0.1 Canonical Factorization Theorem

In section 3, we show a holomorphic function $F \in H^p$, $0 < p \leq \infty$ can be written as $F = BH$, where B is the Blaschke product, H is never zero and $\|H\|_{H^p} = \|F\|_{H^p}$. We will get a finer factorization: canonical factorization. We can factorize F as product of inner and outer function:

$$F(z) = I_F(z)E_F(z)$$

where:

$$I_F(z) = e^{ic}B(z)\exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\sigma(t)\right)$$

and

$$E_F(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log\left|F(e^{it})\right|dt\right)$$

0.1.1 Canonical factorization

For $F \in H^p$, $0 < p \leq \infty$, we use Riesz factorization $F = BH$. We want to keep factorizing non zero function H. Write $\log\left|H(r_je^{it})\right| = \log^+\left|H(r_je^{it})\right| - \log^-\left|H(r_je^{it})\right|$. We know $\log^-\left|H(r_je^{it})\right|$ converges to a positive measure μ_2. And we observe $\log^+\left|H(r_je^{it})\right| \to \log^+\left|H(e^{it})\right|$ in H^1 norm.

Since $\log H(r_jz) = \log|H(r_jz)| + i\theta$, $|\log H(r_jz)| \leq |\log|H(r_jz)|| + |\theta|$. By theorem 3.2 in book, we show that $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\log\left|H(re^{it})\right|\right|dt$ is finite. Thus $\log H(r_jz) \in H^1$.

Now we factorize the non zero function H. by corollary 3.9 in book:

$$\log H(r_jz) = i\arg H(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log\left|H(r_je^{it})\right|dt$$

Using $\log = \log^+ - \log^-$, and let $r_j \to 1$, We have:

$$\log H(z) = ic + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log^+\left|H(e^{it})\right|dt - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\mu_2(t)$$

Write $d\mu_2(t) = g(t)dt + d\sigma(t)$, $k(t) = \log^+\left|H(e^{it})\right| - g(t)$. We finally get canonical theorem:

$$F(z) = I_F(z)E_F(z)$$

where:

$$I_F(z) = e^{ic}B(z)\exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\sigma(t)\right)$$

and

$$E_F(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log\left|F(e^{it})\right|dt\right)$$

Remark 0.1.1.1. *If $p < \infty$, we have $|E_F(z)|^p \leq P(\left|F(e^{it})\right|^p)$, or $\left|E_F(re^{i\theta})\right|^p \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\left|F(e^{it})\right|^p dt$. Integrate both side by θ we have*

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|E_F(re^{i\theta})\right|^p d\theta &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\left|F(e^{it})\right|^p dtd\theta \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|F(e^{it})\right|^p dt\end{aligned}$$

Thus $F \in H^p$ implies $E_F \in H^p$. The $p = \infty$ case is trivial.

$\left|E_F(e^{it})\right| = \left|F(E^{it})\right|$ a.e.t since $I_F(e^it)$ has finite non zero point.

We say $F \in H^p$ is an inner function if and only if $E_F = 1$, and say $F \in H^p$ is an outer function if and only if I_F is constant.

0.1.2 Outer function

We list some conclusions about outer functions. Most of them are criterions of outer function.

Corollary 0.1.2.1. *If $F \in H^p$, $0 < p \leq \infty$, and is not identically zero, then:*

$$\log|F(0)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log\left|F(e^{it})\right|dt$$

and equality holds if and only if F is an outer functions.

The following theorem is an easy consequence of the above corollary.

Theorem 0.1.2.2 (criterion of outer function. theorem 7.5 in book)**.** *Suppose that $F \in H^p$ and $F^{-1} \in H^p$ for some $0 < p \leq \infty$. Then F is an outer function.*

The next theorem states that the limit of sequence of decreasing outer functions is outer function.

Theorem 0.1.2.3. *Let $F_j \in H^p$ be outer functions for $j = 1, 2, \cdots$. Suppose that $|F_1(z)| \geq |F_2(z)| \geq \cdots$ for every $z \in D$ and $F_j(z) \to F(z)$ uniformly over compact subsets of D, as $j \to \infty$. Then, if F is not identically zero, F is an outer function.*

Remark 0.1.2.1 (notes on proof of theorem 7.6 in book)**.**

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{-}\left|F_j(e^{it})\right|dt \to \frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{-}\left|F(e^{it})\right|dt$$

by monotone convergence. But monotone convergence does not ensure limit is finite. I wonder if it can be infinite.

The following theorem is an easy consequence of the above theorem.

Theorem 0.1.2.4 (theorem 7.7 in book)**.** *Let $F \in H^p$, $0 < p \leq \infty$, not identically zero, and such that $\operatorname{Re} F(z) \geq 0$ for every $z \in D$. Then F is an outer function.*

Remark 0.1.2.2 (notes on proof of theorem 7.8 in book)**.**
Proof of $|K(z)|^p = \exp P(\log(\left|K(e^{it})\right|^p)) \leq P(\left|K(e^{it})\right|^p)$ implies $K \in H^p$ and $p = \infty$ case is similar with remark 0.1.1.1.

0.1.3 Inner function

We can prove the following corollary, although sometimes it is considered as definition of inner functions.

Corollary 0.1.3.1 (corollary 7.2 in book)**.** *The inner functions are precisely those functions $F \in H^\infty$ for which $\left|F(e^{it})\right| = 1$ almost everywhere.*

Remark 0.1.3.1 (notes on proof of corollary 7.2 in book)**.** *Since $|E_F(z)| = \exp P(\log\left|F(e^{it})\right|)$. One direction is*

$$E_F = 1 \implies |E_F| = 1 \implies \log\left|F(e^{it})\right| = 0 \implies \left|F(e^{it})\right| = 1$$

Another direction is

$$\left|F(e^{it})\right| = 1 \implies \log\left|F(e^{it})\right| = 0 \implies E_F = 1$$

Consider space $F \cdot \mathscr{P}$, where $\mathscr{P}$ is the space of polynomials. Theorem 7.9 in book shows if F is an outer function, then $F \cdot \mathscr{P}$ is dense in H^p. Corollary 7.11 in book shows if F is just in H^p, then closure of $F \cdot \mathscr{P}$ is $I_F \cdot H^p$. These conclusions show how the inner factor plays in some approximation problems.

Remark 0.1.3.2 (notes on theorem 7.9 in book)**.** *Even if $\mathscr{P}$ is dense in H^p, $F \cdot \mathscr{P}$ may not be dense in H^p. For example if $F(0) = 0$ in a positive measure set.*

By Hahn-Banach theorem, if E is locally convex and F is subspace of E, then F is dense in E if and only if for any $f \in E^*$, $f|_F = 0 \implies f = 0$. Thus for $p \geq 1$, $E_F \cdot \mathscr{P}$ is dense in H^p is equivalent to for any $k(e^{it}) \in L^{p'}$, $\int_{-\pi}^{\pi} E_F(e^{it})e^{ijt}k(e^{it})dt = 0 \implies \int_{-\pi}^{\pi} G(e^{it})k(e^{it})dt = 0$ for each $G \in H^p$. But $\int_{-\pi}^{\pi} E_F(e^{it})e^{ijt}k(e^{it})dt = 0$ also implies the non positive frequencies of $E_F(e^{it})k(e^{it})$ is 0. This means $E_F(e^{it})k(e^{it}) = \sum_{j=1}^{\infty} a_n e^{ijt}$. Thus $E_F(e^{it})k(e^{it}) = e^{it}H(e^{it})$. By Holder inequality, we see $H \in H^1$.

I wonder in which sense $E_F \cdot \mathscr{P}$ is not dense in H^p and how the proof excludes this case.

By Holder inequality, $\int |R - E_K \cdot Q|^p \leq (\int |R - E_K \cdot Q|^{2p})^{\frac{1}{2}}(\int 1)^{\frac{1}{2}}$. Thus $\|R - E_K \cdot Q\|_{H^{2p}} < \epsilon$ implies $\|R - E_K \cdot Q\|_{H^p} < \epsilon$