Flux in 3D example

Google Classroom

After learning about what flux in three dimensions is, here you have the chance to practice with an example.

Background

[Flux in three dimensions](#)
[Unit normal vector](#)
[Surface integral](#)

The steps

In the last article, I talked about how the **flux** of a flowing fluid through a surface is a measure of how much fluid passes through that surface per unit of time. If that fluid flow is represented with a vector field $F(x, y, z)$, and if S represents the surface itself, the flux is computed with the following surface integral:

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, d\Sigma$$

The vector-valued function $\hat{\mathbf{n}}(x, y, z)$ gives the unit normal vector to S. For closed surfaces, you typically choose an outward facing unit normal vector.

In practice, there is quite a lot that goes into solving this integral.

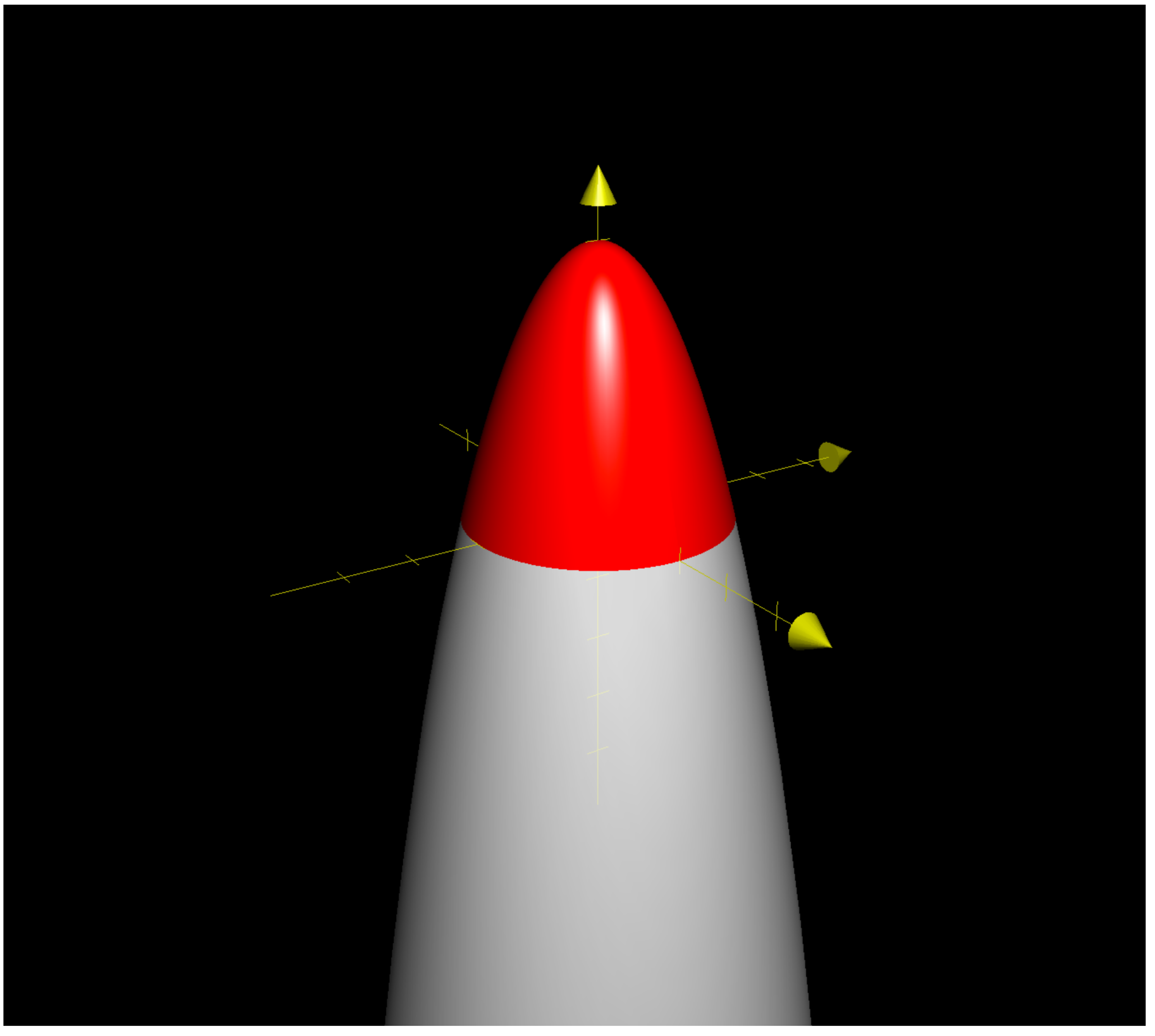
- **Step 1**: Rewrite the integral in terms of a parameterization of S, as you would for any surface integral.
- **Step 2**: Insert the expression for the unit normal vector $\hat{\mathbf{n}}(x, y, z)$. It's best to do this before you actually compute the unit normal vector since part of it cancels out with a term from the surface integral.
- **Step 3**: Simplify the terms inside the integral.
- **Step 4**: Compute the double integral.

The problem

Consider the paraboloid graph defined by the following equation:

$$z = 4 - x^2 - y^2$$

Let S be the portion of this paraboloid which sits **above the xy-plane**:



Whoa, that ended up looking way more like a nuclear warhead than I intended. Ah well, at least it makes clear what surface we're talking about.

For flux integrals, we must specify the orientation of this surface. Let's orient it with **outward-facing normal vectors**, in the sense that the vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are all outward facing from the region under the paraboloid.

Now imagine there is a fluid flowing around in three-dimensional space. Let's say it flows along the vector field defined by the function

$$\mathbf{F}(x, y, z) = \begin{bmatrix} xy \\ xz \\ yz \end{bmatrix}$$

Key question: What is the flux of this flowing fluid through the surface S?

Step 1: Rewrite the flux integral using a parameterization

Right now, the surface S has been defined as a graph, subject to a constraint on z.

Graph: $z = 4 - x^2 - y^2$

Constraint: $z \geq 0$

But for computing surface integrals, we need to describe this surface parametrically. Luckily, this conversion is not to hard. You basically let one parameter play the role of x, while the other parameter plays the role of y:

$$\mathbf{v}(t, s) = \begin{bmatrix} t \\ s \\ 4 - t^2 - s^2 \end{bmatrix}$$

After writing this function, you still need to specify what region of the ts-plane corresponds with our surface S. This requires translating the constraint $z \geq 0$ into a constraint on t and s.

Concept check: What is the constraint on t and s which will ensure that the z-component of $\vec{\mathbf{v}}(t, s)$ is greater than or equal to 0? Write your answer as an inequality.

Check

[Hide explanation]

We take the z-component of the definition and set it to be ≥ 0.

$$4 - t^2 - s^2 \geq 0$$

To better understand what this region looks like, it helps to rearrange it a bit.

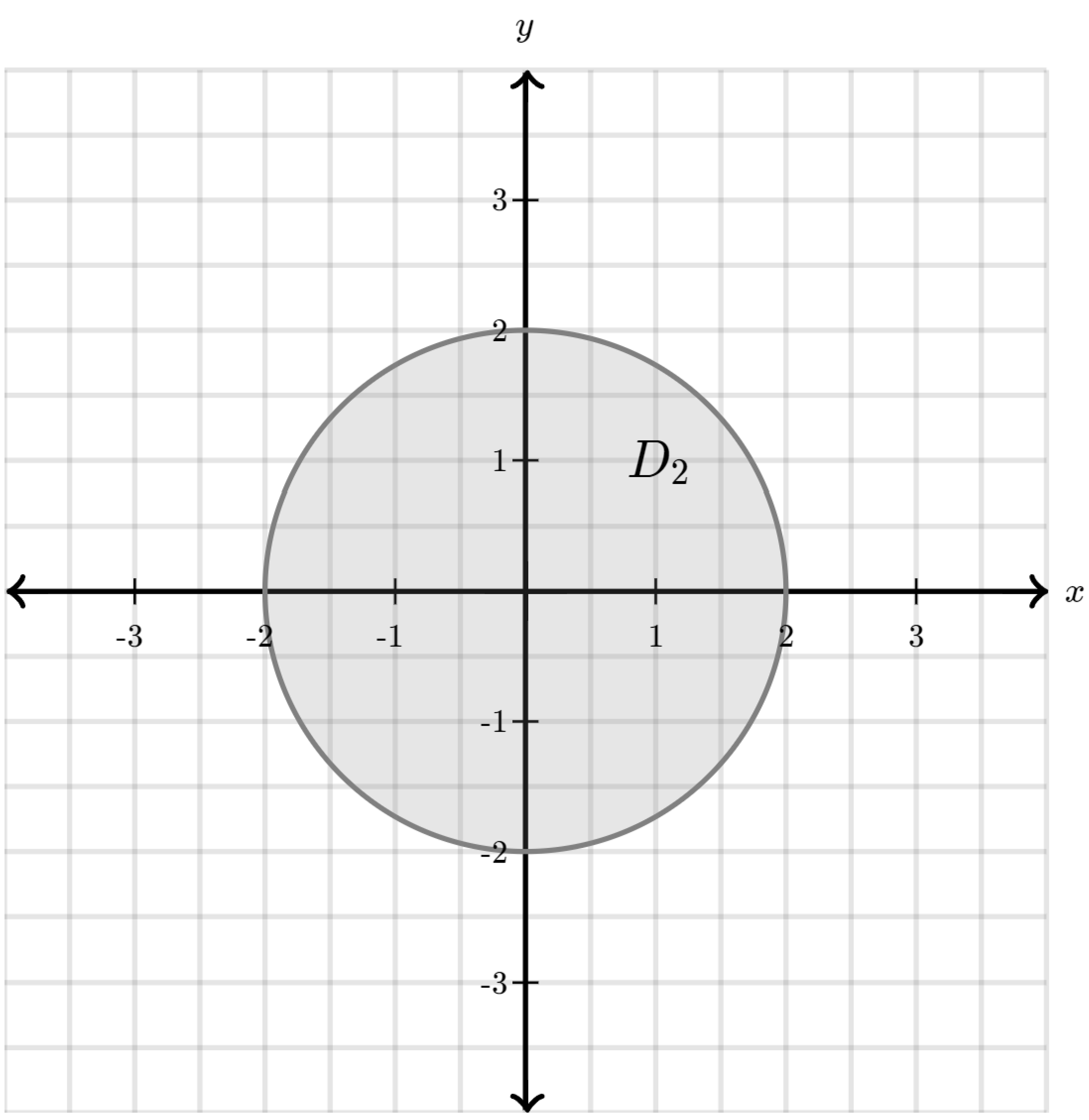
$$4 - t^2 - s^2 \geq 0$$

$$4 \geq t^2 + s^2$$

$$t^2 + s^2 \leq 4$$

Written like this, it is easier to see that the region we care about is a disk of radius 2 centered at the origin of the ts-plane.

Since this region is a disk with radius 2, let's name it D_2.



Later down the road, we'll expand this fully as a set of bounds for t and s, but while we work on all the innards of the integral it helps to just have a symbolic representation.

Writing our flux surface integral as a double integral in the parameter space, here's what we get:

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, d\Sigma$$

$$= \underbrace{\iint_{D_2}}_{\substack{\text{Double integral in} \\ \text{flat parameter space}}} \mathbf{F}(\mathbf{v}(t,s)) \cdot \hat{\mathbf{n}}(\mathbf{v}(t,s)) \underbrace{\left| \frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s} \right| dA}_{d\Sigma}$$

If this transition to a double integral in the parameter space seems unfamiliar, consider reviewing the article on surface integrals, or the one on surface area.

Step 2: Insert the expression for a unit normal vector

In a previous article, I talked about how you can find a function which gives the unit normal vector to a surface based on its parameterization $\mathbf{v}(t,s)$. Basically, you **normalize the cross product of the partial derivatives of $\mathbf{v}(t,s)$** (boy is that a mouthful to say):

$$\frac{\frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s}}{\left| \frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s} \right|}$$

Now, for those of you who just *love* computing the magnitude of cross products of partial derivative vectors, hold off for a moment. When we insert this into the flux integral, that magnitude term cancels out:

$$\iint_{D_2} \mathbf{F}(\mathbf{v}(t,s)) \cdot \hat{\mathbf{n}}(\mathbf{v}(t,s)) \left| \frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s} \right| dA$$

$$= \iint_{D_2} \mathbf{F}(\mathbf{v}(t,s)) \cdot \left(\frac{\frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s}}{\left| \frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s} \right|} \right) \left| \frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s} \right| dA$$

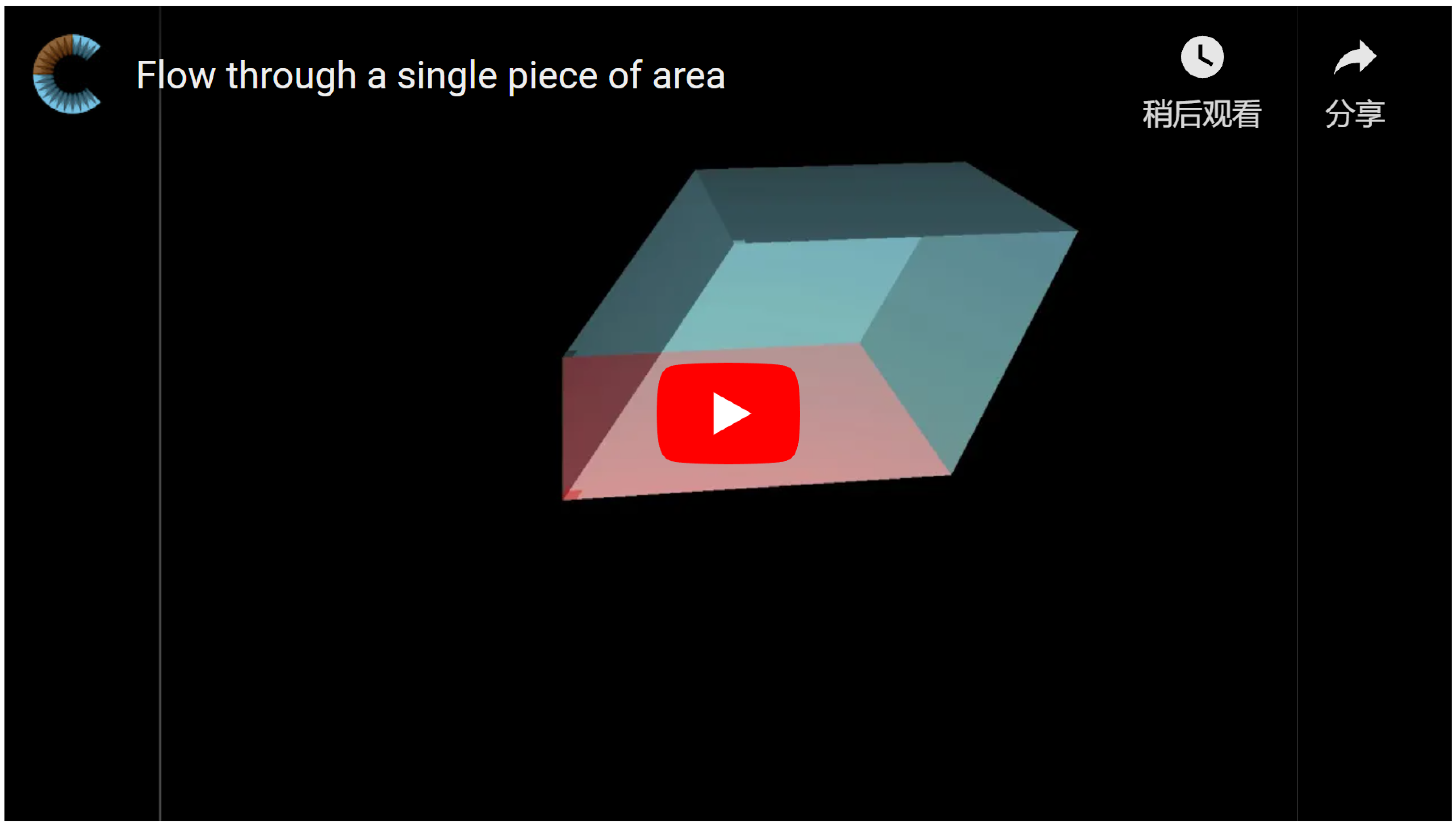
$$= \iint_{D_2} \mathbf{F}(\mathbf{v}(t,s)) \cdot \left(\frac{\frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s}}{\cancel{\left| \frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s} \right|}} \right) \cancel{\left| \frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s} \right|} dA$$

$$= \iint_{D_2} \mathbf{F}(\mathbf{v}(t,s)) \cdot \left(\frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s} \right) dA$$

[Hide explanation]

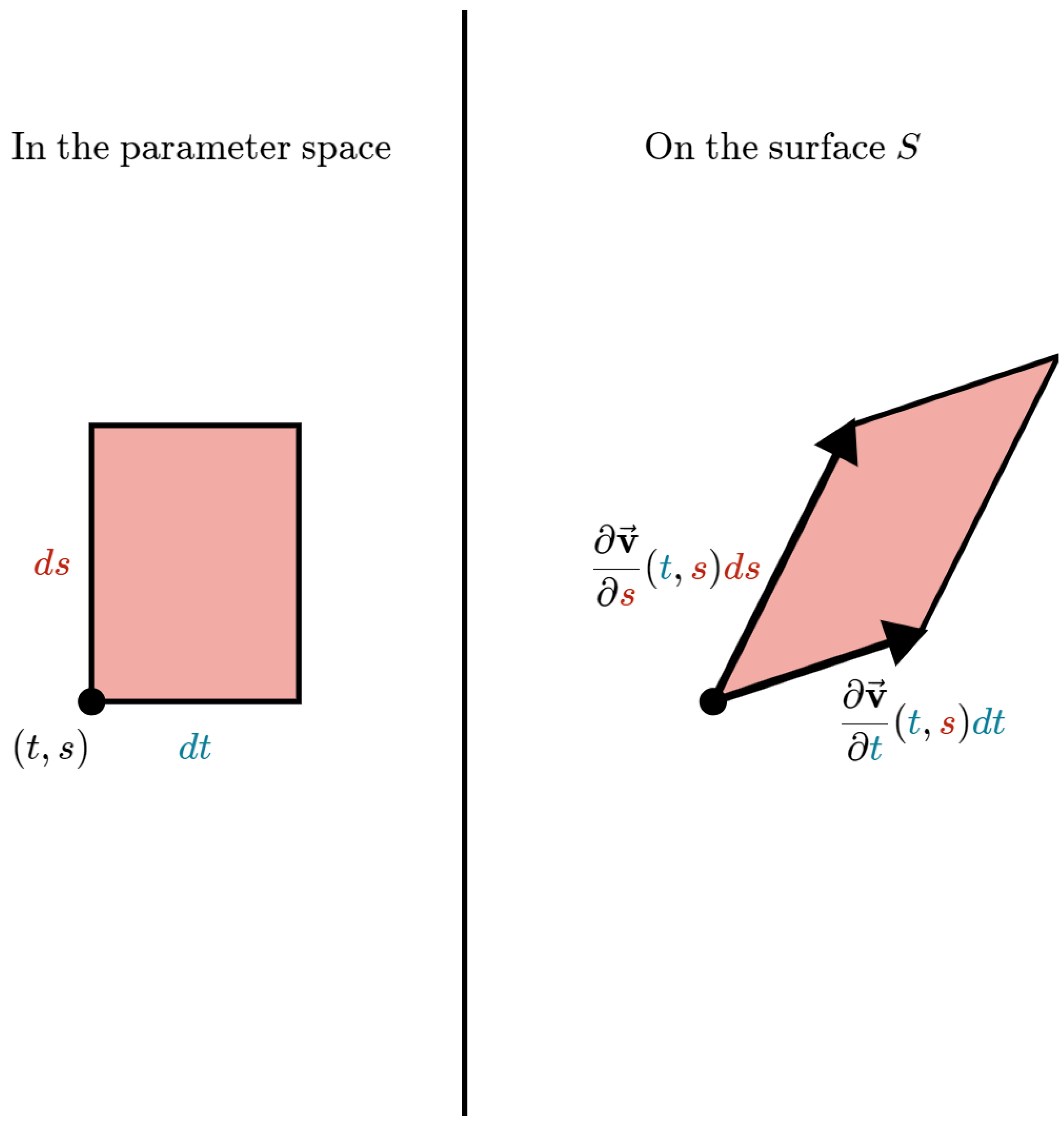
This is a bit detailed, but it's a very satisfying intuition once you wrap your mind around it.

Remember the intuition behind a flux integral: We divide the surface S into many tiny pieces, and look at how much fluid flows through each one of them per unit time:



See video transcript

Typically, we think of one of these tiny pieces of area as being a parallelogram, spanned by the two infinitesimal vectors $\left(\frac{\partial \mathbf{v}}{\partial t}\, dt\right)$ and $\left(\frac{\partial \mathbf{v}}{\partial s}\, ds\right)$.



Because this piece is so small, we can assume that all the fluid particles passing through it move with basically the same velocity vector, $\mathbf{F}(\vec{\mathbf{v}}(t,s))$ for some point (t,s) which lands you inside the piece. This means the volume formed by fluid passing through this piece forms a **parallelepiped** (my favorite word!), which is the three-dimensional analog of a parallelogram.

Let's assume the parallelepiped represents the fluid passing through the parallelogram over one unit of time. Then to find its volume, you want to multiply the **area of the parallelogram** by the **component of the vector $\mathbf{F}(\vec{\mathbf{v}}(t,s))$ which is perpendicular to that parallelogram**. But think about what this cross product represents:

$$\left(\frac{\partial \mathbf{v}}{\partial t}\,dt\right) \times \left(\frac{\partial \mathbf{v}}{\partial s}\,ds\right)$$

It gives a vector perpendicular to the parallelogram, whose magnitude is the area of that parallelogram. So what is the meaning of this dot product:

$$\mathbf{F}(\vec{\mathbf{v}}(t,s)) \cdot \left(\frac{\partial \mathbf{v}}{\partial t}\,dt \times \frac{\partial \mathbf{v}}{\partial s}\,ds\right)$$

It will give the component of $\mathbf{F}(\vec{\mathbf{v}}(t,s))$ that is perpendicular to the parallelogram, multiplied by the area of that parallelogram. But this is exactly the volume we want!

When you pull out the dt and ds terms, whose product is dA, you get the integral that we found above after canceling out the norm of the cross product.

$$\iint_{D_2} \mathbf{F}(\mathbf{v}(t,s)) \cdot \left(\frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s}\right) dA$$

Step 3: Expand the integrand

Let's start by working out the cross product term:

$$\frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s}$$

For reference, this was how I defined $\mathbf{v}(t,s)$:

$$\mathbf{v}(t, s) = \begin{bmatrix} t \\ s \\ 4 - t^2 - s^2 \end{bmatrix}$$

Now compute each partial derivative, then find their cross product.

$$\frac{\partial \mathbf{v}}{\partial t} = \square\, \hat{\mathbf{i}} + \square\, \hat{\mathbf{j}} + \square\, \hat{\mathbf{k}}$$

Check

[Hide explanation]

$$\frac{\partial \mathbf{v}}{\partial t}(t, s) = \begin{bmatrix} \frac{\partial}{\partial t}(t) \\ \frac{\partial}{\partial t}(s) \\ \frac{\partial}{\partial t}(4 - t^2 - s^2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2t \end{bmatrix}$$

$$\frac{\partial \mathbf{v}}{\partial s} = \square\, \hat{\mathbf{i}} + \square\, \hat{\mathbf{j}} + \square\, \hat{\mathbf{k}}$$

Check

[Hide explanation]

$$\frac{\partial \mathbf{v}}{\partial s}(t, s) = \begin{bmatrix} \frac{\partial}{\partial s}(t) \\ \frac{\partial}{\partial s}(s) \\ \frac{\partial}{\partial s}(4 - t^2 - s^2) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2s \end{bmatrix}$$

$$\frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s} = \square\, \hat{\mathbf{i}} + \square\, \hat{\mathbf{j}} + \square\, \hat{\mathbf{k}}$$

Check

[Hide explanation]

To take the cross product, use the usual determinant trick, where the 2^{nd} and 3^{rd} rows are the answers to the previous two questions:

$$\det\left(\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -2t \\ 0 & 1 & -2s \end{bmatrix}\right)$$

For the $\hat{\mathbf{i}}$ component, cross out the top row and left column, then take the determinant:

$$\det\left(\begin{bmatrix} 0 & -2t \\ 1 & -2s \end{bmatrix}\right) = (0)(-2s) - (-2t)(1) = 2t$$

For the $\hat{\mathbf{j}}$ component, cross out the top row and *middle* column, then take the *negative* determinant:

$$-\det\left(\begin{bmatrix} 1 & -2t \\ 0 & -2s \end{bmatrix}\right) = -((-2s)(1) - (-2t)(0)) = 2s$$

Finally, for the $\hat{\mathbf{k}}$ component, cross out the top row and *last* column, then take the determinant:

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$$

Concept check: Does the expression for $\dfrac{\partial \mathbf{v}}{\partial t} \times \dfrac{\partial \mathbf{v}}{\partial s}$ that you just found give an outward-facing or inward-facing normal vector?

Choose 1 answer:

(A) Outward-facing

(B) Inward-facing

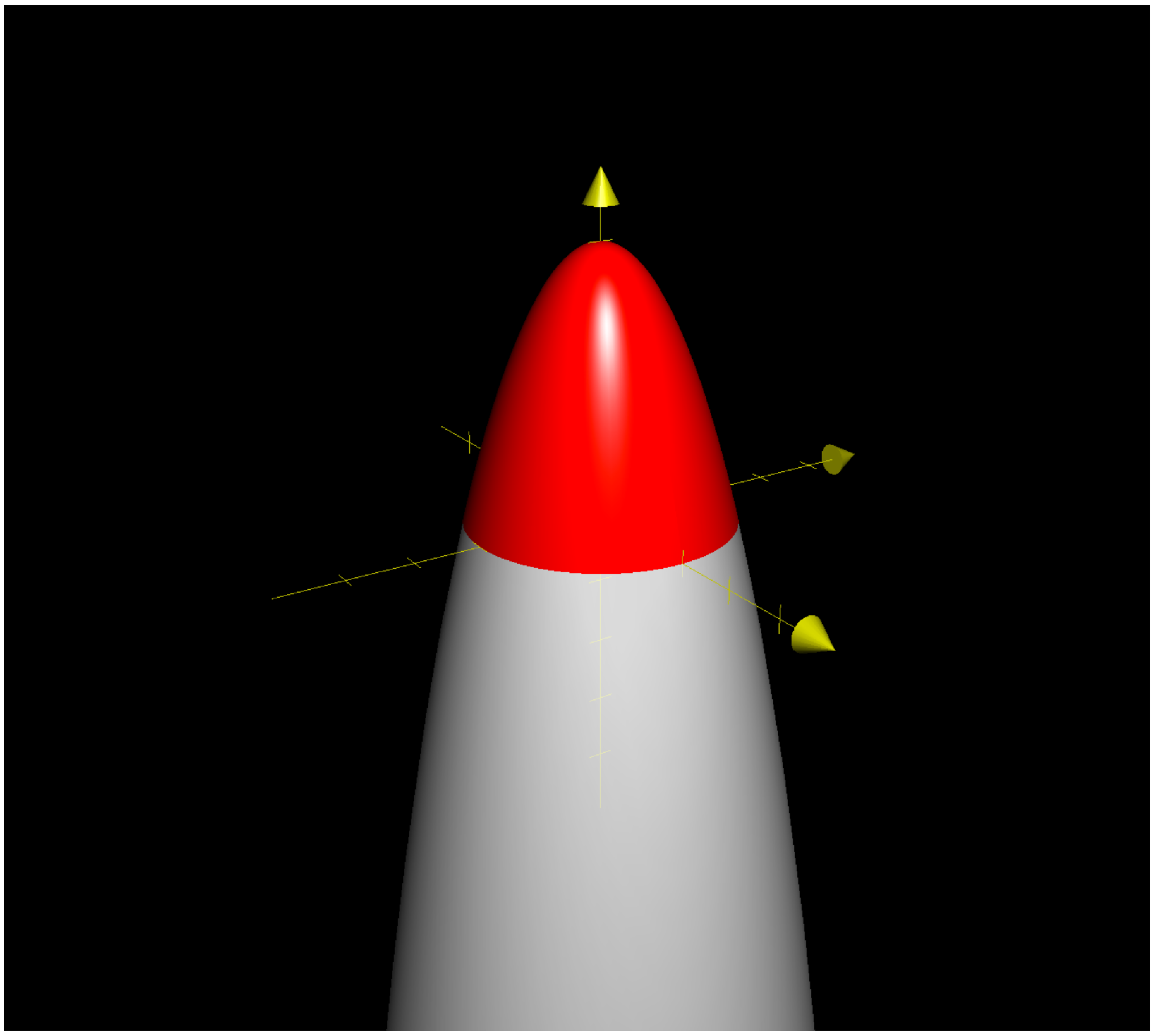
Check

[Hide explanation]

The best way to find out is to just plug in a specific input. The easiest input is probably $(t, s) = (0, 0)$, which corresponds to the very top of our paraboloid surface:

$$\begin{bmatrix} 2t \\ 2s \\ 1 \end{bmatrix} \to \begin{bmatrix} 2(0) \\ 2(0) \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This is a vector pointing straight up, which is outside the region enclosed by the paraboloid. Therefore, it is an outward facing vector. If it was not, we would need to flip the sign of our surface integral.



Next, write out the term $\mathbf{F}(\mathbf{v}(t, s))$ in terms of just t and s. For reference, this is how $\mathbf{F}$ and $\vec{\mathbf{v}}$ are defined:

$$\mathbf{F}(x, y, z) = \begin{bmatrix} xy \\ xz \\ yz \end{bmatrix} \quad \mathbf{v}(t, s) = \begin{bmatrix} t \\ s \\ 4 - t^2 - s^2 \end{bmatrix}$$

$\mathbf{F}(\mathbf{v}(t, s)) =$ ______ $\hat{\mathbf{i}}+$ ______ $\hat{\mathbf{j}}+$ ______ $\hat{\mathbf{k}}$

Check

[Hide explanation]

Let's explicitly write the vector-valued function $\mathbf{v}(t, s)$ as three separate scalar-valued functions:

$$x(t, s) = t$$

$$y(t, s) = s$$

$$z(t, s) = 4 - t^2 - s^2$$

Plugging those into the definition of $\mathbf{F}$, we get the following:

$$\mathbf{F}(\mathbf{v}(t, s)) = \mathbf{F}(x(t, s), y(t, s), z(t, s))$$

$$= \begin{bmatrix} x(t, s)\, y(t, s) \\ x(t, s)\, z(t, s) \\ y(t, s)\, z(t, s) \end{bmatrix}$$

$$= \begin{bmatrix} ts \\ t(4 - t^2 - s^2) \\ s(4 - t^2 - s^2) \end{bmatrix}$$

Great! Now we have all the pieces for the innards of our integral.

$$\iint_{D_2} \mathbf{F}(\mathbf{v}(t, s)) \cdot \left(\frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s} \right) dA$$

By taking the dot product of the previous two answers, write the inside of this integral purely in terms of the parameters t and s. It will help the integral computation in the next section if you simplify your answer as much as possible.

$$\iint_{D_2} \boxed{} \; dA$$

Check

[Hide explanation]

$$\iint_{D_2} \mathbf{F}(\mathbf{v}(t,s)) \cdot \left(\frac{\partial \mathbf{v}}{\partial t} \times \frac{\partial \mathbf{v}}{\partial s} \right) dA$$

$$= \iint_{D_2} \begin{bmatrix} ts \\ t(4-t^2-s^2) \\ s(4-t^2-s^2) \end{bmatrix} \cdot \begin{bmatrix} 2t \\ 2s \\ 1 \end{bmatrix} dA$$

$$= \iint_{D_2} (ts)(2t) + \underbrace{t(4-t^2-s^2)(2s) + s(4-t^2-s^2)(1)}_{\text{Factor out } 4-t^2-s^2} dA$$

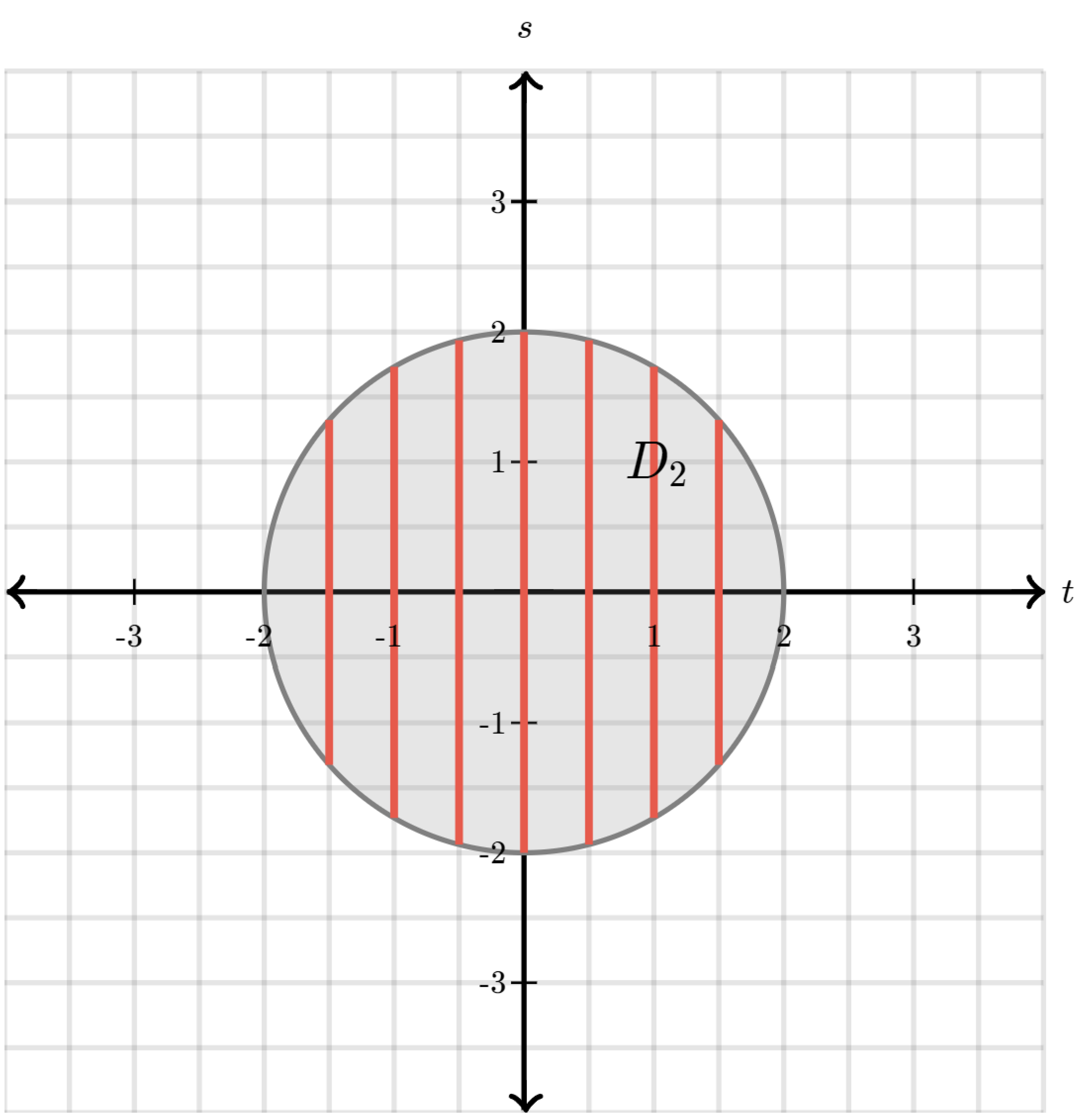
$$= \iint_{D_2} \underbrace{(ts)(2t) + (2ts+s)(4-t^2-s^2)}_{\text{Factor out } s} dA$$

$$= \iint_{D_2} s\left(2t^2 + (2t+1)(4-t^2-s^2)\right) dA$$

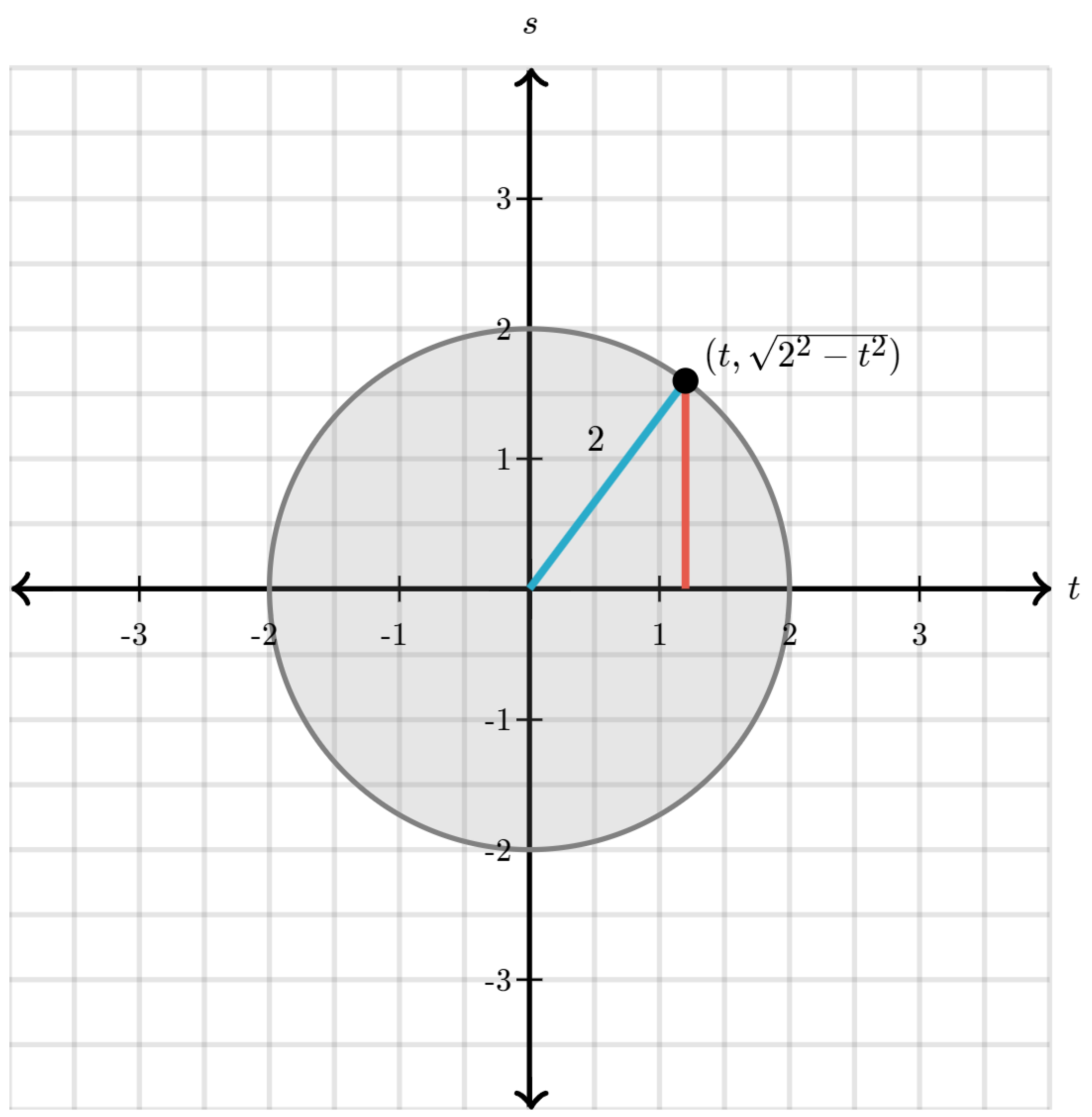
Step 4: Compute the integral

Up until this point, we have been writing a little D_2 under the double integral to indicate that the region of the ts-plane we will be integrating over is the disk with radius 2. Now, as we turn to computing the integral itself, we need to spell this out into concrete bounds on the parameters t and s.

To do this, draw yourself a picture of D_2, and imagine cutting it into vertical stripes:



The value t ranges from -2 to 2. The range for s depends on the value of t, which you can find using the pythagorean theorem.



From the diagram, you can see that s ranges from $-\sqrt{4-t^2}$ to $+\sqrt{4-t^2}$. Applying these bounds to our double integral, here's what we get:

$$\int_0^2 \int_{-\sqrt{4-t^2}}^{+\sqrt{4-t^2}} s\left(2t^2 + (2t+1)(4-t^2-s^2)\right) ds\, dt$$

From here, there are a few ways you might finish the problem

1. **Painfully**: Compute this double integral in full by hand (ugh!).

2. **Pragmatically**: Plug this into a calculator, or a computer algebra tool like Wolfram Alpha.

3. **Cleverly**: You can recognize that the integrand is an *odd function* with respect to s. Distribute the s, and notice that all terms either have an s or an s^3. This means the inner integral on the portion from $-\sqrt{4-t^2}$ to 0 will cancel out with the portion from 0 to $\sqrt{4-t^2}$. Therefore, the integral as a whole is 0.

Summary

A flux integral starts its life looking something like this:

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, d\Sigma$$

Solving this involves the following four steps:

- **Step 1**: Parameterize the surface, and translate this surface integral to a double integral over the parameter space.
- **Step 2**: Apply the formula for a unit normal vector.
- **Step 3**: Simplify the integrand, which involves two vector-valued partial derivatives, a cross product, and a dot product.
- **Step 4**: Compute the double integral (in practice a computer can handle it from here).