Arc length of function graphs, examples

Google Classroom

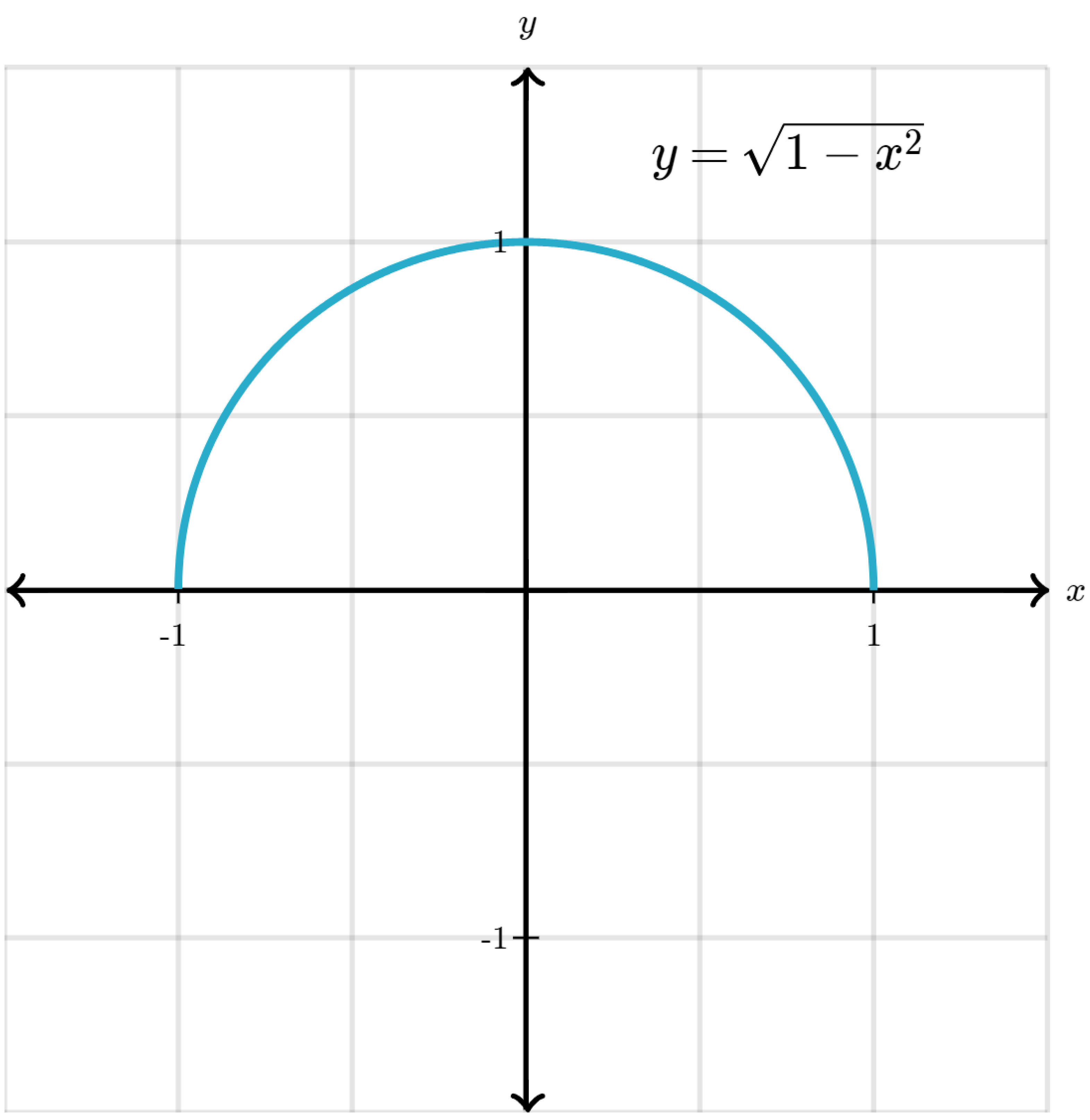
Practice finding the arc length of various function graphs.

Background

- Arc length of function graphs, introduction

Example 1: Practice with a semicircle

Consider a semicircle of radius 1, centered at the origin, as pictured on the right. From geometry, we know that the length of this curve is π. Let's practice our newfound method of computing arc length to rediscover the length of a semicircle.



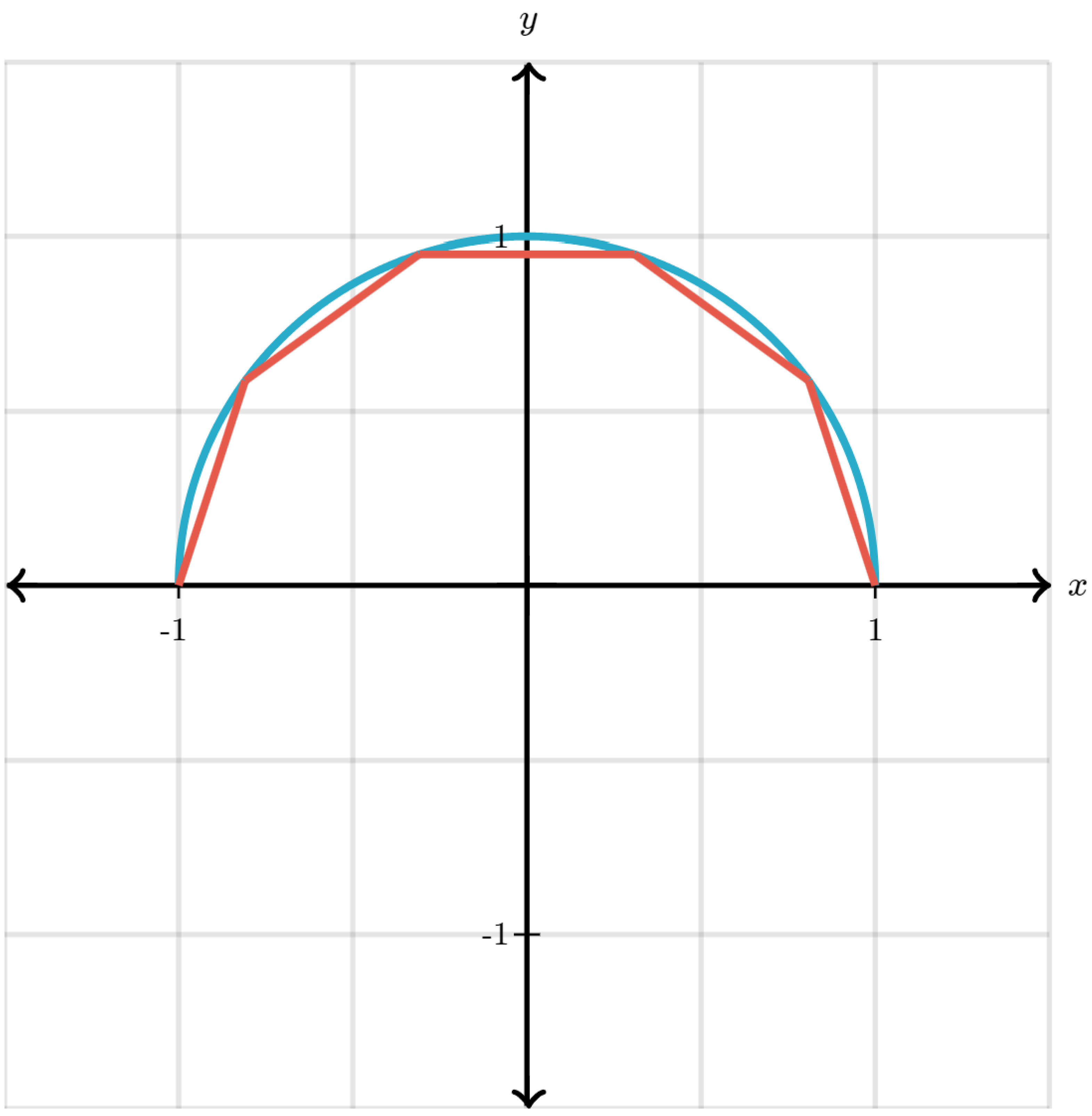
By definition, all points (x, y) on the circle are a distance 1 from the origin, so we have

$$x^2 + y^2 = 1$$

Rearranging to write y as a function of x, we have

$$y = \sqrt{1 - x^2}$$

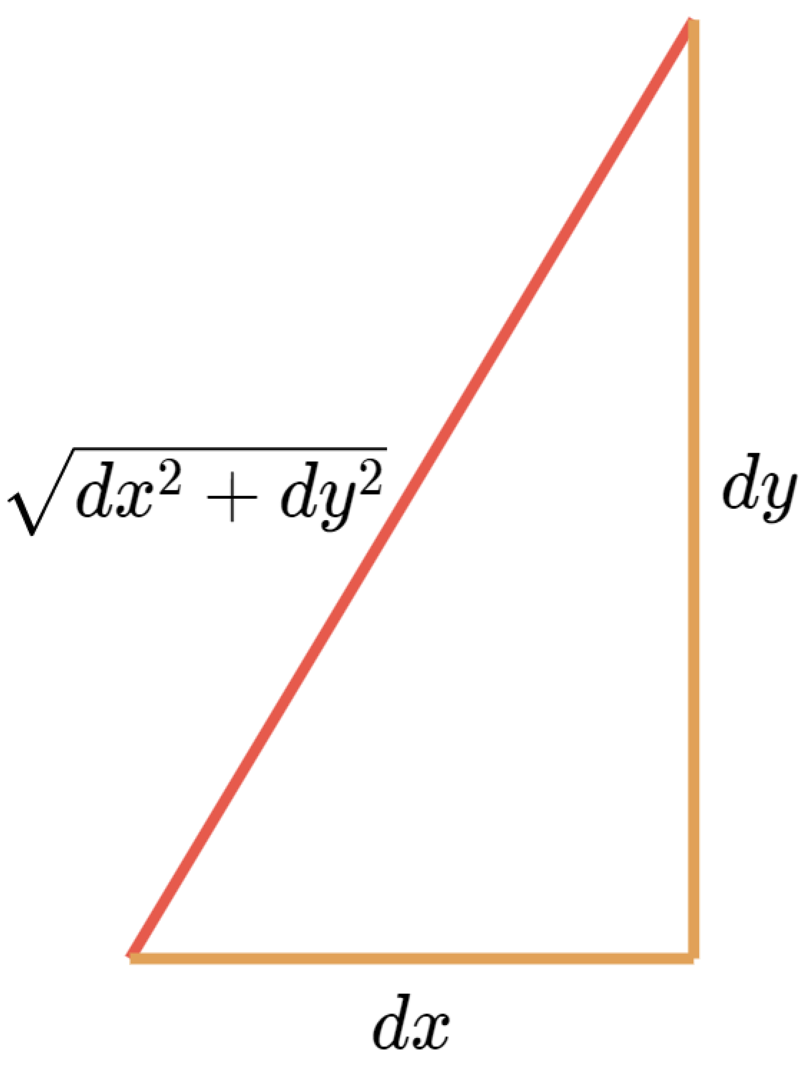
As you set up the arc length integral, it helps to imagine approximating this curve with a bunch of small lines.



Writing down the arc-length integral, ignoring the bounds for just a moment, we get:

$$\int \sqrt{(dx)^2 + (dy)^2}$$

Just as before, we think of the integrand $\sqrt{(dx)^2 + (dy)^2}$ as representing the length of one of these little lines approximating the curve (via the Pythagorean theorem).



Now we start plugging in the definition of our particular curve into the integral.

Step 1: Write dy in terms of dx

Use the fact that $y = \sqrt{1-x^2}$ to write dy in terms of dx.

$dy =$ [ ] dx

Check

[Hide explanation]

We take the derivative of both sides of the equation $y = \sqrt{1-x^2}$, using the chain rule for the right side:

$$d(y) = d(\sqrt{1-x^2})$$

$$dy = \frac{d(1-x^2)}{2\sqrt{1-x^2}}$$

$$dy = \frac{-2x\,dx}{2\sqrt{1-x^2}}$$

$$dy = \frac{-x}{\sqrt{1-x^2}}\,dx$$

Step 2: Replace dy in the integral

Plug this expression for dy into the integral to write the integrand completely in terms of x and dx.

$\int$ [ ] dx

Check

[Hide explanation]

$$\int \sqrt{(dx)^2 + (dy)^2} = \int \sqrt{(dx)^2 + \left(\frac{-x}{\sqrt{1-x^2}}\,dx\right)^2}$$

$$= \int \sqrt{\left(1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2\right)(dx)^2}$$

$$= \int \sqrt{1 + \frac{x^2}{1-x^2}}\,dx$$

$$= \int \sqrt{\frac{1-x^2}{1-x^2} + \frac{x^2}{1-x^2}}\,dx$$

$$= \int \sqrt{\frac{1}{1-x^2}}\,dx$$

Step 3: Place bounds on the integral and solve

Since the curve is defined between when $x = -1$ and $x = 1$, set these values as the bounds of your integral and solve it.

(Sorry, no entry box with a happy green checkmark. We know from geometry that the arc length is π, but the interesting part is to work through it to see how π pops out when using an arc length integral.)

[Hide explanation]

Our integral looks like this:

$$\int_{-1}^{1} \sqrt{\frac{1}{1-x^2}}\,dx$$

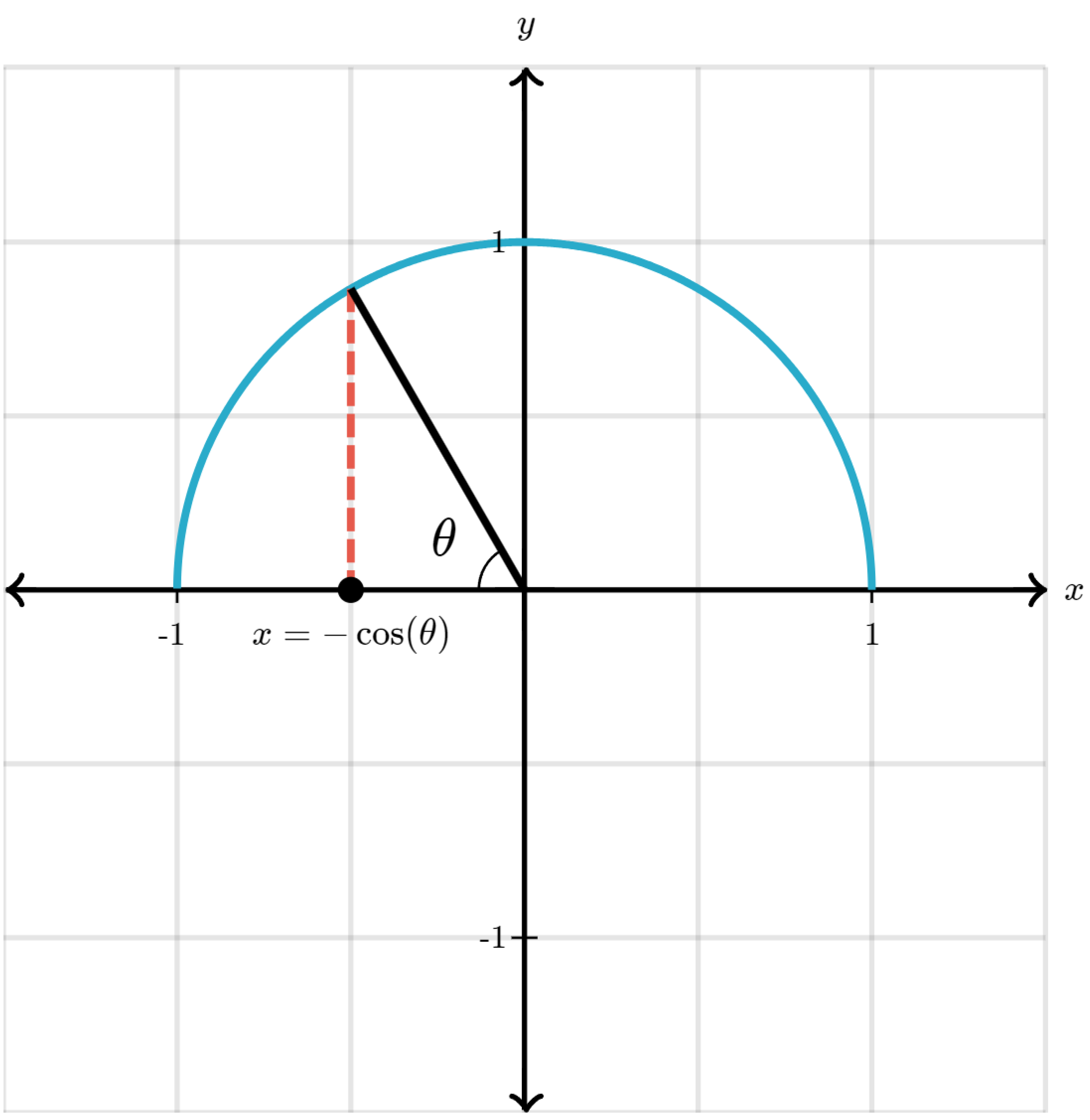
Now, there are a few ways you might proceed from here.

- Plug this integral into a calculator or computer algebra system and watch the answer pop out.
- Somehow remember or recognize that this integral is $\arcsin(x)$
- Solve the integral using trigonometric substitution.

All of these are valid, but the interesting one to focus on is the trigonometric substitution. Rather than just plugging and chugging, I want to shed some light on what this substitution actually means geometrically. Hopefully, this makes clear how and why this technique works, which will help you recognize whether or not such a substitution will work in other integrals that you might encounter.

What does it mean that our integral is currently written in terms of x? It basically has you walking along the x-axis, from -1 to 1, measuring the snippet of arc length on the circle above your head as you go.

However, in this case, it might be natural to imagine walking along the circle itself. Consider the following diagram:



Each point of the circle can be described with a value of the angle θ, ranging from 0 to π. Moreover, since the radius is 1, the value of θ measured in radians actually reflects arc length. A tiny change $d\theta$ will correspond with a tiny change in arc length. Therefore, it should seem reasonable that an alternative integral describing the desired arc length is

$$\int_0^\pi d\theta$$

In a sense, we're done. It's easy to see that this integral evaluates to π, which is indeed the arc length of our semi-circle. But what's the relationship between this and our previous integral?

Since the x value of any point on the circle is

$$x = -\cos(\theta)$$

this suggests that substituting $-\cos(\theta)$ for x will simplify our integral. Let's actually walk through that process. When we apply this substitution, we have to do three things

Step 1: Replace x with $-\cos(\theta)$

Step 2: Replace dx with $d(-\cos(\theta)) = \sin(\theta)\,d\theta$

Step 3: Write the integral's limits $x = -1$ and $x = 1$ in terms of θ.

- The relation $x = -1$ translates to the relation $-\cos(\theta) = -1$. This happens when $\theta = 0$.

- The relation $x = 1$ translates to the relation $-\cos(\theta) = 1$. This happens when $\theta = \pi$.

Applying all of these exchanges, we get

$$\int_{-1}^{1} \sqrt{\frac{1}{1-x^2}}\,dx \to \int_{0}^{\pi} \sqrt{\frac{1}{1-\cos^2(\theta)}}\,\sin(\theta)d\theta$$

Simplifying, using the identity $\sin^2(\theta) + \cos^2(\theta) = 1$, this becomes

$$\int_{0}^{\pi} \sqrt{\frac{1}{1-\cos^2(\theta)}}\,\sin(\theta)d\theta$$

$$\int_{0}^{\pi} \sqrt{\frac{1}{\sin^2(\theta)}}\,\sin(\theta)d\theta$$

$$\int_{0}^{\pi} \frac{1}{\sin(\theta)}\,\sin(\theta)d\theta$$

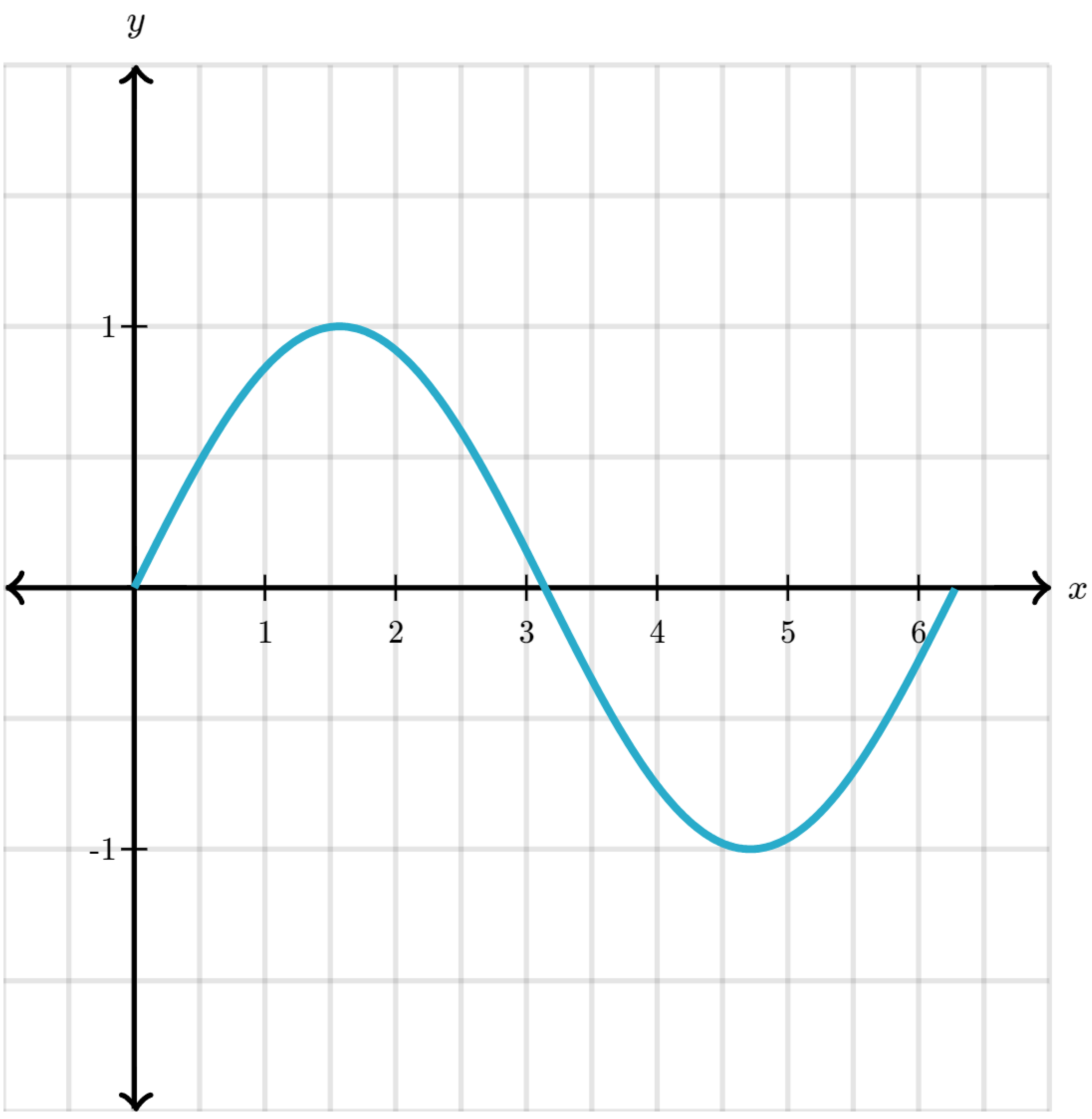
$$\int_{0}^{\pi} d\theta$$

So yes, trigonometric substitution greatly simplifies our integral, but it's not just magic. The reason it makes things simpler is that the curve is more naturally described with the value θ shown in the diagram above.

Practice setting up arc length integrals

The actual integral you get for arc length is often difficult to compute. However, the important skill to practice is setting up that integral. So let's practice that a few times without worrying about computing the final integral (you can use a calculator or wolfram alpha once you get a concrete integral).

Example 2: Sine curve



What integral represents the arc length of the graph of $y = \sin(x)$ between $x = 0$ and $x = 2\pi$?

$$\int_a^b \square \, dx$$

$a = $

$b = $

Check

[Hide explanation]

As always, the integral starts its life looking like this:

$$\int \sqrt{dx^2 + dy^2}$$

Step 1: Write dy in terms of dx

$$y = \sin(x)$$

$$d(y) = d(\sin(x))$$

$$dy = \cos(x)\,dx$$

I think it's enlightening to remind yourself of what this means, beyond just throwing the symbol d in front of everything. If you are sitting at a point (x, y) on the curve, and you take a tiny step dx to the right, the amount you must step up to end up back on the curve happens to be $\cos(x)\,dx$.

Step 2: Plug this value of dy into the integral

$$\int \sqrt{dx^2 + dy^2} = \int \sqrt{dx^2 + (\cos(x)dx)^2}$$

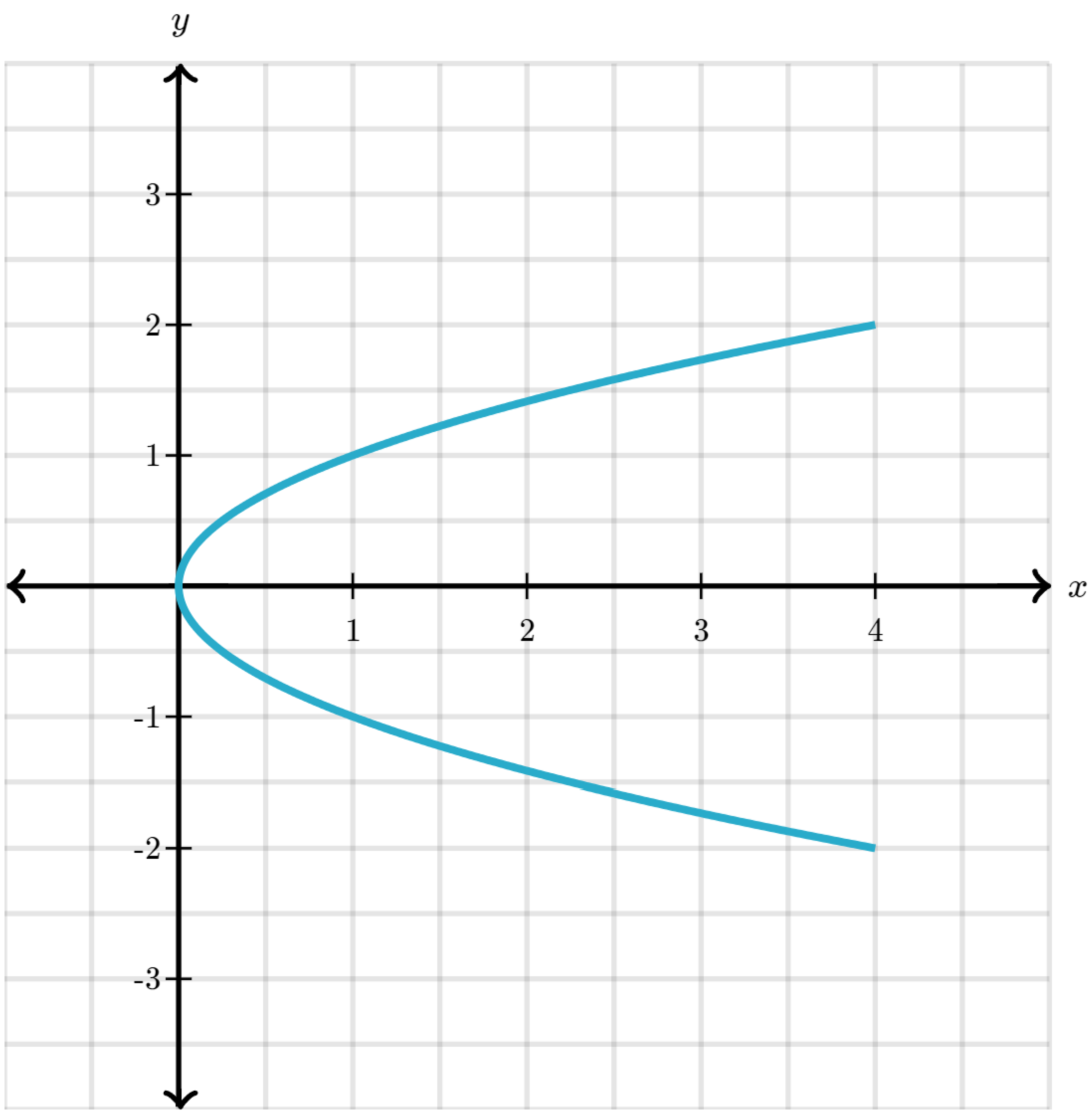
$$= \int \sqrt{1 + \cos^2(x)}dx$$

Step 3: Put bounds on the integral.

In this case, the problem explicitly states that x runs from 0 to 2π.

$$\int_0^{2\pi} \sqrt{1 + \cos^2(x)}dx$$

Example 3: Up, not right



Consider the curve representing

$$y = \pm\sqrt{x}$$

For all values where $x \leq 4$. Find an integral expressing this curve's arc length. But this time, write everything in the integral in terms of y, not x.

$$\int_a^b \square \, dy$$

$a =$

$b =$

Check

[Hide explanation]

Unlike all examples up to this point, the curve is not the graph of a single function of x. Instead, it involves two separate functions:

$$y = \sqrt{x}$$

$$y = -\sqrt{x}$$

However, it is the graph of a function of y with respect to x:

$$x = y^2$$

So rather than constructing two separate integrals in terms of dx, each of which could be thought of as a rightward walk along the graph, we construct a single integral with respect to dy, which can be thought of as walk along the curve from bottom to top.

The steps to setting up this integral are nearly identical to everything we've done up to this point, but pay careful attention to how we are now writing all expressions with x in terms of y, rather than vice versa.

Step 1: Write dx in terms of dy

$$\begin{aligned} x &= y^2 \\ d(x) &= d(y^2) \\ dx &= 2y\,dy \end{aligned}$$

Step 2: Plug this value of dx into the integral

$$\begin{aligned} \int \sqrt{dx^2 + dy^2} &= \int \sqrt{(2y\,dy)^2 + dy^2} \\ &= \int \sqrt{((2y)^2 + 1)dy^2} \\ &= \int \sqrt{4y^2 + 1}\,dy \end{aligned}$$

Step 3: Place bounds on the integral

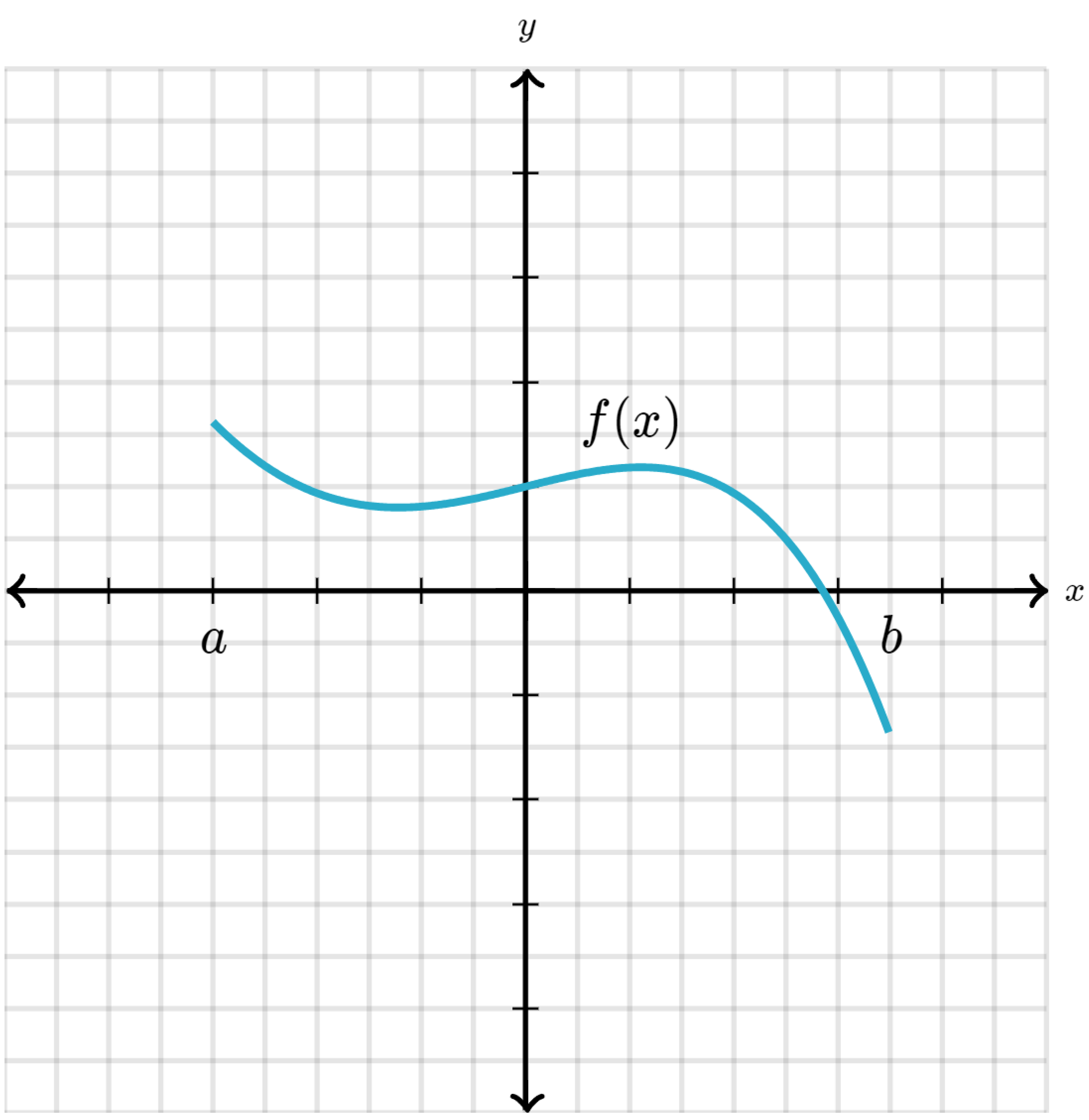
This time, everything in the integral is in terms of y, so the bounds must reflect y values.

Looking at the graph above, we see that the end points of the curve which correspond with $x = 4$ have y values $y = -2$ and $y = 2$.

$$\int_{-2}^{2} \sqrt{4y^2 + 1}\, dy$$

The fundamental reason for writing things in terms of y instead of x is that it's easier to think of moving along this curve from bottom to top, rather than from left to right. In general, and this is an important principle to remember, **when you are integrating along a curve, the formulas and symbols you use should reflect how you would actually walk along the curve.**

Example 4: Full generality



Suppose you have any arbitrary function $f(x)$, with derivative $f'(x)$. Which of the following represents the arc length of the graph

$$y = f(x)$$

between the points $x = a$ and $x = b$?

Choose 1 answer:

Ⓐ $\int_{a}^{b} \sqrt{x + f'(x)} dx$

Ⓑ $\int_a^b \sqrt{x^2 + f'(x)^2} dx$

Ⓒ $\int_a^b \sqrt{1 + f'(x)} dx$

Ⓓ $\int_a^b \sqrt{1 + f'(x)^2} dx$

Check

[Hide explanation]

Step 1: Write dy in terms of dx

$$y = f(x)$$

$$d(y) = d(f(x))$$

$$dy = f'(x) dx$$

Step 2: Plug this expression for dy into the arc length integral

$$\int \sqrt{dx^2 + dy^2}$$

$$= \int \sqrt{dx^2 + (f'(x)\, dx)^2}$$

$$= \int \sqrt{(1 + f'(x)^2) dx^2}$$

$$= \int \sqrt{1 + f'(x)^2}\, dx$$

Step 3: Put bounds on the integral

These are given explicitly, albeit abstractly, as $x = a$ and $x = b$.

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx$$

Oftentimes people will start teaching arc length by presenting this formula. Personally, I think that takes all the fun out of discovering it yourself and getting a genuine feel for what it really represents.

Summary

- We use the words **arc length** to describe how long a curve is. If you imagine the curve as a string, this is the length of that string once you pull it taut.

- You can find the arc length of a curve with an integral of the form

$$\int \sqrt{(dx)^2 + (dy)^2}$$

- If the curve is the graph of a function $y = f(x)$, replace the dy term in the integral with $f'(x)dx$, then factor out the dx. The boundary values of the integral will be the leftmost and rightmost x-values of the curve.

- When setting up an arc length integral, it helps to think about how you would actually choose to walk along the curve.