Contents

Chapter 1

Spatial L^p Spaces

In this chapter, we describe the Connes/Hilsum construction of spatial L^p spaces.

Let M be a von Neumann algebra acting on a Hilbert space H and let ψ_0 be a normal faithful semifinite weight on the commutant M' of M.

The notation is as in Chapter II and III.

Definition 1. *For each positive self-adjoint* (-1)*-homogeneous operator* a *we define the integral with respect to* ψ_0 *by*

$$\int a \mathrm{d}\psi_0 = \varphi(1), \tag{1}$$

where φ *is the (unique) normal semifinite weight on* M *such that* $a = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0}$.

Notation. *For each* $p \in [1, \infty]$, *we denote by*

$$\overline{M}_{-1/p}$$

the set of closed densely defined $(-1/p)$*-homogeneous operators on* H.

Definition 2. *Let* $p \in [1, \infty[$. *We put*

$$L^p(\psi_0) = L^p(M, H, \psi_0) = \{a \in \overline{M}_{-1/p} | \int |a|^p \mathrm{d}\psi_0 < \infty\} \tag{2}$$

and

$$\|a\|_p = \left(\int |a|^p \mathrm{d}\psi_0 \right)^{\frac{1}{p}}, a \in L^p(\psi_0). \tag{3}$$

For $p = \infty$, we put

$$L^\infty(\psi_0) = M \tag{4}$$

and write $\|\cdot\|_\infty$ for the usual operator norm on M.

Note that when a is $(-1/p)$-homogeneous, the operator $|a|^p$ is (-1)-homogeneous so that the integral occurring at the right hand side of (2) is defined.

The spaces $L^p(\psi_0)$ are called spatial L^p spaces (as opposed to the abstract L^p spaces of Haagerup).

We now follow the first part of [10] to describe the relationship between the $L^p(\psi_0)$ and Haagerup's $L^p(M)$.

Let φ_0 be a normal faithful semifinite weight on M. Put

$$d_0 = \frac{\mathrm{d}\varphi_0}{\mathrm{d}\psi_0}. \tag{5}$$

Then

$$\forall t \in \mathbb{R} \forall x \in M : \sigma_t^{\varphi_0}(x) = d_0^{it} x d_0^{-it}. \tag{6}$$

We define a unitary operator u_0 on the Hilbert space $L^2(\mathbb{R}, H)$ by

$$(u_0\xi)(t) = d_0^{it}\xi(t), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}. \tag{7}$$

Recall that the crossed product $N = R(M, \sigma^{\varphi_0})$ is generated by the elements $\pi(x), x \in M$, and $\lambda(s), s \in \mathbb{R}$, as described in the beginning of Chapter II. We shall describe the action of $u_0(\cdot)u_0^*$ on these generating elements.

By $\ell(s), s \in \mathbb{R}$, we denote the operator of translation by s in $L^2(\mathbb{R})$:

$$(\ell(s)f)(t) = f(t - s), f \in L^2(\mathbb{R}), t \in \mathbb{R}.$$

We identify $L^2(\mathbb{R}, H)$ with $H \otimes L^2(\mathbb{R})$ (so that $v \otimes f, v \in H, f \in L^2(\mathbb{R})$, is identified with $\xi \in L^2(\mathbb{R}, H)$ given by $\xi(t) = f(t)v, t \in \mathbb{R}$).

Proposition 3. *1) For all $x \in M$, we have*

$$u_0 \pi(x) u_0^* = x \otimes 1.$$

2) For all $s \in \mathbb{R}$, we have

$$u_0 \lambda(s) u_0^* = d_0^{is} \otimes \ell(s).$$

Proof. Let $\xi \in L^2(\mathbb{R}, H)$. Then

$$\begin{aligned}(u_0 \pi(x) u_0^* \xi)(t) =& d_0^{it} \sigma_{-t}^{\varphi_0}(x) d_0^{-it} \xi(t) \\ =& d_0^{it} d_0^{-it} x d_0^{it} d_0^{-it} \xi(t) \\ =& x\xi(t), t \in \mathbb{R},\end{aligned}$$

and

$$\begin{aligned}(u_0 \lambda(s) u_0^* \xi)(t) =& d_0^{it} (u_0^* \xi)(t-s) \\ =& d_0^{it} d^{-i(t-s)} \xi(t-s) \\ =& d_0^{is} \xi(t-s), t \in \mathbb{R}.\end{aligned}$$

This proves the result since for $\xi = v \otimes f, v \in H, f \in L^2(\mathbb{R})$, we have

$$((x \otimes 1)(v \otimes f))(t) = (xv \otimes f)(t) = f(t)xv = xf(t)v = x\xi(t), t \in \mathbb{R},$$

and

$$\begin{aligned}((d_0^{is} \otimes \ell(s))(v \otimes f))(t) =& (d_0^{is} v \otimes \ell(s) f)(t) \\ =& (\ell(s) f)(t) d_0^{is} v \\ =& f(t-s) d_0^{is} v \\ =& d_0^{is} \xi(t-s), t \in \mathbb{R}.\end{aligned}$$

□

We denote by T the unique positive self-adjoint operator in $L^2(\mathbb{R})$ characterized by

$$\forall s \in \mathbb{R} : T^{is} = \ell(s). \tag{8}$$

For the definition and properties of tensor products of closed operators we refer to [17, Section 9.33].

Proposition 4. *For all normal semifinite weights φ on M we have*

$$u_0 h_\varphi u_0^* = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0} \otimes T. \tag{9}$$

Proof. First suppose that φ is faithful . Then

$$h_\varphi^{it} h_{\varphi_0}^{-it} = (D\tilde{\varphi} : D\tau)_t (D\tau : D\tilde{\varphi}_0)_t = (D\tilde{\varphi} : D\tilde{\varphi}_0)_t = \pi((D\varphi : D\varphi_0)_t)$$

and

$$(D\varphi : D\varphi_0)_t = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0}\right)^{it} \left(\frac{\mathrm{d}\varphi_0}{\mathrm{d}\psi_0}\right)^{-it}$$

for all $t \in \mathbb{R}$, so that by Proposition 3 and the fact that $h_{\varphi_0}^{it} = \lambda(t)$ for all $t \in \mathbb{R}$, we get

$$\begin{aligned} u_0 h_\varphi^{it} u_0^* =& (u_0 h_\varphi^{it} h_{\varphi_0}^{-it} u_0^*)(u_0 h_{\varphi_0}^{-it} u_0^*) \\ =& \left(\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0}^{it}\right)\left(\frac{\mathrm{d}\varphi_0}{\mathrm{d}\psi_0}^{-it}\right) \otimes 1\right)\left(\left(\frac{\mathrm{d}\varphi_0}{\mathrm{d}\psi_0}\right)^{it} \otimes \ell(t)\right) \\ =& \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0}\right)^{it} \otimes T^{it} \end{aligned}$$

for all $t \in \mathbb{R}$, and (9) follows.

In the general case, choose a normal semifinite weight χ with $\operatorname{supp}\chi = 1 - p$ where $p = \operatorname{supp}\varphi$. Then $\varphi^+\chi$ is a normal faithful semifinite weight and hence, by the first part of the proof,

$$u_0 h_{\varphi+\chi} u_0^* = \frac{\mathrm{d}(\varphi+\chi)}{\mathrm{d}\psi_0} \otimes T.$$

Since $p = \operatorname{supp}\frac{\mathrm{d}\varphi}{\mathrm{d}\psi}$ and $\pi(p) = \operatorname{supp} h\varphi$, this implies that

$$\begin{aligned} u_0 h_\varphi u_0^* =& u_0(\pi(p) \cdot h_{\varphi+\chi} \cdot \pi(p)) u_0^* \\ =& u_0 \pi(p) u_0^* \cdot u_0 h_{\varphi+\chi} u_0^* \cdot u_0 \pi(p) u_0^* \\ =& (p \otimes 1) \cdot \left(\frac{\mathrm{d}(\varphi+\chi)}{\mathrm{d}\psi_0} \otimes T\right) \cdot (p \otimes 1) \\ =& \left(p \cdot \frac{\mathrm{d}(\varphi+\chi)}{\mathrm{d}\psi_0} \cdot p\right) \otimes T = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0} \otimes T. \end{aligned}$$

□

Corollary 5. *The mapping*

$$a \mapsto u_0^*(a \otimes T)u_0$$

is a bijection of the set of positive self-adjoint (-1)*-homogeneous operators* a *on* H *onto the set of positive self-adjoint operators* h *affiliated with* $R(M, \sigma^{\varphi_0})$ *satisfying*

$$\forall s \in \mathbb{R} : \theta_s h = e^{-s} h. \tag{10}$$

Furthermore,

$$\int a \mathrm{d}\psi_0 = \mathrm{tr}(u_0^*(a \otimes T)u_0) \tag{11}$$

for all such a.

Proof. Since the mapping in question is nothing but $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0} \mapsto h_\varphi$, it is a bijection by Proposition **??** in Chapter II. By definition, we have $\int \frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0} \mathrm{d}\psi_0 = \varphi(1) = \mathrm{tr}(h_\varphi)$. □

Corollary 6. *Let* $p \in [1, \infty[$*. Let* a *be a closed densely defined operator on* H*. Then*

1) $a \in \overline{M}_{-1/p}$ *if and only if*

$$u_0^*(a \otimes T^{1/p})u_0 \eta R(M, \sigma^{\varphi_0}),$$

2) $a \in L^p(\psi_0)$ *if and only if*

$$u_0^*(a \otimes T^{1/p})u_0 \in L^p(M).$$

For all $a \in L^p(\psi_0)$*, we have*

$$\|a\|_p = \left\|u_0^*(a \otimes T^{1/p})u_0\right\|_p.$$

Corollary 7. *Let* $p \in [1, \infty[$*. Then the mapping*

$$a \mapsto u_0^*(a \otimes T^{1/p})u_0 \tag{12}$$

is a bijection of $\overline{M}_{-1/p}$ *onto the set of closed densely defined operators* h *affiliated with* $R(M, \sigma^{\varphi_0})$ *satisfying*

$$\forall s \in \mathbb{R} : \theta_s h = e^{-s/p} h. \tag{13}$$