Double integrals in polar coordinates

Google Classroom

If you have a two-variable function described using polar coordinates, how do you compute its double integral?

Background

- Polar coordinates (video)
- Double integrals beyond volume

What we're building to

- When you are performing a double integral,

$$\iint_R f\,dA$$

if you wish to express the function f and the bounds for the region R in polar coordinates (r, θ), the way to expand the tiny area dA is

$$dA = r\,d\theta\,dr$$

(Pay attention to the fact that the variable r is part of this expression)

- Beyond that one rule, these double integrals are mostly about being careful to make sure the bounds of your integrals appropriately encode the region R.

- Integrating using polar coordinates is handy whenever your function or your region have some kind of rotational symmetry. For example, polar coordinates are well-suited for integration in a disk, or for functions including the expression $x^2 + y^2$.

Example 1: Tiny areas in polar coordinates

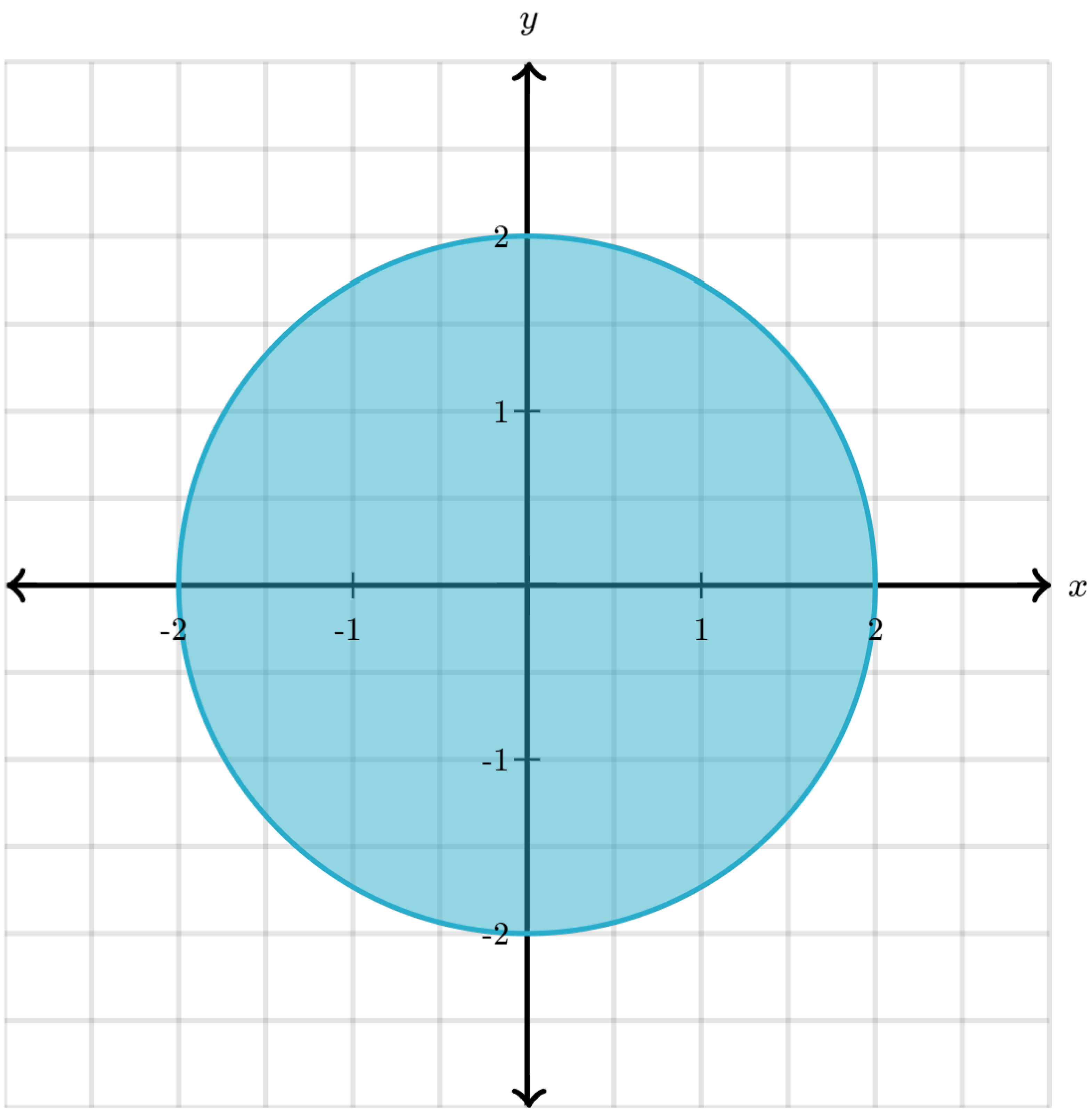
Suppose we have a multivariable function defined using the polar coordinates r and θ,

$$f(r, \theta) = r^2$$

And let's say you want to find the double integral of this function in the region where

$$r \leq 2$$

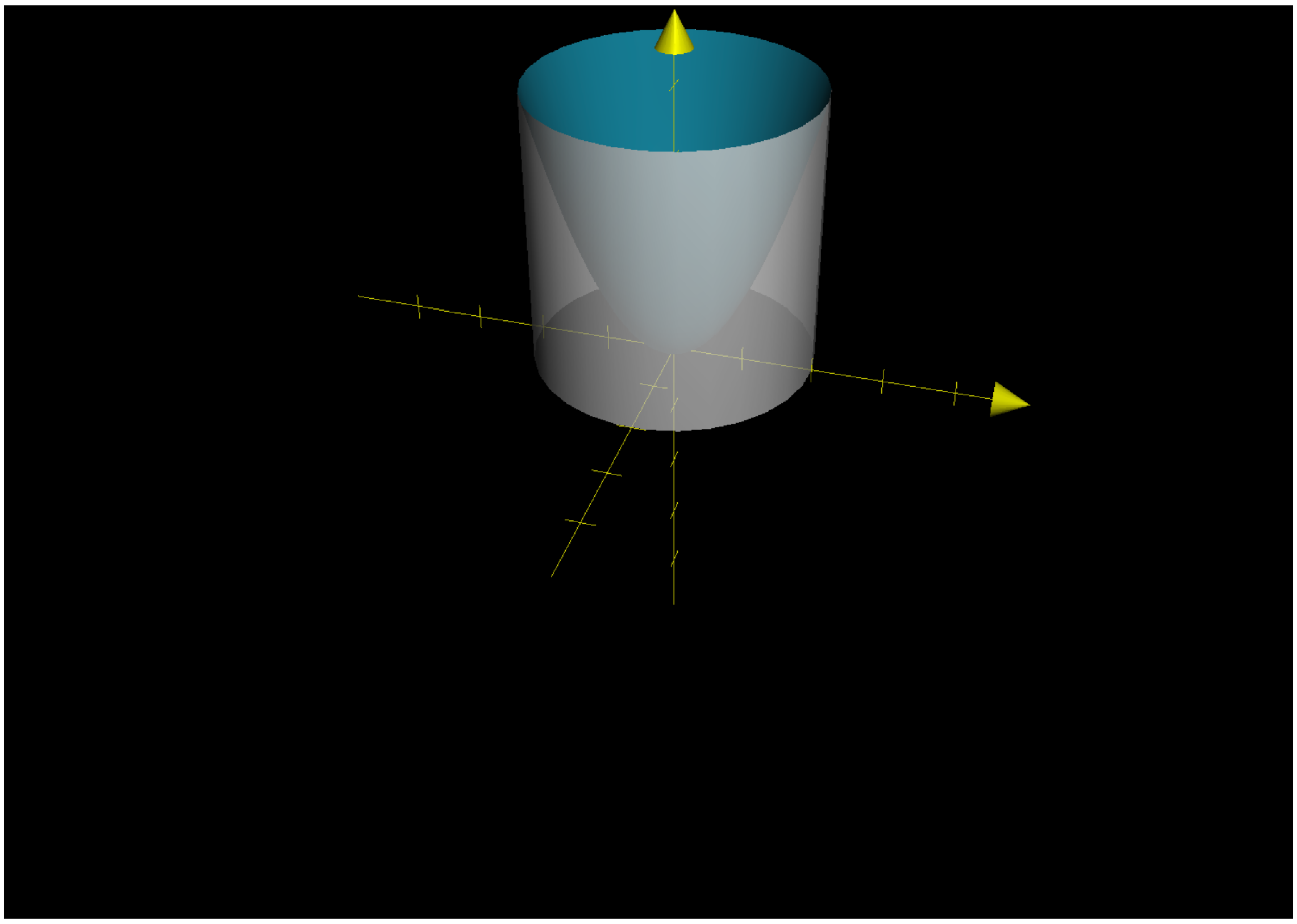
This is a disc of radius 2 centered at the origin.



Written abstractly, here's what this double integral might look like:

$$\iint_{r \leq 2} r^2 \, dA$$

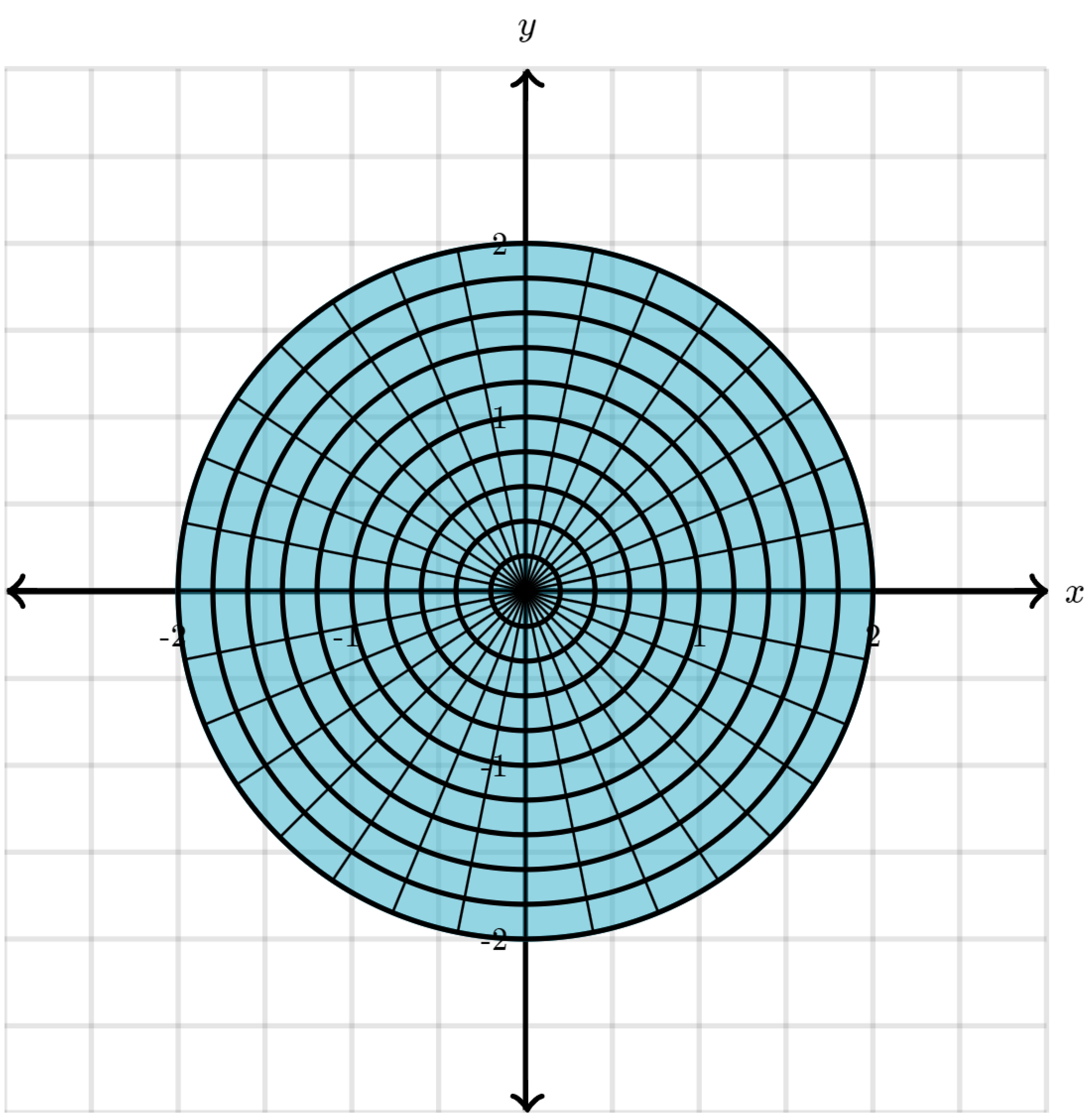
You could interpret this as the volume underneath a paraboloid (the three-dimensional analog of a parabola), as pictured below:



The question is, what do we do with that dA term?

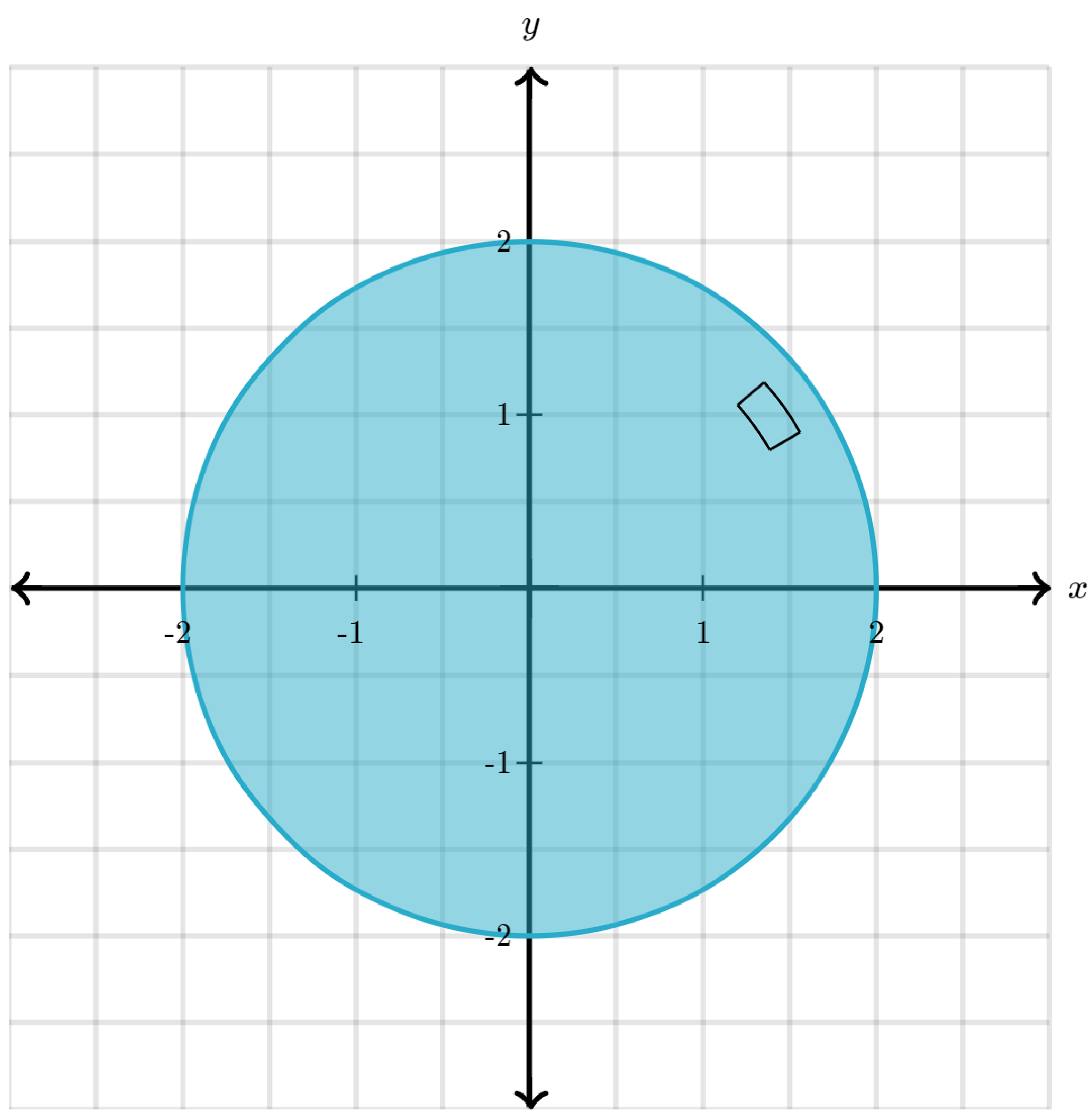
> **Warning!**: You might be tempted to replace dA with $d\theta\, dr$, since in cartesian coordinates we replace it with $dx\, dy$. But this is not correct!

Remember what a double integral is doing: It chops up the region that we are integrating over into tiny pieces, and dA represents the area of each one of those pieces. For example, chopping up our disc of radius 2 might look like this:

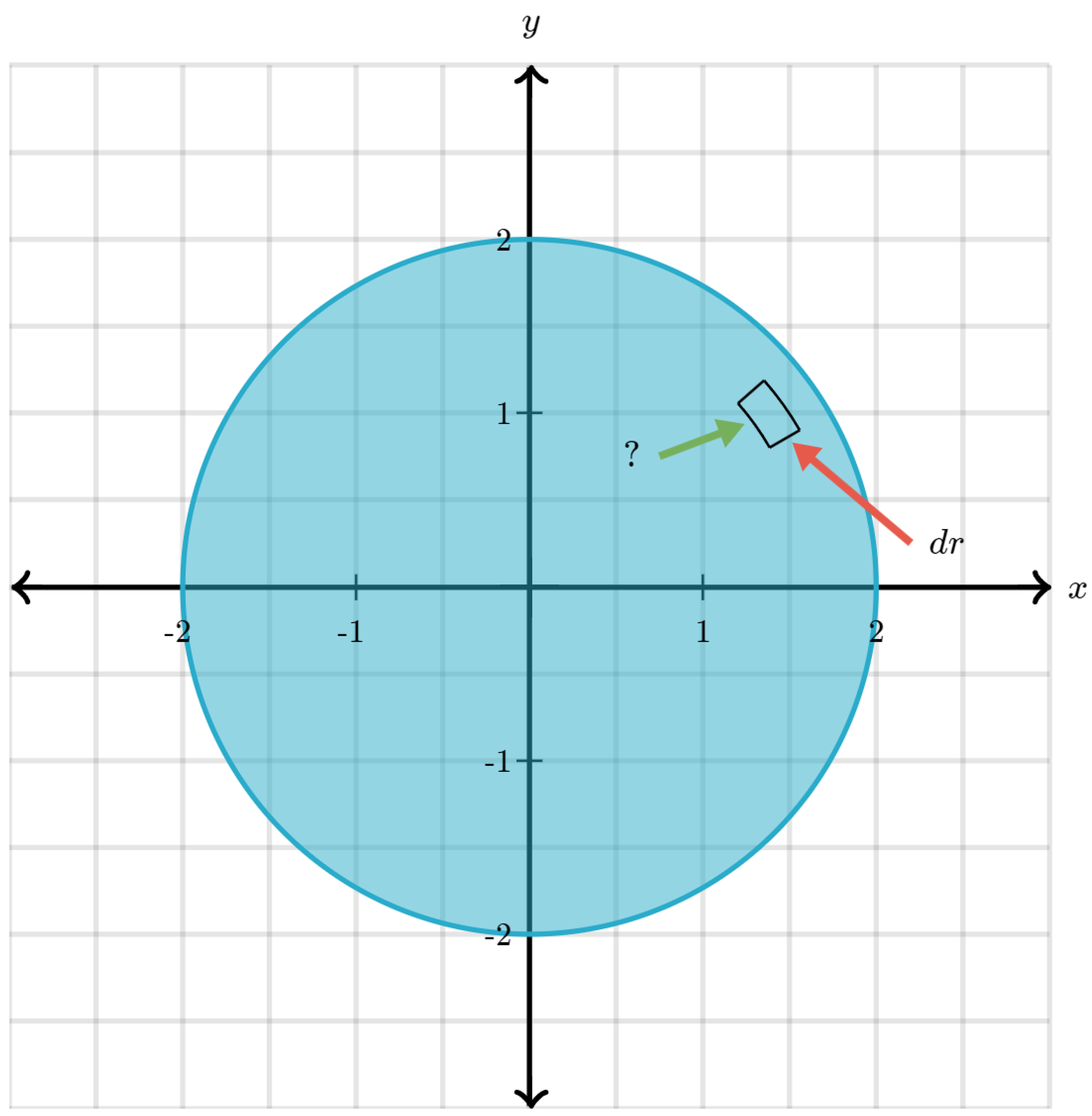


Why did I choose to chop it in this spiderweb pattern, as opposed to using vertical and horizontal lines? Since we are in polar coordinates, it will be easiest to think about the tiny pieces if their edges represent either a constant r value or a constant θ value.

Let's focus on just one of these little chunks:



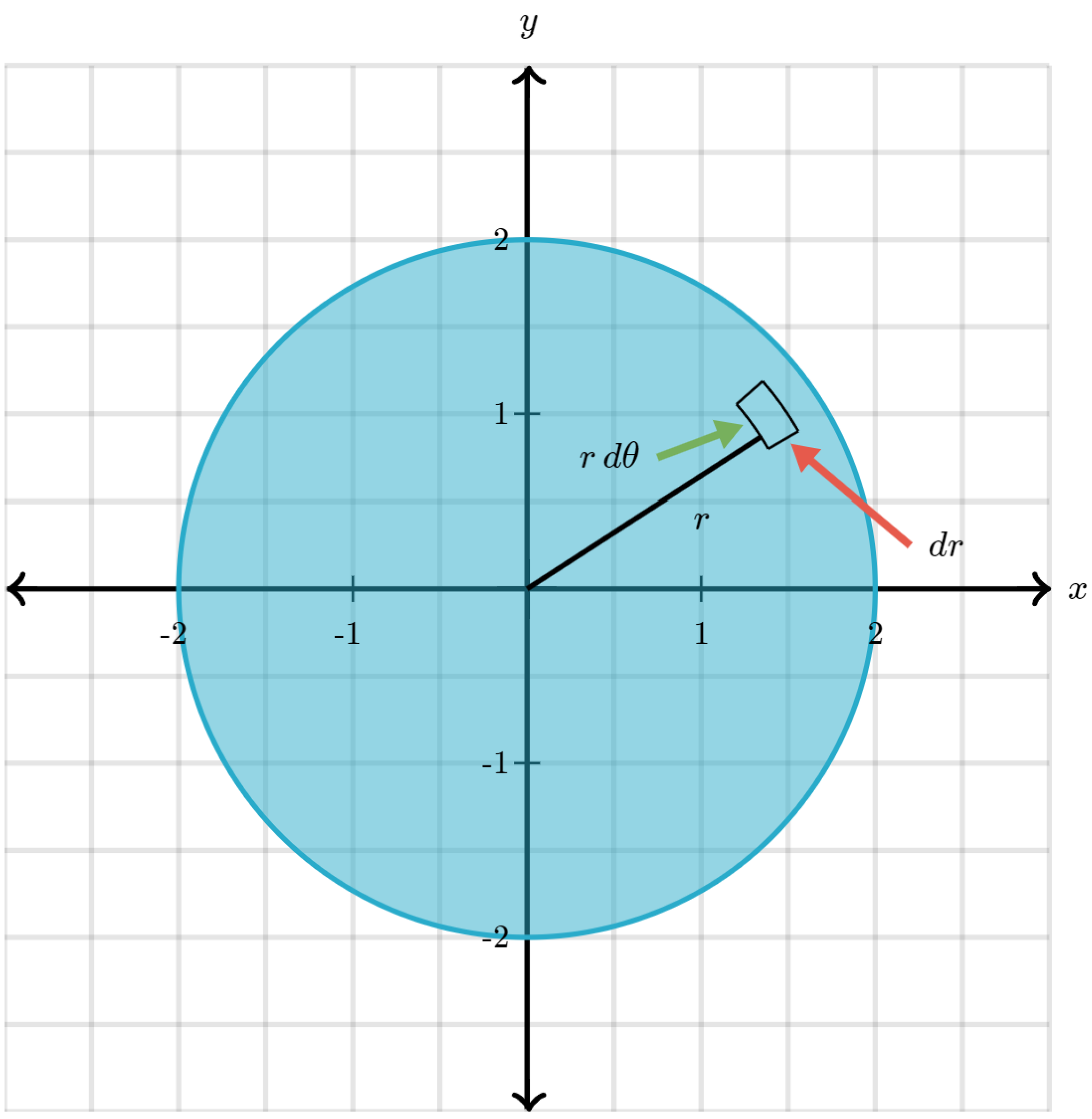
Even though this little piece has a curve shape, if we make finer and finer cuts, we can basically treat it as a rectangle. The length of one side of this "rectangle" can be thought of as dr, a tiny change in the r-coordinate.



Using a differential dr to describe this length emphasizes the fact that we are not really considering a specific piece, but instead we care about what happens as its size approaches 0.

But how long is the other side?

It's not $d\theta$, a tiny change in the angle, because **radians are not a unit of length**. To turn radians into a bit of arc length, we must **multiply by** r.



Therefore, if we treat this tiny chunk as a rectangle, and as dr and $d\theta$ each approach 0 it basically *is* a rectangle, its area is

$$dA = (r\,d\theta)(dr)$$

Plugging this into our original integral, we get

$$\iint_{r\leq 2} r^2\,dA = \iint_{r\leq 2} r^2\,(r\,d\theta)(dr) = \iint_{r\leq 2} r^3\,d\theta\,dr$$

Putting bounds on this region is relatively straight-forward in this example, because circles are naturally suited for polar coordinates. Since we wrote $d\theta$ in front of dr, the inner integral is written with respect to θ. The bounds of this inner integral will reflect the full range of θ as it sweeps once around the circle, going from 0 to 2π. The outer integral is with respect to r, which ranges from 0 to 2.

Concept check: Evaluate this double integral

$$\int_0^2 \int_0^{2\pi} r^3\,d\theta\,dr = \square$$

Check

[Hide explanation]

$$\int_0^2 \underbrace{\left(\int_0^{2\pi} r^3 \, d\theta\right)}_{\substack{r^3 \text{ can be factored} \\ \text{out of the inner integral} \\ \text{with respect to } \theta}} dr$$

$$= \int_0^2 r^3 \left(\int_0^{2\pi} d\theta\right) dr$$

$$= \int_0^2 r^3 (2\pi) dr$$

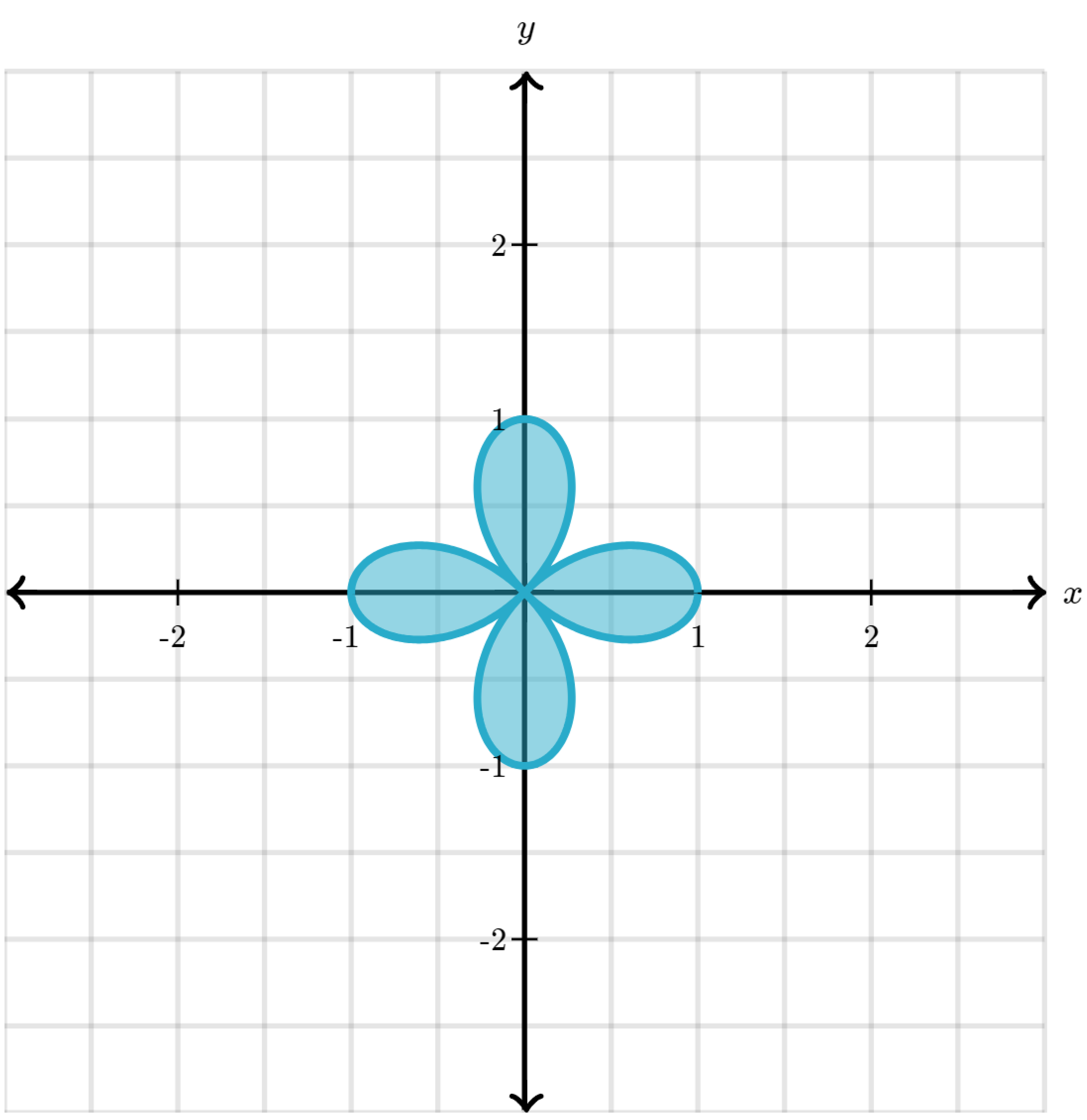
$$= 2\pi \int_0^2 r^3 dr$$

$$= 2\pi \left(\frac{r^4}{4}\right)\Bigg|_0^2$$

$$= 2\pi \left(\frac{2^4}{4} - \frac{0^4}{4}\right)$$

$$= 8\pi$$

Example 2: Integrating over a flower



Define a two-variable function f in polar coordinates as

$$f(r, \theta) = r\sin(\theta)$$

Let R be flower-shaped region, defined by

$$r \leq \cos(2\theta)$$

Solve the double integral

$$\iint_R f\, dA$$

Step 1: Which of the following represents the right way to replace $f\, dA$ in the abstractly written double integral?

Choose 1 answer:

(A) $\iint_R r\sin(\theta)\, dr\, d\theta$

(B) $\iint_R r^2\sin(\theta)\, dr\, d\theta$

Check

[Hide explanation]

The second choice is correct:

$$\iint_R r^2\sin(\theta)\, dr\, d\theta$$

The function f is defined as $r\sin(\theta)$, and the term dA is expanded as $r\, d\theta\, dr$ (don't forget to add the r term here)

$$\iint_R f\, dA$$

$$= \iint_R \underbrace{r\sin(\theta)}_{f(r,\theta)}\, \underbrace{r\, d\theta\, dr}_{dA}$$

$$= \iint_R r^2\sin(\theta)\, dr\, d\theta$$

Step 2: Now we must encode the fact that R is defined as the region where $r \leq \cos(2\theta)$. Which of the following is the right way to put bounds on the double integral?

Choose 1 answer:

(A) $\int_0^{2\pi} \int_0^{\cos(2\theta)} r^2 \sin(\theta)\, dr\, d\theta$

(B) $\int_0^{\arccos(r)/2} \int_0^{2\pi} r^2 \sin(\theta)\, dr\, d\theta$

Check

[Hide explanation]

The first answer choice is correct

$$\int_0^{2\pi} \int_0^{\cos(2\theta)} r^2 \sin(\theta)\, dr\, d\theta$$

The inner integral reflects r bounds, since dr is written before $d\theta$. Our region is explicitly defined such that $r \leq \cos(2\theta)$, and since these are polar coordinates, r can only ever be positive. Therefore, r ranges from 0 to $\cos(2\theta)$.

For the outer integral, θ ranges over its full range from 0 to 2π, since no constraints are given on θ in the definition of R.

Step 3: Solve this integral.

$$\int_0^{2\pi} \int_0^{\cos(2\theta)} r^2 \sin(\theta)\, dr\, d\theta = \square$$

Check

[Hide explanation]

In practice, I would use a calculator or Wolfram Alpha to solve one of these integrals. The hard part, which typically requires more human intelligence, is getting the double integral to the point where all the terms are in place. If you want to solve it by hand, here's how it might go:

$$\int_0^{2\pi} \underbrace{\int_0^{\cos(2\theta)} r^2 \sin(\theta)\, dr}_{\text{Factor out } \sin(\theta) \text{ from } r\text{-integral}} \quad d\theta$$

$$= \int_0^{2\pi} \sin(\theta) \int_0^{\cos(2\theta)} r^2 \, dr \, d\theta$$

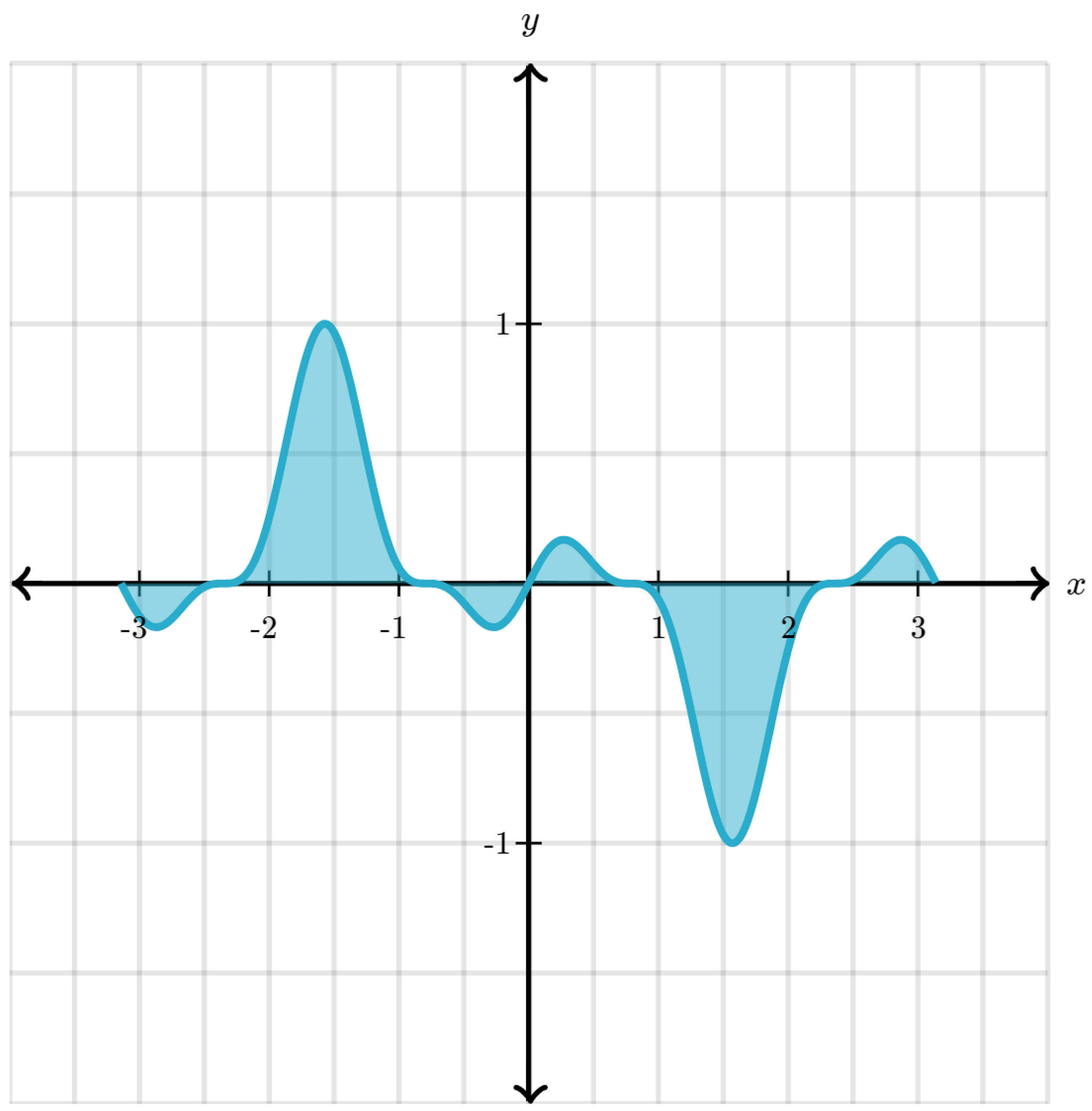
$$= \int_0^{2\pi} \sin(\theta) \left[\frac{r^3}{3}\right]_0^{\cos(2\theta)} d\theta$$

$$= \int_0^{2\pi} \sin(\theta) \left(\frac{\cos(2\theta)^3}{3}\right) d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \sin(\theta) \cos(2\theta)^3 d\theta$$

From here, we can save a lot of computational trouble with an observation: The inside of the integral is an odd function, meaning that when you replace θ by $-\theta$, the value of the expression as a whole becomes negative.

Also, since these trigonometric functions are periodic, integrating between 0 and 2π is the same as integrating between $-\pi$ and π. The integral on the negative half cancels out with the integral on the positive portion, so in total it is 0.



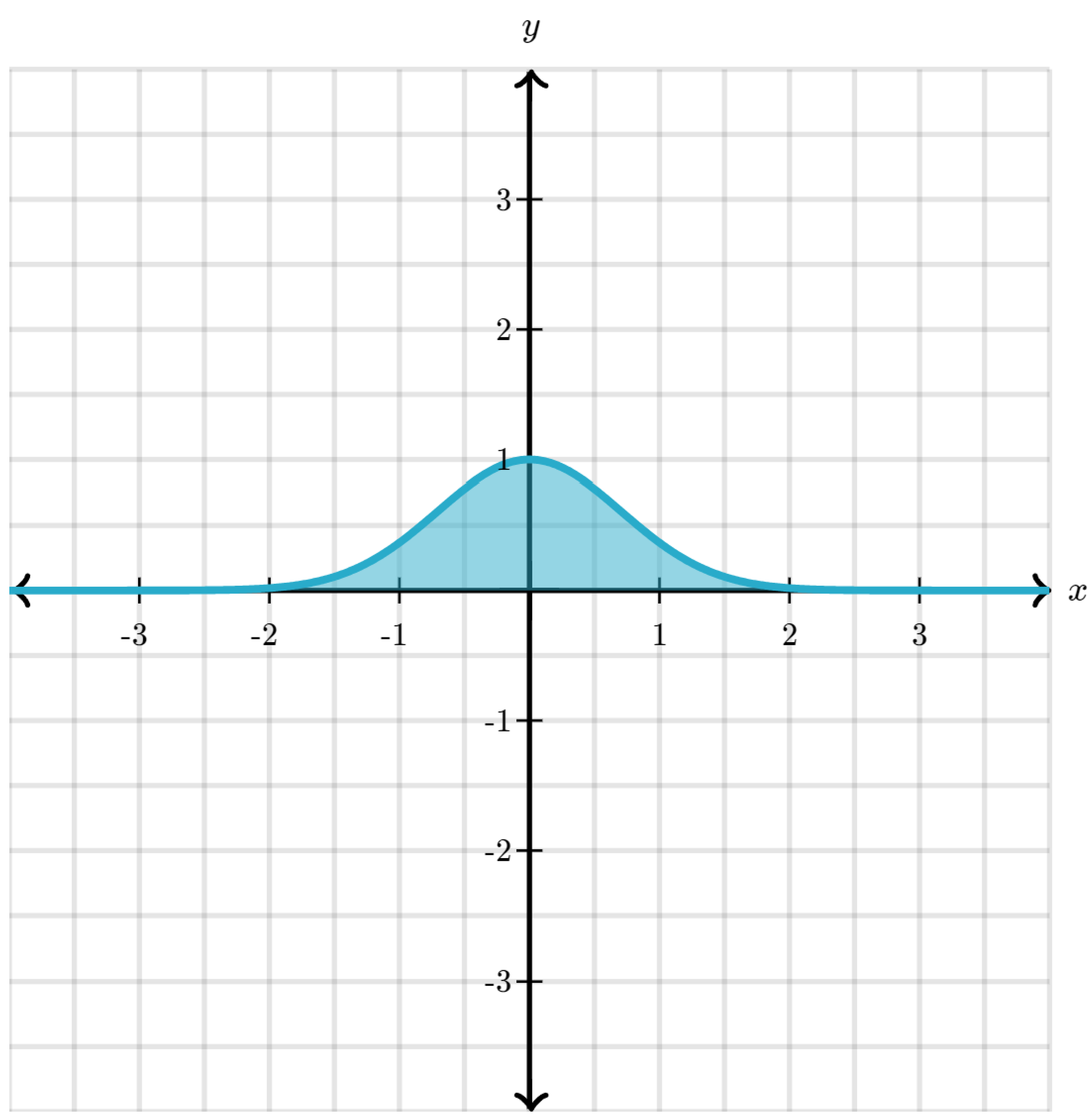
Example 3: The bell curve

Are you ready for one of my favorite results in math? This is really quite clever.

> **Question**: What is the integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$?

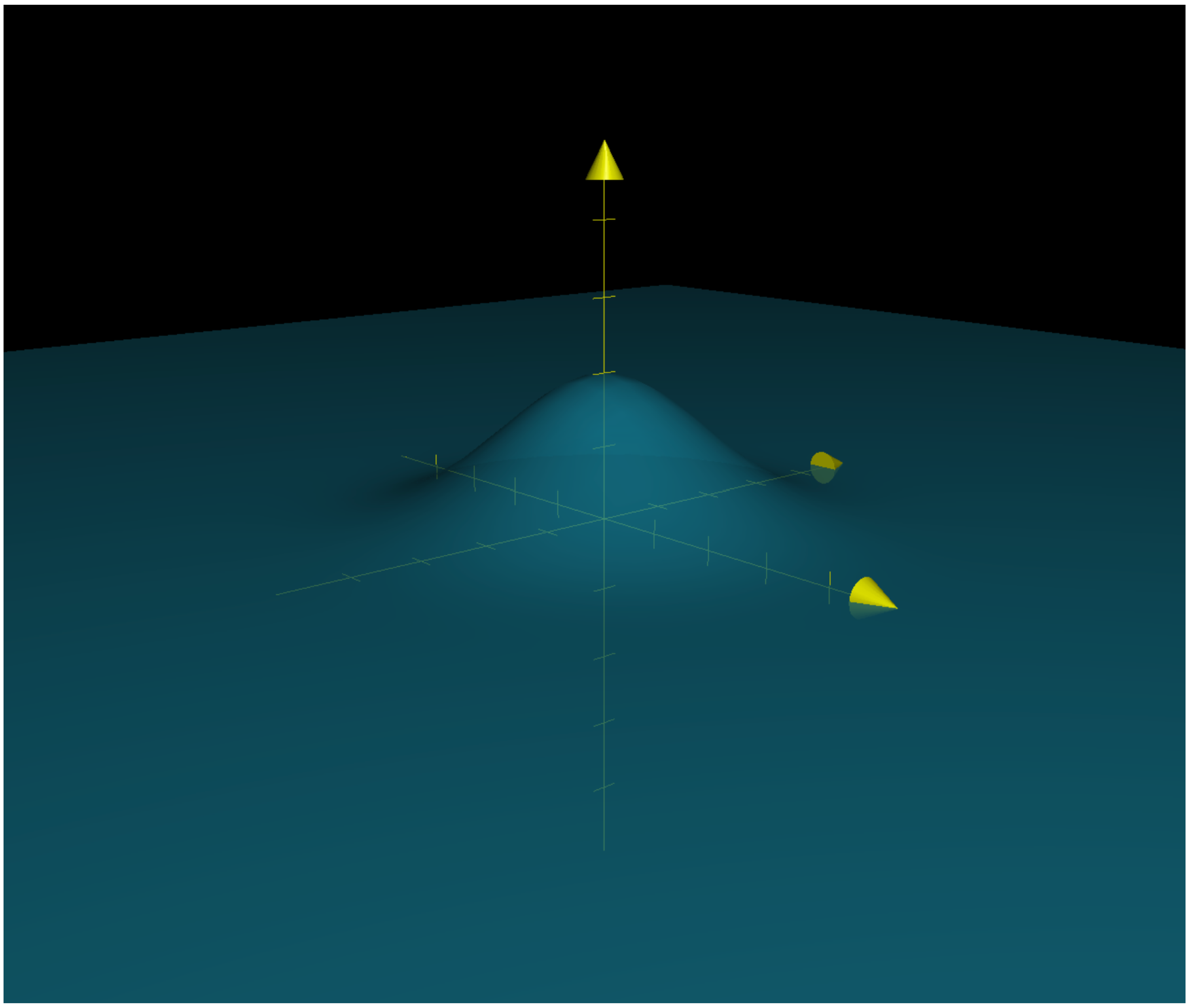
This single integral is hard-to-impossible to compute directly. Just try to find the antiderivative!

This integral is asking about the area under a bell curve, which turns out to be super important for probability and statistics!



"What does this have to do with double integrals in polar coordinates?"

I hear you, my inquisitive friend, it does seem unrelated, doesn't it? Well, this is where someone got super clever.



Surprisingly, it is easier to solve this multi-dimensional analog of this problem. Namely, find the *volume* under a three-dimensional bell curve over the entire xy-plane.

$$\iint_{xy\text{-plane}} e^{-(x^2+y^2)}\, dA$$

If we keep everything in cartesian coordinates, this is as hard to solve as the original single integral.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$$

However, something magical happens when we convert to polar coordinates.

Concept check: Express this double integral using polar coordinates.

Choose 1 answer:

(A) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-r^2} r\,d\theta\,dr$

(B) $\int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2} r\,d\theta\,dr$

(C) $\int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2}\,d\theta\,dr$

Check

[Hide explanation]

The second answer choice is correct.

First, make the following two conversions

$$x^2 + y^2 \to r^2$$

and

$$dy\,dx \to \underbrace{r\,d\theta\,dr}_{\text{Polar form of } dA}$$

So our integral looks like this:

$$\iint_{r\theta\text{-plane}} e^{-r^2} r\,d\theta\,dr$$

To put bounds on the integral, we cover the entire plane by letting θ range from 0 to 2π, and r range from 0 to ∞.

$$\int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2} r\,d\theta\,dr$$

Since the inner integral is with respect to θ, we can factor out everything with an r in it, which in this case is the entire function:

$$\int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2} r\,d\theta\,dr$$

$$= \int_{0}^{\infty} e^{-r^2} r \underbrace{\int_{0}^{2\pi} d\theta} \, dr$$

This evaluates to 2π

$$= \int_0^\infty \left(e^{-r^2} r\right)(2\pi)\, dr$$

$$= 2\pi \int_0^\infty e^{-r^2} r\, dr$$

Concept check: Find the antiderivative of $e^{-r^2} r$, using either u-substitution or the inverse chain rule.

$$\int e^{-r^2} r\, dr = \boxed{}$$

Check

[Hide explanation]

Using u-substitution:

Define u as

$$u = -r^2$$

which means

$$du = -2r dr$$

Our indefinite integral now becomes

$$\int e^{-r^2} r\, dr = \int e^u r \frac{du}{-2r}$$

$$= \int e^u \frac{-1}{2} du$$

$$= -\frac{1}{2} e^u$$

Substituting back to $u = -r^2$, this means our final antiderivative is

$$\int e^{-r^2} r\, dr = -\frac{1}{2} e^{-r^2}$$

Notice, the reason you can now find an antiderivative is because of that little r term that showed up due to the fact that $dA = r\,d\theta\,dr$.

Concept check: Using this antiderivative, finish solving the integral which computes volume under a three-dimensional bell curve.

$$2\pi \int_0^\infty e^{-r^2} r\,dr = \square$$

Check

[Hide explanation]

$$\begin{aligned}
2\pi \int_0^\infty e^{-r^2} r\,dr &= 2\pi \left[-\frac{1}{2}e^{-r^2}\right]_0^\infty \\
&= 2\pi \left[-\frac{1}{2}e^{-(\infty)^2} - \left(-\frac{1}{2}e^{-0^2}\right)\right] \\
&= 2\pi \left[-\frac{1}{2}(0) - \left(-\frac{1}{2}(1)\right)\right] \\
&= 2\pi\frac{1}{2} \\
&= \pi
\end{aligned}$$

Isn't that a beautiful answer? It gets better, you can use this multi-dimensional result to solve the original single integral. Can you see how?

[Hide explanation]

Let $C = \int_{-\infty}^{\infty} e^{-x^2}\,dx$, and try to solve the double integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$ in terms of C

[Hide explanation]

Following the hint, start by saying

$$C = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

Now, as we start chewing on the double integral in cartesian coordinates, even though it cannot be solved directly the way it was in polar form, we can start writing things in terms of C.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy$$

$$= \int_{-\infty}^{\infty} e^{-y^2}\underbrace{\int_{-\infty}^{\infty} e^{-x^2}\,dx}_{\text{This equals } C}\,dy$$

$$= \int_{-\infty}^{\infty} e^{-y^2}C\,dy$$

$$= C\underbrace{\int_{-\infty}^{\infty} e^{-y^2}\,dy}_{\text{This also equals } C}$$

$$= C^2$$

In other words, even if we don't know what the area under a bell curve is, we know that when you square it, you get the volume under a three-dimensional bell curve. But we just solved the volume under three-dimensional bell curve using polar-coordinate integration! We found that the volume was π. Therefore, the original integral is $\sqrt{\pi}$.

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Isn't that crazy? This expression looks like it has nothing to do with π and circles, yet somehow it does. This is just one more of the many places in nature where π and e are connected.

Summary

- The only real thing to remember about double integral in polar coordinates is that

  $$dA = r\,dr\,d\theta$$

  Beyond that, the tricky part is wrestling with bounds, and the nastiness of actually solving the integrals that you get. But those are the same difficulties one runs into with cartesian double integrals.

- The reason this is worth learning is that sometimes double integrals become simpler when you phrase them with polar coordinates, as was the case in the bell curve example.