Curvature

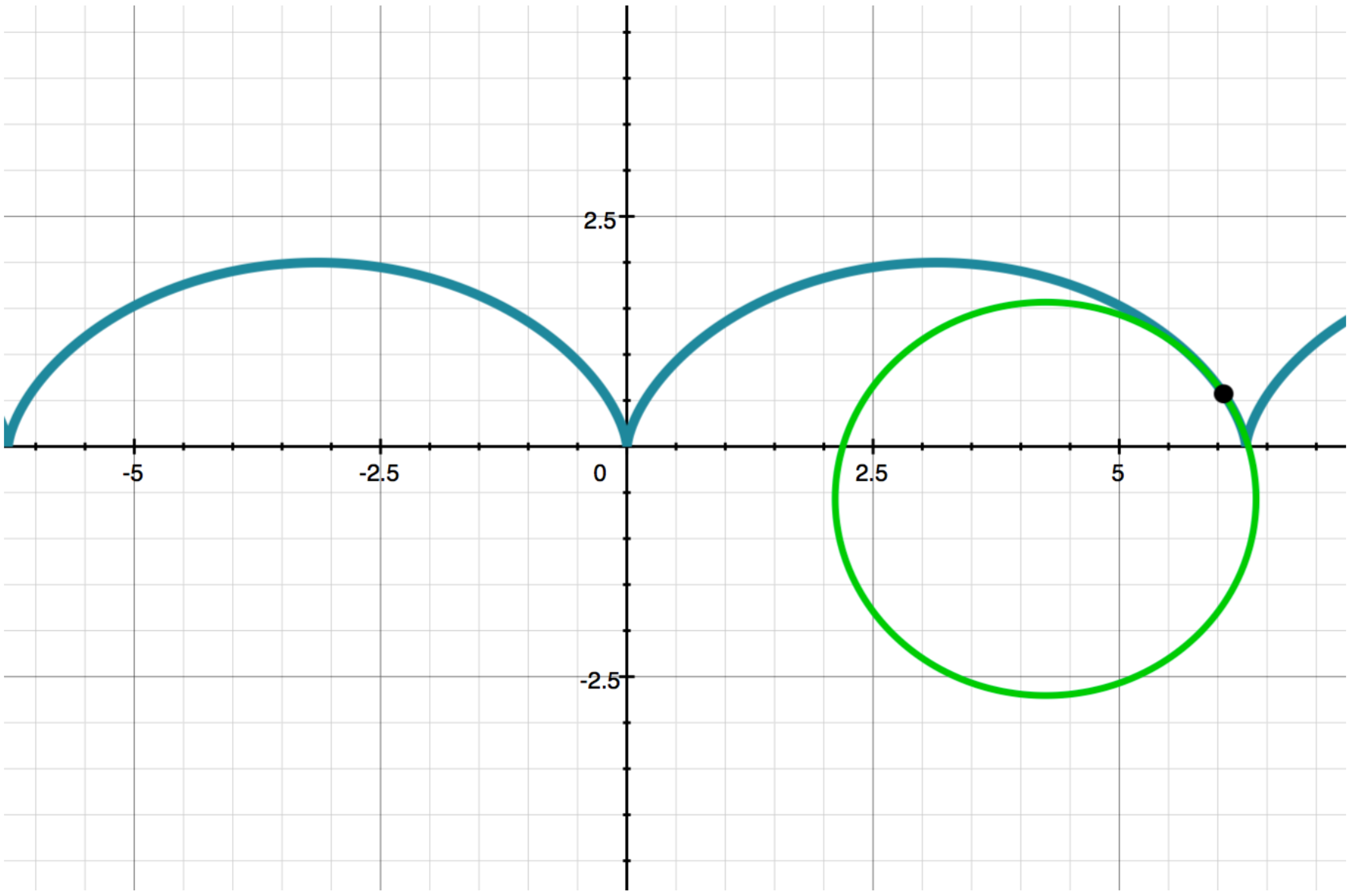
Google Classroom

How do you measure how much a curve actually, you know, curves?

Background

- [Derivatives of vector-valued functions]

What we're building to



- The radius of curvature at a point on a curve is, loosely speaking, the radius of a circle which fits the curve most snugly at that point.

- The **curvature**, denoted κ, is one divided by the radius of curvature.

- In formulas, curvature is defined as the magnitude of the derivative of a unit tangent vector function with respect to arc length:

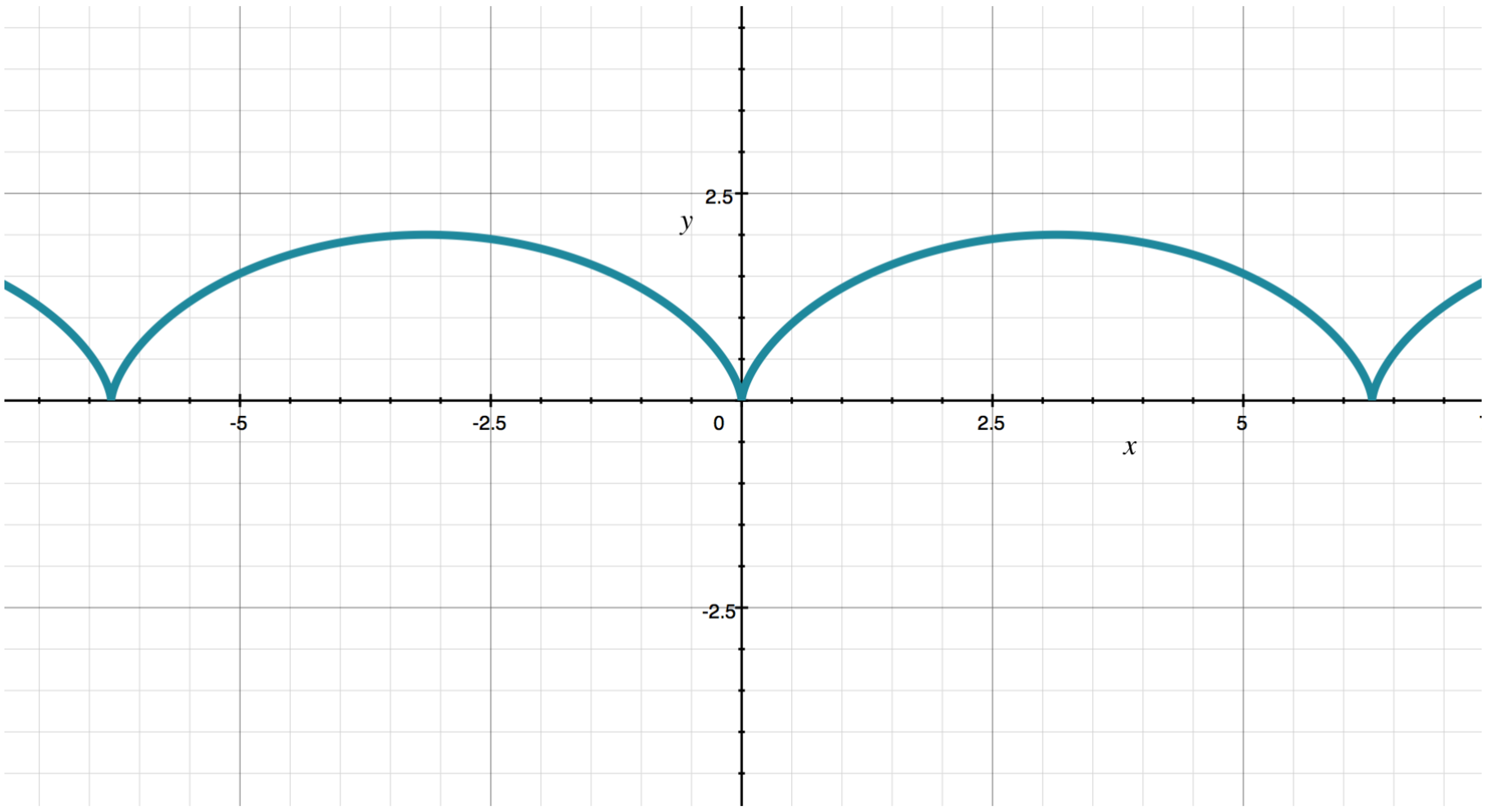
$$\kappa = \left\| \frac{dT}{ds} \right\|$$

  Don't worry, I'll talk about each step of computing this value.

- The intuition here is that the unit tangent vector tells you which direction you are moving, and the rate at which it changes with respect to small steps ds along the curve is a good indication of how quickly you are turning.

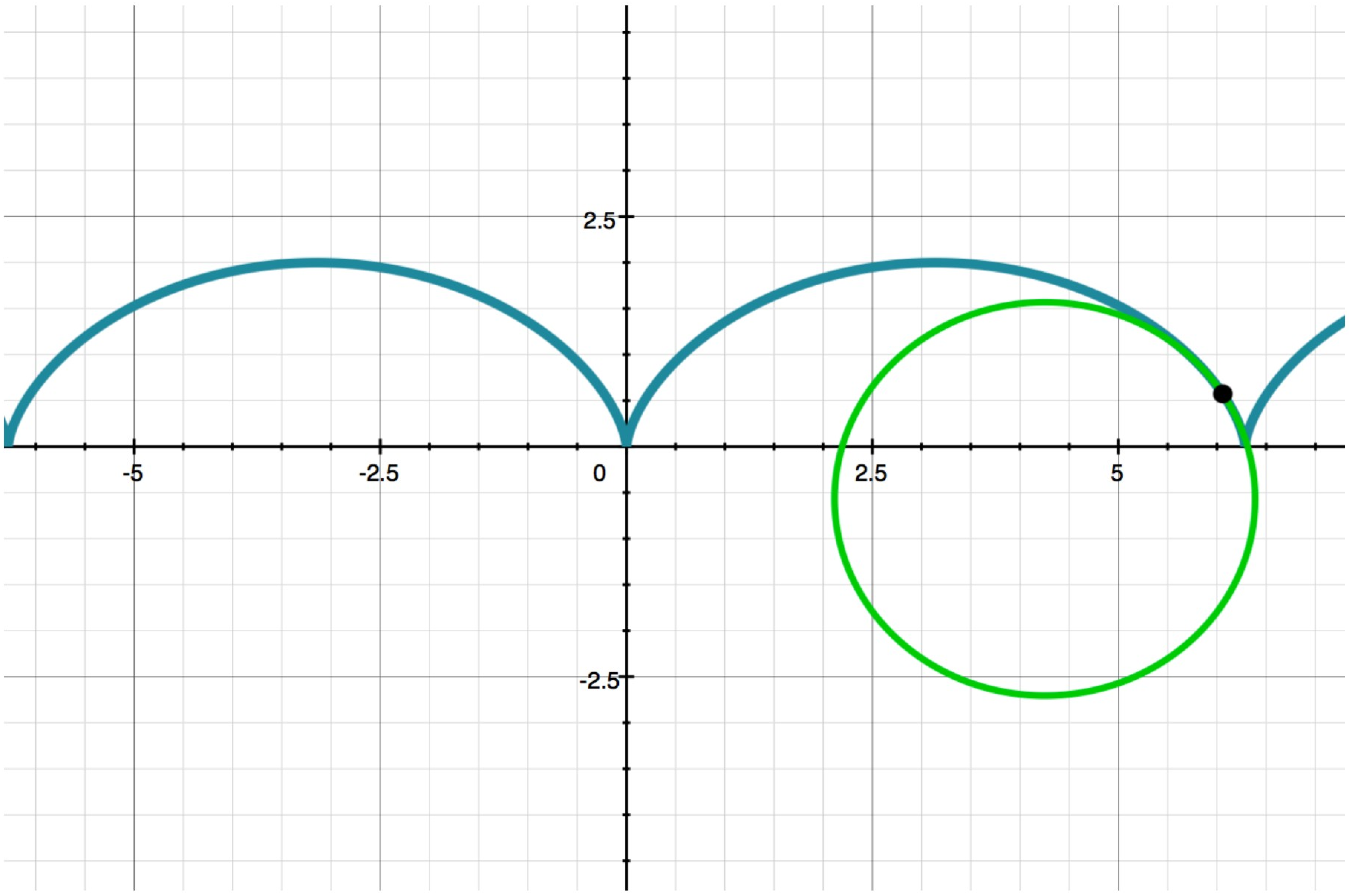
Driving along a curve

Picture some curve in the xy-plane. We'll treat everything with formulas in a bit, but for now, just think of the picture:

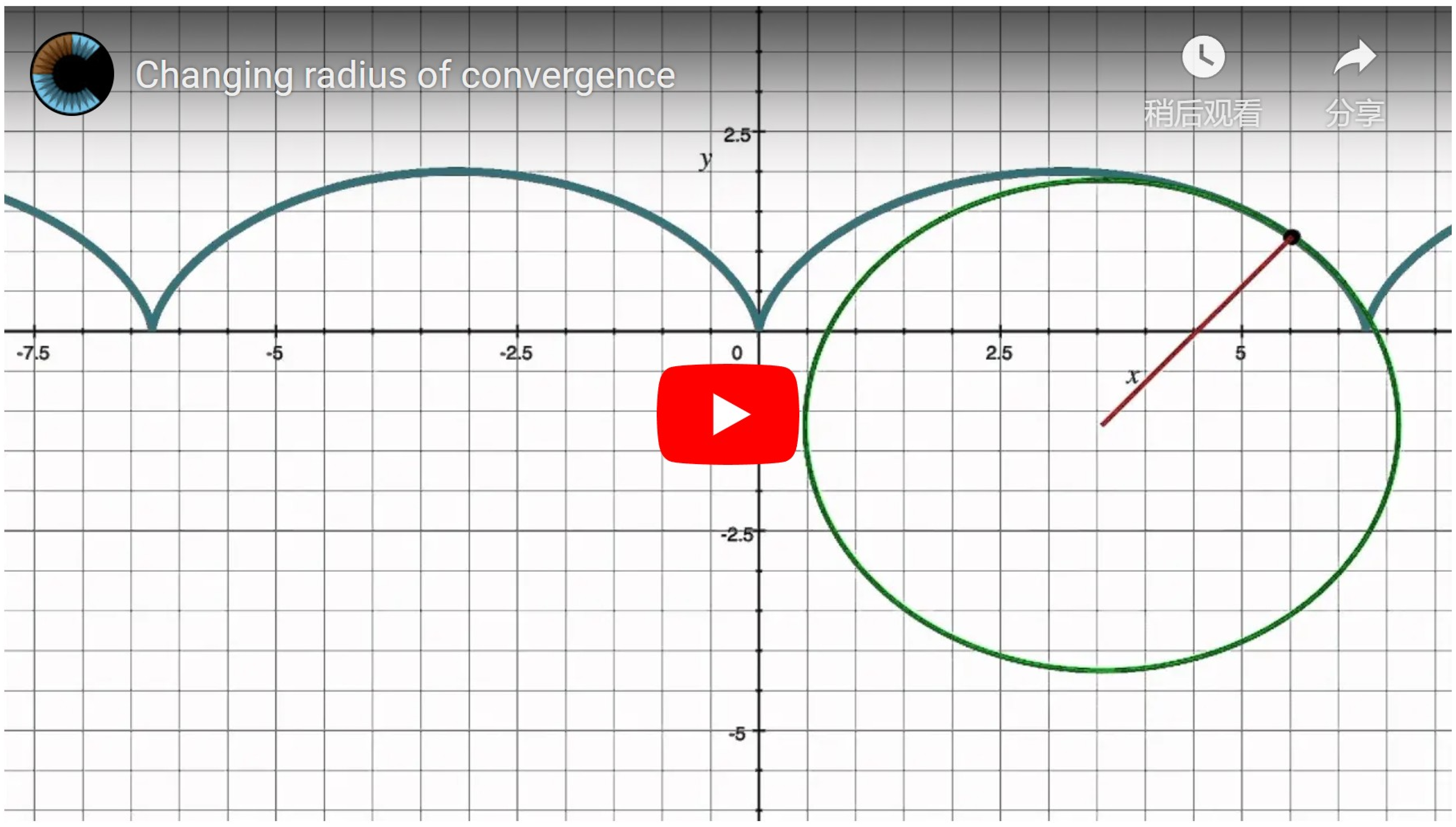


Imagine driving a car along this curve, and think of how much you would be turning the steering wheel at each point. At some points, the road hardly curves, and you are practically driving straight. At others, you have to turn the wheel quite a bit.

Now imagine that at some time while you drive, your steering wheel locks. If you were to keep driving with this locked steering wheel, not at all minding the fact that you're going off the road, your car would trace out some circle, like the one drawn in green below:



If the steering wheel was turned a lot when it froze, that circle would have a relatively small radius. If you were hardly turning at all, the circle would have a very large radius. The following animation shows what those various circles (drawn in green) might look like at different points on the curve. The radius of each circle is drawn in red.



See video transcript

We call the radius of the circle associated with each point the **radius of curvature** at that point. It's a good way to measure how much a curve actually, you know, *curves* at each point. Another way to think about these circles is that they hug the curve more closely than any other circle would.

Another important term is **curvature**, which is just one divided by the radius of curvature. It's typically denoted with the funky-looking little κ symbol:

$$\kappa = \frac{1}{R}$$

Concept check: When a curve is very close to being a straight line, the curvature would be

Choose 1 answer:

- (A) Very large
- (B) Very small

Check

[Hide explanation]

> Imagine if you were driving on a nearly straight road, and the steering wheel got stuck. The circle you would trace out would be enormous.
>
> Therefore, on a nearly straight line, the radius of curvature R would be huge, so the curvature $\kappa = \frac{1}{R}$ would be tiny.

Computing curvature

Suppose you have the function which defines a curve in the xy-plane. For example, the curve I used in the previous section is defined by the following vector-valued function:

$$\vec{\mathbf{s}}(t) = \begin{bmatrix} t - \sin(t) \\ 1 - \cos(t) \end{bmatrix}$$

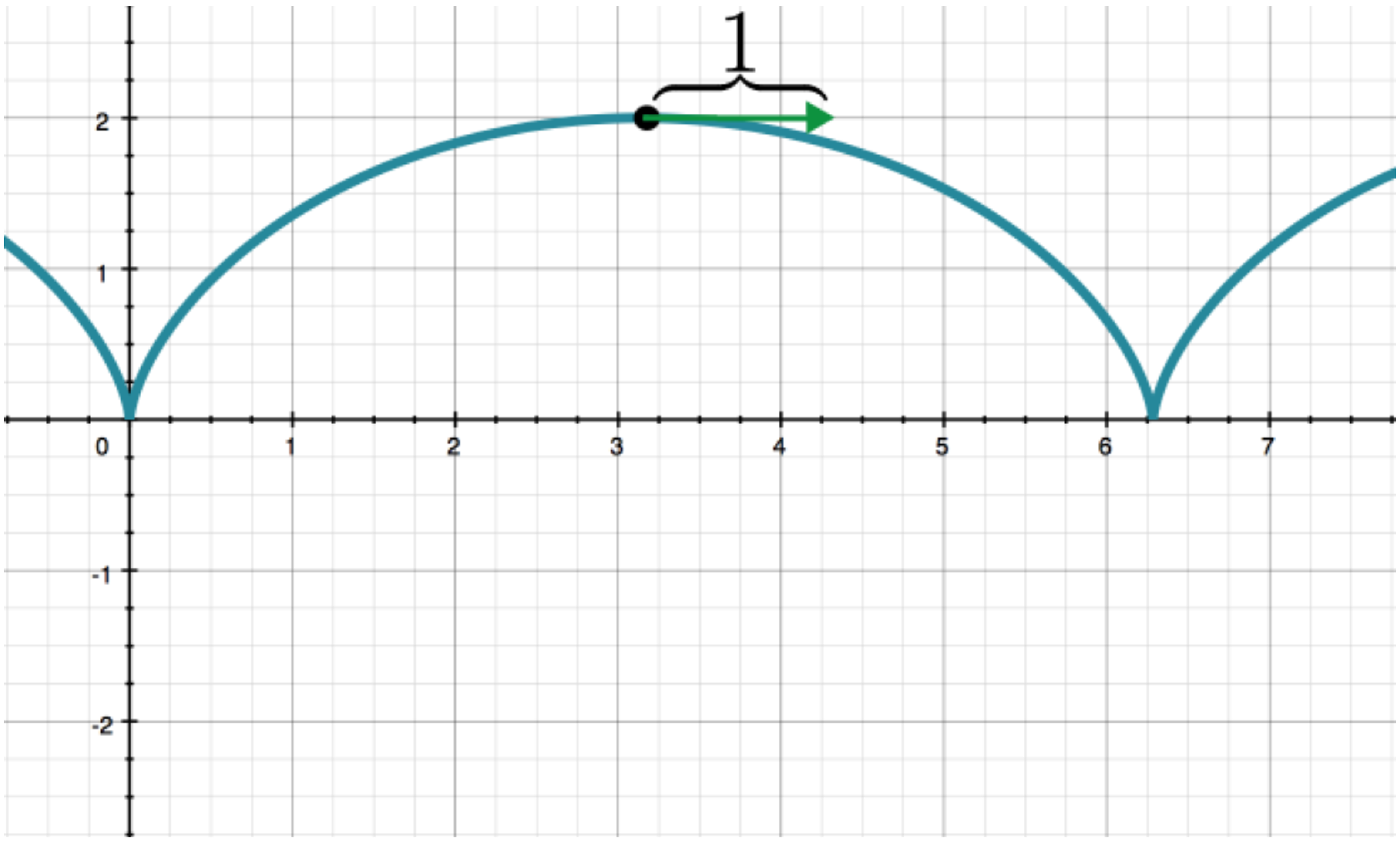
This is the same thing as saying it is defined by the following parametric equations:

$$x(t) = t - \sin(t)$$
$$y(t) = 1 - \cos(t)$$

Computing curvature involves two broad steps:

Step 1: Find a unit tangent vector

A "unit tangent vector" to the curve at a point is, unsurprisingly, a tangent vector with length 1. In the context of a parametric curve defined by $\vec{\mathbf{s}}(t)$, "finding a unit tangent vector" almost always means finding *all* unit tangent vectors. That is to say, defining a vector-valued function $T(t)$, which takes in the same parameter and spits out a unit vector which is tangent to the curve at the point $\vec{\mathbf{s}}(t)$.



Step 2: Find $\dfrac{dT}{ds}$

As you travel along the curve according to $\vec{\mathbf{s}}(t)$, the unit vector changes direction whenever you turn. At sharp turns, it changes a lot, on relatively straight portions, it hardly changes at all. In fact, the curvature κ is *defined* to be the derivative of the unit tangent vector function.

However, it is not the derivative with respect to the parameter t, since that could depend on how quickly you are moving along the curve. It is the derivative with respect to tiny changes in *arc length*, usually represent with the letter s.

$$\kappa = \left|\left|\frac{dT}{ds}\right|\right|$$

Typically, the way to compute this is by first taking the derivative of T with respect to t, then dividing it by $||\frac{d\vec{\mathbf{s}}}{dt}||$.

$$\left|\left|\frac{dT}{ds}\right|\right| = \frac{\left|\left|\frac{dT}{dt}\right|\right|}{\left|\left|\frac{d\vec{\mathbf{s}}}{dt}\right|\right|}$$

Finding the unit tangent vector

Let's take a look at the function we had above:

$$\vec{\mathbf{s}}(t) = \begin{bmatrix} t - \sin(t) \\ 1 - \cos(t) \end{bmatrix}$$

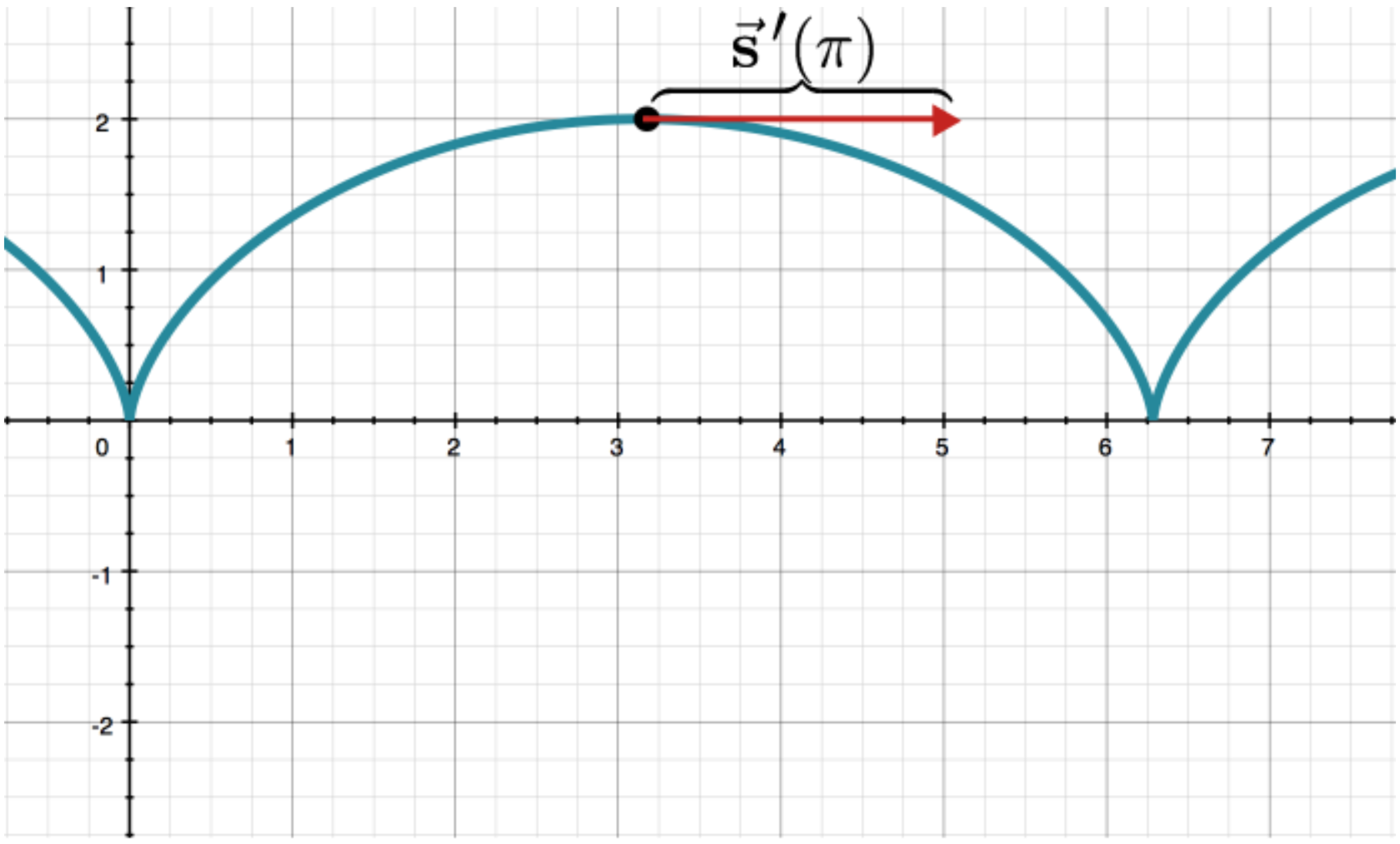
If you read the article on differentiating vector-valued functions, you will know that the derivative of this function can be thought of as a velocity vector.

$$\frac{d\vec{\mathbf{s}}}{dt} = \begin{bmatrix} \frac{d}{dt}(t - \sin(t)) \\ \\ \frac{d}{dt}(1 - \cos(t)) \end{bmatrix} = \begin{bmatrix} 1 - \cos(t) \\ \sin(t) \end{bmatrix}$$

For example, if we evaluate the derivative we just calculated at a specific time, perhaps $t = \pi$, here's the vector we get:

$$\begin{bmatrix} 1 - \cos(\pi) \\ \sin(\pi) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Positioning this vector so that its tail sits on the point $\vec{\mathbf{s}}(\pi)$, where the particle is at time $t = \pi$, it represents the velocity of the particle at that time.



However, we must tweak this function, since we want *unit* tangent vectors. For example, this particular tangent vector has length 2, and $2 \neq 1$[citation needed].

Concept check: Given the formula I just showed for of $\vec{\mathbf{s}}'(t)$,

$$\vec{\mathbf{s}}'(t) = \begin{bmatrix} 1 - \cos(t) \\ \sin(t) \end{bmatrix}$$

what is the magnitude of this vector (as a function of time)?

Choose 1 answer:

- (A) $\sqrt{\cos(t)^2 + \sin(t)^2}$
- (B) $\sqrt{(1 - \cos(t))^2 + \sin(t)^2}$
- (C) $\sqrt{1^2 - \cos(t)^2 + \sin(t)^2}$

Check

[Hide explanation]

> The magnitude of any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\sqrt{x^2 + y^2}$. In this case, that means
>
> $$\sqrt{(1 - \cos(t))^2 + \sin(t)^2}$$

Concept check: What is the unit vector pointing in the same direction as $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$? (This specific vector is not related to our problem at hand, this is just practice with unit vectors).

Choose 1 answer:

- (A) $\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$
- (B) $\begin{bmatrix} 2\sqrt{5} \\ 1\sqrt{5} \end{bmatrix}$

Check

[Hide explanation]

The magnitude of $\vec{\mathbf{v}}$ is $\sqrt{2^2+1^2}=\sqrt{5}$. We can therefore scale the vector to make it have magnitude 1 by multiplying each component by $1/\sqrt{5}$.

$$\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$$

Key question:Which of the following represents the unit tangent vector of a curve parameterized by $\vec{\mathbf{s}}$ (as a function of time)?

Choose 1 answer:

(A) $$T(t)=\frac{\vec{\mathbf{s}}'(t)}{||\vec{\mathbf{s}}'(t)||}$$

(B) $$T(t)=\frac{\vec{\mathbf{s}}(t)+\vec{\mathbf{s}}'(t)}{||\vec{\mathbf{s}}'(t)||}$$

Check

[Hide explanation]

Since the derivative vector $\vec{\mathbf{s}}'(t)$ is tangent to the curve, pointing in the correct direction, we just need to **normalize it**, which means divide it by its own magnitude to force its new magnitude to be 1.

$$\frac{\vec{\mathbf{s}}'(t)}{||\vec{\mathbf{s}}'(t)||}\quad\leftarrow\text{ This is always a unit vector}$$

In practice, this can look pretty wild. For example, with the specific function we had above this would look like

$$\frac{1}{||\vec{\mathbf{s}}'(t)||}\vec{\mathbf{s}}'(t)=\frac{1}{\sqrt{(1-\cos(t))^2+\sin(t)^2}}\begin{bmatrix}1-\cos(t)\\\sin(t)\end{bmatrix}$$

$$=\begin{bmatrix}(1-\cos(t))/\sqrt{(1-\cos(t))^2+\sin(t)^2}\\\sin(t)/\sqrt{(1-\cos(t))^2+\sin(t)^2}\end{bmatrix}$$

Getting curvature from the unit tangent vector

Now we have an expression for the unit tangent vector as a function of time, which I'll denote with a capital T for tangent, (not to be confused with lowercase t for the parameter):

$$T(t) = \frac{\vec{\mathbf{s}}'(t)}{||\vec{\mathbf{s}}'(t)||}$$

The curvature κ is the magnitude of the derivative of this unit tangent vector, but with respect to arc length s, not the parameter t.

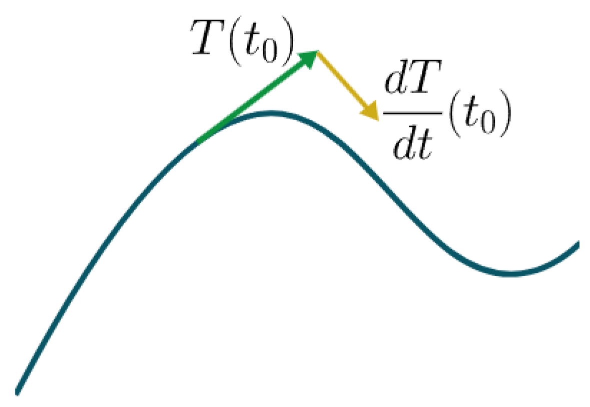
$$\kappa = \left|\left|\frac{dT}{ds}\right|\right|$$

Nevertheless, the typical way to compute this is to first differentiation T with respect to t, then to divide by the magnitude $||\vec{\mathbf{s}}'(t)||$, which you can think of as $\frac{ds}{dt}$.

$$\kappa = \left|\left|\frac{dT}{ds}\right|\right| = \frac{\left|\left|\frac{dT}{dt}\right|\right|}{\left|\left|\frac{d\vec{\mathbf{s}}}{dt}\right|\right|}$$

Intuition

Let's pause for a moment to get an intuition for this. The derivative of $T(t)$ tells us how the unit tangent vector changes over time. Since it's always a *unit* tangent vector, it never changes length, and only changes direction.



At a particular time t_0, you can think of the vector $\frac{dT}{dt}(t_0)$ as sitting at the tip of the vector $T(t_0)$. Imagine the derivative vector trying to pull the vector $T(t_0)$ one way or another, telling it, "Hey! Come point more in this direction."

Since the length of $T(t_0)$ never changes, this derivative vector must always be perpendicular to $T(t_0)$; otherwise. it would "pull" it longer or shorter.

When this derivative vector is long, it's pulling the unit tangent vector *really* hard to change direction. As a result, the curve will change direction more suddenly, meaning it will have a smaller radius of curvature, and hence a very large curvature. Conversely, if the derivative vector is short, it's only half-heartedly pulling on the tangent vector. This translates to a very gentle turn, and hence a large radius of curvature, meaning a small curvature.

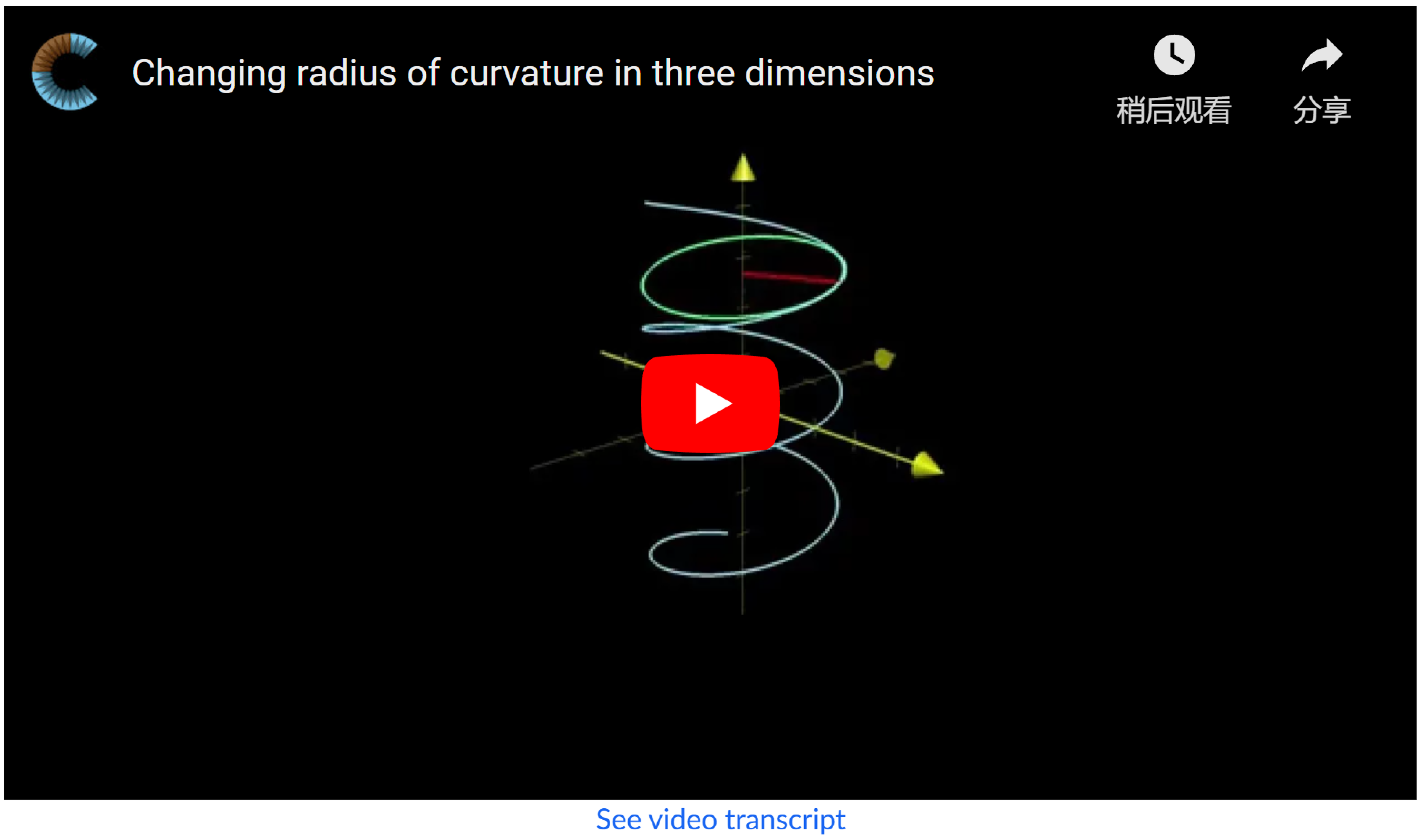
However, we don't want differences in the rate at which we move along the curve to influence the value of curvature since it is a statement about the geometry of the curve itself and not the time-dependent trajectory of whatever particle happens to be traversing it. For this reason, curvature requires differentiating T with respect to arc length, s, instead of the parameter t.

Example: Curvature of a helix

There's nothing specific about two-dimensions in everything we've just done. For example, let's find the curvature of the following three-dimensional function:

$$\vec{\mathbf{v}}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t/5 \end{bmatrix}$$

The animation below shows the shape of this curve, which is known as a **helix**. It also indicates the radius of curvature at each point by showing the circle (drawn in green) which most closely "hugs" the curve at each point, with the radius of each circle drawn in red.



See video transcript

Ah yes, I call this one "vector-calculus hula hoop dance."

One thing you might notice is that the size of these circles doesn't seem to change. This is far from true for most three-dimensional curves, making our example somewhat special.

Concept check: By seeing that the circles above don't change size, what do you expect to be true of our curvature function $\kappa(t)$?

Choose 1 answer:

CORRECT (SELECTED)

$\kappa(t)$ is constant.

(B) $\kappa(t)$ is linear.

Check

If you would like to get some practice with this type of problem, now is the time you should pull out a pencil and paper. We'll go through it together, you and I, but I'll give you the chance to try each step for yourself before the answer is shown.

Step 1: Compute derivative

The first step to finding curvature is to take the derivative of our function,

$$\vec{\mathbf{v}}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t/5 \end{bmatrix}$$

This will give us a tangent vector to the curve which we can then mold into a unit tangent vector. Compute this derivative.

[Hide explanation]

$$\frac{d\vec{\mathbf{v}}}{dt} = \begin{bmatrix} \frac{d}{dt}(\cos(t)) \\ \\ \frac{d}{dt}(\sin(t)) \\ \\ \frac{d}{dt}(t/5) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 1/5 \end{bmatrix}$$

Step 2: Normalize the derivative

To get a **unit** tangent vector we have to normalize this derivative vector, which is to say, divide it by Its magnitude. What is the magnitude of this derivative?

[Hide explanation]

$$\left\|\frac{d\vec{\mathbf{v}}}{dt}\right\| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (1/5)^2}$$

$$= \sqrt{\sin(t)^2 + \cos(t)^2 + 1/25}$$

$$= \sqrt{1 + \frac{1}{25}}$$

$$= \sqrt{\frac{26}{25}}$$

$$= \frac{\sqrt{26}}{5}$$

Thankfully this is a constant. Things are a real pain when it's not. Using the previous two answers, what is our unit tangent vector $T(t)$ as a function of time?

[Hide explanation]

$$T(t) = \frac{\frac{d\vec{\mathbf{v}}}{dt}}{\left\|\frac{d\vec{\mathbf{v}}}{dt}\right\|} = \frac{5}{\sqrt{26}}\begin{bmatrix} -\sin(t) \\ \cos(t) \\ 1/5 \end{bmatrix}$$

Step 3: Take the derivative of the unit tangent

To get curvature, we must find the derivative of this function, with respect to time, then take its magnitude. What is the derivative of $T(t)$ in this case?

[Hide explanation]

$$\frac{d}{dt}\left(\frac{5}{\sqrt{26}}\begin{bmatrix} -\sin(t) \\ \cos(t) \\ 1/5 \end{bmatrix}\right) = \frac{5}{\sqrt{26}}\begin{bmatrix} \frac{d}{dt}(-\sin(t)) \\ \frac{d}{dt}(\cos(t)) \\ \frac{d}{dt}(1/5) \end{bmatrix}$$

$$= \frac{5}{\sqrt{26}}\begin{bmatrix} -\cos(t) \\ -\sin(t) \\ 0 \end{bmatrix}$$

Step 4: Find the magnitude of this value

What is the magnitude of this vector?

[Hide explanation]

$$\begin{aligned}\kappa(t) &= \left|\left|\frac{5}{\sqrt{26}}\begin{bmatrix}-\cos(t)\\ -\sin(t)\\ 0\end{bmatrix}\right|\right| \\ &= \frac{5}{\sqrt{26}}\sqrt{(-\cos(t))^2 + (-\sin(t))^2 + 0^2} \\ &= \frac{5}{\sqrt{26}}\sqrt{1} \\ &= \frac{5}{\sqrt{26}}\end{aligned}$$

Step 5: Divide this value by $||\vec{\mathbf{v}}'(t)||$

To go from $\frac{dT}{dt}$ to $\frac{dT}{ds}$, we must divide this by the magnitude of the derivative of the original parametric function.

[Hide explanation]

Oh hey, we've already computed $||\vec{\mathbf{v}}'(t)||$, how convenient!

$$\left|\left|\frac{d\vec{\mathbf{v}}}{dt}\right|\right| = \frac{\sqrt{26}}{5}$$

This means our curvature is as follows:

$$\kappa = \left|\left|\frac{dT}{ds}\right|\right| = \frac{\left|\left|\frac{dT}{dt}\right|\right|}{\left|\left|\frac{d\vec{\mathbf{v}}}{dt}\right|\right|} = \frac{5/\sqrt{26}}{\sqrt{26}/5} = \frac{25}{26}$$

Notice, this is a constant, so the curvature is the same at all points on the curve.

Summary

- The radius of curvature at a point on a curve is, loosely speaking, the radius of a circle which fits the curve most snugly at that point.

- The **curvature**, denoted κ, is one divided by the radius of curvature.

- To find the curvature given the parametric function $\vec{\mathbf{s}}$ defining a curve:
  - Find the unit tangent vector by normalizing the derivative of $\vec{\mathbf{s}}$:

$$T(t) = \frac{\vec{\mathbf{s}}'(t)}{||\vec{\mathbf{s}}'(t)||}$$

  - Curvature is defined as the magnitude of the derivative of this value with respect to arc length s. You can compute that as follows:

$$\kappa = \left|\left|\frac{dT}{ds}\right|\right| = \frac{\left|\left|\frac{dT}{dt}\right|\right|}{\left|\left|\frac{d\vec{\mathbf{s}}}{dt}\right|\right|}$$

- The intuition here is that the unit tangent vector tells you which direction you are moving, and the rate at which it changes with respect to small steps ds along the curve is a good indication of how quickly you are turning.