Contents

Chapter 1

Spatial L^p Spaces

In this chapter, we describe the Connes/Hilsum construction of spatial L^p spaces.

Let M be a von Neumann algebra acting on a Hilbert space H and let ψ_0 be a normal faithful semifinite weight on the commutant M' of M.

The notation is as in Chapter II and III.

Definition 1. *For each positive self-adjoint* (-1)*-homogeneous operator* a *we define the integral with respect to* ψ_0 *by*

$$\int a \mathrm{d}\psi_0 = \varphi(1), \tag{1}$$

where φ *is the (unique) normal semifinite weight on* M *such that* $a = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0}$.

Notation. *For each* $p \in [1, \infty]$, *we denote by*

$$\overline{M}_{-1/p}$$

the set of closed densely defined $(-1/p)$*-homogeneous operators on* H.

Definition 2. *Let* $p \in [1, \infty[$. *We put*

$$L^p(\psi_0) = L^p(M, H, \psi_0) = \{a \in \overline{M}_{-1/p} | \int |a|^p \mathrm{d}\psi_0 < \infty\} \tag{2}$$

and

$$\|a\|_p = \left(\int |a|^p \mathrm{d}\psi_0 \right)^{\frac{1}{p}}, a \in L^p(\psi_0). \tag{3}$$

For $p = \infty$, we put

$$L^\infty(\psi_0) = M \tag{4}$$

and write $\|\cdot\|_\infty$ for the usual operator norm on M.

Note that when a is $(-1/p)$-homogeneous, the operator $|a|^p$ is (-1)-homogeneous so that the integral occurring at the right hand side of (2) is defined.

The spaces $L^p(\psi_0)$ are called spatial L^p spaces (as opposed to the abstract L^p spaces of Haagerup).

We now follow the first part of [10] to describe the relationship between the $L^p(\psi_0)$ and Haagerup's $L^p(M)$.

Let φ_0 be a normal faithful semifinite weight on M. Put

$$d_0 = \frac{\mathrm{d}\varphi_0}{\mathrm{d}\psi_0}. \tag{5}$$

Then

$$\forall t \in \mathbb{R} \forall x \in M : \sigma_t^{\varphi_0}(x) = d_0^{it} x d_0^{-it}. \tag{6}$$

We define a unitary operator u_0 on the Hilbert space $L^2(\mathbb{R}, H)$ by

$$(u_0\xi)(t) = d_0^{it}\xi(t), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}. \tag{7}$$

Recall that the crossed product $N = R(M, \sigma^{\varphi_0})$ is generated by the elements $\pi(x), x \in M$, and $\lambda(s), s \in \mathbb{R}$, as described in the beginning of Chapter II. We shall describe the action of $u_0(\cdot)u_0^*$ on these generating elements.