Formal definition of curl in two dimensions

Google Classroom

Learn how curl is really defined, which involves mathematically capturing the intuition of fluid rotation. This is good preparation for Green's theorem.

Background

- Curl in two dimensions
- Line integrals in a vector field

If you haven't already, you may also want to read "Why care about the formal definitions of divergence and curl" for motivation.

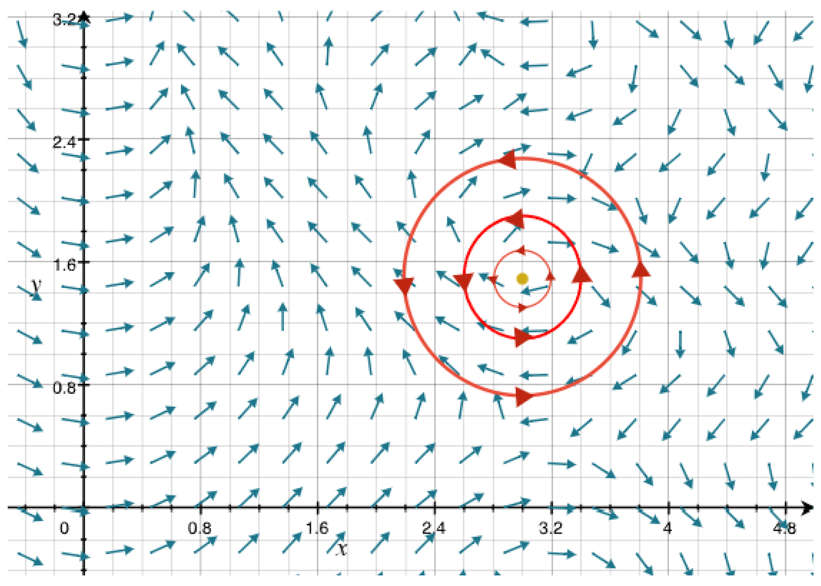
What we're building to

In two dimensions, curl is formally defined as the following limit of a line integral:

$$\text{2d-curl}\,\mathbf{F}(x, y) = \lim_{|A_{(x,y)}| \to 0} \left(\frac{1}{|A_{(x,y)}|} \oint_C \mathbf{F} \cdot d\mathbf{r} \right)$$

[Hide explanation]

- $\mathbf{F}$ is a two-dimensional vector field.



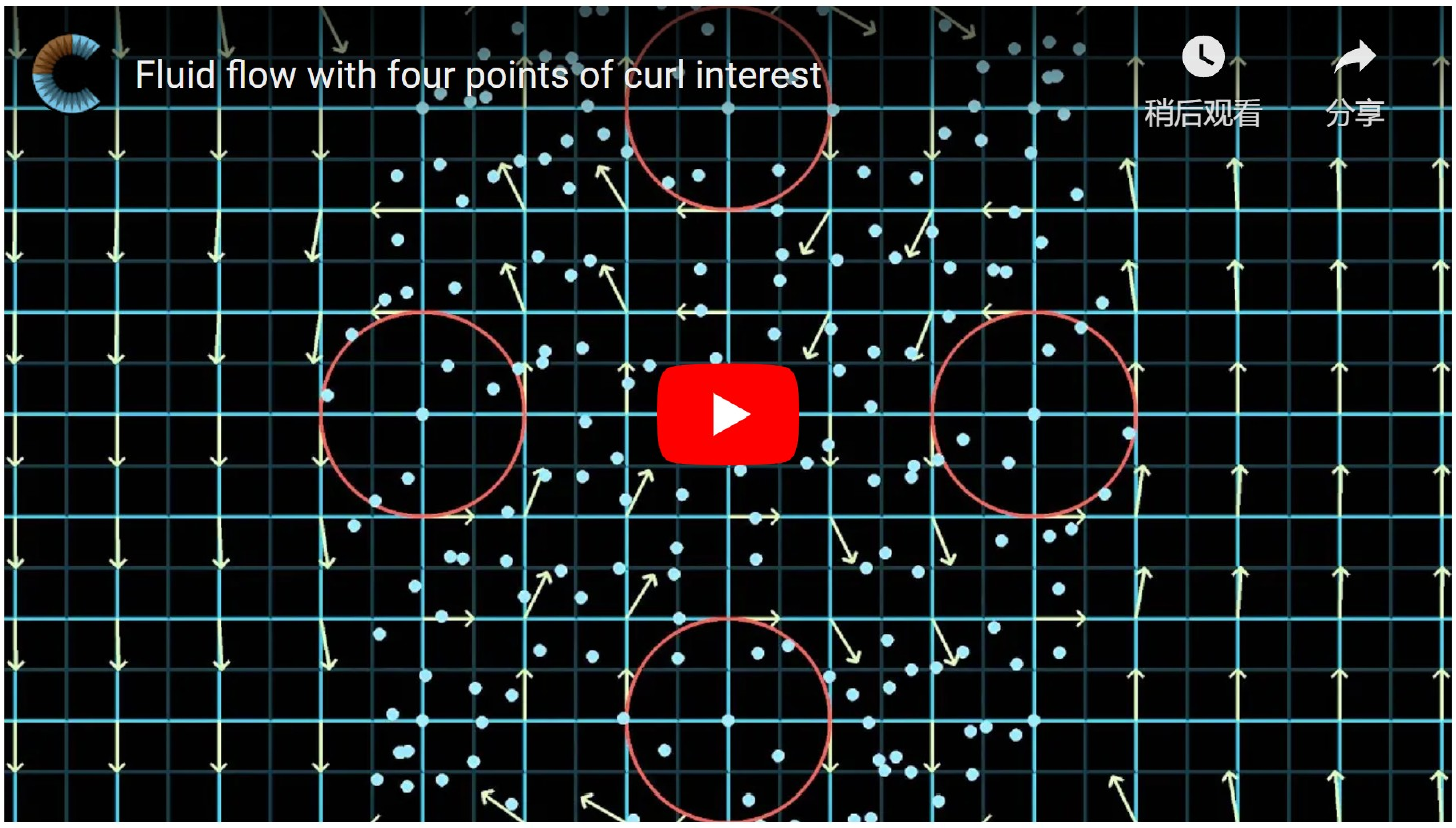
- (x, y) is some specific point in the plane.
- $A_{(x,y)}$ represents some region around the point (x, y). For instance, it could be a circle centered at (x, y).
- $|A_{(x,y)}|$ indicates the area of $A_{(x,y)}$.
- $\lim_{|A_{(x,y)}| \to 0}$ indicates we are considering the limit as the area of $A_{(x,y)}$ goes to zero, meaning the region is shrinking to the point (x, y).
- C is the boundary of $A_{(x,y)}$, oriented counterclockwise.
- $\oint_C$ is the line integral around C, written as $\oint$ instead of $\int$ to emphasize that C is a closed curve.

- The line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ can be thought of as measuring the total fluid rotation around C.

This is complicated, but it will make sense as we build up to it one piece at a time.

Formalizing fluid rotation

Suppose you have a flowing fluid whose velocity is given by a vector field $\mathbf{F}(x, y)$, such as the one we looked at in the two-dimensional curl article.



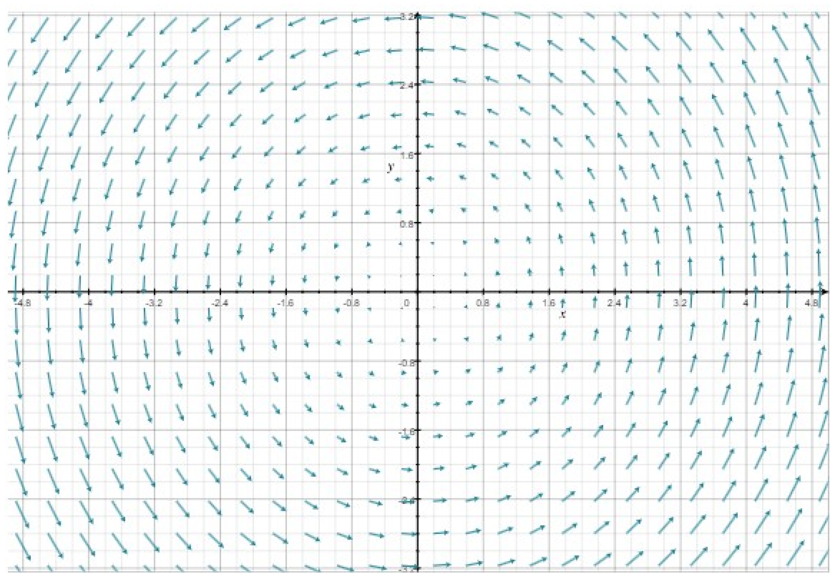
See video transcript

If you didn't already know about curl, but you *did* just learn about line integrals through a vector field, how would you measure fluid rotation in a region?

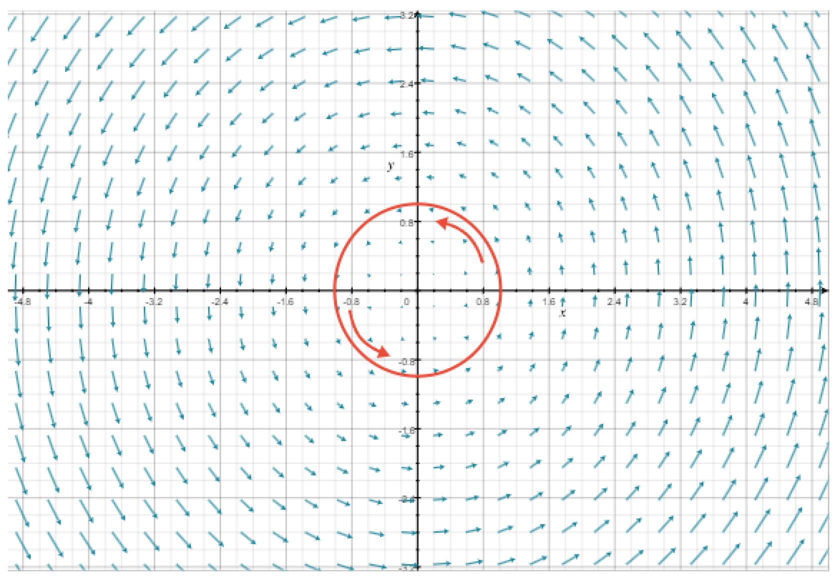
To take a relatively simple example, consider the vector field

$$\mathbf{F}(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$$

This is the quintessential counterclockwise rotation vector field.



How can we make the idea of fluid rotation mathematical (before knowing about curl)? One way to do this is to imagine walking around the perimeter of some region, like a unit circle centered at the origin, and measuring if the fluid seems to flow with you or against you at each point.



Concept check: Let C represent the circumference of a unit circle centered at the origin, oriented counterclockwise. Given the picture of the vector field $\mathbf{F}$ above, consider the following line integral:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

Without calculating it, what is the sign of this integral? (Recall that the symbol $\oint$ just emphasizes the fact that the line integral is being done over a closed loop, but it's computed the same way as any other line integral).

Choose 1 answer:

- (A) Positive
- (B) Negative
- (C) Zero

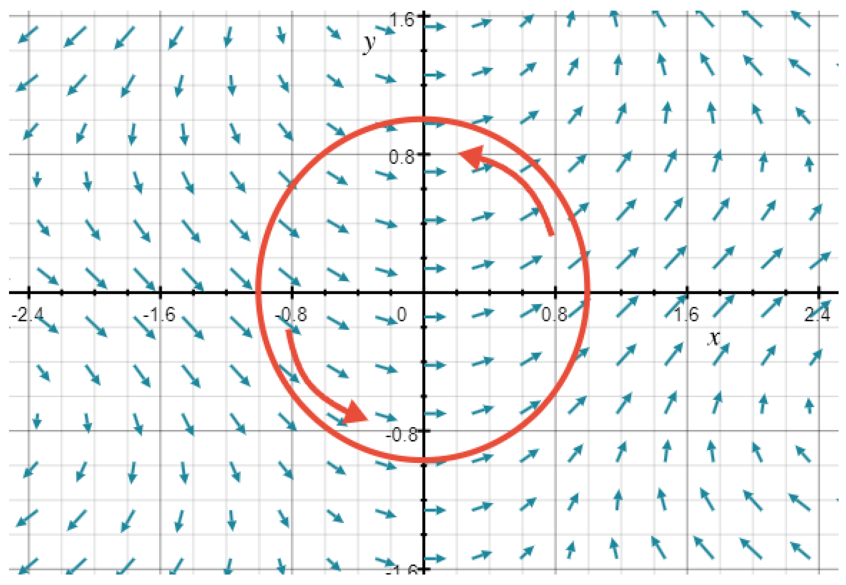
Check

[Hide explanation]

> Positive.
>
> As you walk around the circle, the arrows of the vector field, $\mathbf{F}$, always point in the same direction as your motion, $d\mathbf{r}$. Therefore the dot product $\mathbf{F} \cdot d\mathbf{r}$ will always be positive, and hence the integral as a whole must be positive.

More generally, if a fluid tends to flow counterclockwise around a region, you would expect that the line integral of that fluid's velocity vector field around the perimeter of the region would be positive (when it's oriented counterclockwise).

You could also imagine a more complicated vector field, in which the fluid flows with you at some points on your counterclockwise walk around the circle, but against you at others.



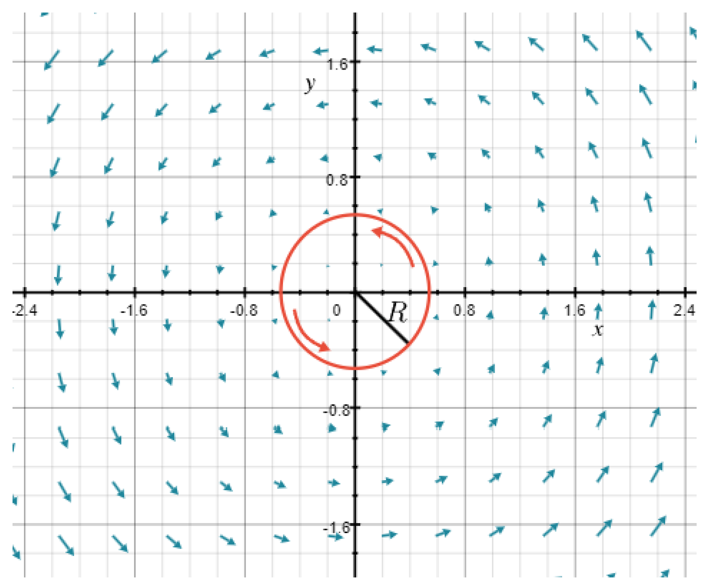
The value $\mathbf{F} \cdot d\mathbf{r}$ will be positive while the flow is with you, and negative when it's against you. In a way, the integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is like a voting system that counts up how much these different directions cancel each other out and which one wins overall.

Letting the size of the region change

So, after mathematically expressing the idea of fluid rotation around a *region*, you might want to capture the more elusive idea of fluid rotation *at a point*. How might you go about that?

You could start by considering smaller and smaller regions around that point, such as circles of smaller and smaller radii, and seeing what the fluid flow around those regions looks like.

Concept check: Back to our vector field $\mathbf{F} = \begin{bmatrix} -y \\ x \end{bmatrix}$, rather than just looking at the unit circle, let C_R represent a circle centered at the origin with radius R. This circle will still be oriented counterclockwise.



Compute the line integral of $\mathbf{F}$ around this circle as a function of R.

$$\oint_{C_R} \mathbf{F} \cdot d\mathbf{r} = \square$$

Check

[Hide explanation]

First we need to parameterize this circle. We can do this with the function

$$\mathbf{r}(t) = \begin{bmatrix} R\cos(t) \\ R\sin(t) \end{bmatrix}$$

For this parameterization to cover the circle exactly once, let t range from 0 to 2π. With this, we expand the line integral as follows:

$$\oint_{C_R} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

Now plug the components of $\mathbf{r}(t)$ into the definition of $\mathbf{F}(x, y)$

$$\begin{aligned} \mathbf{F}(x, y) &= \begin{bmatrix} -y \\ x \end{bmatrix} \\ &\Downarrow \\ \mathbf{F}(\mathbf{r}(t)) &= \mathbf{F}(R\cos(t), R\sin(t)) \\ &= \begin{bmatrix} -R\sin(t) \\ R\cos(t) \end{bmatrix} \end{aligned}$$

Also, take the derivative of $\mathbf{r}(t)$

$$\mathbf{r}'(t) = \begin{bmatrix} \frac{d}{dt} R\cos(t) \\ \frac{d}{dt} R\sin(t) \end{bmatrix} = \begin{bmatrix} -R\sin(t) \\ R\cos(t) \end{bmatrix}$$

The fact that $\mathbf{F}(\mathbf{r}(t)) = \mathbf{r}'(t)$ in this case is a coincidence, peculiar to this example. It makes the completion of the line integral pretty smooth, though.

$$\begin{aligned} \oint_{C_R} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt \\ &= \int_0^{2\pi} \begin{bmatrix} -R\sin(t) \\ R\cos(t) \end{bmatrix} \cdot \begin{bmatrix} -R\sin(t) \\ R\cos(t) \end{bmatrix} dt \\ &= \int_0^{2\pi} (R^2\sin^2(t) + R^2\cos^2(t))dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} R^2 \underbrace{(\sin^2(t) + \cos^2(t))}_{\text{equals } 1} dt \\
&= R^2 \int_0^{2\pi} dt \\
&= 2\pi R^2
\end{aligned}
$$

How does this value relate to the circle C_R?

Choose 1 answer:

(A) It equals the circumference of C_R

(B) It is twice the area enclosed by the circle C_R.

Check

Average rotation per unit area

The answer to this last question suggests something interesting. The rotation around a region seems to be proportional to the area of that region. Of course, you've only shown this for circles centered at the origin, not all possible regions, but it is nevertheless suggestive. This might give you an idea.

> **Key idea**: Maybe if you take $\oint_C \mathbf{F} \cdot d\mathbf{r}$, which measures the fluid flow around a region, and divide it by the area of that region, it can give you some notion of the *average* rotation per unit area.

The idea of "average rotation per unit area" might feel a bit strange, but if you think back to the interpretation of curl, that's kind of what we want curl to represent. Rather than thinking about fluid rotation in a large region, curl is supposed to measure how fluid *tends* to rotate near a point.

Concept check: The vector field from the previous example is a little bit special in that the "rotation-per-unit-area" of circles around the origin is the same value for all circles. What is that value?

Check

[Hide explanation]

> When I say "rotation-per-unit-area", what I mean by "rotation" is the value of the integral
>
> $$\oint_C \mathbf{F} \cdot d\mathbf{r}$$
>
> This is because this integral indicates how much the fluid is flowing counterclockwise around the curve C.
>
> The "unit area" in question is the area enclosed by C.
>
> We found in the previous question that, for our example, the integral equals $2\pi R^2$. Since the area of the circle is πR^2, the ratio is always 2.

Concept check: Recall that the formula for 2d-curl is

$$\text{2d-curl } \mathbf{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

where F_1 and F_2 are the components of $\mathbf{F}$. Given the definition

$$\mathbf{F}(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$$

compute the curl of $\mathbf{F}$.

$$\text{2d-curl } \mathbf{F} = \boxed{}$$

Check

[Hide explanation]

Applying the formula, plug in the following values:

$$F_1 = -y$$

$$F_2 = x,$$

Here's what we get:

$$\begin{aligned} \text{2d-curl}\,\mathbf{F} &= \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \\ &= 1 - (-1) \\ &= 2 \end{aligned}$$

[Hide explanation]

I should point out that the fact that $\text{2d-curl}\,\mathbf{F}$ is constant in this example and the fact that the average rotation per unit area was constant for all circles are both peculiar to this example. In general, the curl varies from point to point, and the average rotation per unit area depends on the size of the area.

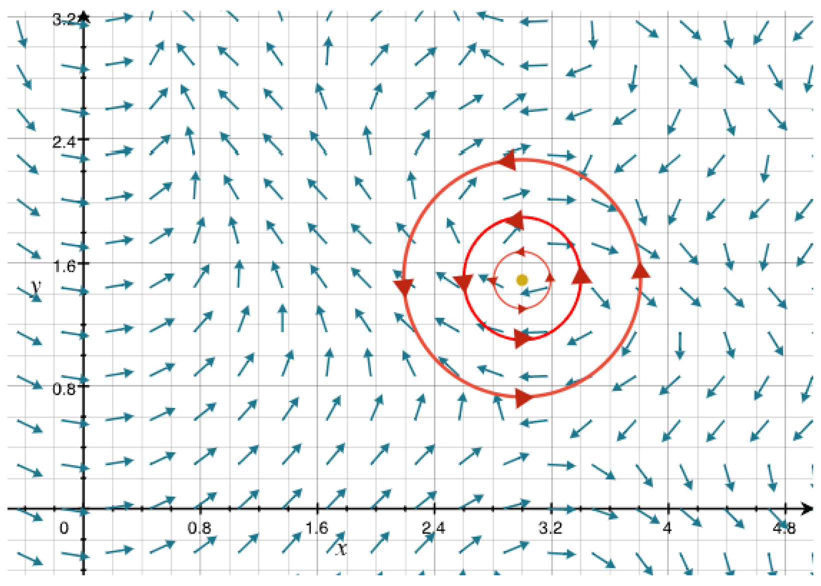
Defining two-dimensional curl

Those last two questions show that the "average rotation per unit area" in circles centered at the origin happens to be the same as the curl of the function, at least for our specific example. This turns out to apply more broadly. In fact, the way we *define* the curl of a vector field $\mathbf{F}$ at a point (x, y) is to be **the limit of this average rotation per unit area in smaller and smaller regions** around the point (x, y).

Specifically, (drumroll please), Here's the formula defining two-dimensional curl:

$$\text{2d-curl}\,\mathbf{F}(x, y) = \lim_{|A_{(x,y)}| \to 0} \underbrace{\left(\frac{1}{|A_{(x,y)}|} \oint_C \mathbf{F} \cdot d\mathbf{r} \right)}_{\text{Average rotation per unit area}}$$

where



- $\mathbf{F}$ is a two-dimensional vector field.
- (x, y) is some specific point in the plane.
- $A_{(x,y)}$ represents some region around the point (x, y). For instance, a circle centered at (x, y).
- $|A_{(x,y)}|$ indicates the area of $A_{(x,y)}$.
- $\lim\limits_{|A_{(x,y)}| \to 0}$ indicates we are considering the limit as the area of $A_{(x,y)}$ goes to 0, meaning this region is shrinking around (x, y).
- C is the boundary of $A_{(x,y)}$, oriented counterclockwise.
- $\oint_C$ is the line integral around C, written as $\oint$ instead of $\int$ to emphasize that C is a closed curve.

This formula is impractical for computation, but the connection between this and fluid rotation is very clear once you wrap your mind around it. It is a very beautiful fact that this definition gives the same thing as the formula used to compute two-dimensional curl.

$$\text{2d-curl } \mathbf{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

One more feature of conservative vector fields

Background: Conservative vector fields

If $\mathbf{F}(x, y)$ is a conservative vector field, all line integrals over closed loops are 0. Looking at the integral above, what does this imply?

Choose 1 answer:

(A) Two-dimensional curl is always 0.

Ⓑ Fluid rotation around region is not necessarily 0, but as the area of that region approaches 0, the integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ approaches 0 faster than the area approaches 0.

Check

[Hide explanation]

The expression $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is always 0, so the definition of two-dimensional curl reads

$$\begin{aligned}
\text{2d-curl}\,\mathbf{F}(x,y) &= \lim_{A(x,y)\to 0}\left(\frac{1}{|A_{(x,y)}|}\overbrace{(0)}^{\text{Line integral}}\right) \\
&= \lim_{A(x,y)\to 0}(0) \\
&= 0
\end{aligned}$$

Hence, the 2d-curl of a conservative vector field is always zero.

This gives an important fact: **If a vector field is conservative, it is *irrotational*, meaning the curl is zero everywhere**.

In particular, since gradient fields are always conservative, **the curl of the gradient is always zero**. That is a fact you could find just by chugging through the formulas. However, I think it gives much more insight to understand it using the definition of curl together with the intuition for why gradient fields are conservative.

What about the converse? If a vector field has zero curl everywhere, does that mean it must be conservative? [Hide explanation]

It does, but unfortunately, you must wait until learning about Green's theorem to see why.

Summary

- If a vector field represents fluid flow, you can quantify "fluid rotation in a region" by taking the line integral of that vector field along the border of that region.

- To go from the idea of fluid rotation in a region to fluid flow around a point (which is what curl measures), we introduce the idea of "average rotation per unit area" in a region. Then consider what this value approaches as your region shrinks around a point.

- In formulas, this gives us the definition of two-dimensional curl as follows:

$$\text{2d-curl}\,\mathbf{F}(x,y) = \lim_{A_{(x,y)}\to 0} \underbrace{\left(\frac{1}{|A_{(x,y)}|}\oint_C \mathbf{F}\cdot d\mathbf{r}\right)}_{\text{Average rotation per unit area}}$$

- This relationship between curl and closed-loop line integrals implies that irrotational fields and conservative fields are one and the same.