Directional derivatives (introduction)

Google Classroom

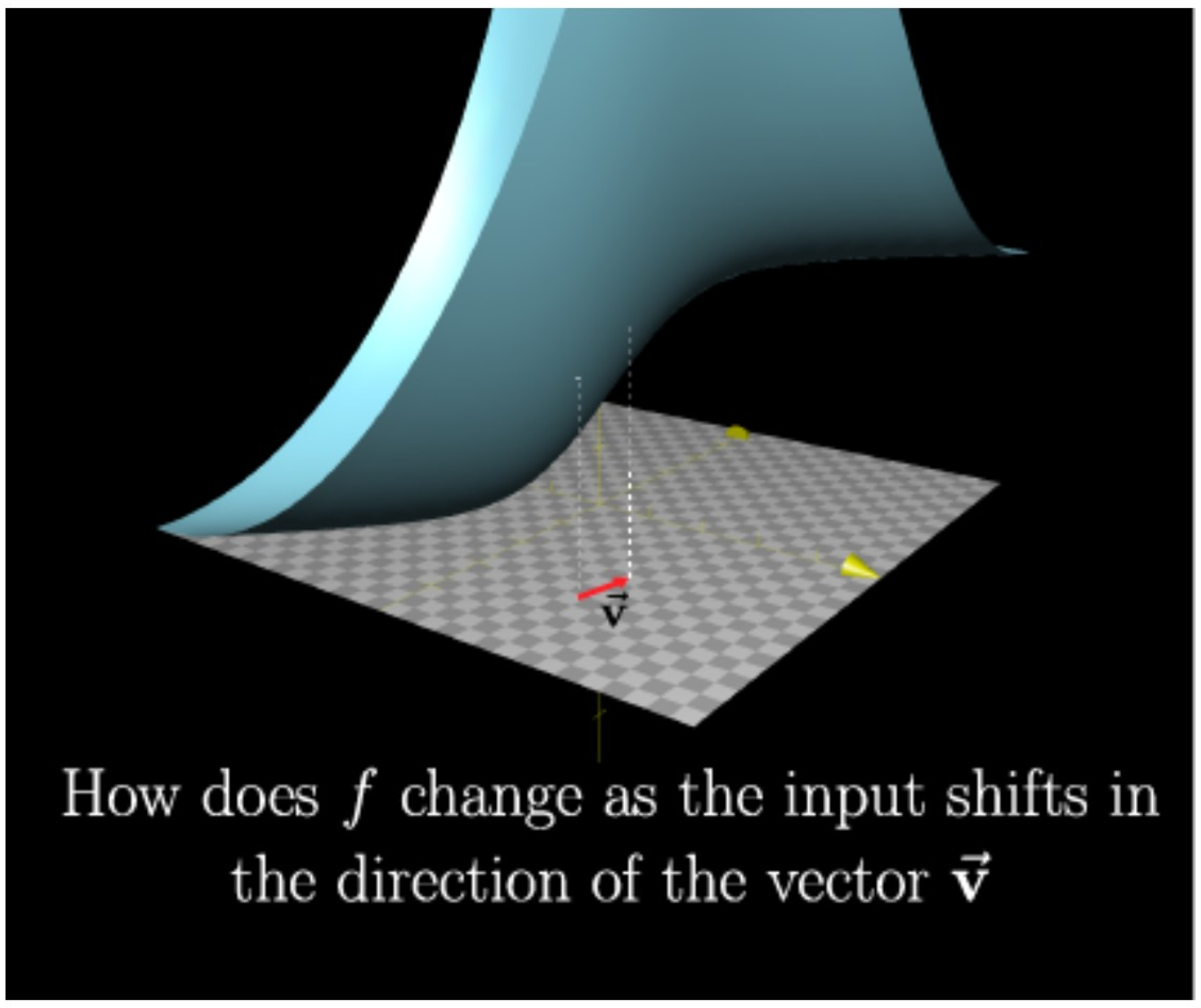
How does the value of a multivariable function change as you nudge the input in a specific direction?

Background

- Partial derivatives
- Gradient

What we're building to

- If you have some multivariable function, $f(x, y)$ and some vector in the function's input space, $\vec{\mathbf{v}}$, the directional derivative of f along $\vec{\mathbf{v}}$ tells you the rate at which f will change while the input moves with velocity vector $\vec{\mathbf{v}}$.



- The notation here is $\nabla_{\vec{\mathbf{v}}} f$, and it is computed by taking the dot product between the gradient of f and the vector $\vec{\mathbf{v}}$, that is, $\nabla f \cdot \vec{\mathbf{v}}$

- When the directional derivative is used to compute slope, be sure to normalize the vector $\vec{\mathbf{v}}$ first.

Generalizing partial derivatives

Consider some multivariable function:

$$f(x, y) = x^2 - xy$$

We know that the partial derivatives with respect to x and y tell us the rate of change of f as we nudge the input either in the x or y direction.

[Hide explanation.]

The partial derivative of f with respect to x is below:

$$\frac{\partial}{\partial x}(x^2 - xy) = 2x - y$$

If we evaluate this at some point—like $(2, 3)$—we get the following:

$$\frac{\partial f}{\partial x}(2, 3) = 2(2) - 3 = 1$$

But what does this actually mean?

Let's say you evaluate the original function f at the point $(2, 3)$:

$$f(2, 3) = (2)^2 - (2)(3) = -2$$

Now nudge the input slightly in the x-direction, perhaps by the value 0.01, moving it to $(2.01, 3)$. The function now evaluates thus:

$$\begin{aligned} f(2.01, 3) &= (2.01)^2 - (2.01)(3) \\ &= 4.0401 - 6.03 \\ &= -1.9899 \end{aligned}$$

The total change in the output, from −2 to −1.9899 is 0.0101. The ratio between this change and the size of our nudge in the input space is below:

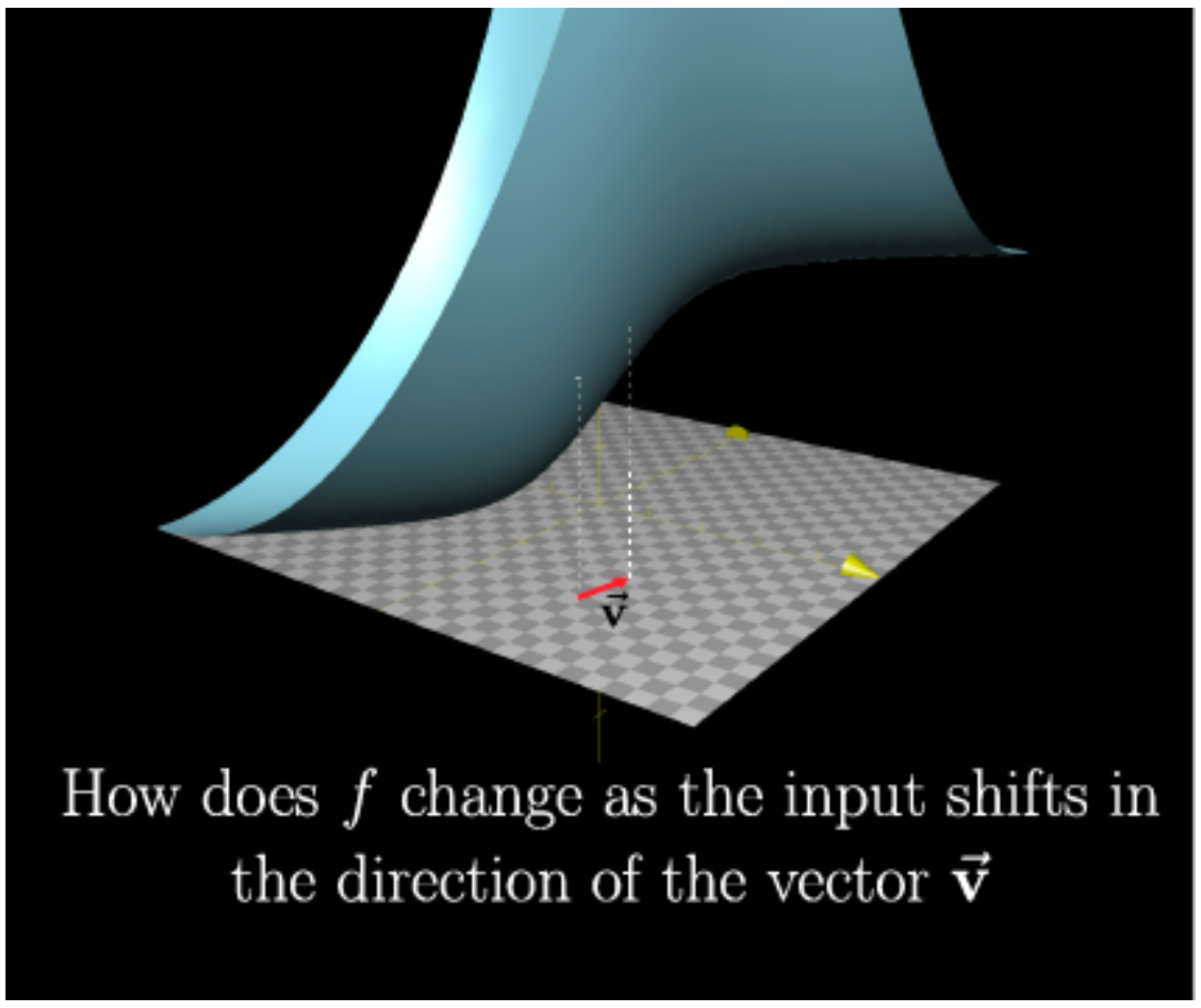
$$\frac{0.0101}{0.01} \approx 1 = \frac{\partial f}{\partial x}(2, 3)$$

The smaller our original nudge, the closer this ratio will be to $\frac{\partial f}{\partial x}(2, 3) = 1$.

The question now is what happens when we nudge the input of f in a direction which is not parallel to the x or y axes.

For example, the image below shows the graph of f along with a small step along a vector $\vec{\mathbf{v}}$ in the input space, meaning the xy-plane in this case. Is there an operation which tells us how the height of the graph above the tip of $\vec{\mathbf{v}}$

compares to the height of the graph above its tail?



As you have probably guessed, there is a new type of derivative, called the **directional derivative**, which answers this question.

Just as the partial derivative is taken with respect to some input variable—e.g., x or y—the directional derivative is taken *along* some vector $\vec{\mathbf{v}}$ in the input space.

One very helpful way to think about this is to picture a point in the input space moving with velocity $\vec{\mathbf{v}}$. The directional derivative of f along $\vec{\mathbf{v}}$ is the resulting rate of change in the output of the function. So, for example, multiplying the vector $\vec{\mathbf{v}}$ by two would double the value of the directional derivative since all changes would be happening twice as fast.

Notation

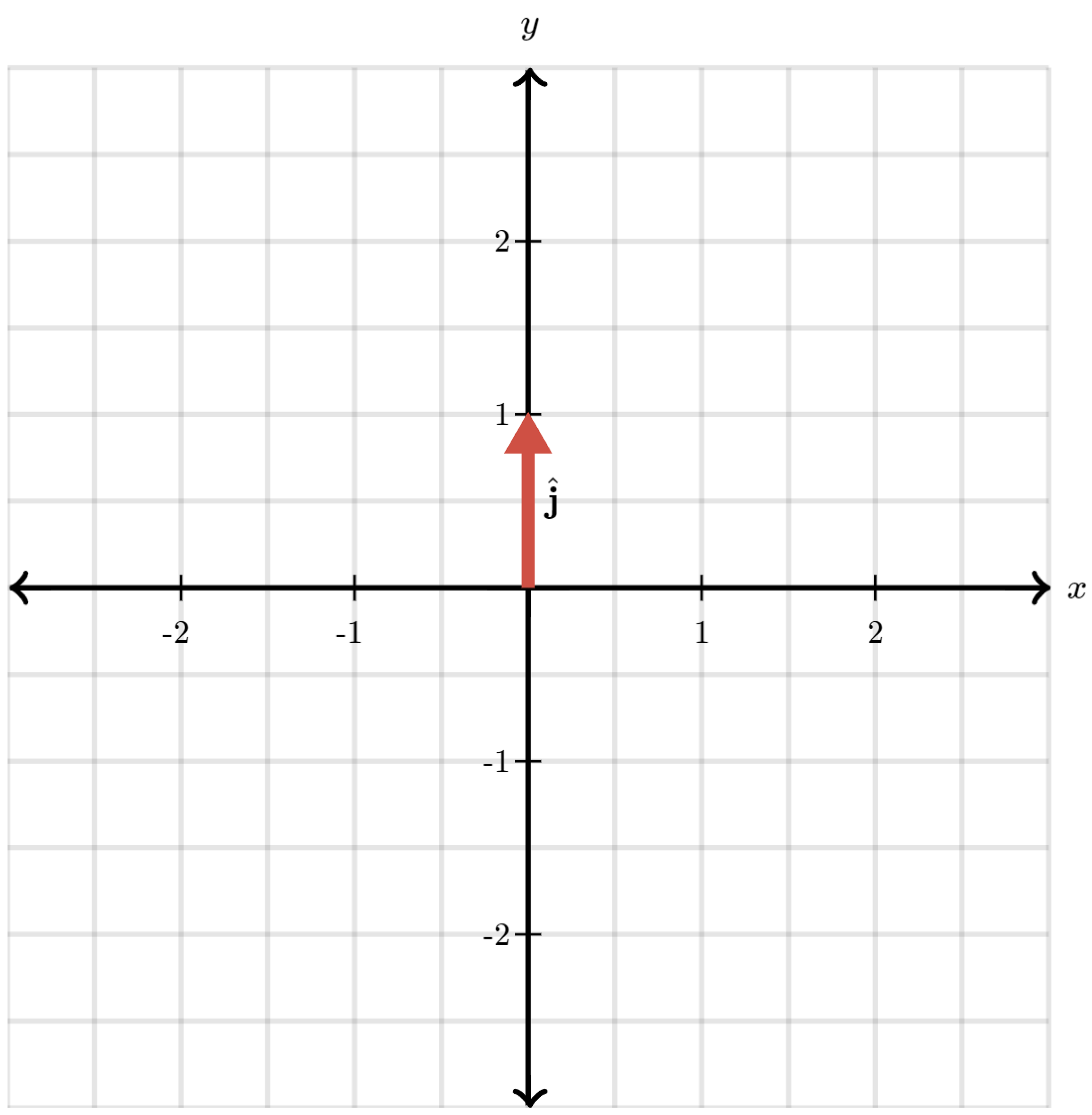
There are quite a few different notations for this one concept:

- $\nabla_{\vec{\mathbf{v}}} f$
- $\dfrac{\partial f}{\partial \vec{\mathbf{v}}}$
- $f'_{\vec{\mathbf{v}}}$
- $D_{\vec{\mathbf{v}}} f$
- $\partial_{\vec{\mathbf{v}}} f$

All of these represent the same thing: the rate of change of f as you nudge the input along the direction of $\vec{\mathbf{v}}$. We'll use the $\nabla_{\vec{\mathbf{v}}} f$ notation, just because it subtly hints at how you compute the directional derivative using the gradient, which you'll see in a moment.

Example 1: $\vec{\mathbf{v}} = \hat{\mathbf{j}}$

Before jumping into the general rule for computing $\nabla_{\vec{\mathbf{v}}} f$, let's look at how we can rewrite the more familiar notion of a partial derivative as a directional derivative.



For example, the partial derivative $\frac{\partial f}{\partial y}$ tells us the rate at which f changes as we nudge the input in the y direction. In other words, as we nudge it along the vector $\hat{\mathbf{j}}$. Therefore, we could equivalently write the partial derivative with respect to y as $\frac{\partial f}{\partial y} = \nabla_{\hat{\mathbf{j}}} f$.

This is all just fiddling with different notation. What's more important is to have a clear mental image of what all this notation *represents*.

Reflection Question: Suppose $\vec{\mathbf{v}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, what is your best guess for $\nabla_{\vec{\mathbf{v}}} f$?

[Hide explanation.]

> $$\nabla_{\vec{\mathbf{v}}} f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

How to compute the directional derivative

Let's say you have a multivariable $f(x, y, z)$ which takes in three variables—x, y and z—and you want to compute its directional derivative along the following vector:

$$\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

The answer, as it turns out, is

$$\nabla_{\vec{\mathbf{v}}} f = 2\frac{\partial f}{\partial x} + 3\frac{\partial f}{\partial y} + (-1)\frac{\partial f}{\partial z}$$

This should make sense because a tiny nudge along $\vec{\mathbf{v}}$ can be broken down into two tiny nudges in the x-direction, $three$ tiny nudges in the y-direction, and a tiny nudge backwards, by -1, in the z-direction. We'll go through the rigorous reasoning behind this much more thoroughly in the next article.

More generally, we can write the vector $\vec{\mathbf{v}}$ abstractly as follows:

$$\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

The directional derivative looks like this:

$$\nabla_{\vec{\mathbf{v}}} f = v_1\frac{\partial f}{\partial x} + v_2\frac{\partial f}{\partial y} + v_3\frac{\partial f}{\partial z}$$

That is, a tiny nudge in the $\vec{\mathbf{v}}$ direction consists of v_1 times a tiny nudge in the x-direction, v_2 times a tiny nudge in the y-direction, and v_3 times a tiny nudge in the z-direction.

This can be written in a super-pleasing compact way using the dot product and the gradient:

$$\nabla_{\vec{\mathbf{v}}} f(x, y, z)$$

$$= v_1\frac{\partial f}{\partial x}(x, y, z) + v_2\frac{\partial f}{\partial y}(x, y, z) + v_3\frac{\partial f}{\partial z}(x, y, z)$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x}(x, y, z) \\ \frac{\partial f}{\partial y}(x, y, z) \\ \frac{\partial f}{\partial z}(x, y, z) \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= \nabla f(x, y, z) \cdot \vec{\mathbf{v}}$$

This is why the notation $\nabla_{\vec{\mathbf{v}}}$ is so suggestive of the way we compute the directional derivative:

$$\nabla_{\vec{\mathbf{v}}} f = \nabla f \cdot \vec{\mathbf{v}}$$

Take a moment to delight in the fact that one single operation, the gradient, packs enough information to compute the rate of change of a function in *every possible direction!* That's so many directions! Left, right, up, down, north-north-east, 34.8° clockwise from the x-axis... Madness!

Example 2:

Problem: Take a look at the following function.

$$f(x, y) = x^2 - xy,$$

What is the directional derivative of f at the point $(2, -3)$ along the vector $\vec{\mathbf{v}} = 0.6\hat{\mathbf{i}} + 0.8\hat{\mathbf{j}}$?

Solution: You can think of the direction derivative either as a weighted sum of partial derivatives, as below:

$$\nabla_{\vec{\mathbf{v}}} f = 0.6\frac{\partial f}{\partial x} + 0.8\frac{\partial f}{\partial y}$$

Or, you can think of it as a dot product with the gradient, as you see here:

$$\nabla_{\vec{\mathbf{v}}} f = \nabla f \cdot \vec{\mathbf{v}}$$

The first is faster, but just for practice, let's see how the gradient interpretation unfolds. We start by computing the gradient itself:

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \\ \dfrac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x}(x^2 - xy) \\ \\ \dfrac{\partial}{\partial y}(x^2 - xy) \end{bmatrix} = \begin{bmatrix} 2x - y \\ -x \end{bmatrix}$$

Next, plug in the point $(x, y) = (2, -3)$ since this is the point the question asks us about.

$$\nabla f(2, -3) = \begin{bmatrix} 2(2) - (-3) \\ \\ -(2) \end{bmatrix} = \begin{bmatrix} 7 \\ \\ -2 \end{bmatrix}$$

To get the desired directional derivative, we take the dot product between this gradient and $\mathbf{v}$:

$$\nabla_{\vec{\mathbf{v}}} f(2, -3) = \nabla f(2, -3) \cdot \left(0.6\hat{\mathbf{i}} + 0.8\hat{\mathbf{j}}\right)$$

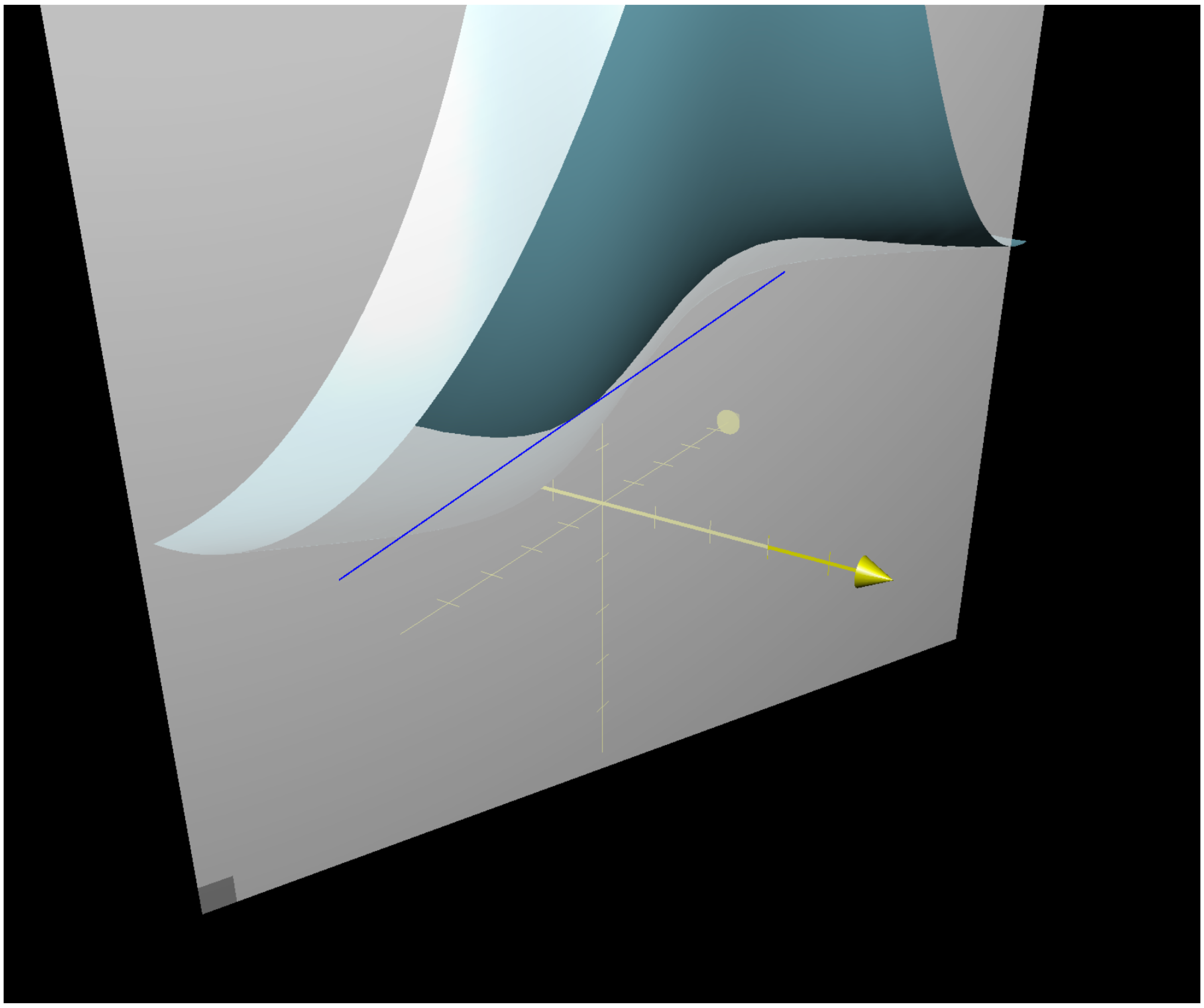
$$= \begin{bmatrix} 7 \\ \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0.6 \\ \\ 0.8 \end{bmatrix}$$

$$= 7(0.6) + (-2)(0.8)$$

$$= 2.6$$

Finding slope

How do you find the slope of a graph intersected with a plane that is not parallel to the x or y axes?



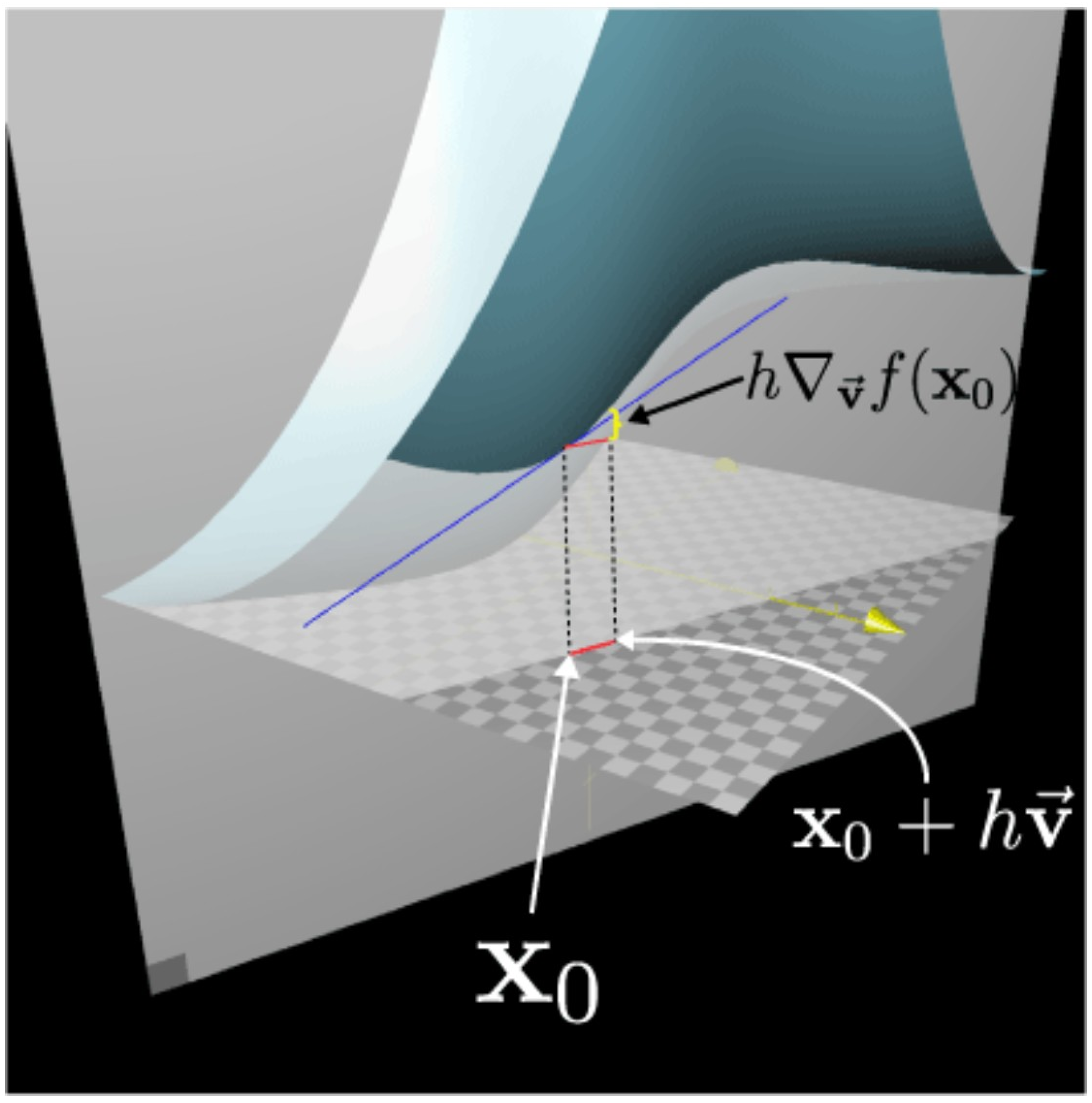
You can use the directional derivative, but there is one important thing to remember:

If the directional derivative is used to compute slope, either $\vec{\mathbf{v}}$ must be a unit vector or you must remember to divide by $||\vec{\mathbf{v}}||$ at the end.

In the definition and computation above, doubling the length of $\vec{\mathbf{v}}$ would double the value of the directional derivative. In terms of the computation, this is because $\nabla f \cdot (2\vec{\mathbf{v}}) = 2(\nabla f \cdot v)$.

However, this might not always be what you want. The slope of a graph in the direction of $\vec{\mathbf{v}}$, for example, depends only on the direction of $\vec{\mathbf{v}}$, not the magnitude $||\vec{\mathbf{v}}||$. Let's see why.

How can we imagine this slope? Slice the graph of f with a vertical plane that cuts the xy-plane in the direction of $\vec{\mathbf{v}}$. The slope in question is that of a line tangent to the resulting curve. As with any slope, we look for the rise over run.



In this case, the run will be the distance of a small nudge in the direction of $\vec{\mathbf{v}}$. We can express such a nudge as an addition of $h\vec{\mathbf{v}}$ to an input point $\mathbf{x}_0$, where h is thought of as some small number. The magnitude of this nudge is $h||\vec{\mathbf{v}}||$.

The resulting change in the output of f can be approximated by multiplying this little value h by the directional derivative:

$$h\nabla_{\vec{\mathbf{v}}}f(x_0, y_0)$$

In fact, the rise of the tangent line—as opposed to the graph of the function—is *precisely* $h\nabla_{\vec{\mathbf{v}}}f(x_0, y_0)$ due to this run of size $h||\vec{\mathbf{v}}||$. For full details on why this is true, see the formal definition of the directional derivative in the next article.

Therefore, the rise-over-run slope of our graph is

$$\frac{h\nabla_{\vec{\mathbf{v}}}f(x_0, y_0)}{h||v||} = \boxed{\frac{\nabla_{\vec{\mathbf{v}}}f(x_0, y_0)}{||v||}}$$

Notice, if $\vec{\mathbf{v}}$ is a unit vector, meaning $||\vec{\mathbf{v}}|| = 1$, then the directional derivative *does* give the slope of a graph along that direction. Otherwise, it is important to remember to divide out by the magnitude of $\vec{\mathbf{v}}$.

Some authors even go so far as to include normalization in the definition of $\nabla_{\vec{\mathbf{v}}}f$.

Alternate definition of directional derivative:

$$\nabla_{\vec{\mathbf{v}}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\vec{\mathbf{v}}) - f(\mathbf{x})}{h||\vec{\mathbf{v}}||}$$

Personally, I think this definition puts too much emphasis on the particular use case of finding slope, so I prefer to use the original definition and normalize $\vec{\mathbf{v}}$ when necessary.

Example 3: Slope

Problem: On the stage for this problem we have three players.

Player 1, the function:

$$f(x, y) = \sin(xy)$$

Player 2, the point:

$$(x_0, y_0) = \left(\frac{\pi}{3}, \frac{1}{2}\right)$$

Player 3, the vector:

$$\vec{\mathbf{v}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$$

What is the slope of the graph of f at the point (x_0, y_0) along the vector $\vec{\mathbf{v}}$?

Answer: Since we are finding slope, we must first normalize the vector in question. The magnitude $||\vec{\mathbf{v}}||$ is $\sqrt{2^2 + 3^2} = \sqrt{13}$, so we divide each term by $\sqrt{13}$ to get the resulting unit vector $\hat{\mathbf{u}}$ in the direction of $\vec{\mathbf{v}}$:

[Hide explanation.]

$$\hat{\mathbf{u}} = \frac{\vec{\mathbf{v}}}{||\vec{\mathbf{v}}||} = \frac{2}{\sqrt{13}}\hat{\mathbf{i}} + \frac{3}{\sqrt{13}}\hat{\mathbf{j}}$$

Next, find the gradient of f:

[Hide explanation.]

$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x}\sin(xy) \\ \\ \frac{\partial}{\partial y}\sin(xy) \end{bmatrix} = \begin{bmatrix} y\cos(xy) \\ x\cos(xy) \end{bmatrix}$$

Plug in the point $(x_0, y_0) = \left(\frac{\pi}{3}, \frac{1}{2}\right)$ to this gradient.

[Hide explanation.]

$$\nabla f\left(\frac{\pi}{3}, \frac{1}{2}\right)$$

$$= \begin{bmatrix} \left(\frac{1}{2}\right)\cos\left(\frac{\pi}{3}\cdot\frac{1}{2}\right) \\ \\ \left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\cdot\frac{1}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ \\ \left(\frac{\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{4} \\ \\ \frac{\pi\sqrt{3}}{6} \end{bmatrix}$$

Finally, take the dot product between $\hat{\mathbf{u}}$ and $\nabla f(\pi/3, 1/2)$:

[Hide explanation.]

$$\nabla f\left(\frac{\pi}{3}, \frac{1}{2}\right) \cdot \hat{\mathbf{u}}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{4} \\ \frac{\pi\sqrt{3}}{6} \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{bmatrix}$$

$$= \frac{\sqrt{3}}{4} \cdot \frac{2}{\sqrt{13}} + \frac{\pi\sqrt{3}}{6} \cdot \frac{3}{\sqrt{13}}$$

$$= \boxed{\frac{(1+\pi)\sqrt{3}}{2\sqrt{13}}}$$

Summary

- If you have some multivariable function, $f(x, y)$ and some vector in the function's input space, $\vec{\mathbf{v}}$, the directional derivative of f along $\vec{\mathbf{v}}$ tells you the rate at which f will change while the input moves with velocity vector $\vec{\mathbf{v}}$.

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