Contents

Chapter 1

L^p Spaces Associated with a Von Neumann Algebra

In this chapter, we present Haagerup's theory of L^p spaces associated with a von Neumann algebra.

Let M be a von Newmann algebra and let φ_0 be a normal faithful semifinite weight on M.

We denote by N the crossed product $R(M, \sigma^{\varphi_0})$ of M by the modular automorphism group σ^{φ_0} associated with φ_0. Recall [18, Section 3; 8, Section 5] that if M is given on a Hilbert space H, then N is the Von Neumann algebra on the Hilbert space $L^2(\mathbb{R}, H)$ generated by the operators $\pi(x), x \in M$, and $\lambda(s)$, $s \in \mathbb{R}$, defined by

$$(\pi(x)\xi)(t) = \sigma_{-t}^{\varphi_0}(x)\xi(t), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}, \tag{1}$$

$$(\lambda(s)\xi)(t) = \xi(t-s), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}. \tag{2}$$

We identify M with its image $\pi(M)$ in N (recall that π normal faithful representation of M).

We denote by θ the dual action of $\mathbb{R}$ in N. The θ_s, $s \in \mathbb{R}$, are automorphisms of N characterized by

$$\theta_s x = x, x \in M \tag{3}$$

$$\theta_s \lambda(t) = e^{-ist}\lambda(t), t \in \mathbb{R}. \tag{4}$$

By (3), M is contained in the set of fixed points under θ. Actually

$$M = \{y \in N | \forall s \in \mathbb{R} : \theta_s y = y\} \tag{5}$$

(see e.g. [5, Lemma 3.6]).

The θ_s, $s \in \mathbb{R}$, naturally extend to automorphisms, still denoted θ_s, of $\hat{N}_+$, the extended positive part of N [7, Section 1]. Recall [8, Lemma 5.2] that the formula

$$Tx = \int_{\mathbb{R}} \theta_s(x)\mathrm{d}s, x \in N_+, \tag{6}$$

defines a normal faithful semifinite operator valued weight T from N to M in the following sense: for each $x \in N_+$, Tx is the element of $\hat{N}_+$ characterized by

$$\langle Tx, \chi\rangle = \int_{\mathbb{R}} \langle \theta_s(x), \chi\rangle \mathrm{d}s \tag{7}$$

for all $x \in N_*^+$. Note that

$$\forall s \in \mathbb{R} : \theta_s \circ T = T. \tag{8}$$

In view of (5), this formula implies that the values of T are actually in $\hat{M}_+$.

For each normal weight φ on M, we put

$$\tilde{\varphi} = \hat{\varphi} \circ T \tag{9}$$

where $\hat{\varphi}$ denotes the extension of φ to a normal weight on $\hat{M}_+$ as described in [7, Proposition 1.10]. Then $\tilde{\varphi}$ is a normal weight on N [7,Proposition 2.3]; $\tilde{\varphi}$ is called the dual weight of φ (see [6, Introduction + Section 1)]. Note that (8) and (9) imply

$$\forall s \in \mathbb{R} : \tilde{\varphi} \circ \theta_s = \tilde{\varphi}. \tag{10}$$

If φ and ψ are normal faithful semifinite weights, then so are $\tilde{\varphi}$ and $\tilde{\psi}$, and we have [7, Theorem 4.7]:

$$\forall t \in \mathbb{R} \forall x \in M : \sigma_t^{\tilde{\varphi}}(x) = \sigma_t^{\varphi}(x), \tag{11}$$

$$\forall t \in \mathbb{R} : (D\tilde{\varphi} : D\tilde{\psi})_t = (D\varphi : D\psi)_t. \tag{12}$$

Lemma 1. *1) The mapping*

$$\varphi \mapsto \tilde{\varphi}$$

is a bijection of the set of all normal semifinite weights on M onto the set of normal semifinite weights ψ on N satisfying

$$\forall s \in \mathbb{R} : \psi \circ \theta_s = \psi. \tag{13}$$

2) For all normal weights φ and ψ on M and all $x \in M$, we have

1. $(\varphi + \psi)^{\sim} = \tilde{\varphi} + \tilde{\psi}$*,*
2. $(x \cdot \varphi \cdot x^*)^{\sim} = x \cdot \tilde{\varphi} \cdot x^*$*,*
3. $\operatorname{supp} \tilde{\varphi} = \operatorname{supp} \varphi$*.*

Proof. That $\tilde{\varphi}$ is semifinite if φ is follows from the proof of [7, Proposition 2.3]. That $\varphi \mapsto \tilde{\varphi}$ is injective follows from the formula

$$\varphi(\dot{T}x) = \tilde{\varphi}(x), x \in m_T,$$

and the fact that $\dot{T}(m_T)$ is σ-weakly dense in M [7, Proposition 2.5].

Now let us prove 2). First observe that $(\varphi + \psi)^{\wedge} = \hat{\varphi} + \hat{\psi}$ since $\hat{\varphi} + \hat{\psi} : \hat{M} \to [0, \infty]$ obviously satisfies the properties that characterize $(\varphi + \psi)^{\wedge}$ ([7, Proposition 1.10]); (a) follows trivially. Similarly, $(x \cdot \varphi \cdot x^*)^{\wedge} = x \cdot \hat{\varphi} \cdot x^*$, whence (b).

To prove (c), put $p_0 = 1 - \operatorname{supp} \varphi$. Then Mp_0 is the σ-weak closure in M of $N_\varphi = \{x \in M | \varphi(x^* x) = 0\}$. Similarly, the σ-weak closure in N of $N_{\tilde{\varphi}} = \{y \in N | \tilde{\varphi}(y^* y) = 0\}$ is Nq_0 where $q_0 = 1 - \operatorname{supp} \tilde{\varphi}$. Now

$$n_T N_\varphi \subset N_{\tilde{\varphi}}$$

since

$$\begin{aligned}\forall y \in n_T \forall x \in N_\varphi : & \tilde{\varphi}(x^* y^* y x) = \varphi(T(x^* y^* y x)) \\ =& \varphi(x^* T(y^* y) x) \leq \|T(y^* y)\| \varphi(x^* x) = 0.\end{aligned}$$

As n_T is σ-weakly dense in N, it follows that

$$N_\varphi \subset \overline{N_{\tilde{\varphi}}}^{\sigma-w}$$

whence

$$p_0 \leq q_0.$$

Note that we must have $q_0 \in M$ since $\tilde{\varphi}$, and hence $\operatorname{supp}\tilde{\varphi}$, is θ-invariant. Thus to conclude that $p_0 = q_0$ we need only show that $\varphi(q_0) = 0$. This follows from

$$\forall x \in m_T : \varphi(q_0\dot{T}(x)q_0) = \varphi(\dot{T}(q_0 x q_0)) = \tilde{\varphi}(q_0 x q_0) = 0$$

and the fact that $\dot{T}(m_T)$ is σ-weakly dense in M [7, Proposition 2.5].

We now return to 1). Let ψ be a normal semifinite weight on N satisfying (13). First suppose that ψ is also faithful. Then by [5, (proof of) Theorem 3.7), it follows that $\psi = \tilde{\varphi}$ for some normal faithful semifinite φ on M.

In the general case, put $q_0 = 1 - \operatorname{supp}\psi$. Then by (13) and (5), we have $q_0 \in M$. Now take any normal semifinite weight χ_0 on M $\square$