Intuition for divergence formula

Google Classroom

Why does adding up certain partial derivatives have anything to do with outward fluid flow?

Background

- Divergence

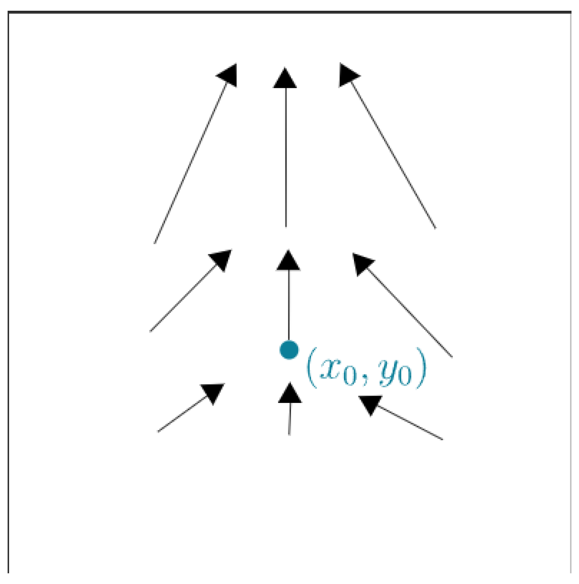
Warmup for the intuition

In the last article, I showed you the formula for divergence, as well as the physical concept it represents. However, you might still be wondering how these two are connected. Before we dive into the intuition, the following questions should help us warm up by thinking of partial derivatives in the context of a vector field.

Reflection question: A two-dimensional vector field is given by a function $\vec{\mathbf{v}}$ defined with two components v_1 and v_2,

$$\vec{\mathbf{v}}(x, y) = \left[\begin{array}{c} v_1(x, y) \\ v_2(x, y) \end{array} \right]$$

A few vectors near a point (x_0, y_0) are sketched below:



- Which of the following describes $v_1(x_0, y_0)$?

Choose 1 answer:

(A) Positive

(B) Zero

(C) Negative

Check

[Hide explanation]

Remember, since v_1 is the first component of $\vec{\mathbf{v}}$, it measures the component of each vector in the x direction, or the horizontal component.

The arrow attached to the point (x_0, y_0) has no horizontal component, since it points straight up, so $v_1(x_0, y_0) = 0$

- Which of the following describes $v_2(x_0, y_0)$?

Choose 1 answer:

(A) Positive

(B) Zero

(C) Negative

Check

[Hide explanation]

Since v_2 is the second component of $\vec{\mathbf{v}}$, it measures the component of each vector in the y direction, or the vertical component.

The arrow attached to (x_0, y_0) points up, hence it has a *positive* vertical component.

- Which of the following describes $\frac{\partial v_1}{\partial x}(x_0, y_0)$?

Choose 1 answer:

(A) Positive

(B) Zero

(C) Negative

Check

[Hide explanation]

> The partial derivative $\frac{\partial v_1}{\partial x}(x_0, y_0)$ measures how much v_1 changes as x changes. In other words, what happens to the horizontal components of vectors as we move from the left to the right.
>
> As you pan from left to right around (x_0, y_0), the arrows go from pointing a little right, to having zero horizontal component, to pointing a little left. Therefore the horizontal component of vectors near (x_0, y_0) decreases as x increases, so $\frac{\partial v_1}{\partial x}(x_0, y_0) < 0$.

- Which of the following describes $\frac{\partial v_2}{\partial y}(x_0, y_0)$?

Choose 1 answer:

(A) Positive

(B) Zero

(C) Negative

Check

[Hide explanation]

> The partial derivative $\frac{\partial v_2}{\partial y}(x_0, y_0)$ measures how much v_2 changes as y changes. In other words, what happens to the vertical components of vectors as we move from the bottom to the top.
>
> As you pan from down to up around (x_0, y_0), the upward component of each vector gets longer and longer. Therefore the vertical component of vectors near (x_0, y_0) increases as y increases, so $\frac{\partial v_2}{\partial y}(x_0, y_0) > 0$.

Intuition behind the divergence formula

Let's limit our view to a two-dimensional vector field,

$$\vec{\mathbf{v}}(x, y) = \begin{bmatrix} v_1(x, y) \\ v_2(x, y) \end{bmatrix}$$

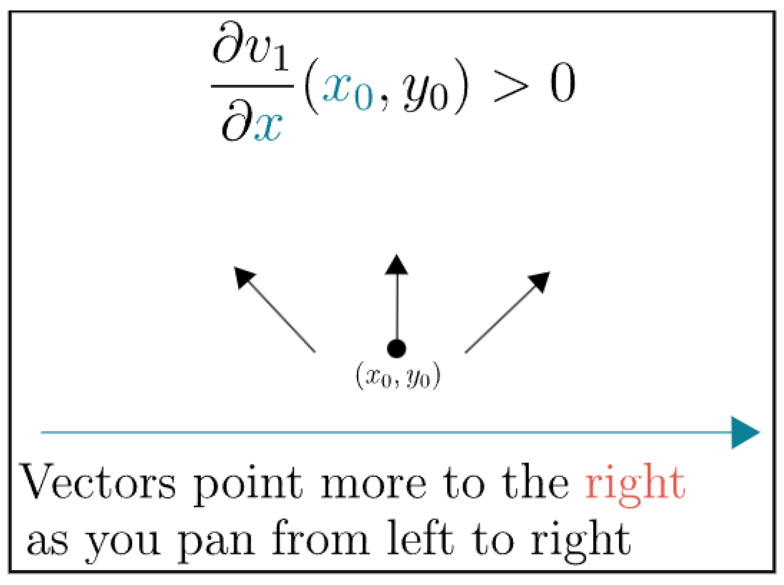
Remember, the formula for divergence looks like this:

$$\nabla \cdot \vec{\mathbf{v}} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$$

Why does this have anything to do with changes in the density of a fluid flowing according to $\vec{\mathbf{v}}(x, y)$?

Let's look at each component separately.

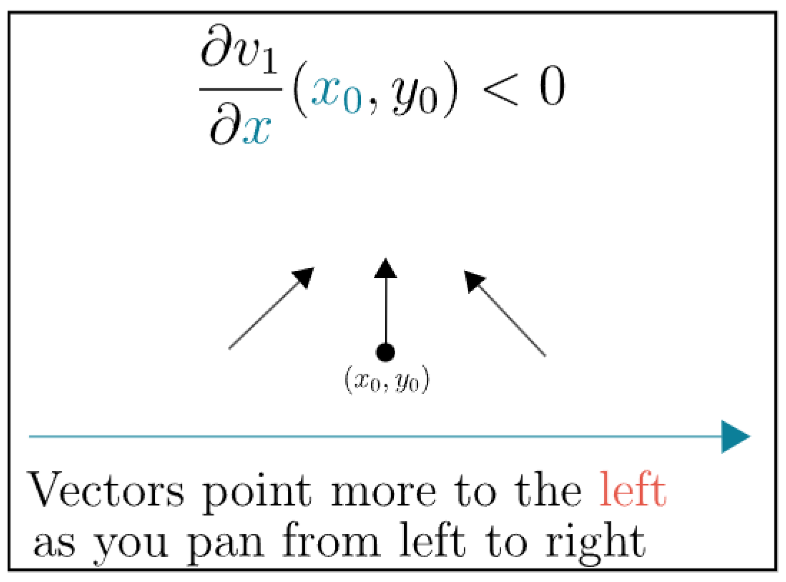
For example, suppose $v_1(x_0, y_0) = 0$, meaning the vector attached to (x_0, y_0) has no horizontal component. And let's say $\frac{\partial v_1}{\partial x}(x_0, y_0)$ happens to be positive. This means that near the point (x_0, y_0), the vector field might look something like this.



- The value of $v_1(x, y_0)$ increases as x grows.
- The value of $v_1(x, y_0)$ decreases as x gets smaller.

Therefore, vectors to the left of (x_0, y_0) will point a little to the left, and vectors to the right of (x_0, y_0) will point a little to the right (see the diagram above). This suggests an outward fluid flow, at least as far as the x-component is concerned.

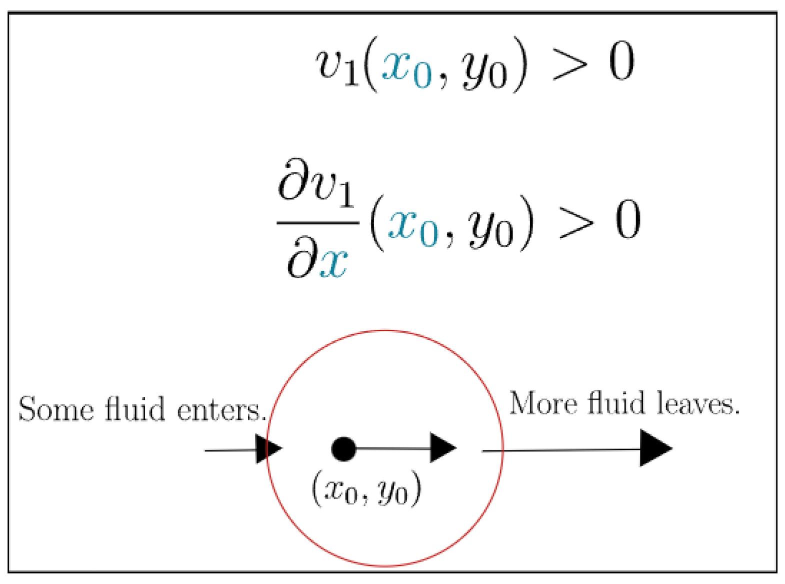
In contrast, here's how it looks if $\frac{\partial v_1}{\partial x}(x_0, y_0)$ is negative:



- The vectors to the left of (x_0, y_0) will point to the right.
- The vectors to the right of (x_0, y_0) will point to the left.

This indicates an inward fluid flow, according to the x-component.

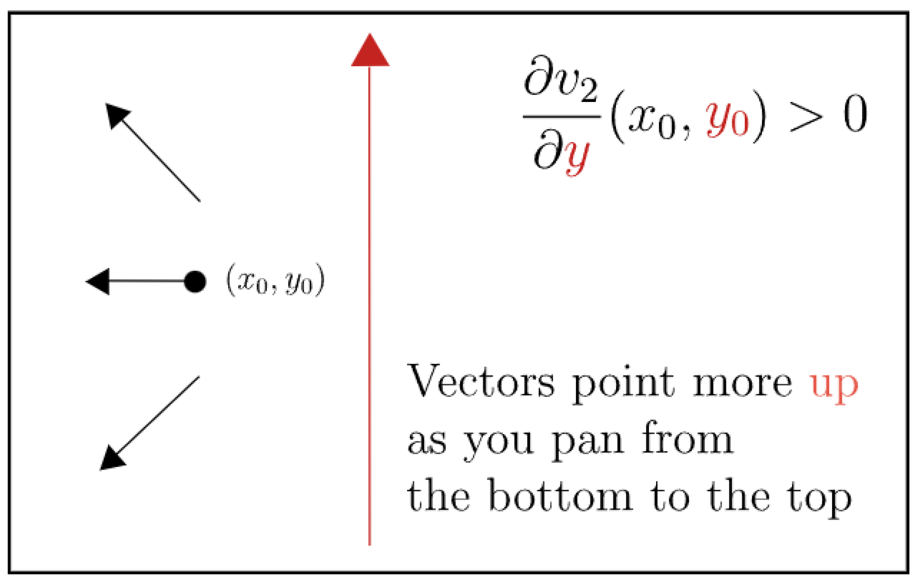
The same intuition applies if $v_1(x_0, y_0)$ is nonzero. For instance, if $v_1(x_0, y_0)$ is positive and $\frac{\partial v_1}{\partial x}(x_0, y_0)$ is also positive, this means all the vectors around (x_0, y_0) point to the right, but they get bigger as we look from left to right. You can imagine the fluid flowing slowly towards (x_0, y_0) from the left, but flowing fast away from it to the right. Since more is leaving than is coming in, the density at this point decreases.



Analyzing the value $\frac{\partial v_2}{\partial y}$ is similar. It indicates the change in the vertical component of vectors, v_2, as one moves up and down in the vector field, changing y.

For example, suppose $v_2(x_0, y_0) = 0$, meaning the vector attached to (x_0, y_0) has no vertical component. Also suppose $\frac{\partial v_2}{\partial y}(x_0, y_0)$ is positive, meaning the vertical component of vectors increases as we move upward.

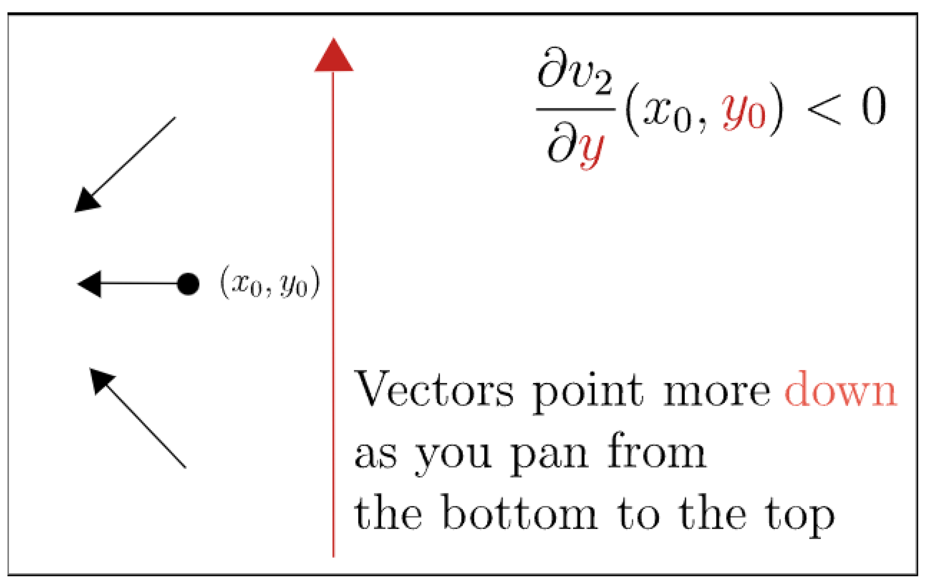
Here's how that might look:



- Vectors below (x_0, y_0) will point slightly downward.
- Vectors above (x_0, y_0) will point slightly upward

This indicates an outward fluid flow, as far as the y-direction is concerned.

Likewise, if $\frac{\partial v_2}{\partial y}(x_0, y_0)$ is negative, it indicates an inward fluid flow near (x_0, y_0) as far as the y-direction is concerned.



Divergence adds these two influences

Adding the two components $\frac{\partial v_1}{\partial x}$ and $\frac{\partial v_2}{\partial y}$ brings together the separate influences of the x and y directions in determining whether fluid-density near a given point increases or decreases.

$$\nabla \cdot \vec{\mathbf{v}} = \overbrace{\frac{\partial v_1}{\partial x}}^{\substack{\text{Change in density} \\ \text{in the } x\text{-direction}}} + \underbrace{\frac{\partial v_2}{\partial y}}_{\substack{\text{Change in density} \\ \text{in the } y\text{-direction}}}$$