Contents

Chapter 1

Spatial Derivatives

Spatial derivatives were introduced by A. Connes in [1]. In this chapter, we give an alternative definition (equivalent to that given in [1]) suggested to us by U. Haagerup, based on the notion of the extended positive part of a von Neumann algebra. This definition permits us to obtain very easily some elementary properties of spatial derivatives. After this, we recall their main modular properties and the characterization as (-1)-homogeneous operators.

1.1 Definition and elementary properties of spatial derivatives

Let M be a von Neumann algebra acting on a Hilbert space H, and let ψ be a normal faithful semifinite weight on the commutant M' of M.

We shall use the following standard notation: $n_\psi = \{y \in M' | \psi(y^*y) < \infty\}$, H_ψ the Hilbert space completion of n_ψ with respect to the inner product $(y_1, y_2) \mapsto \psi(y_2^* y_1)$, Λ_ψ the canonical injection of n_ψ into H_ψ, π_ψ the canonical representation of M' on H_ψ.

Definition 1. *For each $\xi \in H$, we denote by $R^\psi(\xi)$ the (densely defined) operator from H_ψ to H defined by*

$$R^\psi(\xi)\Lambda_\psi(y) = y\xi, y \in n_\psi. \tag{1}$$

Proposition 2. *For all* $\xi, \xi_1, \xi_2 \in H$*,* $x \in M$*, and* $y \in M'$ *we have*

(i) $R^\psi(\xi_1 + \xi_2) = R^\psi(\xi_1) + R^\psi(\xi_2)$*,*

(ii) $R^\psi(x\xi) = xR^\psi(\xi)$*,*

(iii) $yR^\psi(\xi) \subset R^\psi(\xi)\pi_\psi(y)$*,*

and

*(i)** $R^\psi(\xi_1)^* + R^\psi(\xi_2)^* \subset R^\psi(\xi_1 + \xi_2)^*$*,*

*(ii)** $R^\psi(x\xi)^* = R^\psi(\xi)^* x^*$*,*

*(iii)** $\pi_\psi(y)R^\psi(\xi)^* \subset R^\psi(\xi)^* y$*.*

Proof. (i) and (ii) are immediate from Definition 1. (iii): For all $z \in n_\psi$, we have $yR^\psi(\xi)\Lambda_\psi(z) = yz\xi = R^\psi(\xi)\Lambda_\psi(yz) = R^\psi(\xi)\pi_\psi(y)\Lambda_\psi(z)$.

(i)*, (ii)*, and (iii)* follow from (i), (ii), and (iii) using $R^\psi(\xi_1) + R^\psi(\xi_2) \subset (R^\psi(\xi_1) + R^\psi(\xi_2))^*$, $(xR^\psi(\xi))^* = R^\psi(\xi)^* x^*$, and $(y^* R^\psi(\xi))^* = R^\psi(\xi)^* y^*$. $\square$

Definition 3. *A vector* $\xi \in H$ *is called* ψ*-bounded if the operator* $R^\psi(\xi)$ *is bounded. The set of* ψ*-bounded vectors is denoted* $D(H, \psi)$*.*

Notation. *If* $\xi \in D(H, \psi)$*,* $R^\psi(\xi)$ *extends to a bounded operator* $H_\psi \to H$ *which we shall also denote* $R^\psi(\xi)$*.*

Proposition 4. *The set* $D(H, \psi)$ *is an* M*-invariant dense subspace of* H*.*

Proof. That $D(H, \psi)$ is an M-invariant subspace of H follows from Proposition 2, i) and (ii). Denote by e the projection onto $\overline{D(H, \psi)}$; then $e \in M'$. Suppose that $e \neq 1$. Then $\psi(1 - e) > 0$. We can write $\psi = \sum_{i \in I} \omega_{\zeta_i, \zeta_i}$ for certain $\zeta_i \in H$. Then for at least one ζ_i, we have $((1 - e)\zeta_i | \zeta_i) \neq 0$ so that $(1 - e)\zeta_i \neq 0$. On the other hand, we have

$$\forall y \in n_\psi : \|y\zeta_i\|^2 \leq \psi(y^* y) = \|\Lambda_\psi(y)\|^2$$

so that $\zeta_i \in D(H, \psi)$ and hence $e\zeta_i = \zeta_i$. This is a contradiction. Hence we must have $e = 1$ and $D(H, \psi)$ is dense in H. $\square$

Let $\xi \in H$. By Proposition 2, (iii)*, $D(R^{\psi}(\xi)^*)$ is invariant under the action of M'. Hence the projection p onto $\overline{D(R^{\psi}(\xi)^*)}$ is in M. Considered as an operator from pH to H_{ψ}, $R^{\psi}(\xi)^*$ is closed and densely defined and hence $\left|R^{\psi}(\xi)^*\right|^2$ exists as a positive self-adjoint operator on pH which by Proposition 2, (iii)*, is affiliated with pMp. We denote by $\theta^{\psi}(\xi,\xi)$ the element of $\widehat{M}_+$ (the extended positive part of M) associated with the couple $(pH, \left|R^{\psi}(\xi)^*\right|^2)$ as in [7, Example 1.2 and Lemma 1.4], i.e.

Definition 5. *For each $\xi \in H$, we denote by*

$$\theta^{\psi}(\xi,\xi)$$

the element of $\widehat{M}_+$ characterized by

$$\forall \eta \in H : \langle \omega_{\eta,\eta}, \theta^{\psi}(\xi,\xi)\rangle = \begin{cases} \left\|R^{\psi}(\xi)^*\eta\right\|^2 & \text{if } \eta \in D(R^{\psi}(\xi)^*) \\ \infty & \text{otherwise} \end{cases} . \tag{2}$$

Remark 6. *If $\xi \in D(H,\psi)$, we simply have*

$$\theta^{\psi}(\xi,\xi) = R^{\psi}(\xi)R^{\psi}(\xi)^*. \tag{3}$$

Proposition 7. *For all $\xi \in H$ and $x \in M$, we have*

$$\theta^{\psi}(x\xi,x\xi) = x \cdot \theta^{\psi}(\xi,\xi) \cdot x^*.$$

Proof. For all $\eta \in H$, we have, using Proposition 2, (ii)*, and Definition 5

$$\begin{aligned} \langle \omega_{\eta,\eta}, \theta^{\psi}(x\xi,x\xi)\rangle &= \langle \omega_{x^*\eta,x^*\eta}, \theta^{\psi}(\xi,\xi)\rangle \\ &= \langle x^* \cdot \omega_{\eta,\eta} \cdot x, \theta^{\psi}(\xi,\xi)\rangle \\ &= \langle \omega_{\eta,\eta}, x \cdot \theta^{\psi}(\xi,\xi) \cdot x^*\rangle \end{aligned}$$

where the last equality simply follows from the definition of the operation $m \mapsto x \cdot m \cdot x^*$ in $\widehat{M}_+$. □

Recall that by [7, Proposition 1.10], every normal weight φ has a unique extension, also denoted φ, to a normal weight on $\widehat{M}_+$.

Definition 8. *Let φ be a normal weight on M. We define*

$$q_\varphi : H \to [0, \infty]$$

by

$$q_\varphi(\xi) = \langle \varphi, \theta^\psi(\xi, \xi) \rangle, \xi \in H. \tag{4}$$

Proposition 9. *Let φ be a normal weight on M. Then q_φ is a l.s.c. quadratic form on M, i.e.*

(i) $\forall \xi_1, \xi_2 \in H : q_\varphi(\xi_1 + \xi_2) + q_\varphi(\xi_1 - \xi_2) = 2q_\varphi(\xi_1) + 2q_\varphi(\xi_2)$,

(ii) $\forall \xi \in H \forall \lambda \in \mathbb{C} : q_\varphi(\lambda\xi) = |\lambda|^2 q_\varphi(\xi)$,

(iii) q_φ *is lower semi-continuous.*

Proof. (ii) is immediate. For the proof of (i) and (iii), first suppose that $\varphi = \omega_{\eta,\eta}$ for some $\eta \in H$. Then

$$q_\varphi(\xi) = \langle \omega_{\eta,\eta}, \theta^\psi(\xi, \xi) \rangle = \begin{cases} \left\| R^\psi(\xi)^* \eta \right\|^2 & \text{if } \eta \in D(R^\psi(\xi)^*) \\ \infty & \text{otherwise} \end{cases}. \tag{5}$$

Let $\xi_1, \xi_2 \in H$. We shall prove that

$$q_\varphi(\xi_1 + \xi_2) + q_\varphi(\xi_1 - \xi_2) \le 2q_\varphi(\xi_1) + 2q_\varphi(\xi_2). \tag{6}$$

If either $\eta \in D(R^\psi(\xi_1)^*)$ or $\eta \in D(R^\psi(\xi_2)^*)$, the right hand side of (6) is $+\infty$ and hence (6) holds. Now suppose that $\eta \in D(R^\psi(\xi_1)^*)$ and $\eta \in D(R^\psi(\xi_2)^*)$. Then by Proposition 2, (i)*, also $\eta \in D(R^\psi(\xi_1 + \xi_2)^*)$ and $\eta \in D(R^\psi(\xi_1 - \xi_2)^*)$. Furthermore,

$$\begin{aligned} &\left\| R^\psi(\xi_1 + \xi_2)^* \eta \right\|^2 + \left\| R^\psi(\xi_1 - \xi_2)^* \eta \right\|^2 \\ =&\left\| R^\psi(\xi_1)^* \eta + R^\psi(\xi_2)^* \eta \right\|^2 + \left\| R^\psi(\xi_1)^* \eta - R^\psi(\xi_2)^* \eta \right\|^2 \\ =&2\left\| R^\psi(\xi_1)^* \eta \right\|^2 + 2\left\| R^\psi(\xi_2)^* \eta \right\|^2. \end{aligned}$$

Thus we have proved (6) in all cases.

By (6) applied to $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ we get

$$4(q_\varphi(\xi_1) + q_\varphi(\xi_2)) = q_\varphi(2\xi_1) + q_\varphi(2\xi_2) \le 2q_\varphi(\xi_1 + \xi_2) + 2q_\varphi(\xi_1 - \xi_2).$$

In all, we have shown (i).

By (5), we have

$$\begin{aligned}\langle \omega_{\eta,\eta}, \theta^{\psi}(\xi,\xi)\rangle =& \sup\{\left|(R^{\psi}(\xi)^{*}\eta|\zeta)\right|^{2}|\zeta \in D(R^{\psi}(\xi)), \|\zeta\| \leq 1\} \\ =& \sup\{\left|(\eta|R^{\psi}(\xi)\Lambda_{\psi}(y))\right|^{2}|y \in n_{\psi}, \|\Lambda_{\psi}(y)\| \leq 1\} \\ =& \sup\{|(\eta|y\xi)|^{2}|y \in n_{\psi}, \|\Lambda_{\psi}(y)\| \leq 1\}\end{aligned}$$

for all $\xi \in H$. Since each $\xi \mapsto |(\eta|y\xi)|^2$ is continuous, this implies (iii).

Now let φ be an arbitrary normal weight. Then we can write

$$\varphi = \sum_{i \in I} \omega_{\eta_i,\eta_i}$$

and thus (cf. the proof of [7, Proposition 1.10])

$$\forall \xi \in H : q_{\varphi}(\xi) = \langle \varphi, \theta^{\psi}(\xi,\xi)\rangle = \sum_{i=I} \langle \omega_{\eta_i,\eta_i}, \theta^{\psi}(\xi,\xi)\rangle.$$

Now (i) and (iii) follow by the first part of the proof. □

Remark 10. *Let φ be a normal weight on M. Write*

$$\mathrm{Dom}(q_{\varphi}) = \{\xi \in H | q_{\varphi}(\xi) < \infty\}. \tag{7}$$

Then for all $x \in n_{\varphi}$ and $\xi \in D(H,\psi)$, we have

$$x^{*}\xi \in \mathrm{Dom}(q_{\varphi}). \tag{8}$$

Indeed,

$$\begin{aligned}q_{\varphi}(x^{*}\xi) =& \langle \varphi, \theta^{\psi}(x^{*}\xi, x^{*}\xi)\rangle \\ =& \langle \varphi, x^{*} \cdot \theta^{\psi}(\xi,\xi) \cdot x\rangle \\ \leq& \left\|\theta^{\psi}(\xi,\xi)\right\| \langle \varphi, x^{*}x\rangle < \infty.\end{aligned}$$

In particular, if φ is semifinite then $\mathrm{Dom}(q_{\varphi})$ *is dense in H (since n_{φ}^{*} is strongly dense in M).*

Definition 11. *For each normal weight φ on M, we define the spatial derivative $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}$ as the unique element of $\widehat{B(H)}_+$ such that*

$$\forall \xi \in H : \langle \omega_{\xi,\xi}, \frac{\mathrm{d}\varphi}{\mathrm{d}\psi} \rangle = \langle \varphi, \theta^{\psi}(\xi,\xi) \rangle. \tag{9}$$

The existence of $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}$ follows from Proposition 9 and [7, proof of Lemma 1.4].

Remark 12. *If φ is semifinite, $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}$ is simply a positive self-adjoint operator on H (since in this case, $\{\xi \in H | \langle \omega_{\xi,\xi}, \frac{\mathrm{d}\varphi}{\mathrm{d}\psi} \rangle < \infty\} = \mathrm{Dom}(q_\varphi)$ is dense in H). Note that*

$$\forall \xi \in H : q_\varphi(\xi) = \begin{cases} \left\| \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\right)^{\frac{1}{2}} \xi \right\|^2 & if \xi \in D\left(\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\right)^{\frac{1}{2}}\right) \\ \infty & otherwise \end{cases}. \tag{10}$$

We shall see below (Proposition 21 that the definition of $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}$ given here agrees with that given in [1]. (This is not quite obvious. Note that in [1, Lemma 6], the quadratic form q is only defined on the subspace $D(H,\psi)$, and then extended by [1, Lemma 5] to the whole of H.)

Lemma 13. *Let $\varphi_1, \varphi_2, (\varphi_i)_{i\in I}$, and φ be normal weights on M and let $x \in M$. Then*

(i) $\forall m \in \widehat{M}_+ : \langle \varphi_1 + \varphi_2, m \rangle = \langle \varphi_1, m \rangle + \langle \varphi_2, m \rangle$,

(ii) $\forall m \in \widehat{M}_+ : \langle x \cdot \varphi \cdot x^*, m \rangle = \langle \varphi, x^* \cdot m \cdot x \rangle$,

(iii) if $\varphi_i \nearrow \varphi$, *then* $\forall m \in \widehat{M}_+ : \langle \varphi_i, m \rangle \nearrow \langle \varphi, m \rangle$.

Proof. (i) and (ii) are immediate consequences of [7, Proposition 1.10] (or its proof). As for (iii), we have by the proof of [7, Proposition 1.10], using the notation from there,

$$\begin{aligned} \langle \varphi_i, m \rangle =& \sup_n \langle \varphi_i, \int_0^n \lambda \mathrm{d}e_\lambda \rangle + \infty \cdot \varphi_i(p) \\ \nearrow& \sup_n \langle \varphi, \int_0^n \lambda \mathrm{d}e_\lambda \rangle + \infty \cdot \varphi_i(p) = \langle \varphi, m \rangle. \end{aligned}$$

□

Theorem 14. *For all normal weights φ_1, φ_2, and φ on M and all $x \in M$ we have*

(a) $\frac{\mathrm{d}(\varphi_1+\varphi_2)}{\mathrm{d}\psi} = \frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi} + \frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi}$,

(b) $\frac{\mathrm{d}(x\cdot\varphi\cdot x^*)}{\mathrm{d}\psi} = x \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}\psi} \cdot x^*$.

Remark 15. *The sums and products occurring at the right hand side of (a) and (b) are to be understood in the sense of the operations in $\widehat{B(H)}_+$. In particular, if φ_1, φ_2, $\varphi_1+\varphi_2$ are semifinite, $\frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi} + \frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi}$ is the form sum of the positive self-adjoint operators $\frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi}$ and $\frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi}$. Similarly, if $x\cdot\varphi\cdot x^*$ is semifinite, $x\cdot\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\cdot x^*$ is the form product.*

Remark 16. *In [1], the sum property is simply stated without proof. It seems to be difficult to give a proof using only the methods of [1] (one only gets "$\geq$"). - The product property is stated (and proved) only for invertible $x \in M$.*

Proof of Theorem 14. Let $\xi \in H$. Then, using successively Definition 11, Lemma 13, Definition 11 again, and the definition of the sum in $\widehat{B(H)}_+$, we get

$$\begin{aligned}
\langle \omega_{\xi,\xi}, \frac{\mathrm{d}(\varphi_1+\varphi_2)}{\mathrm{d}\psi}\rangle =& \langle \varphi_1+\varphi_2, \theta^\psi(\xi,\xi)\rangle \\
=& \langle \varphi_1, \theta^\psi(\xi,\xi)\rangle + \langle \varphi_2, \theta^\psi(\xi,\xi)\rangle \\
=& \langle \omega_{\xi,\xi}, \frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi}\rangle + \langle \omega_{\xi,\xi}, \frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi}\rangle \\
=& \langle \omega_{\xi,\xi}, \frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi} + \frac{\mathrm{d}\varphi_2}{\mathrm{d}\psi}\rangle.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle \omega_{\xi,\xi}, \frac{\mathrm{d}(x\cdot\varphi\cdot x^*)}{\mathrm{d}\psi}\rangle =& \langle x\cdot\varphi\cdot x^*, \theta^\psi(\xi,\xi)\rangle = \langle \varphi, x^*\cdot\theta^\psi(\xi,\xi)\cdot x\rangle \\
=& \langle \varphi, \theta^\psi(x^*\xi, x^*\xi)\rangle = \langle \omega_{x^*\xi,x^*\xi}, \frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\rangle \\
=& \langle x^*\cdot\omega_{\xi,\xi}\cdot x, \frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\rangle = \langle \omega_{\xi,\xi}, x\cdot\frac{\mathrm{d}\varphi_1}{\mathrm{d}\psi}\cdot x^*\rangle
\end{aligned}$$

where we have used Lemma 13 and Proposition 7. □

Theorem 17. *Let $(\varphi_i)_{i\in I}$ and φ be normal weights on M. Suppose that*

$$\varphi_i \nearrow \varphi.$$

Then

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}\psi} \nearrow \frac{\mathrm{d}\varphi}{\mathrm{d}\psi}.$$

Remark 18. *In particular, if φ is semifinite, we have $\frac{\mathrm{d}\varphi_i}{\mathrm{d}\psi} \nearrow \frac{\mathrm{d}\varphi}{\mathrm{d}\psi}$ in the usual sense of positive self-adjoint operators.*

Proof of Theorem 17. For all $\xi \in H$, we have by Lemma 13

$$\begin{aligned}\langle \omega_{\xi,\xi}, \frac{\mathrm{d}\varphi_i}{\mathrm{d}\psi}\rangle =& \langle \varphi_i, \theta^\psi(\xi,\xi)\rangle \\ & \nearrow \langle \varphi, \theta^\psi(\xi,\xi)\rangle = \langle \omega_{\xi,\xi}, \frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\rangle.\end{aligned}$$

□

Lemma 19. *Let φ be a normal semifinite weight on M. Write $p = \operatorname{supp}\varphi$. Then for all $m \in \widehat{M}_+$, we have*

$$\langle \varphi, m\rangle = 0 \Leftrightarrow p\cdot m\cdot p = 0.$$

Proof. Let $m = \int_0^\infty \lambda \mathrm{d}e_\lambda + \infty\cdot(1-r)$ be the spectral resolution of m. Put $x_n = \int_0^n \lambda \mathrm{d}e_\lambda, n \in \mathbb{N}$. Then

$$\begin{aligned}\langle \varphi, m\rangle = 0 \Leftrightarrow & \forall n \in \mathbb{N} : \langle \varphi, x_n\rangle = 0 \text{ and } \langle \varphi, 1-r\rangle = 0 \\ & \forall n \in \mathbb{N} : p\cdot x_n \cdot p = 0 \text{ and } p\cdot(1-r)\cdot p = 0 \\ & \Leftrightarrow p\cdot m\cdot p = 0.\end{aligned}$$

□

Theorem 20. *Let φ be a normal semifinite weight on M. Then*

$$\operatorname{supp}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\right) = \operatorname{supp}(\varphi). \tag{11}$$

In particular, $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}$ is injective if and only if φ is faithful.

Proof. Put $p = \operatorname{supp}\varphi \in M$. Now for all $\xi \in H$, we have, using Lemma 19 and Proposition 7:

$$\begin{aligned}
\xi \in \ker(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}) &\Leftrightarrow \langle \omega_{\xi,\xi}, \frac{\mathrm{d}\varphi}{\mathrm{d}\psi} \rangle = 0 \\
&\Leftrightarrow \langle \varphi, \theta^{\psi}(\xi,\xi) \rangle = 0 \\
&\Leftrightarrow p \cdot \theta^{\psi}(\xi,\xi) \cdot p = 0 \\
&\Leftrightarrow \theta^{\psi}(p\xi, p\xi) = 0 \\
&\Leftrightarrow p\xi = 0 \\
&\Leftrightarrow \xi \in (1-p)H.
\end{aligned}$$

Since $ker\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\right) = \operatorname{supp}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}\right)^{\perp}$, the result follows. $\square$

Proposition 21.