Contents

Chapter 1

L^p Spaces Associated with a Von Neumann Algebra

In this chapter, we present Haagerup's theory of L^p spaces associated with a von Neumann algebra.

Let M be a von Newmann algebra and let φ_0 be a normal faithful semifinite weight on M.

We denote by N the crossed product $R(M, \sigma^{\varphi_0})$ of M by the modular automorphism group σ^{φ_0} associated with φ_0. Recall [18, Section 3; 8, Section 5] that if M is given on a Hilbert space H, then N is the Von Neumann algebra on the Hilbert space $L^2(\mathbb{R}, H)$ generated by the operators $\pi(x), x \in M$, and $\lambda(s)$, $s \in \mathbb{R}$, defined by

$$(\pi(x)\xi)(t) = \sigma_{-t}^{\varphi_0}(x)\xi(t), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}, \tag{1}$$

$$(\lambda(s)\xi)(t) = \xi(t-s), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}. \tag{2}$$

We identify M with its image $\pi(M)$ in N (recall that π normal faithful representation of M).

We denote by θ the dual action of $\mathbb{R}$ in N. The θ_s, $s \in \mathbb{R}$, are automorphisms of N characterized by

$$\theta_s x = x, x \in M \tag{3}$$

$$\theta_s \lambda(t) = e^{-ist}\lambda(t), t \in \mathbb{R}. \tag{4}$$

By (3), M is contained in the set of fixed points under θ. Actually

$$M = \{y \in N | \forall s \in \mathbb{R} : \theta_s y = y\} \tag{5}$$

(see e.g. [5, Lemma 3.6]).

The θ_s, $s \in \mathbb{R}$, naturally extend to automorphisms, still denoted θ_s, of $\hat{N}_+$, the extended positive part of N [7, Section 1]. Recall [8, Lemma 5.2] that the formula

$$Tx = \int_{\mathbb{R}} \theta_s(x) \mathrm{d}s, x \in N_+, \tag{6}$$

defines a normal faithful semifinite operator valued weight T from N to M in the following sense: for each $x \in N_+$, Tx is the element of $\hat{N}_+$ characterized by

$$\langle Tx, \chi \rangle = \int_{\mathbb{R}} \langle \theta_s(x), \chi \rangle \mathrm{d}s \tag{7}$$

for all $x \in N_*^+$. Note that

$$\forall s \in \mathbb{R} : \theta_s \circ T = T. \tag{8}$$

In view of (5), this formula implies that the values of T are actually in $\hat{M}_+$.

For each normal weight φ on M, we put

$$\tilde{\varphi} = \hat{\varphi} \circ T \tag{9}$$

where $\hat{\varphi}$ denotes the extension of φ to a normal weight on $\hat{M}_+$ as described in [7, Proposition 1.10]. Then $\tilde{\varphi}$ is a normal weight on N [7,Proposition 2.3]; $\tilde{\varphi}$ is called the dual weight of φ (see [6, Introduction + Section 1)]. Note that (8) and (9) imply

$$\forall s \in \mathbb{R} : \tilde{\varphi} \circ \theta_s = \tilde{\varphi}. \tag{10}$$

If φ and ψ are normal faithful semifinite weights, then so are $\tilde{\varphi}$ and $\tilde{\psi}$, and we have [7, Theorem 4.7]:

$$\forall t \in \mathbb{R} \forall x \in M : \sigma_t^{\tilde{\varphi}}(x) = \sigma_t^{\varphi}(x), \tag{11}$$

$$\forall t \in \mathbb{R} : (D\tilde{\varphi} : D\tilde{\psi})_t = (D\varphi : D\psi)_t. \tag{12}$$

Lemma 1. *1) The mapping*

$$\varphi \mapsto \tilde{\varphi}$$

is a bijection of the set of all normal semifinite weights on M onto the set of normal semifinite weights ψ on N satisfying

$$\forall s \in \mathbb{R} : \psi \circ \theta_s = \psi. \tag{13}$$

2) For all normal weights φ and ψ on M and all $x \in M$, we have

1. $(\varphi + \psi)^{\sim} = \tilde{\varphi} + \tilde{\psi}$*,*
2. $(x \cdot \varphi \cdot x^*)^{\sim} = x \cdot \tilde{\varphi} \cdot x^*$*,*
3. $\operatorname{supp} \tilde{\varphi} = \operatorname{supp} \varphi$*.*

Proof. That $\tilde{\varphi}$ is semifinite if φ is follows from the proof of [7, Proposition 2.3]. That $\varphi \mapsto \tilde{\varphi}$ is injective follows from the formula

$$\varphi(\dot{T}x) = \tilde{\varphi}(x), x \in m_T,$$

and the fact that $\dot{T}(m_T)$ is σ-weakly dense in M [7, Proposition 2.5].

Now let us prove 2). First observe that $(\varphi + \psi)^\wedge = \hat{\varphi} + \hat{\psi}$ since $\hat{\varphi} + \hat{\psi} : \hat{M} \to [0, \infty]$ obviously satisfies the properties that characterize $(\varphi+\psi)^\wedge$ ([7, Proposition 1.10]); (a) follows trivially. Similarly, $(x \cdot \varphi \cdot x^*)^\wedge = x \cdot \hat{\varphi} \cdot x^*$, whence (b).

To prove (c), put $p_0 = 1 - \operatorname{supp} \varphi$. Then Mp_0 is the σ-weak closure in M of $N_\varphi = \{x \in M | \varphi(x^*x) = 0\}$. Similarly, the σ-weak closure in N of $N_{\tilde{\varphi}} = \{y \in N | \tilde{\varphi}(y^*y) = 0\}$ is Nq_0 where $q_0 = 1 - \operatorname{supp} \tilde{\varphi}$. Now

$$n_T N_\varphi \subset N_{\tilde{\varphi}}$$

since

$$\begin{aligned} \forall y \in n_T \forall x \in N_\varphi : &\tilde{\varphi}(x^*y^*yx) = \varphi(T(x^*y^*yx)) \\ &= \varphi(x^*T(y^*y)x) \leq \|T(y^*y)\| \varphi(x^*x) = 0. \end{aligned}$$

As n_T is σ-weakly dense in N, it follows that

$$N_\varphi \subset \overline{N_{\tilde{\varphi}}}^{\sigma-w}$$

whence

$$p_0 \leq q_0.$$

Note that we must have $q_0 \in M$ since $\tilde{\varphi}$, and hence $\operatorname{supp}\tilde{\varphi}$, is θ-invariant. Thus to conclude that $p_0 = q_0$ we need only show that $\varphi(q_0) = 0$. This follows from

$$\forall x \in m_T : \varphi(q_0 \dot{T}(x) q_0) = \varphi(\dot{T}(q_0 x q_0)) = \tilde{\varphi}(q_0 x q_0) = 0$$

and the fact that $\dot{T}(m_T)$ is σ-weakly dense in M [7, Proposition 2.5].

We now return to 1). Let ψ be a normal semifinite weight on N satisfying (13). First suppose that ψ is also faithful. Then by [5, (proof of) Theorem 3.7), it follows that $\psi = \tilde{\varphi}$ for some normal faithful semifinite φ on M.

In the general case, put $q_0 = 1 - \operatorname{supp}\psi$. Then by (13) and (5), we have $q_0 \in M$. Now take any normal semifinite weight χ_0 on M such that $\operatorname{supp}\chi_0 = q_0$. Then $\widetilde{\chi}_0$ is a normal faithful semifinite θ-invariant weight on N with $\operatorname{supp}\widetilde{\chi}_0 = q_0$. Hence $\widetilde{\chi}_0 + \psi$ is faithful and thus, as above,

$$\widetilde{\chi}_0 + \psi = \tilde{\varphi}$$

for some normal faithful semifinite weight φ on M. Finally, using (b), we find that

$$\begin{aligned}
\psi =&(1-q_0)\cdot(\widetilde{\chi}_0 + \psi)\cdot(1-q_0) \\
=&(1-q_0)\cdot\tilde{\varphi}\cdot(1-q_0) \\
=&((1-q_0)\cdot\varphi\cdot(1-q_0))^{\sim}.
\end{aligned}$$

□

Denote by τ the normal faithful semifinite trace on N characterized by

$$\forall t \in \mathbb{R} : (D\tilde{\varphi}_0 : D\tau)_t = \lambda(t) \tag{14}$$

(for the existence, see [8, Lemma 5.2]); τ satisfies

$$\forall s \in \mathbb{R} : \tau \circ \theta_s = e^{-s}\tau. \tag{15}$$

With each $h \in \hat{N}_+$ we associate the normal weight $\tau(h\cdot)$ on N as in [8, remarks preceding Proposition 1.11]. When h is simply a positive self-adjoint operator affiliated with N (see [7, Example 1.2]), this definition agrees with that given in [14, Section 4].

We recall some facts about the mapping $h \mapsto \tau(h\cdot)$ (see [7, Theorem 1.12 (and its proof) and Preposition 1.11, (4)]):

Lemma 2. *1) The mapping*

$$h \mapsto \tau(h\cdot)$$

is a bijection of $\hat{N}_+$ onto the set of normal weights on N. In particular, it is a bijection of the positive self-adjoint operators affiliated with N onto the normal semifinite weights on N.

2) For all $h, k \in \hat{N}_+$ and all $x \in N$, we have

1. $\tau((h\dot{+}k)\cdot) = \tau(h\cdot) + \tau(k\cdot)$,
2. $\tau((x \cdot h \cdot x^*)\cdot) = x \cdot \tau(h\cdot) \cdot x^*$,
3. $\operatorname{supp} \tau(h\cdot) = \operatorname{supp} h$.

Here, $h\dot{+}k$ and $x{\cdot}h{\cdot}x^*$ denote the operations in $\hat{N}_+$ introduced in [7, Definition 1.3]. If h and k are positive self-adjoint operators such that $D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}})$ is dense, then $h\dot{+}k$ is the simply the form sum of h and k [2, Corollary 4.13]. If h is a positive self-adjoint operator and x a bounded operator such that $D(h^{\frac{1}{2}}x^*)$ is dense, then $x \cdot h \cdot x^* = \left|h^{\frac{1}{2}}x^*\right|^2$.

Definition 3. *For each normal weight φ on M we define h_φ as the unique element of $\hat{N}_+$ given by*

$$\tilde{\varphi} = \tau(h_\varphi\cdot). \tag{16}$$

Proposition 4. *1) The mapping*

$$\varphi \mapsto h_\varphi$$

is a bijection of the set of all normal semifinite weights on M onto the set of all positive self-adjoint operators h affiliated with N satisfying

$$\forall s \in \mathbb{R} : \theta_s h = e^{-s} h. \tag{17}$$

(2) For all normal weights φ and ψ on M and all $x \in M$, we have

1. $h_{\varphi+\psi} + h_\varphi \dot{+} h_\psi$,
2. $h_{x\cdot\varphi\cdot x^*} = x \cdot h_\varphi \cdot x^*$,
3. $\operatorname{supp} h_\varphi = \operatorname{supp} \varphi$.

Proof. This proposition is an immediate consequence of Lemma 1 and 2. We just need to prove that a positive self-adjoint operator h affiliated with N satisfies (17) if and only if the corresponding weight $\tau(h\cdot)$ is θ-invariant. This follows easily from (15). Indeed, for all $s \in \mathbb{R}$ we have

$$\tau(e^s \theta_s(h)\cdot) = e^s (\tau \circ \theta_s)(h\theta_{-s}(\cdot)) = \tau(h\theta_{-s}(\cdot)) = \tau(h\cdot) \circ \theta_{-s},$$

whence

$$e^s \theta_s(h) = h \Leftrightarrow \tau(e^s \theta_s(h)\cdot) = \tau(h\cdot) \Leftrightarrow \tau(h\cdot) = \tau(h\cdot) \circ \theta_{-s}.$$

The equivalence of (17) and

$$\forall s \in \mathbb{R} : \tau(h\cdot) = \tau(h\cdot) \circ \theta_s$$

follows. □

The next lemma is essential. It will permit us apply results on τ-measurable operators.

As usual, $\chi_{]\gamma,\infty[}$ denotes the characteristic function for the interval $]\gamma, \infty[$.

Lemma 5. *Let φ be a normal semifinite weight on M. Then for all $\gamma \in \mathbb{R}_+$, we have*

$$\tau(\chi_{]\gamma,\infty[}(h_\varphi)) = \frac{1}{\gamma}\varphi(1).$$

Proof. First let us prove the formula in the case $\gamma = 1$.

Let $s \in \mathbb{R}$. Then since θ_s is an automorphism and $\theta_s h_\varphi = e^{-s} h_\varphi$ we have

$$\theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi)) = e^s h_\varphi^{-1}\chi_{]1,\infty[}(e^{-s}h_\varphi).$$

Now let $h_\varphi = \int \lambda \mathrm{d}e_\lambda$ be the spectral decomposition of h_φ. Then for any vector functional $\omega_{\xi,\xi}$, where ξ is a unit vector, we have

$$\begin{aligned}
\langle \int_{\mathbb{R}} \theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))\mathrm{d}s, \omega_{\xi,\xi}\rangle &= \int_{\mathbb{R}} \langle e^s h_\varphi^{-1}\chi_{]1,\infty[}(e^{-s}h_\varphi), \omega_{\xi,\xi}\rangle \mathrm{d}s \\
&= \int_{\mathbb{R}} \int_{]0,\infty[} e^s \lambda^{-1}\chi_{]1,\infty[}(e^{-s}\lambda)\, \mathrm{d}(e_\lambda\xi|\xi)\, \mathrm{d}s \\
&= \int_{]0,\infty[} \lambda^{-1} \left(\int_{]-\infty,\log\lambda[} e^s \mathrm{d}s \right) \mathrm{d}(e_\lambda\xi|\xi) \\
&= \int_{]0,\infty[} \lambda^{-1}\lambda\, \mathrm{d}(e_\lambda\xi|\xi) \\
&= \|(\operatorname{supp} h_\varphi)\xi\|^2
\end{aligned}$$

So that

$$\int_{\mathbb{R}} \theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))\mathrm{d}s = \operatorname{supp} h_\varphi = \operatorname{supp}\varphi.$$

Finally, since $\tilde{\varphi} = \tau(h_\varphi \cdot)$ we have

$$\begin{aligned}
\tau(\chi_{]1,\infty[}(h_\varphi)) &= \tau(h_\varphi^{\frac{1}{2}}(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))h_\varphi^{\frac{1}{2}}) \\
&= \tilde{\varphi}(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi)) \\
&= \varphi\left(\int \theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))\mathrm{d}s \right) = \varphi(\operatorname{supp}\varphi) = \varphi(1).
\end{aligned}$$

This completes the proof in the case $\gamma = 1$. In the general case, write $\gamma = e^s$, $s \in \mathbb{R}$. Then by (15)

$$\begin{aligned}\tau(\chi_{]e^s,\infty[}(h_\varphi)) =&\tau(\chi_{]1,\infty[}(e^{-s}h_\varphi))\\ =&\tau(\theta_s(\chi_{]1,\infty[}(h_\varphi)))\\ =&e^{-s}\tau(\chi_{]1,\infty[}(h_\varphi)) = e^{-s}\varphi(1).\end{aligned}$$

□

By Chapter I, Proposition **??**, we have

Corollary 6. *Let φ be a normal semifinite weight on M. Then h_φ is τ-measurable if and only if $\varphi \in M_*$.*

We denote by $\tilde{N}$ the set of all τ-measurable closed densely defined operators affiliated with N. Recall (Chapter I) that $\tilde{N}$ is a topological *-algebra with respect to strong sum and product. Sums and products of elements in $\tilde{N}$ will always be understood to be in the strong sense although we do not emphasize it in the notation.

We denote by $\tilde{N}_+$ the subset of all positive self-adjoint elements of $\tilde{N}$.

Note that the θ_s, $s \in \mathbb{R}$, extend to continuous *-automorphisms, still denoted θ_s, of $\tilde{N}$. We have

$$\forall s \in \mathbb{R} \forall \epsilon, \delta \in \mathbb{R}_+ : \theta_s(N(\epsilon, \delta)) = N(\epsilon, e^{-s}\delta) \tag{18}$$

Since for all $a \in \tilde{N}_+$

$$\tau(\chi_{]\epsilon,\infty[}(\theta_s a)) = \tau(\theta_s(\chi_{]\epsilon,\infty[}(a))) = e^{-s}\tau(\chi_{]\epsilon,\infty[}(a))$$

(for the definition and properties of the 0-neighbourhoods $N(\epsilon, \delta)$, we refer to Chapter I).

Theorem 7. *1) The mapping*

$$\varphi \mapsto h_\varphi$$

extends to a linear bijection, still denoted $\varphi \mapsto h_\varphi$, of M_* onto the subspace

$$\{h \in \tilde{N} | \forall s \in \mathbb{R} : \theta_s h = e^{-s}h\} \tag{19}$$

of N.

2) For all $\varphi \in M_*$ and $x, y \in M$, we have

$$h_{x \cdot \varphi \cdot y^*} = x h_\varphi y^* \tag{20}$$

and

$$h_{\varphi^*} = h_\varphi^*. \tag{21}$$

3) If $\varphi = u|\varphi|$ is the polar decomposition of φ, then $h = uh_{|\varphi|}$ ($h_\varphi = uh_{|\varphi|}$) is the polar decomposition of h_φ. In particular,

$$|h_\varphi| = h_{|\varphi|}. \tag{22}$$

The proof will be based on Corollary 6, Proposition 4, and the following lemma.

Lemma 8. *1) Let h and k be positive self-adjoint operators such that $D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}})$ is dense. Then*

$$h + k \subset h \dot{+} k.$$

If $h + k$ is essentially self-adjoint, then its unique self-adjoint extension is precisely $h \dot{+} k$.

2) Let h be a positive self-adjoint operator and x a bounded operator such that $D(h^{\frac{1}{2}} x^*)$ is dense. Then

$$x h x^* \subset x \cdot h \cdot x^*.$$

If xhx^* is essentially self-adjoint, then its unique self-adjoint extension is precisely $x \cdot h \cdot x^*$.

Proof. 1) Recall that by definition $h \dot{+} k$ is the unique positive self-adjoint operator characterized by $D((h \dot{+} k)^{\frac{1}{2}}) = D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}})$ and

$$\forall \xi \in D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}}) : \left\| (h \dot{+} k)^{\frac{1}{2}} \xi \right\|^2 = \left\| h^{\frac{1}{2}} \xi \right\|^2 + \left\| k^{\frac{1}{2}} \xi \right\|^2. \tag{23}$$

By polarization, it follows that

$$\forall \xi \in D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}}) : ((h \dot{+} k)^{\frac{1}{2}} \xi | (h \dot{+} k)^{\frac{1}{2}} \eta) = (h^{\frac{1}{2}} \xi | h^{\frac{1}{2}} \eta) + (k^{\frac{1}{2}} \xi | k^{\frac{1}{2}} \eta).$$

Now let $\xi \in D(h+k) = D(h) \cap D(k)$ and $\eta \in D(h\dotplus k)$. Then also $\xi \in D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}})$ and $\eta \in D((h\dotplus k)^{\frac{1}{2}}) = D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}})$ so that

$$\begin{aligned}((h+k)\xi|\eta) =&(h\xi|\eta) + (k\xi|\eta)\\ =&(h^{\frac{1}{2}}\xi|h^{\frac{1}{2}}\eta) + (k^{\frac{1}{2}}\xi|k^{\frac{1}{2}}\xi)\\ =&((h\dotplus k)^{\frac{1}{2}}\xi|(h\dotplus k)^{\frac{1}{2}}\eta)\\ =&(\xi|(h\dotplus k)\eta).\end{aligned}$$

It follows that

$$h + k \subset (h\dotplus k)^* = (h\dotplus k).$$

Hence $h+k$ is preclosed and $[h+k] \subset h\dotplus k$. If $[h+k]$ is self-adjoint, we must have $[h+k] = h\dotplus k$.

2) Recall that $x \cdot h \cdot x^* = \left|h^{\frac{1}{2}}x^*\right|^2$. Now let $\xi \in D(xhx^*) = D(hx^*)$ and $\eta \in D(x \cdot h \cdot x^*)$. Then also $\xi \in D(h^{\frac{1}{2}}x^*)$ and $\eta \in D((x \cdot h \cdot x^*)^{\frac{1}{2}}) = D(h^{\frac{1}{2}}x^*)$ so that

$$(xhx^*\xi|\eta) = (hx^*\xi|x^*\eta) = (h^{\frac{1}{2}}x^*\xi|h^{\frac{1}{2}}x^*\eta) = (\xi|(x \cdot h \cdot x^*)\eta).$$

It follows that

$$xhx^* \subset (x \cdot h \cdot x^*)^* = x \cdot h \cdot x^*.$$

Hence xhx^* is preclosed and $[xhx^*] \subset x \cdot h \cdot x^*$. If $[xhx^*]$ is self-adjoint, we must have $[xhx^*] = x \cdot h \cdot x^*$. $\square$

Proof of Theorem 7. Let $\varphi, \psi \in M_*^+$. Then h_φ and h_ψ are positive self-adjoint and τ-measurable so that their strong sum exists and is again a positive self-adjoint τ-measurable operator. By Lemma 8, this sum then coincides with $h_\varphi \dotplus h_\psi$. Then Proposition 4 yields

$$h_{\varphi+\psi} = h_\varphi + h_\psi,$$

where the sum at the right hand side is now the sum in $\tilde{N}$.

Similarly for all $\varphi \in M_*^+$ and $x \in M$ we get

$$h_{x \cdot \varphi \cdot \xi^*} = xh_\varphi x^*. \tag{24}$$

Now the additive and homogeneous mapping $\varphi \mapsto h_\varphi$ of M_*^+ onto $\{h \in \tilde{N}_+ | \forall s \in \mathbb{R} : \theta_s h = e^{-s} h\}$ extends by linearity to a linear mapping $\varphi \mapsto h_\varphi$ of M_* onto the subspace of $\tilde{N}$ spanned by $\{h \in \tilde{N}_+ | \forall s \in \mathbb{R} : \theta_s h = e^{-s} h\}$, i.e. onto the subspace (19) (evidently, (19) is stable under $h \mapsto h^*$ and $h \mapsto |h|$ and hence spanned by its positive elements).

By linearity, we must have (21) for a11 $\varphi \in M_*$. Also by linearity, (24) holds for all $\varphi \in M_*$ and $x \in M$; by polarization the equation (20) follows for all $\varphi \in M_*$ and $x, y \in M$.

In particular, if $\varphi \in u|\varphi|$ is the polar decomposition of φ, we have

$$h_\varphi = h_{u|\varphi|} = u h_{|\varphi|}.$$

That this relation is the polar decomposition of h_φ follows from the fact that the initial projection for the partial isometry u is $\operatorname{supp} |\varphi| = \operatorname{supp} h_{|\varphi|}$.

Finally, $\varphi \mapsto h_\varphi$ is injective: if $h_\varphi = 0$, then $h_{|\varphi|} = |h_\varphi| = 0$, whence $|\varphi| = 0$ and $\varphi = 0$. □

Motivated by Theorem 7, we now give the following definition:

Definition 9. *For each $p \in [1, \infty]$, we let*

$$L^p(M) = \{a \in \tilde{N} | \forall s \in \mathbb{R} : \theta_s a = e^{-\frac{s}{p}} a\}.$$

Note that the $L^p(M)$ are linear subspaces of $\tilde{N}$ and that they are linearly spanned by their positive part $L^p(M)_+ = L^p(M) \cap \tilde{N}_+$.

By Theorem 7, we know that $L^1(M) \cong M_*$. And:

Proposition 10. *We have $L^\infty(M) = M$.*

Proof. In view of (5), we just need to show that every $a \in L^\infty(M)$ is bounded. Let $a \in L^\infty(M)$. Then for all $s \in \mathbb{R}$ and all $\lambda \in \mathbb{R}_+$ we have

$$\begin{aligned}\tau(\chi_{]\lambda,\infty[}(|a|)) =& \tau(\chi_{]\lambda,\infty[}(\theta_s |a|)) \\ =& \tau(\theta_s(\chi_{]\lambda,\infty[}(|a|))) = e^{-s} \tau(\chi_{]\lambda,\infty[}(|a|)).\end{aligned}$$

Hence for all $\lambda \in \mathbb{R}_+$ we must have

$$\tau(\chi_{]\lambda,\infty[}(|a|)) = 0 \text{ or } \tau(\chi_{]\lambda,\infty[}(|a|)) = \infty.$$

Since a is τ-measurable, we have $\tau(\chi_{]\lambda,\infty[}(|a|)) < \infty$ for some λ. Hence $\tau(\chi_{]\lambda,\infty[}(|a|)) = 0$ and thus $\chi_{]\lambda,\infty[}(|a|) = 0$ since τ is faithful. This means that a is bounded. $\square$

Remark 11. *We have seen that all elements of $L^\infty(M)$ are bounded. In contrast to this, all non-zero elements of $L^p(M)$, where $p < \infty$, are unbounded. To see this, let $a \in L^p(M)$ and suppose that $a \neq 0$. Then for some $\lambda \in \mathbb{R}_+$ we have $\chi_{]\lambda,\infty[}(|a|) \neq 0$ and hence $\tau(\chi_{]\lambda,\infty[}(|a|)) \neq 0$. Then for all $\mu \in \mathbb{R}_+$ we have*

$$\tau(\chi_{]\mu,\infty[}(|a|)) \neq 0$$

since for all $s \in \mathbb{R}$

$$\begin{aligned}\tau(\chi_{]e^{\frac{s}{p}}\lambda,\infty[}(|a|)) =& \tau(\chi_{]\lambda,\infty[}(e^{-\frac{s}{p}}|a|)) \\ =& \tau(\chi_{]\lambda,\infty[}(\theta_s|a|)) \\ =& \tau(\theta_s\chi_{]\lambda,\infty[}(|a|)) \\ =& e^{-s}\tau(\chi_{]\lambda,\infty[}(|a|)) \neq 0.\end{aligned}$$

It follows that $|a|$ must be unbounded.

Proposition 12. *Let a be a closed densely defined operator affiliated with N with polar decomposition $a = u|a|$. Let $p \in [1,\infty[$. Then*

$$a \in L^p(M)$$

if and only if

$$u \in M \text{ and } |a|^p \in L^1(M).$$

Proof. Recall that $a \in \tilde{N}$ if and only if $|a| \in \tilde{N}$. Furthermore, $|a| \in \tilde{N}$ if and only if $|a|^p \in \tilde{N}$ since $\tau(\chi_{]\lambda,\infty[}(|a|)) = \tau(\chi_{]\lambda^p,\infty[}(|a|^p))$ for all $\lambda \in \mathbb{R}_+$. For all such a and all $s \in \mathbb{R}$ we have

$$\theta_s a = e^{-\frac{s}{p}}a \Leftrightarrow \theta_s u = u \text{ and } \theta_s|a|^p = e^{-s}|a|^p.$$

The result follows by Definition 9 and Proposition 10. $\square$

A similar result holds for the right polar decomposition.

Definition 13. *We define a linear functional* tr *on* $L^1(M)$ *by*

$$\operatorname{tr}(h_\varphi) = \varphi(1), \varphi \in M_*.$$

Note that

$$\operatorname{tr}(|h_\varphi|) = \operatorname{tr}\big(h_{|\varphi|}\big) = |\varphi|(1) = \|\varphi\| \tag{25}$$

for all $\varphi \in M_*$. This implies that

$$|\operatorname{tr}(a)| \leq \operatorname{tr}(|a|) \tag{26}$$

for all $a \in L^1(M)$ and that the mapping $a \mapsto \operatorname{tr}(|a|)$ is a norm on $L^1(M)$.

Definition 14. *Let* $p \in [1, \infty[$. *Then we define* $\|\cdot\|_p$ *on* $L^p(M)$ *by*

$$\|a\|_p = \operatorname{tr}(|a|^p)^{\frac{1}{p}}, a \in L^p(M).$$

For $p = \infty$, *we put*

$$\|a\|_\infty = \|a\|, a \in L^\infty(M).$$

We shall see that for all p, $\|\cdot\|_p$ is a norm on $L^p(M)$.

By (26), we have

Proposition 15. *The mapping*

$$\varphi \mapsto h_\varphi : M_* \to L^1(M)$$

is an isometry of M_* *onto* $L^1(M)$.

Lemma 16. *Let* $p \in [1, \infty[$ *and* $\epsilon, \delta \in \mathbb{R}_+$. *Then*

$$N(\epsilon, \delta) \cap L^p(M) = \{a \in L^p(M) | \|a\|_p \leq \epsilon\delta^{\frac{1}{p}}\}.$$

Proof. Let $a \in L^p(M)$. Then $|a|^p \in L^1(M)_+$ and hence $|a|^p = h_\varphi$ for some $\varphi \in M_*^+$. Now

$$\begin{aligned}\tau(\chi_{]\epsilon,\infty[}(|a|)) =& \tau(\chi_{]\epsilon^p,\infty[}(|a|^p)) \\ =& \frac{1}{\epsilon^p}\varphi(1) \\ =& \frac{1}{\epsilon^p}\||a|^p\|_1 = \frac{1}{\epsilon^p}\|a\|_p^p\end{aligned}$$

Using this we get

$$\begin{aligned}
a \in N(\epsilon,\delta) \Leftrightarrow & |a| \in N(\epsilon,\delta) \\
\Leftrightarrow & \tau(\chi_{]\epsilon,\infty[}(|a|)) \leq \delta \\
\Leftrightarrow & \frac{1}{\epsilon^p}\|a\|_p^p \leq \delta \\
\Leftrightarrow & \|a\|_p \leq \epsilon\delta^{\frac{1}{p}}.
\end{aligned}$$

□

Corollary 17. *On $L^1(M)$ the norm topology is exactly the topology induced from $\tilde{N}$.*

We denote by $\mathbb{C}_+$ the closed half-plane $\{a \in \mathbb{C} | \operatorname{Re} a \geq 0\}$ and by $\mathbb{C}_+^\circ$ the corresponding open half-plane.

Lemma 18. *Let $h \in \tilde{N}_+$. Then the mapping*

$$\alpha \mapsto h^\alpha : \mathbb{C}_+^\circ \to \tilde{N}$$

is differentiable.

Proof. First note that all h^α, $\alpha \in \mathbb{C}_+^\circ$, are actually τ-measurable since h is τ-measurable.

1) Suppose that h is bounded, i.e. $h \in N_+$. Then the mapping

$$\alpha \mapsto h^\alpha : \mathbb{C}_+^\circ \to N$$

is differentiable with respect to the norm topology on N and

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} h^\alpha = h^\alpha \log h \tag{27}$$

(note that the expression at the right hand side is defined for any positive $h \in N$ since the function $\lambda \mapsto \lambda^\alpha \log \lambda$ is continuous on the closed half-plane $\mathbb{C}_+$). This follows from spectral theory using the fact that for all $\alpha_0 \in \mathbb{C}_+^\circ$ we have

$$\begin{aligned}
\frac{1}{\alpha-\alpha_0}(\lambda^\alpha - \lambda^{\alpha_0}) - \lambda^{\alpha_0}\log\lambda = & \frac{1}{\alpha-\alpha_0}(e^{\alpha\log\lambda} - e^{\alpha_0\log\lambda}) - \log\lambda e^{\alpha_0\log\lambda} \\
\to & 0 \text{ as } \alpha \to \alpha_0 \text{ uniformly in } \lambda \in]0, \|h\|].
\end{aligned}$$

2) Now let h be any element of $\tilde{N}_+$. We claim that $\alpha \mapsto h^\alpha : \mathbb{C}_+^\circ \to \tilde{N}$ is differentiable with respect to the topology on $\tilde{N}$ and that (27) still holds (as above, $h^\alpha \log h$ is a well-defined positive self-adjoint operator and, by spectral theory, it is τ-measurable). Now let $\epsilon, \delta \in \mathbb{R}_+$. Take $\lambda \in \mathbb{R}_+$ such that $\tau(\chi_{]\lambda,\infty[}(h)) \leq \delta$. Put $p = \chi_{[0,\lambda]}(h)$. Then hp is bounded and by the first part of the proof

$$
\begin{aligned}
&\left\| \left(\frac{1}{\alpha - \alpha_0}(h^\alpha - h^{\alpha_0}) - h^{\alpha_0} \log h \right) p \right\| \\
=&\left\| \frac{1}{\alpha - \alpha_0}((hp)^\alpha - (hp)^{\alpha_0}) - (hp)^{\alpha_0} \log(hp) \right\| \leq \epsilon
\end{aligned}
$$

Origin article here is $(hp)^\alpha \log(hp)$ for all $\alpha \in \mathbb{C}_+^\circ$ sufficiently close to α_0. Thus

$$
\frac{1}{\alpha - \alpha_0}(h^\alpha - h^{\alpha_0}) - h^{\alpha_0} \log h \in N(\epsilon, \delta)
$$

for α sufficiently close to α_0. This proves the lemma. $\square$

We denote by S the closed complex strip $\{\alpha \in \mathbb{C} | 0 \leq \operatorname{Re} \alpha \leq 1\}$ and by S° the corresponding open strip.

Lemma 19. *Let $h, k \in L^1(M)_+$. Then for $\alpha \in S^\circ$ we have*

$$
h^\alpha k^{1-\alpha} \in L^1(M),
$$

and the mapping

$$
\alpha \mapsto h^\alpha k^{1-\alpha} : S^\circ \to L^1(M) \tag{28}
$$

is analytic.

Proof. That $h^\alpha k^{1-\alpha} \in L^1(M)$ follows from Definition 9 since

$$
\begin{aligned}
\forall s \in \mathbb{R} : \theta_s(h^\alpha k^{1-\alpha}) =&(\theta_s h)^\alpha (\theta_s k)^{1-\alpha} \\
=&e^{-\alpha s} h^\alpha e^{-(1-\alpha)s} k^{1-\alpha} = e^{-s} h^\alpha k^{1-\alpha}.
\end{aligned}
$$

Origin article here is $e^{-s} h^\alpha h^{1-\alpha}$ we want to prove that the mapping (28) is differentiable. In view of Corollary 17 we may as well prove that (28) is differentiable as a mapping into $\tilde{N}$. Now by the preceding lemma, the functions $f, g : S^\circ \mapsto \tilde{N}$ defined by $f(\alpha) = h^\alpha$ and $g(\alpha) = k^{1-\alpha}$. are differentiable. It follows that for all $\alpha_0 \in S^\circ$ we have

$$\begin{aligned}
&\frac{1}{\alpha - \alpha_0}(f(\alpha)g(\alpha) - f(\alpha_0)g(\alpha_0)) \\
=&\frac{1}{\alpha - \alpha_0} f(\alpha)(g(\alpha) - g(\alpha_0)) + \frac{1}{\alpha - \alpha_0}(f(\alpha) - f(\alpha_0))g(\alpha_0) \\
\to& f(\alpha_0)g'(\alpha_0) + f'(\alpha_0)g(\alpha_0) \text{ as } \alpha \to \alpha_0
\end{aligned}$$

so that also $f \cdot g : S^\circ \to \tilde{N}$ is differentiable. □

Lemma 20. *Let $t \in \mathbb{R}$ and put*

$$\tilde{N}_{\frac{1}{2}+it} = \{a \in \tilde{N} | \forall s \in \mathbb{R} : \theta_s a = e^{-(\frac{1}{2}+it)} a\}. \tag{29}$$

Let $a, b \in \tilde{N}_{\frac{1}{2}+it}$. Then $b^* a, ab^* \in L^1(M)$ and

$$\operatorname{tr}(b^* a) = \operatorname{tr}(ab^*). \tag{30}$$

Proof. That $b^* a, ab^* \in L^1(M)$ follows from Definition 9 and (29).

To prove (30), suppose first that $a = b$. Then by Definition 13 and Lemma 5

$$\operatorname{tr}(a^* a) = \tau(\chi_{]1,\infty[}(a^* a)) = \tau(\chi_{]1,\infty[}(aa^*)) = \operatorname{tr}(aa^*).$$

In the general case, note that $a + ib \in \tilde{N}_{\frac{1}{2}+it}$ and

$$b^* a = \frac{1}{4} \sum_{k=0}^{3} i^k (a + i^k b)^* (a + i^k b)$$

$$ab^* = \frac{1}{4} \sum_{k=0}^{3} i^k (a + i^k b)(a + i^k b)^*$$

The result follows since tr is linear. □

Proposition 21. *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $a \in L^p(M)$ and $b \in L^q(M)$. Then $ab, ba \in L^1(M)$ and*

$$\operatorname{tr}(ab) = \operatorname{tr}(ba).$$

Proof. If $p = 1$ we have $a = h_\varphi$ for some $\varphi \in M_*$ and the result follows by Theorem 7:

$$\operatorname{tr}(h_\varphi b) = \operatorname{tr}(h_{\varphi\cdot b}) = (\varphi \cdot b)(1) = (b \cdot \varphi)(1) = \operatorname{tr}(h_{b\cdot\varphi}) = \operatorname{tr}(bh_\varphi)$$

Now suppose that $p, q \in]1, \infty[$. As usual, we easily see that ab and ba are in $L^1(M)$. By linearity, we may assume that $a \in L^p(M)_+$ and $b \in L^q(M)_+$. Now $a^p, b^q \in L^1(M)_+$ and by Lemma 19 the functions F and G on S° defined by $F(\alpha) = \operatorname{tr}\big(a^{p\alpha}b^{q(1-\alpha)}\big)$ and $G(\alpha) = \operatorname{tr}\big(b^{q(1-\alpha)}a^{p\alpha}\big)$ are analytic. For all $t \in \mathbb{R}$, we have $a^{p(\frac{1}{2}+it)} \in \tilde{N}_{\frac{1}{2}+it}$ and $b^{q(\frac{1}{2}+it)} \in \tilde{N}_{\frac{1}{2}+it}$ so that by Lemma 20

$$\begin{aligned} F(\frac{1}{2} + it) &= \operatorname{tr}\Big(a^{p(\frac{1}{2}+it)}b^{q(\frac{1}{2}-it)}\Big) = \operatorname{tr}\Big(a^{p(\frac{1}{2}+it)}(b^{q(\frac{1}{2}-it)})^*\Big) \\ &= \operatorname{tr}\Big((b^{q(\frac{1}{2}-it)})^*a^{p(\frac{1}{2}+it)}\Big) = \operatorname{tr}\Big(b^{q(\frac{1}{2}-it)}a^{p(\frac{1}{2}+it)}\Big) = G(\frac{1}{2} + it) \end{aligned}$$

We conclude that $F = G$. In particular,

$$\operatorname{tr}(ab) = F(\frac{1}{p}) = G(\frac{1}{p}) = \operatorname{tr}(ba).$$

$\square$

The proof of the next lemma is based on the 3 lines theorem for analytic functions (see e.g. [23, p.93]). The 3 lines theorem also holds for analytic functions F with values in a Banach space (to see this, apply it to the scalar-valued functions $\alpha \mapsto v(F(\alpha))$, where v is in the dual of the given Banach space).

Lemma 22. *Let $h, k \in L^1(M)_+$ and suppose that $\|h\|_1 = \|k\|_1 = 1$. Then for all $\alpha \in S^\circ$, we have*

$$\left\|h^\alpha k^{1-\alpha}\right\|_1 \leq 1$$

Proof. Write $s = \operatorname{Re}\alpha$, $t = \operatorname{Im}\alpha$. Then $h^s \in L^{\frac{1}{s}}(M)$ with $\|h^s\|_{\frac{1}{s}} = 1 = s^{-s} \cdot s^s$, whence by Lemma 16

$$h^s \in N(s^{-s}, s).$$

Similarly,

$$k^{1-s} \in N((1-s)^{-(1-s)}, 1-s).$$

It follows that

$$\begin{aligned} h^s k^{1-s} \in & N(s^{-s}, s) \cdot N((1-s)^{-(1-s)}, 1-s) \\ \subset & N(s^{-s}(1-s)^{-(1-s)}, s + (1-s)) \end{aligned}$$

whence also

$$h^\alpha k^{1-\alpha} = h^{it} h^s k^{1-s} k^{-it} \in N(s^{-s}(1-s)^{-(1-s)}, 1)$$

Again by Lemma 16,

$$\left\| h^\alpha k^{1-\alpha} \right\|_1 \leq s^{-s}(1-s)^{-(1-s)}$$

Since $s \mapsto s^{-s}(1-s)^{-(1-s)}$ is bounded, the function $\alpha \mapsto h^\alpha k^{1-\alpha} : S^\circ \to L^1(M)$ is bounded. It is analytic by Lemma 19. Hence we can apply the 3 lines theorem on each closed strip $\{a \in \mathbb{C} | \epsilon \leq \operatorname{Re} \alpha \leq 1 - \epsilon\}$ and we obtain

$$\sup_{t \leq \operatorname{Re} \alpha \leq 1-\epsilon} \left\| h^\alpha k^{1-\alpha} \right\|_1 \leq \epsilon^{-\epsilon}(1-\epsilon)^{-(1-\epsilon)}.$$

Hence for fixed $a \in S^\circ$, the inequality

$$\left\| h^\alpha k^{1-\alpha} \right\|_1 \leq \epsilon^{-\epsilon}(1-\epsilon)^{-(1-\epsilon)}$$

holds for all $\epsilon \in \mathbb{R}_+$ such that $\epsilon \leq \operatorname{Re} \alpha \leq 1 - \epsilon$. Since

$$\epsilon^{-\epsilon}(1-\epsilon)^{-(1-\epsilon)} = e^{-\epsilon \log \epsilon} e^{-(1-\epsilon)\log(1-\epsilon)} \to 1 \text{ as } \epsilon \to 0,$$

it follows that

$$\left\| h^\alpha k^{1-\alpha} \right\|_1 \leq 1$$

This proves the lemma. $\square$

Theorem 23 (Hölder's inequality)**.** *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $a \in L^p(M)$ and $b \in L^q(M)$. Then*

$$\|ab\|_1 \leq \|a\|_p \|b\|_q.$$

Proof. If $p = 1$, we have $a = h_\varphi$ for some $\varphi \in M_*$ and

$$\|h_\varphi b\|_1 = \|h_{\varphi \cdot b}\|_1 = \|\varphi \cdot b\| \leq \|\varphi\| \|b\|_\infty = \|h_\varphi\|_1 \cdot \|b\|_\infty$$

for all $b \in L^\infty(M) = M$. The case $q = 1$ is quite similar to this.

Now assume $p, q \in]1, \infty[$, and $\|a\|_p = 1$, $\|b\|_q = 1$. Let $a = u|a|$ be the (usual) polar decomposition of a and $b = |b^*|v$ the right polar decomposition of b. Then $|a|^p, |b^*|^q \in L^1(M)$ with $\||a|^p\| = \||b^*|_r^q\|_1 = 1$ and Lemma 22 applies:

$$\begin{aligned} \|ab\|_1 =& \|u|a||b^*|v\|_1 \leq \||a||b^*|\|_1 \\ =& \left\| |a|^{\frac{p}{p}} |b^*|^{\frac{q}{q}} \right\|_1 \leq 1. \end{aligned}$$

□

Proposition 24. *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $a \in L^p(M)$. Then*

$$\|a\|_p = \sup\{|\mathrm{tr}(ab)| | b \in L^q(M), \|b\|_q \leq 1\}.$$

Proof. If $p = 1$ or $p = \infty$ this is well-known (since $\mathrm{tr}(ch_\varphi) = \mathrm{tr}(h_\varphi c) = \varphi(c)$ for all $\varphi \in M_*$ and $c \in M$). Suppose that $1 < p < \infty$. We may assume that $\|a\|_p = 1$. Then putting $b = |a|^{\frac{p}{q}} u^*$, where $a = u|a|$ is the polar decomposition of a, we have $b \in L^q(M)$ with $\|b\|_q = \left\| |a|^{\frac{p}{q}} u^* \right\|_q = \mathrm{tr}(|a|^p)^{\frac{1}{q}} = 1$ and

$$\mathrm{tr}(ab) = \mathrm{tr}\left(u|a||a|^{\frac{p}{q}} u^*\right) = \mathrm{tr}(|a|^p) = 1.$$

Hence

$$\|a\|_p = 1 \leq \sup\{|\mathrm{tr}(ab)| | b \in L^q(M), \|b\|_q \leq 1\}.$$

The converse inequality follows from Hölder's inequality (together with (26)). □

Corollary 25. $\|\cdot\|_p$ *is a norm on $L^p(M)$.*

Proof. The inequality

$$\|a + b\|_p \leq \|a\|_p + \|b\|_p$$

follows immediately from Proposition 24. □

Proposition 26. *On $L^p(M)$, the norm topology is exactly the topology induced from $\tilde{N}$.*

Proof. Now that we know that $\|\cdot\|_p$ is a norm, this is a corollary of Lemma 16. $\square$

Corollary 27. $(L^p(M), \|\cdot\|_p)$ *is a Banach space.*

Proof. From the definition of $L^p(M)$ it follows that it is a closed subspace of the complete topological vector space $\tilde{N}$. Hence it is complete for the uniform structure induced from $\tilde{N}$. By Lemma 16, this is simply the uniform structure coming from the norm. Hence $L^p(M)$ is a complete normed space. $\square$

Corollary 28. $(L^2(M), \|\cdot\|_2)$ *is a Hilbert space with the inner product*

$$(a|b)_{L^2(M)} = \mathrm{tr}(b^*a) \quad (= \mathrm{tr}(ab^*)), a, b \in L^2(M).$$

Proof. That $(a, b) \mapsto (a|b)_{L^2(M)}$ is an inner product defining the norm $\|\cdot\|_2$ is easily verified. By Corollary 27, $L^2(M)$ is complete. $\square$

Remark 29. *Let $t \in \mathbb{R}$. Define $\tilde{N}_{\frac{1}{2}+it}$ as in Lemma 20. Then*

$$(a, b) \mapsto \mathrm{tr}(b^*a)$$

is an inner product on $\tilde{N}_{\frac{1}{2}+it}$ and

$$a \mapsto \mathrm{tr}(a^*a)^{\frac{1}{2}}$$

is a norm which we shall denote by $\|\cdot\|_2$ (as in the case $t = 0$ where $\tilde{N}_{\frac{1}{2}} = L^2(M)$). Note that

$$|\mathrm{tr}(b^*a)| \leq \|a\|_2 \|b\|_2$$

and

$$\|a + b\|_2^2 + \|a - b\|_2^2 = 2\|a\|_2^2 + 2\|b\|_2^2$$

for all $a, b \in \tilde{N}_{\frac{1}{2}+it}$.

Remark 30. *Let $p \in [1, \infty[$. Then we have a natural identification*

$$L^p(M \oplus M) \cong L^p(M) \times L^p(M) \tag{31}$$

such that

$$\forall (a,b) \in L^p(M) \times L^p(M) \cong L^p(M \oplus M) : \|(a,b)\|_p = (\|a\|_p^p + \|b\|_p^p)^{\frac{1}{p}}. \tag{32}$$

To see this, write $M^{(2)} = M \oplus M$ and define the normal faithful semifinite weight $\varphi_0^{(2)}$ on $M^{(2)}$ by $\varphi_0^{(2)} = \varphi_0 \oplus \varphi_0$, i.e.

$$\varphi_0^{(2)} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \varphi_0(x) + \varphi_0(y), x, y \in M_+.$$

Let us denote by $N^{(2)}$, $\tau^{(2)}$ etc. the objects associated with $(M^{(2)}, \varphi_0^{(2)})$ analogous to N, τ etc. associated with (M, φ_0). Then one easily verifies that $N^{(2)} \cong N \oplus N$, $\tau^{(2)} \cong \tau \oplus \tau$, $(M^{(2)})_* \cong M_* \oplus M_*$, $h^{(2)}_{\varphi \oplus \psi} \cong h_\varphi \oplus h_\psi$, $\theta_s^{(2)} \cong \theta_s \oplus \theta_s$, $\tilde{N}^{(2)} \cong \tilde{N} \oplus \tilde{N}$, and finally (31). Furthermore, $\mathrm{tr}^{(2)} \cong \mathrm{tr} \oplus \mathrm{tr}$ so that

$$\begin{aligned} \|(a,b)\|_p^p &= \mathrm{tr}^{(2)} \left(\left| \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right|^p \right) = \mathrm{tr}^{(2)} \begin{pmatrix} |a|^p & 0 \\ 0 & |b|^p \end{pmatrix} \\ &= \mathrm{tr}(|a|^p) + \mathrm{tr}(|b|^p) = \|a\|_p^p + \|b\|_p^p \end{aligned}$$

for all $a, b \in L^p(M)$. This proves (32).

Proposition 31 (Clarkson's inequality)**.** *Let $p \in [2, \infty[$. Then for all $a, b \in L^p(M)$ we have*

$$\|a + b\|_p^p + \|a - b\|_p^p \leq 2^{p-1}(\|a\|_p^p + \|b\|_p^p).$$

Proof. Using Remark 30 we may reformulate the inequality to be proved as

$$\|(a + b, a - b)\|_p \leq 2^{\frac{1}{q}} \|(a,b)\|_p \tag{33}$$

where we have put $\frac{1}{q} = 1 - \frac{1}{p}$.

Let $(a,b) \in L^p(M \oplus M)$ and $(c,d) \in L^q(M \oplus M)$ such that

$$\|(a,b)\|_p = 1 \text{ and } \|(c,d)\|_q = 1. \tag{34}$$

Let

$$a = uh^{\frac{1}{p}}, b = vk^{\frac{1}{p}}$$

be the polar decompositions of a and b, and

$$c = f^{\frac{1}{q}}w, d = g^{\frac{1}{q}}z$$

the right polar decompositions of c and d. Then $h, k, f, g \in L^1(M)_+$ and

$$\|(h,k)\|_1 = 1, \|(f,g)\|_1 = 1.$$

For each $a \in S^\circ$, put

$$F(\alpha) = \mathrm{tr}\big((uh^\alpha + vk^\alpha)f^{1-\alpha}w + (uh^\alpha - vk^\alpha)g^{1-\alpha}z\big).$$

Then

$$F(\frac{1}{p}) = \mathrm{tr}((a+b)c + (a-b)d).$$

For all $a \in S^\circ$, we have

$$\begin{aligned} F(\alpha) = &\mathrm{tr}^{(2)}\left(\begin{pmatrix} u & 0 \\ 0 & -v \end{pmatrix}\begin{pmatrix} h^\alpha & 0 \\ 0 & k^\alpha \end{pmatrix}\begin{pmatrix} f^{1-\alpha} & 0 \\ 0 & g^{1-\alpha} \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}\right) \\ &+ \mathrm{tr}^{(2)}\left(\begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} k^\alpha & 0 \\ 0 & h^\alpha \end{pmatrix}\begin{pmatrix} f^{1-\alpha} & 0 \\ 0 & g^{1-\alpha} \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}\right) \end{aligned}$$

By Lemma 19 and 22 applied to $(h,k) \in L^1(M \oplus M)$ and $(f,g) \in L^1(M \oplus M)$ we conclude that F is analytic and

$$\forall \alpha \in S^\circ : |F(\alpha)| \leq 2. \tag{35}$$

we claim that

$$\forall t \in \mathbb{R} : \left| F(\frac{1}{2} + it) \right| \leq \sqrt{2}. \tag{36}$$

For the proof we app1y first the Cauchy-Schwarz inequality in

$\tilde{N}^{(2)}_{\frac{1}{2}+it}$, next the parallelogram law in $\tilde{N}_{\frac{1}{2}+it}$ (cf. Remark 29):

$$\begin{aligned}
&\left|F(\frac{1}{2}+it)\right|^2 \\
=&\left|\mathrm{tr}^{(2)}\left(\begin{pmatrix} uh^{\frac{1}{2}+it}+vk^{\frac{1}{2}+it} & 0 \\ 0 & uh^{\frac{1}{2}+it}-vk^{\frac{1}{2}+it}\end{pmatrix}\begin{pmatrix} f^{\frac{1}{2}-it}w & 0 \\ 0 & g^{\frac{1}{2}-it}z\end{pmatrix}\right)\right|^2 \\
\leq&\left\|\begin{pmatrix} uh^{\frac{1}{2}+it}+vk^{\frac{1}{2}+it} & 0 \\ 0 & uh^{\frac{1}{2}+it}-vk^{\frac{1}{2}+it}\end{pmatrix}\right\|_2^2\left\|\begin{pmatrix} f^{\frac{1}{2}-it}w & 0 \\ 0 & g^{\frac{1}{2}-it}z\end{pmatrix}^*\right\|_2^2 \\
=&(\left\|uh^{\frac{1}{2}+it}+vk^{\frac{1}{2}+it}\right\|_2^2+\left\|uh^{\frac{1}{2}+it}-vk^{\frac{1}{2}+it}\right\|_2^2)(\left\|f^{\frac{1}{2}-it}w\right\|_2^2+\left\|g^{\frac{1}{2}-it}z\right\|_2^2) \\
=&(2\left\|uh^{\frac{1}{2}+it}\right\|_2^2+2\left\|vk^{\frac{1}{2}+it}\right\|_2^2)(\left\|f^{\frac{1}{2}}\right\|_2^2+\left\|g^{\frac{1}{2}}\right\|_2^2)=2(\left\|h^{\frac{1}{2}}\right\|_2^2+\left\|k^{\frac{1}{2}}\right\|_2^2)=2
\end{aligned}$$

Finally, by the 3 lines theorem applied to each strip $\{\alpha\in\mathbb{C}|\epsilon\leq \operatorname{Re}\alpha\leq\frac{1}{2}\}$ where $0<\epsilon<\frac{1}{p}$, 35 and 36 give

$$\begin{aligned}
|\mathrm{tr}((a+b)c+(a-b)d)|=&\left|F(\frac{1}{p})\right| \\
\leq&2^{(\frac{1}{2}-\frac{1}{p})/(\frac{1}{2}-\epsilon)}\cdot(\sqrt{2})^{((\frac{1}{p})-\epsilon)/(\frac{1}{2}-\epsilon)} \\
\to&2^{1-\frac{2}{p}}\cdot 2^{\frac{1}{p}}=2^{\frac{1}{q}}\text{ as }\epsilon\to\infty.
\end{aligned}$$

Hence

$$\left|\mathrm{tr}^{(2)}\left(\begin{pmatrix} a+b & 0 \\ 0 & a-b\end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d\end{pmatrix}\right)\right|\leq 2^{\frac{1}{q}}$$

for all $(a,b)\in L^p(M\oplus M)$ and $(c,d)\in L^q(M\oplus M)$ satisfying (34). By Proposition 24 applied to $L^p(M\oplus M)$ this implies that

$$\|(a+b,a-b)\|_p\leq 2^{\frac{1}{q}}$$

for all $(a,b)\in L^p(M\oplus M)$ with $\|(a,b)\|_p=1$. (33) follows. □

By Clarkson's inequality, the Banach space $L^p(M)$, where $2\leq p<\infty$, is uniformly convex. Hence it is reflexive (see e.g. [22, p. 127, Theorem 2]).

Theorem 32. *Let $p \in [1, \infty[$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

1) Let $a \in L^q(M)$. Then φ_a defined by

$$\varphi_a(b) = \operatorname{tr}(ab), b \in L^p(M),$$

is a bounded linear functional on $L^p(M)$.

2) The mapping

$$a \mapsto \varphi_a$$

is an isometric isomorphism of $L^q(M)$ onto the dual Banach space of $L^p(M)$.

Proof. By Proposition 24, $a \mapsto \varphi_a$ is an isometry of $L^q(M)$ onto a subspace of the dual $L^p(M)^*$ of $L^p(M)$. Since $L^q(M)$ is complete, this subspace is closed. It follows from Proposition 24 that it is w*-dense (its orthogonal in $L^p(M)$ vanishes).

Now if $p \geq 2$, the space $L^p(M)$ is reflexive. Hence $L^p(M)^*$ is also reflexive and thus the w*-closure of the subspace $L^q(M)$ is equal to its norm closure. Hence $L^q(M) = L^p(M)^*$.

If $p < 2$, we have $q \geq 2$ and thus $L^p(M) \cong L^q(M)^*$ via tr. It follows that $L^p(M)^* \cong L^q(M)^{**} \cong L^q(M)$ (via tr). $\square$

Proposition 33. *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $a \in L^q(M)$. Then $a \geq 0$ if and only if*

$$\forall b \in L^p(M)_* : \operatorname{tr}(ab) \geq 0 \tag{37}$$

Proof. If $p, q \in \{1, \infty\}$, the result is well-known. Now assume that $p, q \in]1, \infty[$. If $a \in L^q(M)_+$, then $a^{\frac{1}{2}} b a^{\frac{1}{2}} \in L^1(M) \cap \tilde{N}_+ = L^1(M)_+$ and hence

$$\operatorname{tr}(ab) = \operatorname{tr}\left(a^{\frac{1}{2}} a^{\frac{1}{2}} b\right) = \operatorname{tr}\left(a^{\frac{1}{2}} b a^{\frac{1}{2}}\right) \geq 0$$

Conversely, suppose that $a \in L^q(M)$ satisfies (37). Then $a = a^*$ since

$$\operatorname{tr}(ab) = \overline{\operatorname{tr}(ab)} = \operatorname{tr}((ab)^*) = \operatorname{tr}(ba^*) = \operatorname{tr}(a^* b)$$

for all $b \in L^p(M)_+$. Put $a_+ = (a + |a|)/2, a_- = (a - |a|)/2 \in L^q(M)_+$. Then $a = a_+ - a_-$ and $a_+ a_- = 0$. Put $b = a_-^{\frac{q}{p}}$. Then $b \in L^p(M)_+$ so that $\operatorname{tr}(ab) \geq 0$. Now

$$\operatorname{tr}(ab) = \operatorname{tr}(a_+ b) - \operatorname{tr}(a_- b) = -\operatorname{tr}(a_- b) = -\operatorname{tr}(a_-^q).$$

It follows that $\mathrm{tr}(a_-^q) = 0$ whence $a_- = 0$ and $a = a_+ \in L^q(M)_+$.
$\square$

For each $p \in [1, \infty]$ we define left and right actions λ_p and ρ_p on $L^p(M)$ by

$$\lambda_p(x)a = xa, a \in L^p(M), \tag{38}$$

$$\rho_p(x)a = ax, a \in L^p(M), \tag{39}$$

for all $x \in M$. That $\lambda_p(x)$ and $\rho_p(x)$ map $L^p(M)$ into itself follows immediately from Definition 9. From Lemma 16 and the fact that $xN(\epsilon, \delta) \subset N(\|x\|\epsilon, \delta)$ for all $x \in M$ and $\epsilon, \delta \in \mathbb{R}_+$, we get

$$\forall x \in M \forall a \in L^p(M) : \|xa\|_p \leq \|x\|_\infty \|a\|_p. \tag{40}$$

Since $ax = (x^* a^*)^*$, we also have

$$\forall x \in M \forall a \in L^p(M) : \|ax\|_p \leq \|x\|_\infty \|a\|_p. \tag{41}$$

Hence $\lambda_p(x)$ and $\rho_p(x)$ are bounded linear operators on $L^p(M)$.

Proposition 34. *Let $p \in [1, \infty]$.*

1) λ_p (resp.ρ_p) is a faithful representation (resp. anti-representation) of M on the Banach space $L^p(M)$.

2) For all $x \in M$, we have

$$J_p \lambda_p(x) J_p = \rho_p(x^*),$$

where J_p denotes the conjugate linear isometric involution $a \mapsto a^*$ of $L^p(M)$.

3) Let z be an element of the center of M. Then

$$\lambda_p(z) = \rho_p(z).$$

Proof. 1) Suppose that $\lambda_p(x) = 0$. Then

$$\forall a \in L^p(M) \forall b \in L^q(M) : \mathrm{tr}(xab) = \mathrm{tr}((\lambda_p(x)a)b) = 0.$$

Since $L^1(M) = L^p(M) \cdot L^q(M)$, x must be 0.

2) For all $a \in L^p(M)$, we have

$$(J_p\lambda_p(x)J_p)(a) = (xa^*)^* = ax^* = \rho_p(x^*)a.$$

3) Clearly, $\lambda_\infty(z) = \rho_\infty(z)$. It follows that

$$\forall a \in L^1(M) \forall b \in L^\infty(M) : \mathrm{tr}(zab) = \mathrm{tr}(abz) = \mathrm{tr}(azb)$$

whence $\lambda_1(z) = \rho_1(z)$. In particular

$$\forall a \in L^1(M)_+ : za = az,$$

whence by spectral theory

$$\forall a \in L^1(M)_+ : za^{\frac{1}{p}} = a^{\frac{1}{p}}z.$$

Thus $\lambda_p(z)$ and $\rho_p(z)$ coincide on $L^p(M)_+$. Hence $\lambda_p(z) = \rho_p(z)$. $\square$

Proposition 35. *For all $p \in [1, \infty]$, we have*

$$\lambda_p(M) = \rho_p(M)' \text{ and } \rho_p(M) = \lambda_p(M)' \tag{42}$$

(where $\rho_p(M)'$, resp. $\lambda_p(M)'$, denotes the set of bounded linear operators on $L^p(M)$ commuting with all $\rho_p(x), x \in M$, resp. all $\lambda_p(x), x \in M$).

Proof. Obviously

$$\lambda_p(M) \subset \rho_p(M)' \text{ and } \rho_p(M) \subset \lambda_p(M)'.$$

To show (42) we need only prove either $\lambda_p(M) \supset \rho_p(M)'$ or $\rho_p(M) \supset \lambda_p(M)'$. Then the other one follows by Proposition 34, 2).

(i) First suppose that $p = \infty$. Let $T \in \lambda_\infty(M)'$. Then

$$\forall a \in L^\infty(M) : T(a) = T(a1) = aT(1)$$

whence $T = \rho_\infty(T(1)) \in \rho_\infty(M)$.

(ii) Next we consider the case $p = 1$. Let $S \in \lambda_1(M)'$. Denote by $T : L^\infty(M) \to L^\infty(M)$ the transpose of S given by

$$\mathrm{tr}(T(a)b) = \mathrm{tr}(aS(b)), a \in L^\infty(M), b \in L^1(M).$$

Now

$$\begin{aligned}\forall x \in M \forall a \in L^\infty(M) \forall b \in L^1(M) : \operatorname{tr}(T(ax)b) &= \operatorname{tr}(axS(b)) \\ &= \operatorname{tr}(aS(xb)) \\ &= \operatorname{tr}(T(a)xb).\end{aligned}$$

Thus $T \in \rho_\infty(M)'$ and hence $T = \lambda_\infty(y)$ for some $y \in M$. It follows that

$$\begin{aligned}\forall a \in L^\infty(M) \forall b \in L^1(M) : \operatorname{tr}(aS(b)) &= \operatorname{tr}(T(a)b) \\ &= \operatorname{tr}(yab) = \operatorname{tr}(aby),\end{aligned}$$

whence $S = \rho_1(y) \in \rho_1(M)$.

(iii) Now let $p \in]1, \infty[$. Let $T \in \lambda_p(M)'$. We want to define a linear mapping $S : L^1(M) \to L^1(M)$ by

$$S\left(\sum_{i=1}^n b_i a_i\right) = \sum_{i=1}^n b_i T(a_i) \tag{43}$$

for all $a_1, \ldots, a_n \in L^p(M)$ and $b_1, \ldots, b_n \in L^q(M)$. First let us show that

$$\sum_{i=1}^n b_i a_i = 0 \Rightarrow \sum_{i=1}^n b_i T(a_i) = 0 \tag{44}$$

=0 so that S is well-defined.

Suppose that $\sum_{i=1}^n b_i a_i = 0$. Put $a = (\sum_{i=1}^n a_i^* a_i)^{\frac{1}{2}} \in L^p(M)_+$. Then all $a_i^* a_i \leq a^2$. Hence there exist $x_1, \ldots, x_n \in M$ such that

$$a_i = x_i a \text{ and } \sum_{i=1}^n x_i^* x_i = \operatorname{supp} a.$$

Then

$$\left(\sum_{i=1}^n b_i x_i\right) a = \sum_{i=1}^n b_i a_i = 0$$

and

$$\operatorname{supp}\left(\sum_{i=1}^n b_i x_i\right) \leq \operatorname{supp} a$$

whence

$$\sum_{i=1}^{n} b_i x_i = 0.$$

It follows that

$$\sum_{i=1}^{n} b_i T(a_i) = \sum_{i=1}^{n} b_i T(x_i a) = \sum_{i=1}^{n} b_i x_i T(a) = \left(\sum_{i=1}^{n} b_i x_i\right) T(a) = 0$$

as wanted.

We have shown that $S : L^1(M) \mapsto L^1(M)$ is a well-defined linear map. It is also bounded. Indeed, any $c \in L^1(M)$ may be written as a product $c = ba$ where $a \in L^p(M), b \in L^q(M)$, and $\|c\|_1 = \|b\|_q \|a\|_p$. Then

$$\|S(c)\|_1 = \|bT(a)\|_1 \leq \|b\|_q \|T(a)\|_p \leq \|b\|_q \|T\| \|a\|_p = \|T\| \|c\|_1.$$

Finally, since

$$\forall x \in M \forall b \in L^q(M) \forall a \in L^p(M) : S(xba) = xbT(a) = xS(ba)$$

we have $S \in \lambda_1(M)'$. Hence $S = \rho_1(y)$ for some $y \in M$.

Now

$$bT(a) = S(ba) = bay = b\rho_p(y)a$$

for all $b \in L^q(M)$ and $a \in L^p(M)$. It follows that $T = \rho_p(y) \in \rho_p(M)$ as wanted. □

We shall denote λ_2 and ρ_2 simply by λ and ρ, and J_2 by J (i.e. $Ja = a^*$ for all $a \in L^2(M)$).

Theorem 36. *1) λ (resp. ρ) is a normal faithful representation (resp. anti-representation) of M on the Hilbert space $L^2(M)$.*

2) The von Neumann algebras $\lambda(M)$ and $\rho(M)$ are commutants of each other, and

$$\rho(M) = J\lambda(M)J$$

3) $(\lambda(M), L^2(M), J, L^2(M)_+)$ is a standard form of M in the sense of [4, Definition 2.1].

Proof. For all $x \in M$ and $a, b \in L^2(M)$ we have

$$(\lambda(x)a|b)_{L^2(M)} = \mathrm{tr}(b^*xa) = \mathrm{tr}((x^*b)^*a) = (a|\lambda(x^*)b)_{L^2(M)}$$

so that λ is a *-representation.

Suppose that $x_i \nearrow x \in M$. Then for all $a \in L^2(M)$, we have

$$\begin{aligned}(\lambda(x_i)a|a)_{L^2(M)} &= \mathrm{tr}(a^*x_ia) = \mathrm{tr}(x_iaa^*) = \langle x_i, aa^*\rangle \\ &\nearrow \langle x, aa^*\rangle = \mathrm{tr}(xaa^*) = \mathrm{tr}(a^*xa) = (\lambda(x)x|a)_{L^2(M)}.\end{aligned}$$

2) follows immediately from Proposition 35 and Proposition 34, 2).

3) That $L^2(M)_+$ is a self-dual cone follows from Proposition 33. Now

1. $J\lambda(M)J = \rho(M) = \lambda(M)'$;
2. $J\lambda(z)J = \rho(x^*) = \lambda(z^*) = \lambda(z)^*$ for all z in the center of M;
3. for all $a \in L^2(M)_+$, we have $a^* = a$;
4. for all $a \in L^2(M)_+$ and $x \in M$, we have $(\lambda(x)J\lambda(x)J)a = \lambda(x)\rho(x^*)a = xax^* \in L^2(M)_+$.

□

1.1 Independence of the choice of φ_0

The spaces $L^p(M)$ and their relations are independent of the choice of φ_0 (and hence canonically associated with M). This is a consequence of the following theorem and its corollary when we recall that the spaces $(L^p(M), \|\cdot\|_p)$ are defined in terms of $\tilde{N}$, $(\theta_s)_{s\in\mathbb{R}}$, and τ.

Let φ_0 and φ_1 be normal faithful semifinite weights on M. We view the crossed products $N_0 = R(M, \sigma^{\varphi_0})$ and $N_1 = R(M, \sigma^{\varphi_1})$ as von Neumann algebras on $L^2(\mathbb{R}, H)$. They are generated by $\pi_0(x), x \in M$, (resp. $\pi_1(x), x \in M$) and $\lambda(s), s \in \mathbb{R}$, where

$$(\pi_0(x)\xi)(t) = \sigma^{\varphi_0}_{-t}(x)\xi(t), (\pi_1(x)\xi)(t) = \sigma^{\varphi_1}_{-t}(x)\xi(t),$$

$$(\lambda(s)\xi)(t) = \xi(t-s)$$

for all $\xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}$.

Denote by $s \mapsto \theta_s$ the dual action of $\mathbb{R}$ in N_0 and N_1. Recall [18, Section 4] that each θ_s has the form

$$\theta_s(y) = \mu_s y \mu_s^{-1} \tag{45}$$

where μ_s is the unitary on $L^2(\mathbb{R}, H)$ given by

$$(\mu_s\xi)(t) = e^{-ist}\xi(t), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}. \tag{46}$$

Denote by τ_0, resp. τ_1, the trace on N_0, resp. N_1, given by (14).

Theorem 37. *There exists an isomorphism*

$$\kappa : N_0 \to N_1$$

such that

$$\forall s \in \mathbb{R} : \kappa \circ \theta_s \circ \kappa^{-1} = \theta_s \tag{47}$$

and

$$\tau_1 = \tau_0 \circ \kappa^{-1}. \tag{48}$$

Proof. (cf. [18,Proposition 3.5]). We define a unitary u on $L^2(\mathbb{R}, H)$ by

$$(u\xi)(t) = (D\varphi_1 : D\varphi_0)_{-t}\xi(t), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}.$$

Now

$$\forall x \in M : u\pi_0(x)u^* = \pi_1(x) \tag{49}$$

and

$$\forall s \in \mathbb{R} : u\lambda(s)u^* = \pi_1((D\varphi_1, D\varphi_0)_s^*)\lambda(s) \tag{50}$$

since

$$\begin{aligned}(u\pi_0(x)u^*\xi)(t) =&(D\varphi_1 : D\varphi_0)_{-t}\sigma_{-t}^{\varphi_0}(x)(D\varphi_1 : D\varphi_0)_{-t}^*\xi(t)\\ =&\sigma_{-t}^{\varphi_1}(x)\xi(t), t \in \mathbb{R},\end{aligned}$$

and

$$\begin{aligned}(u\lambda(s)u^*\xi)(t) =&(D\varphi_1 : D\varphi_0)_{-t}(D\varphi_1 : D\varphi_0)_{-(t-s)}^*\xi(t-s)\\ =&(D\varphi_1 : D\varphi_0)_{-t}((D\varphi_1 : D\varphi_0)_{-t}\sigma_{-t}^{\varphi_0}((D\varphi_1 : D\varphi_0)_s))^*\xi(t-s)\\ =&(D\varphi_1 : D\varphi_0)_{-t}\sigma_{-t}^{\varphi_0}((D\varphi_1 : D\varphi_0)_s^*)(D\varphi_1 : D\varphi_0)_{-t}^*\xi(t-s)\\ =&(\sigma_{-t}^{\varphi_1}((D\varphi_1 : D\varphi_0)_s^*)\lambda(s)\xi)(t), t \in \mathbb{R},\end{aligned}$$

for all $x \in M, s \in \mathbb{R}$, and $\xi \in L^2(\mathbb{R}, H)$.

Hence $\kappa = u(\cdot)u^*$ maps N_0 into N_1. Similarly, $u^*(\cdot)u$ maps N_1 into N_0. In all, we have shown that

$$\kappa : N_0 \to N_1$$

is an isomorphism of N_0 onto N_1. $\square$