Double integrals

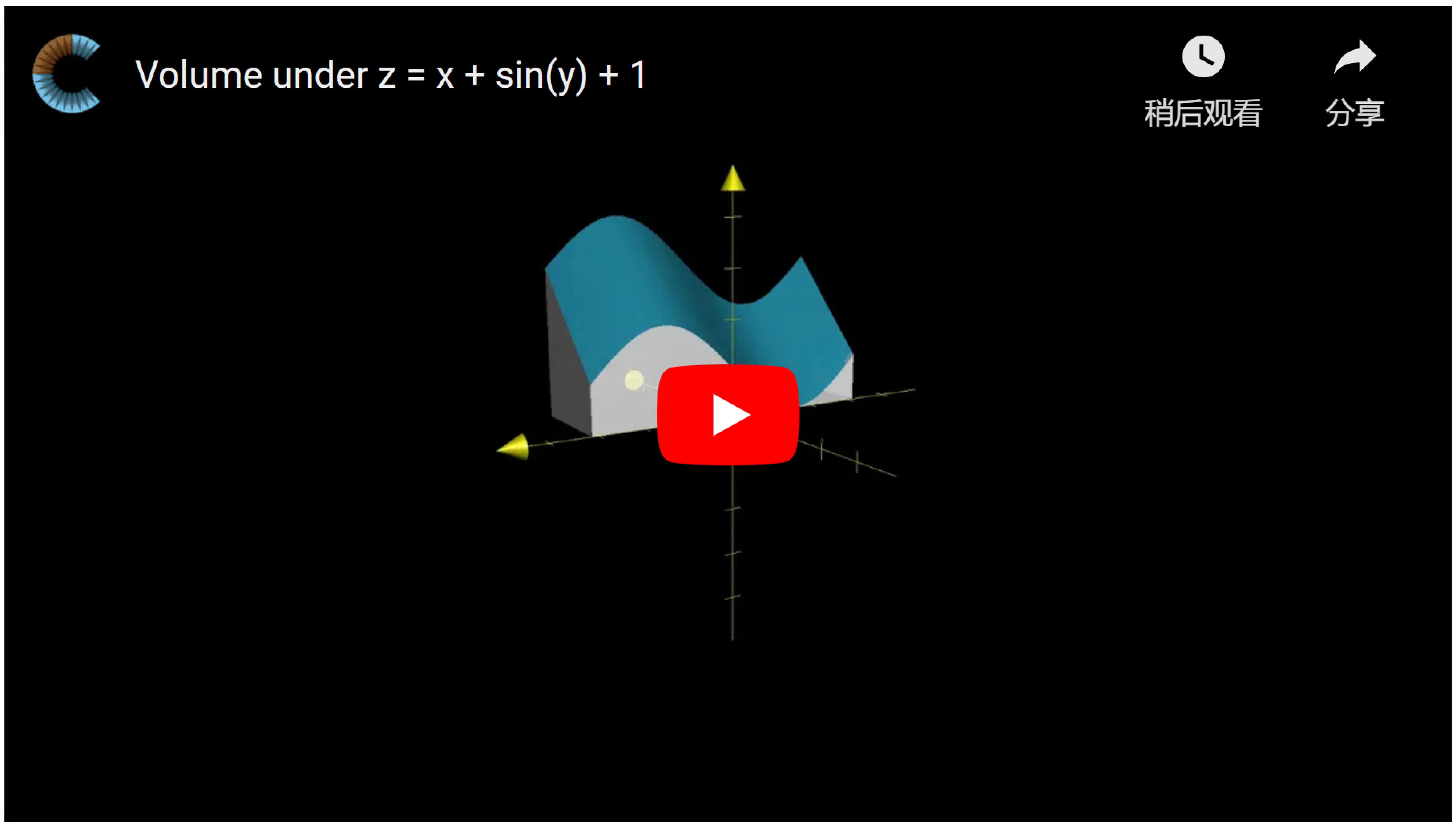
Google Classroom

Double integrals are a way to integrate over a two-dimensional area. Among other things, they lets us compute the volume under a surface.

Background

- Ordinary integrals
- Graphs of multivariable functions

What we're building to



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- Given a two-variable function $f(x, y)$, you can find the volume between this graph and a rectangular region of the xy-plane by taking an integral *of an integral*,

$$\int_{y_1}^{y_2} \overbrace{\left(\int_{x_1}^{x_2} f(x, y)dx \right)}^{\text{This is a function of } y} dy$$

  This is called a **double integral**.

- You can compute this same volume by changing the order of integration:

$$\int_{x_1}^{x_2} \overbrace{\left(\int_{y_1}^{y_2} f(x, y)dy \right)}^{\text{This is a function of } x} dx$$

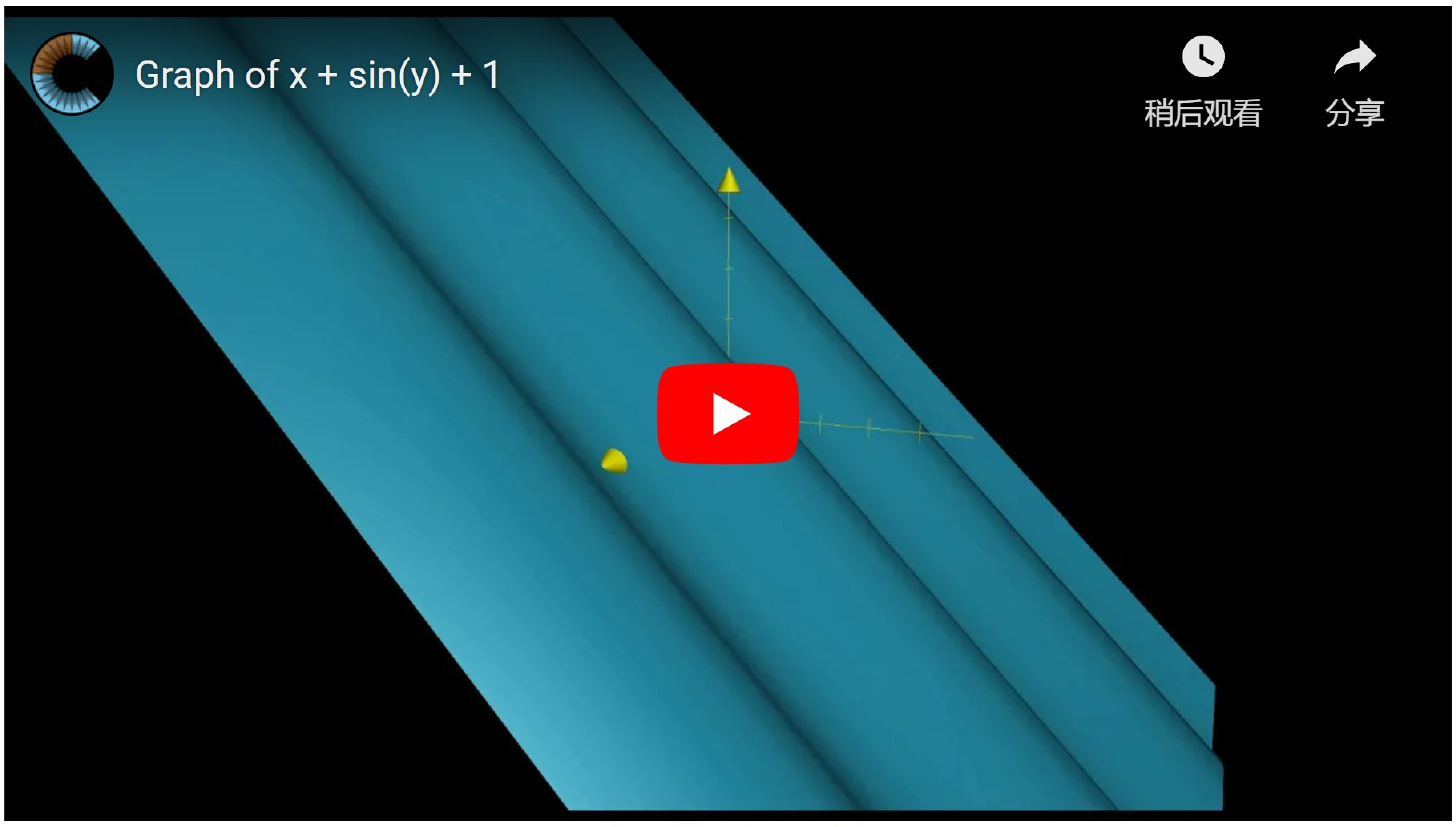
The computation will look and feel very different, but it still gives the same result.

Volume under a surface

Consider the function

$$f(x, y) = x + \sin(y) + 1$$

Its graph looks like this:



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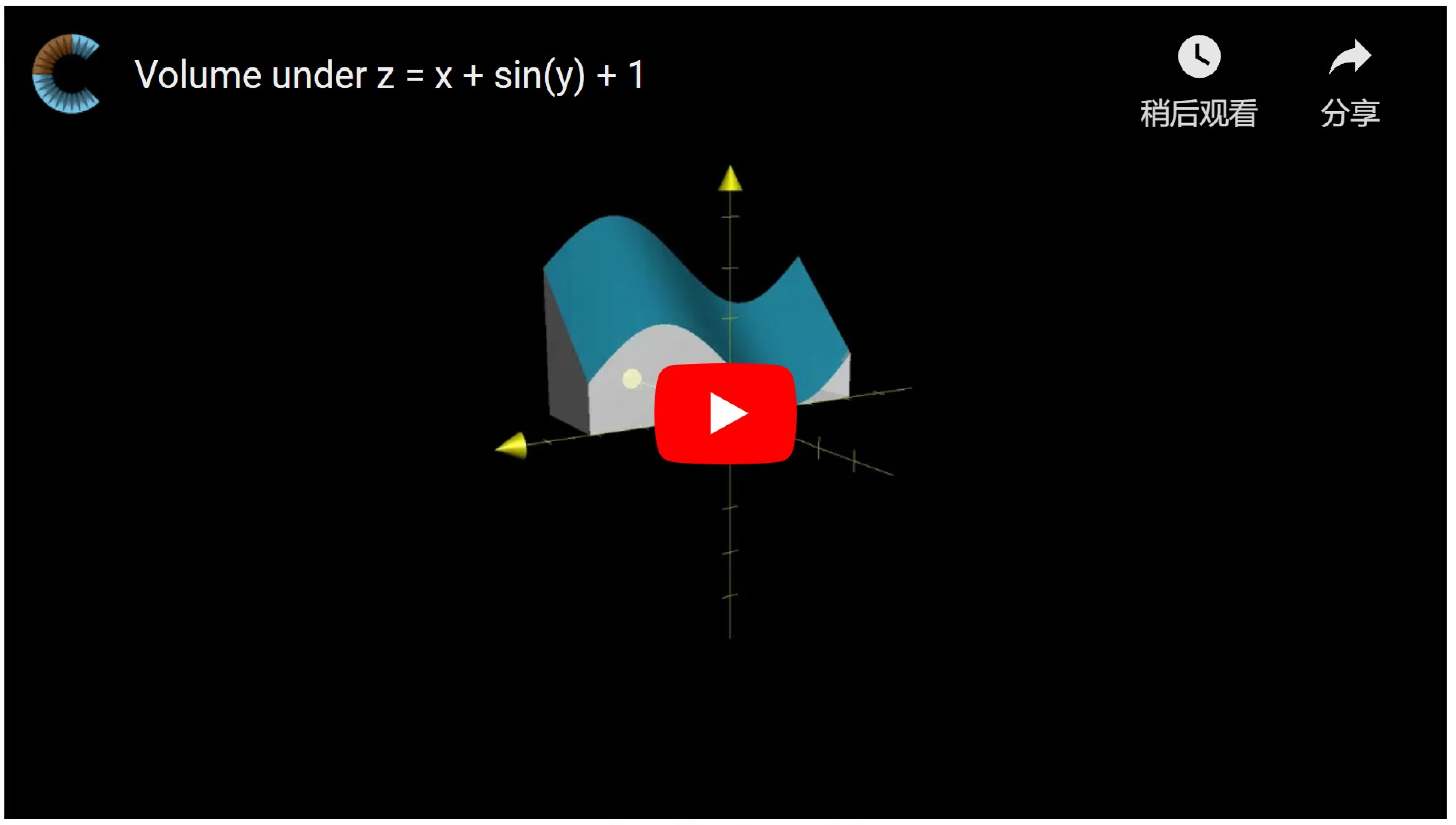
Now consider the rectangle on the xy-plane defined by

$$0 < x < 2$$

and

$$-\pi < y < \pi$$

What is the volume above this rectangle, and under the graph of $f(x, y)$?



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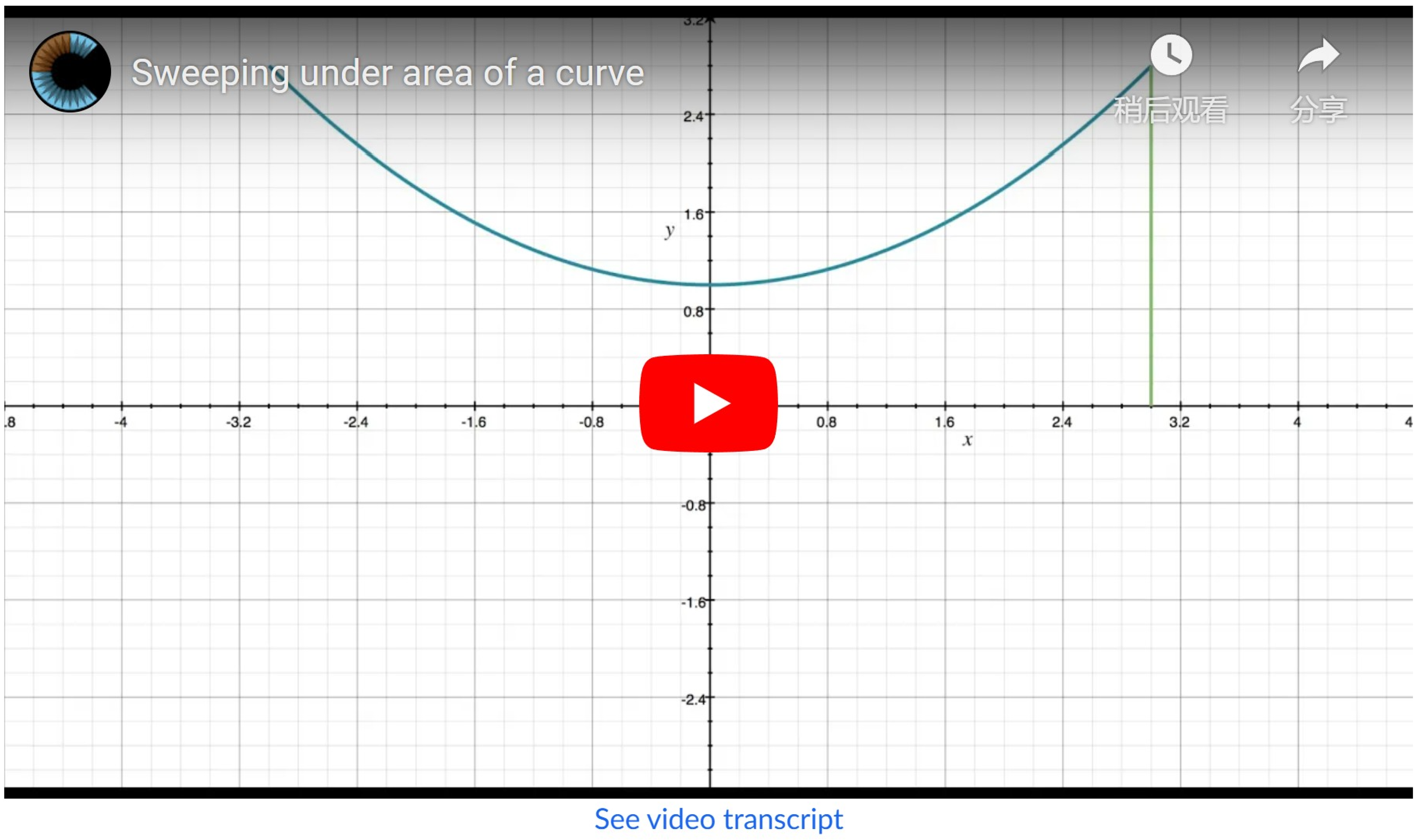
Quick refresh of area under curve

From single variable calculus, we know that integrals let us compute the *area* under a curve. For example, the area under the graph of $y = \frac{1}{4}x^2 + 1$ between the values $x = -3$ and $x = 3$ is

$$\int_{-3}^{3} \left(\frac{1}{4}x^2 + 1\right) \, dx$$

A nice way to think about this is to imagine adding the areas of infinitely many, infinitely thin rectangles which sweep under the curve in the specified region:



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You can think of the value of the function $g(x) = \frac{1}{4}x^2 + 1$ as being the height of each rectangle, dx as being the infinitesimal width, and $\int$ as being a pumped-up summing machine that's able to handle the idea of infinitely many infinitely small things. Written more abstractly, this looks like

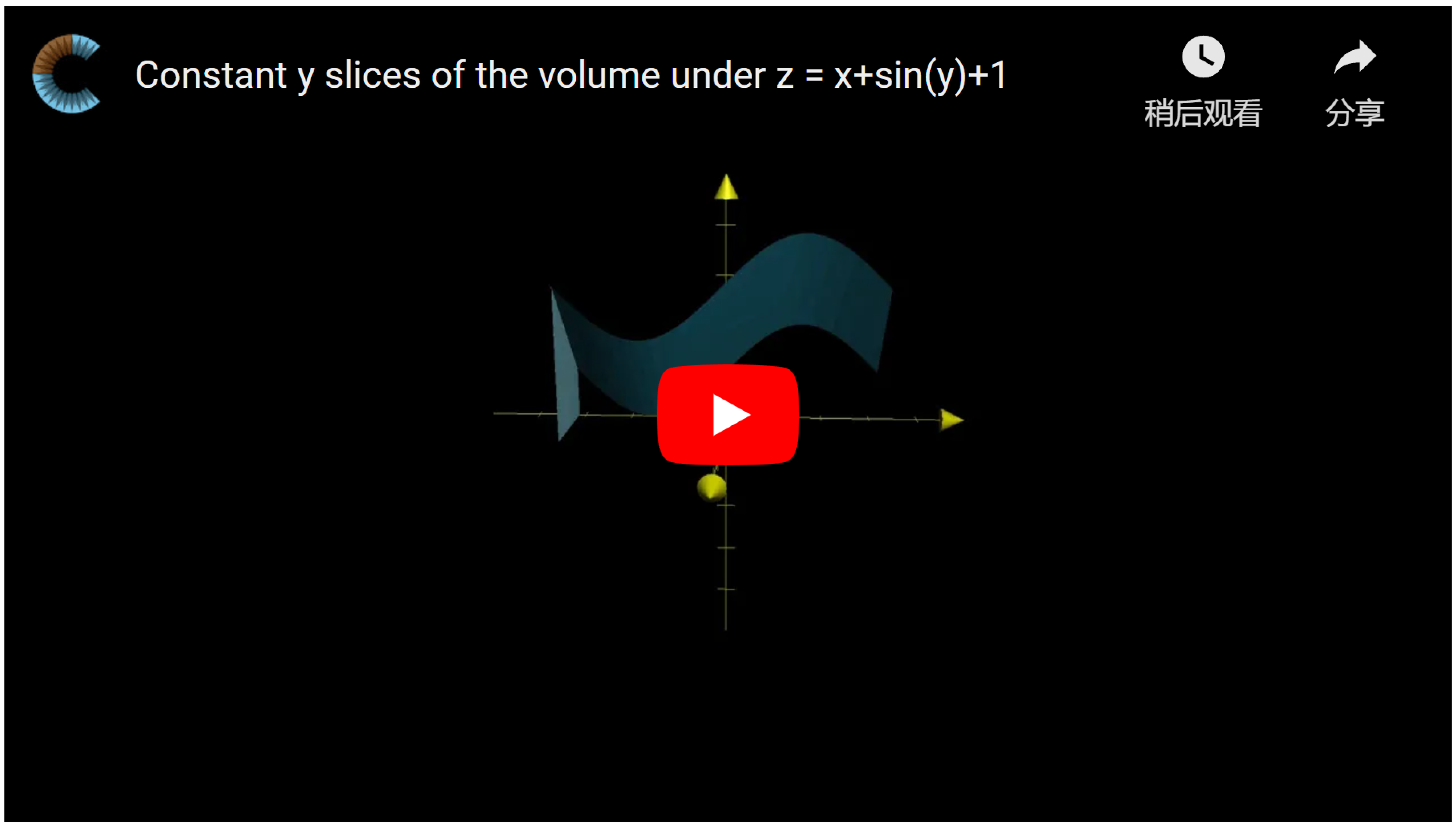
$$\int_{x_1}^{x_2} g(x)\, dx$$

Sweeping area under a volume

For our volume problem, we can do something similar. Our strategy will be to

1. Subdivide the volume into slices with two-dimensional area
2. Compute the areas of these slices
3. Combine them all together to get the volume as a whole.

Think of two-dimensional slices of the volume under the graph of $f(x, y)$. Specifically, take all the slices representing a constant value of y:



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Consider just one of those slices, such as the one representing $y = \frac{\pi}{2}$. The area of that slice is given by the integral

$$\int_0^2 f\left(x, \frac{\pi}{2}\right)\, dx = \int_0^2 \left(x + \sin\left(\frac{\pi}{2}\right) + 1\right)\, dx$$

Written more abstractly, for a given value of y, the area of that slice is

$$\int_0^2 f(x, y)\, dx$$

Notice, this is an integral with respect to x, as indicated by the dx, so as far as the integral is concerned, the symbol "y" represents a constant.

When you perform this integral, it will be some expression of y.

> **Try it for yourself**: Perform the integral to compute the area of these constant-y-value slices:
>
> $$\int_0^2 (x + \sin(y) + 1)\, dx = \square$$
>
> Check
>
> [Hide explanation]
>
> $$\int_0^2 f(x,y)\, dx = \int_0^2 (x + \sin(y) + 1)\ dx$$
>
> $$= \int_0^2 x\, dx + \int_0^2 \underbrace{\sin(y)}_{\substack{\text{Constant with} \\ \text{respect to } x}} dx + \int_0^2 1\, dx$$
>
> $$= \left[\frac{x^2}{2}\right]_0^2 + 2\sin(y) + 2$$
>
> $$= 2 + 2\sin(y) + 2$$
>
> $$= 4 + 2\sin(y)$$

When you plug in some value of y to this expression, such as $y = \frac{\pi}{2}$, you get the area of a slice of our volume representing that y-value.

Now if we multiply the area of each one of these slices by dy, a tiny change in the y-direction, we will get a tiny slice of volume. For example, $4 + 2\sin(y)$ might represent the area of a slice, but $(4 + 2\sin(y))dy$ represents the infinitesimal volume of that slice.

Using yet another integral, this time with respect to y, we can effectively sum up all those tiny volume slices to get the total volume under the surface:

$$\text{Volume under } f(x,y)$$

$$\overbrace{\int_{-\pi}^{\pi}\left(\underbrace{\int_0^2 f(x,y)dx}_{\text{Area of a slice}}\right)dy}$$

Try it yourself! What do you get when you plug in the expression for $\int_0^2 f(x,y)\,dx$ that you found above, and solve the second integral?

$$\int_{-\pi}^{\pi}\left(\int_0^2 (x+\sin(y)+1)dx\right)dy = \boxed{}$$

Check

[Hide explanation]

Start by plugging in the value for the inner integral that you found above,

$$\int_{-\pi}^{\pi}\left(\int_0^2 (x+\sin(y)+1)dx\right)dy = \int_{-\pi}^{\pi}(4+2\sin(y))\,dy$$

Then compute the integral with respect to y

$$\int_{-\pi}^{\pi}(4+2\sin(y))\,dy$$

$$= \Big[4y+2(-\cos(y))\Big]_{-\pi}^{\pi}$$

$$= (4\pi - 2\cos(\pi)) - (4(-\pi) - 2\cos(-\pi))$$

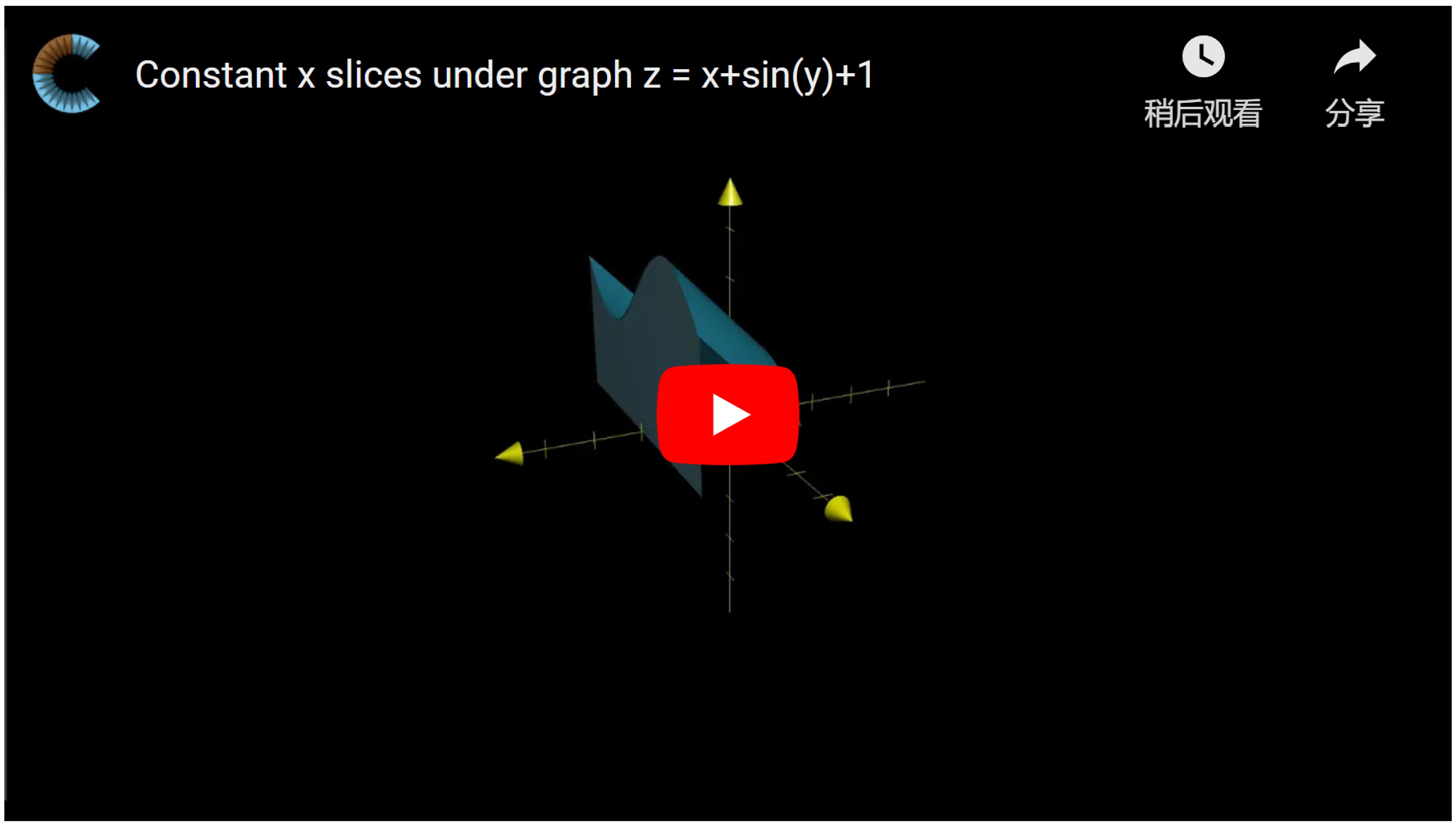
$$= (4\pi - 2(-1)) - (-4\pi - 2(-1))$$

$$= 8\pi$$

So evidently the volume we were looking for is 8π.

Two choices in direction

You could also dissect the volume we are trying to find a different way. Take slices which represent constant x values, instead of constant y values, and add up the volume slices.



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Concept check: Which of the following integrals represents the area of a constant-x-value slice?

Choose 1 answer:

(A) $\int_0^2 f(x, y)\, dx$

(B) $\int_{-\pi}^{\pi} f(x, y)\, dy$

Check

[Hide explanation]

The second answer is correct:

$$\int_{-\pi}^{\pi} f(x, y)\, dy$$

To get the area of a slice representing a constant x-value, you integrate with respect to y. Some people call this "integrating the y-out", since you will be left with a function of x.

Now imagine multiplying each of these areas by dx, a tiny step in the x-direction, which is perpendicular to the slice. This will give some kind of infinitesimal volume. By adding up all those infinitesimal volumes as x ranges from 0 to 2, we will get the volume under the surface.

Concept check: Which of the following double-integrals represents the volume under the graph of our function

$$f(x, y) = x + \sin(y) + 1$$

in the region where

$$0 \leq x \leq 2 \text{ and } -\pi \leq y \leq \pi?$$

Choose 1 answer:

(A) $\int_0^2 \left(\int_{-\pi}^{\pi} (x + \sin(y) + 1)\, dx \right) dy$

(B) $\int_0^2 \left(\int_{-\pi}^{\pi} (x + \sin(y) + 1)\, dy \right) dx$

Check

[Hide explanation]

The second answer is correct

$$\int_0^2 \left(\int_{-\pi/2}^{\pi/2} (x + \sin(y) + 1)\, dy \right) dx$$

It correctly matches the bounds of the inner dy integral with the y-bounds of the region where we are integrating, and likewise it matches the bounds of the outer dx-integral with the x-bounds of our region.

Try it yourself!: Perform the double integral to compute the volume under the surface. (Of course, you already found the volume in the previous section, but it is edifying to see how it can be computed a second way).

$$\int_0^2 \left(\int_{-\pi}^{\pi} (x + \sin(y) + 1) dy \right) dx = \underline{\qquad}$$

Check

[Hide explanation]

$$\int_0^2 \left(\underbrace{\int_{-\pi}^{\pi} (x + \sin(y) + 1) dy}_{\text{Integrate with respect to } y} \right) dx$$

$$= \int_0^2 \left(xy - \cos(y) + y \right)\Big|_{y=-\pi}^{y=\pi} dx$$

$$= \int_0^2 \underbrace{(x\pi - \cos(\pi) + \pi) - (x(-\pi) - \cos(-\pi) + (-\pi))}_{\text{Consolidate terms}} dx$$

$$= \int_0^2 (2x\pi + 2\pi) \, dx$$

$$= \left(2\frac{x^2}{2}\pi + 2\pi x \right)\Big|_0^2$$

$$= \left(2\frac{(2)^2}{2}\pi + 2\pi(2) \right) - \left(2\frac{(0)^2}{2}\pi + 2\pi(0) \right)$$

$$= 4\pi + 4\pi - (0 + 0)$$

$$= 8\pi$$

Thankfully, this computation gives the same volume that we found in the previous section. Something would have to be wrong with our reasoning if it didn't.

In short, **the order of integration does not matter**. On the one hand, this might seem obvious, since either way you are computing the same volume. However, these are two *genuinely different* computations, so the fact that they equal each other turns out to be a useful mathematical trick.

For example, several proofs in probability theory involve showing that two quantities are equal by showing that both result from the same double integral, just computed in a different order.

[Hide explanation]

> Technically, I should mention that there exist some exotic double integrals where swapping the order of integration gives a different value. The relevant piece of mathematics describing when you can and cannot swap integrals is "Fubini's Theorem".
>
> You may learn the full details of Fubini's Theorem in an analysis course, but as an introduction to double integrals, you really needn't worry about it. For most, if not all, the function you deal with in practice, you can happily swap around the integrals to your heart's content!

Another example

Consider the function

$$f(x, y) = \cos(y)x^2 + 1$$

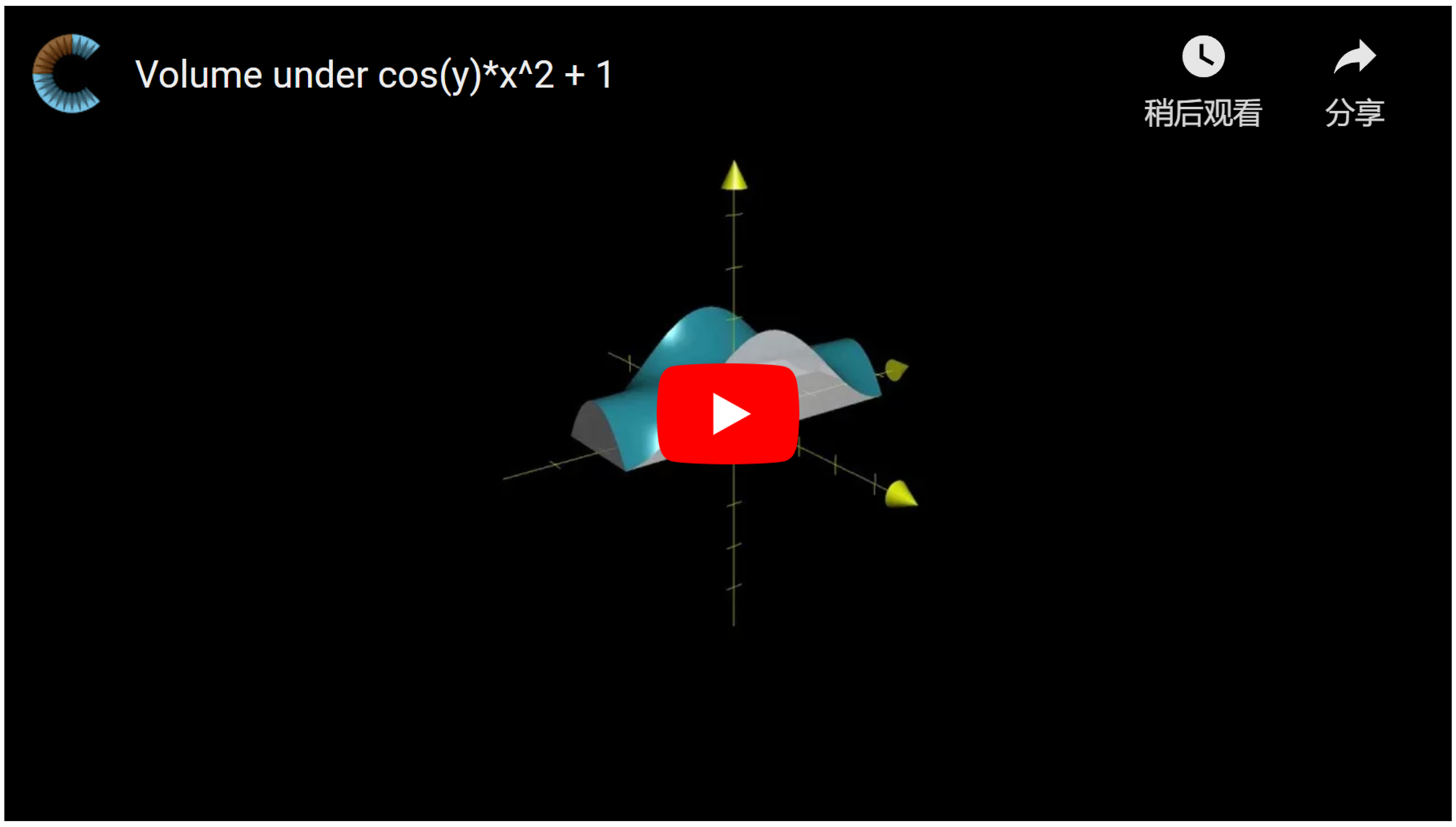
What is the volume under the graph of this function in the region where

$$-1 \leq x \leq 1$$

and

$$-\pi \leq y \leq \pi?$$

Here what this volume looks like:



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Concept check: Imagine cutting this volume under $f(x, y) = \cos(y)x^2 + 1$ along the plane representing $y = 1$. Which of the following integrals represents the area of that slice?

Choose 1 answer:

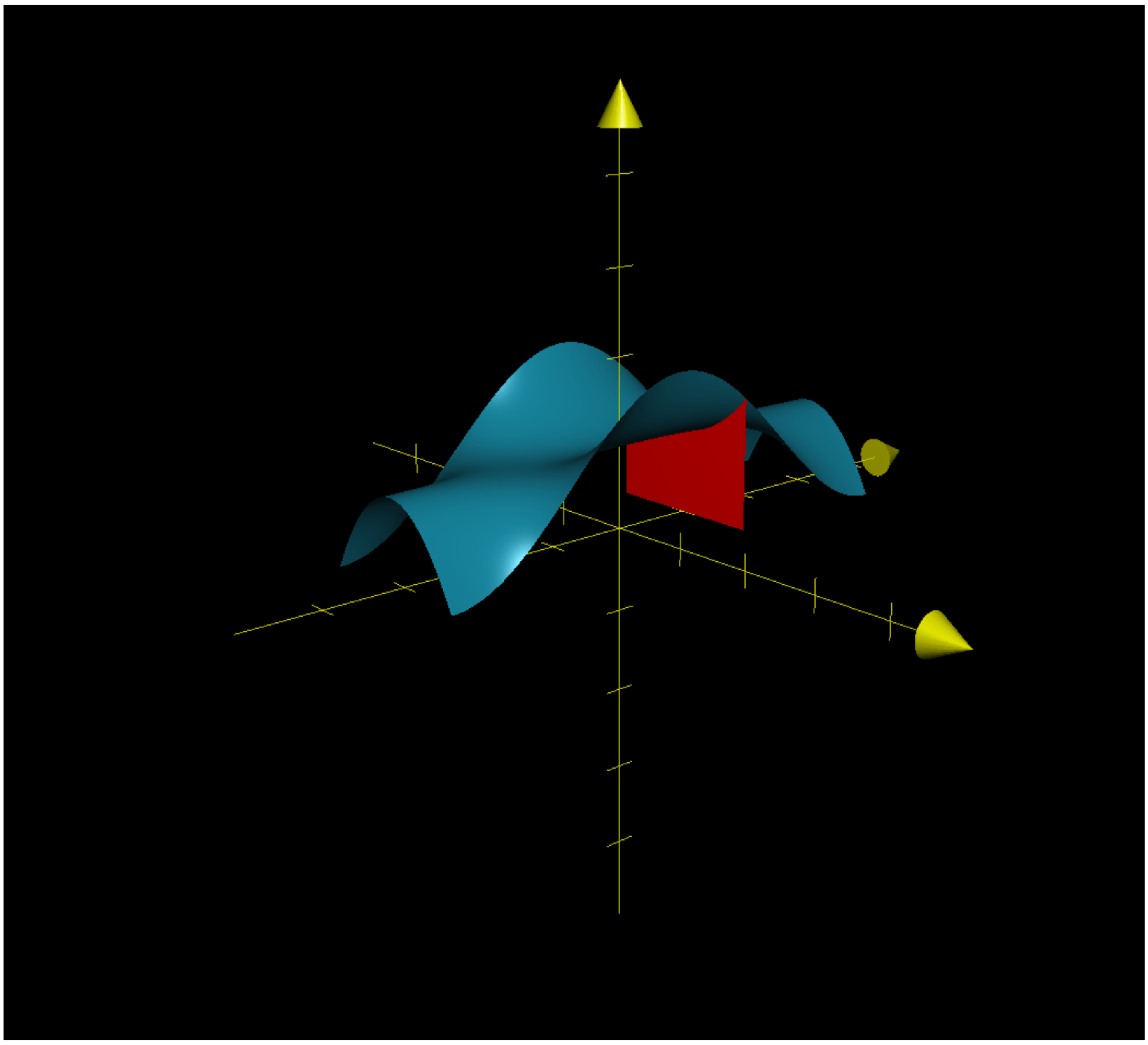
(A) $\int_{-1}^{1} \left(\cos(1)x^2 + 1\right) dx$

(B) $\int_{-\pi}^{\pi} \left(\cos(y)(1)^2 + 1\right) dy$

Check

[Hide explanation]

Here is a picture of the slice in question (in red)



To find its area, you integrate from -1 to 1 in the x-direction. Therefore, the first answer is correct.

$$\int_{-1}^{1} \left(\cos(1)x^2 + 1\right) dx$$

Practice: What do you get when you compute this integral for a general value of y, not just $y = 1$?

$$\int_{-1}^{1} \left(\cos(y)x^2 + 1\right) dx = \boxed{}$$

Check

[Hide explanation]

$$\begin{aligned}\int_{-1}^{1} \left(\cos(y)x^2 + 1\right) dx &= \left(\cos(y)\frac{x^3}{3} + x\right)\Big|_{-1}^{1} \\ &= \left(\cos(y)\frac{(1)^3}{3} + (1)\right) - \left(\cos(y)\frac{(-1)^3}{3} + (-1)\right) \\ &= \frac{2}{3}\cos(y) + 2\end{aligned}$$

More practice: The expression you just found represents the area of slices of our volume representing constant y-values. Using this expression, set up an integral to find the volume under the surface, and solve the integral.

Final volume: $\boxed{}$

Check

[Hide explanation]

We just found that the area of a constant-y-value slice of our volume between the values $x = -1$ and $x = 1$ is

$$\frac{2}{3}\cos(y) + 2$$

You can imagine adding a tiny bit of depth to this area by multiplying it by dy, a tiny step in the y direction.

$$\left(\frac{2}{3}\cos(y) + 2\right) dy$$

You can think of this as an infinitesimal volume.

Since our volume is defined in the region where $-\pi \leq y \leq \pi$, we add up these infinitesimal volumes within that range:

$$\int_{-\pi}^{\pi}\left(\frac{2}{3}\cos(y)+2\right)dy = \left(\frac{2}{3}\sin(y)+2y\right)\Bigg|_{-\pi}^{\pi}$$

$$= \left(\frac{2}{3}\sin(\pi)+2\pi\right) - \left(\frac{2}{3}\sin(-\pi)+2(-\pi)\right)$$

$$= (0+2\pi)-(0-2\pi)$$

$$= 4\pi$$

Summary

- Given a two-variable function $f(x, y)$, you can find the volume between its graph and a rectangular region of the xy-plane by taking an integral *of an integral*,

  $$\int_{y_1}^{y_2}\overbrace{\left(\int_{x_1}^{x_2} f(x,y)dx\right)}^{\text{This is a function of } y} dy$$

  This is called a **double integral**.

- You can compute this same volume by changing the order of integration:

  $$\int_{x_1}^{x_2}\overbrace{\left(\int_{y_1}^{y_2} f(x,y)dy\right)}^{\text{This is a function of } x} dx$$

  The computation will look and feel very different, but it still gives the same result.