Contents

Chapter 1

Spatial L^p Spaces

In this chapter, we describe the Connes/Hilsum construction of spatial L^p spaces.

Let M be a von Neumann algebra acting on a Hilbert space H and let ψ_0 be a normal faithful semifinite weight on the commutant M' of M.

The notation is as in Chapter II and III.

Definition 1. *For each positive self-adjoint (-1)-homogeneous operator a we define the integral with respect to ψ_0 by*

$$\int a \mathrm{d}\psi_0 = \varphi(1), \tag{1}$$

where φ is the (unique) normal semifinite weight on M such that $a = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0}$.

Notation. *For each $p \in [1, \infty]$, we denote by*

$$\overline{M}_{-1/p}$$

the set of closed densely defined $(-1/p)$-homogeneous operators on H.

Definition 2. *Let $p \in [1, \infty[$. We put*

$$L^p(\psi_0) = L^p(M, H, \psi_0) = \{a \in \overline{M}_{-1/p} | \int |a|^p \mathrm{d}\psi_0 < \infty\} \tag{2}$$

and

$$\|a\|_p = \left(\int |a|^p \mathrm{d}\psi_0\right)^{\frac{1}{p}}, a \in L^p(\psi_0). \tag{3}$$

For $p = \infty$, we put

$$L^\infty(\psi_0) = M \tag{4}$$

and write $\|\cdot\|_\infty$ for the usual operator norm on M.

Note that when a is $(-1/p)$-homogeneous, the operator $|a|^p$ is (-1)-homogeneous so that the integral occurring at the right hand side of (2) is defined.

The spaces $L^p(\psi_0)$ are called spatial L^p spaces (as opposed to the abstract L^p spaces of Haagerup).

We now follow the first part of [10] to describe the relationship between the $L^p(\psi_0)$ and Haagerup's $L^p(M)$.

Let φ_0 be a normal faithful semifinite weight on M. Put

$$d_0 = \frac{\mathrm{d}\varphi_0}{\mathrm{d}\psi_0}. \tag{5}$$

Then

$$\forall t \in \mathbb{R} \forall x \in M : \sigma_t^{\varphi_0}(x) = d_0^{it} x d_0^{-it}. \tag{6}$$

We define a unitary operator u_0 on the Hilbert space $L^2(\mathbb{R}, H)$ by

$$(u_0\xi)(t) = d_0^{it}\xi(t), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}. \tag{7}$$

Recall that the crossed product $N = R(M, \sigma^{\varphi_0})$ is generated by the elements $\pi(x), x \in M$, and $\lambda(s), s \in \mathbb{R}$, as described in the beginning of Chapter II. We shall describe the action of $u_0(\cdot)u_0^*$ on these generating elements.

By $\ell(s), s \in \mathbb{R}$, we denote the operator of translation by s in $L^2(\mathbb{R})$:

$$(\ell(s)f)(t) = f(t-s), f \in L^2(\mathbb{R}), t \in \mathbb{R}.$$

We identify $L^2(\mathbb{R}, H)$ with $H \otimes L^2(\mathbb{R})$ (so that $v \otimes f, v \in H, f \in L^2(\mathbb{R})$, is identified with $\xi \in L^2(\mathbb{R}, H)$ given by $\xi(t) = f(t)v, t \in \mathbb{R}$).

Proposition 3. *1) For all $x \in M$, we have*

$$u_0 \pi(x) u_0^* = x \otimes 1.$$

2) For all $s \in \mathbb{R}$, we have

$$u_0 \lambda(s) u_0^* = d_0^{is} \otimes \ell(s).$$

Proof. Let $\xi \in L^2(\mathbb{R}, H)$. Then

$$\begin{aligned}(u_0 \pi(x) u_0^* \xi)(t) =& d_0^{it} \sigma_{-t}^{\varphi_0}(x) d_0^{-it} \xi(t) \\ =& d_0^{it} d_0^{-it} x d_0^{it} d_0^{-it} \xi(t) \\ =& x\xi(t), t \in \mathbb{R},\end{aligned}$$

and

$$\begin{aligned}(u_0 \lambda(s) u_0^* \xi)(t) =& d_0^{it} (u_0^* \xi)(t-s) \\ =& d_0^{it} d^{-i(t-s)} \xi(t-s) \\ =& d_0^{is} \xi(t-s), t \in \mathbb{R}.\end{aligned}$$

This proves the result since for $\xi = v \otimes f, v \in H, f \in L^2(\mathbb{R})$, we have

$$((x \otimes 1)(v \otimes f))(t) = (xv \otimes f)(t) = f(t)xv = xf(t)v = x\xi(t), t \in \mathbb{R},$$

and

$$\begin{aligned}((d_0^{is} \otimes \ell(s))(v \otimes f))(t) =& (d_0^{is} v \otimes \ell(s) f)(t) \\ =& (\ell(s) f)(t) d_0^{is} v \\ =& f(t-s) d_0^{is} v \\ =& d_0^{is} \xi(t-s), t \in \mathbb{R}.\end{aligned}$$

□

We denote by T the unique positive self-adjoint operator in $L^2(\mathbb{R})$ characterized by

$$\forall s \in \mathbb{R} : T^{is} = \ell(s). \tag{8}$$

For the definition and properties of tensor products of closed operators we refer to [17, Section 9.33].

Proposition 4. *For all normal semifinite weights φ on M we have*

$$u_0 h_\varphi u_0^* = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0} \otimes T. \tag{9}$$

Proof. First suppose that φ is faithful . Then

$$h_\varphi^{it} h_{\varphi_0}^{-it} = (D\tilde{\varphi} : D\tau)_t (D\tau : D\tilde{\varphi}_0)_t = (D\tilde{\varphi} : D\tilde{\varphi}_0)_t = \pi((D\varphi : D\varphi_0)_t)$$

and

$$(D\varphi : D\varphi_0)_t = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0}\right)^{it} \left(\frac{\mathrm{d}\varphi_0}{\mathrm{d}\psi_0}\right)^{-it}$$

for all $t \in \mathbb{R}$, so that by Proposition 3 and the fact that $h_{\varphi_0}^{it} = \lambda(t)$ for all $t \in \mathbb{R}$, we get

$$\begin{aligned} u_0 h_\varphi^{it} u_0^* =& (u_0 h_\varphi^{it} h_{\varphi_0}^{-it} u_0^*)(u_0 h_{\varphi_0}^{-it} u_0^*) \\ =& \left(\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0}^{it} \right) \left(\frac{\mathrm{d}\varphi_0}{\mathrm{d}\psi_0}^{-it} \right) \otimes 1 \right) \left(\left(\frac{\mathrm{d}\varphi_0}{\mathrm{d}\psi_0} \right)^{it} \otimes \ell(t) \right) \\ =& \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0} \right)^{it} \otimes T^{it} \end{aligned}$$

for all $t \in \mathbb{R}$, and (9) follows.

In the general case, choose a normal semifinite weight χ with $\operatorname{supp} \chi = 1 - p$ where $p = \operatorname{supp} \varphi$. Then $\varphi^+ \chi$ is a normal faithful semifinite weight and hence, by the first part of the proof,

$$u_0 h_{\varphi+\chi} u_0^* = \frac{\mathrm{d}(\varphi + \chi)}{\mathrm{d}\psi_0} \otimes T.$$

Since $p = \operatorname{supp} \frac{\mathrm{d}\varphi}{\mathrm{d}\psi}$ and $\pi(p) = \operatorname{supp} h\varphi$, this implies that

$$\begin{aligned} u_0 h_\varphi u_0^* =& u_0(\pi(p) \cdot h_{\varphi+\chi} \cdot \pi(p)) u_0^* \\ =& u_0 \pi(p) u_0^* \cdot u_0 h_{\varphi+\chi} u_0^* \cdot u_0 \pi(p) u_0^* \\ =& (p \otimes 1) \cdot \left(\frac{\mathrm{d}(\varphi+\chi)}{\mathrm{d}\psi_0} \otimes T \right) \cdot (p \otimes 1) \\ =& \left(p \cdot \frac{\mathrm{d}(\varphi+\chi)}{\mathrm{d}\psi_0} \cdot p \right) \otimes T = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0} \otimes T. \end{aligned}$$

□

Corollary 5. *The mapping*

$$a \mapsto u_0^*(a \otimes T)u_0$$

is a bijection of the set of positive self-adjoint (-1)-homogeneous operators a on H onto the set of positive self-adjoint operators h affiliated with $R(M, \sigma^{\varphi_0})$ satisfying

$$\forall s \in \mathbb{R} : \theta_s h = e^{-s} h. \tag{10}$$

Furthermore,

$$\int a \mathrm{d}\psi_0 = \mathrm{tr}(u_0^*(a \otimes T)u_0) \tag{11}$$

for all such a.

Proof. Since the mapping in question is nothing but $\frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0} \mapsto h_\varphi$, it is a bijection by Proposition **??** in Chapter II. By definition, we have $\int \frac{\mathrm{d}\varphi}{\mathrm{d}\psi_0} \mathrm{d}\psi_0 = \varphi(1) = \mathrm{tr}(h_\varphi)$. □

Corollary 6. *Let $p \in [1, \infty[$. Let a be a closed densely defined operator on H. Then*

1) $a \in \overline{M}_{-1/p}$ if and only if

$$u_0^*(a \otimes T^{1/p})u_0 \eta R(M, \sigma^{\varphi_0}),$$

2) $a \in L^p(\psi_0)$ if and only if

$$u_0^*(a \otimes T^{1/p})u_0 \in L^p(M).$$

For all $a \in L^p(\psi_0)$, we have

$$\|a\|_p = \left\|u_0^*(a \otimes T^{1/p})u_0\right\|_p.$$

Corollary 7. *Let $p \in [1, \infty[$. Then the mapping*

$$a \mapsto u_0^*(a \otimes T^{1/p})u_0 \tag{12}$$

is a bijection of $\overline{M}_{-1/p}$ onto the set of closed densely defined operators h affiliated with $R(M, \sigma^{\varphi_0})$ satisfying

$$\forall s \in \mathbb{R} : \theta_s h = e^{-s/p} h. \tag{13}$$

Proof of Corollary 6 and 7. Let a be a closed densely defined operator on H with polar decomposition $a = u|a|$. Then

$$h = u_0^*(u \otimes 1)u_0(u_0^*(|a| \otimes T)u_0)^{1/p}$$

is the polar decomposition of $h = u_0^*(a \otimes T^{1/p})u_0$. Corollary 6, 1), and Corollary 7 now follow from Corollary 5 and Proposition 3, 1) (and the fact that $a \mapsto a \otimes T^{1/p}$ is injective). The rest of Corollary 6 follows from the equation $\int |a|^p \mathrm{d}\psi_0 = \mathrm{tr}\big(\big|u_0^*(|a| \otimes T^{1/p})u_0\big|^p\big)$. $\square$

Proposition 8. *Let $p \in [1, \infty]$. Then for all $a \in L^p(\psi_0)$, we have $a^* \in L^p(\psi_0)$ and*

$$\|a^*\|_p = \|a\|_p.$$

Proof. Let $a \in L^p(\psi_0)$. Then $a \otimes T^{1/p} \in u_0 L^p(M)u_0^*$. Hence also $a^* \otimes T^{1/p} = (a \otimes T^{1/p})^* \in u_0 L^p(M)u_0^*$. Thus $a^* \in L^p(\psi_0)$ by Corollary 6 and $\|a^*\|_p = \big\|u_0^*(a^* \otimes T^{1/p})u_0\big\|_p = \big\|u_0^*(a \otimes T^{1/p})u_0\big\|_p = \|a\|_p$. $\square$

If we identify $L^2(\mathbb{R})$ with $L^2(\mathbb{R})$ via Fourier transformation, T is simply the multiplication operator in $L^2(\mathbb{R})$ given by multiplication by $t \mapsto e^t$, and similarly, for each $p \in [1, \infty[$, $T^{1/p}$ is simply multiplication by $t \mapsto e^{t/p}$. This observation will permit us to obtain information about operators a on H from information about the tensor products $a \otimes T^{1/p}$. First we have:

Lemma 9. *Let a be a closed densely defined operator on H and f a Borel function on $\mathbb{R}$, and denote by m_f the corresponding multiplication operator on $L^2(\mathbb{R})$. Write*

$$D = \{\xi \in L^2(\mathbb{R}, H) | \xi(t) \in D(a) \textit{for a.a.} t \in \mathbb{R} \\ \textit{and} \int \|f(t)a\xi(t)\|^2 \mathrm{d}t < \infty\}.$$

Then $D(a \otimes m_f) = D$ and

$$((a \otimes m_f)\xi)(t) = f(t)a\xi(t), \xi \in D, t \in \mathbb{R}.$$

Proof. Denote by $m(a,f)$ the operator in $L^2(\mathbb{R},H)$ given by

$$D(m(a,f)) = D$$

and

$$(m(a,f)\xi)(t) = f(t)a\xi(t), \xi \in D, t \in \mathbb{R}.$$

Then $m(a,f)$ is a closed operator and

$$m(a^*,\bar{f}) \subset m(a,f)^*$$

(in fact, equality holds). Now evidently

$$a \odot m_f \subset m(a,f),$$

where $a \odot m_f$ denotes the algebraic tensor product of a and m_f, and hence

$$a \otimes m_f = [a \odot m_f] \subset m(a,f).$$

Applying this to a^* and $\bar{f}$, we get

$$a^* \otimes m_{\bar{f}} \subset m(a^*,\bar{f}).$$

Combining this, and using that $(A \otimes B)^* = A^* \otimes B^*$, we find that

$$m(a,f) \subset m(a^*,\bar{f})^* \subset (a^* \otimes m_{\bar{f}})^* = a \otimes m_f.$$

In all, we have shown that $a \otimes m_f = m(a,f)$. $\square$

Lemma 10. *Let $p \in [1,\infty]$ and $a,b \in L^p(\psi_0)$. Then $a+b$ is densely defined and preclosed, and*

$$[a+b] \in L^p(\psi_0).$$

Proof. 1) Denote by e the projection onto $\overline{D(a) \cap D(b)}$. Then

$$\begin{aligned}
&(e \otimes 1)L^2(\mathbb{R},H)\\
=&\{\xi \in L^2(\mathbb{R},H) | \xi(t) = e\xi(t) \text{ for a.a. } t \in \mathbb{R}\}\\
\supset&\{\xi \in L^2(\mathbb{R},H) | \xi(t) \in D(a) \cap D(b) \text{ for a.a. } t \in \mathbb{R}\}
\end{aligned}$$

By Lemma 9, this set contains

$$D(a \otimes T^{1/p}) \cap D(b \otimes T^{1/p}).$$

Now since $a \otimes T^{1/p}, b \otimes T^{1/p} \in u_0 L^p(\psi) u_0^*$, their sum is densely defined. Hence $D(a \otimes T^{1/p}) \cap D(b \otimes T^{1/p})$ is dense in $L^2(\mathbb{R}, H)$. It follows that $e = 1$. Hence $D(a + b) = D(a) \cap D(b)$ is dense in H.

2) Now let us show that $a + b$ is preclosed. By Proposition 8, a^* and b^* are in $L^p(\psi_0)$ and hence by the first part of proof, $a^* + b^*$ is densely defined. Since $a + b \subset (a^* + b^*)^*$, $a + b$ is preclosed.

3) Finally, let us show that

$$[a + b] \otimes T^{1/p} = [(a \otimes T^{1/p}) + (b \otimes T^{1/p})]. \tag{14}$$

First, by the characterization of $a \otimes T^{1/p}$ given in Lemma 9 we obviously have

$$(a \otimes T^{1/p}) + (b \otimes T^{1/p}) \subset [a + b] \otimes T^{1/p},$$

whence

$$[(a \otimes T^{1/p}) + (b \otimes T^{1/p})] \subset [a + b] \otimes T^{1/p}.$$

On the other hand, again by that characterization,

$$[a + b] \otimes T^{1/p} \subset ((a^* \otimes T^{1/p}) + (b^* \otimes T^{1/p}))^*,$$

and finally

$$((a^* \otimes T^{1/p}) + (b^* \otimes T^{1/p}))^* = [(a \otimes T^{1/p}) + (b \otimes T^{1/p})]$$

since * is an involution in $L^p(M)$ (and hence respects the strong sum). In all, we have proved (14). Now the right hand side of (14) is in $u_0 L^p(M) u_0^*$. Hence by Corollary 6, $[a+b] \in L^p(\psi_0)$. $\square$

Lemma 11. *Let $p, p_1, p_2 \in [1, \infty]$ such that $1/p = 1/p_1 + 1/p_2$. Let $a \in L^{p_1}(\psi_0)$ and $b \in L^{p_2}(\psi_0)$. Then ab is densely defined and preclosed and*

$$[ab] \in L^p(\psi_0).$$

Proof. 1) Denote by e the projection onto $D(ab)$. Then, using Lemma 9, we have

$$\begin{aligned}
&D((a \otimes T^{1/p})(b \otimes T^{1/p})) \\
\subset &\{\xi \in D(b \otimes T^{1/p}) | b\xi(t) \in D(a) \text{ for a.a. } t \in \mathbb{R}\} \\
\subset &\{\xi \in L^2(\mathbb{R}, H) | \xi(t) \in D(b) \text{ for a.a. } t \in \mathbb{R} \\
&\quad \text{and } b\xi(t) \in D(a) \text{ for a.a. } t \in \mathbb{R}\} \\
\subset &\{\xi \in L^2(\mathbb{R}, H) | \xi(t) \in D(ab) \text{ for a.a. } t \in \mathbb{R}\} \\
\subset &\{\xi \in L^2(\mathbb{R}, H) | \xi(t) = e\xi(t) \text{ for a.a. } t \in \mathbb{R}\} \\
= &(e \otimes 1)L^2(\mathbb{R}, H).
\end{aligned}$$

Hence $e = 1$ and ab is densely defined.

2) By 1) applied to b^* and a^*, b^*a^* is densely defined. Since $ab \subset (b^*a^*)^*$, ab is preclosed. $\square$