Line integrals in a vector field

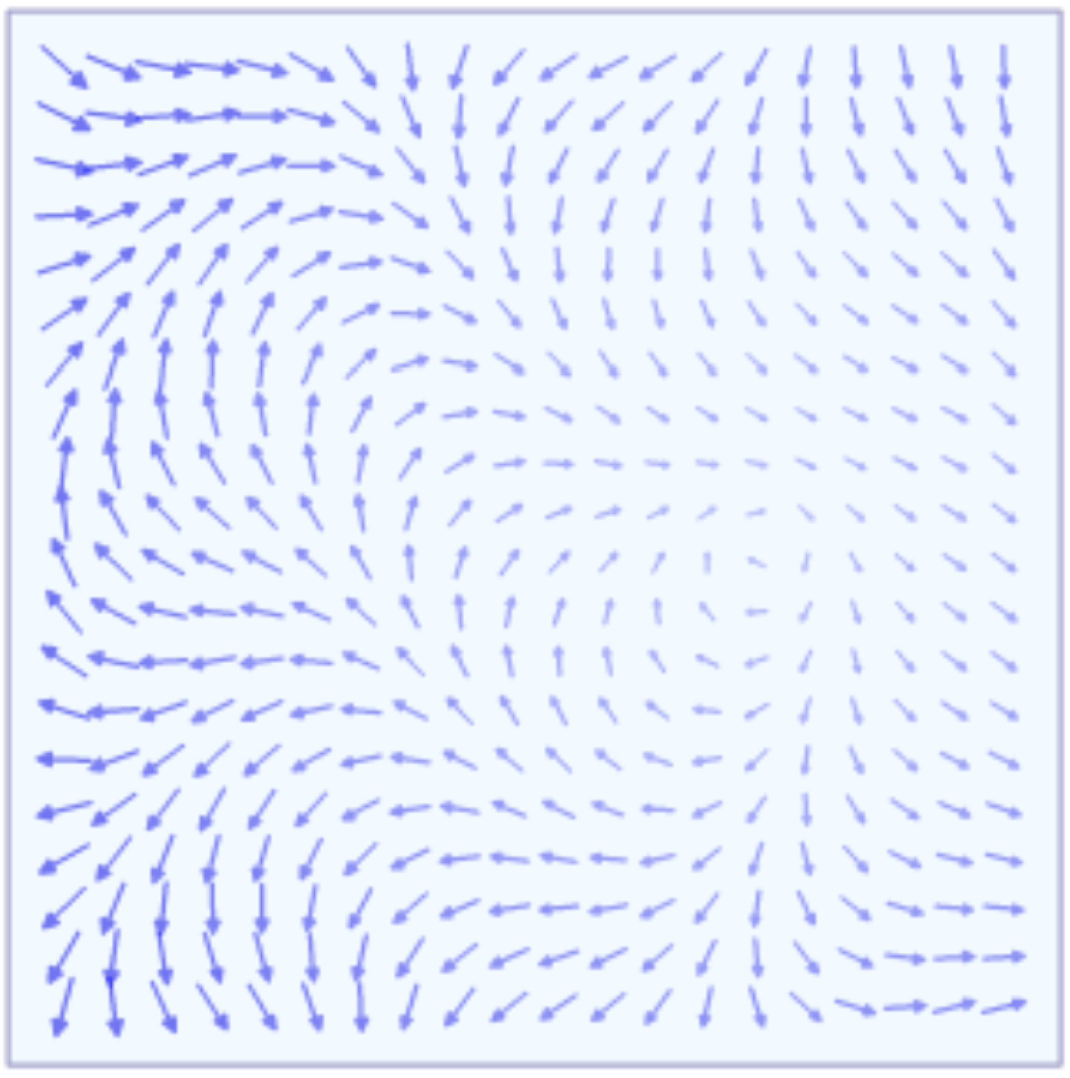
Google Classroom

After learning about line integrals in a scalar field, learn about how line integrals work in vector fields.

Background

- Line integrals in scalar fields
- Vector fields
- Interpreting the derivative of a vector-valued function

What we are building to



$\mathbf{F}$

This animation will be described in more detail below.

Animation credit: By Lucas V. Barbosa (Own work) [Public domain], via Wikimedia Commons

Let's say there is some vector field $\mathbf{F}$ and a curve C wandering through that vector field. Imagine walking along the curve, and at each step taking the dot product between the following two vectors:

- The vector from the field $\mathbf{F}$ at the point where you are standing.
- The displacement vector associated with the next step you take along this curve.

If you add up those dot products, you have just approximated the **line integral of $\mathbf{F}$ along C**

The shorthand notation for this line integral is

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

(Pay special attention to the fact that this is a dot product)

The more explicit notation, given a [parameterization]() $\mathbf{r}(t)$ of C, is

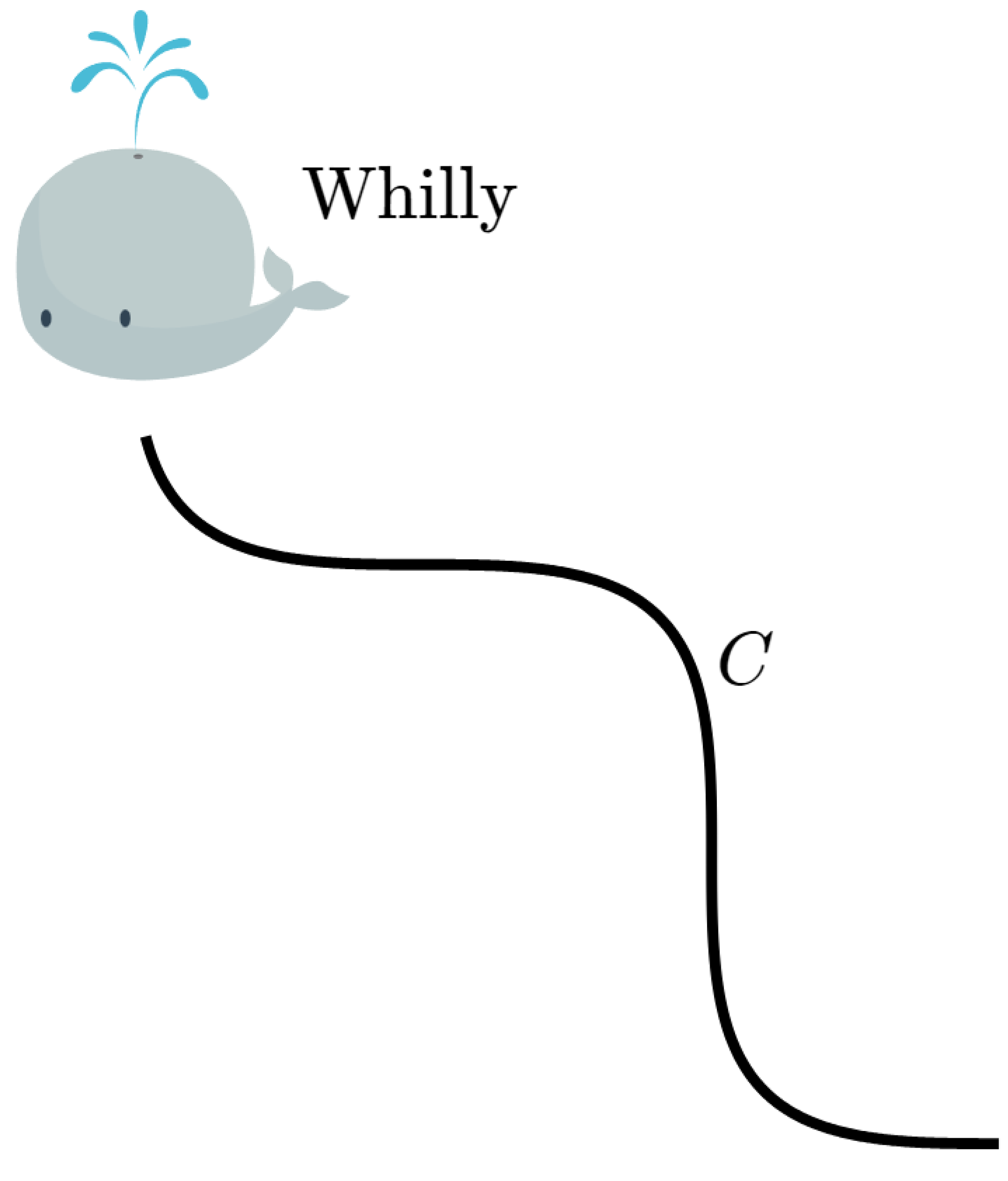
$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

Line integrals are useful in physics for computing the work done by a force on a moving object.

If you parameterize the curve such that you move in the opposite direction as t increases, the value of the line integral is multiplied by -1.

Whale falling from the sky

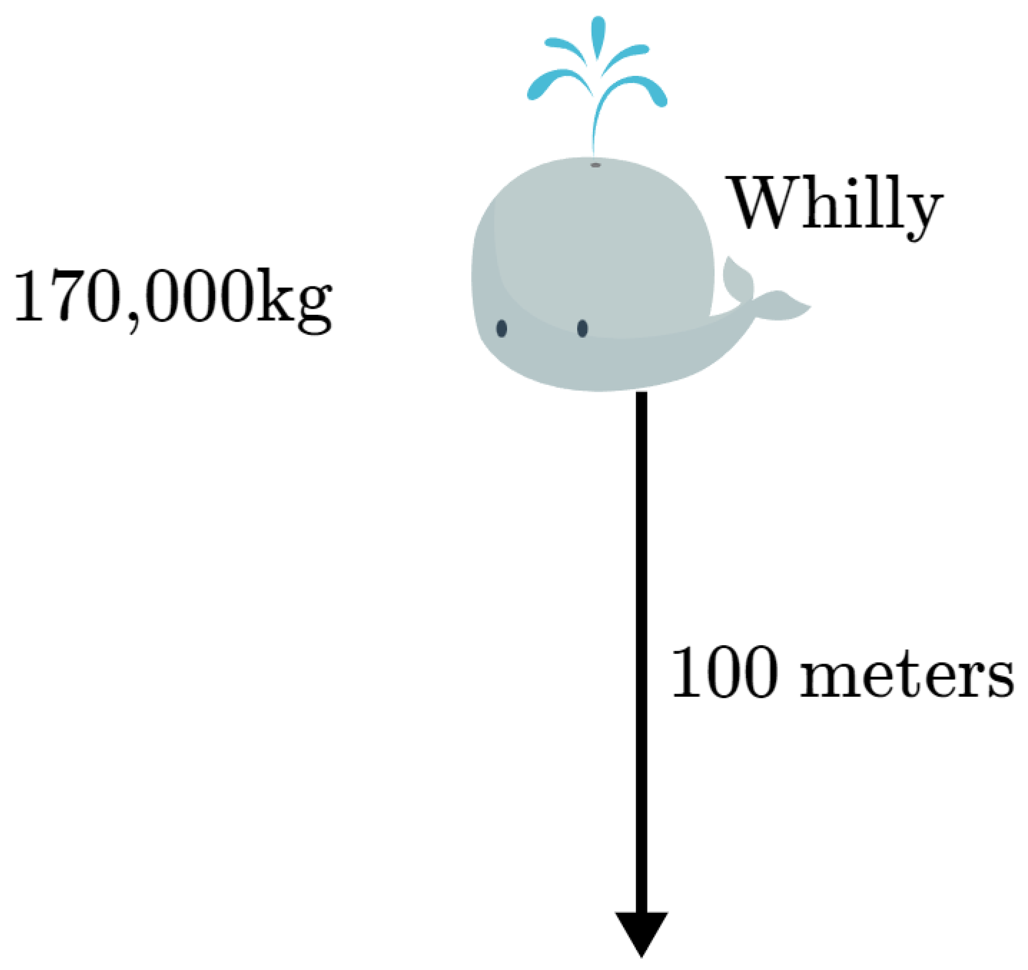
Let's say we have a whale, whom I'll name Whilly, falling from the sky. Suppose he falls along a curved path, perhaps because the air currents push him this way and that.



In this example, I am assuming you are familiar with the idea from physics that a force does work on a moving object, and that work is defined as the dot product between the force vector and the displacement vector.

[Hide explanation]

When there is a force, such as gravity, and an object moving in the region where the force acts, the force is said to "do work" on the object. For example, suppose Whilly falls straight down 100m from the sky (over some specified period of time):



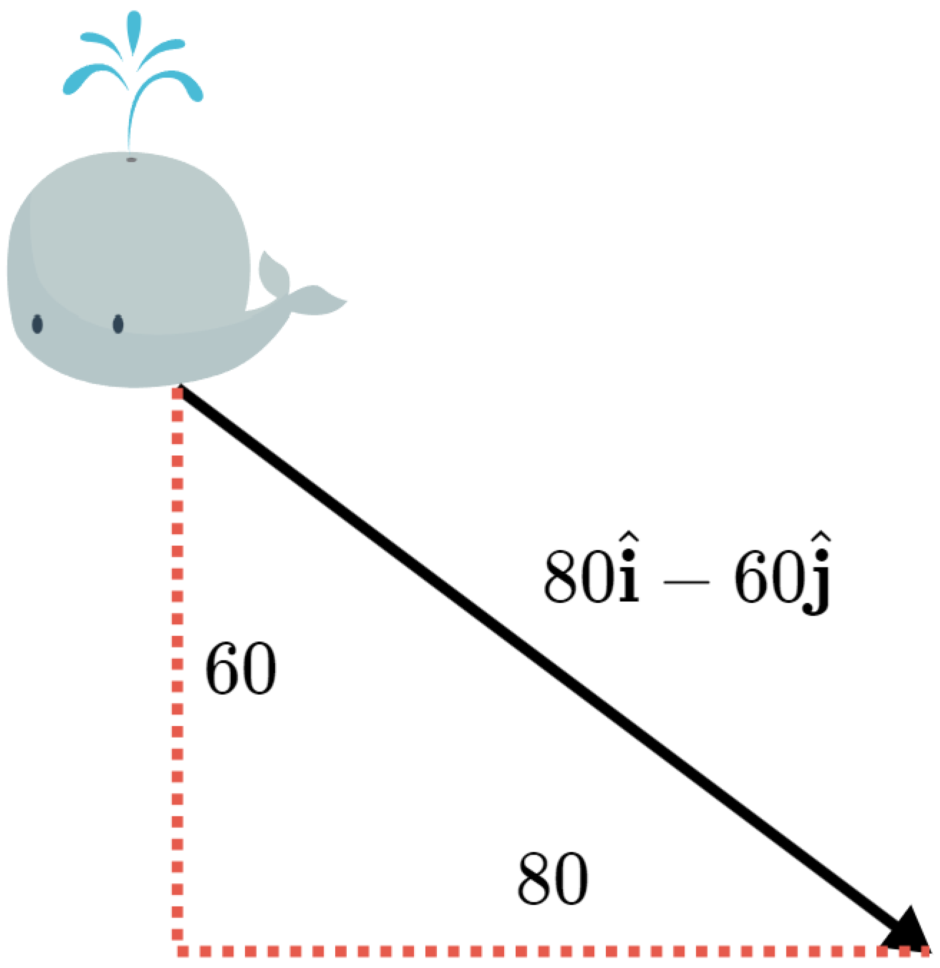
The force of gravity on this whale is

$$\begin{aligned} F &= mg \\ &= (170{,}000\text{kg})\left(9.8\frac{\text{m}}{\text{s}^2}\right) \end{aligned}$$

And the **work** that gravity does on Whilly **is the force times displacement**:

$$\begin{aligned} W &= Fs \\ &= \underbrace{(170{,}000\text{kg})\left(9.8\frac{\text{m}}{\text{s}^2}\right)}_{\text{Force}} \overbrace{(100\text{m})}^{\text{displacement}} \end{aligned}$$

Actually, this formula is not quite right. Suppose Whilly does not move straight down. Perhaps instead his displacement vector is down and to the right, with a y-component of -60m, and an x-component of 80m:



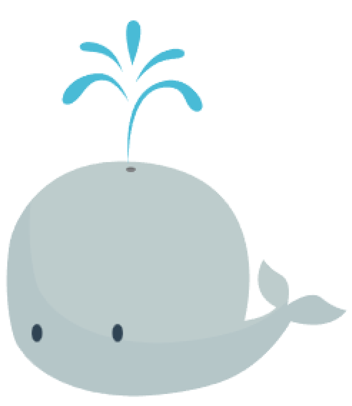
Gravity still does work on Whilly, but all that matters is the component of the Whilly's displacement vector in the direction of gravity. In other words, work is the **dot product** between the force vector and the displacement vector.

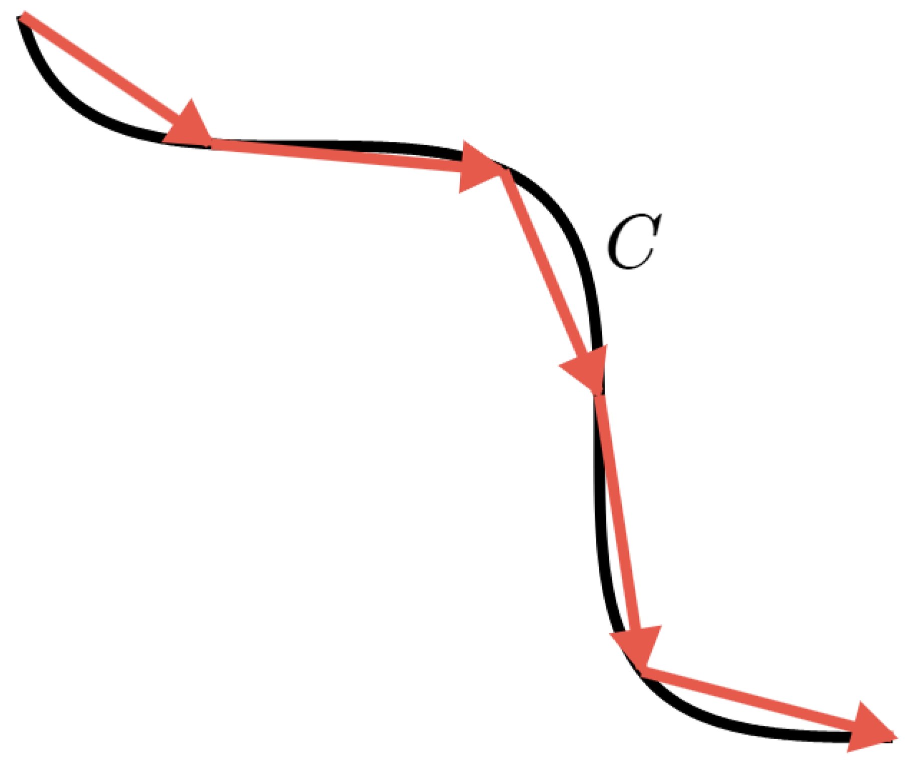
$$
\begin{aligned}
W &= \vec{F} \cdot \vec{s} \\
&= \left(-(170{,}000\text{kg})\left(9.8\frac{\text{m}}{\text{s}^2}\right)\hat{\mathbf{j}}\right) \cdot \left(80\text{m}\hat{\mathbf{i}} - 60\text{m}\hat{\mathbf{j}}\right) \\
&= -(170{,}000\text{kg})\left(9.8\frac{\text{m}}{\text{s}^2}\right)(-60\text{m}) \\
&= (170{,}000)(9.8)(60)\frac{\text{kg}\,\text{m}^2}{\text{s}^2}
\end{aligned}
$$

In this case, since gravity points purely in the $-\hat{\mathbf{j}}$ direction, performing the dot product ends up being the same as pulling out the vertical component from the displacement vector.

Key question: What is the work done on Whilly by gravity as he falls along the curved path C?

Usually, computing work is done with respect to a straight force vector and a straight displacement vector, so what can we do with this curved path? You can start by imagining the curve is broken up into many little displacement vectors:





Go ahead and give each one of these displacement vectors a name,

$\vec{\Delta s_1}, \vec{\Delta s_2}, \vec{\Delta s_3}, \ldots$

The work done by gravity along each one of these displacement vectors is the gravity force vector, which I'll denote $\vec{F_g}$, dotted with the displacement vector itself:

$$\vec{F_g} \cdot \vec{\Delta s_i}$$

The total work done by gravity along the entire curve is then estimated by

$$\sum_{n=1}^{N} \vec{F_g} \cdot \vec{\Delta s_n}$$

But of course, this is calculus, so we don't just look at a specific number of finite steps along the curve C. We consider what limiting value this sum approaches as the size of those steps shrinks smaller and smaller. This is captured with the following integral:

$$\int_C \vec{F_g} \cdot \vec{ds}$$

This is very similar to [line integration in a scalar field](), but there is the key difference: **The tiny step $\vec{ds}$ is now thought of as a vector, not a scalar length.** In the integral above, I wrote both $\vec{F_g}$ and $\vec{ds}$ with little arrows on top to emphasize that they are vectors. A more subtle and more common way to emphasize that these are vector quantities is to write the variable in bold:

$$\int_C \mathbf{F}_g \cdot d\mathbf{s}$$

Key takeaway: The thing we're adding up as we wander along C is not the full value of $\mathbf{F}_g$ at each point, but the component of $\mathbf{F}_g$ pointed in the same direction as the vector $d\mathbf{s}$. That is, the component of force in the direction of the curve.

Example 1: Putting numbers on Whilly's fall.

Let's see how this plays out when we go through the computation.

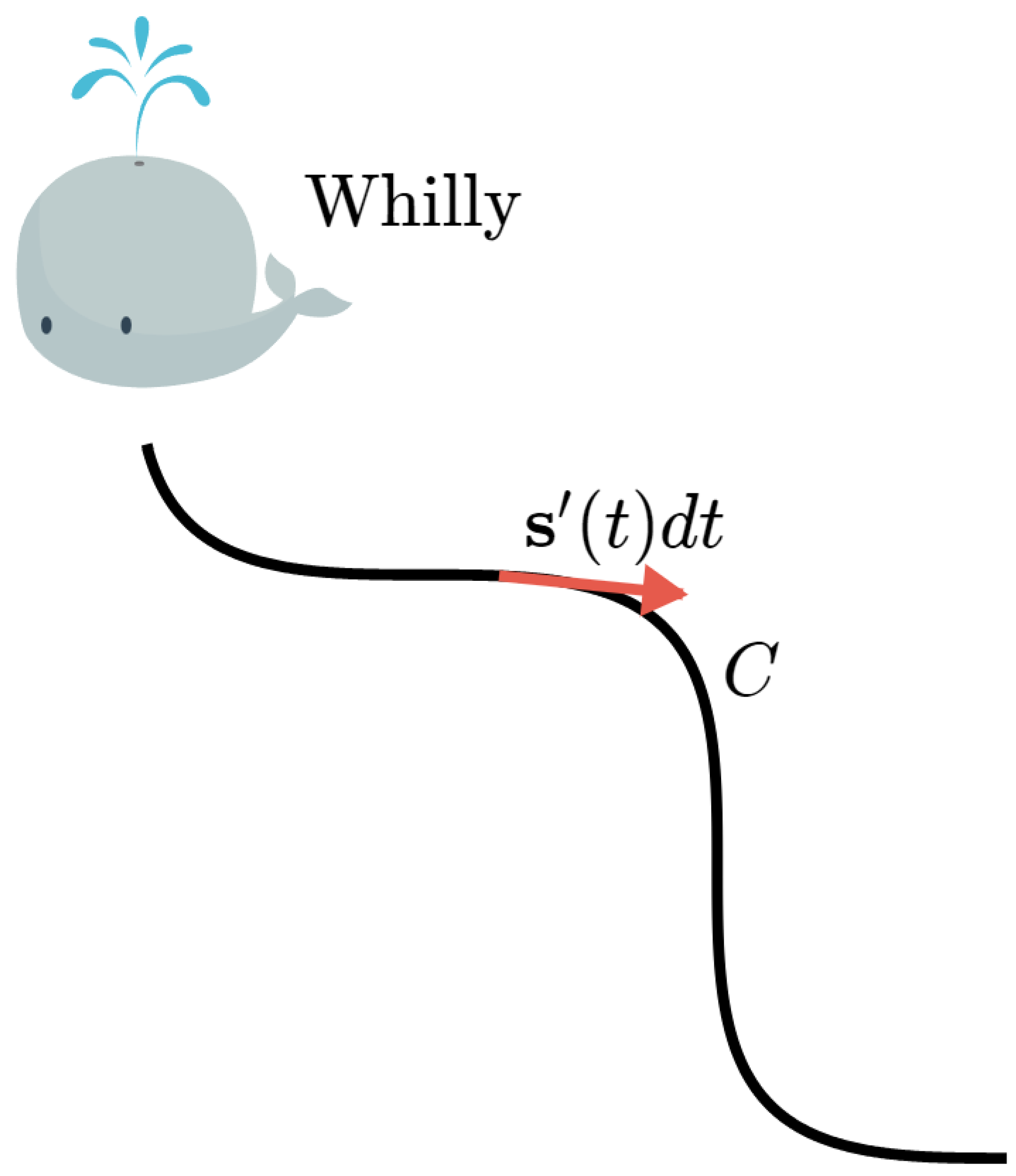
Suppose the curve of Whilly's fall is described by the parametric function

$$\mathbf{s}(t) = \begin{bmatrix} 100(t - \sin(t)) \\ 100(-t - \sin(t)) \end{bmatrix}$$

The vector $d\mathbf{s}$ representing a tiny step along the curve can be given as the derivative of this function, times dt:

$$d\mathbf{s} = \frac{d\mathbf{s}}{dt}dt = \mathbf{s}'(t)dt$$

If these seem unfamiliar, consider taking a look at the [article describing derivatives of parametric functions](). The way to visualize this is to think of a tiny increase to the parameter t of size dt. This results in a tiny nudge along the curve described by $\mathbf{s}(t)$, which is given by the vector $\mathbf{s}'(t)dt$.



Evaluating this derivative vector simply requires taking the derivative of each component:

$$\frac{d\mathbf{s}}{dt} = \begin{bmatrix} \frac{d}{dt}100(t - \sin(t)) \\ \\ \frac{d}{dt}100(-t - \sin(t)) \end{bmatrix}$$

$$\frac{d\mathbf{s}}{dt} = \begin{bmatrix} 100(1 - \cos(t)) \\ \\ 100(-1 - \cos(t)) \end{bmatrix}$$

The force of gravity is given by the acceleration $9.8\frac{\text{m}}{\text{s}^2}$ times the mass of Whilly. Not that it matters, but I looked up the typical mass of a blue whale, and it's around $170{,}000\,\text{kg}$, so let's use that number.

Since this force is directed purely downward, gravity as a force vector looks like this:

$$\mathbf{F}_g = \begin{bmatrix} 0 \\ -(170{,}000)(9.8) \end{bmatrix}$$

Let's say we want to find the work done by gravity between times $t = 0$ and $t = 10$. What do you get when you plug in all this information to the integral integral $\int_C \mathbf{F}_g \cdot d\mathbf{s}$ and evaluate the integral? Take a moment to try writing this out for yourself before peeking at the answer.

[Hide explanation]

$$W = \int_C \mathbf{F}_g \cdot d\mathbf{s}$$

$$= \int_0^{10} \mathbf{F}_g \cdot \mathbf{s}'(t)dt$$

$$= \int_0^{10} \begin{bmatrix} 0 \\ -(170{,}000)(9.8) \end{bmatrix} \cdot \begin{bmatrix} 100(1 - \cos(t)) \\ 100(-1 - \cos(t)) \end{bmatrix} dt$$

$$= \int_0^{10} -(170{,}000)(9.8)100(-1 - \cos(t))dt$$

$$= 166{,}600{,}000 \int_0^{10} (1 + \cos(t))dt$$

$$= 166{,}600{,}000\,[t + \sin(t)]_0^{10}$$

$$= 166{,}600{,}000(10 + \sin(10) - (0 + \sin(0))$$

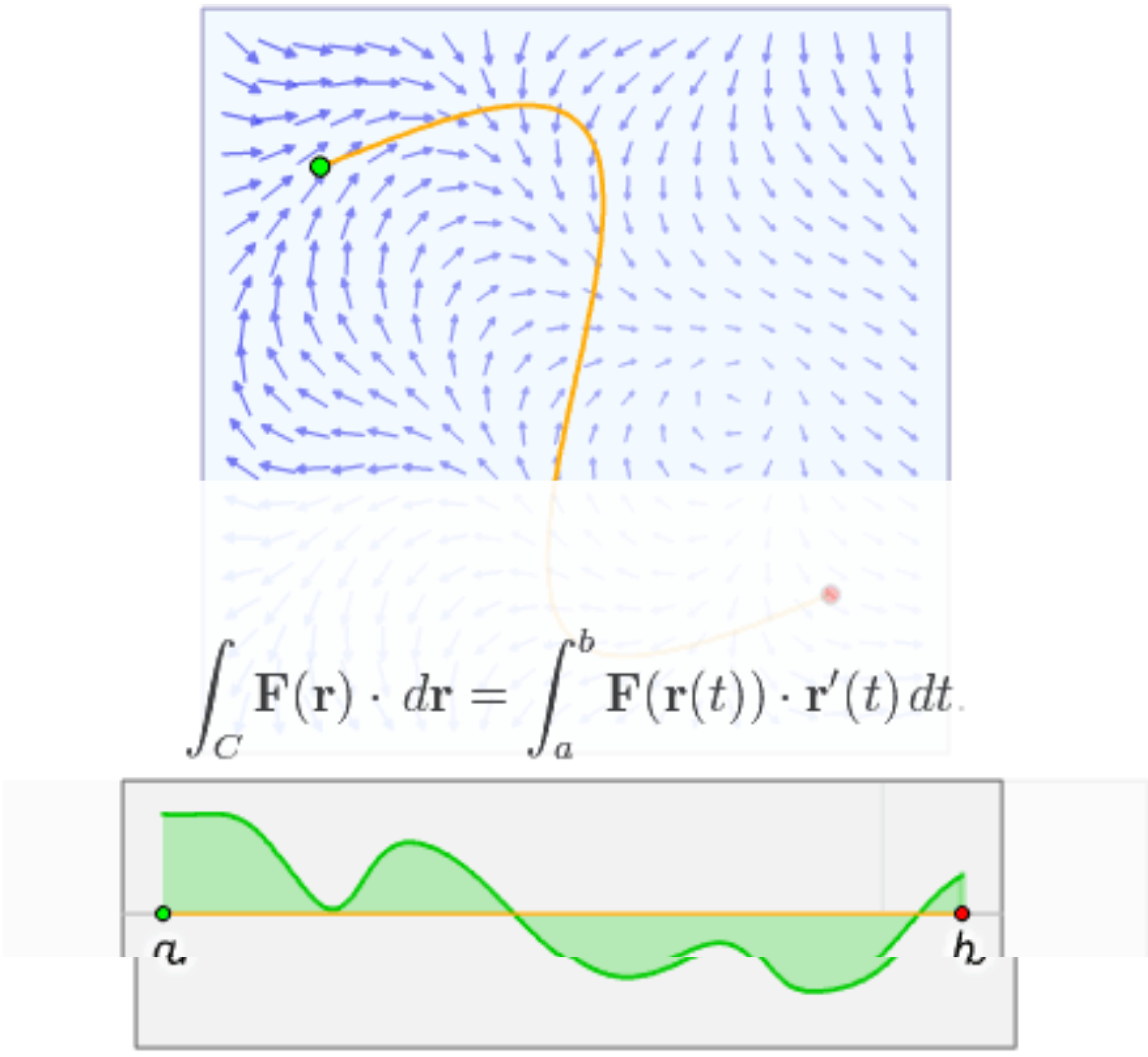
$$\approx 1.575 \times 10^9$$

(To those physics students among you who notice that it would be easier to just compute the gravitational potential of Whilly at the start and end of his fall and find the difference, you are going to love the topic of conservative fields!)

Visualizing more general line integrals through a vector field

In the previous example, the gravity vector field is constant. Gravity points straight down with the same magnitude everywhere. With most line integrals through a vector field, the vectors in the field are different at different points in space, so the value dotted against $d\mathbf{s}$ changes. The following animation shows what this might look like.

(Note, the animation uses the variable $\mathbf{r}$ instead of $\mathbf{s}$ to parameterize the curve, but of course, it does not make a difference.)



Animation credit: By Lucas V. Barbosa (Own work) [Public domain], via Wikimedia Commons

Let's dissect what's going on here. The line integral itself is written as

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

where

- $\mathbf{F}$ is a vector field, associating each point in space with a vector. You can think of this as a force field.

- C is a curve through space.

- $\mathbf{r}(t)$ is a vector-valued function parameterizing the curve C in the range $a \leq t \leq b$

- $\mathbf{r}'(t)$ is the derivative of $\mathbf{r}$, representing the velocity vector of a particle whose position is given by $\mathbf{r}(t)$ while t increases at a constant rate. When you multiply this by a tiny step in time, dt, it gives a tiny displacement vector, which I like to think of as a tiny step along the curve. Technically it is a tiny step in the tangent direction to the curve, but for small enough dt this amounts to the same thing.

- Note, in this animation the length of $\mathbf{r}'(t)$ stays constant. This is not necessarily true for most parameterizations of C, which may have you speeding up or slowing down as your position varies according to $\mathbf{r}$. For example, Whilly was probably speeding up during his fall, making the velocity vector grow over time.

- The rotating circle in the bottom right of the diagram is a bit confusing at first. It represents the extent to which the vector $\mathbf{F}(\mathbf{r}(t))$ lines up with the tangent vector $\mathbf{r}'(t)$. The grey x and y vectors are shown to see how these vectors are oriented relative to xy-plane as a whole.

Concept check: What does the dot product $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$ represent?

Choose 1 answer:

(A) The rate at which the force $\mathbf{F}(\mathbf{r}(t))$ is changing with respect to t.

(B) The component of $\mathbf{F}(\mathbf{r}(t))$ pointing in the same direction as a tiny step along the curve at the point $\mathbf{r}(t)$, multiplied by the size of that tiny step.

Check

Hide explanation

The dot product of any two vectors

$$\mathbf{w} \cdot \mathbf{v}$$

gives the component of $\mathbf{w}$ pointing in the same direction as $\mathbf{v}$ multiplied by the magnitude of $\mathbf{v}$.

In this case, $\mathbf{r}'(t)dt$ represents a tiny step along the curve.

In physics terms, you can think about this dot product

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

as

$$dW$$

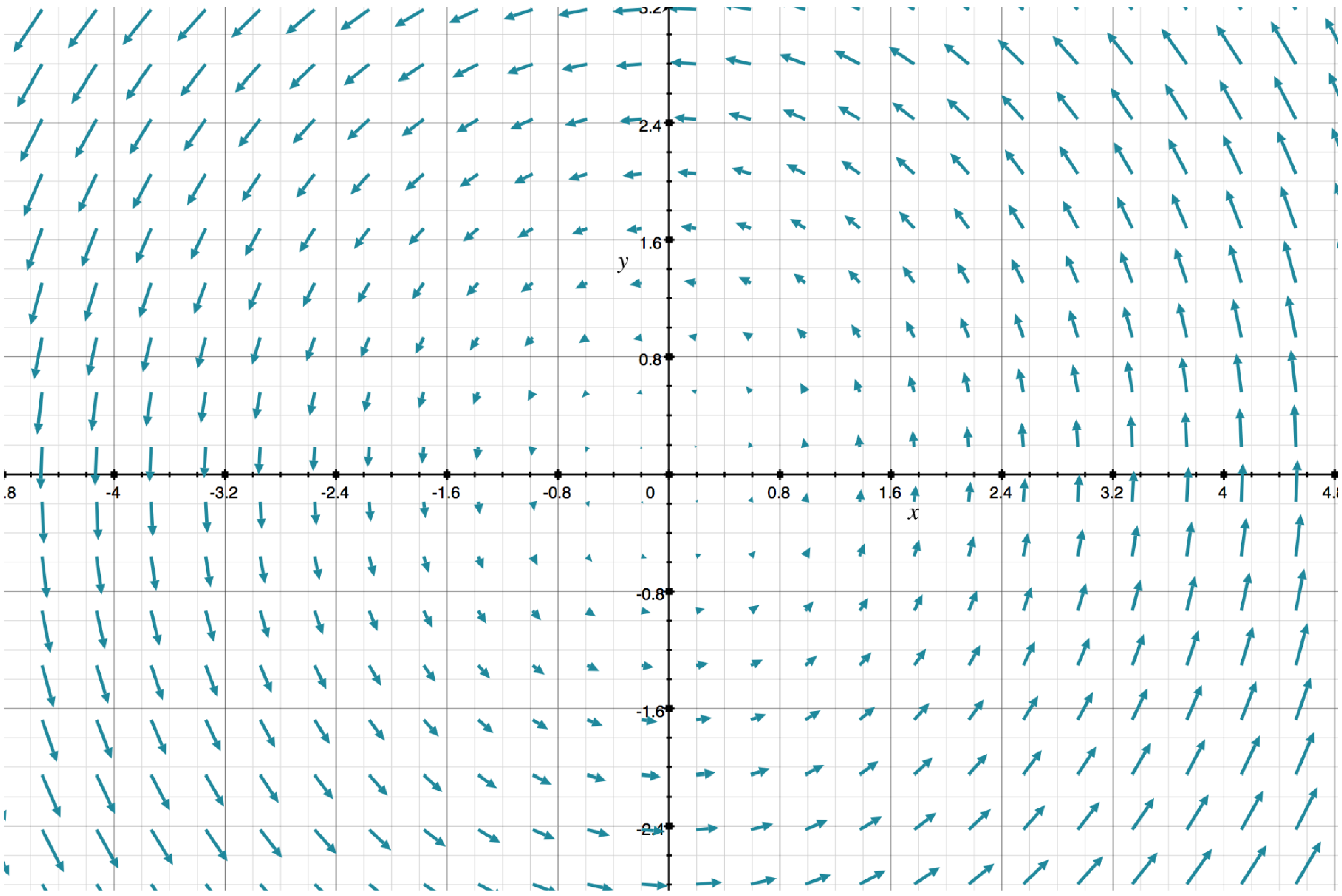
That is, a tiny amount of work done by the force field $\mathbf{F}$ on a particle moving along C.

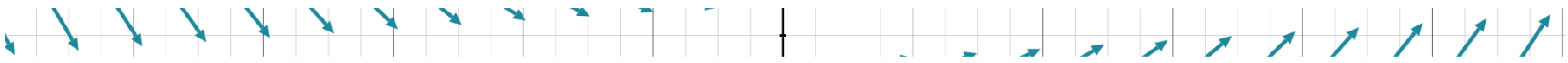
Example 2: Work done by a tornado

Consider the vector field described by the function

$$\mathbf{F}(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$$

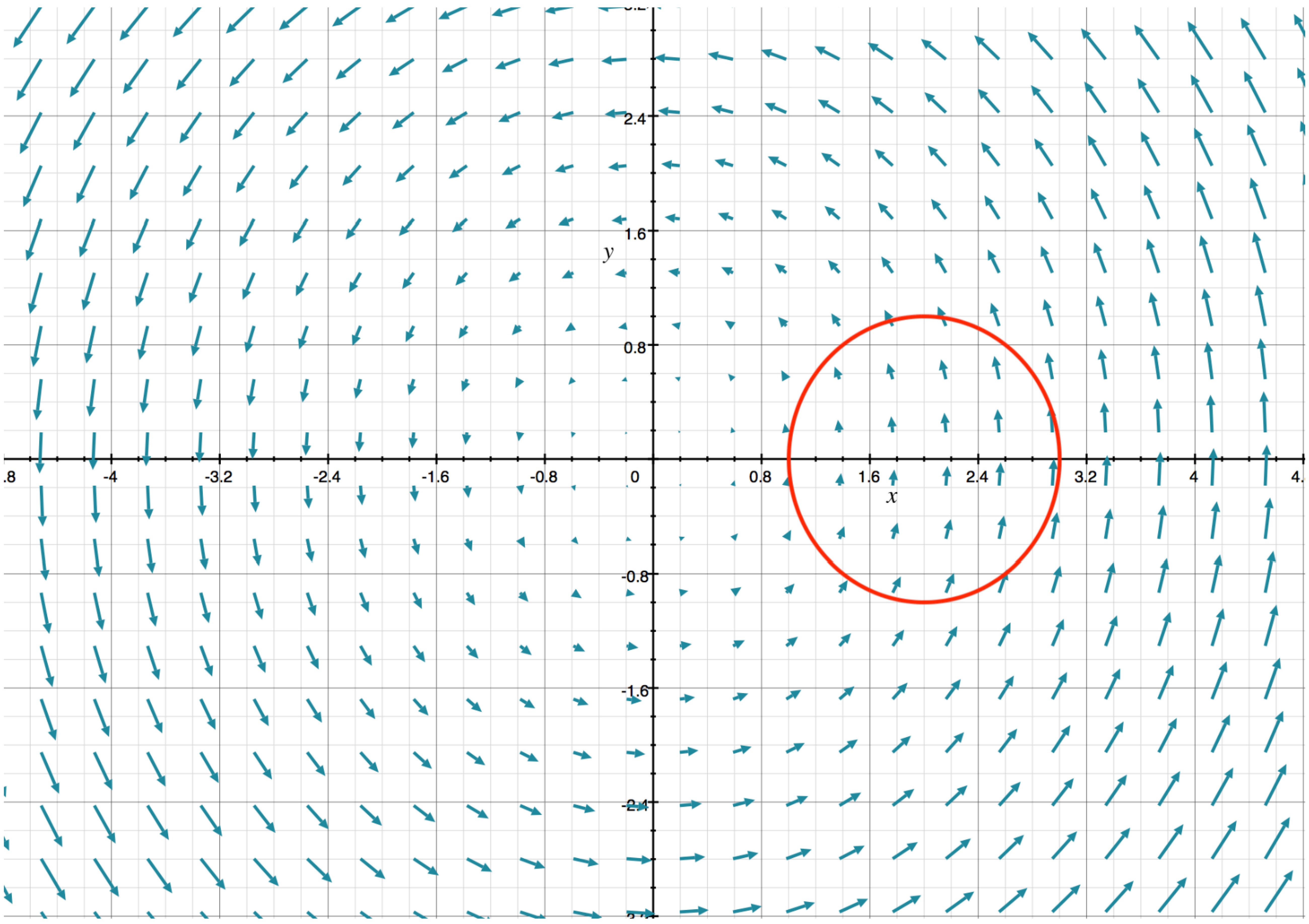
The vector field looks like this:





Thought of as a force, this vector field pushes objects in the counterclockwise direction about the origin. For example, maybe this represents the force due to air resistance inside a tornado. This is a little unrealistic because it would imply that force continually gets stronger as you move away from the tornado's center, but we can just euphemistically say it's a "simplified model" and continue on our merry way.

Suppose we want to compute a line integral through this vector field along a circle or radius 1 centered at $(2, 0)$.



I should point out that orientation matters here. The work done by the tornado force field as we walk counterclockwise around the circle could be different from the work done as we walk clockwise around it (we'll see this explicitly in a bit).

If we choose to consider a counterclockwise walk around this circle, we can parameterize the curve with the function.

$$\mathbf{r}(t) = \begin{bmatrix} \cos(t) + 2 \\ \sin(t) \end{bmatrix}$$

where t ranges from 0 to 2π.

Again, to set up the line integral representing work, you consider the force vector at each point, $\mathbf{F}(x, y)$, and you dot it with a tiny step along the curve,

$d\mathbf{r}$:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

Step 1: Expand the integral

Concept check: Which of the following integrals represents the same thing as $\int_C \mathbf{F} \cdot d\mathbf{r}$?

Choose 1 answer:

(A) $\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$

(B) $\int_a^b \mathbf{F}(\mathbf{r}(t), \mathbf{r}'(t)) dt$

Check

Hide explanation

See the previous section.

Step 2: Expand each component

Concept check: Based on the definitions above, what is $\mathbf{F}(\mathbf{r}(t))$?

Choose 1 answer:

(A) $\begin{bmatrix} \cos(t) + 2 \\ \sin(t) \end{bmatrix}$

(B) $\begin{bmatrix} -\sin(t) \\ \cos(t) + 2 \end{bmatrix}$

Check

Hide explanation

$F(x, y)$ is defined as

$$\mathbf{F}(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$$

and $\mathbf{r}(t)$ is defined as

$$\mathbf{r}(t) = \begin{bmatrix} \cos(t) + 2 \\ \sin(t) \end{bmatrix}$$

For $\mathbf{F}(\mathbf{r}(t))$, plug in the components of $\mathbf{r}(t)$ as the inputs x and y of $\mathbf{F}$:

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) &= \mathbf{F}(\cos(t) + 2, \sin(t)) \\ &= \begin{bmatrix} -\sin(t) \\ \cos(t) + 2 \end{bmatrix} \end{aligned}$$

Concept check: What is $\mathbf{r}'(t)$?

Choose 1 answer:

(A) $\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$

(B) $\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$

Check

Hide explanation

$$\mathbf{r}'(t) = \begin{bmatrix} \frac{d}{dt}[\cos(t) + 2] \\ \\ \frac{d}{dt}\sin(t) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

Step 3: Solve the integral

Concept check: Put the last three answers together to solve the integral.

$\int_C \mathbf{F} \cdot d\mathbf{r} =$

Check

Hide explanation

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{2\pi} \begin{bmatrix} -\sin(t) \\ \cos(t) + 2 \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} dt \\ &= \int_0^{2\pi} \left(\underbrace{\sin^2(t) + \cos^2(t)}_{\text{equals } 1} + 2\cos(t) \right) dt \end{aligned}$$

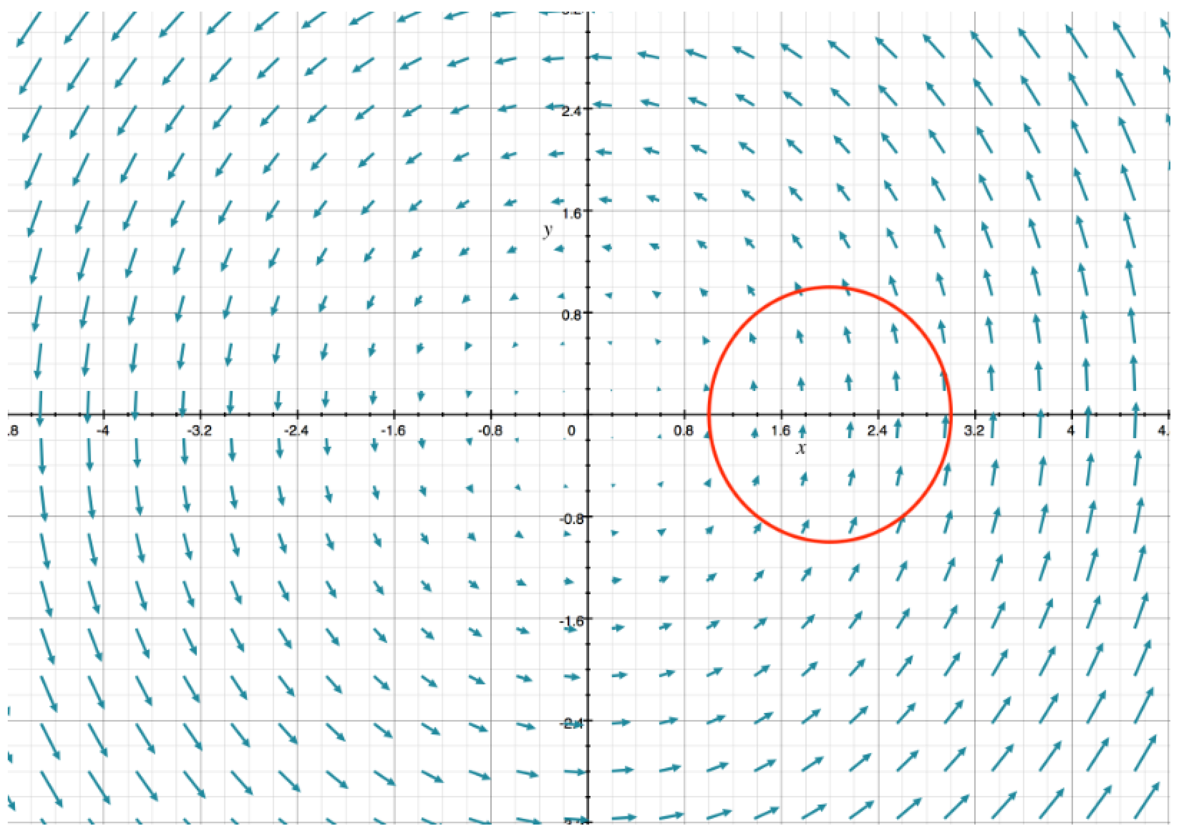
$$
\begin{aligned}
&= \int_0^{2\pi} (1 + 2\cos(t))dt \\
&= \left(t + 2\sin(t)\right)\Big|_0^{2\pi} \\
&= (2\pi + 2\sin(2\pi)) - (0 + 2\sin(0)) \\
&= 2\pi
\end{aligned}
$$

This final answer gives the amount of work that the tornado force field does on a particle moving counterclockwise around the circle pictured above.

Reflection question: Why should it be intuitive that this answer is positive?

[Hide explanation]

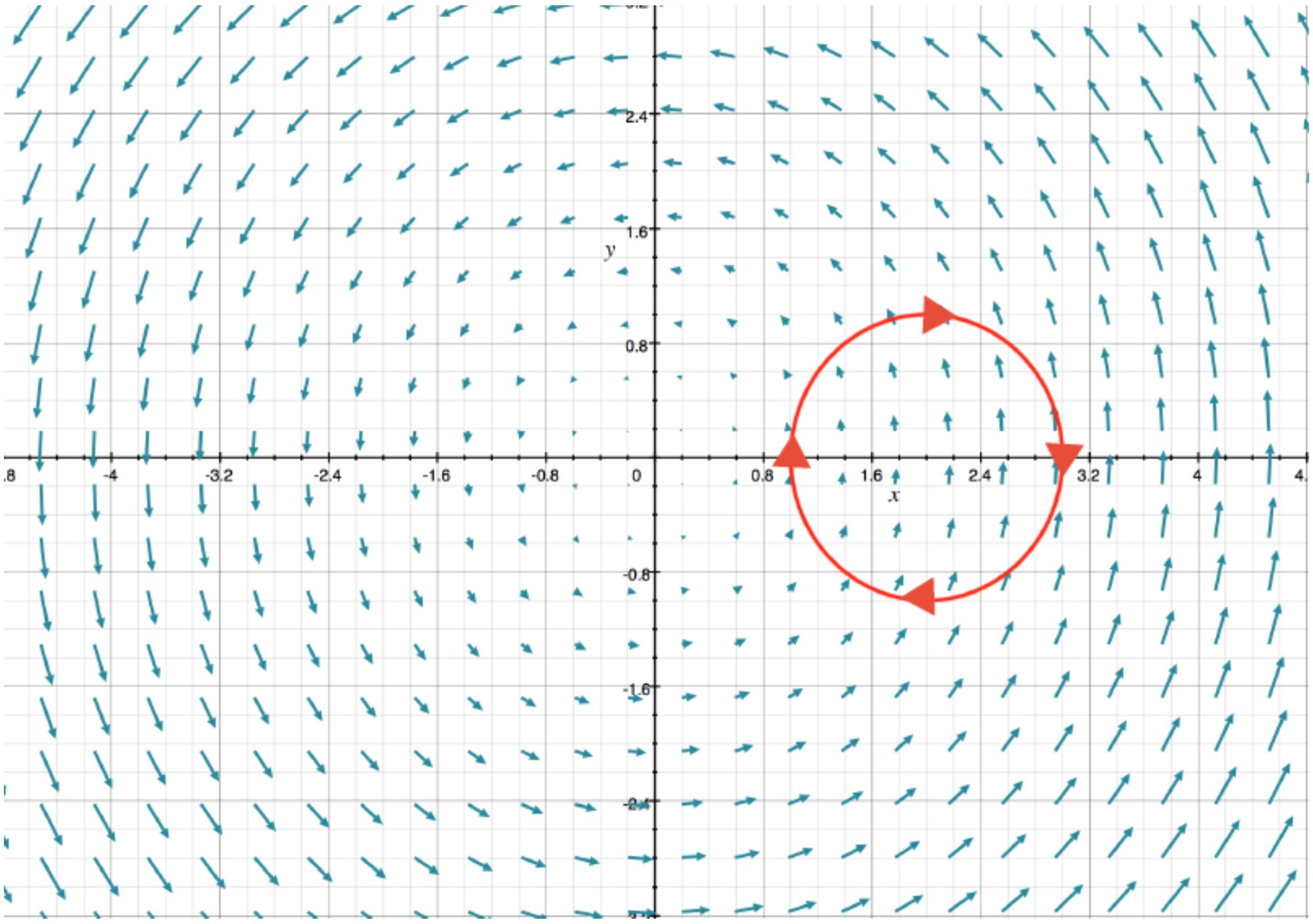
Since the circle is oriented counterclockwise, you walk up the right half, and down the left half.



The vectors touching the right half of the circle are relatively long, and pointing roughly in the same direction that you walk, thus contributing a lot of positive work.

The vectors touching the left half of the circle are still pointing roughly up, which is now against the direction you are walking, and hence contribute negative work. However, these vectors are relatively short, so they do not cancel out the positive work done while walking up the right half.

Orientation matters



What would have happened if in the preceding example, we had oriented the circle clockwise? For instance, we could have parameterized it with the function

$$\mathbf{r}(t) = \begin{bmatrix} \cos(t) + 2 \\ -\sin(t) \end{bmatrix}$$

You can, if you want, plug this in and work through all the computations to see what happens. However, there is a simpler way to reason about what will happen. In the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

each vector $d\mathbf{r}$ representing a tiny step along the curve will get turned around to point in the opposite direction.

Concept check: Suppose you have two vectors $\mathbf{v}$ and $\mathbf{w}$, and $\mathbf{v} \cdot \mathbf{w} = 3$. You turn $\mathbf{v}$ around to point in the opposite direction, getting a new vector $\mathbf{v}_{\text{new}} = -\mathbf{v}$. What happens to the dot product?

$\mathbf{v}_{\text{new}} \cdot \mathbf{w} =$

Check

Hide explanation

$\mathbf{v}_{\text{new}} \cdot \mathbf{w} = (-\mathbf{v}) \cdot \mathbf{w}$

$$= -(\mathbf{v} \cdot \mathbf{w})$$
$$= -3$$

Since the dot product inside the integral gets multiplied by -1 when you swap the direction of each $d\mathbf{r}$, we can conclude the following:

Key Takeaway: The line integral through a vector field gets multiplied by -1 when you reverse the orientation of a curve.

Summary

- The shorthand notation for a line integral through a vector field is

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

- The more explicit notation, given a parameterization $\mathbf{r}(t)$ of C, is

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

- Line integrals are useful in physics for computing the work done by a force on a moving object.

- If you parameterize the curve such that you move in the opposite direction as t increases, the value of the line integral is multiplied by -1.