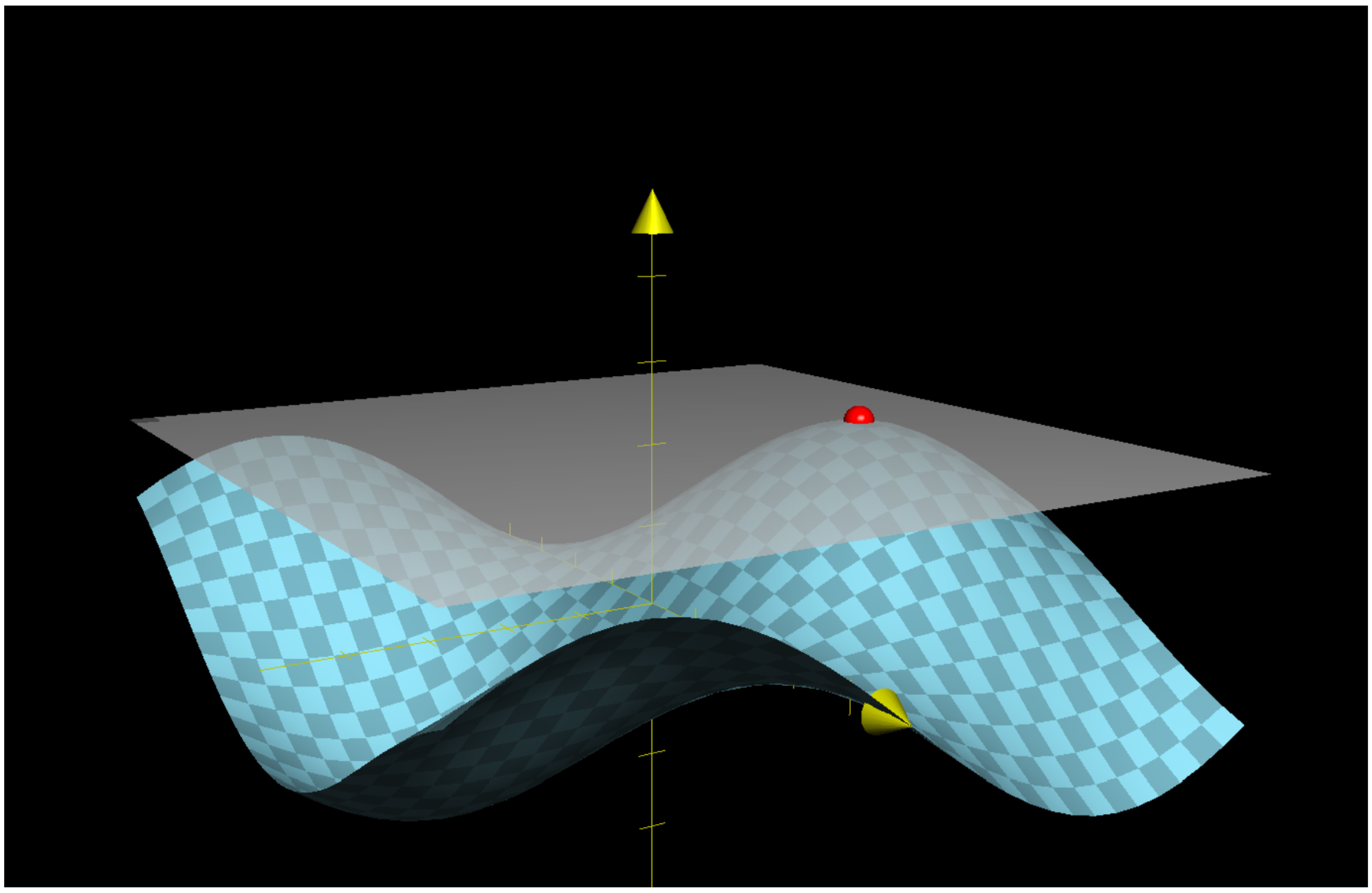
Also, if you are a little rusty on the second derivative test from single-variable calculus, you might want to quickly review it here since it's a good comparison for the second *partial* derivative test. *[Quick review of second derivative test]*

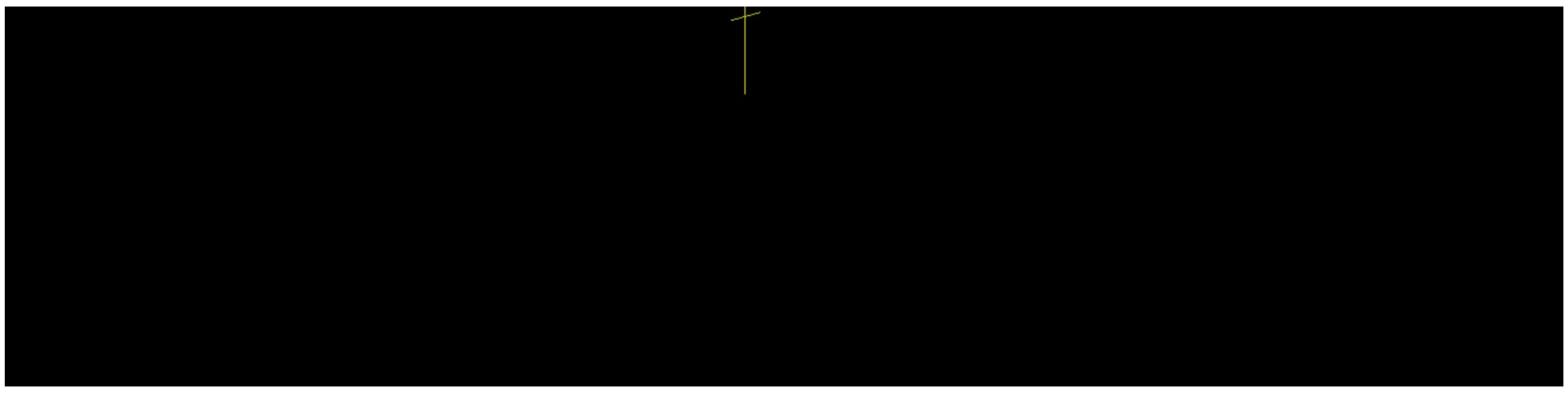
The statement of the second partial derivative test

If you are looking for the local maxima/minima of a two-variable function $f(x, y)$, the first step is to find input points (x_0, y_0) where the gradient is the $\mathbf{0}$ vector.

$$\nabla f(x_0, y_0) = \mathbf{0}$$

These are basically points where the tangent plane on the graph of f is flat.
[Hide explanation]



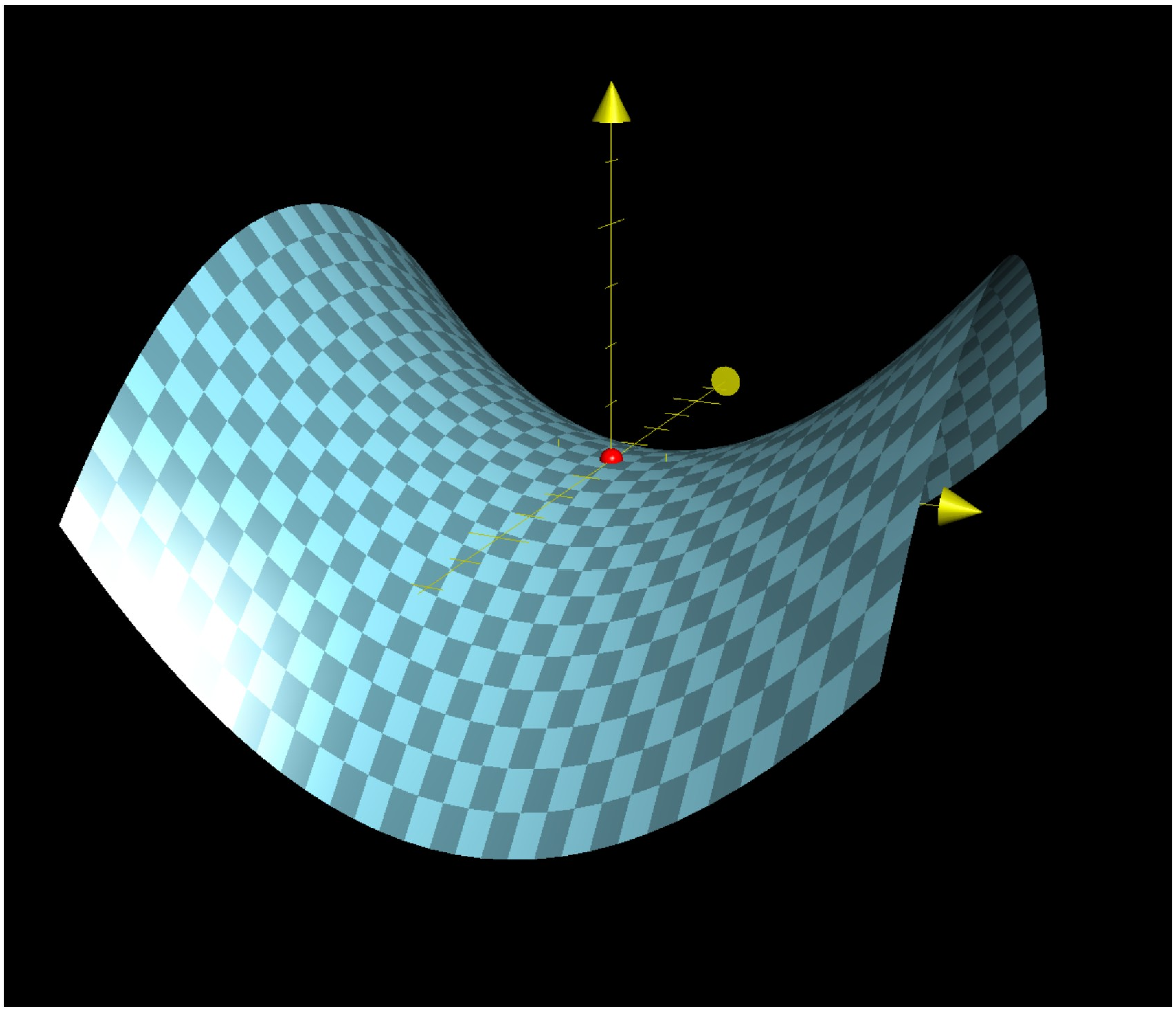


The **second partial derivative test** tells us how to verify whether this stable point is a local maximum, local minimum, or a saddle point. Specifically, you start by computing this quantity:

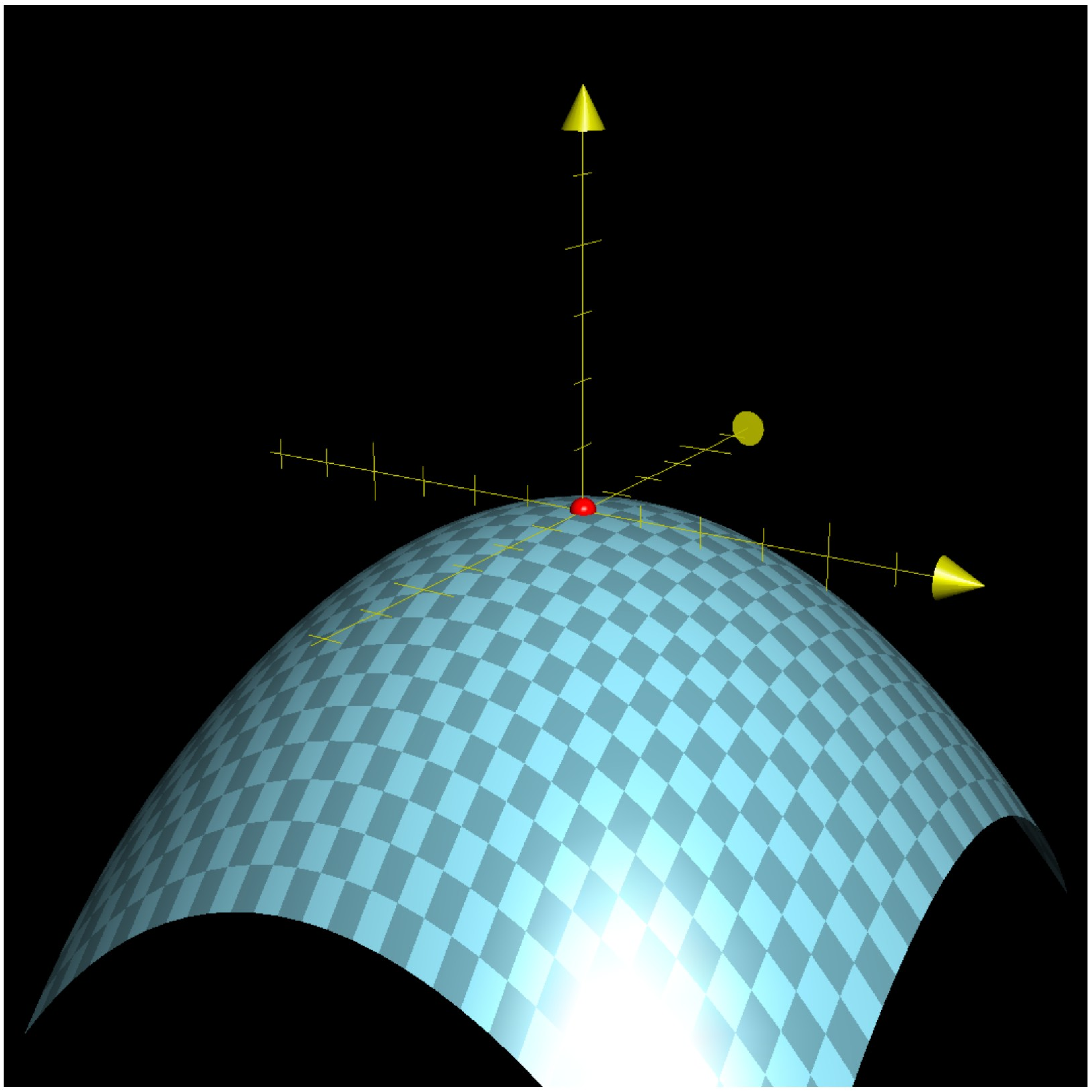
$$H = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$$

Then the second partial derivative test goes as follows:

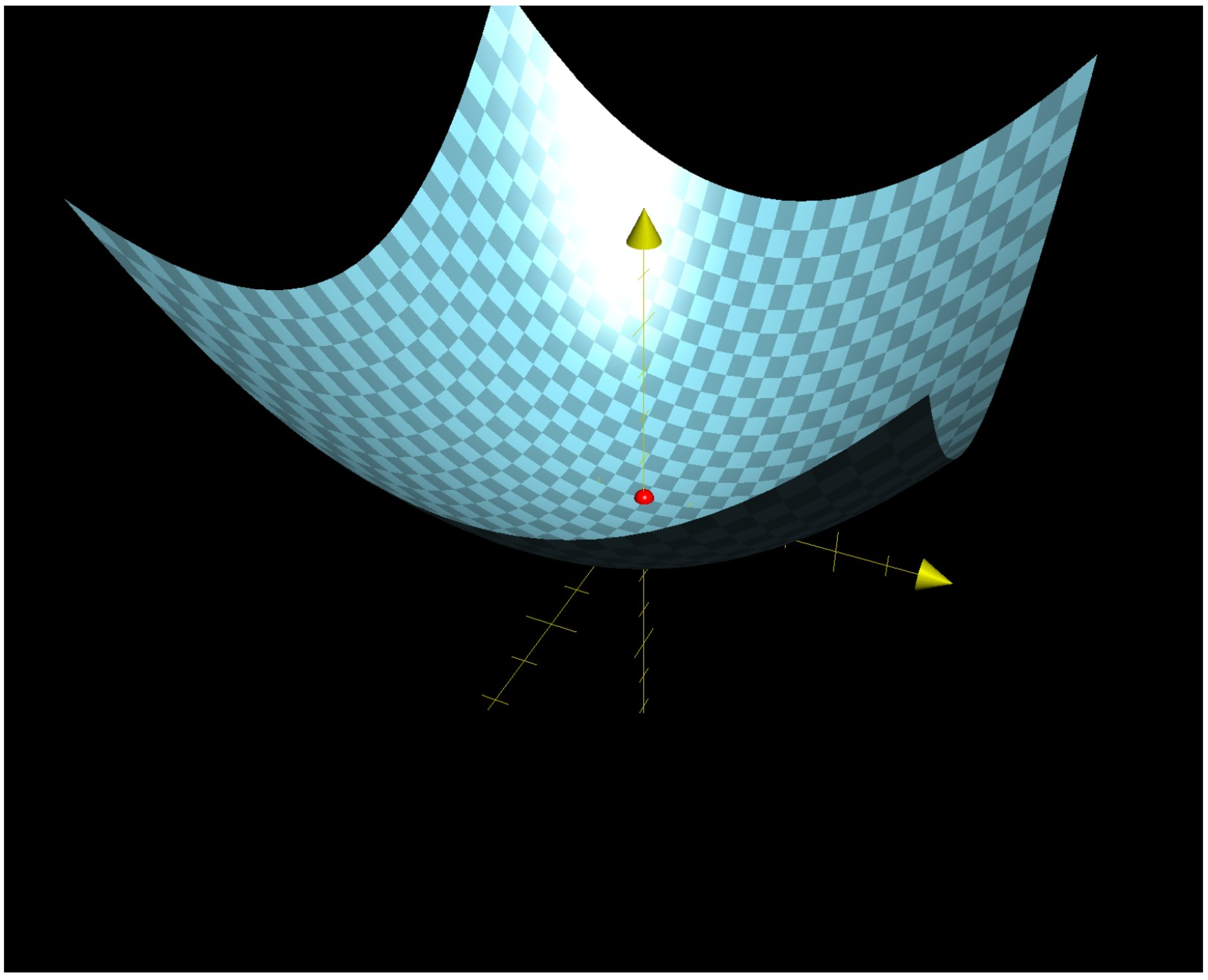
- If $H < 0$, then (x_0, y_0) is a saddle point. [Hide explanation]



- If $H > 0$, then (x_0, y_0) is either a maximum or a minimum point, and you ask one more question:
  - If $f_{xx}(x_0, y_0) < 0$, (x_0, y_0) is a local maximum point. [Hide explanation]



- If $f_{xx}(x_0, y_0) > 0$, (x_0, y_0) is a local minimum point.
  [Hide explanation]



(You could also use $f_{yy}(x_0, y_0)$ instead of $f_{xx}(x_0, y_0)$, it actually doesn't matter)

- If $H = 0$, we do not have enough information to tell.

[Hide explanation]

Hessian determinant

Believe it or not, this key expression $f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$ was not just handed down from the gods of math. It is actually the determinant of the Hessian matrix:

$$\begin{aligned} \det\left(\mathbf{H}f(x_0, y_0)\right) &= \det\left(\begin{bmatrix} f_{xx}(x_0, y_0) & f_{yx}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}\right) \\ &= f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0) f_{yx}(x_0, y_0) \\ &= \boxed{f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2} \end{aligned}$$

In a different tutorial, when I talk about the connections between multivariable calculus and linear algebra, you'll see why this determinant has anything to do with maxima and minima . For now, knowing this fact does two things:

- It can help you remember what the key expression for the second derivative test is without making little mistakes (e.g. do you add the square of the mixed partial derivative or subtract it?).

- It lets us write the second derivative test much more elegantly:

  - $\det\left(\mathbf{H}f(x_0, y_0)\right) < 0$ means f has a saddle point.
  - $\det\left(\mathbf{H}f(x_0, y_0)\right) > 0$ means f has a local maximum or minimum.

  - $\det\left(\mathbf{H}f(x_0, y_0)\right) = 0$ means second derivatives alone don't tell us enough to draw a conclusion.

Even without linear algebra, we can get a loose intuition for why this expression works as we want it to.

Loose intuition

$$\underset{\text{in x-direction}}{\text{Concavity}} \quad \underset{\text{in y-direction}}{\text{Concavity}}$$

$$\underbrace{\overbrace{f_{xx}(x_0, y_0)}\,\overbrace{f_{yy}(x_0, y_0)}}_{\substack{\text{Positive only when x and y} \\ \text{directions agree on concavity direction}}} \quad - \quad \underbrace{f_{xy}(x_0, y_0)^2}_{\substack{\text{How much } f \text{ looks} \\ \text{like } g(x,\, y) \;=\; xy}}$$

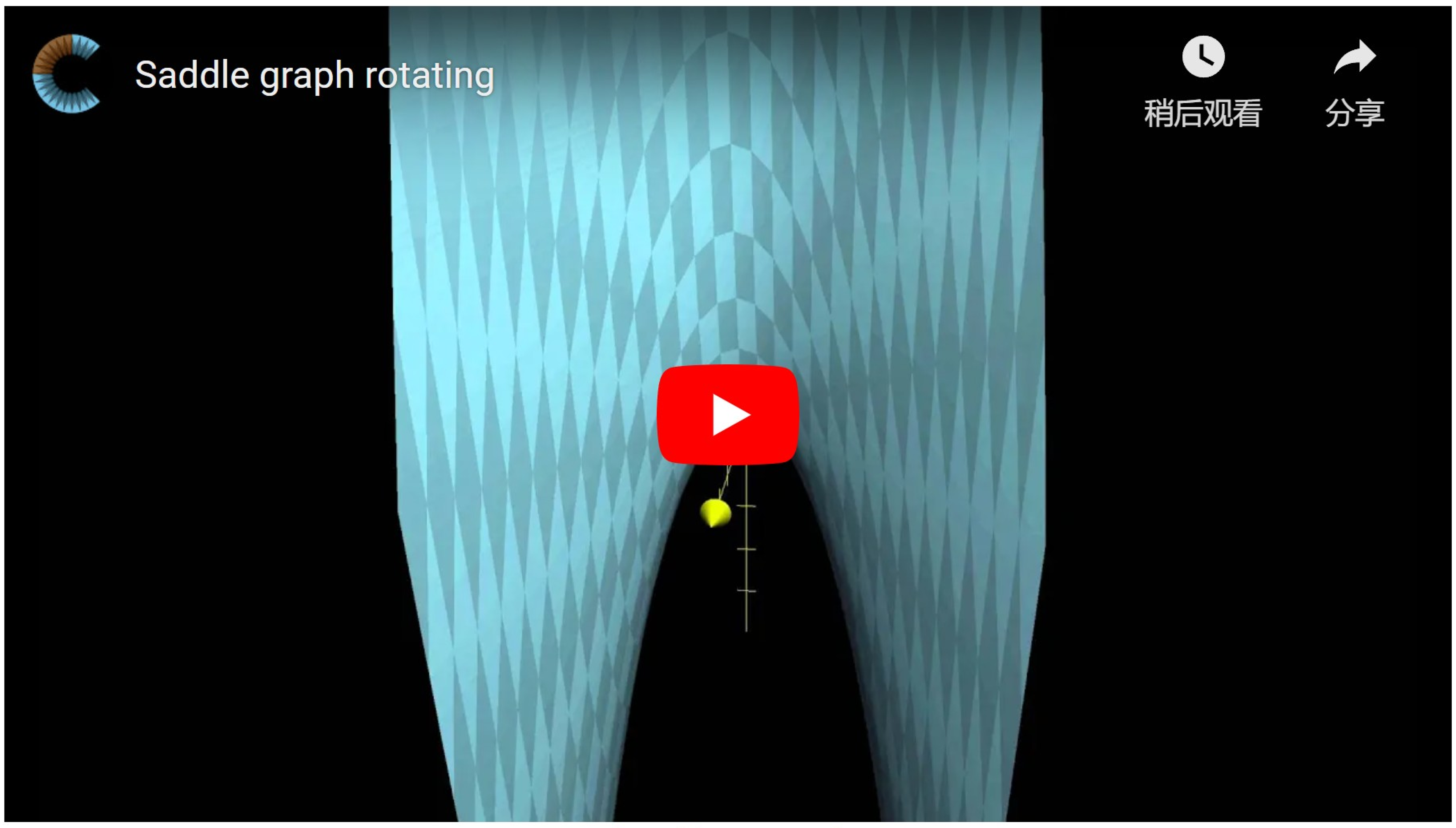
Focus first on this term:

$$f_{xx}(x_0, y_0) f_{yy}(x_0, y_0)$$

You can think of it as cleverly encoding whether or not the concavity of f's graph is the same in both the x and y directions.

For example, look at the function

$$f(x, y) = x^2 - y^2$$



See video transcript

This function has a saddle point at $(x, y) = (0, 0)$. The second partial derivative with respect to x is a positive constant:

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\frac{\partial}{\partial x}(x^2 - y^2)$$

$$= \frac{\partial}{\partial x} 2x$$

$$= 2 > 0$$

In particular, $f_{xx}(0, 0) = 2 > 0$, and the fact that this is positive means $f(x, y)$ looks like it has upward concavity as we travel in the x-direction. On the other hand, the second partial derivative with respect to y is a negative constant:

$$\begin{aligned} f_{yy}(x, y) &= \frac{\partial}{\partial y}\frac{\partial}{\partial y}(x^2 - y^2) \\ &= \frac{\partial}{\partial y} - 2y \\ &= -2 < 0 \end{aligned}$$

This indicates downward concavity as we travel in the y-direction. This mismatch means we must have a saddle point, and it is encoded as the product of the two second partial derivatives:

$$f_{xx}(0,0)f_{yy}(0,0) = (2)(-2) = -4 < 0$$

Since $f_{xy}(0{,}0)^2$ can only be positive, subtracted it will only make the full expression more negative.

$$f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0{,}y_0)^2$$

On the other hand, when the signs of $f_{xx}(x_0, y_0)$ and $f_{yy}(y_0, y_0)$ are either both positive or both negative, the x and y directions agree about what the concavity of f should be. In either of these cases, the term $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0)$ will be positive.

But this is not enough!

The f_{xy}^2 term

Consider the function

$$f(x, y) = x^2 + y^2 + pxy$$

where p is some constant.

Concept check: With this definition of f, compute its second derivatives:

$f_{xx}(x, y) =$ ☐

$f_{yy}(x, y) =$ ☐

$f_{xy}(x, y) =$

Check

Hide explanation

$$f_{xx}(x, y) = \frac{\partial}{\partial x}\frac{\partial}{\partial x}(x^2 + y^2 + pxy)$$

$$= \frac{\partial}{\partial x}(2x + py)$$

$$= 2$$

$$f_{yy}(x, y) = \frac{\partial}{\partial y}\frac{\partial}{\partial y}(x^2 + y^2 + pxy)$$

$$= \frac{\partial}{\partial y}(2y + px)$$

$$= 2$$

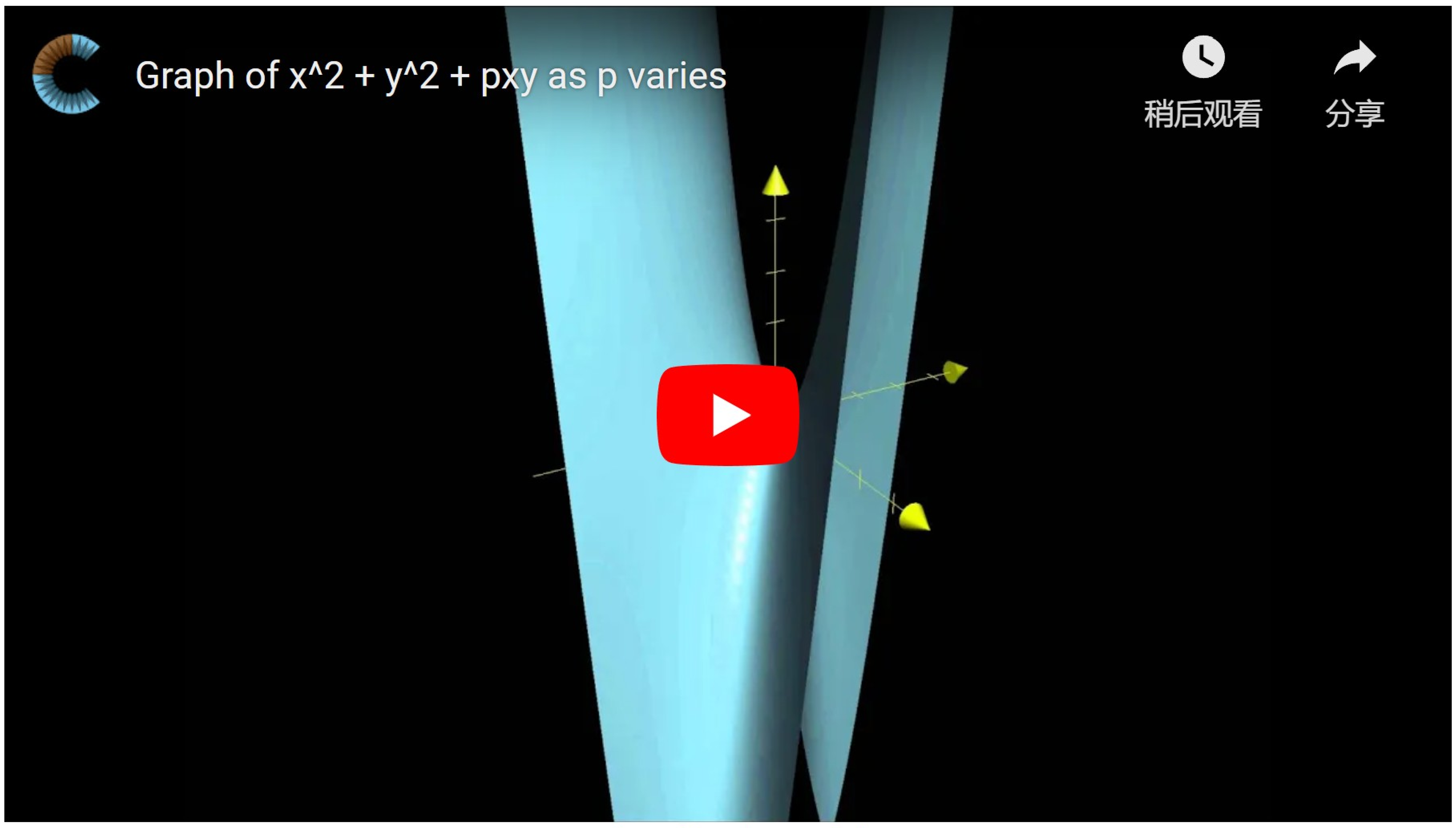
$$f_{xy}(x, y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x}(x^2 + y^2 + pxy)$$

$$= \frac{\partial}{\partial y}(2x + py)$$

$$= p$$

Because the second derivatives $f_{xx}(0, 0)$ and $f_{yy}(0, 0)$ are both positive, the graph will appear concave up as we travel in either the pure x direction or the pure y direction (no matter what p is).

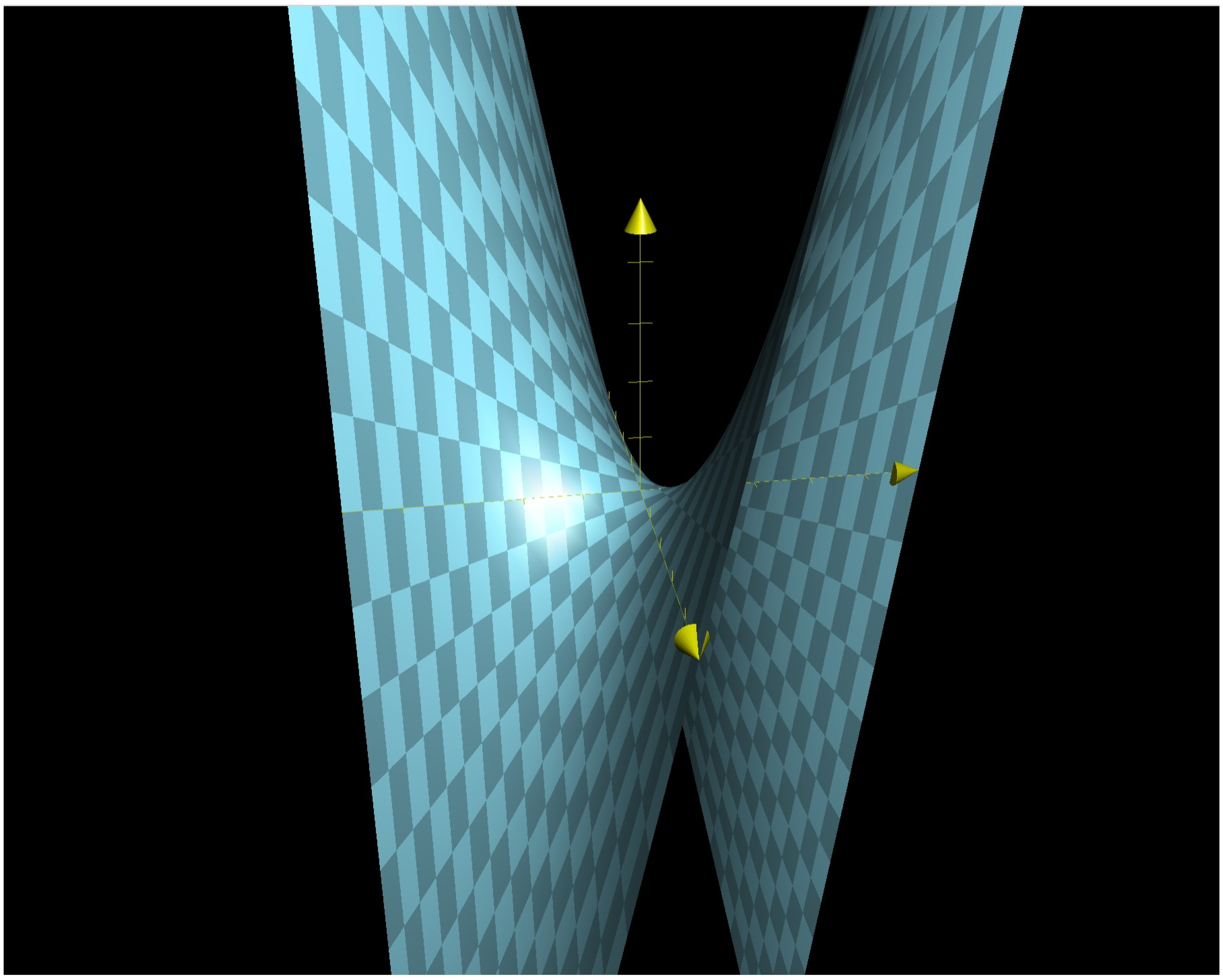
However, watch the following video where we show how this graph changes as we let the constant p vary from 1 to 3, then back to 1:



See video transcript

What's going on here? How can the graph have a saddle point even though it is concave up in both the x and y directions? The short answer is that other directions matter too, and in this case, they are captured by the term pxy.

For example, if we isolate this xy term and look at the graph of $g(x, y) = xy$, here's what it looks like:



Graph of g(x, y) = xy. Very similar to the graph of $x^2 - y^2$, but rotated 45° and expanded a bit.

It has a saddle point at $(0,0)$. This is not because the x and y directions disagree about concavity, but instead because the concavity appears positive along the diagonal direction $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and negative in the direction $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Let's see what the second derivative test tells us about the function $f(x,y) = x^2 + y^2 + pxy$. Using the values for the second derivatives you were asked to compute above, Here's what we get:

$$f_{xx}(0,0) f_{yy}(0,0) - f_{xy}(0,0)^2 = (2)(2) - p^2$$

When $p > 2$, this is negative, so f has a saddle point. When $p < 2$, it is positive, so f has a local minimum.

You can think of the quantity $f_{xy}(x_0, y_0)$ as measuring how much the function f looks like the graph of $g(x,y) = xy$ near the point (x_0, y_0).

Considering how many directions have to agree with each other, it is actually quite surprising that we only need to consider three values, $f_{xx}(0,0)$, $f_{yy}(0,0)$ and $f_{xy}(0,0)$.

The next article gives more detailed reasoning behind the second partial derivative test.

Summary

- Once you find a point where the gradient of a multivariable function is the zero vector, meaning the tangent plane of the graph is flat at this point, the second partial derivative test is a way to tell if that point is a local maximum, local minimum, or a saddle point.

- The key term of the second partial derivative test is this:

$$H = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$$

- If $H > 0$, the function definitely has a local maximum/minimum at the point (x_0, y_0).
  - If $f_{xx}(x_0, y_0) > 0$, it is a minimum.
  - If $f_{xx}(x_0, y_0) < 0$, it is a maximum.

- If $H < 0$, the function definitely has a saddle point at (x_0, y_0).

- If $H = 0$, there is not enough information to tell.