Contents

Chapter 1

Spatial Derivatives

Spatial derivatives were introduced by A. Connes in [1]. In this chapter, we give an alternative definition (equivalent to that given in [1]) suggested to us by U. Haagerup, based on the notion of the extended positive part of a von Neumann algebra. This definition permits us to obtain very easily some elementary properties of spatial derivatives. After this, we recall their main modular properties and the characterization as (-1)-homogeneous operators.

1.1 Definition and elementary properties of spatial derivatives

Let M be a von Neumann algebra acting on a Hilbert space H, and let ψ be a normal faithful semifinite weight on the commutant M' of M.

We shall use the following standard notation: $n_\psi = \{y \in M' | \psi(y^*y) < \infty\}$, H_ψ the Hilbert space completion of n_ψ with respect to the inner product $(y_1, y_2) \mapsto \psi(y_2^* y_1)$, Λ_ψ the canonical injection of n_ψ into H_ψ, π_ψ the canonical representation of M' on H_ψ.

Definition 1. *For each $\xi \in H$, we denote by $R^\psi(\xi)$ the (densely defined) operator from H_ψ to H defined by*

$$R^\psi(\xi)\Lambda_\psi(y) = y\xi, y \in n_\psi. \tag{1}$$

Proposition 2. *For all $\xi, \xi_1, \xi_2 \in H$, $x \in M$, and $y \in M'$ we have*

(i) $R^\psi(\xi_1 + \xi_2) = R^\psi(\xi_1) + R^\psi(\xi_2)$,

(ii) $R^\psi(x\xi) = xR^\psi(\xi)$,

(iii) $yR^\psi(\xi) \subset R^\psi(\xi)\pi_\psi(y)$,

and

*(i)** $R^\psi(\xi_1)^* + R^\psi(\xi_2)^* \subset R^\psi(\xi_1 + \xi_2)^*$,

*(ii)** $R^\psi(x\xi)^* = R^\psi(\xi)^* x^*$,

*(iii)** $\pi_\psi(y)R^\psi(\xi)^* \subset R^\psi(\xi)^* y$.

Proof. (i) and (ii) are immediate from Definition 1. (iii): For all $z \in n_\psi$, we have $yR^\psi(\xi)\Lambda_\psi(z) = yz\xi = R^\psi(\xi)\Lambda_\psi(yz) = R^\psi(\xi)\pi_\psi(y)\Lambda_\psi(z)$.

(i)*, (ii)*, and (iii)* follow from (i), (ii), and (iii) using $R^\psi(\xi_1) + R^\psi(\xi_2) \subset (R^\psi(\xi_1) + R^\psi(\xi_2))^*$, $(xR^\psi(\xi))^* = R^\psi(\xi)^* x^*$, and $(y^* R^\psi(\xi))^* = R^\psi(\xi)^* y^*$. □

Definition 3. *A vector $\xi \in H$ is called ψ-bounded if the operator $R^\psi(\xi)$ is bounded. The set of ψ-bounded vectors is denoted $D(H, \psi)$.*

Notation. *If $\xi \in D(H, \psi)$, $R^\psi(\xi)$ extends to a bounded operator $H_\psi \to H$ which we shall also denote $R^\psi(\xi)$.*

Proposition 4. *The set $D(H, \psi)$ is an M-invariant dense subspace of H.*