Contents

Chapter 1

Measurable Operators with Respect to a Trace

In this chapter, we define the notion of measurability with respect to a trace τ on a von Neumann algebra M and show that the set M of τ-measurable operators is a complete topological *-algebra. Our presentation is a modified version of that given by E. Nelson [13].

Let M be a - necessarily semifinite - von Neumann algebra acting on a Hilbert space H and let τ be a normal faithful semifinite trace on M.

For the convenience of the reader, we immediately give the definition of τ-measurability and state the main theorem about τ-measurable operators.

Definition 14: A closed densely defined operator a affiliated with M is called τ-measurable if for all $\delta \in \mathbb{R}_+$ there exists a projection $p \in M$ such that

$$pH \subset D(a) \text{ and } \tau(1-p) \leq \delta$$

For a characterization of τ-measurable operators in terms of the spectral projections of their absolute value, see Proposition 21 below.

We denote by $\widetilde{M}$ the set of τ-measurable closed densely defined operators.

Theorem 28. 1) $\widetilde{M}$ is a *-algebra with respect to strong sum, strong product, and adjoint operation.

2) The sets

$$N(\epsilon,\delta)=\{a\in\widetilde{M}|\exists p\in M_{\text{proj}}:pH\subset D(a),\|ap\|\leq\epsilon,\tau(1-p)\leq\delta\},$$

where $\epsilon,\delta\in\mathbb{R}_+$, form a basis for the neighbourhoods of 0 for a topology on $\widetilde{M}$ that turns $\widetilde{M}$ into a topological vector space.

3) $\widetilde{M}$ is a complete Hausdorff topological * -algebra and M is a dense subset of $\widetilde{M}$.

Once this theorem has been proven, we can freely add and multiply operators from $\widetilde{M}$, the operations being understood in the strong sense (see the definition below). Until then, we have to deal with unbounded operators in the usual careful way.

Although we are mainly interested in closed densely defined opera tors it will be convenient for us to work with more general kinds of unbounded operators. We therefore start by recalling some basic facts on arbitrary unbounded operators. Next, we recall some properties of the lattice M_{proj} of projections in M. After this, we go on to develop the theory of τ-measurability.

1.1 Preliminaries on unbounded operators

Recall that for any (linear) operators a and b on H we can define the sum $a+b$ and the product ab as operators on H with domains

$$D(a+b)=D(a)\cap D(b), \tag{1}$$

$$D(ab)=\{\xi\in D(b)|b\xi\in D(a)\}. \tag{2}$$

These operations are associative so that $a+b+c$ and abc are well-defined operators. Furthermore, for all a, b and c we have

$$(a+b)c=ac+bc \text{ and } c(a+b)\supset ca+cb \tag{3}$$

(with equality if $D(c)=H$).

We shall use the following terminology: an operator a on H is closed if its graph $G(a)$ is closed in $H \otimes H$; a is preclosed if the closure $\overline{G(a)}$ of its graph is the graph of some - necessarily closed - operator (the closure of a, denoted $[a]$; a is densely defined if $D(a)$ is dense in H.

If a, b and ab are densely defined, then

$$(ab)^* \supset b^* a^* \tag{4}$$

with equality if a is bounded and everywhere defined.

A closed densely defined operator a has a unique polar decomposition

$$a = u|a| \tag{5}$$

where $|a|$ is a positive self-adjoint operator and u a partial isometry with $\mathrm{supp}(a)$ as its initial projection and $r(a)$, the projection onto the closure of the range of a, as its final projection.

If the sum $a + b$ of two closed densely defined operators a and b is preclosed and densely defined, then the closure $[a+b]$ is called the strong sum of a and b. Similarly, the strong product is the closure $[ab]$ if ab is preclosed and densely defined.

We shall write

$$\|a\| = \sup\{\|a\xi\| | \|\xi\| \leq 1\}$$

for all everywhere defined operators a on H, bounded or not.For all such operators, the usual norm estimates hold:

$$\|a + b\| \leq \|a\| + \|b\|, \|ab\| \leq \|a\| \|b\|.$$

Denote by M' the commutant of M.

Definition 1. *A linear operator a on H is said to be affiliated with M (and we write $a\eta H$) if*

$$\forall y \in M' : ya \subset ay \tag{6}$$

Remark 2. *Using (3), (4) and (5) one easily verifies that*

1. if $a, b\eta M$, then $a + b\eta M$ and $ab\eta H$;

2. if a is preclosed, resp. densely defined, and $a\eta M$, then $[a]\eta M$, resp. $a^*\eta M$;

3. if a is a closed densely defined operator with polar decomposition $a = u|a|$, then $a\eta M$ if and only if $u \in M$.

Notation. We denote by $\overline{M}$ the set of closed densely defined operators affiliated with M.

1.2 Preliminaries on projections

We denote by M_{proj} the lattice of (orthogonal) projections in M. For a family $(p_i)_{i\in I}$ of projections in M, $\wedge_{i\in I} p_i$ (resp. $\vee_{i\in I} p_i$) is the projection onto $\cap_{i\in I} p_i H$ (resp. $\overline{\cup_{i\in I} p_i H}$).

Recall that

$$(\wedge_{i\in I} p_i)^{\perp} = \vee_{i\in I} p_i^{\perp}, (\vee_{i\in I} p_i)^{\perp} = \wedge_{i\in I} p_i^{\perp} \tag{7}$$

where $p^{\perp} = 1 - p$ is the projection orthogonal to p

Two projections p and q are equivalent if $p = u^*u$ and $q = uu^*$ for some $u \in M$. We denote equivalence by $\sim$. Equivalent projections have the same trace.

By the polar decomposition theorem, we have

Lemma 3. *Let a be a closed densely defined operator affiliated with M. Then*

$$\mathrm{supp}(a) \sim \mathrm{r}(a)$$

where $\mathrm{r}(a)$ denotes the projection onto the closure of the range of a.

For any projections $p, q \in M$ we have

$$(p \vee q) - p \sim q - (p \wedge q). \tag{8}$$

It follows that

$$\tau(p \vee q) \le \tau(p) + \tau(q). \tag{9}$$

More generally,

$$\tau(\vee_{i\in I} p_i) \le \sum_{i\in I} \tau(p_i) \tag{10}$$

for any family $(p_i)_{i\in I}$ of projections in M (if I is finite, this follows by induction from (9); for the general case, use the normality of τ).

Another consequence of (8) is this:

$$\forall p, q \in M_{\text{proj}} : p \wedge q = 0 \Rightarrow p \precsim 1 - q \tag{11}$$

(where $\precsim$ means: "is equivalent to a subprojection of"). Indeed,

$$p = 1 - p^\perp = (p \wedge q)^\perp - p^\perp = (p^\perp \vee q^\perp) - p^\perp \sim q^\perp - (p^\perp \wedge q^\perp) \leq q^\perp = 1 - q.$$

1.3 The theory of τ-measurable operators

Definition 4. *Let $\epsilon, \delta \in \mathbb{R}_+$. Then we denote by $D(\epsilon, \delta)$ the set of all operators $a\eta M$ for which there exists a projection $p \in M$ such that*

1. *$pH \subset D(a)$ and $\|ap\| \leq \epsilon$ and*
2. *$\tau(1-p) \leq \delta$.*

When $pH \subset D(a)$, the operator ap is everywhere defined. The requirement $\|ap\| \leq \epsilon$ in particular implies that ap is bounded.

Note that we do not require a to be densely defined, closed or preclosed.

Proposition 5. *Let $\epsilon_1, \epsilon_2, \delta_1, \delta_2 \in \mathbb{R}_+$. Then*

1. *$D(\epsilon_1, \delta_1) + D(\epsilon_2, \delta_2) \subset D(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$,*
2. *$D(\epsilon_1, \delta_1)D(\epsilon_2, \delta_2) \subset D(\epsilon_1\epsilon_2, \delta_1 + \delta_2)$.*

Proof. (1) Let $a \in D(\epsilon_1, \delta_1)$ and $b \in D(\epsilon_2, \delta_2)$. Then there exist projections $p, q \in M$ such that

$$pH \subset D(a), \|ap\| \leq \epsilon_1, \text{ and } \tau(1-p) \leq \delta_1,$$
$$qH \subset D(b), \|bq\| \leq \epsilon_2, \text{ and } \tau(1-q) \leq \delta_2.$$

Put $r = p \wedge q$. Then

$$rH = pH \cap qH \subset D(a) \cap D(b) = D(a+b)$$

and

$$\|(a+b)r\| = \|ar + br\| \leq \|ar\| + \|br\| \leq \|ap\| + \|bq\| \leq \epsilon_1 + \epsilon_2$$

Furthermore,

$$\tau(1-r) = \tau((p \wedge q)^{\perp}) = \tau(p^{\perp} \vee q^{\perp}) \leq \tau(1-p) + \tau(1-q) \leq \delta_1 + \delta_2$$

This proves (1).

To prove (2), let $a \in D(\epsilon_1, \delta_1)$, $b \in D(\epsilon_2, \delta_2)$ and take $p, q \in M_{\mathrm{proj}}$ as above. Then bq, and hence $(1-p)bq$, is bounded. Denote by s the projection onto its null space:

$$sH = N((1-p)bq).$$

Then $bq\xi \in pH \subset D(a)$ for all $\xi \in sH$, so that $sH \subset D(abq)$ and hence

$$(q \wedge s)H \subset D(ab)$$

Also, $abqs = apbqs$ so that

$$ab(q \wedge s) = abqs(q \wedge s) = apbq(q \wedge s)$$

and thus

$$\|ab(q \wedge s)\| \leq \|ap\|\|bq\| \leq \epsilon_1 \epsilon_2.$$

On the other hand, using that

$$1 - s = \mathrm{supp}((1-p)bq) \sim \mathrm{r}((1-p)bq) \leq 1 - p,$$

we have

$$\begin{aligned}\tau(1 - (q \wedge s)) = \tau((1-q) \vee (1-s)) \leq& \tau(1-q) + \tau(1-s) \\ \leq& \tau(1-q) + \tau(1-p) \leq \delta_1 + \delta_2.\end{aligned}$$

This completes the proof. □

Proposition 6. *Let* $\epsilon, \delta \in \mathbb{R}_+$.

1. *Let a be a preclosed operator. Then*

$$a \in D(\epsilon, \delta) \Rightarrow [a] \in D(\epsilon, \delta).$$

2. *Let a be a closed densely defined operator with polar decomposition $a = u|a|$. Then*

$$a \in D(\epsilon, \delta) \Leftrightarrow u \in M \text{ and } |a| \in D(\epsilon, \delta).$$

Proof. (1): trivial. (2): trivial, since $a = u|a|$, $|a| = u^*a$, and $\|u\| \leq 1$. $\square$

Lemma 7. *Let $a \in \overline{M}$ and $\epsilon, \delta \in \mathbb{R}_+$. Then*

$$a \in D(\epsilon, \delta) \Leftrightarrow \tau(\chi_{]\epsilon,\infty[}(|a|)) \leq \delta$$

(where $\chi_{]\epsilon,\infty[}(|a|)$ denotes the spectral projection of $|a|$ corresponding to the interval $]\epsilon, \infty[$).

Proof. "$\Leftarrow$": Put $p = \chi_{[0,\epsilon]}(|a|)$. Then $pH \subset D(|a|)$ and $\||a|p\| \leq \epsilon$.

"$\Rightarrow$": For some $p \in M_{\text{proj}}$, we have

$$\||a|p\| \leq \epsilon \text{ and } \tau(1-p) \leq \delta.$$

Let $|a| = \int_0^\infty \lambda \mathrm{d}e_\lambda$ be the spectral decomposition of $|a|$. Now for all $\xi \in pH$ we have

$$\||a|\xi\|^2 \leq \epsilon^2 \|\xi\|^2,$$

and for all $\xi \in (1-e_\epsilon)H \setminus \{0\}$ we have

$$\||a|\xi\|^2 > \epsilon^2 \|\xi\|^2$$

since

$$\||a|\xi\|^2 = \int_0^\infty \lambda^2 \, \mathrm{d}(e_\lambda \xi | \xi) = \int_{]\epsilon,\infty[} \lambda^2 \, \mathrm{d}(e_\lambda \xi | \xi) \, .$$

Hence $(1-e_\epsilon)H \cap pH$ must be $\{0\}$, i.e. $(1-e_\epsilon) \wedge p = 0$. By (11) we conclude that $1 - e_\epsilon \precsim 1 - p$, whence $\tau(1-e_\epsilon) \leq \delta$. $\square$

Proposition 8. *Let $a \in \overline{M}$ and $\epsilon, \delta \in \mathbb{R}_+$. Then*

$$a \in D(\epsilon, \delta) \Leftrightarrow a^* \in D(\epsilon, \delta)$$

Proof. Let $a = u|a|$ be the polar decomposition of a. Then u is an isometry of $\chi_{]0,\infty[}(|a|) = \mathrm{supp}(a)$ onto $\chi_{]0,\infty[}(|a^*|) = \mathrm{supp}(a^*) = \mathrm{r}(a)$. By uniqueness of the spectral decomposition, u induces for each $\lambda \in \mathbb{R}_+$ an isometry of $\chi_{]\lambda,\infty[}(|a|)$ onto $\chi_{]\lambda,\infty[}(|a^*|)$. The result follows by Lemma 7. □

Definition 9. *A subspace E of H is called τ-dense if for all $\delta \in \mathbb{R}_+$, there exists a projection $p \in M$ such that*

$$pH \subset E \text{ and } \tau(1-p) \leq \delta.$$

Proposition 10. *Let E be a τ-dense subspace of H . Then there exists an increasing sequence $(p_n)_{n\in\mathbb{N}}$ of projections in M with*

$$p_n \nearrow 1, \tau(1-p_n) \to 0, \text{ and } \cup_{n=1}^{\infty} p_n H \subset E.$$

Proof. Take projections $q_k \in M$, $k \in N$, such that

$$q_k H \subset E \text{ and } \tau(1-q_k) \leq 2^{-k}.$$

For each $n \in N$, put

$$p_n = \wedge_{k=n+1}^{\infty} q_k.$$

Then

$$p_n H = \cap_{k=n+1}^{\infty} q_k H \subset E$$

and

$$\tau(1-p_n) = \tau\left(\vee_{k=n+1}^{\infty}(1-q_k)\right) \leq \sum_{k=n+1}^{\infty} \tau(1-q_k) \leq \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n}$$

It follows that

$$p_n \nearrow 1;$$

indeed, denoting by p the supremum of the increasing sequence p_n, we have

$$\forall n \in \mathbb{N} : \tau(1-p) \leq \tau(1-p_n) \leq 2^{-n}$$

whence $\tau(1-p)=0$ and $p=1$.

Furthermore,

$$\cup_{n=1}^{\infty} p_n H \subset E.$$

$\square$

Corollary 11. *Let E be a τ-dense subspace of H. Then E is dense in H.*

An important property of τ-dense subspaces is the following:

Proposition 12. *Let $a, b \in \overline{M}$ and let E be a τ-dense subspace of H contained in $D(a) \cap D(b)$. Suppose that*

$$a|_E = b|_E.$$

Then $a=b$.

The proof is based on the following lemma:

Lemma 13. *1) Let $p_0 \in M_{proj}$. Suppose that*

$$\forall \delta \in \mathbb{R}_+ \exists p \in M_{proj} : p_0 \wedge p = 0 \text{ and } \tau(1-p) \leq \delta.$$

Then $p_0 = 0$.

2) Let $p_1, p_2 \in M_{proj}$. Suppose that

$$\forall \delta \in \mathbb{R}_+ \exists p \in M_{proj} : p_1 \wedge p = p_2 \wedge p \text{ and } \tau(1-p) \leq \delta.$$

and $p_1 = p_2$.

Proof. 1) Let $\delta \in \mathbb{R}_+$. Then $\tau(p_0) \leq \delta$. (indeed, for some $p \in M_{\text{proj}}$ we have $p_0 \wedge p = 0$ and $\tau(1-p) \leq \delta$, whence $p_0 \lesssim 1-p$ and $\tau(p_0) \leq \tau(1-p) \leq \delta$). Hence $\tau(p_0)=0$ and $p_0=0$.

2) Put $p_0 = p_1 - (p_1 \wedge p_2)$. Now $p_1 \wedge p = p_2 \wedge p$ implies $p_1 \wedge p = (p_1 \wedge p_2) \wedge p$ and hence $p_0 \wedge p = 0$, so that 1) applies to p_0. Hence $p_0 = 0$, i.e. $p_1 = p_1 \wedge p_2$. Similarly, $p_2 = p_1 \wedge p_2$. In all, $p_1 = p_2$. $\square$

Proof of Proposition 12. Consider in the Hilbert space $H_2 = H \oplus H$ the von Neumann algebra $M_2 = \begin{bmatrix} M & M \\ M & M \end{bmatrix}$ equipped with the normal faithful semifinite trace τ_2 defined by

$$\tau \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \tau(x_{11}) + \tau(x_{22}).$$

Denote by p_a and p_b the projections onto the graphs $G(a)$ and $G(b)$ of a and b. Since a and b are affiliated with M, $G(a)$ and $G(b)$ are invariant under all elements of $M_2' = \{ \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} : y \in M' \}$ and thus $p_a, p_b \in M_2$.

Let $\delta \in \mathbb{R}_+$. Then there exists a projection $p \in M$ with $pH \subset E$ and $\tau(1-p) \leq \frac{\delta}{2}$. Put $p_2 = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$. Then $\tau_2(1-p_2) \leq \delta$.

Furthermore,

$$p_a \wedge p_2 = p_b \wedge p_2$$

since a and b agree on $pH \subset E$ and thus

$$\begin{aligned} G(a) \cap (pH \oplus pH) =& \{\langle \xi, a\xi \rangle, \xi \in pH, a\xi \in pH\} \\ =& \{\langle \xi, b\xi \rangle, \xi \in pH, b\xi \in pH\} = G(b) \cap (pH \oplus pH). \end{aligned}$$

By Lemma 13, we conclude that $p_a = p_b$, whence $a = b$. $\square$

Definition 14. *An operator $a \in \overline{M}$ is called τ-measurable if $D(a)$ is τ-dense, i.e. if for all $\delta \in \mathbb{R}_+$ there exists a projection $p \in M$ such that*

$$pH \subset D(a) \text{ and } \tau(1-p) \leq \delta. \tag{12}$$

The set of τ-measurable operators $a \in \overline{M}$ is denoted $\widetilde{M}$.

Corollary 15. *1) Let $a, b \in \widetilde{M}$. If*

$$a \subset b$$

then

$$a = b.$$

2) Let $a \in \widetilde{M}$. If a is symmetric (in particular, if a is positive), then a is self-adjoint.

Proof. Immediate from Definition 14 and Proposition 12 (for 2), use that $a \subset a^*$). $\square$

Note that when a is closed and $p \in M_{\text{proj}}$ is such that $pH \subset D(a)$, then the everywhere defined operator ap is also closed and hence - by the closed graph theorem - automatically bounded. Therefore the following definition is a generalization of Definition 14.

Definition 16. *Any operator $a\eta M$ is called τ-premeasurable if for all $\delta \in \mathbb{R}_+$ there exists a projection $p \in M$ such that*

$$pH \subset D(a), \|ap\| < \infty, \text{ and } \tau(1-p) \leq \delta. \tag{13}$$

By definition of the $D(\epsilon, \delta)$, this may be reformulated as:

Remark 17. *Let $a\eta M$. Then a is τ-premeasurable if and only if*

$$\forall \delta \in \mathbb{R}_+ \exists \epsilon \in \mathbb{R}_+ : a \in D(\epsilon, \delta).$$

Also note

Proposition 18. *Let $a\eta M$. If a is τ-premeasurable, then a is densely defined.*

Proof. $D(a)$ is τ-dense. $\square$

Proposition 19. *Let $a\eta M$. Suppose that a is τ-premeasurable and preclosed. Then*

$$[a] \in \widetilde{M}.$$

Proof. Trivial. $\square$

Proposition 20. *Let $a, b\eta M$ be τ-premeasurable. Then $a + b$ and ab are also τ-premeasurable.*

Proof. Combine Remark 17 and Proposition 5. $\square$

We have the following characterization of τ-measurable operators:

Proposition 21. *Let $a \in \overline{M}$ with polar decomposition $a = u|a|$. Then the following assertions are equivalent:*

1. *a is τ-measurable,*

2. *$|a|$ is τ-measurable,*

3. *$\forall\delta \in \mathbb{R}_+ \exists\epsilon \in \mathbb{R}_+ : a \in D(\epsilon, \delta)$,*

4. *$\forall\delta \in \mathbb{R}_+ \exists\epsilon \in \mathbb{R}_+ : \tau(\chi_{]\epsilon,\infty[}(|a|)) \leq \delta$,*

5. *$\tau(\chi_{]\lambda,\infty[}(|a|)) \to 0$ as $\lambda \to \infty$,*

6. *$\forall\lambda \in \mathbb{R}_+ : \tau(\chi_{]\lambda,\infty[}(|a|)) < \infty$.*

Proof. The equivalence of (i), (ii), and (iii), follows from Lemma 7. Now note that

$$\tau(\chi_{]\lambda,\infty[}(|a|)) \searrow \emptyset \text{ as } \lambda \to \infty$$

so that, by the normality of τ,

$$\tau(\chi_{]\lambda,\infty[}(|a|)) \searrow 0 \text{ as } \lambda \to \infty$$

if $\tau(\chi_{]\lambda_0,\infty[}(|a|)) < \infty$ for some λ_0. The equivalence of (iii), (iv), (v), and (vi) follows. □

Corollary 22. *We have $M \subset \widetilde{M}$.*

Proof. If a is bounded, then $\tau(\chi_{]\|a\|,\infty[}(|a|)) = 0$. □

Proposition 23. *Let $a \in \widetilde{M}$ Then also $a^* \in \widetilde{M}$.*

Proof. Combine Proposition 8 and Proposition 21, (i) $\Leftrightarrow$ (iii). □

Proposition 24. *1) Let $a, b \in \widetilde{M}$. Then $a+b$ and ab are densely defined and preclosed, and $[a+b] \in \widetilde{M}$, $[ab] \in \widetilde{M}$.*

*2) $\widetilde{M}$ is a *-algebra with respect to strong sum and strong product.*

Proof. 1) Let $a, b \in \widetilde{M}$. Then also $a^*, b^* \in \widetilde{M}$. By Proposition 20, $a+b$ and a^*+b^* are τ-premeasurable. In particular, they are densely defined. Hence $(a^*+b^*)^*$ exists and $a+b \subset (a^*+b^*)^*$, whence $a+b$ is also preclosed. By Proposition 19, $[a+b] \in \widetilde{M}$.

A quite analogous reasoning gives the result on ab.

2) Let $a, b, c \in \widetilde{M}$. Then by Proposition 20 the operators

$$a + b + c, abc, ac + bc, ca + cb, a^* + b^*, b^*a^*$$

are all τ-premeasurale. Hence by Proposition 12, each of them admits at most one extension in $\widetilde{M}$. It follows that

$$\begin{aligned} [[a + b] + c] = [a + [b + c]], [[ab]c] = [a[bc]], \\ [[a + b]c] = [[ac] + [bc]], [c[a + b]] = [[ca] + [cb]], \\ [a + b]^* = [a^* + b^*], [ab]^* = [b^*a^*]. \end{aligned}$$

□

Notation. *From now on, we will omit the [ ] in the notation for strong sum and strong product.*

Definition 25. *For all $\epsilon, \delta \in \mathbb{R}_+$, we put*

$$N(\epsilon, \delta) = \widetilde{M} \cap D(\epsilon, \delta),$$

i.e. $N(\epsilon, \delta)$ is the set of τ-measurable $a \in \widetilde{M}$ for which there exists a projection $p \in M$ such that

$$\|ap\| \leq \epsilon \text{ and } \tau(1 - p) \leq \delta.$$

Lemma 26. *For all $\epsilon, \epsilon_1, \epsilon_2, \delta, \delta_1, \delta_2 \in \mathbb{R}_+$ and $\lambda \in \mathbb{C}$ we have*

1. $N(\epsilon, \delta)^* = N(\epsilon, \delta)$,
2. $N(|\lambda|\epsilon, \delta) = \lambda N(\epsilon, \delta)$,
3. $\epsilon_1 \leq \epsilon_2$, $\delta_1 \leq \delta_2 \Rightarrow N(\epsilon_1, \delta_1) \subset N(\epsilon_2, \delta_2)$,
4. $N(\epsilon_1, \delta_1) \cap N(\epsilon_2, \delta_2) \supset N(\epsilon_1 \wedge \epsilon_2, \delta_1 \wedge \delta_2)$,
5. $N(\epsilon_1, \delta_1) + N(\epsilon_2, \delta_2) \subset N(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$,
6. $N(\epsilon_1, \delta_1)N(\epsilon_2, \delta_2) \subset N(\epsilon_1\epsilon_2, \delta_1\delta_2)$,

Proof. (ii), (iii), (iv) are easily verified. (i) follows from Proposition 8 and (v), (vi) follow from Proposition 5 and Proposition 6, (i) ((v) and (vi) are to be understood in the strong sense). □

Proposition 27. *The $N(\epsilon, \delta)$, $\epsilon, \delta \in \mathbb{R}_+$, form a basis for the neighbourhoods of* 0 *for a topological vector space topology on $\widetilde{M}$.*

Proof. This follows from Lemma 26, (ii), (iii), (iv) and (v). $\square$

Theorem 28. *$\widetilde{M}$ is a complete Hausdorff topological *-algebra in which M is dense.*

Proof. 1) To show that $\widetilde{M}$ is Hausdorff, we shall prove that

$$\cap_{\epsilon,\delta\in\mathbb{R}_+} N(\epsilon, \delta) = \{0\}.$$

Let $a \in \cap_{\epsilon,\delta\in\mathbb{R}_+} N(\epsilon, \delta)$. Then

$$\forall\delta \in \mathbb{R}_+ \forall\epsilon \in \mathbb{R}_+ : \tau(\chi_{]\epsilon,\infty[}(|a|)) \leq \delta.$$

Since τ is faithful, this implies that all $\chi_{]\epsilon,\infty[}(|a|) = 0$, whence $a = 0$.

2) Next let us prove that $\widetilde{M}$ is a topological *-algebra. By Lemma 26, (i), the adjoint operation is continuous. Now let $a_0, b_0 \in \widetilde{M}$ and let $\epsilon, \delta \in \mathbb{R}_+$. Take $\mu, \lambda \in \mathbb{R}_+$ such that

$$a_0 \in N(\mu, \delta), b_0 \in N(\lambda, \delta).$$

Then for all $a, b \in \widetilde{M}$ such that $a - a_0 \in N(\epsilon, \delta)$ and $b - b_0 \in N(\epsilon, \delta)$, we have

$$\begin{aligned} ab - a_0b_0 =&(a - a_0)(b - b_0) + a_0(b - b_0) + (a - a_0)b_0 \\ \in&N(\epsilon, \delta)N(\epsilon, \delta) + N(\mu, \delta)N(\epsilon, \delta) + N(\epsilon, \delta)N(\lambda, \delta) \\ \subset&N(\epsilon^2, 2\delta) + N(\mu\epsilon, 2\delta) + N(\lambda\epsilon, 2\delta) \\ \subset&N(\epsilon(\epsilon + \lambda + \mu), 6\delta). \end{aligned}$$

It follows that

$$(a, b) \mapsto (ab)$$

is continuous.

3) M is dense in $\widetilde{M}$. Indeed, let $a \in \widetilde{M}$ and take projections $p_n \in M$ such that

$$p_n \nearrow 1, \tau(1 - p_n) \to 0, \text{ and } \cup_{n\in\mathbb{N}}\, p_nH \subset D(a)$$

(possible by Proposition 10). Then $ap_n \in M$ and

$$ap_n \to a \text{ in } \widetilde{M}$$

since $\|(ap_n - a)p_m\| = 0$ for all $m \geq n$ and $\tau(1 - p_m) \to 0$ as $m \to \infty$.

4) Finally, we shall prove that the topological vector space $\widetilde{M}$ is complete.

Since $\widetilde{M}$ has a countable basis for the neighbourhoods of 0 (use e.g. the $N(1/n, 1/m)$, $n, m \in \mathbb{N}$), we just have to show that every Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ in $\widetilde{M}$ converges. So let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\widetilde{M}$.

Since M is dense in $\widetilde{M}$, we may assume that all $a_n \in M$ (if not, replace each a_n by $a'_n \in M$ such that $a_n - a'_n \in N(1/n, 1/n)$, and observe that $(a'_n)_{n\in\mathbb{N}}$ converges if and only if $(a_n)_{n\in\mathbb{N}}$ converges). Furthermore, we may assume that

$$\forall n \in \mathbb{N} : a_{n+1} - a_n \in N(2^{-(n+1)}, 2^{-n})$$

(since a subsequence of the given sequence has this property).

Now take projections $p_n \in M$ such that

$$\|(a_{n+1} - a_n)p_n\| \leq 2^{-(n+1)} \text{ and } \tau(1 - p_n) \leq 2^{-n}.$$

For each $n \in \mathbb{N}$, put

$$q_n = \wedge_{k=n+1}^{\infty} p_k.$$

Then

$$\tau(1-q_n) = \tau\left(\vee_{k=n-1}^{\infty}(1 - p_k)\right) \leq \sum_{k=n+1}^{\infty} \tau(1-p_k) \leq \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n};$$

and

$$\forall m \geq n + 1 \ \forall l \in \mathbb{N} : \|(a_{m+l} - a_m)q_n\| \leq 2^{-m} \tag{14}$$

since $q_n \leq p_k$ for all $k \geq m \geq n + 1$ and hence

$$\begin{aligned} \|(a_{m+l} - a_m)q_n\| &\leq \sum_{k=m}^{m+l-1} \|(a_{k+1} - a_k)q_n\| \\ &\leq \sum_{k=m}^{m+l-1} \|(a_{k+1} - a_k)p_k\| \leq \sum_{k=m}^{m+l-1} 2^{-(k+1)} \leq 2^{-m}. \end{aligned}$$

Let $\xi \in \cup_{n\in\mathbb{N}} q_n H$. Then $\xi \in q_n H$ for some $n \in \mathbb{N}$ and hence by (14), the sequence $(a_m\xi)_{m\in\mathbb{N}}$ is Cauchy. Put

$$a\xi = \lim_{m\to\infty} a_m\xi$$

We have now defined an operator a with $D(a) = \cup_{n\in\mathbb{N}} q_n H$ (note that $D(a)$ is a linear subspace because $(q_n)_{n\in\mathbb{N}}$ is an increasing sequence of projections).

By construction, a is τ-premeasurable: for all $n \in \mathbb{N}$, we have $q_n H \subset D(a)$ and $\tau(1-q_n) \leq 2^{-n}$. We claim that a is also preclosed. To see this, apply the preceding arguments to $(a_n^*)_{n\in\mathbb{N}}$. Hence there exists a τ-premeasurable operator b such that

$$b\eta = \lim_{m\to\infty} a_n^*\eta, \eta \in D(b).$$

Then

$$\forall \xi \in D(a)\, \forall \eta \in D(b) : (a\xi|\eta) = \lim(a_m\xi|\eta) = \lim(\xi|a_m^*\eta) = (\xi|b\eta),$$

whence

$$a \in b^*.$$

Hence a is preclosed. By Proposition 19 we then have $[a] \in \widetilde{M}$. Write $a_0 = [a]$.

Finally we shall prove that actually

$$a_n \to a_0 \text{ in } \widetilde{M}. \tag{15}$$

Let $\epsilon, \delta \in \mathbb{R}_+$. Take $n_0 \in \mathbb{N}$ such that $2^{-(n_0+1)} \leq \epsilon$ and $2^{-n_0} \leq \delta$. Then for all $m \geq n_0 + 1$ we have

$$\|(a_0 - a_m)q_{n_0}\| \leq 2^{-(n_0+1)} \leq \epsilon$$

and

$$\tau(1 - q_{n_0}) \leq 2^{-n_0} \leq \delta$$

since

$$\forall \xi \in H : (a_0 - a_m)q_{n_0}\xi = \lim_{l\to\infty}(a_{m+l}a_m)q_{n_0}\xi$$

and

$$\|(a_{m+l} - a_m)q_{n_0}\| \leq 2^{-m} \leq 2^{-(n_0+1)} \leq \epsilon.$$

Hence

$$\forall m \geq n_0 + 1 : a_0 - a_m \in N(\epsilon, \delta).$$

This proves (1.3). $\square$

Example. *1) If τ is finite, then $\widetilde{M} = \overline{M}$, i.e. all closed densely defined operators affiliated with M are τ-measurable (by Proposition 21, (vi)).*

2) If $M = B(H)$ and τ is the usual trace Tr*, then $\widetilde{M} = M$ (by Proposition 21, (iv), and the fact that* $\mathrm{Tr}(x) < 1, x \geq 0$*, implies $x = 0$).*

3) If (X, μ) is a measure space, $M = L^\infty(x, \mu)$ and $\tau = \int \cdot \mathrm{d}\mu$, then $\widetilde{M}$ is the closure of $L^\infty(x, \mu)$ for the topology of convergence in measure.

1.4 L^p spaces with respect to a trace

For any positive self-adjoint operator a affiliated with M, we put

$$\tau(a) = \sup_{n \in \mathbb{N}} \tau \left(\int_0^n \lambda \mathrm{d}e_\lambda \right)$$

where

$$a = \int_0^\infty \lambda \mathrm{d}e_\lambda$$

is the spectral representation of a . Then for each $p \in [1, \infty[$, we can define

$$L^p(M, \tau) = \{a \in \overline{M} | \tau(|a|^p) < \infty\}$$

and

$$\|a\|_p = \tau(|a|^p)^{\frac{1}{p}}, a \in L^p(M, \tau).$$

The $(L^p(M, \tau), \|\cdot\|_p)$ are Banach spaces in which $I = \{x \in M | \tau(|x|) < \infty\}$ is dense, and they are all contained in (and even continuously embedded in) $\widetilde{M}$ (for this and further results, see [13]; see also [3], [12], [21], and Chapter IV).

1.5 Notes and comments

The notion of measurable operators was introduced by I.E. Segal [15] and formed the basis for investigations in non-commutative integration theory, i.e. a theory of "integration" where $L^{\infty}(X, \mu)$ (corresponding to a measure space (X, μ) is replaced by a more general von Neumann algebra. Among other things, this theory provided a framework for constructing L^p spaces associated with (semifinite) von Neumann algebras as concrete spaces of (closed densely defined) operators ([12], [21]) (isomorphic to J. Dixmier's "abstract" L^p spaces [3]).

In [13], E. Nelson gave a new approach - requiring less knowledge of von Newmann algebra techniques - to the theory, based on the notion of measurability with respect to a trace (inspired by the notion of convergence in measure introduced by W. F. Stinespring in [16]). Any τ-measurable operator is also measurable in the sense of [15, Definition 2.1], whereas the converse is not in general true. The set of τ-measurable operators is, howe ver, big enough to contain the L^p spaces with respect to τ.

In our presentation, we have followed [13] with some modifications. In [13], $\widetilde{M}$ is defined as the (abstract) completion of M with respect to a certain (measure) topology on M (given by the 0-neighbourhoods $N(\epsilon, \delta) \cap M$, there simply called $N(\epsilon, \delta)$; afterwards, $\widetilde{M}$ is identified with a subset of the closed densely defined operators affiliated with M. As a tool, the completion of the Hilbert space H with respect to a certain (measure) topology is considered. - We have preferred to work with operators on H right from the beginning and to introduce the measure topology directly on the whole of $\widetilde{M}$. When doing so, we do not need a new topology on H.

Chapter 2

L^p Spaces Associated with a Von Neumann Algebra

In this chapter, we present Haagerup's theory of L^p spaces associated with a von Neumann algebra.

Let M be a von Newmann algebra and let φ_0 be a normal faithful semifinite weight on M.

We denote by N the crossed product $R(M, \sigma^{\varphi_0})$ of M by the modular automorphism group σ^{φ_0} associated with φ_0. Recall [18, Section 3; 8, Section 5] that if M is given on a Hilbert space H, then N is the Von Neumann algebra on the Hilbert space $L^2(\mathbb{R}, H)$ generated by the operators $\pi(x), x \in M$, and $\lambda(s)$, $s \in \mathbb{R}$, defined by

$$(\pi(x)\xi)(t) = \sigma_{-t}^{\varphi_0}(x)\xi(t), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}, \tag{1}$$

$$(\lambda(s)\xi)(t) = \xi(t-s), \xi \in L^2(\mathbb{R}, H), t \in \mathbb{R}. \tag{2}$$

We identify M with its image $\pi(M)$ in N (recall that π normal faithful representation of M).

We denote by θ the dual action of $\mathbb{R}$ in N. The θ_s, $s \in \mathbb{R}$, are automorphisms of N characterized by

$$\theta_s x = x, x \in M \tag{3}$$

$$\theta_s \lambda(t) = e^{-ist}\lambda(t), t \in \mathbb{R}. \tag{4}$$

By (3), M is contained in the set of fixed points under θ. Actually

$$M = \{y \in N | \forall s \in \mathbb{R} : \theta_s y = y\} \tag{5}$$

(see e.g. [5, Lemma 3.6]).

The θ_s, $s \in \mathbb{R}$, naturally extend to automorphisms, still denoted θ_s, of $\hat{N}_+$, the extended positive part of N [7, Section 1]. Recall [8, Lemma 5.2] that the formula

$$Tx = \int_{\mathbb{R}} \theta_s(x) \mathrm{d}s, x \in N_+, \tag{6}$$

defines a normal faithful semifinite operator valued weight T from N to M in the following sense: for each $x \in N_+$, Tx is the element of $\hat{N}_+$ characterized by

$$\langle Tx, \chi \rangle = \int_{\mathbb{R}} \langle \theta_s(x), \chi \rangle \mathrm{d}s \tag{7}$$

for all $x \in N_*^+$. Note that

$$\forall s \in \mathbb{R} : \theta_s \circ T = T. \tag{8}$$

In view of (5), this formula implies that the values of T are actually in $\hat{M}_+$.

For each normal weight φ on M, we put

$$\tilde{\varphi} = \hat{\varphi} \circ T \tag{9}$$

where $\hat{\varphi}$ denotes the extension of φ to a normal weight on $\hat{M}_+$ as described in [7, Proposition 1.10]. Then $\tilde{\varphi}$ is a normal weight on N [7,Proposition 2.3]; $\tilde{\varphi}$ is called the dual weight of φ (see [6, Introduction + Section 1)]. Note that (8) and (9) imply

$$\forall s \in \mathbb{R} : \tilde{\varphi} \circ \theta_s = \tilde{\varphi}. \tag{10}$$

If φ and ψ are normal faithful semifinite weights, then so are $\tilde{\varphi}$ and $\tilde{\psi}$, and we have [7, Theorem 4.7]:

$$\forall t \in \mathbb{R} \forall x \in M : \sigma_t^{\tilde{\varphi}}(x) = \sigma_t^{\varphi}(x), \tag{11}$$

$$\forall t \in \mathbb{R} : (D\tilde{\varphi} : D\tilde{\psi})_t = (D\varphi : D\psi)_t. \tag{12}$$

Lemma 1. *1) The mapping*

$$\varphi \mapsto \tilde{\varphi}$$

is a bijection of the set of all normal semifinite weights on M onto the set of normal semifinite weights ψ on N satisfying

$$\forall s \in \mathbb{R} : \psi \circ \theta_s = \psi. \tag{13}$$

2) For all normal weights φ and ψ on M and all $x \in M$, we have

1. $(\varphi + \psi)^{\sim} = \tilde{\varphi} + \tilde{\psi}$,
2. $(x \cdot \varphi \cdot x^*)^{\sim} = x \cdot \tilde{\varphi} \cdot x^*$,
3. $\operatorname{supp} \tilde{\varphi} = \operatorname{supp} \varphi$.

Proof. That $\tilde{\varphi}$ is semifinite if φ is follows from the proof of [7, Proposition 2.3]. That $\varphi \mapsto \tilde{\varphi}$ is injective follows from the formula

$$\varphi(\dot{T}x) = \tilde{\varphi}(x), x \in m_T,$$

and the fact that $\dot{T}(m_T)$ is σ-weakly dense in M [7, Proposition 2.5].

Now let us prove 2). First observe that $(\varphi + \psi)^\wedge = \hat{\varphi} + \hat{\psi}$ since $\hat{\varphi} + \hat{\psi} : \hat{M} \to [0, \infty]$ obviously satisfies the properties that characterize $(\varphi + \psi)^\wedge$ ([7, Proposition 1.10]); (a) follows trivially. Similarly, $(x \cdot \varphi \cdot x^*)^\wedge = x \cdot \hat{\varphi} \cdot x^*$, whence (b).

To prove (c), put $p_0 = 1 - \operatorname{supp} \varphi$. Then Mp_0 is the σ-weak closure in M of $N_\varphi = \{x \in M | \varphi(x^*x) = 0\}$. Similarly, the σ-weak closure in N of $N_{\tilde{\varphi}} = \{y \in N | \tilde{\varphi}(y^*y) = 0\}$ is Nq_0 where $q_0 = 1 - \operatorname{supp} \tilde{\varphi}$. Now

$$n_T N_\varphi \subset N_{\tilde{\varphi}}$$

since

$$\begin{aligned} \forall y \in n_T \forall x \in N_\varphi : & \tilde{\varphi}(x^*y^*yx) = \varphi(T(x^*y^*yx)) \\ = & \varphi(x^*T(y^*y)x) \leq \|T(y^*y)\| \varphi(x^*x) = 0. \end{aligned}$$

As n_T is σ-weakly dense in N, it follows that

$$N_\varphi \subset \overline{N_{\tilde{\varphi}}}^{\sigma-w}$$

whence

$$p_0 \leq q_0.$$

Note that we must have $q_0 \in M$ since $\tilde{\varphi}$, and hence $\operatorname{supp}\tilde{\varphi}$, is θ-invariant. Thus to conclude that $p_0 = q_0$ we need only show that $\varphi(q_0) = 0$. This follows from

$$\forall x \in m_T : \varphi(q_0 \dot{T}(x) q_0) = \varphi(\dot{T}(q_0 x q_0)) = \tilde{\varphi}(q_0 x q_0) = 0$$

and the fact that $\dot{T}(m_T)$ is σ-weakly dense in M [7, Proposition 2.5].

We now return to 1). Let ψ be a normal semifinite weight on N satisfying (13). First suppose that ψ is also faithful. Then by [5, (proof of) Theorem 3.7), it follows that $\psi = \tilde{\varphi}$ for some normal faithful semifinite φ on M.

In the general case, put $q_0 = 1 - \operatorname{supp}\psi$. Then by (13) and (5), we have $q_0 \in M$. Now take any normal semifinite weight χ_0 on M such that $\operatorname{supp}\chi_0 = q_0$. Then $\widetilde{\chi}_0$ is a normal faithful semifinite θ-invariant weight on N with $\operatorname{supp}\widetilde{\chi}_0 = q_0$. Hence $\widetilde{\chi}_0 + \psi$ is faithful and thus, as above,

$$\widetilde{\chi}_0 + \psi = \tilde{\varphi}$$

for some normal faithful semifinite weight φ on M. Finally, using (b), we find that

$$\begin{aligned}\psi =&(1-q_0)\cdot(\widetilde{\chi}_0+\psi)\cdot(1-q_0)\\ =&(1-q_0)\cdot\tilde{\varphi}\cdot(1-q_0)\\ =&((1-q_0)\cdot\varphi\cdot(1-q_0))^{\sim}.\end{aligned}$$

□

Denote by τ the normal faithful semifinite trace on N characterized by

$$\forall t \in \mathbb{R} : (D\tilde{\varphi}_0 : D\tau)_t = \lambda(t) \tag{14}$$

(for the existence, see [8, Lemma 5.2]); τ satisfies

$$\forall s \in \mathbb{R} : \tau \circ \theta_s = e^{-s}\tau. \tag{15}$$

With each $h \in \hat{N}_+$ we associate the normal weight $\tau(h\cdot)$ on N as in [8, remarks preceding Proposition 1.11]. When h is simply a positive self-adjoint operator affiliated with N (see [7, Example 1.2]), this definition agrees with that given in [14, Section 4].

We recall some facts about the mapping $h \mapsto \tau(h\cdot)$ (see [7, Theorem 1.12 (and its proof) and Preposition 1.11, (4)]):

Lemma 2. *1) The mapping*

$$h \mapsto \tau(h\cdot)$$

is a bijection of $\hat{N}_+$ onto the set of normal weights on N. In particular, it is a bijection of the positive self-adjoint operators affiliated with N onto the normal semifinite weights on N.

2) For all $h, k \in \hat{N}_+$ and all $x \in N$, we have

1. $\tau((h\dot{+}k)\cdot) = \tau(h\cdot) + \tau(k\cdot)$,
2. $\tau((x \cdot h \cdot x^*)\cdot) = x \cdot \tau(h\cdot) \cdot x^*$,
3. $\operatorname{supp} \tau(h\cdot) = \operatorname{supp} h$.

Here, $h\dot{+}k$ and $x{\cdot}h{\cdot}x^*$ denote the operations in $\hat{N}_+$ introduced in [7, Definition 1.3]. If h and k are positive self-adjoint operators such that $D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}})$ is dense, then $h\dot{+}k$ is the simply the form sum of h and k [2, Corollary 4.13]. If h is a positive self-adjoint operator and x a bounded operator such that $D(h^{\frac{1}{2}}x^*)$ is dense, then $x \cdot h \cdot x^* = \left|h^{\frac{1}{2}}x^*\right|^2$.

Definition 3. *For each normal weight φ on M we define h_φ as the unique element of $\hat{N}_+$ given by*

$$\tilde{\varphi} = \tau(h_\varphi\cdot). \tag{16}$$

Proposition 4. *1) The mapping*

$$\varphi \mapsto h_\varphi$$

is a bijection of the set of all normal semifinite weights on M onto the set of all positive self-adjoint operators h affiliated with N satisfying

$$\forall s \in \mathbb{R} : \theta_s h = e^{-s} h. \tag{17}$$

(2) For all normal weights φ and ψ on M and all $x \in M$, we have

1. $h_{\varphi+\psi} + h_\varphi \dotplus h_\psi$,
2. $h_{x\cdot\varphi\cdot x^*} = x \cdot h_\varphi \cdot x^*$,
3. $\operatorname{supp} h_\varphi = \operatorname{supp} \varphi$.

Proof. This proposition is an immediate consequence of Lemma 1 and 2. We just need to prove that a positive self-adjoint operator h affiliated with N satisfies (17) if and only if the corresponding weight $\tau(h\cdot)$ is θ-invariant. This follows easily from (15). Indeed, for all $s \in \mathbb{R}$ we have

$$\tau(e^s\theta_s(h)\cdot) = e^s(\tau \circ \theta_s)(h\theta_{-s}(\cdot)) = \tau(h\theta_{-s}(\cdot)) = \tau(h\cdot) \circ \theta_{-s},$$

whence

$$e^s\theta_s(h) = h \Leftrightarrow \tau(e^s\theta_s(h)\cdot) = \tau(h\cdot) \Leftrightarrow \tau(h\cdot) = \tau(h\cdot) \circ \theta_{-s}.$$

The equivalence of (17) and

$$\forall s \in \mathbb{R} : \tau(h\cdot) = \tau(h\cdot) \circ \theta_s$$

follows. □

The next lemma is essential. It will permit us apply results on τ-measurable operators.

As usual, $\chi_{]\gamma,\infty[}$ denotes the characteristic function for the interval $]\gamma, \infty[$.

Lemma 5. *Let φ be a normal semifinite weight on M. Then for all $\gamma \in \mathbb{R}_+$, we have*

$$\tau(\chi_{]\gamma,\infty[}(h_\varphi)) = \frac{1}{\gamma}\varphi(1).$$

Proof. First let us prove the formula in the case $\gamma = 1$.

Let $s \in \mathbb{R}$. Then since θ_s is an automorphism and $\theta_s h_\varphi = e^{-s} h_\varphi$ we have

$$\theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi)) = e^s h_\varphi^{-1}\chi_{]1,\infty[}(e^{-s}h_\varphi).$$

Now let $h_\varphi = \int \lambda \mathrm{d}e_\lambda$ be the spectral decomposition of h_φ. Then for any vector functional $\omega_{\xi,\xi}$, where ξ is a unit vector, we have

$$\begin{aligned}
\langle \int_{\mathbb{R}} \theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))\mathrm{d}s, \omega_{\xi,\xi}\rangle &= \int_{\mathbb{R}} \langle e^s h_\varphi^{-1}\chi_{]1,\infty[}(e^{-s}h_\varphi), \omega_{\xi,\xi}\rangle \mathrm{d}s \\
&= \int_{\mathbb{R}} \int_{]0,\infty[} e^s \lambda^{-1}\chi_{]1,\infty[}(e^{-s}\lambda)\, \mathrm{d}(e_\lambda \xi|\xi)\, \mathrm{d}s \\
&= \int_{]0,\infty[} \lambda^{-1}\left(\int_{]-\infty,\log\lambda[} e^s \mathrm{d}s\right) \mathrm{d}(e_\lambda\xi|\xi) \\
&= \int_{]0,\infty[} \lambda^{-1}\lambda\, \mathrm{d}(e_\lambda\xi|\xi) \\
&= \|(\operatorname{supp} h_\varphi)\xi\|^2
\end{aligned}$$

So that

$$\int_{\mathbb{R}} \theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))\mathrm{d}s = \operatorname{supp} h_\varphi = \operatorname{supp}\varphi.$$

Finally, since $\tilde{\varphi} = \tau(h_\varphi \cdot)$ we have

$$\begin{aligned}
\tau(\chi_{]1,\infty[}(h_\varphi)) &= \tau(h_\varphi^{\frac{1}{2}}(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))h_\varphi^{\frac{1}{2}}) \\
&= \tilde{\varphi}(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi)) \\
&= \varphi\left(\int \theta_s(h_\varphi^{-1}\chi_{]1,\infty[}(h_\varphi))\mathrm{d}s\right) = \varphi(\operatorname{supp}\varphi) = \varphi(1).
\end{aligned}$$

This completes the proof in the case $\gamma = 1$. In the general case, write $\gamma = e^s$, $s \in \mathbb{R}$. Then by (15)

$$\begin{aligned}\tau(\chi_{]e^s,\infty[}(h_\varphi)) =&\tau(\chi_{]1,\infty[}(e^{-s}h_\varphi))\\ =&\tau(\theta_s(\chi_{]1,\infty[}(h_\varphi)))\\ =&e^{-s}\tau(\chi_{]1,\infty[}(h_\varphi)) = e^{-s}\varphi(1).\end{aligned}$$

□

By Chapter I, Proposition 21, we have

Corollary 6. *Let φ be a normal semifinite weight on M. Then h_φ is τ-measurable if and only if $\varphi \in M_*$.*

We denote by $\tilde{N}$ the set of all τ-measurable closed densely defined operators affiliated with N. Recall (Chapter I) that $\tilde{N}$ is a topological *-algebra with respect to strong sum and product. Sums and products of elements in $\tilde{N}$ will always be understood to be in the strong sense although we do not emphasize it in the notation.

We denote by $\tilde{N}_+$ the subset of all positive self-adjoint elements of $\tilde{N}$.

Note that the θ_s, $s \in \mathbb{R}$, extend to continuous *-automorphisms, still denoted θ_s, of $\tilde{N}$. We have

$$\forall s \in \mathbb{R} \forall \epsilon, \delta \in \mathbb{R}_+ : \theta_s(N(\epsilon, \delta)) = N(\epsilon, e^{-s}\delta) \tag{18}$$

Since for all $a \in \tilde{N}_+$

$$\tau(\chi_{]\epsilon,\infty[}(\theta_s a)) = \tau(\theta_s(\chi_{]\epsilon,\infty[}(a))) = e^{-s}\tau(\chi_{]\epsilon,\infty[}(a))$$

(for the definition and properties of the 0-neighbourhoods $N(\epsilon, \delta)$, we refer to Chapter I).

Theorem 7. *1) The mapping*

$$\varphi \mapsto h_\varphi$$

extends to a linear bijection, still denoted $\varphi \mapsto h_\varphi$, of M_* onto the subspace

$$\{h \in \tilde{N} | \forall s \in \mathbb{R} : \theta_s h = e^{-s} h\} \tag{19}$$

of N.

2) For all $\varphi \in M_*$ and $x, y \in M$, we have

$$h_{x\cdot\varphi\cdot y^*} = xh_\varphi y^* \tag{20}$$

and

$$h_{\varphi^*} = h_\varphi^*. \tag{21}$$

3) If $\varphi = u|\varphi|$ is the polar decomposition of φ, then $h = uh_{|\varphi|}$ ($h_\varphi = uh_{|\varphi|}$) is the polar decomposition of h_φ. In particular,

$$|h_\varphi| = h_{|\varphi|}. \tag{22}$$

The proof will be based on Corollary 6, Proposition 4, and the following lemma.

Lemma 8. *1) Let h and k be positive self-adjoint operators such that $D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}})$ is dense. Then*

$$h + k \subset h\dot{+}k.$$

If $h + k$ is essentially self-adjoint, then its unique self-adjoint extension is precisely $h\dot{+}k$.

2) Let h be a positive self-adjoint operator and x a bounded operator such that $D(h^{\frac{1}{2}}x^*)$ is dense. Then

$$xhx^* \subset x \cdot h \cdot x^*.$$

If xhx^* is essentially self-adjoint, then its unique self-adjoint extension is precisely $x \cdot h \cdot x^*$.

Proof. 1) Recall that by definition $h\dot{+}k$ is the unique positive self-adjoint operator characterized by $D((h\dot{+}k)^{\frac{1}{2}}) = D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}})$ and

$$\forall \xi \in D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}}) : \left\|(h\dot{+}k)^{\frac{1}{2}}\xi\right\|^2 = \left\|h^{\frac{1}{2}}\xi\right\|^2 + \left\|k^{\frac{1}{2}}\xi\right\|^2. \tag{23}$$

By polarization, it follows that

$$\forall \xi \in D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}}) : ((h\dot{+}k)^{\frac{1}{2}}\xi|(h\dot{+}k)^{\frac{1}{2}}\eta) = (h^{\frac{1}{2}}\xi|h^{\frac{1}{2}}\eta) + (k^{\frac{1}{2}}\xi|k^{\frac{1}{2}}\eta).$$

Now let $\xi \in D(h+k) = D(h) \cap D(k)$ and $\eta \in D(h\dot{+}k)$. Then also $\xi \in D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}})$ and $\eta \in D((h\dot{+}k)^{\frac{1}{2}}) = D(h^{\frac{1}{2}}) \cap D(k^{\frac{1}{2}})$ so that

$$\begin{aligned}((h+k)\xi|\eta) =&(h\xi|\eta) + (k\xi|\eta)\\ =&(h^{\frac{1}{2}}\xi|h^{\frac{1}{2}}\eta) + (k^{\frac{1}{2}}\xi|k^{\frac{1}{2}}\xi)\\ =&((h\dot{+}k)^{\frac{1}{2}}\xi|(h\dot{+}k)^{\frac{1}{2}}\eta)\\ =&(\xi|(h\dot{+}k)\eta).\end{aligned}$$

It follows that

$$h + k \subset (h\dot{+}k)^* = (h\dot{+}k).$$

Hence $h+k$ is preclosed and $[h+k] \subset h\dot{+}k$. If $[h+k]$ is self-adjoint, we must have $[h+k] = h\dot{+}k$.

2) Recall that $x \cdot h \cdot x^* = \left|h^{\frac{1}{2}}x^*\right|^2$. Now let $\xi \in D(xhx^*) = D(hx^*)$ and $\eta \in D(x \cdot h \cdot x^*)$. Then also $\xi \in D(h^{\frac{1}{2}}x^*)$ and $\eta \in D((x \cdot h \cdot x^*)^{\frac{1}{2}}) = D(h^{\frac{1}{2}}x^*)$ so that

$$(xhx^*\xi|\eta) = (hx^*\xi|x^*\eta) = (h^{\frac{1}{2}}x^*\xi|h^{\frac{1}{2}}x^*\eta) = (\xi|(x \cdot h \cdot x^*)\eta).$$

It follows that

$$xhx^* \subset (x \cdot h \cdot x^*)^* = x \cdot h \cdot x^*.$$

Hence xhx^* is preclosed and $[xhx^*] \subset x \cdot h \cdot x^*$. If $[xhx^*]$ is self-adjoint, we must have $[xhx^*] = x \cdot h \cdot x^*$. □

Proof of Theorem 7. Let $\varphi, \psi \in M_*^+$. Then h_φ and h_ψ are positive self-adjoint and τ-measurable so that their strong sum exists and is again a positive self-adjoint τ-measurable operator. By Lemma 8, this sum then coincides with $h_\varphi \dot{+} h_\psi$. Then Proposition 4 yields

$$h_{\varphi+\psi} = h_\varphi + h_\psi,$$

where the sum at the right hand side is now the sum in $\tilde{N}$.

Similarly for all $\varphi \in M_*^+$ and $x \in M$ we get

$$h_{x\cdot\varphi\cdot\xi^*} = xh_\varphi x^*. \tag{24}$$

Now the additive and homogeneous mapping $\varphi \mapsto h_\varphi$ of M_*^+ onto $\{h \in \tilde{N}_+ | \forall s \in \mathbb{R} : \theta_s h = e^{-s} h\}$ extends by linearity to a linear mapping $\varphi \mapsto h_\varphi$ of M_* onto the subspace of $\tilde{N}$ spanned by $\{h \in \tilde{N}_+ | \forall s \in \mathbb{R} : \theta_s h = e^{-s} h\}$, i.e. onto the subspace (19) (evidently, (19) is stable under $h \mapsto h^*$ and $h \mapsto |h|$ and hence spanned by its positive elements).

By linearity, we must have (21) for a11 $\varphi \in M_*$. Also by linearity, (24) holds for all $\varphi \in M_*$ and $x \in M$; by polarization the equation (20) follows for all $\varphi \in M_*$ and $x, y \in M$.

In particular, if $\varphi \in u|\varphi|$ is the polar decomposition of φ, we have

$$h_\varphi = h_{u|\varphi|} = u h_{|\varphi|}.$$

That this relation is the polar decomposition of h_φ follows from the fact that the initial projection for the partial isometry u is $\operatorname{supp}|\varphi| = \operatorname{supp} h_{|\varphi|}$.

Finally, $\varphi \mapsto h_\varphi$ is injective: if $h_\varphi = 0$, then $h_{|\varphi|} = |h_\varphi| = 0$, whence $|\varphi| = 0$ and $\varphi = 0$. $\square$

Motivated by Theorem 7, we now give the following definition:

Definition 9. *For each $p \in [1, \infty]$, we let*

$$L^p(M) = \{a \in \tilde{N} | \forall s \in \mathbb{R} : \theta_s a = e^{-\frac{s}{p}} a\}.$$

Note that the $L^p(M)$ are linear subspaces of $\tilde{N}$ and that they are linearly spanned by their positive part $L^p(M)_+ = L^p(M) \cap \tilde{N}_+$.

By Theorem 7, we know that $L^1(M) \cong M_*$. And:

Proposition 10. *We have $L^\infty(M) = M$.*

Proof. In view of (5), we just need to show that every $a \in L^\infty(M)$ is bounded. Let $a \in L^\infty(M)$. Then for all $s \in \mathbb{R}$ and all $\lambda \in \mathbb{R}_+$ we have

$$\begin{aligned}\tau(\chi_{]\lambda,\infty[}(|a|)) =& \tau(\chi_{]\lambda,\infty[}(\theta_s|a|)) \\ =& \tau(\theta_s(\chi_{]\lambda,\infty[}(|a|))) = e^{-s}\tau(\chi_{]\lambda,\infty[}(|a|)).\end{aligned}$$

Hence for all $\lambda \in \mathbb{R}_+$ we must have

$$\tau(\chi_{]\lambda,\infty[}(|a|)) = 0 \text{ or } \tau(\chi_{]\lambda,\infty[}(|a|)) = \infty.$$

Since a is τ-measurable, we have $\tau(\chi_{]\lambda,\infty[}(|a|)) < \infty$ for some λ. Hence $\tau(\chi_{]\lambda,\infty[}(|a|)) = 0$ and thus $\chi_{]\lambda,\infty[}(|a|) = 0$ since τ is faithful. This means that a is bounded. □

Remark 11. *We have seen that all elements of $L^\infty(M)$ are bounded. In contrast to this, all non-zero elements of $L^p(M)$, where $p < \infty$, are unbounded. To see this, let $a \in L^p(M)$ and suppose that $a \neq 0$. Then for some $\lambda \in \mathbb{R}_+$ we have $\chi_{]\lambda,\infty[}(|a|) \neq 0$ and hence $\tau(\chi_{]\lambda,\infty[}(|a|)) \neq 0$. Then for all $\mu \in \mathbb{R}_+$ we have*

$$\tau(\chi_{]\mu,\infty[}(|a|)) \neq 0$$

since for all $s \in \mathbb{R}$

$$\begin{aligned}\tau(\chi_{]e^{\frac{s}{p}}\lambda,\infty[}(|a|)) =& \tau(\chi_{]\lambda,\infty[}(e^{-\frac{s}{p}}|a|)) \\ =& \tau(\chi_{]\lambda,\infty[}(\theta_s|a|)) \\ =& \tau(\theta_s\chi_{]\lambda,\infty[}(|a|)) \\ =& e^{-s}\tau(\chi_{]\lambda,\infty[}(|a|)) \neq 0.\end{aligned}$$

It follows that $|a|$ must be unbounded.

Proposition 12. *Let a be a closed densely defined operator affiliated with N with polar decomposition $a = u|a|$. Let $p \in [1,\infty[$. Then*

$$a \in L^p(M)$$

if and only if

$$u \in M \text{ and } |a|^p \in L^1(M).$$

Proof. Recall that $a \in \tilde{N}$ if and only if $|a| \in \tilde{N}$. Furthermore, $|a| \in \tilde{N}$ if and only if $|a|^p \in \tilde{N}$ since $\tau(\chi_{]\lambda,\infty[}(|a|)) = \tau(\chi_{]\lambda^p,\infty[}(|a|^p))$ for all $\lambda \in \mathbb{R}_+$. For all such a and all $s \in \mathbb{R}$ we have

$$\theta_s a = e^{-\frac{s}{p}}a \Leftrightarrow \theta_s u = u \text{ and } \theta_s|a|^p = e^{-s}|a|^p.$$

The result follows by Definition 9 and Proposition 10. □

A similar result holds for the right polar decomposition.

Definition 13. *We define a linear functional* tr *on* $L^1(M)$ *by*

$$\operatorname{tr}(h_\varphi) = \varphi(1), \varphi \in M_*.$$

Note that

$$\operatorname{tr}(|h_\varphi|) = \operatorname{tr}\big(h_{|\varphi|}\big) = |\varphi|(1) = \|\varphi\| \tag{25}$$

for all $\varphi \in M_*$. This implies that

$$|\operatorname{tr}(a)| \leq \operatorname{tr}(|a|) \tag{26}$$

for all $a \in L^1(M)$ and that the mapping $a \mapsto \operatorname{tr}(|a|)$ is a norm on $L^1(M)$.

Definition 14. *Let* $p \in [1, \infty[$. *Then we define* $\|\cdot\|_p$ *on* $L^p(M)$ *by*

$$\|a\|_p = \operatorname{tr}(|a|^p)^{\frac{1}{p}}, a \in L^p(M).$$

For $p = \infty$, *we put*

$$\|a\|_\infty = \|a\|, a \in L^\infty(M).$$

We shall see that for all p, $\|\cdot\|_p$ is a norm on $L^p(M)$.

By (26), we have

Proposition 15. *The mapping*

$$\varphi \mapsto h_\varphi : M_* \to L^1(M)$$

is an isometry of M_* *onto* $L^1(M)$.

Lemma 16. *Let* $p \in [1, \infty[$ *and* $\epsilon, \delta \in \mathbb{R}_+$. *Then*

$$N(\epsilon, \delta) \cap L^p(M) = \{a \in L^p(M) | \|a\|_p \leq \epsilon\delta^{\frac{1}{p}}\}.$$

Proof. Let $a \in L^p(M)$. Then $|a|^p \in L^1(M)_+$ and hence $|a|^p = h_\varphi$ for some $\varphi \in M_*^+$. Now

$$\begin{aligned}\tau(\chi_{]\epsilon,\infty[}(|a|)) =& \tau(\chi_{]\epsilon^p,\infty[}(|a|^p)) \\ =& \frac{1}{\epsilon^p}\varphi(1) \\ =& \frac{1}{\epsilon^p}\||a|^p\|_1 = \frac{1}{\epsilon^p}\|a\|_p^p\end{aligned}$$

Using this we get

$$
\begin{aligned}
a \in N(\epsilon,\delta) \Leftrightarrow & |a| \in N(\epsilon,\delta) \\
\Leftrightarrow & \tau(\chi_{]\epsilon,\infty[}(|a|)) \leq \delta \\
\Leftrightarrow & \frac{1}{\epsilon^p}\|a\|_p^p \leq \delta \\
\Leftrightarrow & \|a\|_p \leq \epsilon\delta^{\frac{1}{p}}.
\end{aligned}
$$

□

Corollary 17. *On $L^1(M)$ the norm topology is exactly the topology induced from $\tilde{N}$.*

We denote by $\mathbb{C}_+$ the closed half-plane $\{a \in \mathbb{C} \,|\, \operatorname{Re} a \geq 0\}$ and by $\mathbb{C}_+^\circ$ the corresponding open half-plane.

Lemma 18. *Let $h \in \tilde{N}_+$. Then the mapping*

$$\alpha \mapsto h^\alpha : \mathbb{C}_+^\circ \to \tilde{N}$$

is differentiable.

Proof. First note that all h^α, $\alpha \in \mathbb{C}_+^\circ$, are actually τ-measurable since h is τ-measurable.

1) Suppose that h is bounded, i.e. $h \in N_+$. Then the mapping

$$\alpha \mapsto h^\alpha : \mathbb{C}_+^\circ \to N$$

is differentiable with respect to the norm topology on N and

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} h^\alpha = h^\alpha \log h \tag{27}$$

(note that the expression at the right hand side is defined for any positive $h \in N$ since the function $\lambda \mapsto \lambda^\alpha \log\lambda$ is continuous on the closed half-plane $\mathbb{C}_+$). This follows from spectral theory using the fact that for all $\alpha_0 \in \mathbb{C}_+^\circ$ we have

$$
\begin{aligned}
\frac{1}{\alpha-\alpha_0}(\lambda^\alpha - \lambda^{\alpha_0}) - \lambda^{\alpha_0}\log\lambda = & \frac{1}{\alpha-\alpha_0}(e^{\alpha\log\lambda} - e^{\alpha_0\log\lambda}) - \log\lambda e^{\alpha_0\log\lambda} \\
\to & 0 \text{ as } \alpha \to \alpha_0 \text{ uniformly in } \lambda \in]0, \|h\|].
\end{aligned}
$$

2) Now let h be any element of $\tilde{N}_+$. We claim that $\alpha \mapsto h^\alpha : \mathbb{C}_+^\circ \to \tilde{N}$ is differentiable with respect to the topology on $\tilde{N}$ and that (27) still holds (as above, $h^\alpha \log h$ is a well-defined positive self-adjoint operator and, by spectral theory, it is τ-measurable). Now let $\epsilon, \delta \in \mathbb{R}_+$. Take $\lambda \in \mathbb{R}_+$ such that $\tau(\chi_{]\lambda,\infty[}(h)) \leq \delta$. Put $p = \chi_{[0,\lambda]}(h)$. Then hp is bounded and by the first part of the proof

$$\begin{aligned}&\left\|\left(\frac{1}{\alpha-\alpha_0}(h^\alpha - h^{\alpha_0}) - h^{\alpha_0}\log h\right)p\right\| \\ =&\left\|\frac{1}{\alpha-\alpha_0}((hp)^\alpha - (hp)^{\alpha_0}) - (hp)^{\alpha_0}\log(hp)\right\| \leq \epsilon\end{aligned}$$

Origin article here is $(hp)^\alpha \log(hp)$ for all $\alpha \in \mathbb{C}_+^\circ$ sufficiently close to α_0. Thus

$$\frac{1}{\alpha-\alpha_0}(h^\alpha - h^{\alpha_0}) - h^{\alpha_0}\log h \in N(\epsilon, \delta)$$

for α sufficiently close to α_0. This proves the lemma. $\square$

We denote by S the closed complex strip $\{\alpha \in \mathbb{C} | 0 \leq \operatorname{Re}\alpha \leq 1\}$ and by S° the corresponding open strip.

Lemma 19. *Let $h, k \in L^1(M)_+$. Then for $\alpha \in S^\circ$ we have*

$$h^\alpha k^{1-\alpha} \in L^1(M),$$

and the mapping

$$\alpha \mapsto h^\alpha k^{1-\alpha} : S^\circ \to L^1(M) \tag{28}$$

is analytic.

Proof. That $h^\alpha k^{1-\alpha} \in L^1(M)$ follows from Definition 9 since

$$\begin{aligned}\forall s \in \mathbb{R} : \theta_s(h^\alpha k^{1-\alpha}) =&(\theta_s h)^\alpha (\theta_s k)^{1-\alpha} \\ =&e^{-\alpha s} h^\alpha e^{-(1-\alpha)s} k^{1-\alpha} = e^{-s} h^\alpha k^{1-\alpha}.\end{aligned}$$

Origin article here is $e^{-s} h^\alpha h^{1-\alpha}$ we want to prove that the mapping (28) is differentiable. In view of Corollary 17 we may as well prove that (28) is differentiable as a mapping into $\tilde{N}$. Now by the preceding lemma, the functions $f, g : S^\circ \mapsto \tilde{N}$ defined by $f(\alpha) = h^\alpha$ and $g(\alpha) = k^{1-\alpha}$. are differentiable. It follows that for all $\alpha_0 \in S^\circ$ we have

$$\begin{aligned}
&\frac{1}{\alpha - \alpha_0}(f(\alpha)g(\alpha) - f(\alpha_0)g(\alpha_0)) \\
=&\frac{1}{\alpha - \alpha_0}f(\alpha)(g(\alpha) - g(\alpha_0)) + \frac{1}{\alpha - \alpha_0}(f(\alpha) - f(\alpha_0))g(\alpha_0) \\
\to& f(\alpha_0)g'(\alpha_0) + f'(\alpha_0)g(\alpha_0) \text{ as } \alpha \to \alpha_0
\end{aligned}$$

so that also $f \cdot g : S^\circ \to \tilde{N}$ is differentiable. □

Lemma 20. *Let $t \in \mathbb{R}$ and put*

$$\tilde{N}_{\frac{1}{2}+it} = \{a \in \tilde{N} | \forall s \in \mathbb{R} : \theta_s a = e^{-(\frac{1}{2}+it)}a\}. \tag{29}$$

Let $a, b \in \tilde{N}_{\frac{1}{2}+it}$. Then $b^*a, ab^* \in L^1(M)$ and

$$\mathrm{tr}(b^*a) = \mathrm{tr}(ab^*). \tag{30}$$

Proof. That $b^*a, ab^* \in L^1(M)$ follows from Definition 9 and (29).

To prove (30), suppose first that $a = b$. Then by Definition 13 and Lemma 5

$$\mathrm{tr}(a^*a) = \tau(\chi_{]1,\infty[}(a^*a)) = \tau(\chi_{]1,\infty[}(aa^*)) = \mathrm{tr}(aa^*).$$

In the general case, note that $a + ib \in \tilde{N}_{\frac{1}{2}+it}$ and

$$b^*a = \frac{1}{4}\sum_{k=0}^{3} i^k(a + i^k b)^*(a + i^k b)$$

$$ab^* = \frac{1}{4}\sum_{k=0}^{3} i^k(a + i^k b)(a + i^k b)^*$$

The result follows since tr is linear. □

Proposition 21. *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $a \in L^p(M)$ and $b \in L^q(M)$. Then $ab, ba \in L^1(M)$ and*

$$\mathrm{tr}(ab) = \mathrm{tr}(ba).$$

Proof. If $p = 1$ we have $a = h_\varphi$ for some $\varphi \in M_*$ and the result follows by Theorem 7:

$$\operatorname{tr}(h_\varphi b) = \operatorname{tr}(h_{\varphi\cdot b}) = (\varphi \cdot b)(1) = (b \cdot \varphi)(1) = \operatorname{tr}(h_{b\cdot\varphi}) = \operatorname{tr}(bh_\varphi)$$

Now suppose that $p, q \in]1, \infty[$. As usual, we easily see that ab and ba are in $L^1(M)$. By linearity, we may assume that $a \in L^p(M)_+$ and $b \in L^q(M)_+$. Now $a^p, b^q \in L^1(M)_+$ and by Lemma 19 the functions F and G on S° defined by $F(\alpha) = \operatorname{tr}\big(a^{p\alpha} b^{q(1-\alpha)}\big)$ and $G(\alpha) = \operatorname{tr}\big(b^{q(1-\alpha)} a^{p\alpha}\big)$ are analytic. For all $t \in \mathbb{R}$, we have $a^{p(\frac{1}{2}+it)} \in \tilde{N}_{\frac{1}{2}+it}$ and $b^{q(\frac{1}{2}+it)} \in \tilde{N}_{\frac{1}{2}+it}$ so that by Lemma 20

$$\begin{aligned} F(\frac{1}{2} + it) &= \operatorname{tr}\Big(a^{p(\frac{1}{2}+it)} b^{q(\frac{1}{2}-it)}\Big) = \operatorname{tr}\Big(a^{p(\frac{1}{2}+it)} (b^{q(\frac{1}{2}-it)})^*\Big) \\ &= \operatorname{tr}\Big((b^{q(\frac{1}{2}-it)})^* a^{p(\frac{1}{2}+it)}\Big) = \operatorname{tr}\Big(b^{q(\frac{1}{2}-it)} a^{p(\frac{1}{2}+it)}\Big) = G(\frac{1}{2} + it) \end{aligned}$$

We conclude that $F = G$. In particular,

$$\operatorname{tr}(ab) = F(\frac{1}{p}) = G(\frac{1}{p}) = \operatorname{tr}(ba).$$

□

The proof of the next lemma is based on the 3 lines theorem for analytic functions (see e.g. [23, p.93]). The 3 lines theorem also holds for analytic functions F with values in a Banach space (to see this, apply it to the scalar-valued functions $\alpha \mapsto v(F(\alpha))$, where v is in the dual of the given Banach space).

Lemma 22. *Let $h, k \in L^1(M)_+$ and suppose that $\|h\|_1 = \|k\|_1 = 1$. Then for all $\alpha \in S^\circ$, we have*

$$\left\|h^\alpha k^{1-\alpha}\right\|_1 \leq 1$$

Proof. Write $s = \operatorname{Re}\alpha$, $t = \operatorname{Im}\alpha$. Then $h^s \in L^{\frac{1}{s}}(M)$ with $\|h^s\|_{\frac{1}{s}} = 1 = s^{-s} \cdot s^s$, whence by Lemma 16

$$h^s \in N(s^{-s}, s).$$

Similarly,

$$k^{1-s} \in N((1-s)^{-(1-s)}, 1-s).$$

It follows that

$$\begin{aligned} h^s k^{1-s} \in & N(s^{-s}, s) \cdot N((1-s)^{-(1-s)}, 1-s) \\ \subset & N(s^{-s}(1-s)^{-(1-s)}, s + (1-s)) \end{aligned}$$

whence also

$$h^\alpha k^{1-\alpha} = h^{it} h^s k^{1-s} k^{-it} \in N(s^{-s}(1-s)^{-(1-s)}, 1)$$

Again by Lemma 16,

$$\left\| h^\alpha k^{1-\alpha} \right\|_1 \leq s^{-s}(1-s)^{-(1-s)}$$

Since $s \mapsto s^{-s}(1-s)^{-(1-s)}$ is bounded, the function $\alpha \mapsto h^\alpha k^{1-\alpha} : S^\circ \to L^1(M)$ is bounded. It is analytic by Lemma 19. Hence we can apply the 3 lines theorem on each closed strip $\{a \in \mathbb{C} | \epsilon \leq \operatorname{Re} \alpha \leq 1 - \epsilon\}$ and we obtain

$$\sup_{t \leq \operatorname{Re} \alpha \leq 1-\epsilon} \left\| h^\alpha k^{1-\alpha} \right\|_1 \leq \epsilon^{-\epsilon}(1-\epsilon)^{-(1-\epsilon)}.$$

Hence for fixed $a \in S^\circ$, the inequality

$$\left\| h^\alpha k^{1-\alpha} \right\|_1 \leq \epsilon^{-\epsilon}(1-\epsilon)^{-(1-\epsilon)}$$

holds for all $\epsilon \in \mathbb{R}_+$ such that $\epsilon \leq \operatorname{Re} \alpha \leq 1 - \epsilon$. Since

$$\epsilon^{-\epsilon}(1-\epsilon)^{-(1-\epsilon)} = e^{-\epsilon \log \epsilon} e^{-(1-\epsilon)\log(1-\epsilon)} \to 1 \text{ as } \epsilon \to 0,$$

it follows that

$$\left\| h^\alpha k^{1-\alpha} \right\|_1 \leq 1$$

This proves the lemma. □

Theorem 23 (Hölder's inequality)**.** *Let* $p, q \in [1, \infty]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $a \in L^p(M)$ *and* $b \in L^q(M)$. *Then*

$$\|ab\|_1 \leq \|a\|_p \|b\|_q.$$

Proof. If $p = 1$, we have $a = h_\varphi$ for some $\varphi \in M_*$ and

$$\|h_\varphi b\|_1 = \|h_{\varphi\cdot b}\|_1 = \|\varphi \cdot b\| \leq \|\varphi\| \|b\|_\infty = \|h_\varphi\|_1 \cdot \|b\|_\infty$$

for all $b \in L^\infty(M) = M$. The case $q = 1$ is quite similar to this.

Now assume $p, q \in]1, \infty[$, and $\|a\|_p = 1$, $\|b\|_q = 1$. Let $a = u|a|$ be the (usual) polar decomposition of a and $b = |b^*|v$ the right polar decomposition of b. Then $|a|^p, |b^*|^q \in L^1(M)$ with $\||a|^p\| = \||b^*|^q_r\|_1 = 1$ and Lemma 22 applies:

$$\begin{aligned}\|ab\|_1 =& \|u|a||b^*|v\|_1 \leq \||a||b^*|\|_1 \\ =& \left\| |a|^{\frac{p}{p}} |b^*|^{\frac{q}{q}} \right\|_1 \leq 1.\end{aligned}$$

□

Proposition 24. *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $a \in L^p(M)$. Then*

$$\|a\|_p = \sup\{|\mathrm{tr}(ab)| | b \in L^q(M), \|b\|_q \leq 1\}.$$

Proof. If $p = 1$ or $p = \infty$ this is well-known (since $\mathrm{tr}(ch_\varphi) = \mathrm{tr}(h_\varphi c) = \varphi(c)$ for all $\varphi \in M_*$ and $c \in M$). Suppose that $1 < p < \infty$. We may assume that $\|a\|_p = 1$. Then putting $b = |a|^{\frac{p}{q}} u^*$, where $a = u|a|$ is the polar decomposition of a, we have $b \in L^q(M)$ with $\|b\|_q = \left\| |a|^{\frac{p}{q}} u^* \right\|_q = \mathrm{tr}(|a|^p)^{\frac{1}{q}} = 1$ and

$$\mathrm{tr}(ab) = \mathrm{tr}\left(u|a||a|^{\frac{p}{q}} u^*\right) = \mathrm{tr}(|a|^p) = 1.$$

Hence

$$\|a\|_p = 1 \leq \sup\{|\mathrm{tr}(ab)| | b \in L^q(M), \|b\|_q \leq 1\}.$$

The converse inequality follows from Hölder's inequality (together with (26)). □

Corollary 25. $\|\cdot\|_p$ *is a norm on* $L^p(M)$.

Proof. The inequality

$$\|a + b\|_p \leq \|a\|_p + \|b\|_p$$

follows immediately from Proposition 24. □

Proposition 26. *On $L^p(M)$, the norm topology is exactly the topology induced from $\tilde{N}$.*

Proof. Now that we know that $\|\cdot\|_p$ is a norm, this is a corollary of Lemma 16. $\square$

Corollary 27. $(L^p(M), \|\cdot\|_p)$ *is a Banach space.*

Proof. From the definition of $L^p(M)$ it follows that it is a closed subspace of the complete topological vector space $\tilde{N}$. Hence it is complete for the uniform structure induced from $\tilde{N}$. By Lemma 16, this is simply the uniform structure coming from the norm. Hence $L^p(M)$ is a complete normed space. $\square$

Corollary 28. $(L^2(M), \|\cdot\|_2)$ *is a Hilbert space with the inner product*

$$(a|b)_{L^2(M)} = \operatorname{tr}(b^*a) \quad (= \operatorname{tr}(ab^*)), a, b \in L^2(M).$$

Proof. That $(a, b) \mapsto (a|b)_{L^2(M)}$ is an inner product defining the norm $\|\cdot\|_2$ is easily verified. By Corollary 27, $L^2(M)$ is complete. $\square$

Remark 29. *Let $t \in \mathbb{R}$. Define $\tilde{N}_{\frac{1}{2}+it}$ as in Lemma 20. Then*

$$(a, b) \mapsto \operatorname{tr}(b^*a)$$

is an inner product on $\tilde{N}_{\frac{1}{2}+it}$ and

$$a \mapsto \operatorname{tr}(a^*a)^{\frac{1}{2}}$$

is a norm which we shall denote by $\|\cdot\|_2$ (as in the case $t = 0$ where $\tilde{N}_{\frac{1}{2}} = L^2(M)$). Note that

$$|\operatorname{tr}(b^*a)| \leq \|a\|_2 \|b\|_2$$

and

$$\|a + b\|_2^2 + \|a - b\|_2^2 = 2\|a\|_2^2 + 2\|b\|_2^2$$

for all $a, b \in \tilde{N}_{\frac{1}{2}+it}$.