Arc length of function graphs, introduction

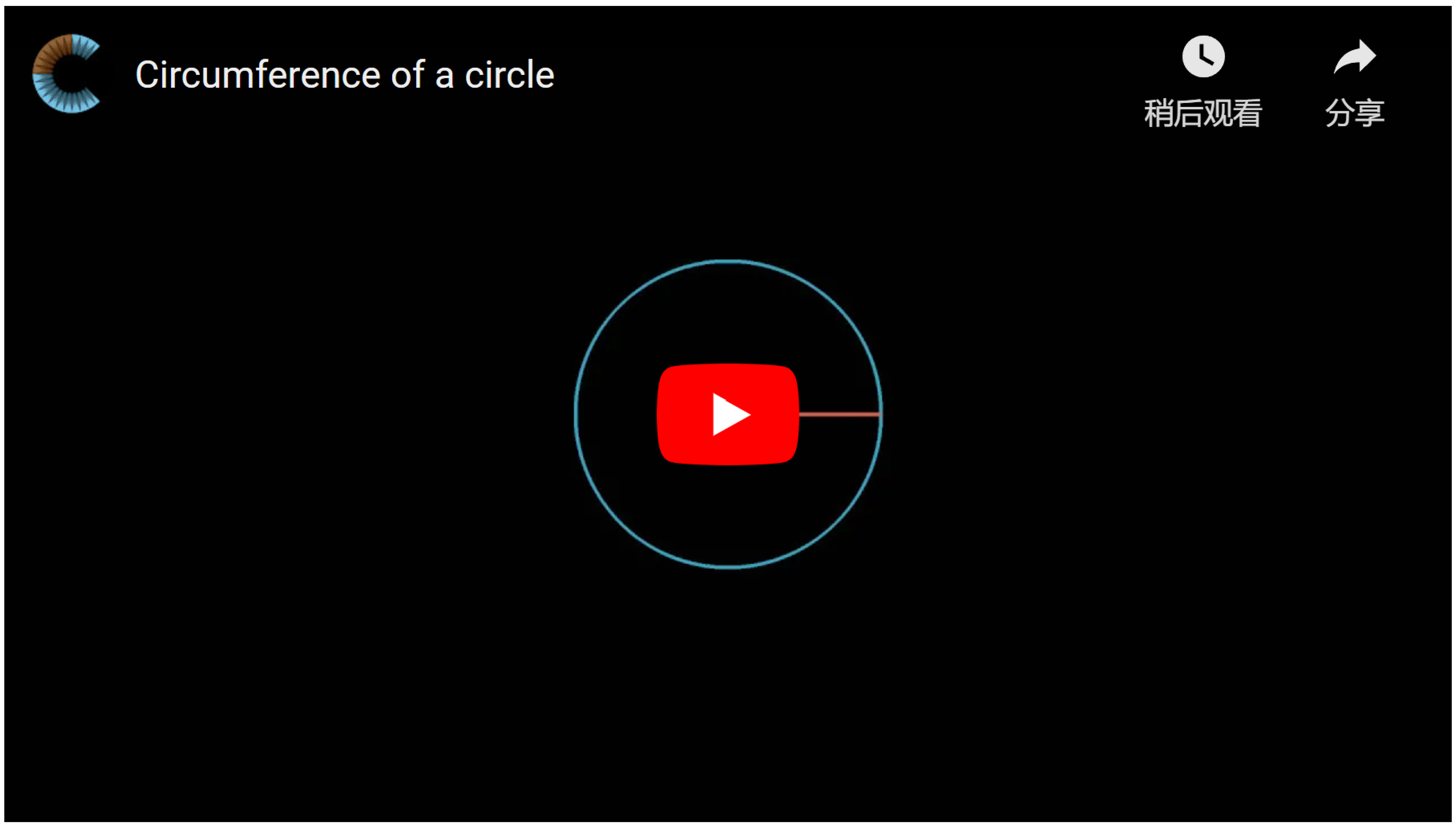
Google Classroom

The length of a curve, called its "arc length", can be found using a certain integral.

Background

- Ordinary integrals

What is arc length?



See video transcript

We usually measure length with a straight line, but curves have length too. A familiar example is the circumference of a circle, which has length $2\pi r$ for radius r. In general, the length of a curve is called the **arc length**. But how do you find the arc length of an arbitrary curve? Let's find out.

What we're building to

- You can find the arc length of a curve with an integral that looks something like this:

$$\int \sqrt{(dx)^2 + (dy)^2}$$

  The bounds of this integral depend on how you define the curve.

- If the curve is the graph of a function $y = f(x)$, replace the dy term in the integral with $f'(x)dx$, then factor out the dx.

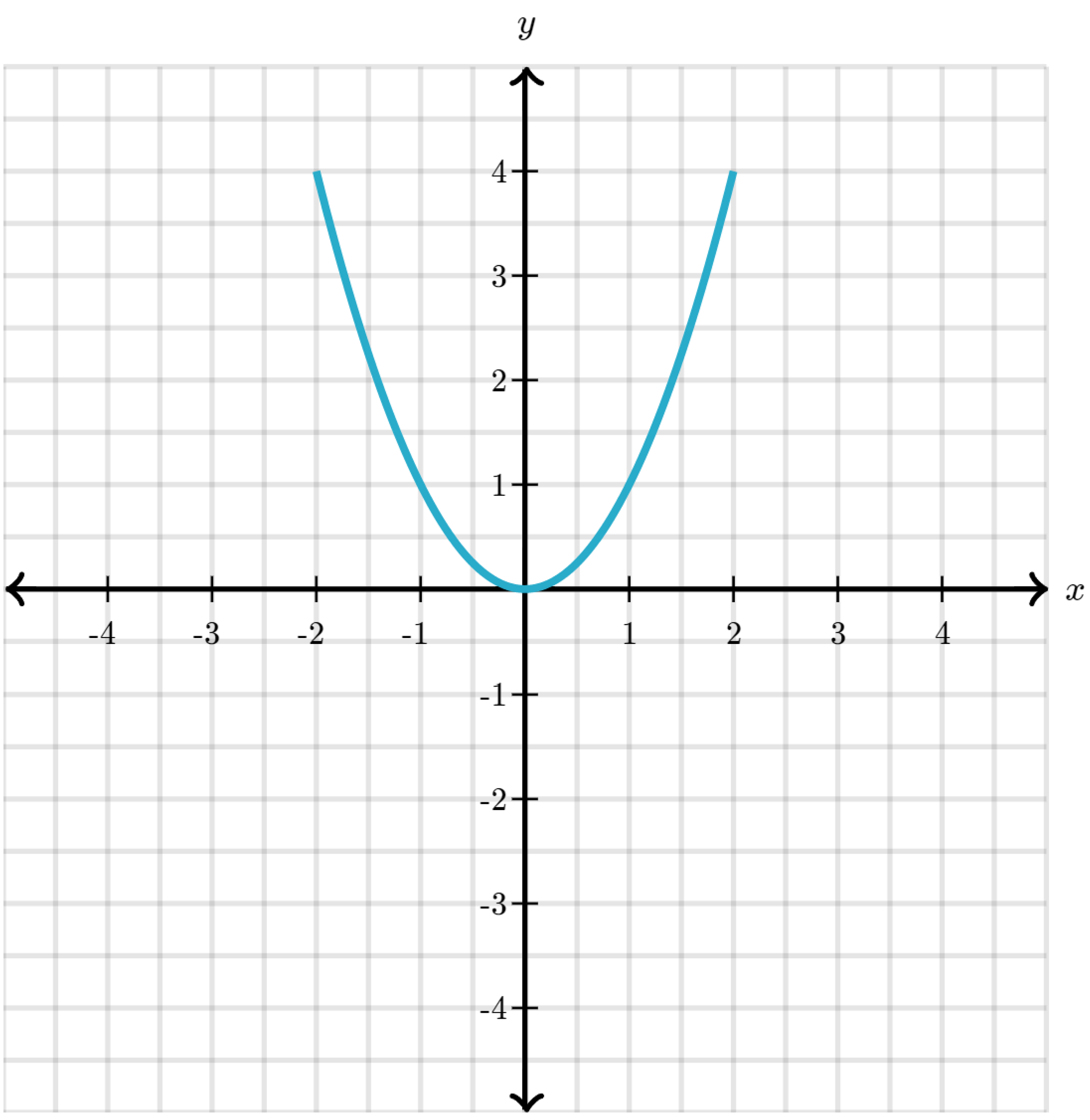
Warmup: Approximating arc length

Let's look at the parabola defined by the following equation:

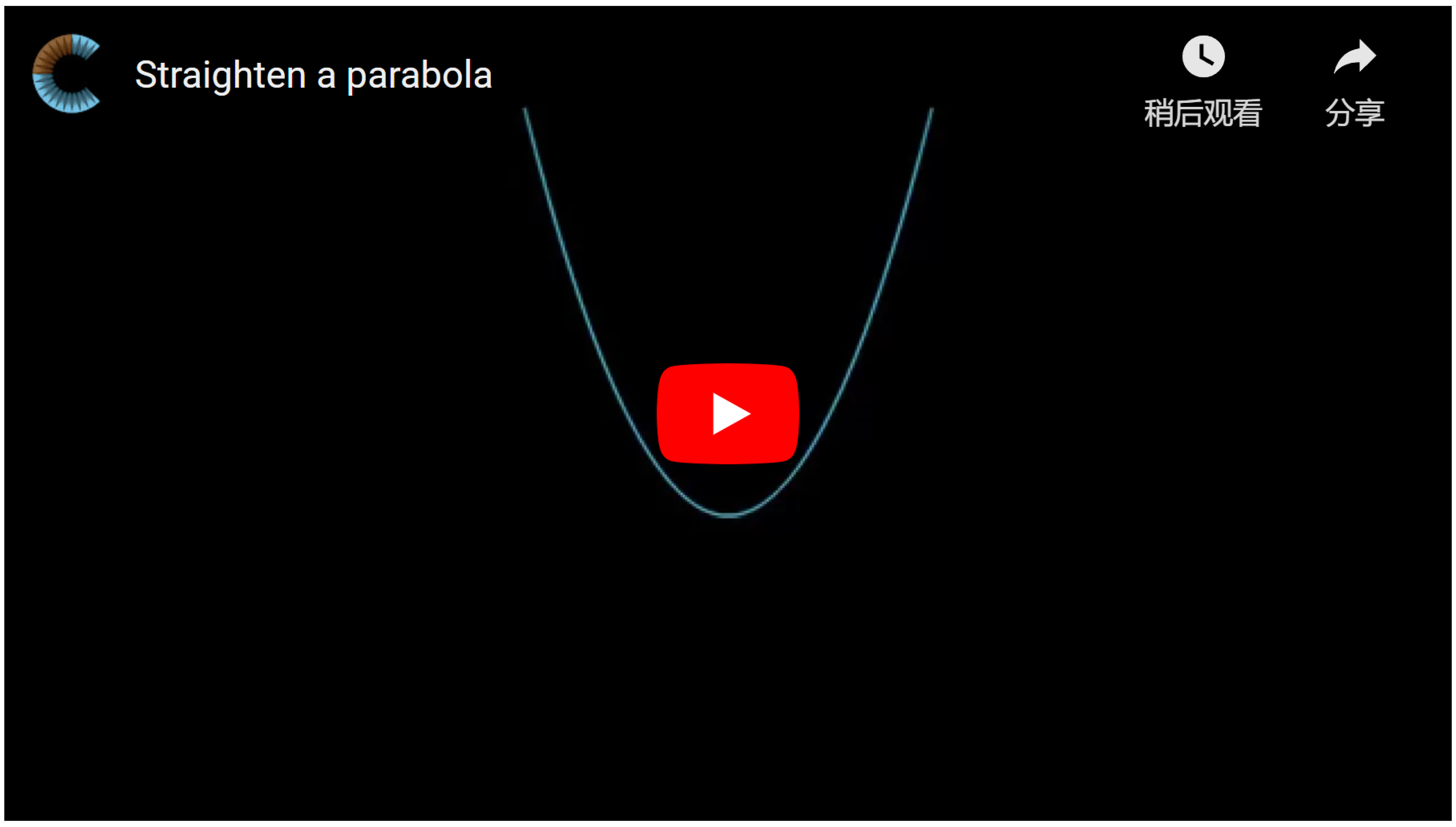
$$y = f(x) = x^2$$

Consider the portion of the curve between $x = -2$ and $x = 2$.

> **Key question**: What is the arc length of this curve?



Just so the question is clear, imagine the curve was a piece of string. You pull this string straight and measure it with a ruler.



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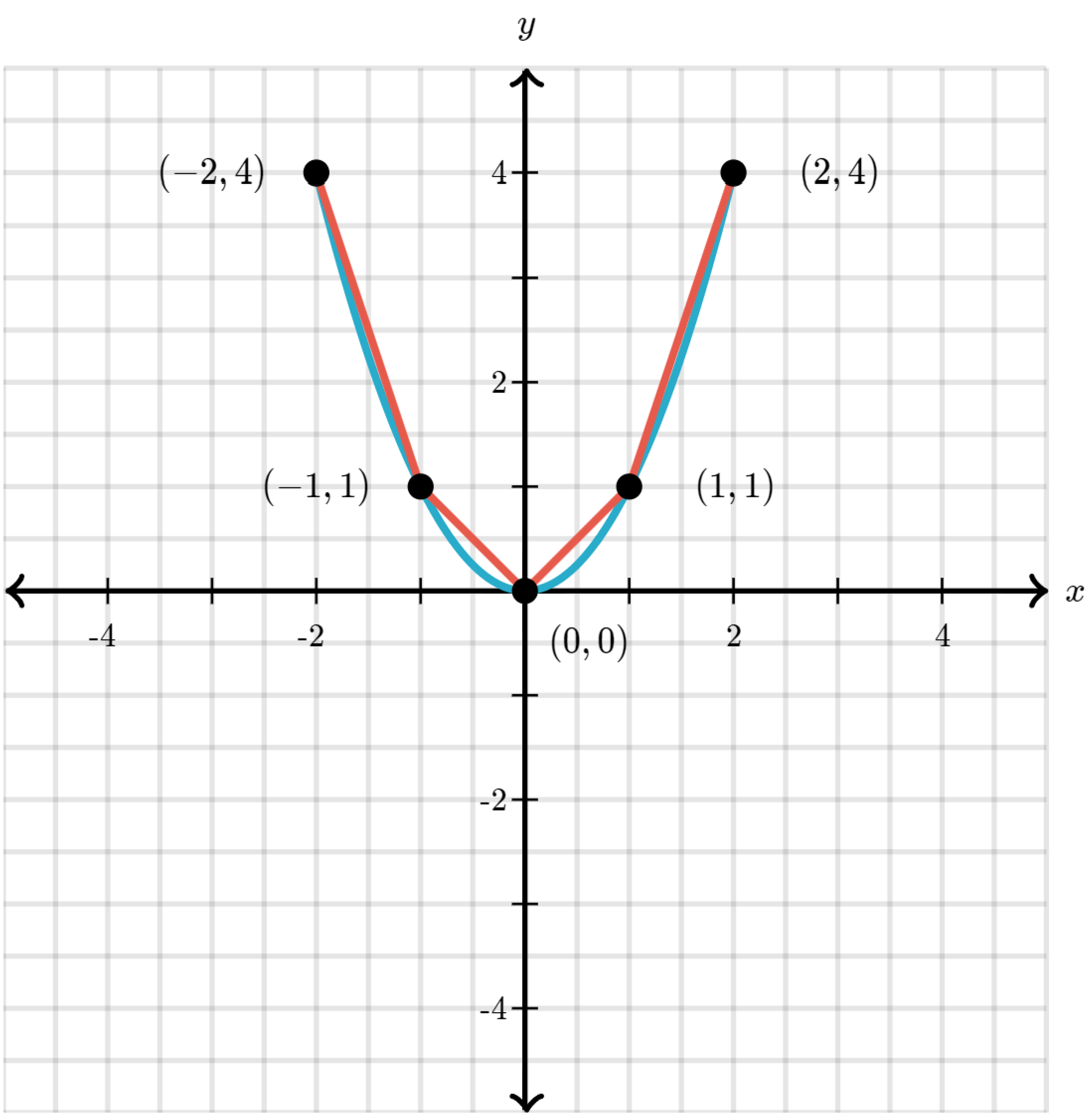
If you had to guess, you could start by approximating this curve with some lines. Here's how that might look:



- A line from $(-2, 4)$ to $(-1, 1)$
- A line from $(-1, 1)$ to $(0, 0)$
- A line from $(0, 0)$ to $(1, 1)$
- A line from $(1, 1)$ to $(2, 4)$

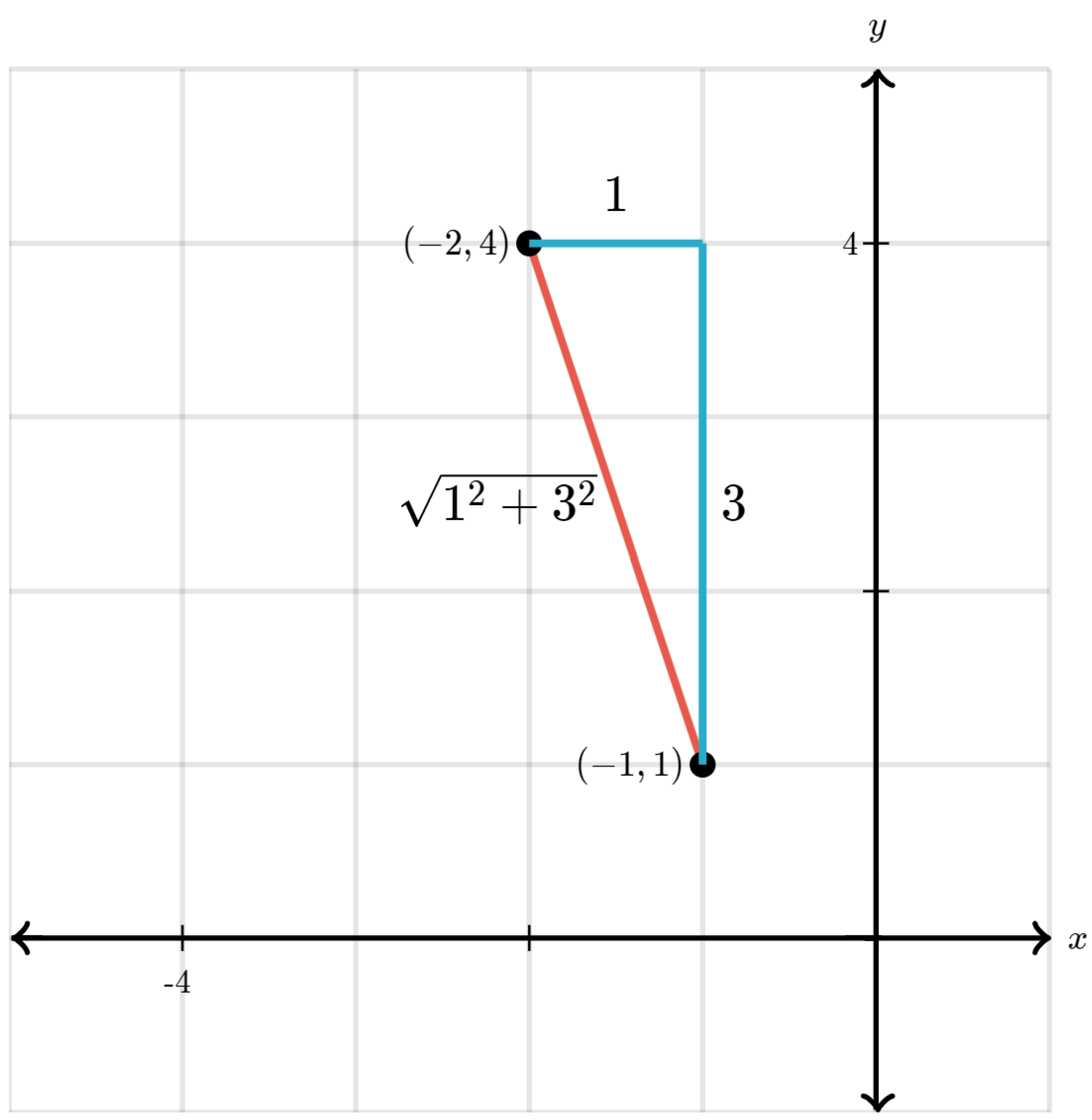
It would be tedious, but you could calculate the length of each line segment using the Pythagorean theorem, then add them up.

Concept check: What is the length of the line from $(-2, 4)$ to $(-1, 1)$?
Enter answer in exact form, with a square root:

Check

[Hide explanation]

Consider the triangle pictured here:



The hypotenuse is the line whose length we wish to calculate. One leg is parallel to the x-axis and has length

$$-1-(-2)=1,$$

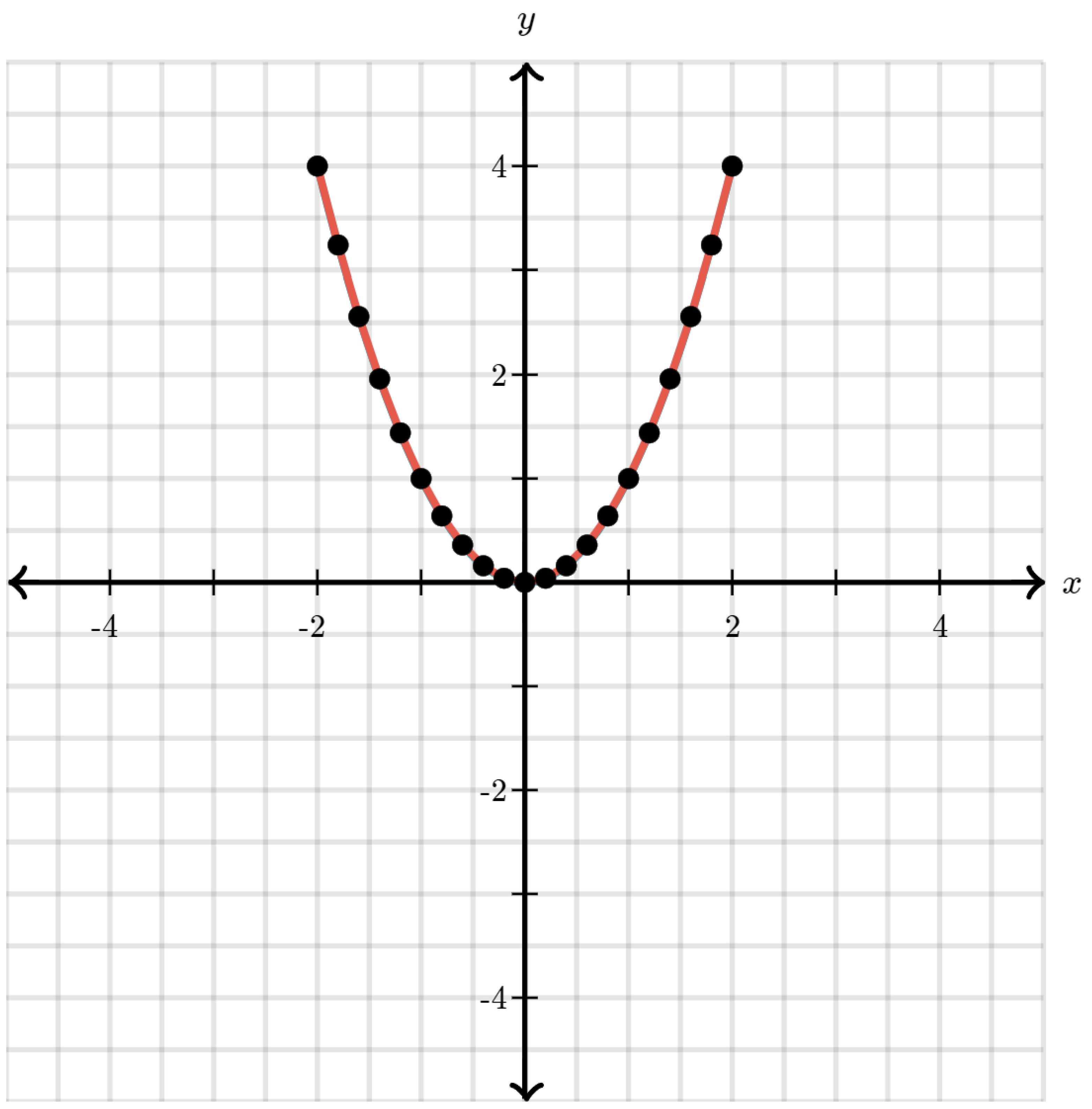
and the other leg is parallel to the y-axis and has length

$$4-1=3.$$

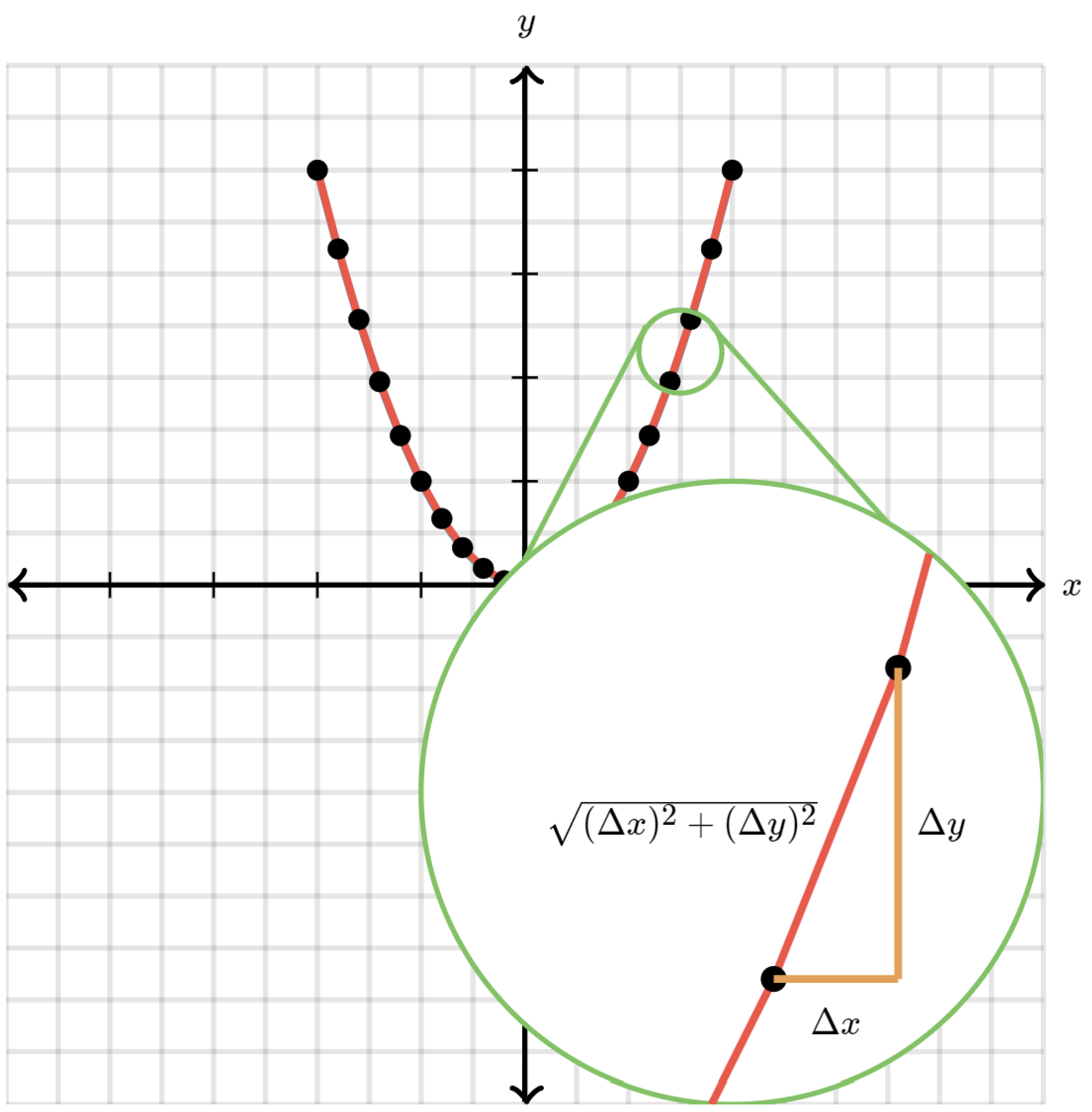
The length of the hypotenuse is therefore

$$\sqrt{1^2+3^2}=\sqrt{10}$$

For an even more accurate estimate, you could approximate the curve with many tiny lines.



Computing all their lengths and adding them up would be painfully mind-numbing, but let's break down what it would actually look like. Zoom in on one of the little lines.



First look at the change in the x value from the start of the line to its end. Let's call that Δx. Similarly, let's say the change in the y-value is Δy. Then, using the Pythagorean theorem, we can write the length of the line as

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Our approximation for the length of the curve will then be the sum of the lengths of all these little lines. When expressing an idea like this with symbols, it's common for writers to be a little loose with the notation and write something like this:

$\sum_{\text{all little lines}} \sqrt{(\Delta x)^2 + (\Delta y)^2}$

[Hide explanation]

You might be used to reading sums like this:

$$\sum_{n=1}^{100} n^2$$

- The variable n is what's known as an **indexing variable**.
- This indexing variable has a specified range, in this case from 1 to 100.
- The portion inside the sum, in this case n^2, is called the **summand**, and it is written in terms of n.

In comparison, the sum I wrote to approximate the length of the parabola is not at all rigorous.

$$\sum_{\text{all little lines}} \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

- It has no indexing variable,
- There is no specified range,
- The summand certainly isn't written in terms of an indexing variable (since there isn't one).

Out of laziness, it is left to you, the intelligent human reader, to understand it in context. Namely, in the context of our discussion, there were many tiny lines approximating the parabola, so I put the words "all little lines" under the Σ. It should also be understood that the value of the summand $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ depends on which specific little line is being considered. Also, the expressions Δx and Δy are not given using formulas, but were instead defined using English words from the paragraphs preceding.

It is important to be able to read loose notation like this and understand what it means. Throughout multivariable calculus, something very similar happens with how integrals are written, where the terms and bounds reference the context of a discussion. In fact, you're about to see one such example in the next section.

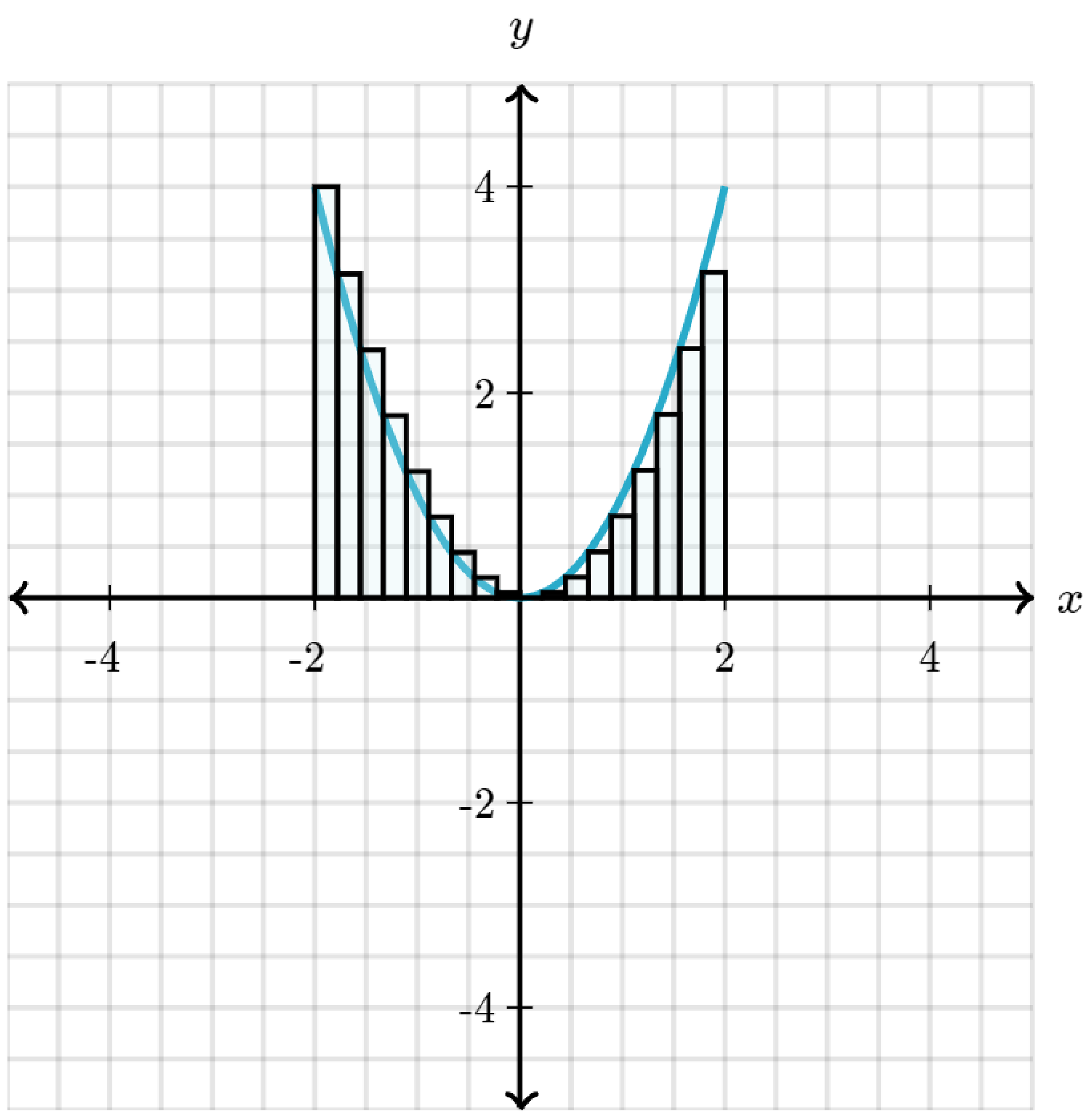
Writers use loose notation like this when the key takeaway point is a broader conceptual idea, not a specific computation. When the time comes to do a computation, though, this kind of notation gets tightened up into something rigorously defined.

Bringing in integrals

Hmm, let's see... we are approximating a curve with many tiny steps, then adding up a very large number of very small things. With tinier steps and a larger sum, we will get a better approximation. Sound familiar?

Problems like this one are exactly what **integrals** were made for.

Most people first learn about integration in the context of computing the area under a curve. You imagine approximating that area with a bunch of thin rectangles. The width of each one is thought of as "dx", some tiny change in the x-value. The height of a rectangle at a given x-value is $f(x)$. Therefore, the area of each rectangle is



$$\overbrace{f(x)}^{\text{height}}\underbrace{dx}_{\text{width}}$$

The full area under the curve is then expressed with an integral:

$$\int_a^b f(x)dx$$

This integral is a powerful machine, like a Σ on steroids. It does not merely sum over the values $f(x)dx$ for a particular tiny value of dx; it considers the limiting value of such a sum as the tiny width dx tends toward 0. In other words, as the approximation using rectangles comes closer and closer to the true area under the curve.

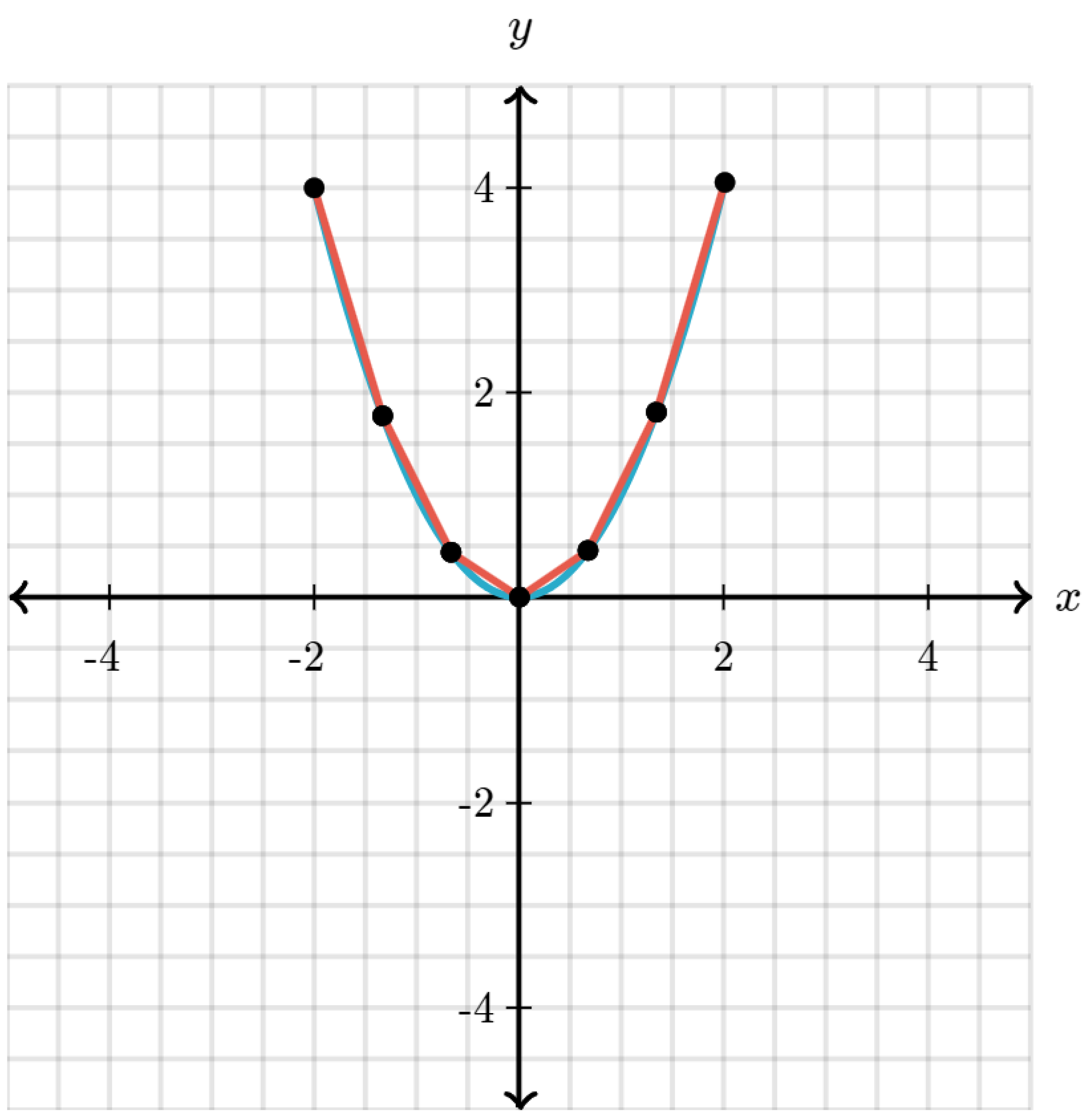
But this powerful machine can be used in many contexts unrelated to the area under a curve. Anytime you get that sensation of wanting to add a very large number of very small things, the integral swoops in to simultaneously make things less tedious and more accurate.

For example, we get that sensation when approximating arc length with the vague sum:

$$\sum_{\text{all little lines}} \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

So we turn it into an integral:

$$\int \sqrt{(dx)^2 + (dy)^2}$$



One thing this notation does not communicate well is that dy, the change in height across one of our little approximation lines, is dependent on dx, the horizontal component of that line. Specifically, since the curve is defined by the relation $y = x^2$, we can take the derivative of each side to see how dy depends on dx,

$$\begin{aligned} y &= x^2 \\ d(y) &= d(x^2) \\ dy &= 2x\,dx \end{aligned}$$

[Hide explanation]

> If this is very strange, consider reviewing implicit differentiation.
>
> If you prefer, you can imagine taking the $\frac{d}{dx}$-style derivative of each side, then rearranging the "fraction" $\frac{dy}{dx}$.
>
> $$\frac{d}{dx}y = \frac{d}{dx}x^2$$
>
> $$\frac{dy}{dx} = 2x$$
>
> $$dy = 2x\,dx$$

When we put this into our integral, it unfolds to look a bit more familiar.

$$\begin{aligned} \int \sqrt{(dx)^2 + (dy)^2} &= \int \sqrt{(dx)^2 + (2x\,dx)^2} \\ &= \int \sqrt{(1 + (2x)^2)(dx)^2} \\ &= \int \sqrt{1 + 4x^2}dx \end{aligned}$$

I've been purposefully lazy about placing bounds on this integral, but now that everything *inside* the integral is in terms of x, with no dy's mucking it up, it makes sense to define the bounds of integration in terms of the x value, which in this case is from -2 to 2.

$$\int_{-2}^{2} \sqrt{1 + 4x^2} dx$$

This looks like something we can compute. Well, actually, in this case, it turns out to be quite a tricky integral, but in this day and age, we can just plug integrals into a computer if we need to. The point is that the idea of approximating our curve's length with little lines, which was at first vaguely written with loose notation, has now turned into a concrete, computable integral.

For now, rather than getting bogged down with the details of this integral (there's plenty of that coming in the next article), I want to highlight some points from this example.

Takeaways

- The central expression to remember is $\sqrt{dx^2 + dy^2}$, which represents a tiny unit of arc length in terms of x and y.

- The arc length integral you set up starts its life looking like this:

$$\int \sqrt{(dx)^2 + (dy)^2}$$

- Before computing the integral, we had to write the differential dy in terms of the differential dx. To do this, we took the derivative of the function defining the curve.

- In general, an integral can only be computed with respect to a single differential, and finding relations between differentials can be done using the derivative.

- Maybe the most important lesson to take away from this is that integrals can be used to do things *other* than finding the area under a curve.

Practice

To solidify your understanding, you can practice more arc length problems in the next article.