Contents

Chapter 1

Spatial Derivatives

Spatial derivatives were introduced by A. Connes in [1]. In this chapter, we give an alternative definition (equivalent to that given in [1]) suggested to us by U. Haagerup, based on the notion of the extended positive part of a von Neumann algebra. This definition permits us to obtain very easily some elementary properties of spatial derivatives. After this, we recall their main modular properties and the characterization as (-1)-homogeneous operators.

1.1 Definition and elementary properties of spatial derivatives

Let M be a von Neumann algebra acting on a Hilbert space H, and let ψ be a normal faithful semifinite weight on the commutant M' of M.

We shall use the following standard notation: $n_\psi = \{y \in M' | \psi(y^*y) < \infty\}$, H_ψ the Hilbert space completion of n_ψ with respect to the inner product $(y_1, y_2) \mapsto \psi(y_2^* y_1)$, Λ_ψ the canonical injection of n_ψ into H_ψ, π_ψ the canonical representation of M' on H_ψ.

Definition 1. *For each $\xi \in H$, we denote by $R^\psi(\xi)$ the (densely defined) operator from H_ψ to H defined by*

$$R^\psi(\xi)\Lambda_\psi(y) = y\xi, y \in n_\psi. \tag{1}$$

Proposition 2. *For all* $\xi, \xi_1, \xi_2 \in H$*,* $x \in M$*, and* $y \in M'$ *we have*

(i) $R^\psi(\xi_1 + \xi_2) = R^\psi(\xi_1) + R^\psi(\xi_2)$,

(ii) $R^\psi(x\xi) = xR^\psi(\xi)$,

(iii) $yR^\psi(\xi) \subset R^\psi(\xi)\pi_\psi(y)$,

and

*(i)** $R^\psi(\xi_1)^* + R^\psi(\xi_2)^* \subset R^\psi(\xi_1 + \xi_2)^*$,

*(ii)** $R^\psi(x\xi)^* = R^\psi(\xi)^* x^*$,

*(iii)** $\pi_\psi(y)R^\psi(\xi)^* \subset R^\psi(\xi)^* y$.

Proof. (i) and (ii) are immediate from Definition 1. (iii): For all $z \in n_\psi$, we have $yR^\psi(\xi)\Lambda_\psi(z) = yz\xi = R^\psi(\xi)\Lambda_\psi(yz) = R^\psi(\xi)\pi_\psi(y)\Lambda_\psi(z)$.

(i)*, (ii)*, and (iii)* follow from (i), (ii), and (iii) using $R^\psi(\xi_1) + R^\psi(\xi_2) \subset (R^\psi(\xi_1) + R^\psi(\xi_2))^*$, $(xR^\psi(\xi))^* = R^\psi(\xi)^* x^*$, and $(y^* R^\psi(\xi))^* = R^\psi(\xi)^* y^*$. □

Definition 3. *A vector* $\xi \in H$ *is called* ψ*-bounded if the operator* $R^\psi(\xi)$ *is bounded. The set of* ψ*-bounded vectors is denoted* $D(H, \psi)$*.*

Notation. *If* $\xi \in D(H, \psi)$*,* $R^\psi(\xi)$ *extends to a bounded operator* $H_\psi \to H$ *which we shall also denote* $R^\psi(\xi)$*.*

Proposition 4. *The set* $D(H, \psi)$ *is an* M*-invariant dense subspace of* H*.*

Proof. That $D(H, \psi)$ is an M-invariant subspace of H follows from Proposition 2, i) and (ii). Denote by e the projection onto $\overline{D(H, \psi)}$; then $e \in M'$. Suppose that $e \neq 1$. Then $\psi(1 - e) > 0$. We can write $\psi = \sum_{i \in I} \omega_{\zeta_i, \zeta_i}$ for certain $\zeta_i \in H$. Then for at least one ζ_i, we have $((1 - e)\zeta_i | \zeta_i) \neq 0$ so that $(1 - e)\zeta_i \neq 0$. On the other hand, we have

$$\forall y \in n_\psi : \|y\zeta_i\|^2 \leq \psi(y^* y) = \|\Lambda_\psi(y)\|^2$$

so that $\zeta_i \in D(H, \psi)$ and hence $e\zeta_i = \zeta_i$. This is a contradiction. Hence we must have $e = 1$ and $D(H, \psi)$ is dense in H. □

Let $\xi \in H$. By Proposition 2, (iii)*, $D(R^\psi(\xi)^*)$ is invariant under the action of M'. Hence the projection p onto $\overline{D(R^\psi(\xi)^*)}$ is in M. Considered as an operator from pH to H_ψ, $R^\psi(\xi)^*$ is closed and densely defined and hence $\left|R^\psi(\xi)^*\right|^2$ exists as a positive self-adjoint operator on pH which by Proposition 2, (iii)*, is affiliated with pMp. We denote by $\theta^\psi(\xi,\xi)$ the element of $\widehat{M}_+$ (the extended positive part of M) associated with the couple $(pH, \left|R^\psi(\xi)^*\right|^2)$ as in [7, Example 1.2 and Lemma 1.4], i.e.

Definition 5. *For each $\xi \in H$, we denote by*

$$\theta^\psi(\xi,\xi)$$

the element of $\widehat{M}_+$ characterized by

$$\forall \eta \in H : \langle \omega_{\eta,\eta}, \theta^\psi(\xi,\xi)\rangle = \begin{cases} \left\|R^\psi(\xi)^*\eta\right\|^2 & \text{if } \eta \in D(R^\psi(\xi)^*) \\ \infty & \text{otherwise} \end{cases}. \tag{2}$$

Remark 6. *If $\xi \in D(H,\psi)$, we simply have*

$$\theta^\psi(\xi,\xi) = R^\psi(\xi)R^\psi(\xi)^*. \tag{3}$$

Proposition 7. *For all $\xi \in H$ and $x \in M$, we have*

$$\theta^\psi(x\xi, x\xi) = x \cdot \theta^\psi(\xi,\xi) \cdot x^*.$$

Proof. For all $\eta \in H$, we have, using Proposition 2, (ii)*, and Definition 5

$$\begin{aligned} \langle \omega_{\eta,\eta}, \theta^\psi(x\xi, x\xi)\rangle &= \langle \omega_{x^*\eta, x^*\eta}, \theta^\psi(\xi,\xi)\rangle \\ &= \langle x^* \cdot \omega_{\eta,\eta} \cdot x, \theta^\psi(\xi,\xi)\rangle \\ &= \langle \omega_{\eta,\eta}, x \cdot \theta^\psi(\xi,\xi) \cdot x^*\rangle \end{aligned}$$

where the last equality simply follows from the definition of the operation $m \mapsto x \cdot m \cdot x^*$ in $\widehat{M}_+$. □

Recall that by [7, Proposition 1.10], every normal weight φ has a unique extension, also denoted φ, to a normal weight on $\widehat{M}_+$.

Definition 8. *Let φ be a normal weight on M. We define*

$$q_\varphi : H \to [0, \infty]$$

by

$$q_\varphi(\xi) = \langle \varphi, \theta^\psi(\xi, \xi) \rangle, \xi \in H. \tag{4}$$

Proposition 9. *Let φ be a normal weight on M. Then q_φ is a l.s.c. quadratic form on M, i.e.*

(i) $\forall \xi_1, \xi_2 \in H : q_\varphi(\xi_1 + \xi_2) + q_\varphi(\xi_1 - \xi_2) = 2q_\varphi(\xi_1) + 2q_\varphi(\xi_2)$,

(ii) $\forall \xi \in H \forall \lambda \in \mathbb{C} : q_\varphi(\lambda\xi) = |\lambda|^2 q_\varphi(\xi)$,

(iii) q_φ *is lower semi-continuous.*

Proof. (ii) is immediate. For the proof of (i) and (iii), first suppose that $\varphi = \omega_{\eta,\eta}$ for some $\eta \in H$. Then

$$q_\varphi(\xi) = \langle \omega_{\eta,\eta}, \theta^\psi(\xi, \xi) \rangle = \begin{cases} \left\| R^\psi(\xi)^* \eta \right\|^2 & \text{if } \eta \in D(R^\psi(\xi)^*) \\ \infty & \text{otherwise} \end{cases}. \tag{5}$$

Let $\xi_1, \xi_2 \in H$. We shall prove that

$$q_\varphi(\xi_1 + \xi_2) + q_\varphi(\xi_1 - \xi_2) \leq 2q_\varphi(\xi_1) + 2q_\varphi(\xi_2). \tag{6}$$

If either $\eta \in D(R^\psi(\xi_1)^*)$ or $\eta \in D(R^\psi(\xi_2)^*)$, the right hand side of (6) is $+\infty$ and hence (6) holds. Now suppose that $\eta \in D(R^\psi(\xi_1)^*)$ and $\eta \in D(R^\psi(\xi_2)^*)$. Then by Proposition 2, (i)*, also $\eta \in D(R^\psi(\xi_1 + \xi_2)^*)$ and $\eta \in D(R^\psi(\xi_1 - \xi_2)^*)$. Furthermore,

$$\begin{aligned} &\left\| R^\psi(\xi_1 + \xi_2)^* \eta \right\|^2 + \left\| R^\psi(\xi_1 - \xi_2)^* \eta \right\|^2 \\ =&\left\| R^\psi(\xi_1)^* \eta + R^\psi(\xi_2)^* \eta \right\|^2 + \left\| R^\psi(\xi_1)^* \eta - R^\psi(\xi_2)^* \eta \right\|^2 \\ =&2\left\| R^\psi(\xi_1)^* \eta \right\|^2 + 2\left\| R^\psi(\xi_2)^* \eta \right\|^2. \end{aligned}$$

Thus we have proved (6) in all cases.

By (6) applied to $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ we get

$$4(q_\varphi(\xi_1) + q_\varphi(\xi_2)) = q_\varphi(2\xi_1) + q_\varphi(2\xi_2) \leq 2q_\varphi(\xi_1 + \xi_2) + 2q_\varphi(\xi_1 - \xi_2).$$

In all, we have shown (i).

By (5), we have

$$\begin{aligned}\langle\omega_{\eta,\eta},\theta^{\psi}(\xi,\xi)\rangle &= \sup\{\left|(R^{\psi}(\xi)^{*}\eta|\zeta)\right|^{2}|\zeta\in D(R^{\psi}(\xi)),\|\zeta\|\leq 1\}\\ &= \sup\{\left|(\eta|R^{\psi}(\xi)\Lambda_{\psi}(y))\right|^{2}|y\in n_{\psi},\|\Lambda_{\psi}(y)\|\leq 1\}\\ &= \sup\{|(\eta|y\xi)|^{2}|y\in n_{\psi},\|\Lambda_{\psi}(y)\|\leq 1\}\end{aligned}$$

for all $\xi\in H$. Since each $\xi\mapsto|(\eta|y\xi)|^{2}$ is continuous, this implies (iii).

Now let φ be an arbitrary normal weight. Then we can write

$$\varphi=\sum_{i\in I}\omega_{\eta_i,\eta_i}$$

and thus (cf. the proof of [7, Proposition 1.10])

$$\forall\xi\in H: q_{\varphi}(\xi)=\langle\varphi,\theta^{\psi}(\xi,\xi)\rangle=\sum_{i=I}\langle\omega_{\eta_i,\eta_i},\theta^{\psi}(\xi,\xi)\rangle.$$

Now (i) and (iii) follow by the first part of the proof. □

Remark 10. *Let φ be a normal weight on M. Write*

$$\mathrm{Dom}(q_{\varphi})=\{\xi\in H|q_{\varphi}(\xi)<\infty\}. \tag{7}$$

Then for all $x\in n_{\varphi}$ and $\xi\in D(H,\psi)$, we have

$$x^{*}\xi\in\mathrm{Dom}(q_{\varphi}). \tag{8}$$

Indeed,

$$\begin{aligned}q_{\varphi}(x^{*}\xi) &=\langle\varphi,\theta^{\psi}(x^{*}\xi,x^{*}\xi)\rangle\\ &=\langle\varphi,x^{*}\cdot\theta^{\psi}(\xi,\xi)\cdot x\rangle\\ &\leq\left\|\theta^{\psi}(\xi,\xi)\right\|\langle\varphi,x^{*}x\rangle<\infty.\end{aligned}$$

In particular, if φ is semifinite then $\mathrm{Dom}(q_{\varphi})$ *is dense in H (since n_{φ}^{*} is strongly dense in M).*