Stokes' theorem

Google Classroom

This is the 3d version of Green's theorem, relating the surface integral of a curl vector field to a line integral around that surface's boundary.

Background

- Green's theorem
- Flux in three dimensions
- Curl in three dimensions

Not strictly required, but very helpful for a deeper understanding:

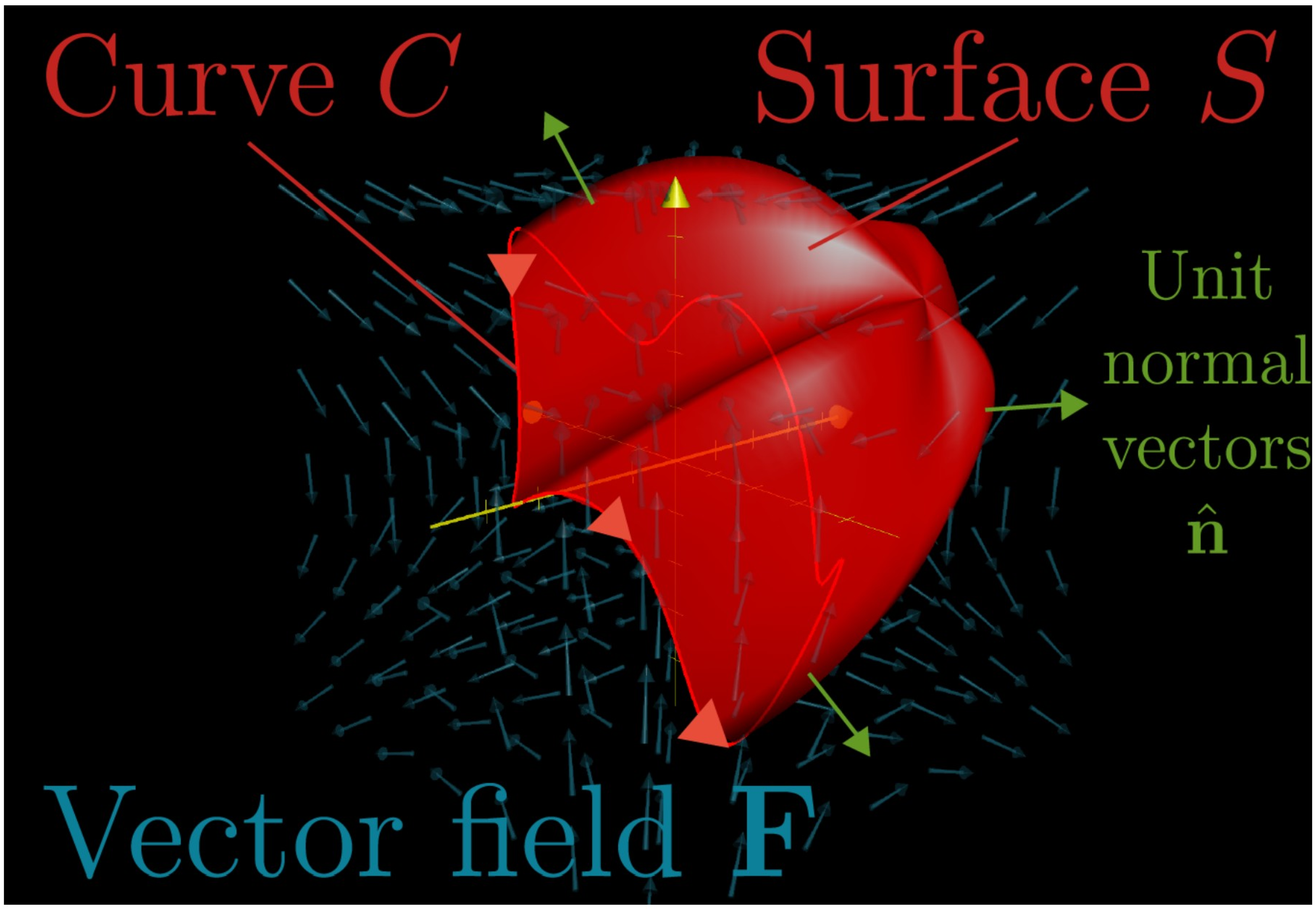
- Formal definition of curl in three dimensions

This article is for physical intuition

If you would like examples of using Stokes' theorem for computations, you can find them in the next article. Here, the goal is to present the theorem in such a way that you can get a gut feeling for what it is really saying, and why it is true.

What we're building to

- Stokes' theorem is the 3D version of Green's theorem.



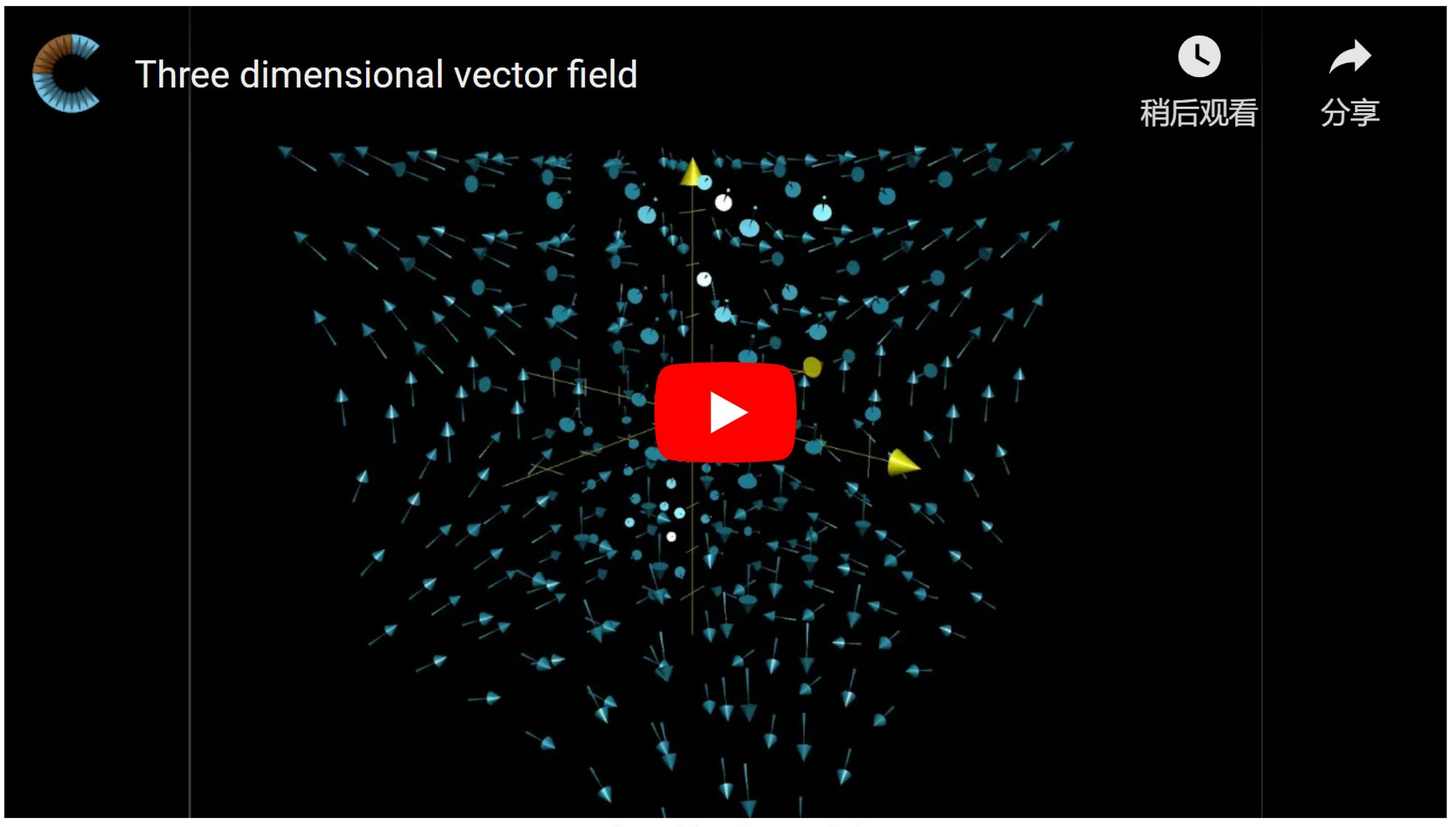
- It relates the surface integral of the curl of a vector field with the line integral of that same vector field around the boundary of the surface:

$$\overbrace{\underbrace{\iint_S}_{S\text{ is a surface in 3D}} (\text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}})\,d\Sigma}^{\text{Surface integral of a curl vector field}} = \underbrace{\int_C \mathbf{F}\cdot d\mathbf{r}}_{\text{Line integral around boundary of surface}}$$

[Breakdown of terms]

Interpreting a line integral in 3D

Let $\mathbf{F}(x, y, z)$ represent a three-dimensional vector field.



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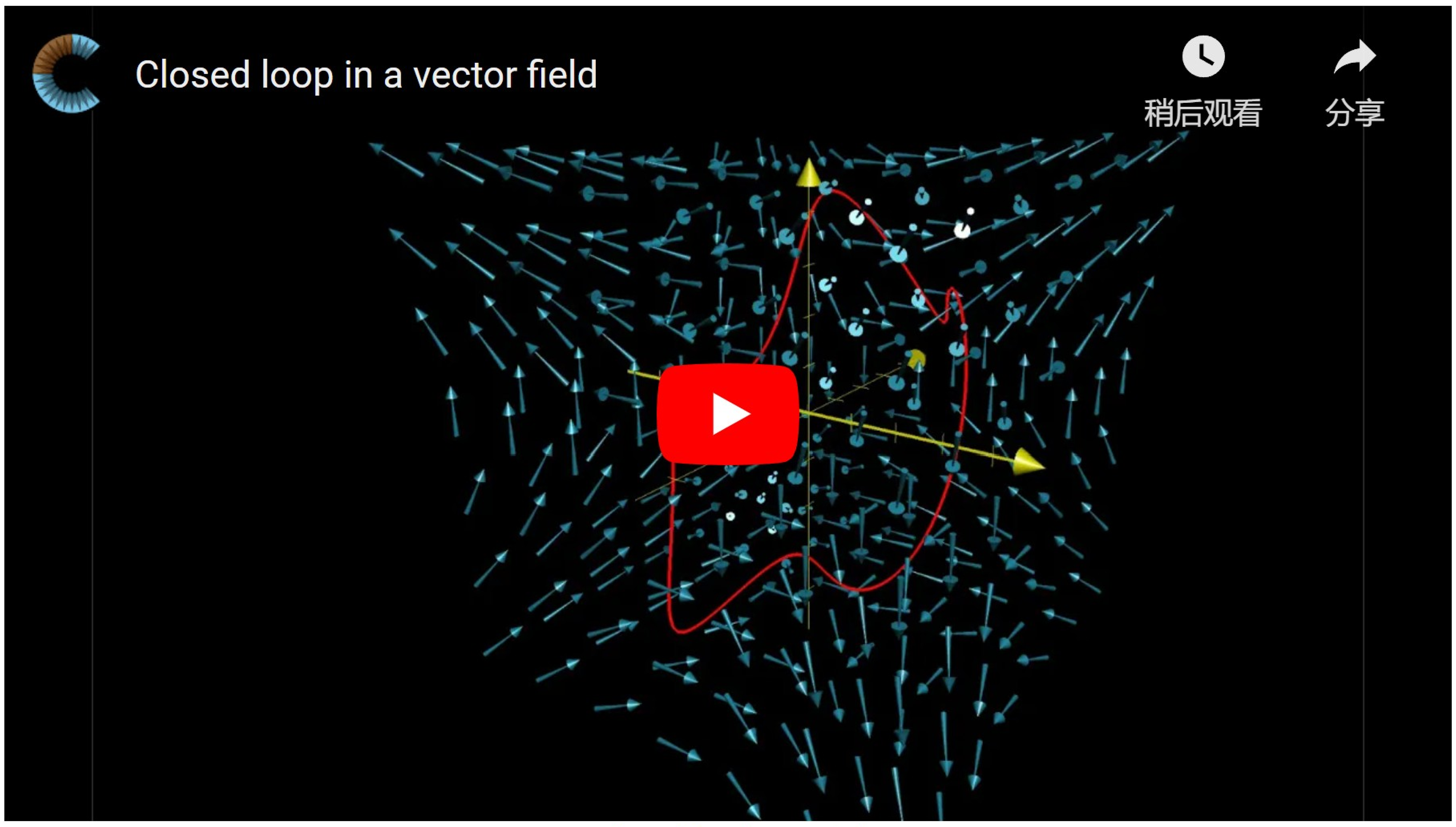
Think of this vector field as being the velocity vector of some gas, whooshing about through space.

Now let C be some closed curve inside this vector field.



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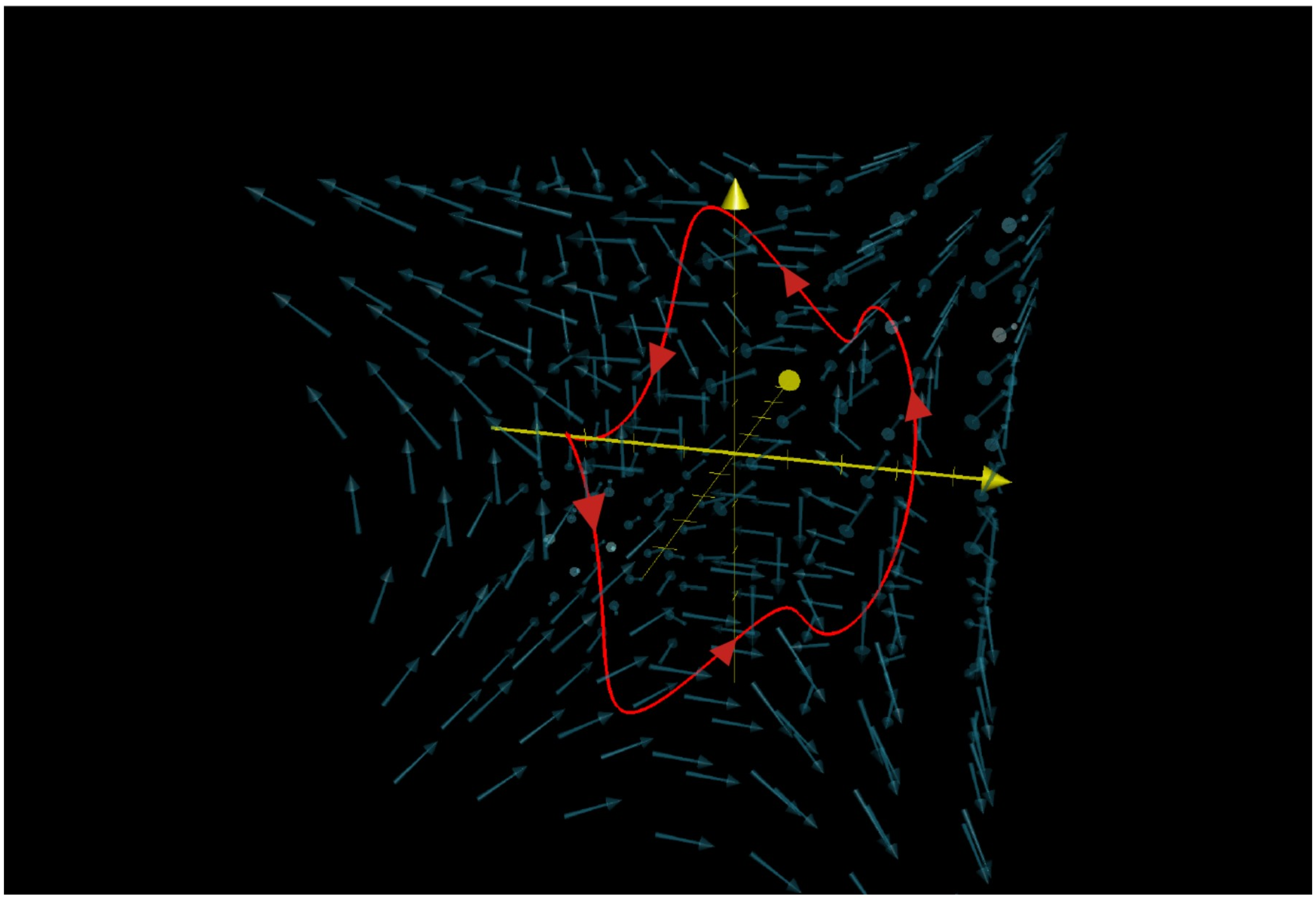
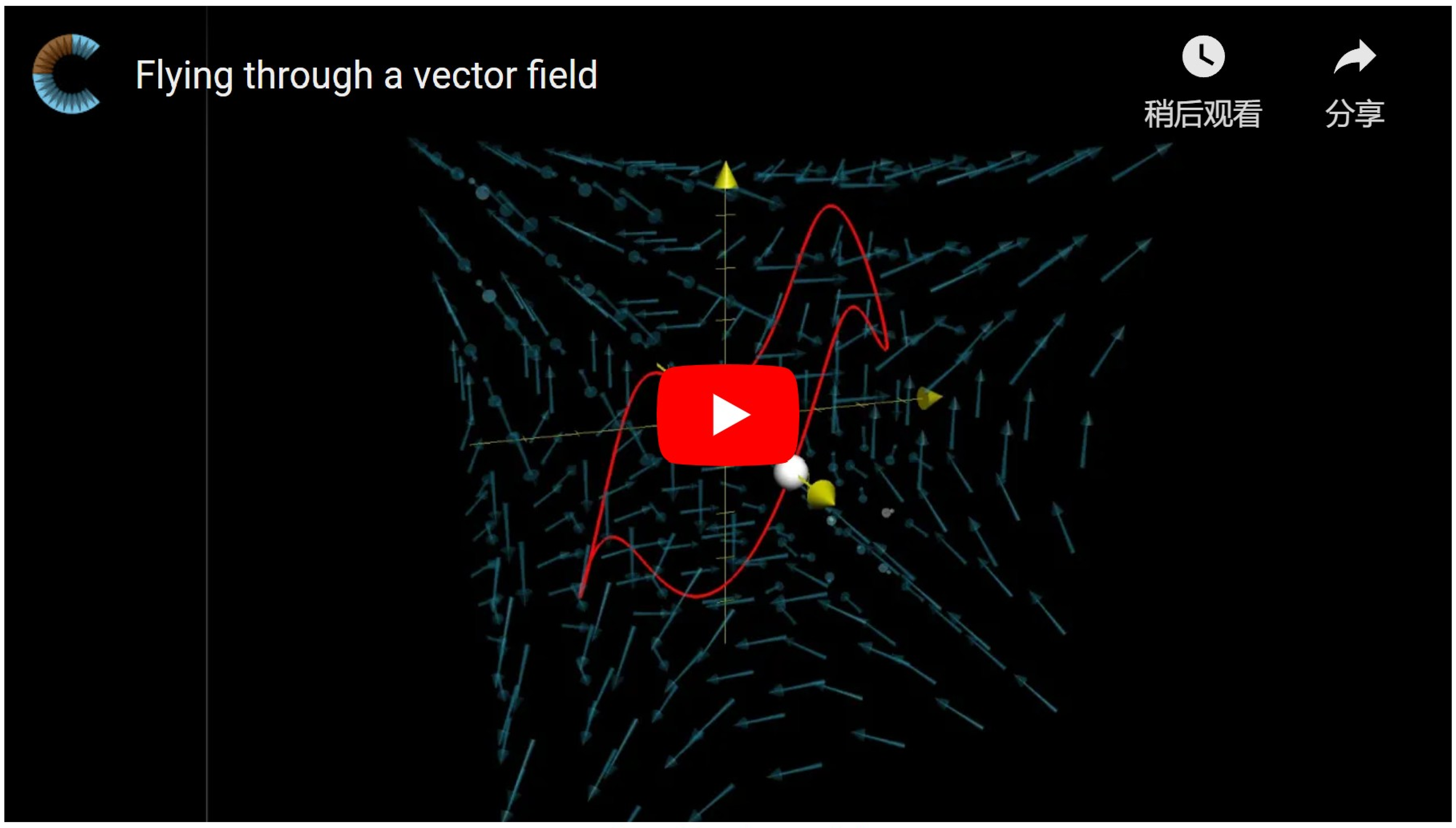
How can you interpret the line integral of $\mathbf{F}$ around C?

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

Well, first of all, this integral doesn't make sense until the curve is oriented. The differential vector $d\mathbf{r}$ represents a tiny step along the curve, but in which direction? In three dimensions, you can't just say "clockwise" or "counterclockwise", since that will depend on where you are in space when you look at the curve. I'll address how we specify orientation mathematically below, but for now, it's easier to just draw an orientation:



Imagine you are a bird, flying through space along the curve C while the wind blows according to the vector field $\mathbf{F}$. (For the purposes of this animation, you are a sphere-shaped bird).



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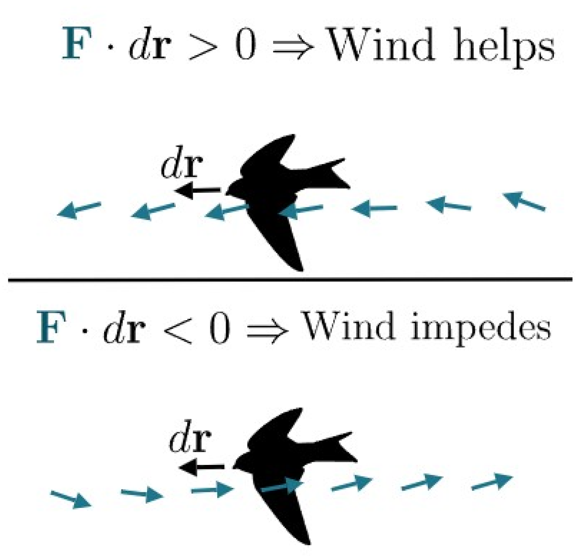
Think of each step (wing-flap?) of your motion along C as being the tiny vector $d\mathbf{r}$. Consider the dot product between $d\mathbf{r}$ and the wind-velocity-vector from the field $\mathbf{F}$ where you are. It will be positive when the wind is helping you, and negative when it is in your face.



Now look back at the line integral I originally asked about:

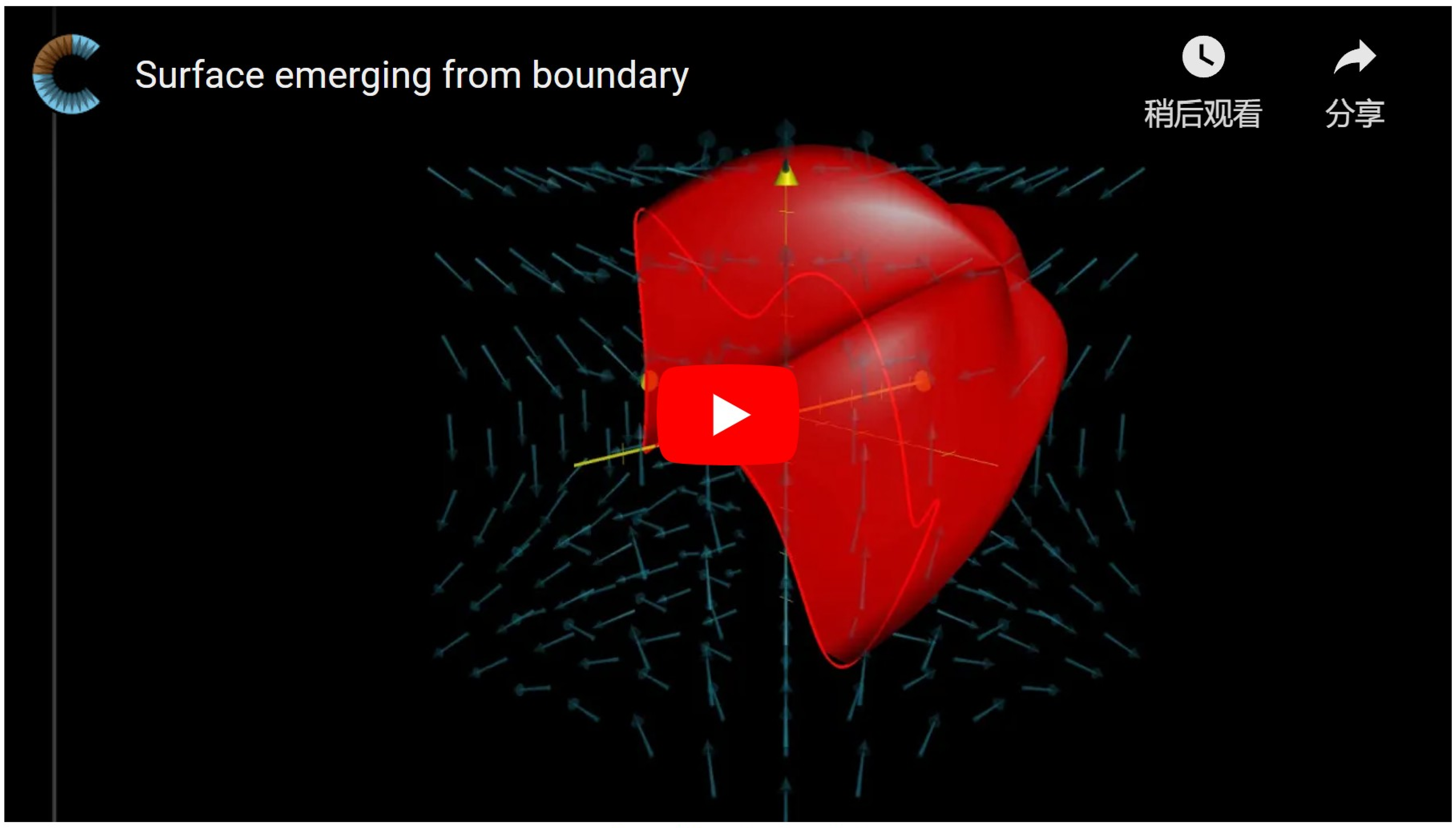
$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

You can think of this as adding up how helpful or burdensome the wind was during your flight. **It measures the tendency of the fluid flow to circulate around** C. If it is positive, the wind was generally helpful, and you could say that it tends to circulate around C in the direction of your specified orientation. If it is negative, you could say it tends to circulate the other way.

Chopping up a surface

Those of you who read the Green's theorem article will find what follows very familiar.

Consider a surface S in space whose boundary is the curve C, as if C was a wire loop that you just dipped in soap, and S is the beginnings of a soap bubble emerging from the loop.



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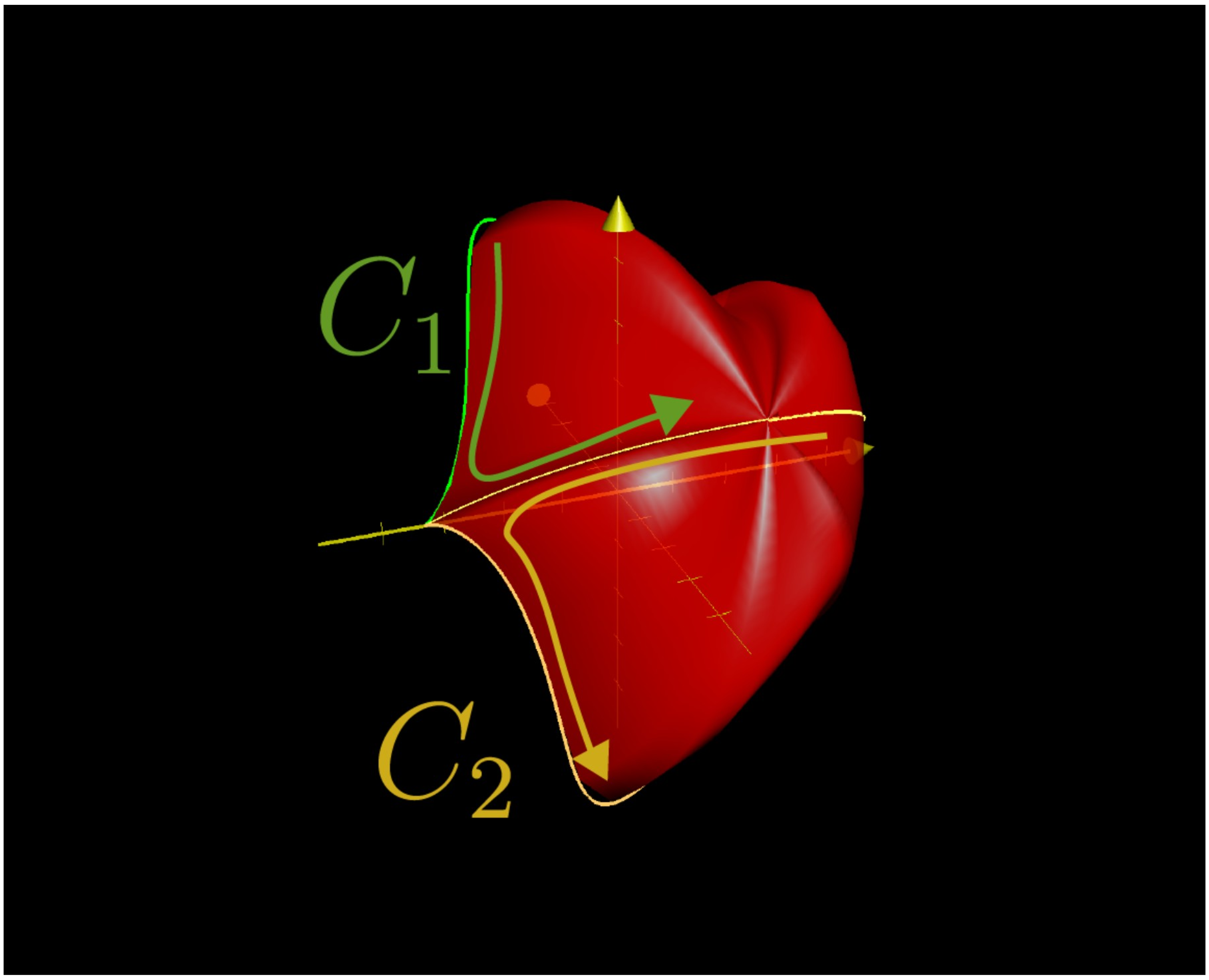
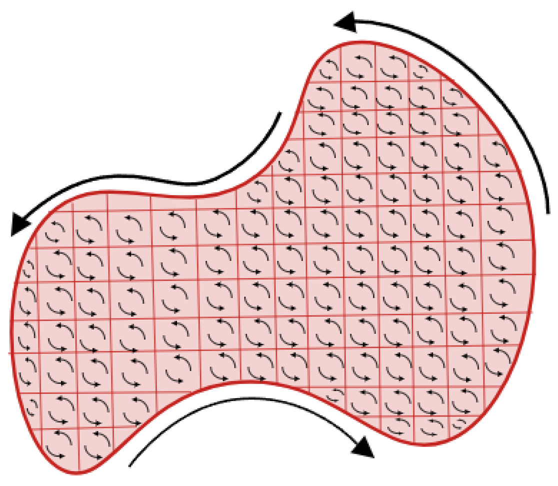
Slice this surface in half, and name the boundaries of the two resulting pieces C_1 and C_2. If they are each oriented the same way C was, the line integrals (of the same vector field $\mathbf{F}$) around each of these smaller curves cancel out along the slice that you made:



The portions of C_1 and C_2 which remain make up the original boundary C. So the sum of the line integrals around the smaller pieces equals the full line integral around C:

$$\underbrace{\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r}}_{\text{Cancel out along slice through } S} = \oint_{C} \mathbf{F} \cdot d\mathbf{r}$$

More generally, imagine slicing up S into many, many really small pieces, name their boundaries $C_1, \dots, C_n$, and orient them all the same way as C. It gets messy drawing this in 3D, so I'll just steal an image from the Green's theorem article showing the 2D version, which has essentially the same intuition.

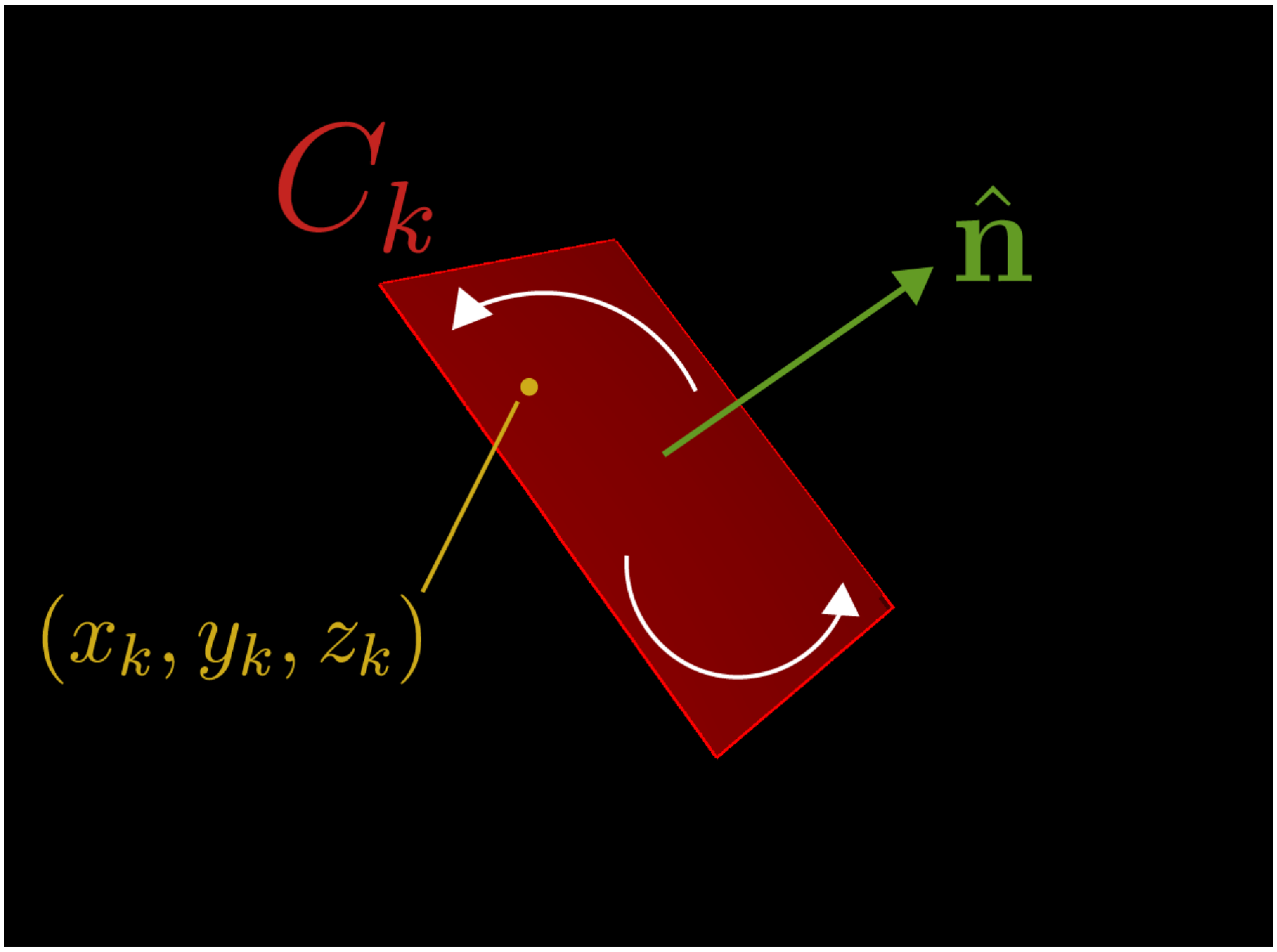


The line integrals around all of these little loops will cancel out along the slices within C, leaving only something equal to the line integral around C itself.

$$\underbrace{\sum_{k=1}^{n} \oint_{C_k} \mathbf{F} \cdot d\mathbf{r}}_{\text{Cancel out along slice through } S} = \oint_{C} \mathbf{F} \cdot d\mathbf{r}$$

Curl on each piece

The reason for chopping up S like this is that the line integral around a very small loop can be approximated using curl. Specifically, zoom in on a specific one of those pieces. If it's small enough, you can think of it as basically being flat.

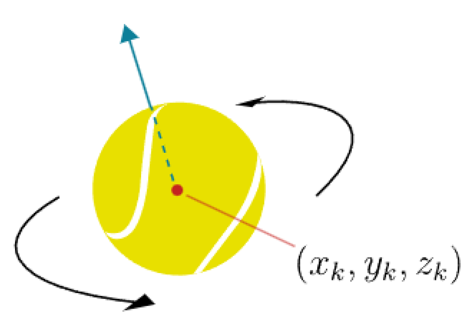


- Name the boundary of this piece "C_k".
- Choose some point (x_k, y_k, z_k) on the surface, inside this little loop.
- Let $\hat{\mathbf{n}}$ be a unit normal vector to the surface at the point (x_k, y_k, z_k). "Pointing which way?", you might ask. Curl the fingers of your right hand around the little loop C_k so that they align with its orientation. Stick out your thumb, and this is the direction of $\hat{\mathbf{n}}$.
- Let $d\Sigma$ represent the area of this little piece (in anticipation of using an infinitesimal area for a surface integral in just a bit).

Then the line integral of $\mathbf{F}$ around C_k can be approximated as follows:

$$\underbrace{\oint_{C_k} \mathbf{F} \cdot d\mathbf{r}}_{\substack{\text{Integral around} \\ \text{boundary of piece}}} \approx \overbrace{\left((\text{curl}\,\mathbf{F}(x_k, y_k, z_k)) \cdot \hat{\mathbf{n}}\right)}^{\text{Component of curl perpendicular to piece}} \underbrace{d\Sigma}_{\text{Area of piece}}$$

If you feel uneasy about your intuition for what curl means, or how a vector can represent rotation, consider reviewing [this article on curl](#).



Here's the loose intuition for why this approximation works: $\text{curl}\,\mathbf{F}(x_k, y_k, z_k)$ is a vector which tells you how the fluid flowing along the vector field $\mathbf{F}$ tends to rotate near the point (x_k, y_k, z_k). For example, if you imagine a small tennis ball floating in space, centered at the point (x_k, y_k, z_k), the vector $\text{curl}\,\mathbf{F}(x_k, y_k, z_k)$ describes the way it will tend to spin due to the wind blowing around it. That is to say, the vector is directed along the axis of rotation, and its magnitude is proportional to the rate of rotation.

When we take the dot product between this curl vector and $\hat{\mathbf{n}}$, the unit normal vector to the surface, it extract the component of the curl vector which is perpendicular to the surface. This will describe the rate of fluid rotation *on the surface itself*. On the other hand, that little bit of fluid rotation is also described by the line integral around the boundary of the tiny piece: $\oint_{C_k} \mathbf{F} \cdot d\mathbf{r}$.

Actually, that line integral produces a really small number (since C_k is very short), but $\text{curl}\,\mathbf{F}(x_k, y_k, z_k)$ produces a number which doesn't care about the size of the piece containing (x_k, y_k, z_k). This is why we scale down the relevant component of curl by the area of the tiny piece.

$$\underbrace{\oint_{C_k} \mathbf{F} \cdot d\mathbf{r}}_{\substack{\text{Integral around} \\ \text{boundary of piece}}} \approx \overbrace{\left((\text{curl}\,\mathbf{F}(x_k, y_k, z_k)) \cdot \hat{\mathbf{n}} \right)}^{\text{Component of curl perpendicular to piece}} \underbrace{d\Sigma}_{\text{Area of piece}}$$

(For a deeper understanding of this approximation, take a look at the formal definition of curl in three dimensions.)

Surface integral of curl

Combining the ideas of the last two sections, here's what we get:

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\substack{\text{Integral around boundary} \\ \text{of the entire surface}}}$$

$$\downarrow$$

$$\underbrace{\sum_{k=1}^{n} \oint_{C_k} \mathbf{F} \cdot d\mathbf{r}}_{\text{Sum of integrals around tiny pieces}}$$

$$\downarrow$$

$$\underbrace{\sum_{k=1}^{n} \operatorname{curl} \mathbf{F}(x_k, y_k, z_k) \cdot \hat{\mathbf{n}}\, d\Sigma}_{\text{Apply curl approximation to each piece}}$$

As we chop things up more and more finely, this last sum approaches the surface integral of $(\operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}})$ over the surface S. (If this does not make sense to you, consider reviewing the article on surface integrals).

$$\sum_{k=1}^{n} \operatorname{curl} \mathbf{F}(x_k, y_k, z_k) \cdot \hat{\mathbf{n}}\, d\Sigma$$

$$\downarrow \text{As } S \text{ is chopped more and more finely}$$

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}\, d\Sigma$$

Putting this together, we have the following marvelous equation, known as Stokes' theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}\, d\Sigma$$

[Hide explanation]

Okay, a few of you may raise the objection that I started with an approximation for the line integral around each piece, and am now making a conclusion using an approximation-free equality. And you'd be right to do so!

In the article on Green's theorem, which involves a nearly identical line of reasoning, but in two dimensions, I offered a couple notes to go into the details of how the approximation disappears. If you are curious, I encourage you to go back through that same line of reasoning and think about how it works for Stokes' theorem and surface integrals.

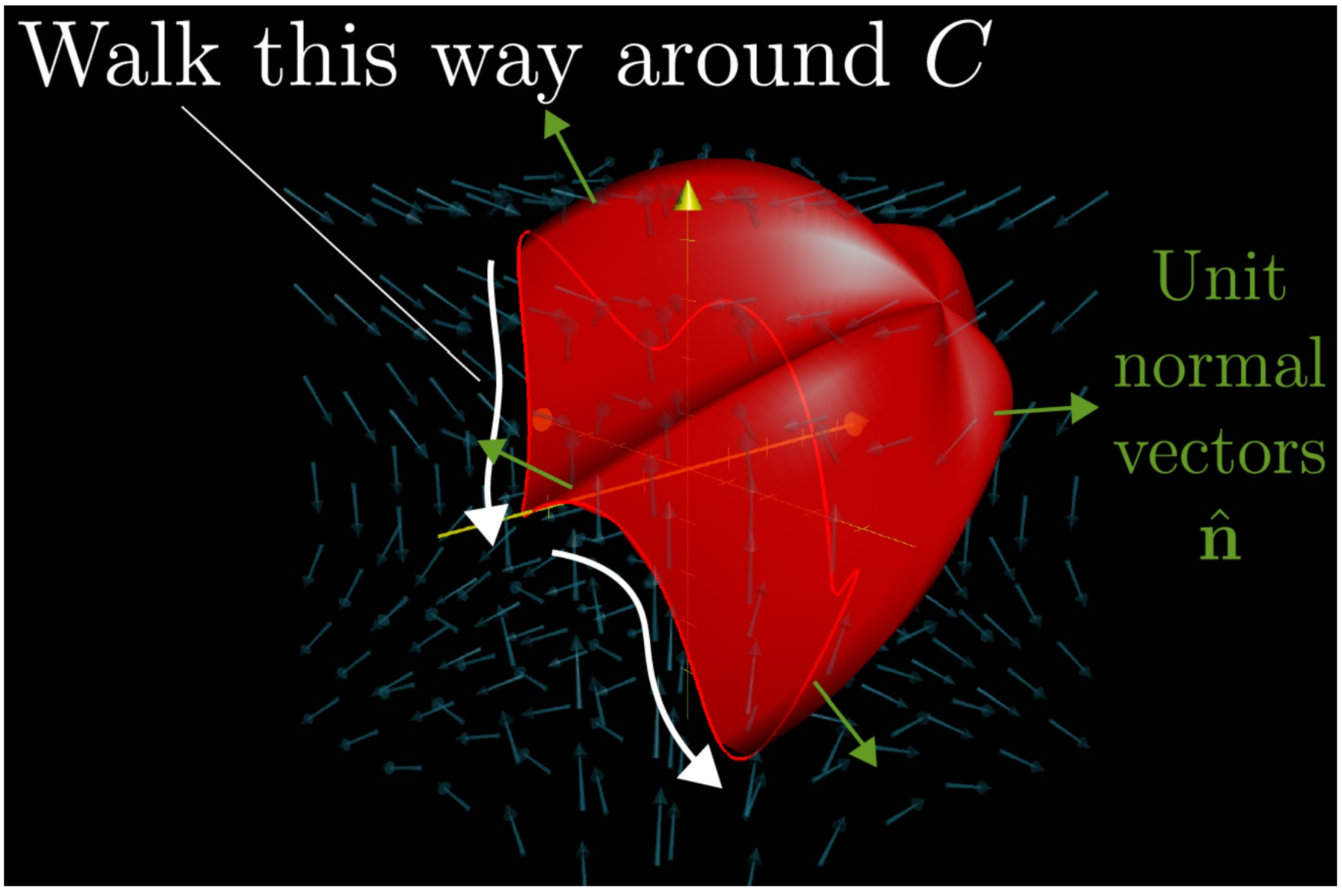
This exercise will also be made all the more enlightening if you go in armed with the formal definition of curl in three dimensions.

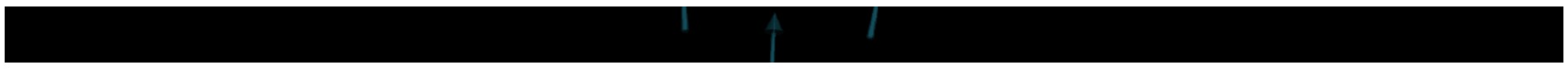
Aligning orientation

Surfaces are oriented by the chosen direction for their unit normal vectors. For example, you will often see a surface oriented using *outward-facing* unit normal vectors (although not all surfaces have a notion of outward-facing vs. inward-facing unit normal vectors).

Curves are oriented by the chosen direction for their tangent vectors.

For Stokes' theorem to work, the orientation of the surface and its boundary must "match up" in the right way. Otherwise, the equation will be off by a factor of -1. Here are several different ways you will hear people describe what this matching up looks like; all are describing the same thing:





- If you look at the surface in such a way that the unit normal vectors are all pointed towards your face, the curve should be oriented counterclockwise.

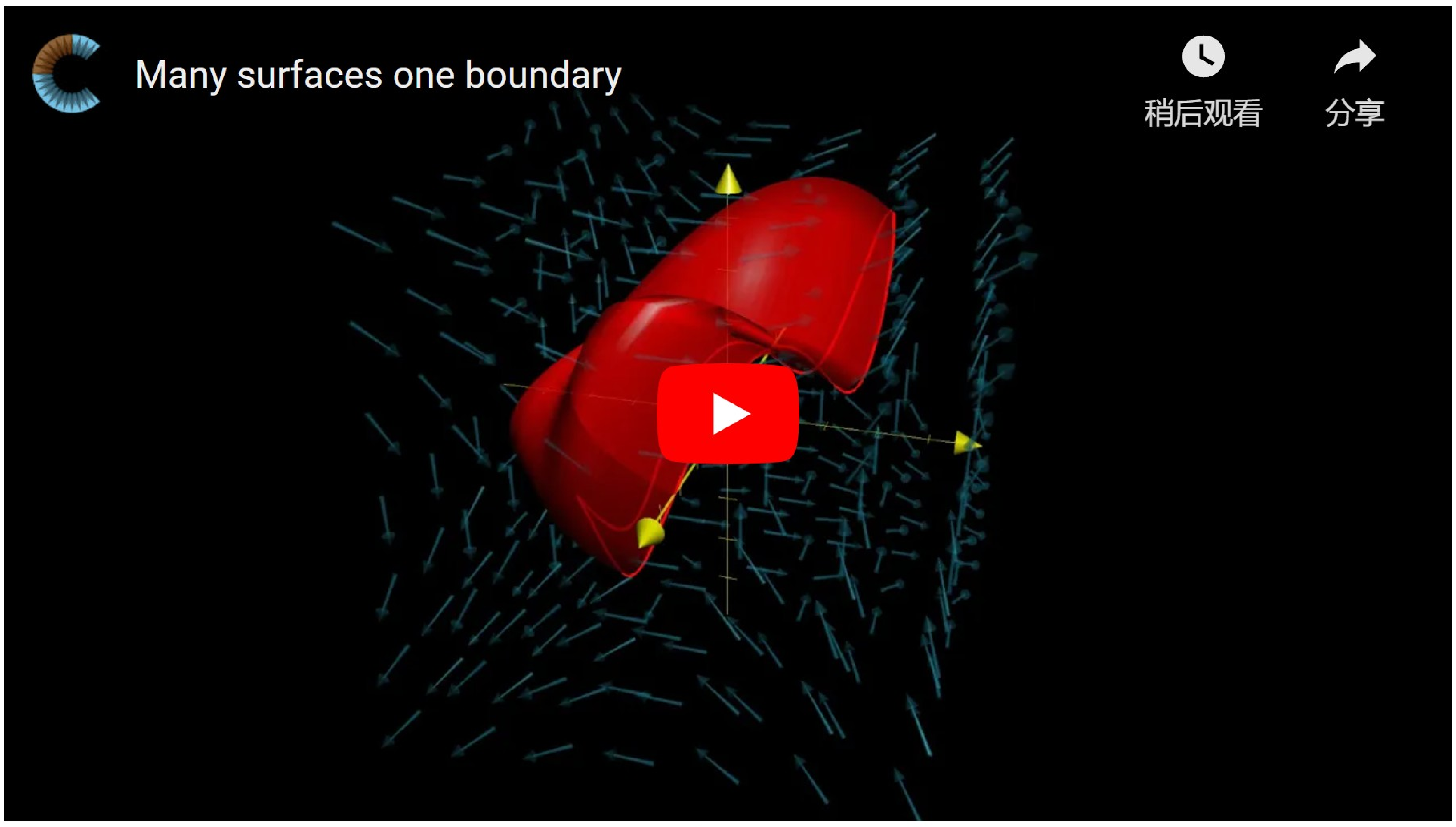
- The curve's orientation should follow the right-hand rule, in the sense that if you stick the thumb of your right hand in the direction of a unit normal vector near the edge of the surface, and curl your fingers, the direction they point on the curve should match its orientation.

- When you are walking along the boundary curve with your body pointing out in the direction of the unit normal vector, you should be walking in such a way that the surface is to your *left* side.

Blowing bubbles

Here's something pretty awesome about Stokes' theorem: **The surface itself doesn't matter, all that matters is what its boundary is.**

For example, imagine a particular loop through space, and think about all the different surfaces that could have this loop as a boundary; all the different soap bubbles which could emerge from this one loop:



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For any given vector field $\mathbf{F}(x, y, z)$, the surface integral $\iint_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}\, d\Sigma$ will be the same for each one of these surfaces. Isn't that crazy! These surface integrals involve adding up completely different values at completely different points in space, yet they turn out to be the same simply because they share a boundary.

What this tells you is just how special curl vector fields are, since with most vector fields, the surface integral *absolutely* depends on the specific surface at hand. If you learned about conservative vector fields, this is analogous to path independence, and how it indicates just how special gradient vector fields are.

What if there is no boundary?

If you have a closed surface, like a sphere or a torus, then there is no boundary. This means the "line integral over the boundary" is zero, and Stokes' theorem reads as follows:

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}} \, d\Sigma = 0$$

If you think back to chopping up the surface to get many tiny little line integrals, this basically says all those little line integrals cancel out with nothing left to show for their work.

Summary

- Stokes' theorem is the 3D version of Green's theorem.

$$\overbrace{\underbrace{\iint_S}_{S \text{ is a surface in 3D}} (\operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}) d\Sigma}^{\text{Surface integral of a curl vector field}} = \underbrace{\int_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Line integral around boundary of surface}}$$

- The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ tells you how much a fluid flowing along $\mathbf{F}$ tends to circulate around the boundary C of the surface S.

- The left-hand side surface integral can be seen as adding up all the little bits of fluid rotation on the surface S itself. The vector $\operatorname{curl} \mathbf{F}$ describes the fluid rotation at each point, and dotting it with a unit normal vector to the surface, $\hat{\mathbf{n}}$, extracts the component of that fluid rotation which happens on the surface itself.

- $\mathbf{F}(x, y, z)$ is a three-dimensional vector field.
- $\operatorname{curl} \mathbf{F}$ means the same thing as $\nabla \times \mathbf{F}$. It is the three-dimensional curl of $\mathbf{F}$, which is a vector field.
- S is a surface in three dimensions.
- $\hat{\mathbf{n}}$ represents a function that gives unit normal vectors to S.
- C is the boundary of S
- C is oriented using the right-hand rule, meaning if you point the thumb of your right hand in the direction of a unit normal vector $\hat{\mathbf{n}}$ near the edge of S and curl your fingers, the direction they point indicates the direction you should integrate around C.