Contents

Introduction

The main part of these notes (Chapter II) is devoted to a complete and detailed exposition of the theory of abstract L^p spaces associated with von Neumann algebras. This theory was developed by U. Haagerup some seven years ago and outlined in a preprint (which now appears in [9]). Unfortunately, in spite of his intentions, Haagerup has not yet had the time for writing down his theory in full. This is our motivation for writing these notes.

The proofs that we give are (close to) those that Haagerup originally had in mind and which he has told us at various occasions.

Essential for the construction of the L^p spaces is the theory of measurable operators with respect to a trace on a von Neumann algebra (due to E. Nelson [13] and inspired by [15] and [16]); we treat this in Chapter I. Other prerequisites are the basic facts on crossed products of a von Neumann algebra with a modular automorphism group and some results on operator valued weights and the extended positive part of a von Neumann algebra; we have not included this in the text but we give detailed references, especially to parts of ([7] and [8], at the places where it is needed.

After the appearance of Haagerup's L^p spaces, A. Connes proposed a definition of spatial L^p spaces based on the notion of spatial derivatives [1]. These spaces have been studied by H. Hilsum [10]. We include a discussion of them and show how their main properties follow easily from the corresponding properties of Haagerup's spaces (thus our presentation is complementary to Hilsum's work [10] where the objective is to develop the theory directly based on properties of spatial derivatives, avoiding as far as possible the dependence of Haagerup's construction). This is contained in Chapter IV.

Before this, we recall the main properties of spatial derivatives (Chapter III). We profit from this occasion to present a definition (due to U. Haagerup) of spatial derivatives that is slightly different from that given in [1] and to show how certain properties (such as the sum property) of spatial derivatives are almost immediate consequences of this new definition.

The reader will notice that these notes do not contain a special chapter on the - now classic - theory of spaces with respect to a trace, due - in various formulations - to J. Dixmier [3] and R. A. Kunze [12] (see also [21] and [13]). Although this important particular case has been motivating for the development of the more general theory, we do not directly need it in our preliminaries.For the sake of completeness, however, we give the definition of L^p spaces with respect to a trace at the end of Chapter I, and in the following chapters, we point out how results concerning the trace case are related to the general results.

Another omission in these notes is the recent definition of L^p spaces as complex interpolation spaces. For this, we simply refer to [11] and [20].

Chapter 1

Measurable Operators with Respect to a Trace

In this chapter, we define the notion of measurability with respect to a trace τ on a von Neumann algebra M and show that the set M of τ-measurable operators is a complete topological *-algebra. Our presentation is a modified version of that given by E. Nelson [13].

Let M be a - necessarily semifinite - von Neumann algebra acting on a Hilbert space H and let τ be a normal faithful semifinite trace on M.

For the convenience of the reader, we immediately give the definition of τ-measurability and state the main theorem about τ-measurable operators.

Definition 14: A closed densely defined operator a affiliated with M is called τ-measurable if for all $\delta \in \mathbb{R}_+$ there exists a projection $p \in M$ such that

$$pH \subset D(a) \text{ and } \tau(1-p) \leq \delta$$

For a characterization of τ-measurable operators in terms of the spectral projections of their absolute value, see Proposition 21 below.

We denote by $\widetilde{M}$ the set of τ-measurable closed densely defined operators.

Theorem 28. 1) $\widetilde{M}$ is a *-algebra with respect to strong sum, strong product, and adjoint operation.

2) The sets

$$N(\epsilon, \delta) = \{a \in \widetilde{M} | \exists p \in M_{\text{proj}} : pH \subset D(a), \|ap\| \leq \epsilon, \tau(1-p) \leq \delta\},$$

where $\epsilon, \delta \in \mathbb{R}_+$, form a basis for the neighbourhoods of 0 for a topology on $\widetilde{M}$ that turns $\widetilde{M}$ into a topological vector space.

3) $\widetilde{M}$ is a complete Hausdorff topological * -algebra and M is a dense subset of $\widetilde{M}$.

Once this theorem has been proven, we can freely add and multiply operators from $\widetilde{M}$, the operations being understood in the strong sense (see the definition below). Until then, we have to deal with unbounded operators in the usual careful way.

Although we are mainly interested in closed densely defined opera tors it will be convenient for us to work with more general kinds of unbounded operators. We therefore start by recalling some basic facts on arbitrary unbounded operators. Next, we recall some properties of the lattice M_{proj} of projections in M. After this, we go on to develop the theory of τ-measurability.

Preliminaries on unbounded operators.

Recall that for any (linear) operators a and b on H we can define the sum $a+b$ and the product ab as operators on H with domains

$$D(a+b) = D(a) \cap D(b), \tag{1}$$

$$D(ab) = \{\xi \in D(b) | b\xi \in D(a)\}. \tag{2}$$

These operations are associative so that $a + b + c$ and abc are well-defined operators. Furthermore, for all a, b and c we have

$$(a + b)c = ac + bc \text{ and } c(a + b) \supset ca + cb \tag{3}$$

(with equality if $D(c) = H$).

We shall use the following terminology: an operator a on H is closed if its graph $G(a)$ is closed in $H \otimes H$; a is preclosed if the closure $\overline{G(a)}$ of its graph is the graph of some - necessarily closed - operator (the closure of a, denoted $[a]$; a is densely defined if $D(a)$ is dense in H.

If a, b and ab are densely defined, then

$$(ab)^* \supset b^* a^* \tag{4}$$

with equality if a is bounded and everywhere defined.

A closed densely defined operator a has a unique polar decomposition

$$a = u|a| \tag{5}$$

where $|a|$ is a positive self-adjoint operator and u a partial isometry with $\mathrm{supp}(a)$ as its initial projection and $r(a)$, the projection onto the closure of the range of a, as its final projection.

If the sum $a + b$ of two closed densely defined operators a and b is preclosed and densely defined, then the closure $[a+b]$ is called the strong sum of a and b. Similarly, the strong product is the closure $[ab]$ if ab is preclosed and densely defined.

We shall write

$$\|a\| = \sup\{\|a\xi\| | \|\xi\| \leq 1\}$$

for all everywhere defined operators a on H, bounded or not.For all such operators, the usual norm estimates hold:

$$\|a + b\| \leq \|a\| + \|b\|, \|ab\| \leq \|a\|\|b\|.$$

Denote by M' the commutant of M.

Definition 1. *A linear operator a on H is said to be affiliated with M (and we write $a\eta H$) if*

$$\forall y \in M' : ya \subset ay \tag{6}$$

Remark 2. *Using (3), (4) and (5) one easily verifies that*

1. if $a, b\eta M$, then $a + b\eta M$ and $ab\eta H$;

2. if a is preclosed, resp. densely defined, and $a\eta M$, then $[a]\eta M$, resp. $a^*\eta M$;

3. if a is a closed densely defined operator with polar decomposition $a = u|a|$, then $a\eta M$ if and only if $u \in M$.

Notation. We denote by $\overline{M}$ the set of closed densely defined operators affiliated with M.

Preliminaries on projections.

We denote by M_{proj} the lattice of (orthogonal) projections in M. For a family $(p_i)_{i\in I}$ of projections in M, $\wedge_{i\in I}p_i$ (resp. $\vee_{i\in I}p_i$) is the projection onto $\cap_{i\in I}p_iH$ (resp. $\overline{\cup_{i\in I}p_iH}$).

Recall that

$$(\wedge_{i\in I}p_i)^\perp = \vee_{i\in I}p_i^\perp, (\vee_{i\in I}p_i)^\perp = \wedge_{i\in I}p_i^\perp \tag{7}$$

where $p^\perp = 1 - p$ is the projection orthogonal to p

Two projections p and q are equivalent if $p = u^*u$ and $q = uu^*$ for some $u \in M$. We denote equivalence by $\sim$. Equivalent projections have the same trace.

By the polar decomposition theorem, we have

Lemma 3. *Let a be a closed densely defined operator affiliated with M. Then*

$$\operatorname{supp}(a) \sim \mathrm{r}(a)$$

where $\mathrm{r}(a)$ denotes the projection onto the closure of the range of a.

For any projections $p, q \in M$ we have

$$(p \vee q) - p \sim q - (p \wedge q). \tag{8}$$

It follows that

$$\tau(p \vee q) \leq \tau(p) + \tau(q). \tag{9}$$

More generally,

$$\tau(\vee_{i\in I}p_i) \leq \sum_{i\in I} \tau(p_i) \tag{10}$$

for any family $(p_i)_{i\in I}$ of projections in M (if I is finite, this follows by induction from (9); for the general case, use the normality of τ).

Another consequence of (8) is this:

$$\forall p, q \in M_{\text{proj}} : p \wedge q = 0 \Rightarrow p \lesssim 1 - q \tag{11}$$

(where $\lesssim$ means: "is equivalent to a subprojection of"). Indeed,

$$p = 1-p^{\perp} = (p\wedge q)^{\perp}-p^{\perp} = (p^{\perp}\vee q^{\perp})-p^{\perp} \sim q^{\perp}-(p^{\perp}\wedge q^{\perp}) \leq q^{\perp} = 1-q.$$

The theory of τ-measurable operators.

Definition 4. *Let $\epsilon, \delta \in \mathbb{R}_+$. Then we denote by $D(\epsilon, \delta)$ the set of all operators $a\eta M$ for which there exists a projection $p \in M$ such that*

1. *$pH \subset D(a)$ and $\|ap\| \leq \epsilon$ and*
2. *$\tau(1-p) \leq \delta$.*

When $pH \subset D(a)$, the operator ap is everywhere defined. The requirement $\|ap\| \leq \epsilon$ in particular implies that ap is bounded.

Note that we do not require a to be densely defined, closed or preclosed.

Proposition 5. *Let $\epsilon_1, \epsilon_2, \delta_1, \delta_2 \in \mathbb{R}_+$. Then*

1. *$D(\epsilon_1, \delta_1) + D(\epsilon_2, \delta_2) \subset D(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$,*
2. *$D(\epsilon_1, \delta_1)D(\epsilon_2, \delta_2) \subset D(\epsilon_1\epsilon_2, \delta_1 + \delta_2)$.*

Proof. (1) Let $a \in D(\epsilon_1, \delta_1)$ and $b \in D(\epsilon_2, \delta_2)$. Then there exist projections $p, q \in M$ such that

$$pH \subset D(a), \|ap\| \leq \epsilon_1, \text{ and } \tau(1-p) \leq \delta_1,$$
$$qH \subset D(b), \|bq\| \leq \epsilon_2, \text{ and } \tau(1-q) \leq \delta_2.$$

Put $r = p \wedge q$. Then

$$rH = pH \cap qH \subset D(a) \cap D(b) = D(a+b)$$

and

$$\|(a+b)r\| = \|ar + br\| \leq \|ar\| + \|br\| \leq \|ap\| + \|bq\| \leq \epsilon_1 + \epsilon_2$$

Furthermore,

$$\tau(1-r) = \tau((p \wedge q)^{\perp}) = \tau(p^{\perp} \vee q^{\perp}) \leq \tau(1-p) + \tau(1-q) \leq \delta_1 + \delta_2$$

This proves (1).

To prove (2), let $a \in D(\epsilon_1, \delta_1)$, $b \in D(\epsilon_2, \delta_2)$ and take $p, q \in M_{\text{proj}}$ as above. Then bq, and hence $(1-p)bq$, is bounded. Denote by s the projection onto its null space:

$$sH = N((1-p)bq).$$

Then $bq\xi \in pH \subset D(a)$ for all $\xi \in sH$, so that $sH \subset D(abq)$ and hence

$$(q \wedge s)H \subset D(ab)$$

Also, $abqs = apbqs$ so that

$$ab(q \wedge s) = abqs(q \wedge s) = apbq(q \wedge s)$$

and thus

$$\|ab(q \wedge s)\| \leq \|ap\|\|bq\| \leq \epsilon_1 \epsilon_2.$$

On the other hand, using that

$$1 - s = \operatorname{supp}((1-p)bq) \sim \mathrm{r}((1-p)bq) \leq 1 - p,$$

we have

$$\begin{aligned} \tau(1 - (q \wedge s)) = \tau((1-q) \vee (1-s)) \leq & \tau(1-q) + \tau(1-s) \\ \leq & \tau(1-q) + \tau(1-p) \leq \delta_1 + \delta_2. \end{aligned}$$

This completes the proof. □

Proposition 6. *Let $\epsilon, \delta \in \mathbb{R}_+$.*

1. *Let a be a preclosed operator. Then*

$$a \in D(\epsilon,\delta) \Rightarrow [a] \in D(\epsilon,\delta).$$

2. *Let a be a closed densely defined operator with polar decomposition $a = u|a|$. Then*

$$a \in D(\epsilon,\delta) \Leftrightarrow u \in M \text{ and } |a| \in D(\epsilon,\delta).$$

Proof. (1): trivial. (2): trivial, since $a = u|a|$, $|a| = u^*a$, and $\|u\| \leq 1$. $\square$

Lemma 7. *Let $a \in \overline{M}$ and $\epsilon, \delta \in \mathbb{R}_+$. Then*

$$a \in D(\epsilon,\delta) \Leftrightarrow \tau(\chi_{]\epsilon,\infty[}(|a|)) \leq \delta$$

(where $\chi_{]\epsilon,\infty[}(|a|)$ denotes the spectral projection of $|a|$ corresponding to the interval $]\epsilon,\infty[$).

Proof. "$\Leftarrow$": Put $p = \chi_{[0,\epsilon]}(|a|)$. Then $pH \subset D(|a|)$ and $\||a|p\| \leq \epsilon$.

"$\Rightarrow$": For some $p \in M_{\mathrm{proj}}$, we have

$$\||a|p\| \leq \epsilon \text{ and } \tau(1-p) \leq \delta.$$

Let $|a| = \int_0^\infty \lambda \mathrm{d}e_\lambda$ be the spectral decomposition of $|a|$. Now for all $\xi \in pH$ we have

$$\||a|\xi\|^2 \leq \epsilon^2 \|\xi\|^2,$$

and for all $\xi \in (1-e_\epsilon)H \setminus \{0\}$ we have

$$\||a|\xi\|^2 > \epsilon^2 \|\xi\|^2$$

since

$$\||a|\xi\|^2 = \int_0^\infty \lambda^2 \,\mathrm{d}(e_\lambda \xi|\xi) = \int_{]\epsilon,\infty[} \lambda^2 \,\mathrm{d}(e_\lambda \xi|\xi)\,.$$

Hence $(1-e_\epsilon)H \cap pH$ must be $\{0\}$, i.e. $(1-e_\epsilon) \wedge p = 0$. By (11) we conclude that $1-e_\epsilon \precsim 1-p$, whence $\tau(1-e_\epsilon) \leq \delta$. $\square$

Proposition 8. *Let $a \in \overline{M}$ and $\epsilon, \delta \in \mathbb{R}_+$. Then*

$$a \in D(\epsilon, \delta) \Leftrightarrow a^* \in D(\epsilon, \delta)$$

Proof. Let $a = u|a|$ be the polar decomposition of a. Then u is an isometry of $\chi_{]0,\infty[}(|a|) = \mathrm{supp}(a)$ onto $\chi_{]0,\infty[}(|a^*|) = \mathrm{supp}(a^*) = \mathrm{r}(a)$. By uniqueness of the spectral decomposition, u induces for each $\lambda \in \mathbb{R}_+$ an isometry of $\chi_{]\lambda,\infty[}(|a|)$ onto $\chi_{]\lambda,\infty[}(|a^*|)$. The result follows by Lemma 7. □

Definition 9. *A subspace E of H is called τ-dense if for all $\delta \in \mathbb{R}_+$, there exists a projection $p \in M$ such that*

$$pH \subset E \text{ and } \tau(1-p) \leq \delta.$$

Proposition 10. *Let E be a τ-dense subspace of H . Then there exists an increasing sequence $(p_n)_{n\in\mathbb{N}}$ of projections in M with*

$$p_n \nearrow 1, \tau(1-p_n) \to 0, \text{ and } \cup_{n=1}^{\infty} p_n H \subset E.$$

Proof. Take projections $q_k \in M$, $k \in N$, such that

$$q_k H \subset E \text{ and } \tau(1-q_k) \leq 2^{-k}.$$

For each $n \in N$, put

$$p_n = \wedge_{k=n+1}^{\infty} q_k.$$

Then

$$p_n H = \cap_{k=n+1}^{\infty} q_k H \subset E$$

and

$$\tau(1-p_n) = \tau\left(\vee_{k=n+1}^{\infty}(1-q_k)\right) \leq \sum_{k=n+1}^{\infty} \tau(1-q_k) \leq \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n}$$

It follows that

$$p_n \nearrow 1;$$

indeed, denoting by p the supremum of the increasing sequence p_n, we have

$$\forall n \in \mathbb{N} : \tau(1-p) \leq \tau(1-p_n) \leq 2^{-n}$$

whence $\tau(1-p)=0$ and $p=1$.

Furthermore,

$$\cup_{n=1}^{\infty} p_n H \subset E.$$

$\square$

Corollary 11. *Let E be a τ-dense subspace of H. Then E is dense in H.*

An important property of τ-dense subspaces is the following:

Proposition 12. *Let $a, b \in \overline{M}$ and let E be a τ-dense subspace of H contained in $D(a) \cap D(b)$. Suppose that*

$$a|_E = b|_E.$$

Then $a = b$.

The proof is based on the following lemma:

Lemma 13. *1) Let $p_0 \in M_{proj}$. Suppose that*

$$\forall \delta \in \mathbb{R}_+ \exists p \in M_{proj} : p_0 \wedge p = 0 \text{ and } \tau(1-p) \leq \delta.$$

Then $p_0 = 0$.

2) Let $p_1, p_2 \in M_{proj}$. Suppose that

$$\forall \delta \in \mathbb{R}_+ \exists p \in M_{proj} : p_1 \wedge p = p_2 \wedge p \text{ and } \tau(1-p) \leq \delta.$$

and $p_1 = p_2$.

Proof. 1) Let $\delta \in \mathbb{R}_+$. Then $\tau(p_0) \leq \delta$. (indeed, for some $p \in M_{\text{proj}}$ we have $p_0 \wedge p = 0$ and $\tau(1-p) \leq \delta$, whence $p_0 \precsim 1-p$ and $\tau(p_0) \leq \tau(1-p) \leq \delta$). Hence $\tau(p_0) = 0$ and $p_0 = 0$.

2) Put $p_0 = p_1 - (p_1 \wedge p_2)$. Now $p_1 \wedge p = p_2 \wedge p$ implies $p_1 \wedge p = (p_1 \wedge p_2) \wedge p$ and hence $p_0 \wedge p = 0$, so that 1) applies to p_0. Hence $p_0 = 0$, i.e. $p_1 = p_1 \wedge p_2$. Similarly, $p_2 = p_1 \wedge p_2$. In all, $p_1 = p_2$. $\square$

Proof of Proposition 12. Consider in the Hilbert space $H_2 = H \oplus H$ the von Neumann algebra $M_2 = \begin{bmatrix} M & M \\ M & M \end{bmatrix}$ equipped with the normal faithful semifinite trace τ_2 defined by

$$\tau \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \tau(x_{11}) + \tau(x_{22}).$$

Denote by p_a and p_b the projections onto the graphs $G(a)$ and $G(b)$ of a and b. Since a and b are affiliated with M, $G(a)$ and $G(b)$ are invariant under all elements of $M_2' = \{ \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} : y \in M' \}$ and thus $p_a, p_b \in M_2$.

Let $\delta \in \mathbb{R}_+$. Then there exists a projection $p \in M$ with $pH \subset E$ and $\tau(1-p) \leq \frac{\delta}{2}$. Put $p_2 = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$. Then $\tau_2(1-p_2) \leq \delta$.

Furthermore,

$$p_a \wedge p_2 = p_b \wedge p_2$$

since a and b agree on $pH \subset E$ and thus

$$\begin{aligned} G(a) \cap (pH \oplus pH) =& \{\langle \xi, a\xi \rangle, \xi \in pH, a\xi \in pH\} \\ =& \{\langle \xi, b\xi \rangle, \xi \in pH, b\xi \in pH\} = G(b) \cap (pH \oplus pH). \end{aligned}$$

By Lemma 13, we conclude that $p_a = p_b$, whence $a = b$. $\square$

Definition 14. *An operator $a \in \overline{M}$ is called τ-measurable if $D(a)$ is τ-dense, i.e. if for all $\delta \in \mathbb{R}_+$ there exists a projection $p \in M$ such that*

$$pH \subset D(a) \text{ and } \tau(1-p) \leq \delta. \tag{12}$$

The set of τ-measurable operators $a \in \overline{M}$ is denoted $\widetilde{M}$.

Corollary 15. *1) Let $a, b \in \widetilde{M}$. If*

$$a \subset b$$

then

$$a = b.$$

2) Let $a \in \widetilde{M}$. If a is symmetric (in particular, if a is positive), then a is self-adjoint.

Proof. Immediate from Definition 14 and Proposition 12 (for 2), use that $a \subset a^*$). $\square$

Note that when a is closed and $p \in M_{\text{proj}}$ is such that $pH \subset D(a)$, then the everywhere defined operator ap is also closed and hence - by the closed graph theorem - automatically bounded. Therefore the following definition is a generalization of Definition 14.

Definition 16. *Any operator $a\eta M$ is called τ-premeasurable if for all $\delta \in \mathbb{R}_+$ there exists a projection $p \in M$ such that*

$$pH \subset D(a), \|ap\| < \infty, \text{ and } \tau(1-p) \leq \delta. \tag{13}$$

By definition of the $D(\epsilon, \delta)$, this may be reformulated as:

Remark 17. *Let $a\eta M$. Then a is τ-premeasurable if and only if*

$$\forall \delta \in \mathbb{R}_+ \exists \epsilon \in \mathbb{R}_+ : a \in D(\epsilon, \delta).$$

Also note

Proposition 18. *Let $a\eta M$. If a is τ-premeasurable, then a is densely defined.*

Proof. $D(a)$ is τ-dense. $\square$

Proposition 19. *Let $a\eta M$. Suppose that a is τ-premeasurable and preclosed. Then*

$$[a] \in \widetilde{M}.$$

Proof. Trivial. $\square$

Proposition 20. *Let $a, b\eta M$ be τ-premeasurable. Then $a + b$ and ab are also τ-premeasurable.*

Proof. Combine Remark 17 and Proposition 5. $\square$

We have the following characterization of τ-measurable operators:

Proposition 21. *Let $a \in \overline{M}$ with polar decomposition $a = u|a|$. Then the following assertions are equivalent:*

1. *a is τ-measurable,*

2. *$|a|$ is τ-measurable,*

3. $\forall \delta \in \mathbb{R}_+ \exists \epsilon \in \mathbb{R}_+ : a \in D(\epsilon, \delta)$,

4. $\forall \delta \in \mathbb{R}_+ \exists \epsilon \in \mathbb{R}_+ : \tau(\chi_{]\epsilon,\infty[}(|a|)) \leq \delta$,

5. $\tau(\chi_{]\lambda,\infty[}(|a|)) \to 0$ *as* $\lambda \to \infty$,

6. $\forall \lambda \in \mathbb{R}_+ : \tau(\chi_{]\lambda,\infty[}(|a|)) < \infty$.

Proof. The equivalence of (i), (ii), and (iii), follows from Lemma 7. Now note that

$$\tau(\chi_{]\lambda,\infty[}(|a|)) \searrow \emptyset \text{ as } \lambda \to \infty$$

so that, by the normality of τ,

$$\tau(\chi_{]\lambda,\infty[}(|a|)) \searrow 0 \text{ as } \lambda \to \infty$$

□