Divergence

Google Classroom

Divergence measures the change in density of a fluid flowing according to a given vector field.

Background

- Partial derivatives
- Vector fields

What we're building to

- Interpret a vector field as representing a fluid flow.

- The divergence is an operator, which takes in the vector-valued function defining this vector field, and outputs a scalar-valued function measuring the change in density of the fluid at each point.

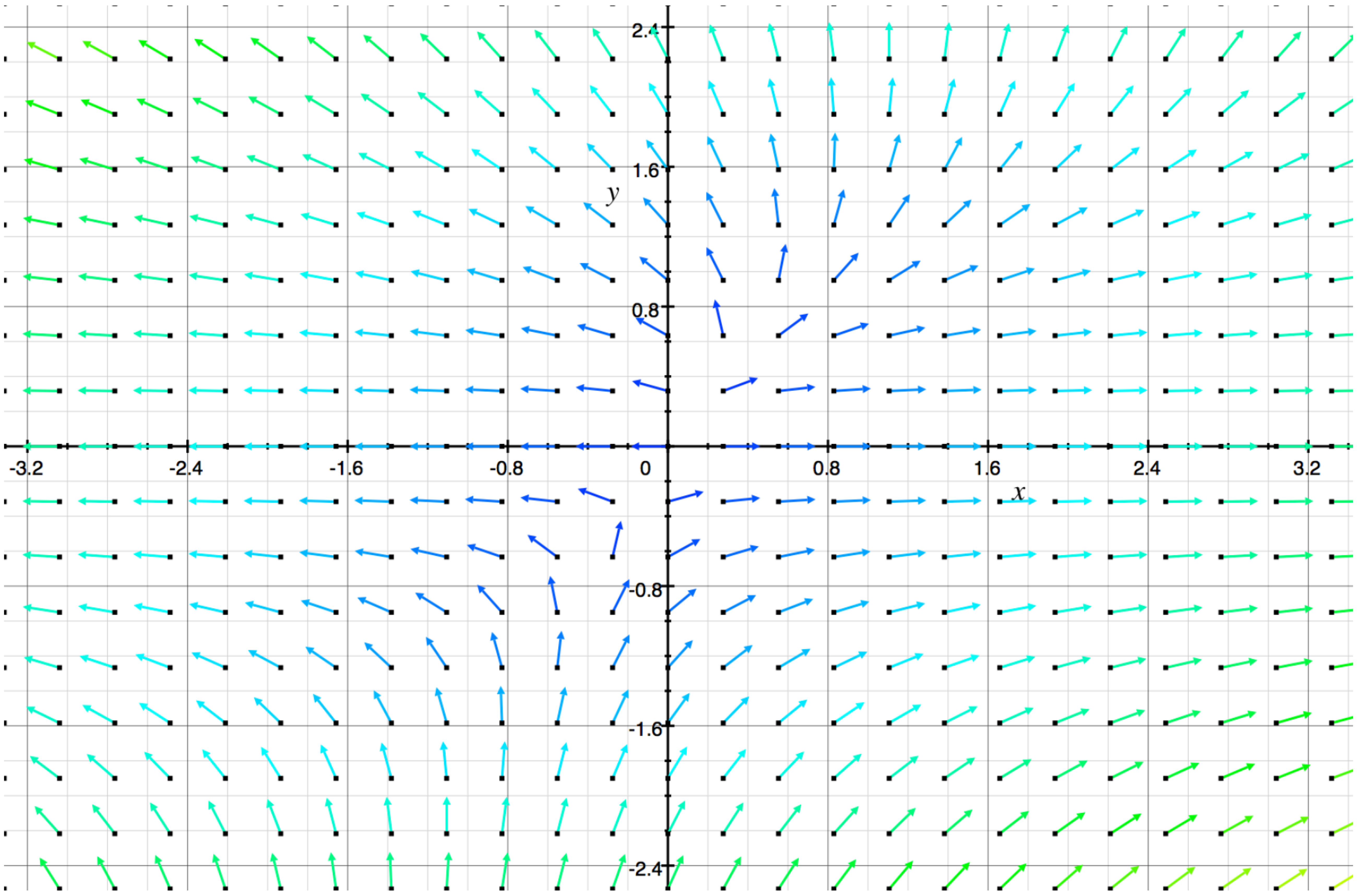
- This is the formula for divergence:

$$\operatorname{div} \vec{\mathbf{v}} = \nabla \cdot \vec{\mathbf{v}} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \cdots$$

  Here, $v_1, v_2, \dots$ are the component functions of $\vec{\mathbf{v}}$.

Changing density in fluid flow

Take a look at the following vector field:



[Hide explanation]

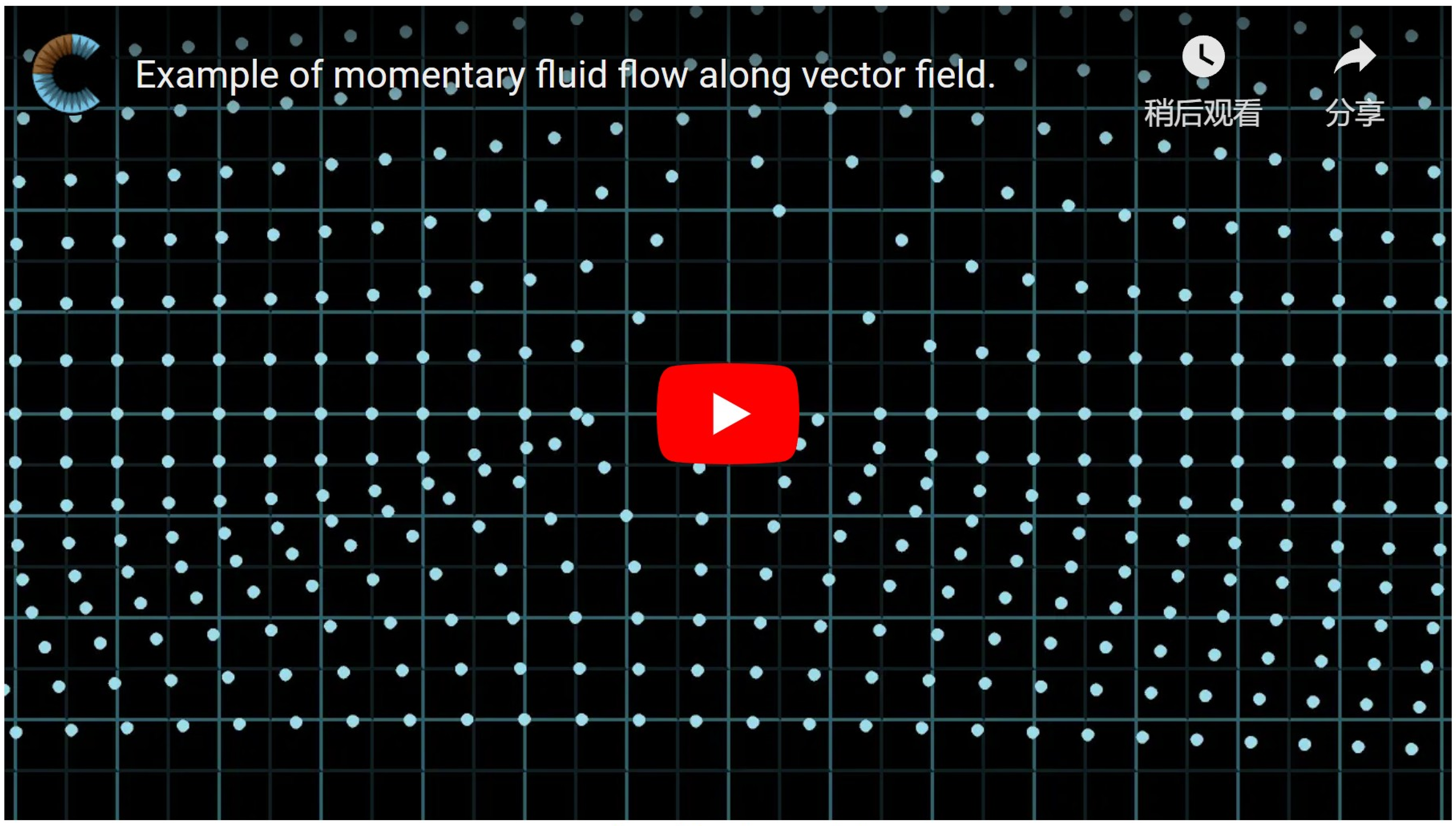
> Notice, this particular drawing of the vector field is color-coded, in the sense you should interpret blue vectors as being shorter, and greenish-yellow vectors as being longer, even though technically they are all drawn with the same length.

So that's the picture, but what's the function?

$$\vec{\mathbf{v}}(x, y) = \begin{bmatrix} 2x - y \\ y^2 \end{bmatrix}$$

The inputs to $\vec{\mathbf{v}}$ are points in two-dimensional space, (x, y), and the outputs are two-dimensional vectors, which in the vector field are attached to the corresponding point (x, y).

A nice way to think about vector fields is to imagine the **fluid flow** they could represent. Specifically, for each point (x, y) in two-dimensional space, imagine a particle sitting at (x, y) flowing in the direction of the vector attached to that point, $\vec{\mathbf{v}}(x, y)$. Moreover, suppose the speed of the particle's movement is determined by the length of that vector. The following animation shows what this might look like for our given function $\vec{\mathbf{v}}$ for just a brief instant:



See video transcript

Notice, during this fluid flow, some regions tend to become less dense with dots as particles flow away, such as the upper middle section. On the other hand, down and to the left of that region, particles tend to flow towards each other and the dots get more dense.

Key question: For a given vector-valued function $\vec{\mathbf{v}}(x, y)$, how can we measure the **change in density** of particles around a point (x, y) as these particles flow along the vectors given by $\vec{\mathbf{v}}(x, y)$?

We can answer this question using a variation of the derivative called **divergence**. We'll talk more about fluid flow below, but first, let's establish the notation and formula used to express this concept.

Notation and formula for divergence

The notation for divergence uses the same symbol "∇" which you may be familiar with from the gradient. As with the gradient, we think of this symbol loosely as representing a vector of partial derivative symbols.

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial y} \\ \\ \vdots \end{bmatrix}$$

We write the divergence of a vector-valued function $\vec{\mathbf{v}}(x, y, \dots)$ like this

$$\nabla \cdot \vec{\mathbf{v}} \quad \leftarrow \text{Divergence of } \vec{\mathbf{v}}$$

This is mildly nonsensical since ∇ isn't *really* a vector. Its entries are operators, not numbers. Nevertheless, using this dot product notation is super helpful for remembering how to compute divergence, just take a look:

$$\nabla \cdot \vec{\mathbf{v}} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \\ \frac{\partial}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} 2x - y \\ y^2 \end{bmatrix}$$

$$= \frac{\partial}{\partial x}(2x - y) + \frac{\partial}{\partial y}(y^2)$$
$$= 2 + 2y$$

More generally, the divergence can apply to vector-fields of any dimension. This means $\vec{\mathbf{v}}$ can have any number of input variables, as long as its output has the same dimensions. Otherwise, it couldn't represent a vector field. If we write $\vec{\mathbf{v}}$ component-wise like this:

$$\vec{\mathbf{v}}(x_1, \dots, x_n) = \begin{bmatrix} v_1(x_1, \dots, x_n) \\ \vdots \\ v_n(x_1, \dots, x_n) \end{bmatrix}$$

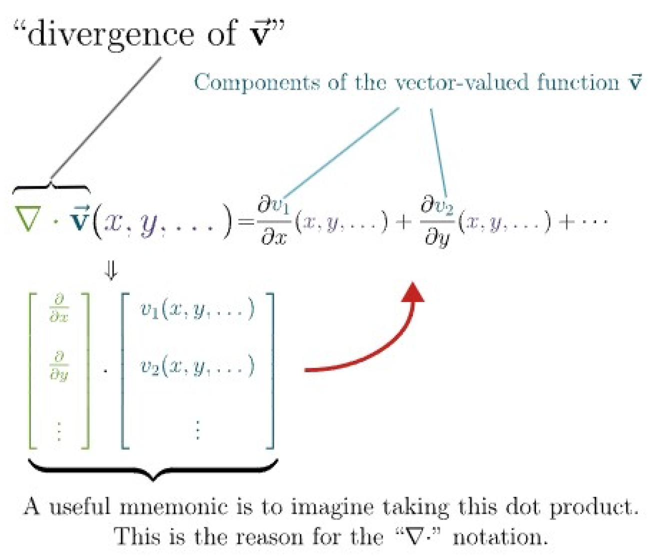
Then the divergence of $\vec{\mathbf{v}}$ looks like this:

$$\nabla \cdot \vec{\mathbf{v}} = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n}$$

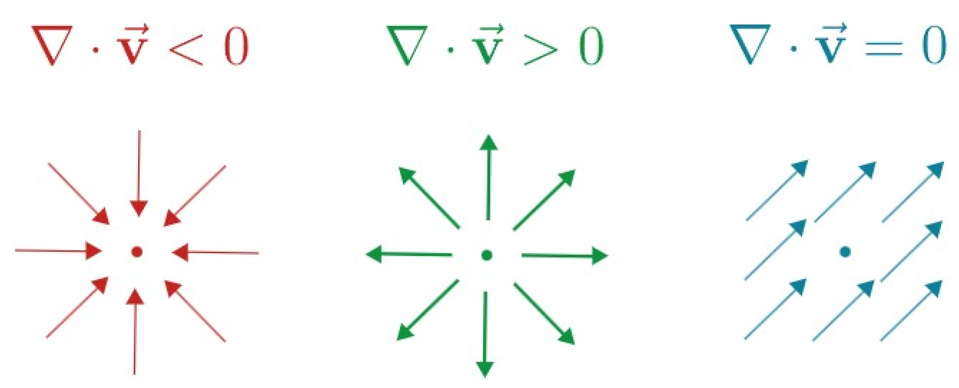
[Hide explanation]

> You could imagine taking the matrix of all possible partial derivatives (we could be fancy and call this the Jacobian), and adding all the diagonal elements:
>
> $$\begin{bmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} & \frac{\partial v_1}{\partial z} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} & \frac{\partial v_2}{\partial z} \\ \frac{\partial v_3}{\partial x} & \frac{\partial v_3}{\partial y} & \frac{\partial v_3}{\partial z} \end{bmatrix} \rightarrow \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Let's summarize this with a quick diagram:



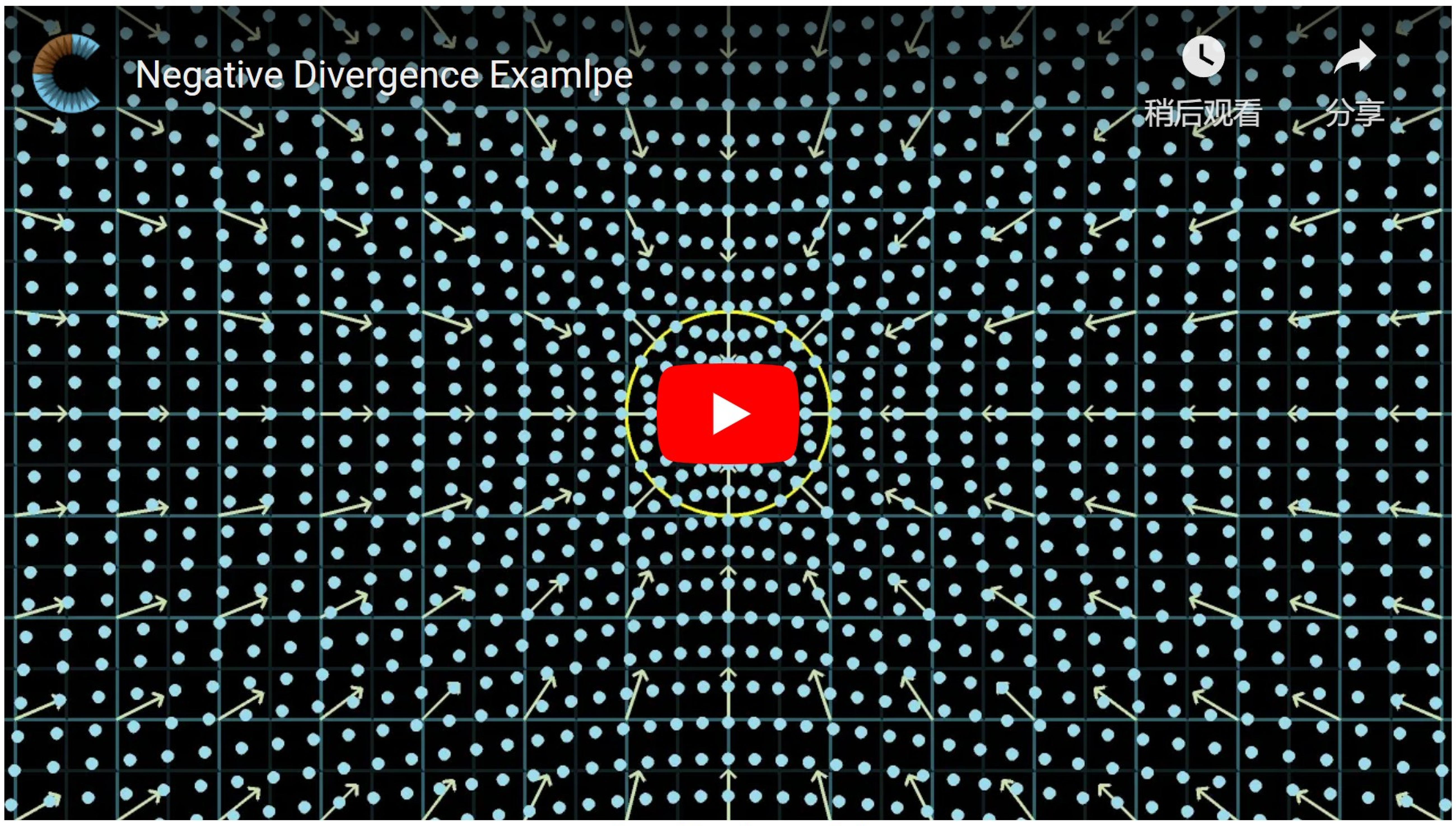
Interpretation of divergence



Let's say you evaluate the divergence of a function $\vec{\mathbf{v}}$ at some point (x_0, y_0), and it comes out **negative**.

$$\nabla \cdot \vec{\mathbf{v}}(x_0, y_0) < 0$$

This means a fluid flowing along the vector field defined by $\vec{\mathbf{v}}$ would tend to become **more dense** at the point (x_0, y_0). For example, the following animation shows a vector field with negative divergence at the origin.

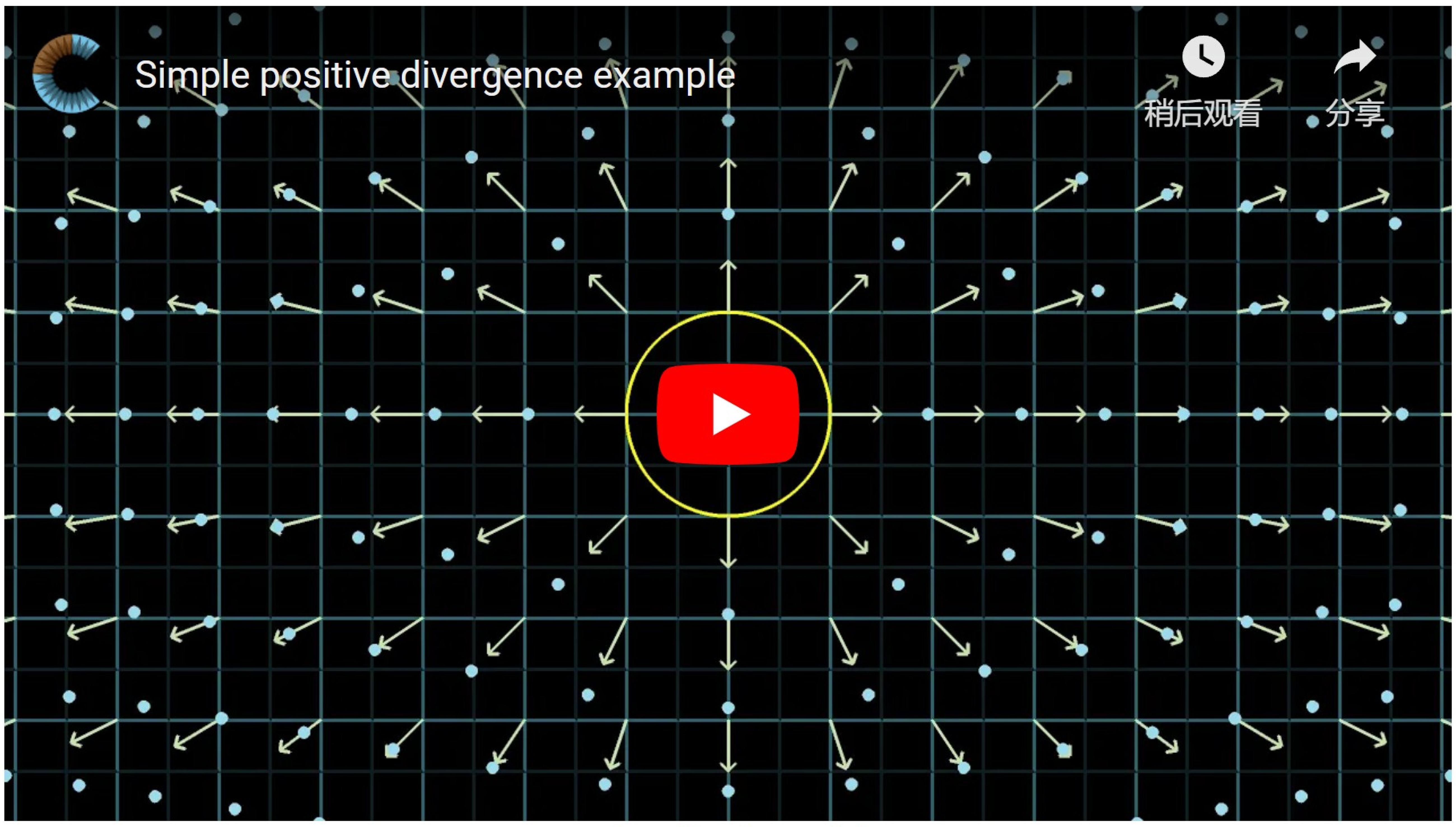


See video transcript

On the other hand, if the divergence at a point (x_0, y_0) is **positive**,

$$\nabla \cdot \vec{\mathbf{v}}(x_0, y_0) > 0$$

the fluid flowing along the vector field becomes **less dense** around (x_0, y_0). Here's an example:

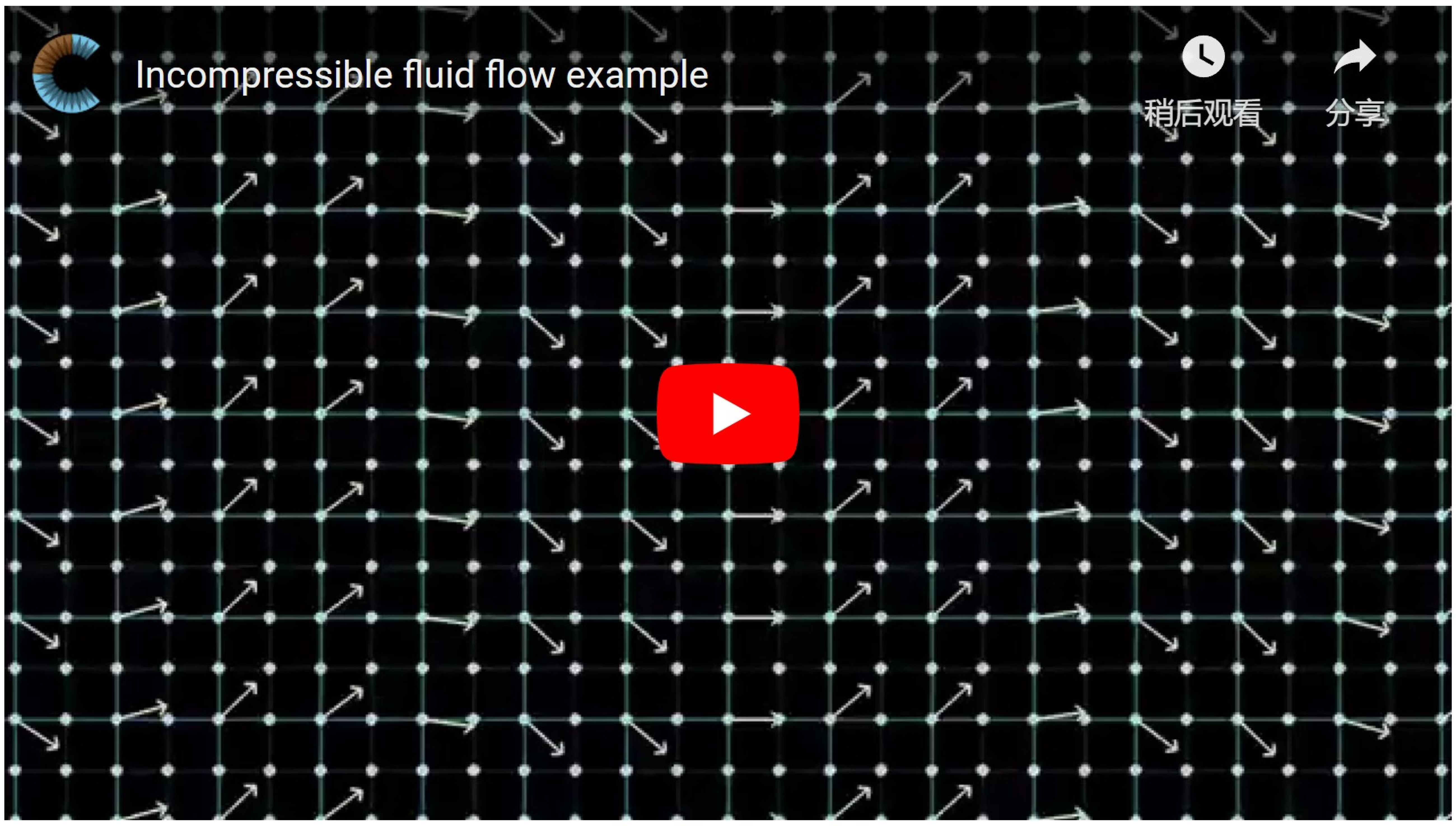


See video transcript

Finally, the concept of **zero-divergence** is very important in fluid dynamics and electrodynamics. It indicates that even though a fluid flows freely, its **density stays constant**. This is particularly handy when modeling incompressible fluids, such as water. In fact, the very idea that a fluid is incompressible can be tightly communicated with the following equation:

$$\nabla \cdot \vec{\mathbf{v}} = 0$$

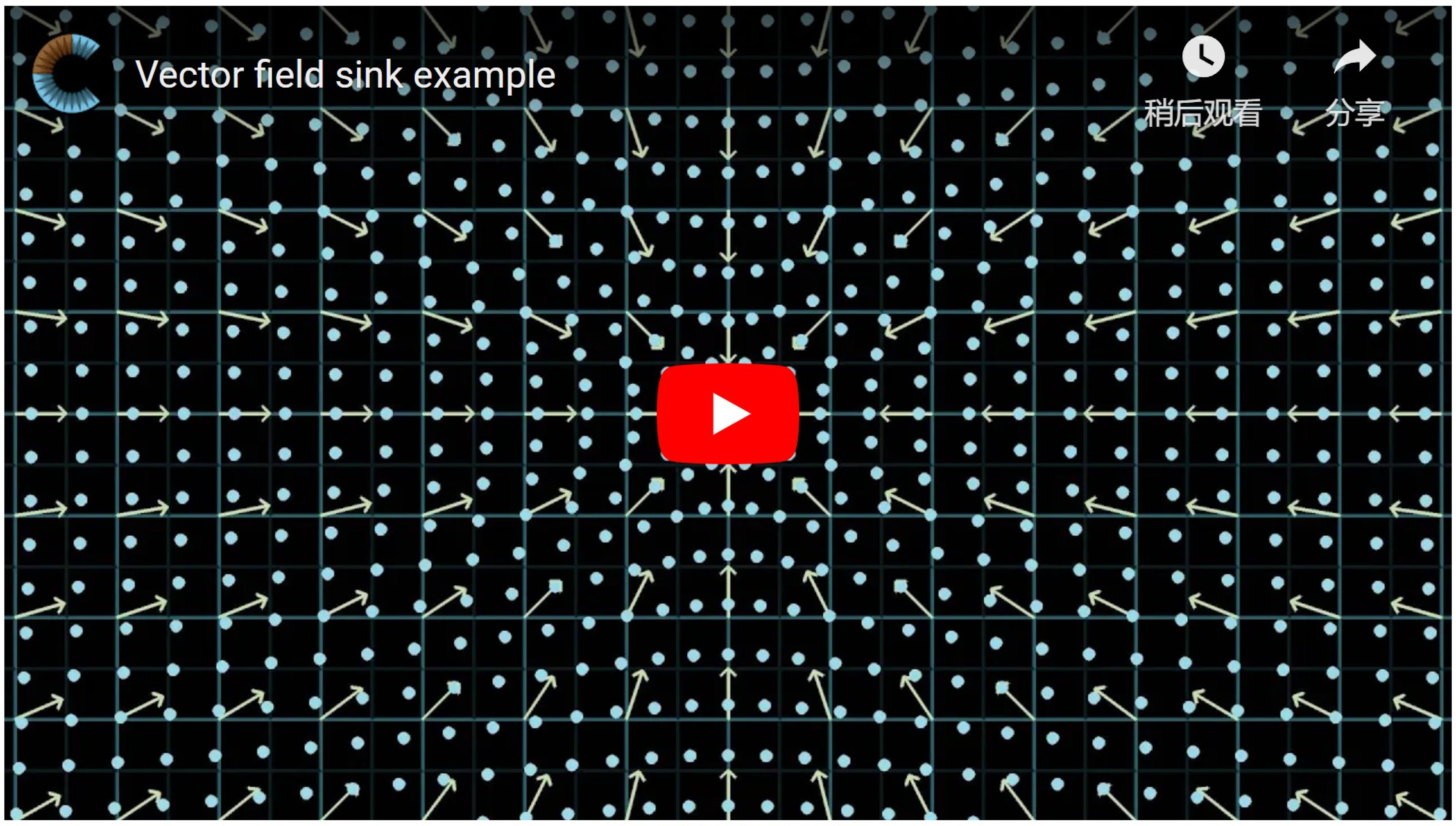
Such vector fields are called "**divergence-free**." Here's an example of what that might look like:



See video transcript

Sources and sinks

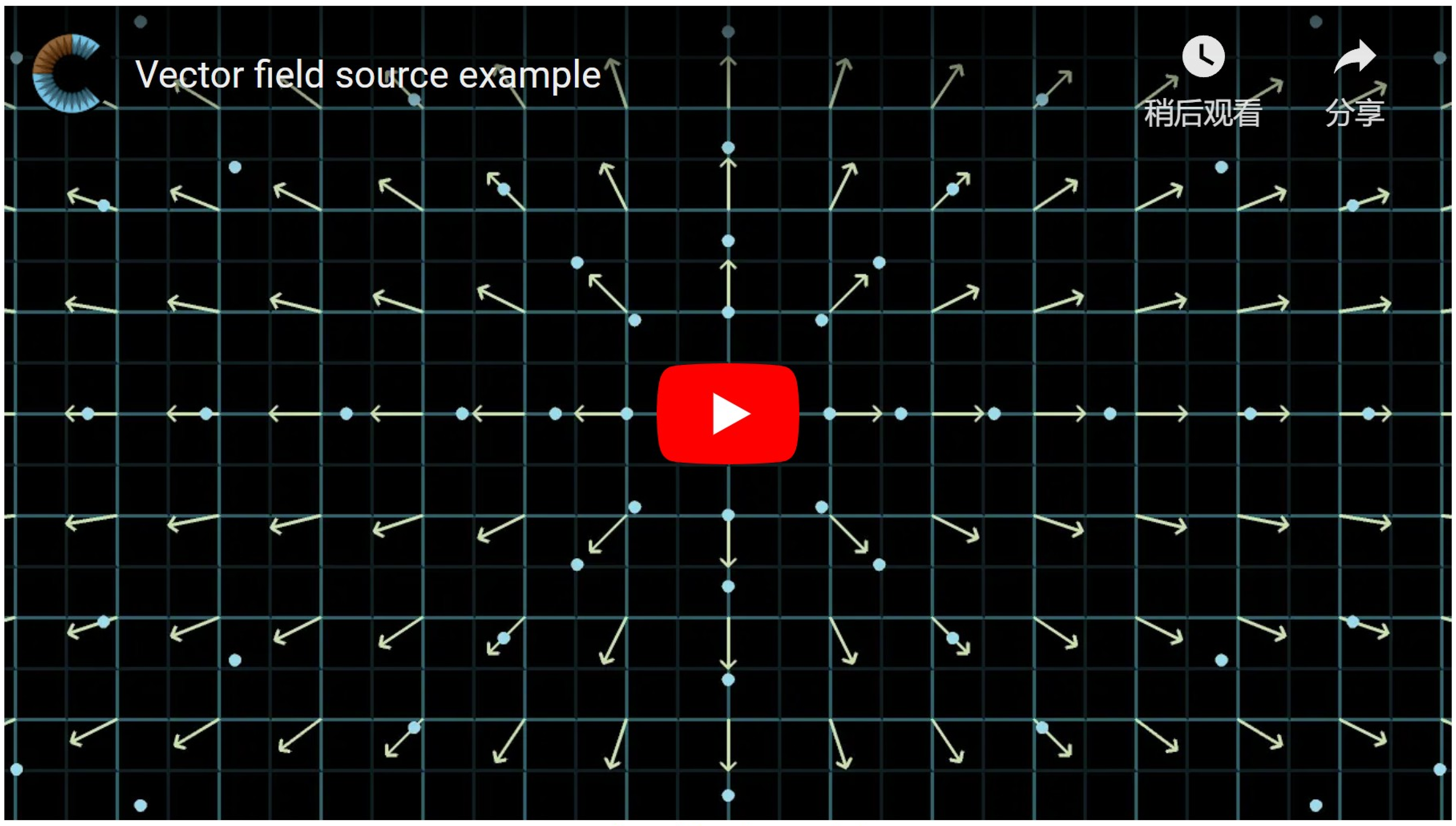
Sometimes, for points with negative divergence, instead of thinking about a fluid getting more dense after a momentary fluid motion, some people imagine the fluid draining at that point while the fluid flows constantly. Here's what this might look like:



See video transcript

As such, points of negative divergence are often called "**sinks**."

Likewise, instead of thinking of points with positive divergence as becoming less dense during a momentary motion, these points might be thought of as "**sources**" constantly generating more fluid particles.



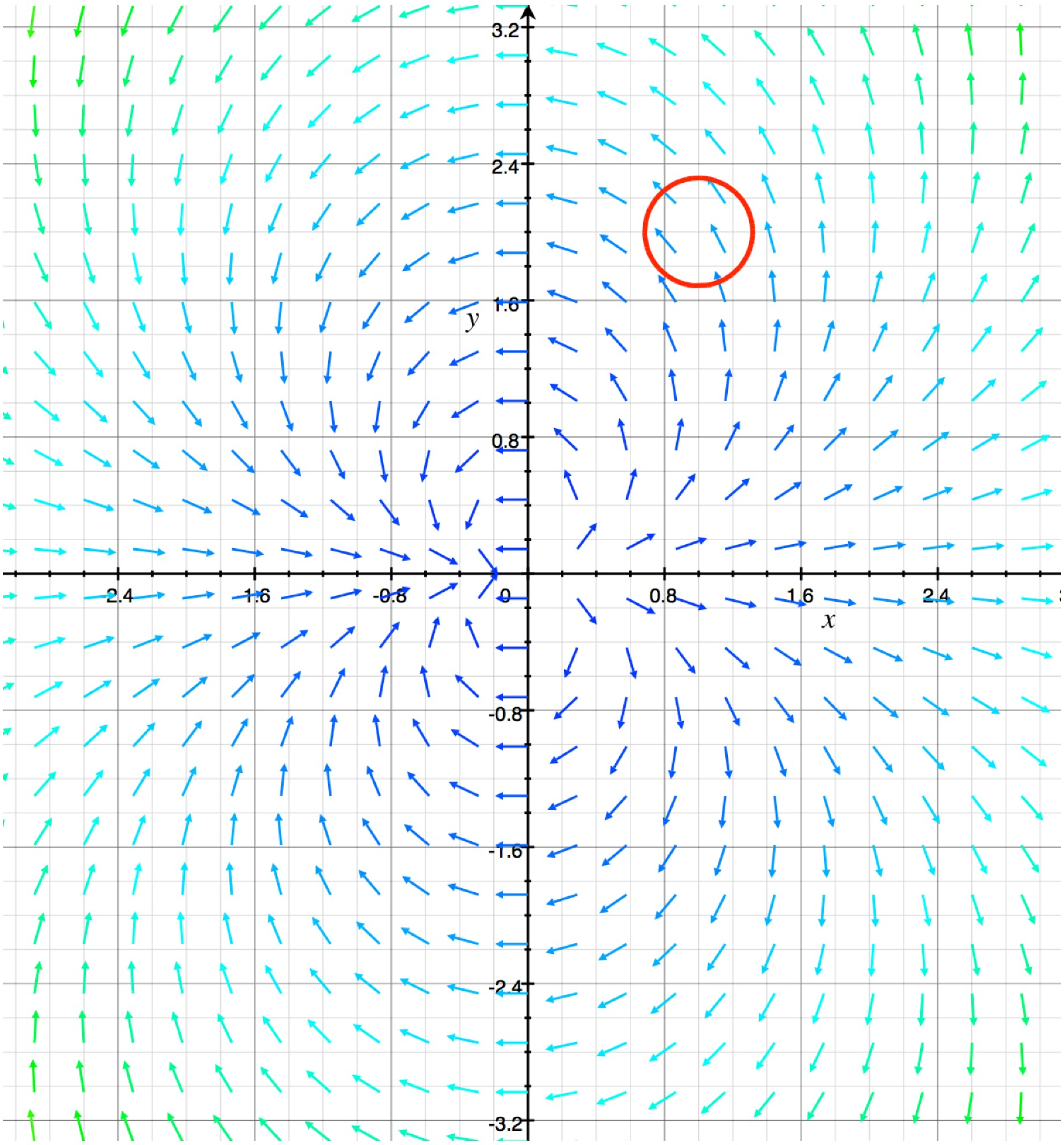
See video transcript

Divergence in higher dimensions

Although all the diagrams and animations I'm making show the two-dimensional case, you should understand that all these concepts could apply to three or more dimensions as well.

Try this as a good mental exercise to test if you understand what divergence represents: Imagine a three-dimensional vector field, and picture what points of positive, negative, and zero divergences might look like.

Example 1: Compute and interpret divergence



Vector field for Example 1

Problem: Define a vector field by

$$\vec{\mathbf{v}}(x, y) = (x^2 - y^2)\hat{\mathbf{i}} + 2xy\hat{\mathbf{j}}$$

Compute the divergence, and determine whether the point $(1, 2)$ is more of a source or a sink.

Step 1: Compute the divergence.

$\nabla \cdot \vec{\mathbf{v}} =$ [ ]

Check

[Hide explanation]

We compute divergence by applying the formula. Add the partial derivative with respect to x of the first component to the partial derivative with respect to y to the second component.

$$\begin{aligned} \nabla \cdot \vec{\mathbf{v}} &= \frac{\partial}{\partial x}(x^2 - y^2) + \frac{\partial}{\partial y}(2xy) \\ &= 2x + 2x \\ &= \boxed{4x} \end{aligned}$$

Step 2: Plug in $(1, 2)$.

$\nabla \cdot \vec{\mathbf{v}}(1, 2) =$ [ ]

Check

[Hide explanation]

Evaluating this function at the point $(1, 2)$, we get

$$\nabla \cdot \vec{\mathbf{v}}(1, 2) = 4(1) = 4$$

Step 3: Interpret. Is the fluid more of a source or a sink at $(1, 2)$?

Choose 1 answer:

(A) Source

(B) Sink

(C) Neither

Check

[Hide explanation]

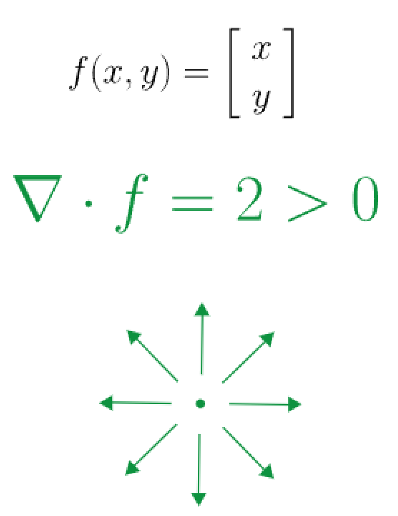
> Because this is positive, the density of a fluid flowing along the vector field given by $\vec{\mathbf{v}}(x, y)$ decreases at the point $(1, 2)$. Therefore, it is more of a source.

Confusing signs

It always trips me up that *positive* divergence indicates a *negative* change in density, and that a *negative* divergence indicates a *positive* change in density. Isn't that confusing? The sources/sinks interpretation helps a bit, because points of positive divergences are generating more fluid, while points of negative divergence are sucking it away.

Personally, the way I always remember is to think of the case when f is the identity function, taking the point (x, y) to the vector $\left[\begin{array}{c} x \\ y \end{array}\right]$. The resulting vector field has all vectors pointing away from the origin (can you see why?), and it's relatively quick to compute $\nabla \cdot f$.

$$\nabla \cdot f = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 1 + 1 = 2$$



So each time I return to divergence after not having seen it for a while and think "hmm, is it positive or negative divergence that indicates a loss in density," I go through this little exercise and remember, "Ah yes, that's how it goes, positive divergence indicates an outward flow."

Further resources

In the [next article](), I'll give an intuition for why the formula for divergence has anything to do with fluid flow.

Later on, once line integrals and surface integrals are covered, I talk about the [formal definition of divergence]().

Summary

- Interpret a vector field as representing a fluid flow.

- The divergence is an operator, which takes in the vector-valued function defining this vector field, and outputs a scalar-valued function measuring the change in density of the fluid at each point.

- The formula for divergence is

$$\operatorname{div} \vec{\mathbf{v}} = \nabla \cdot \vec{\mathbf{v}} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \cdots$$

  where $v_1, v_2, \ldots$ are the component functions of $\vec{\mathbf{v}}$.

Keep in mind, though, divergence is used in all sorts of contexts which can have nothing to do with fluid. Electrodynamics is a big one, for example. The fluid flow interpretation is very useful, and gives a much stronger intuition than a blind use of symbols would, but it should be taken with a grain of salt from time to time.