Surface area example

Google Classroom

Here you have the chance to practice computing surface area, using the example of a torus.

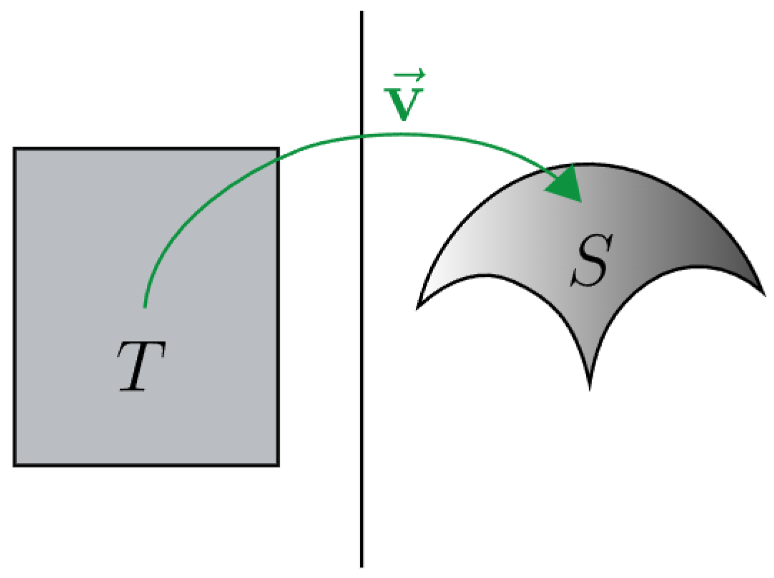
Who is this for?

This article is meant for anyone who read the last article on computing the surface area of parametric surfaces using a certain double integral, and who wants to practice this concept. You will compute the surface area of a torus (a doughnut shape) using this method, which requires no small amount of computation.

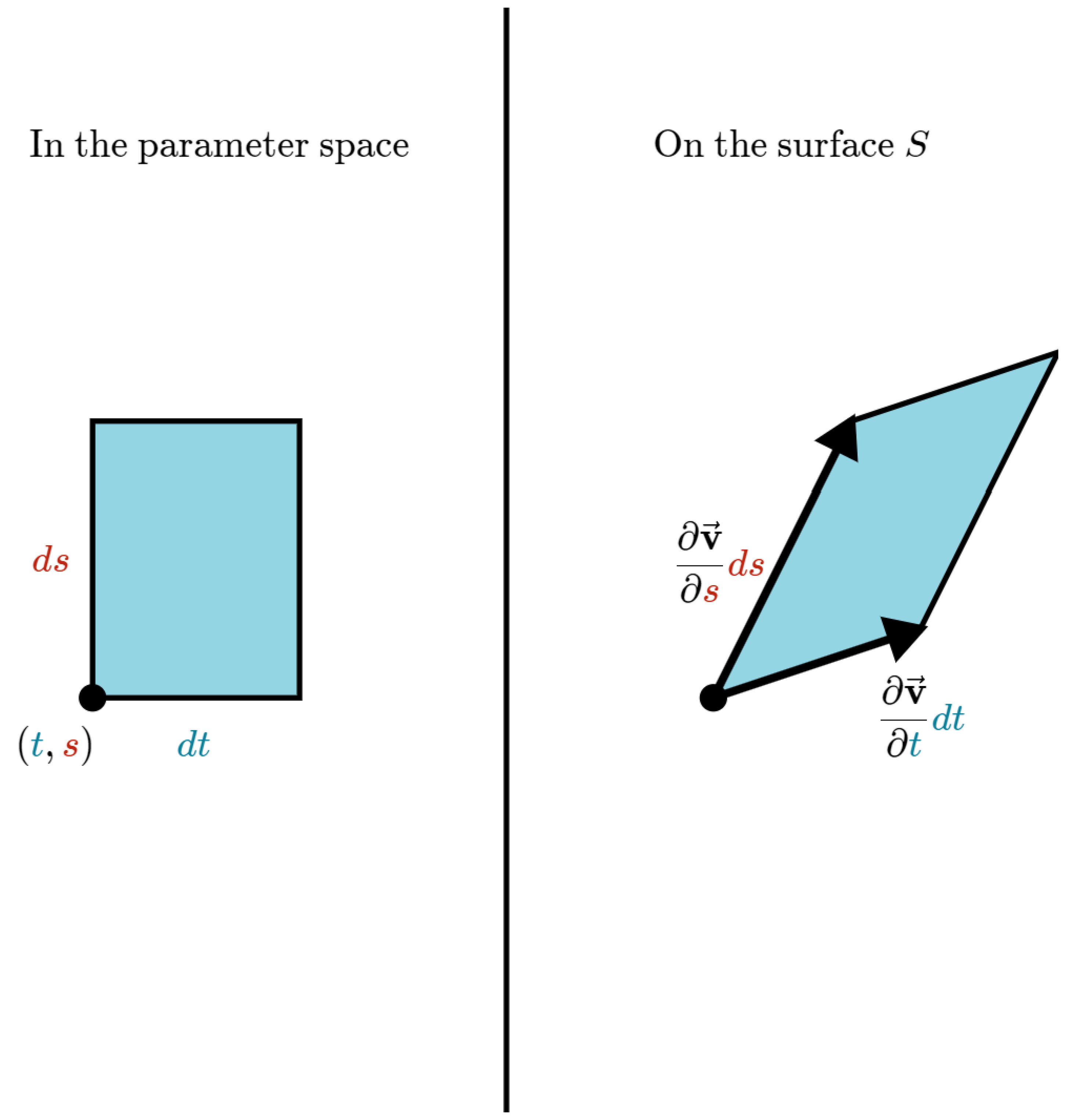
If you neither want or need practice with this computation, and you feel comfortable with the general concept of how these surface area integrals work, feel free to skip ahead to the next article.

Quick recap of the surface area integral

Before diving into the example, let's quickly remind ourselves of the method described for finding the area of a surface discussed in the last article.



- **Parameterize the surface.** In other words, find a vector-valued function $\vec{\mathbf{v}}(t, s)$ which maps some region T of the two-dimensional ts-plane onto your surface in three dimensions. Sometimes this parameterization will be given to you, if that's how your surface is defined. Other times the surface is defined some other way, and you have to find it yourself.



- **Imagine chopping up the parameter space** with horizontal and vertical lines, thus dividing your region T into little rectangles. Each of these rectangles gets mapped onto a little piece of your surface which is well-approximated by a parallelogram. If your little rectangle sits at the point (t, s), and it has width dt and height ds, you can approximate its area with the following expression:

$$\left|\frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s}\right| dt\, ds$$

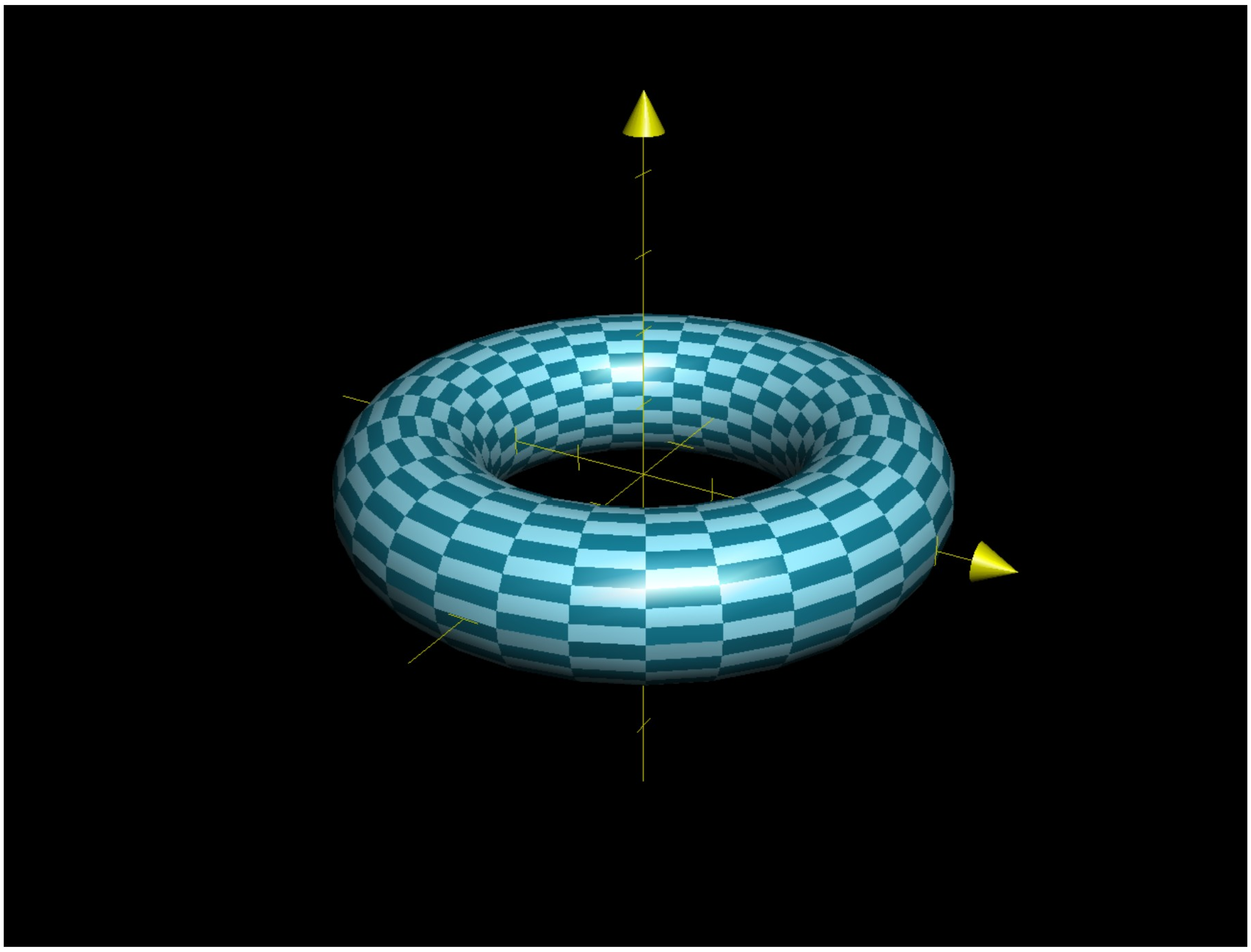
The smaller your initial rectangles, the more closely the corresponding piece of your surface resembles an actual flat parallelogram, and the closer this expression is to giving the true area of that piece.

- **Add up the areas of these pieces with a double integral**:

$$\iint_T \left|\frac{\partial \vec{\mathbf{v}}}{\partial t} \times \frac{\partial \vec{\mathbf{v}}}{\partial s}\right| dt\, ds$$

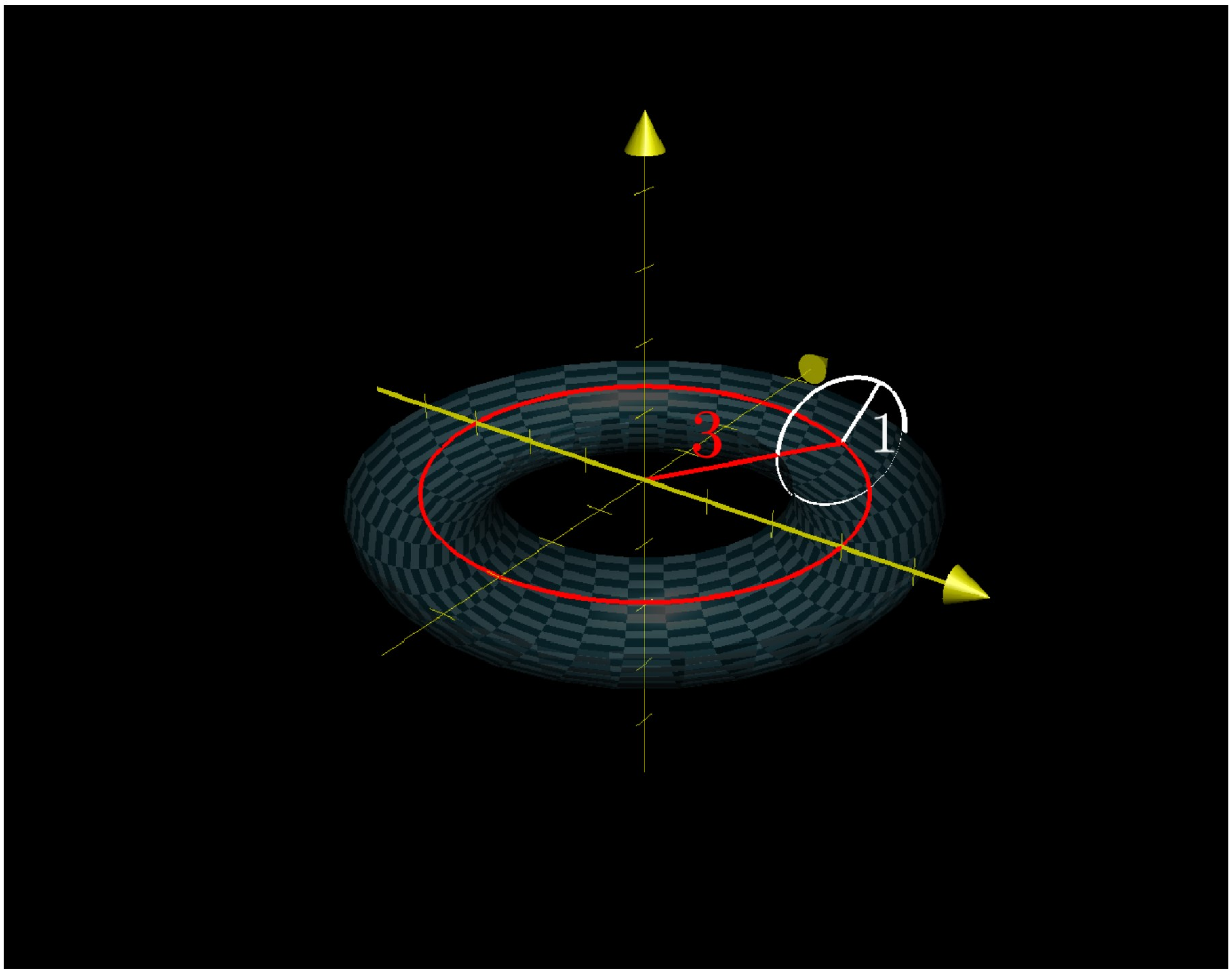
Surface area of a torus

The goal of this article is to find the surface area of a torus:



English makes it hard to describe the dimensions of this torus, but I'll give it a shot with the help of some doughnut terminology. Imagine this torus as the glaze on a doughnut.

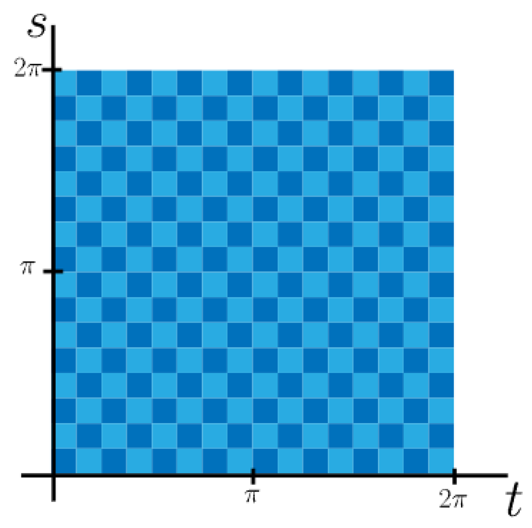
- Let's say the distance between the origin and the innermost part of this jelly filling is 3. Call this the "outer radius".
- Let's also say the distance between the innermost part of the jelly filling and the glaze itself is 1. Call this the "inner radius"



[redacted]

With these dimensions, the torus (i.e. glaze) can be parameterized with the following function:

$$\vec{\mathbf{v}}(t, s) = \begin{bmatrix} 3\cos(t) + \cos(s)\cos(t) \\ 3\sin(t) + \cos(s)\sin(t) \\ \sin(s) \end{bmatrix}$$



For this parameterization to cover the torus once and only once, apply it to the region of the ts-plane where

$$0 \le t \le 2\pi$$
$$0 \le s \le 2\pi$$

For a description of where this parameterization comes from, check out the last example in [this article].

Step 1: Compute each partial derivative

$$\frac{\partial \vec{\mathbf{v}}}{\partial t}(t, s) = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}} + \square\,\hat{\mathbf{k}}$$

Check

Hide explanation

$$\frac{\partial}{\partial t}\vec{\mathbf{v}}(t, s) = \begin{bmatrix} \frac{\partial}{\partial t}(3\cos(t) + \cos(s)\cos(t)) \\ \frac{\partial}{\partial t}(3\sin(t) + \cos(s)\sin(t)) \\ \frac{\partial}{\partial t}\sin(s) \end{bmatrix} = \begin{bmatrix} -3\sin(t) - \cos(s)\sin(t) \\ 3\cos(t) + \cos(s)\cos(t) \\ 0 \end{bmatrix}$$

$$\frac{\partial \vec{\mathbf{v}}}{\partial s}(t, s) = \square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}} + \square\,\hat{\mathbf{k}}$$

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$$\frac{\partial}{\partial s}\vec{\mathbf{v}}(t,s) = \begin{bmatrix} \frac{\partial}{\partial s}(3\cos(t) + \cos(s)\cos(t)) \\ \frac{\partial}{\partial s}(3\sin(t) + \cos(s)\sin(t)) \\ \frac{\partial}{\partial s}\sin(s) \end{bmatrix} = \begin{bmatrix} -\sin(s)\cos(t) \\ -\sin(s)\sin(t) \\ \cos(s) \end{bmatrix}$$

Remember, you should think of these vectors as representing the edges of little parallelograms, which piece together to make the torus as a whole. More accurately, you must multiply the first one by dt and the second one by ds to scale them down to the infinitesimal size of one of these parallelograms.

As it so happens, these vectors are perpendicular to each other (you can check by taking their dot product). This implies all the little parallelograms making up the torus happen to be rectangles, at least when we use this particular parameterization. You can see this in the picture of the torus above.

[Hide explanation]

> In the following steps, you will compute the area of one of these parallelograms by taking the cross product of the two vectors you just found, and computing its magnitude. This is good practice for more general surface area computations.
>
> If you wanted to be clever, you could exploit the fact that these vectors happen to be perpendicular to each other. Specifically, since the "parallelogram" they span will always be a rectangle, you can compute its area by taking the magnitude of these two partial derivative vectors, and multiplying them together.

Step 2: Compute the cross product

To find the area of a parallelogram spanned by the two vectors you just found, the first step is to take their cross product. (Warning: This one gets hairy)

$$\frac{\partial \vec{\mathbf{v}}}{\partial t}(t,s) \times \frac{\partial \vec{\mathbf{v}}}{\partial s}(t,s) =$$

$\square\,\hat{\mathbf{i}} + \square\,\hat{\mathbf{j}} + \square\,\hat{\mathbf{k}}$

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$$\frac{\partial \vec{\mathbf{v}}}{\partial t}(t, s) \times \frac{\partial \vec{\mathbf{v}}}{\partial s}(t, s)$$

$$= \begin{bmatrix} -3\sin(t) - \cos(s)\sin(t) \\ 3\cos(t) + \cos(s)\cos(t) \\ 0 \end{bmatrix} \times \begin{bmatrix} -\sin(s)\cos(t) \\ -\sin(s)\sin(t) \\ \cos(s) \end{bmatrix}$$

$$= \det\left(\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3\sin(t) - \cos(s)\sin(t) & 3\cos(t) + \cos(s)\cos(t) & 0 \\ -\sin(s)\cos(t) & -\sin(s)\sin(t) & \cos(s) \end{bmatrix}\right)$$

It's best to calculate one component at a time.

$\hat{\mathbf{i}}$-component: Cross out top row and left column, then take the determinant:

$$\det\left(\begin{bmatrix} 3\cos(t) + \cos(s)\cos(t) & 0 \\ -\sin(s)\sin(t) & \cos(s) \end{bmatrix}\right)$$

$$= (3\cos(t) + \cos(s)\cos(t))(\cos(s)) - 0$$

$$= (3 + \cos(s))\cos(t)\cos(s)$$

$\hat{\mathbf{j}}$-component: Cross out top row and middle column, then take the *negative* determinant:

$$-\det\left(\begin{bmatrix} -3\sin(t) - \cos(s)\sin(t) & 0 \\ -\sin(s)\cos(t) & \cos(s) \end{bmatrix}\right)$$

$$= -(-3\sin(t) - \cos(s)\sin(t))(\cos(s)) - 0$$

$$= (3 + \cos(s))\sin(t)\cos(s)$$

$\hat{\mathbf{k}}$-component: This one is nastier than the last two. Cross out top row and right column, then take the determinant:

$$\det\left(\begin{bmatrix} -3\sin(t)-\cos(s)\sin(t) & 3\cos(t)+\cos(s)\cos(t) \\ -\sin(s)\cos(t) & -\sin(s)\sin(t) \end{bmatrix}\right)$$

$$= (-3\sin(t)-\cos(s)\sin(t))(-\sin(s)\sin(t)) -$$

$$(3\cos(t)+\cos(s)\cos(t))(-\sin(s)\cos(t))$$

$$= 3\sin^2(t)\sin(s)+\cos(s)\sin^2(t)\sin(s)+$$

$$3\cos^2(t)\sin(s)+\cos(s)\cos^2(t)\sin(s)$$

$$= 3\sin(s)(\underbrace{\sin^2(t)+\cos^2(t)}_{1})+\cos(s)\sin(s)(\underbrace{\sin^2(t)+\cos^2(t)}_{1})$$

$$= 3\sin(s)+\cos(s)\sin(s)$$

$$= \sin(s)(3+\cos(s))$$

Step 3: Find the magnitude of this cross product

The cross product you just computed is a vector. In order to find the area of a paralellogram spanned by the two partial derivative vectors, we must find its magnitude. (Warning: This one gets even more hairy).

$$\left|\frac{\partial\vec{\mathbf{v}}}{\partial t}(t,s)\times\frac{\partial\vec{\mathbf{v}}}{\partial s}(t,s)\right| = \square$$

Check

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Plugging in the answer we found in the last example, we get something which is at first enourmous, but then simplifies down quite nicely.

$$\left|\frac{\partial\vec{\mathbf{v}}}{\partial t}(t,s)\times\frac{\partial\vec{\mathbf{v}}}{\partial s}(t,s)\right|$$

$$= \left\|\begin{bmatrix} (3+\cos(s))\cos(t)\cos(s) \\ (3+\cos(s))\sin(t)\cos(s) \\ \sin(s)(3+\cos(s)) \end{bmatrix}\right\|$$

$$= \sqrt{\underbrace{(3+\cos(s))^2\cos^2(t)\cos^2(s) + (3+\cos(s))^2\sin^2(t)\cos^2(s)}_{\text{Factor out } (3+\cos(s))^2\cos^2(s)} + \sin^2(s)(3+\cos}$$

$$= \sqrt{(3+\cos(s))^2\cos^2(s)\underbrace{(\cos^2(t)+\sin^2(t))}_{1} + \sin^2(s)(3+\cos(s))^2}$$

$$= \sqrt{\underbrace{(3+\cos(s))^2\cos^2(s) + \sin^2(s)(3+\cos(s))^2}_{\text{Factor out } (3+\cos(s))^2}}$$

$$= \sqrt{(3+\cos(s))^2(\cos^2(s)+\sin^2(s))}$$

$$= \sqrt{(3+\cos(s))^2}$$

$$= 3+\cos(s)$$

Once you scale this down by $dt\,ds$, this tells you the area of each of the little parallelograms making up the torus, as a function of s and t. Well, in this case, the answer is just a function of s, which means the area of these parallelograms doesn't change as you let t vary.

Step 4: Set up the appropriate double integral

Which of the following represents the right bounds to place on the double integral representing surface area for this torus?

Choose 1 answer:

(A) $\int_0^{2\pi}\int_0^{6\pi}\left|\frac{\partial\vec{\mathbf{v}}}{\partial t}(t,s)\times\frac{\partial\vec{\mathbf{v}}}{\partial s}(t,s)\right|\,dt\,ds$

(B) $\int_0^{6\pi}\int_0^{2\pi}\left|\frac{\partial\vec{\mathbf{v}}}{\partial t}(t,s)\times\frac{\partial\vec{\mathbf{v}}}{\partial s}(t,s)\right|\,dt\,ds$

(C) $\int_0^{2\pi}\int_0^{2\pi}\left|\frac{\partial\vec{\mathbf{v}}}{\partial t}(t,s)\times\frac{\partial\vec{\mathbf{v}}}{\partial s}(t,s)\right|\,dt\,ds$

Check

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The third answer choice is correct:

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{\partial\vec{\mathbf{v}}}{\partial t}(t,s)\times\frac{\partial\vec{\mathbf{v}}}{\partial s}(t,s)\right|\,dt\,ds$$

This comes from the definition for our parameterization of this torus given at the start of this section. In particular, both t and s range from 0 to 2π.

$$0 \leq t \leq 2\pi$$
$$0 \leq s \leq 2\pi$$

Step 5: Compute the double integral

Surface area of this torus:

Check

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First take the result from step 3:

$$\left| \frac{\partial \vec{\mathbf{v}}}{\partial t}(t, s) \times \frac{\partial \vec{\mathbf{v}}}{\partial s}(t, s) \right| = 3 + \cos(s)$$

Then applying the double integral from step 4:

$$\int_0^{2\pi} \int_0^{2\pi} \left| \frac{\partial \vec{\mathbf{v}}}{\partial t}(t, s) \times \frac{\partial \vec{\mathbf{v}}}{\partial s}(t, s) \right| dt\, ds$$

$$= \int_0^{2\pi} \underbrace{\int_0^{2\pi} (3 + \cos(s))\, dt}_{\text{Integrating a constant with respect to } t} ds$$

$$= \int_0^{2\pi} 2\pi(3 + \cos(s))\, ds$$

$$= \Big[2\pi(3s + \sin(s))\Big]_{s=0}^{s=2\pi}$$

$$= 2\pi\left((3(2\pi) + \sin(2\pi)) - (3(0) + \sin(0))\right)$$

$$= 2\pi(3(2\pi))$$

$$= 12\pi^2$$

There is something quite nice about this result. The specific torus we chose has an outer radius of 3 and an inner radius of 1. This gives the feeling that if you cut the torus and "unroll" it, you will get a flat rectangle with side lengths $2\pi(3)$ and $2\pi(1)$, since the circumference of a circle with radius r is $2\pi r$

Of course, this "unrolling" process involves stretching the inner parts of the torus, and squishing its outer parts, so there is no guarantee that surface area will be preserved. Nevertheless, the area of that rectangle is

$$2\pi(3) \cdot 2\pi = 12\pi^2$$

This is the same as the surface area of our torus. This means that, in a sense, the stretching and squishing of this unrolling process perfectly cancel out!

Congratulations

These integrals are a lot of work, so pat yourself on the back for working all the way through this!