Partial derivatives of parametric surfaces

Google Classroom

If you have a function representing a surface in three dimensions, you can take its partial derivative. Here we see what that looks like, and how to interpret it.

Background

- Interpreting derivatives of vector valued functions
- Partial derivatives
- Parametric surfaces

What we're building to

- As setup, we have some vector-valued function with a two-dimensional input and a three-dimensional output:

$$\vec{\mathbf{v}}(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix}$$

Its partial derivatives are computed by taking the partial derivative of each component:

$$\frac{\partial \vec{\mathbf{v}}}{\partial t}(s,t) = \begin{bmatrix} \frac{\partial x}{\partial t}(s,t) \\ \frac{\partial y}{\partial t}(s,t) \\ \frac{\partial z}{\partial t}(s,t) \end{bmatrix}$$

$$\frac{\partial \vec{\mathbf{v}}}{\partial s}(s,t) = \begin{bmatrix} \frac{\partial x}{\partial s}(s,t) \\ \frac{\partial y}{\partial s}(s,t) \\ \frac{\partial z}{\partial s}(s,t) \end{bmatrix}$$

- You can interpret these partial derivatives as giving vectors tangent to the parametric surface defined by $\vec{\mathbf{v}}$.

The goal

Suppose I were to give you a function with a two-dimensional input, and a three-dimensional output, like this one:

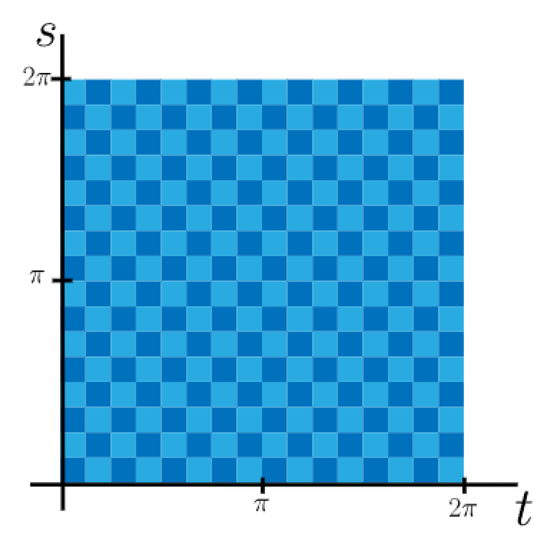
$$\vec{\mathbf{v}}(t, s) = \begin{bmatrix} 3\cos(t) + \cos(t)\cos(s) \\ 3\sin(t) + \sin(t)\cos(s) \\ \sin(s) \end{bmatrix}$$

Since the input is multi-dimensional, you cannot take the ordinary derivative of this function, but you can take a partial derivative. The focus of this article is on getting an intuitive feel for what those partial derivatives mean.

Interpret the function as a surface

The function itself actually has a very nice geometric meaning. Since it has a two-coordinate input and a three-coordinate output, we can visualize it as a parametric surface.

Specifically, consider all inputs (t, s) such that $0 \le t \le 2\pi$ and $0 \le s \le 2\pi$. This can be seen as a square in the "ts-plane". I'll draw this with a checkerboard pattern since it makes things easier to follow later on.



For any given point (t, s), the value $\vec{\mathbf{v}}(t, s)$ is some point in three-dimensional space.

Concept check: Evaluate $\vec{\mathbf{v}}(\pi, \pi)$. In other words, where does the function $\vec{\mathbf{v}}$ take the input $(t, s) = (\pi, \pi)$? [Hide explanation]

$$\vec{\mathbf{v}}(t, s) = \begin{bmatrix} 3\cos(t) + \cos(t)\cos(s) \\ 3\sin(t) + \sin(t)\cos(s) \\ \sin(s) \end{bmatrix}$$

Choose 1 answer:

(A) $\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$

(B) $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$

Check

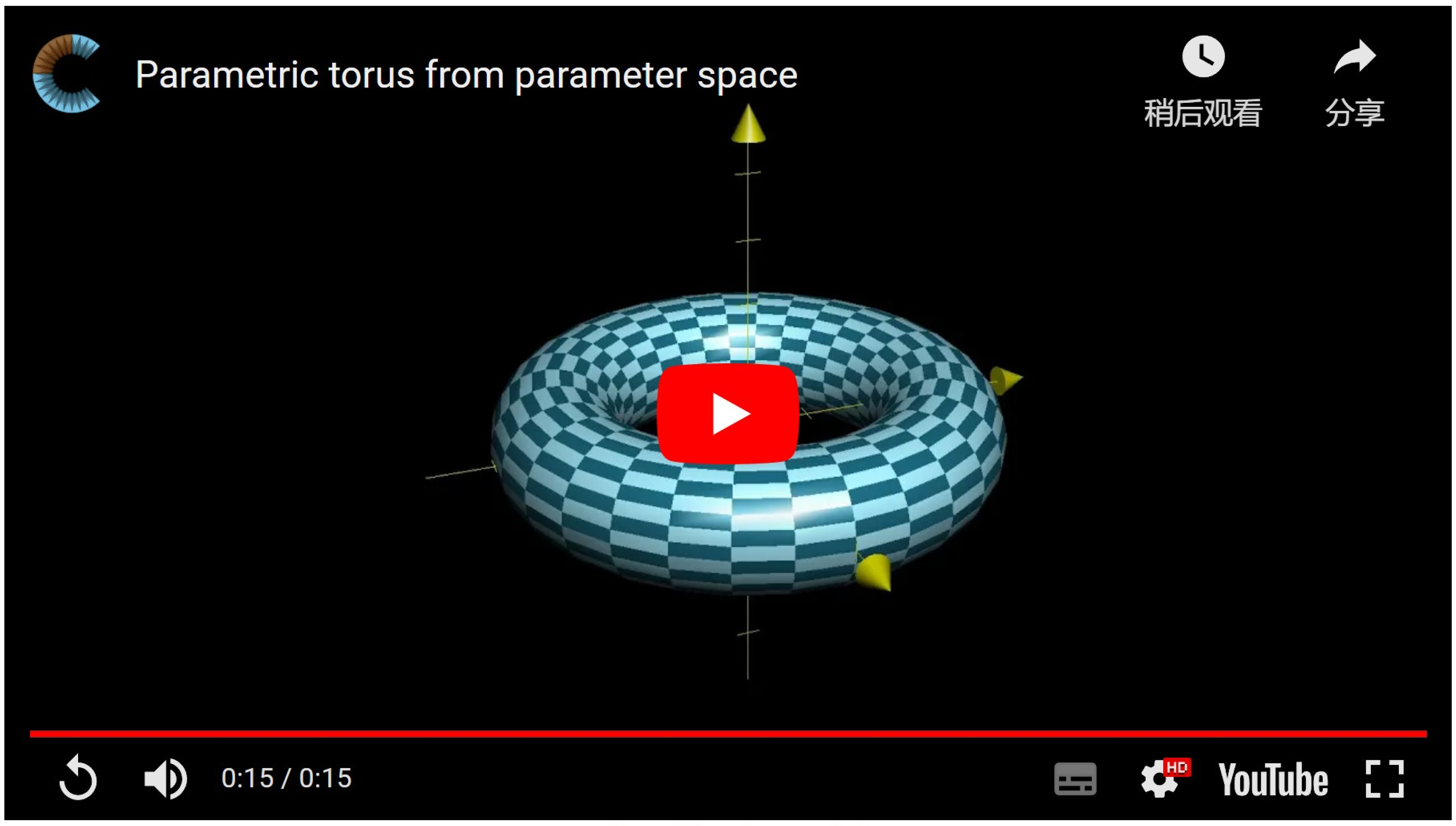
[Hide explanation]

$$\vec{\mathbf{v}}(t, s) = \begin{bmatrix} 3\cos(\pi) + \cos(\pi)\cos(\pi) \\ 3\sin(\pi) + \sin(\pi)\cos(\pi) \\ \sin(\pi) \end{bmatrix}$$

$$= \begin{bmatrix} 3(-1) + (-1)(-1) \\ 3(0) + (0)(-1) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

If you imagine doing this computation for all inputs (t, s) in the square, getting some point in three-dimensional space each time, all of the resulting outputs will form a two-dimensional surface in three-dimensional space. I like to imagine each point of the square moving to its appropriate location in space.



See video transcript

The result is a doughnut shape! Math folk call this a torus.

Interpreting the partial derivatives

Differentiate with respect to t

To compute a partial derivative of this function, say $\frac{\partial \vec{\mathbf{v}}}{\partial t}$, you take the partial derivative of each individual component.

$$\frac{\partial \vec{\mathbf{v}}}{\partial t}(t, s) = \frac{\partial}{\partial t}\begin{bmatrix} 3\cos(t) + \cos(t)\cos(s) \\ 3\sin(t) + \sin(t)\cos(s) \\ \sin(s) \end{bmatrix}$$

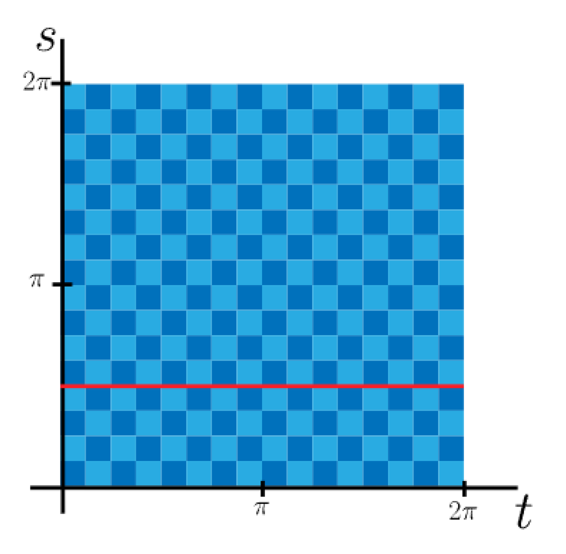
$$= \begin{bmatrix} \frac{\partial}{\partial t}(3\cos(t) + \cos(t)\cos(s)) \\ \frac{\partial}{\partial t}(3\sin(t) + \sin(t)\cos(s)) \\ \frac{\partial}{\partial t}(\sin(s)) \end{bmatrix}$$

$$= \begin{bmatrix} -3\sin(t) - \sin(t)\cos(s) \\ 3\cos(t) + \cos(t)\cos(s) \\ 0 \end{bmatrix}$$

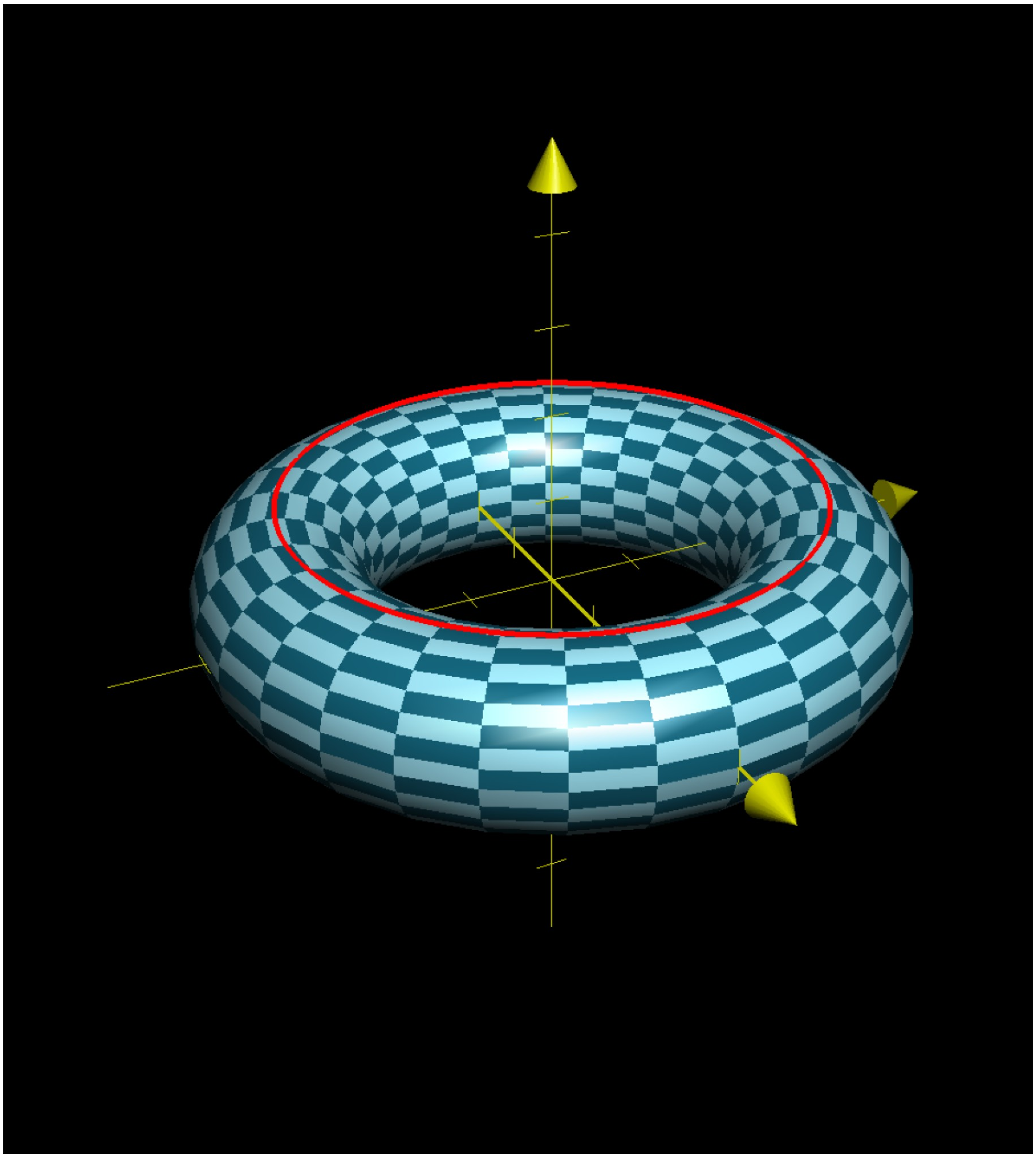
So...what does this new vector-valued function actually mean?

Well, computing this partial derivative requires treating the variable s as if it was constant. What does this mean geometrically?

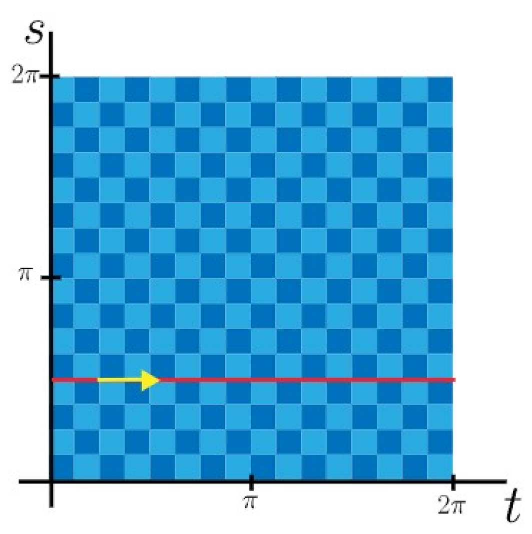
In the ts-plane, a constant value of s corresponds with a horizontal line. Here's one such line representing $s = \pi/2$, drawn in red:

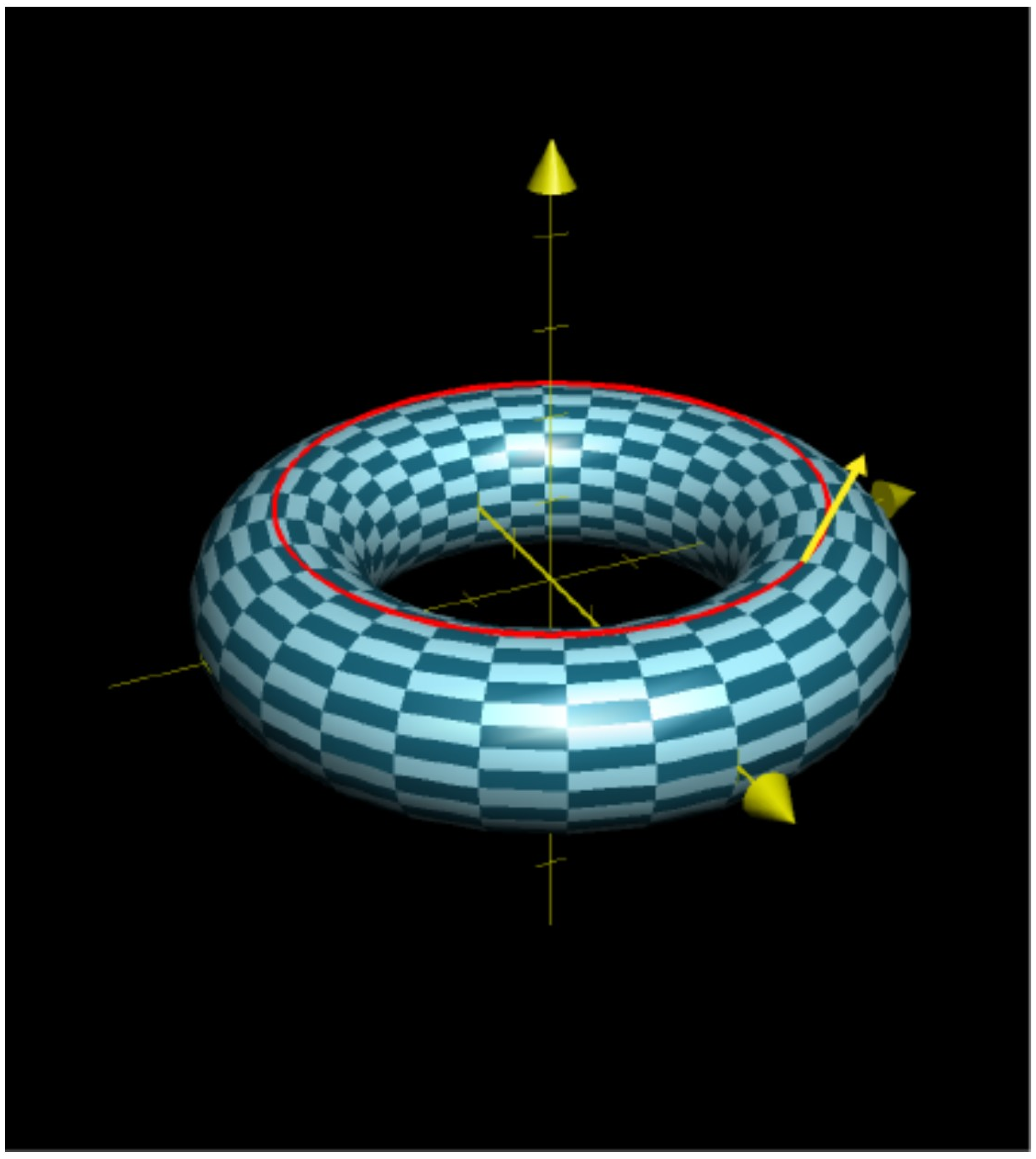


After this square gets warped and morphed into the torus, this red line gets turned into some circle which goes the long way around the torus:



The partial derivative $\frac{\partial \vec{\mathbf{v}}}{\partial t}$ tells us how the output changes slightly when we nudge the input in the t-direction. In this case, the vector representing that nudge (drawn in yellow below) gets transformed into a vector tangent to the red circle which represents a constant value of s on the surface:





Specifically, the input point used for the pictures above is $(t_0, s_0) = \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. This means the point on the torus is

$$\vec{\mathbf{v}}\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = \begin{bmatrix} 3\cos(\pi/4) + \cos(\pi/4)\cos(\pi/2) \\ 3\sin(\pi/4) + \sin(\pi/4)\cos(\pi/2) \\ \sin(\pi/2) \end{bmatrix}$$

$$= \begin{bmatrix} 3\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(0) \\ 3\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(0) \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{3\sqrt{2}}{2} \\ \\ \dfrac{3\sqrt{2}}{2} \\ \\ 1 \end{bmatrix}$$

And the tangent vector is

$$\frac{\partial \vec{\mathbf{v}}}{\partial t}\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = \begin{bmatrix} -3\sin(\pi/4) - \sin(\pi/4)\cos(\pi/2) \\ 3\cos(\pi/4) + \cos(\pi/4)\cos(\pi/2) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(0) \\ 3\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(0) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3\sqrt{2}}{2} \\ \frac{3\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

Concept check: Why does it make sense that the z-component of this tangent vector is 0?

Choose 1 answer:

(A) The z-coordinate of all the points on the red circle representing $s = \pi/2$ is always 0.

CORRECT (SELECTED)

The z-coordinate of all the points on the red circle representing $s = \pi/2$ does not change.

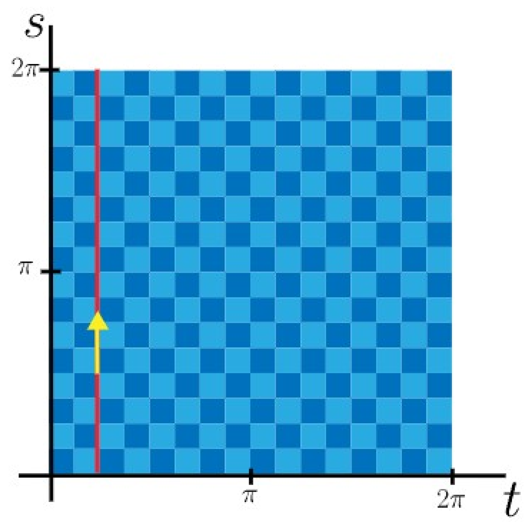
Check

Differentiate with respect to s

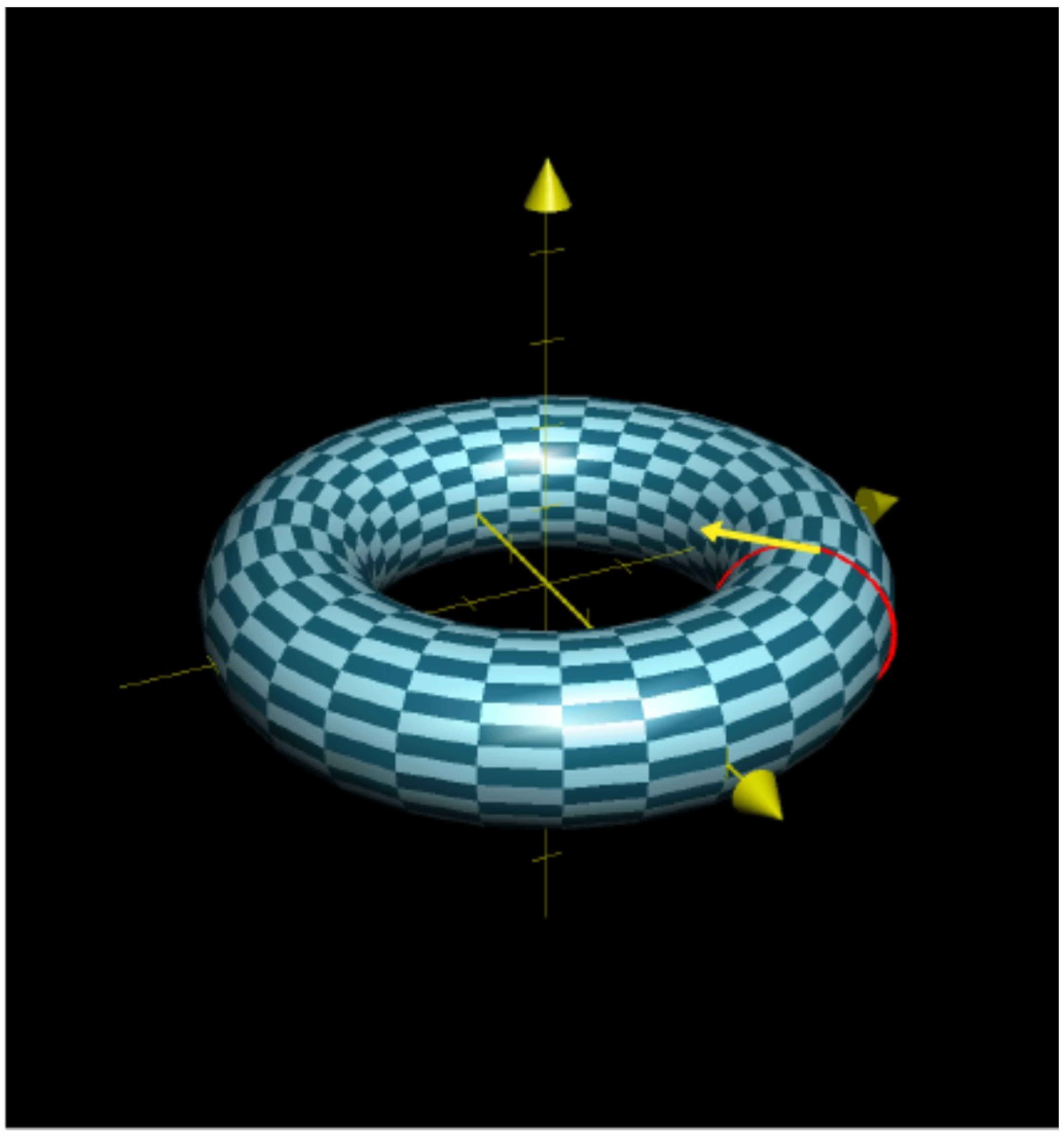
The partial derivative with respect to s is similar. You compute it by taking the partial derivative of each component in the definition of $\vec{\mathbf{v}}$:

$$\frac{\partial \vec{\mathbf{v}}}{\partial s}(t, s) = \frac{\partial}{\partial s} \begin{bmatrix} 3\cos(t) + \cos(t)\cos(s) \\ 3\sin(t) + \sin(t)\cos(s) \\ \sin(s) \end{bmatrix}$$
$$= \begin{bmatrix} -\cos(t)\sin(s) \\ -\sin(t)\sin(s) \\ \cos(s) \end{bmatrix}$$

This time, we can imagine holding t constant to get some vertical line in the parameter space.



The yellow arrow represents some velocity vector as a particle travels up along this line. Which is to say, as you vary s while holding t constant. After the square turns into the torus via the function $\vec{\mathbf{v}}$, the red line and the yellow velocity vector might look something like this:



The partial derivative $\dfrac{\partial \vec{\mathbf{v}}}{\partial s}$ can be interpreted as this resulting velocity vector on the torus.

Summary

- As setup, we have some vector-valued function with a two-dimensional input and a three-dimensional output:

$$\vec{\mathbf{v}}(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix}$$

  Its partial derivatives are computed by taking the partial derivative of each component:

$$\frac{\partial \vec{\mathbf{v}}}{\partial t}(s,t) = \begin{bmatrix} \frac{\partial x}{\partial t}(s,t) \\ \frac{\partial y}{\partial t}(s,t) \\ \frac{\partial z}{\partial t}(s,t) \end{bmatrix}$$

$$\frac{\partial \vec{\mathbf{v}}}{\partial s}(s,t) = \begin{bmatrix} \frac{\partial x}{\partial s}(s,t) \\ \frac{\partial y}{\partial s}(s,t) \\ \frac{\partial z}{\partial s}(s,t) \end{bmatrix}$$

- You can interpret these partial derivatives as giving vectors tangent to the parametric surface defined by $\vec{\mathbf{v}}$.

- For example, imagine nudging a point in the input space along the t direction, say from the coordinates (s,t) to the coordinates $(s, t+h)$ for some small h. This results in some small nudge in the output along the surface, which is represented by the vector $h\frac{\partial \vec{\mathbf{v}}}{\partial t}(s,t)$.