Stokes' theorem and the fundamental theorem of calculus

Google Classroom

Both Green's theorem and Stokes' theorem are higher-dimensional versions of the fundamental theorem of calculus, see how!

Background

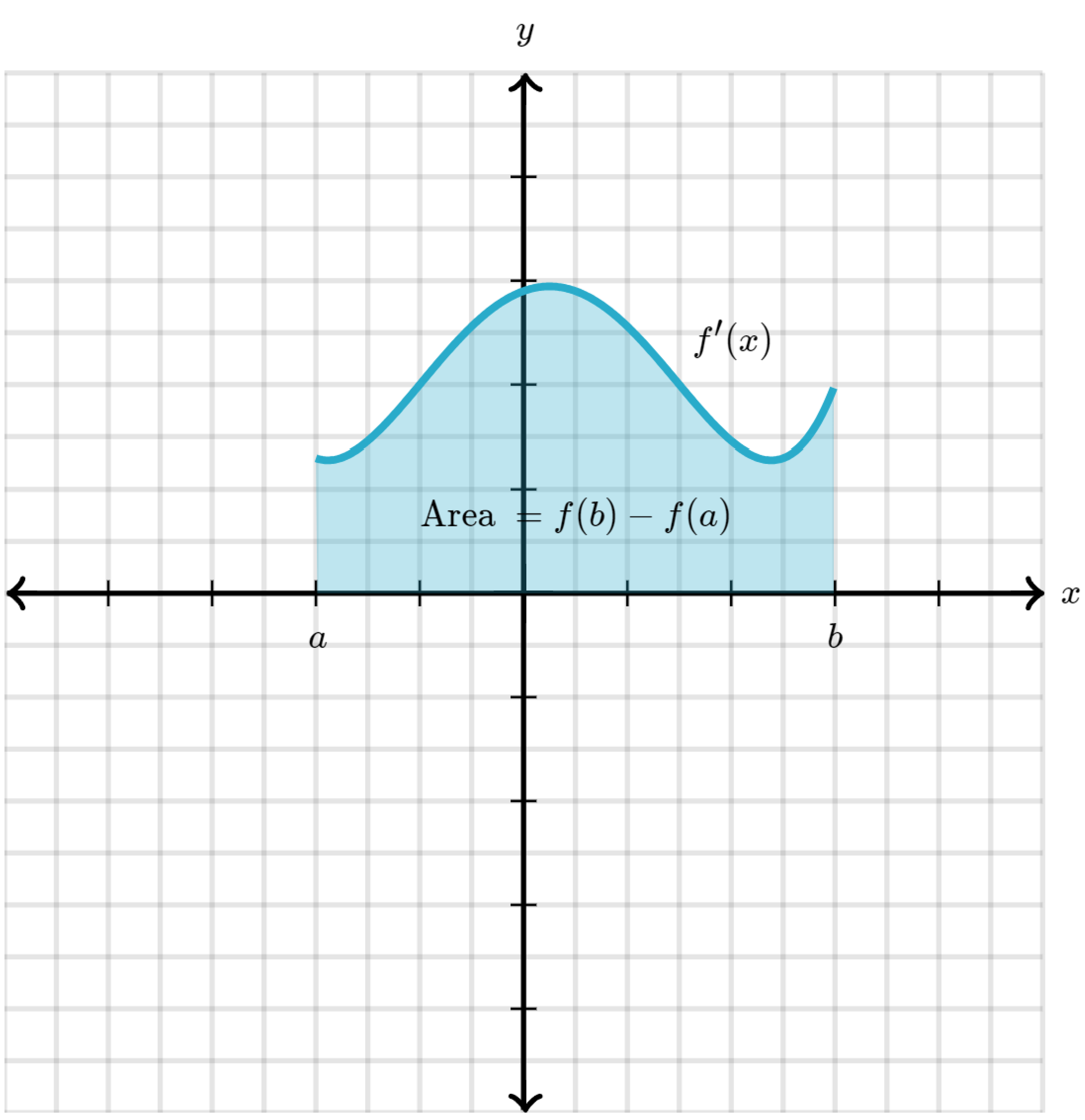
- Fundamental theorem of calculus (video)
- Green's theorem
- Stokes' theorem

What we're building to

- Both Green's theorem and Stokes' theorem, as well as several other multivariable calculus results, are really just higher dimensional analogs of the fundamental theorem of calculus.

Quick review of the fundamental theorem of calculus

Remember the fundamental theorem of calculus?



Here's what it says:

$$\int_a^b f'(x) = f(b) - f(a)$$

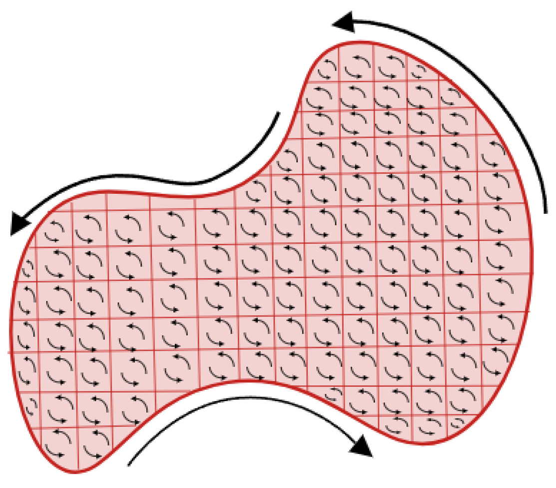
In other words, when you integrate the *derivative* of a function on a region $[a, b]$ of the number line, it's the same as evaluating the function itself on the boundary of that region, meaning the numbers a and b, and taking the difference.

Green's theorem

Green's theorem can be seen as completely analogous to the fundamental theorem, but for two dimensions.

$$\iint_R \text{2d-curl}\, \mathbf{F}\, dA = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

- Instead of taking the **derivative** of a single-variable function f, it involves the 2d-curl of a two-variable vector-valued function $\mathbf{F}(x, y)$.



- Instead of integrating this over a region $[a, b]$ of the number line, take its double integral over a region R of the xy-plane.

- The boundary of the one-dimensional range $[a, b]$ is just the pair of points a and b. But because R is two-dimensional, its boundary is a curve C.

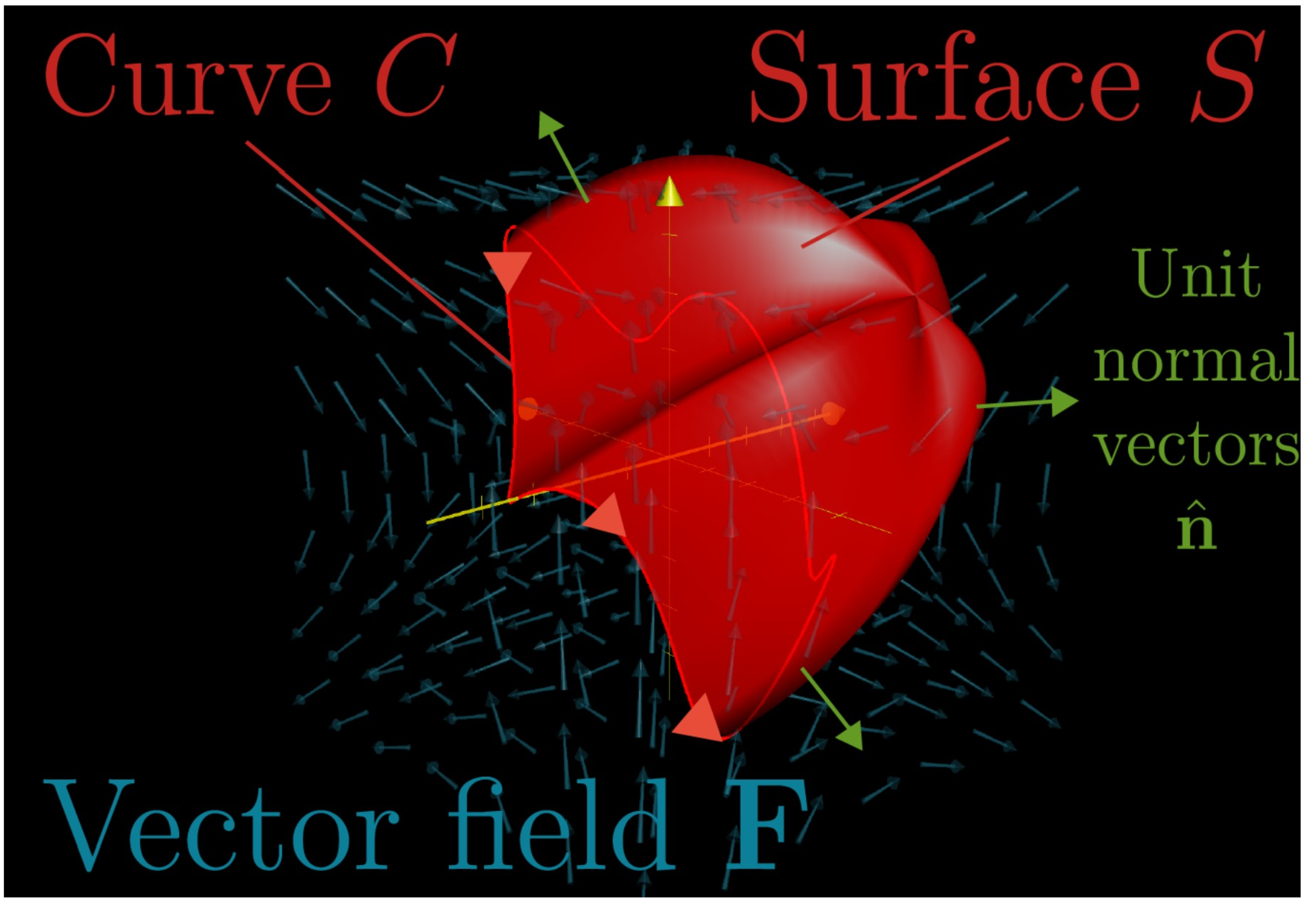
- Instead of evaluating f at the two boundary points a and b and taking the difference, take the *line integral* of $\mathbf{F}$ around the boundary C oriented counterclockwise.

The underlying idea here is that when you integrate the "derivative" of a thing over a region, the value only depends on the value of that thing on the boundary of the region. It's just that in two dimensions, the relevant notion of a derivative is 2d-curl, and the boundary of a region involves an entire curve rather than a pair of points.

Stoke's theorem

Stokes' theorem takes this to three dimensions. Instead of just thinking of a flat region R on the xy-plane, you think of a surface S living in space. This time, let C represent the boundary to this surface.

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}\, d\Sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$$



- Instead of a single variable function f, or a two-dimensional vector field, $\mathbf{F}(x, y, z)$ is a three-dimensional vector field.

- Instead of taking the derivative $f'(x)$, or the 2d-curl, take the full-blown three-dimensional curl of $\mathbf{F}$.

- Instead of taking the single integral over an interval $[a, b]$, or a double integral in a two-dimensional region, take the surface integral over S in three-dimensions. Taking the surface integral of a vector field involves dotting that vector field with unit normal vectors.

- On the right-hand side, instead of writing $f(b) - f(a)$, which involves evaluating f on the boundary of the interval $[a, b]$ and taking the difference, we have the line integral of our function $\mathbf{F}$ around the boundary C of the surface S, just as we did for Green's theorem.

More generalizations

The divergence theorem, covered in just a bit, is yet another version of this phenomenon. It relates the triple integral of the divergence of a three-dimensional vector field in a three-dimensional volume to the surface integral of that vector field on the boundary of that volume.

The fundamental theorem of line integrals also falls under the same overarching principle, relating the line integral of the gradient of a function to the values of that function on the bounds of the line.

In general, it seems that the universe is trying to tell us that when you integrate the "derivative" of a function within a region, where the type of integration/derivative/region/function involved might be multidimensional, you get something that just depends on the value of that function on the boundary of that region. I think this is just one of the most beautiful things in math.

The generalized Stokes' theorem

In case you are curious, pure mathematics does have a deeper theorem which captures all these theorems (and more) in a very compact formula. It is called the **generalized Stokes' theorem**. The language to describe it is a bit technical, involving the ideas of "differential forms" and "manifolds", so I won't go into it here. But if you understand all the examples above, you already understand the underlying intuition and beauty of this unifying theorem.