Contents

Chapter 1

L^p Spaces Associated with a Von Neumann Algebra

In this chapter, we present Haagerup's theory of L^p spaces associated with a von Neumann algebra.

Let M be a von Newmann algebra and let φ_0 be a normal faithful semifinite weight on M.

We denote by N the crossed product $R(M, \sigma^{\varphi_0})$ of M by the modular automorphism group σ^{φ_0} associated with φ_0. Recall [18, Section 3; 8, Section 5] that if M is given on a Hilbert space H, then N is the Von Neumann algebra on the Hilbert space $L^2(\mathbb{R}, H)$ generated by the operators $\pi(x), x \in M$, and $\lambda(s)$, $s \in \mathbb{R}$, defined by

$$(\pi(x)\xi)(t) = \sigma_{-t}^{\varphi_0}(x)\xi(t), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}, \tag{1}$$

$$(\lambda(s)\xi)(t) = \xi(t-s), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}. \tag{2}$$

We identify M with its image $\pi(M)$ in N (recall that π normal faithful representation of M).

We denote by θ the dual action of $\mathbb{R}$ in N. The θ_s, $s \in \mathbb{R}$, are automorphisms of N characterized by

$$\theta_s x = x, x \in M \tag{3}$$

$$\theta_s \lambda(t) = e^{-ist}\lambda(t), t \in \mathbb{R}. \tag{4}$$

By (3), M is contained in the set of fixed points under θ. Actually

$$M = \{y \in N | \forall s \in \mathbb{R} : \theta_s y = y\} \tag{5}$$

(see e.g. [5, Lemma 3.6]).

The θ_s, $s \in \mathbb{R}$, naturally extend to automorphisms, still denoted θ_s, of $\hat{N}_+$, the extended positive part of N [7, Section 1]. Recall [8, Lemma 5.2] that the formula

$$Tx = \int_{\mathbb{R}} \theta_s(x) \mathrm{d}s, x \in N_+, \tag{6}$$

defines a normal faithful semifinite operator valued weight T from N to M in the following sense: for each $x \in N_+$, Tx is the element of $\hat{N}_+$ characterized by

$$\langle Tx, \chi \rangle = \int_{\mathbb{R}} \langle \theta_s(x), \chi \rangle \mathrm{d}s \tag{7}$$

for all $x \in N_*^+$. Note that

$$\forall s \in \mathbb{R} : \theta_s \circ T = T. \tag{8}$$

In view of (5), this formula implies that the values of T are actually in $\hat{M}_+$.

For each normal weight φ on M, we put

$$\tilde{\varphi} = \hat{\varphi} \circ T \tag{9}$$

where $\hat{\varphi}$ denotes the extension of φ to a normal weight on $\hat{M}_+$ as described in [7, Proposition 1.10]. Then $\tilde{\varphi}$ is a normal weight on N [7,Proposition 2.3]; $\tilde{\varphi}$ is called the dual weight of φ (see [6, Introduction + Section 1)]. Note that (8) and (9) imply

$$\forall s \in \mathbb{R} : \tilde{\varphi} \circ \theta_s = \tilde{\varphi}. \tag{10}$$

If φ and ψ are normal faithful semifinite weights, then so are $\tilde{\varphi}$ and $\tilde{\psi}$, and we have [7, Theorem 4.7]:

$$\forall t \in \mathbb{R} \forall x \in M : \sigma_t^{\tilde{\varphi}}(x) = \sigma_t^{\varphi}(x), \tag{11}$$

$$\forall t \in \mathbb{R} : (D\tilde{\varphi} : D\tilde{\psi})_t = (D\varphi : D\psi)_t. \tag{12}$$

Lemma 1. *1) The mapping*

$$\varphi \mapsto \tilde{\varphi}$$

is a bijection of the set of all normal semifinite weights on M onto the set of normal semifinite weights ψ on N satisfying

$$\forall s \in \mathbb{R} : \psi \circ \theta_s = \psi. \tag{13}$$

2) For all normal weights φ and ψ on M and all $x \in M$, we have

1. $(\varphi + \psi)^{\sim} = \tilde{\varphi} + \tilde{\psi}$,
2. $(x \cdot \varphi \cdot x^*)^{\sim} = x \cdot \tilde{\varphi} \cdot x^*$,
3. $\operatorname{supp} \tilde{\varphi} = \operatorname{supp} \varphi$.

Proof. That $\tilde{\varphi}$ is semifinite if φ is follows from the proof of [7, Proposition 2.3]. That $\varphi \mapsto \tilde{\varphi}$ is injective follows from the formula

$$\varphi(\dot{T}x) = \tilde{\varphi}(x), x \in m_T,$$

and the fact that $\dot{T}(m_T)$ is σ-weakly dense in M [7, Proposition 2.5].

Now let us prove 2). First observe that $(\varphi + \psi)^{\wedge} = \hat{\varphi} + \hat{\psi}$ since $\hat{\varphi} + \hat{\psi} : \hat{M} \to [0, \infty]$ obviously satisfies the properties that characterize $(\varphi+\psi)^{\wedge}$ ([7, Proposition 1.10]); (a) follows trivially. Similarly, $(x \cdot \varphi \cdot x^*)^{\wedge} = x \cdot \hat{\varphi} \cdot x^*$, whence (b).

To prove (c), put $p_0 = 1 - \operatorname{supp} \varphi$. Then Mp_0 is the σ-weak closure in M of $N_\varphi = \{x \in M | \varphi(x^*x) = 0\}$. Similarly, the σ-weak closure in N of $N_{\tilde{\varphi}} = \{y \in N | \tilde{\varphi}(y^*y) = 0\}$ is Nq_0 where $q_0 = 1 - \operatorname{supp} \tilde{\varphi}$. Now

$$n_T N_\varphi \subset N_{\tilde{\varphi}}$$

since

$$\begin{aligned}\forall y \in n_T \forall x \in N_\varphi : &\tilde{\varphi}(x^*y^*yx) = \varphi(T(x^*y^*yx)) \\ &= \varphi(x^*T(y^*y)x) \leq \|T(y^*y)\| \varphi(x^*x) = 0.\end{aligned}$$

As n_T is σ-weakly dense in N, it follows that

$$N_\varphi \subset \overline{N_{\tilde{\varphi}}}^{\sigma-w}$$

whence

$$p_0 \leq q_0.$$

Note that we must have $q_0 \in M$ since $\tilde{\varphi}$, and hence $\operatorname{supp}\tilde{\varphi}$, is θ-invariant. Thus to conclude that $p_0 = q_0$ we need only show that $\varphi(q_0) = 0$. This follows from

$$\forall x \in m_T : \varphi(q_0 \dot{T}(x) q_0) = \varphi(\dot{T}(q_0 x q_0)) = \tilde{\varphi}(q_0 x q_0) = 0$$

and the fact that $\dot{T}(m_T)$ is σ-weakly dense in M [7, Proposition 2.5].

We now return to 1). Let ψ be a normal semifinite weight on N satisfying (13). First suppose that ψ is also faithful. Then by [5, (proof of) Theorem 3.7), it follows that $\psi = \tilde{\varphi}$ for some normal faithful semifinite φ on M.

In the general case, put $q_0 = 1 - \operatorname{supp}\psi$. Then by (13) and (5), we have $q_0 \in M$. Now take any normal semifinite weight χ_0 on M such that $\operatorname{supp}\chi_0 = q_0$. Then $\widetilde{\chi_0}$ is a normal faithful semifinite θ-invariant weight on N with $\operatorname{supp}\widetilde{\chi_0} = q_0$. Hence $\widetilde{\chi_0} + \psi$ is faithful and thus, as above,

$$\widetilde{\chi_0} + \psi = \tilde{\varphi}$$

for some normal faithful semifinite weight φ on M. Finally, using (b), we find that

$$\begin{aligned} \psi =& (1 - q_0) \cdot (\widetilde{\chi_0} + \psi) \cdot (1 - q_0) \\ =& (1 - q_0) \cdot \tilde{\varphi} \cdot (1 - q_0) \\ =& ((1 - q_0) \cdot \varphi \cdot (1 - q_0))^{\sim}. \end{aligned}$$

$\square$

Denote by τ the normal faithful semifinite trace on N characterized by

$$\forall t \in \mathbb{R} : (D\tilde{\varphi}_0 : D\tau)_t = \lambda(t) \tag{14}$$

(for the existence, see [8, Lemma 5.2]); τ satisfies

$$\forall s \in \mathbb{R} : \tau \circ \theta_s = e^{-s}\tau. \tag{15}$$

With each $h \in \hat{N}_+$ we associate the normal weight $\tau(h\cdot)$ on N as in [8, remarks preceding Proposition 1.11]. When h is simply a positive self-adjoint operator affiliated with N (see [7, Example 1.2]), this definition agrees with that given in [14, Section 4].

We recall some facts about the mapping $h \mapsto \tau(h\cdot)$ (see [7, Theorem 1.12 (and its proof) and Preposition 1.11, (4)]):

Lemma 2. *1) The mapping*

$$h \mapsto \tau(h\cdot)$$

is a bijection of $\hat{N}_+$ onto the set of normal weights on N. In particular, it is a bijection of the positive self-adjoint operators affiliated with N onto the normal semifinite weights on N.

2) For all $h, k \in \hat{N}_+$ and all $x \in N$, we have

1. $\tau((h\dot{+}k)\cdot) = \tau(h\cdot) + \tau(k\cdot)$,
2. $\tau((x \cdot h \cdot x^*)\cdot) = x \cdot \tau(h\cdot) \cdot x^*$,
3. $\operatorname{supp} \tau(h\cdot) = \operatorname{supp} h$.

Here, $h\dot{+}k$ and $x{\cdot}h{\cdot}x^*$ denote the operations in $\hat{N}_+$ introduced in [7, Definition 1.3]. If h and k are positive self-adjoint operators such that $D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}})$ is dense, then $h\dot{+}k$ is the simply the form sum of h and k [2, Corollary 4.13]. If h is a positive self-adjoint operator and x a bounded operator such that $D(h^{\frac{1}{2}}x^*)$ is dense, then $x \cdot h \cdot x^* = \left|h^{\frac{1}{2}}x^*\right|^2$.

Definition 3. *For each normal weight φ on M we define h_φ as the unique element of $\hat{N}_+$ given by*

$$\tilde{\varphi} = \tau(h_\varphi\cdot). \tag{16}$$

Proposition 4. *1) The mapping*

$$\varphi \mapsto h_\varphi$$

is a bijection of the set of all normal semifinite weights on M onto the set of all positive self-adjoint operators h affiliated with N satisfying

$$\forall s \in \mathbb{R} : \theta_s h = e^{-s} h. \tag{17}$$

(2) For all normal weights φ and ψ on M and all $x \in M$, we have

1. $h_{\varphi+\psi} + h_\varphi \dot{+} h_\psi$,
2. $h_{x\cdot\varphi\cdot x^*} = x \cdot h_\varphi \cdot x^*$,
3. $\operatorname{supp} h_\varphi = \operatorname{supp} \varphi$.

Proof. This proposition is an immediate consequence of Lemma 1 and 2. We just need to prove that a positive self-adjoint operator h affiliated with N satisfies (17) if and only if the corresponding weight $\tau(h\cdot)$ is θ-invariant. This follows easily from (15). Indeed, for all $s \in \mathbb{R}$ we have

$$\tau(e^s\theta_s(h)\cdot) = e^s(\tau \circ \theta_s)(h\theta_{-s}(\cdot)) = \tau(h\theta_{-s}(\cdot)) = \tau(h\cdot) \circ \theta_{-s},$$

whence

$$e^s\theta_s(h) = h \Leftrightarrow \tau(e^s\theta_s(h)\cdot) = \tau(h\cdot) \Leftrightarrow \tau(h\cdot) = \tau(h\cdot) \circ \theta_{-s}.$$

The equivalence of (17) and

$$\forall s \in \mathbb{R} : \tau(h\cdot) = \tau(h\cdot) \circ \theta_s$$

follows. □

The next lemma is essential. It will permit us apply results on τ-measurable operators.

As usual, $\chi_{]\gamma,\infty[}$ denotes the characteristic function for the interval $]\gamma, \infty[$.

Lemma 5. *Let φ be a normal semifinite weight on M. Then for all $\gamma \in \mathbb{R}_+$, we have*

$$\tau(\chi_{]\gamma,\infty[}(h_\varphi)) = \frac{1}{\gamma}\varphi(1).$$

Proof. First let us prove the formula in the case $\gamma = 1$.

Let $s \in \mathbb{R}$. Then since θ_s is an automorphism and $\theta_s h_\varphi = e^{-s}h_\varphi$ we have

$$\theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi)) = e^s h_\varphi^{-1}\chi_{]1,\infty[}(e^{-s}h_\varphi).$$

Now let $h_\varphi = \int \lambda \mathrm{d}e_\lambda$ be the spectral decomposition of h_φ. Then for any vector functional $\omega_{\xi,\xi}$, where ξ is a unit vector, we have

$$\begin{aligned}
\langle \int_{\mathbb{R}} \theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))\mathrm{d}s, \omega_{\xi,\xi}\rangle &= \int_{\mathbb{R}} \langle e^s h_\varphi^{-1}\chi_{]1,\infty[}(e^{-s}h_\varphi), \omega_{\xi,\xi}\rangle \mathrm{d}s \\
&= \int_{\mathbb{R}}\int_{]0,\infty[} e^s\lambda^{-1}\chi_{]1,\infty[}(e^{-s}\lambda)\,\mathrm{d}(e_\lambda\xi|\xi)\,\mathrm{d}s \\
&= \int_{]0,\infty[} \lambda^{-1}\left(\int_{]-\infty,\log\lambda[} e^s\mathrm{d}s\right)\mathrm{d}(e_\lambda\xi|\xi) \\
&= \int_{]0,\infty[} \lambda^{-1}\lambda\,\mathrm{d}(e_\lambda\xi|\xi) \\
&= \|(\operatorname{supp} h_\varphi)\xi\|^2
\end{aligned}$$

So that

$$\int_{\mathbb{R}} \theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))\mathrm{d}s = \operatorname{supp} h_\varphi = \operatorname{supp}\varphi.$$

Finally, since $\tilde{\varphi} = \tau(h_\varphi\cdot)$ we have □