Contents

Chapter 1

Classical Theory of Hardy Space

1.1 Harmonic Functions, Poisson Representation

In plane case, we can rewrite harmonic condition: $\Delta F = 0$ as $\Delta F = (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})F = 0$. Notices the equation $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 0$ is equivalent to the Cauchy-Riemann equations for function $F = u + iv$.

> **Remark 1.1.0.1** (holomorphic indicates harmonic)**.** If function F satisfies C-R equations, then F satisfies Laplace equation for $\mathbb{C}$. And if F satisfies $\Delta = 0$, so does $\bar{F}$. Thus F and $\bar{F}$ are all harmonic functions. Besides, the real and imaginary part of F, u and v also satisfy $\Delta = 0$, means u and v are harmonic functions.

> **Remark 1.1.0.2** (harmonic indicates holomorphic)**.** Assume that u satisfies $\Delta = 0$. If we let $v_x = -u_y$ and $v_y = u_x$, then we have $v_{xy} = v_{yx}$, which indicates exists of v (equivalent of Fubini's Theorem and the equality of the mixed partial derivatives). And $v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$ indicates v is also harmonic. v is determined up to an additive constant, and $F = u + iv$ is holomorphic.

> **Remark 1.1.0.3.** Later we will see holomorphic function is the special case of harmonic function.

If u is a real harmonic function, by above remark, we know $u = \operatorname{Re} F$ for some holomorphic function $F(z) = \sum_{k=0}^{\infty} c_k z^k$. Then we can derive the series representation of u:

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta} \tag{1.1.1}$$

and it converges uniformly on compact subsets of $D(0, R)$.

> **Remark 1.1.0.4** (uniformly converge on compact subsets)**.** Using mean value property, sequence of holomorphic functions which uniformly converge on compact subsets, the limit is a holomorphic function.
>
> The partial sum of $u_n(re^{i\theta}) = \sum_{k=-n}^{n} a_k r^{|k|} e^{ik\theta}$ is obviously harmonic, and it converges uniformly on compact subsets of $D(0, R)$. Thus $u(re^{i\theta})$ is harmonic.

1.1.1 Harmonic function to Poisson integral (or Poisson representation of harmonic function)

Assumed u is harmonic in $D(0, R)$.If $R > 1$, we can represent a_k in equation (1.1.1) by $u(e^{it})$ using Fourier analysis. Finally we derive the Poisson kernel and the Poisson representation:

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) u(e^{it}) dt \tag{1.1.2}$$

If u is only harmonic on $D(0, 1)$, the equation (1.1.2) still can be valid in some sense once we add some restriction on u.

Theorem 1.1.1.1 (theorem 1.3 in book)**.** *Let u be a harmonic function in D such that*

$$\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} \left| u(re^{it}) \right|^p dt < \infty \tag{1.1.3}$$

for some $p > 1$. Then there is a function $f \in L^p$ such that

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(t) dt \tag{1.1.4}$$

This theorem still holds for $p = \infty$ if we replace inequality (1.1.3) by $\sup_{0 \leq r < 1} \left| u(re^{it}) \right| < \infty$.

In later section, we will see left side of inequality (1.1.3) is H^p norm. The proof of Theorem 1.1.1.1 shows that how dual space, w*-topology and Banach-Alaoglu theorem perform in integral representation. The tricky part is using representation $u(r_n re^{i\theta})$. In the proof we also need $P_r(\theta) \in L^{p'}$ which is easy since $\|P_r(\theta)\|_\infty = \frac{1+r}{1-r}$ and $L^\infty([-\pi, \pi]) \subset L^{p'}([-\pi, \pi])$.

> **Remark 1.1.1.1** (some details of proof of theorem 1.1.1.1)**.** In metrizable space, compact and sequentially compact are equivalent.
>
> Let $f_n(t) = u(r_n e^{it})$, $(f_n) \subset L^{p'*}$ is in a close ball w.r.t p-norm and Banach-Alaoglu theorem indicates this ball is w* compact. Since w*-compact is equivalent to w*-sequentially compact if this close ball is metrizable in w*-topology, we need to prove this close ball is metrizable in w*-topology.
>
> Here is a direct proof. Since $L^{p'}$ is separable there is a countable dense

set $(g_n) \in L^{p'}$. For every $g \in L^{p'}$, let $\hat{g}(f) = f(g)$ where $f \in L^{p'*}$. Any pair of points $f_1, f_2 \in L^{p'*}$ can be separate by some $g_i \in L^{p'}$ (since every $\hat{g_n}$ is w*-continuous, density of (g_n) ensures f_1, f_2 can be separate), Thus $(\hat{g_n})$ is a countable family of continuous functions that separates points in $L^{p'*}$. Then we can use metric: $d(f_1, f_2) = \sum_{n=0}^{\infty} 2^{-n} |f_1(g_n) - f_2(g_n)|$.

We have proved that $L^{p'*}$ is metrizable in w*-topology. Here is a topological thought. By Nagata-Smirnov metrization theorem, $L^{p'*}$ is regular. we don't know how to shown this property directly. Also, $L^{p'*}$ a basis countably locally finite (by Nagata-Smirnov metrization theorem), we also want to show this property directly. Besides, consider Urysohn metrization theorem, if we have shown $L^{p'*}$ is regular, $L^{p'*}$ is metrizable once we show there is a countable basis for $L^{p'*}$. We don't know if it is possible.

Now (f_n) is w* sequentially compact in $L^{p'*}$. This means for any $g \in L^{p'}$, there is a subsequence (f_{n_k}) and $f \in L^{p'*}$ such that $\int g f_{n_k} \to \int g f$.

Remark 1.1.1.2. $L^{p'*} \cong L^p$. Thus $L^{p'*}$ is a normed space and $L^{p'*}$ is metrizable.

The $p = 1$ case is failed since we can not use w*-topology. L^1 is not a separate dual space of some space (Assume $L^1 = X^*$. L^1 is separable in w*-topology implies X is separable). One method showing L^1 is not a separate dual space of some space by showing L^1 does not have the Radon-Nikodym property (Radon-Nikodym theorem is valid).

Remark 1.1.1.3. If X is a Banach space, then X^* has the Radon-Nikodym property (RNP) if (and only if) every separable, linear subspace of X has a separable dual (Charles Stegall: The Radon-Nikodym property in Conjugate Banach Space. II).

Thus for $p = 1$ case, we can not use same argument as Theorem 1.1.1.1. However there is a relative result. L^1 can be isometrically imbedded in to M, the space of Borel measures with bounded variation, which is dual of continuous function with compact support space C. Thus we can use same argument to M and C and introduce Poisson-Stieltjes integral.

When we use Borel measures case in Theorem 1.1.1.1, we get *Poisson integral of positive measure for harmonic functions.*

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{\pi}^{\pi} P_r(\theta - t) d\mu(t) \tag{1.1.5}$$

Here $d\mu(t)$ is the w*-limit of $u(r_n e^{it})dt$. The difference between (1.1.2) and (1.1.5) is that in (1.1.2), u needs to be harmonic in a little larger disk $D(0, R)$, $R > 1$, but in (1.1.5) u needs not. However, this is not hold for all harmonic functions since we need constraints.

1.1.2 Poisson integral indicates harmonic

Using Fourier series, we show that given a function $f \in L^p$, $1 \leq p \leq \infty$ (or a complex Borel measure μ), Poisson integral u=$P(f)$ (or u=$P(\mu)$) is real part of a holomorphic function. Thus it is harmonic (Theorem 1.11 (or 1.14) in book). And we give bound of norm of u by norm of f (or measure μ):

- $\int_{-\pi}^{\pi} \left|u(re^{it})\right|^p dt \leq \int_{-\pi}^{\pi} |f(t)|^p dt \; for \; p < \infty$
- $|u(z)| \leq \|f(t)\|_\infty \; for \; p = \infty$
- $\int_{-\pi}^{\pi} \left|u(re^{it})\right| dt \leq \int_{-\pi}^{\pi} d\,|\mu|\,(t)$

For $f \in L^p$, $p \leq \infty$ case, the equality holds when $r \to 1$ or take sup on right hand side. We prove this in section 3.

1.1.3 Boundary behavior of Poisson integral (or norm convergence and pointwise convergence)

Definition 1.1.3.1 (Dirichlet problem)**.** *Given a continuous function f on ∂D, we want to find a continuous function on $\bar{D}$, which is harmonic in D and coincides with f on ∂D.*

From section 1.1.2, we can see $u(re^{it}) = P(f)$ is harmonic function. Roughly speaking, if $u(e^{it}) = f$ (Actually the domain of u is D, so $u(e^{it})$ is not defined), the classical Dirichlet problem is solved. Thus we need to study the boundary behavior of Poisson integral. *The key idea is approximate identity.* Using this idea, we prove the following theorem.

Theorem 1.1.3.2 (Theorem 1.16 in book)**.** *Let ϕ_α be an approximate identity on the torus T. Then:*

1. *If $f \in L^p([-\pi, \pi])$ with $1 \leq p < \infty$ and f_α stands for convolution*

$$f_\alpha(\theta) = (f * \phi_\alpha)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta - t)\phi_\alpha(t)dt$$

   it follows that $f_\alpha \to f$ in L^p, i.e.:

$$\int_{-\pi}^{\pi} |f_\alpha(t) - f(t)|^p \, dt \to 0$$

2. *If f is a continuous 2π-periodic function, we have $f_\alpha \to f$ uniformly on T.*

Theorem 1.1.3.2 shows that for $f \in L^p$, $1 \leq p < \infty$ or $f \in C$, Poisson integral converges to boundary in norm. Following the proof of theorem 1.1.3.2, we can show another two case: $p = \infty$ and $f \in M$. The convergence becomes w* convergence (Corollary 1.19 in book).

Here comes another topic: What about pointwise convergence for $P(\mu)$ on boundary if $P(\mu)$ is Poisson-Stieltjes integral? We need a new concept: non-tangentially converge. We show that $P(\mu)(z) \to F'(\theta_1)$ as $z \to e^{i\theta_1}$ N.T., where $F(\theta) = \int_0^\theta d\mu(t)$ and $F'(\theta_1)$ exists and finite (Theorem 1.20 in book).

You can omit the following remark if you choose not to check the proof in book.

> **Remark 1.1.3.1** (Proof of Theorem 1.20)**.** First we take $c > 0$, since we need proof for any $c > 0$. This c decides the approach region.
> Then given $\epsilon > 0$, we take δ small enough s.t. $|F(t)| < \epsilon |t|$ when $|t| < \delta$. This δ can be taken since $F(0) = 0$ and $F'(0) = 0$.
> If we take r large enough and $re^{i\theta}$ in the region. then $|\theta|$ can be less than $\frac{\delta}{4}$.
> The estimation of $u(re^{i\theta})$ is (we use $F(t) = \int_{-\pi}^{\pi} d\mu(t)$ in the third line):
>
> $$\begin{aligned}\left|u(re^{i\theta})\right| &= \left|\frac{1}{2\pi}\int_{\delta<|t|\le\pi} P_r(\theta-t)d\mu(t) + \frac{1}{2\pi}\int_{-\delta}^{\delta} P_r(\theta-t)d\mu(t)\right| \\ &\le \left|\frac{1}{2\pi}\int_{\delta<|t|\le\pi} P_r(\theta-t)d\mu(t)\right| + \left|\frac{1}{2\pi}\int_{-\delta}^{\delta} P_r(\theta-t)d\mu(t)\right| \\ &\le (\sup_{|t|>\delta} P_r(\theta-t)) \cdot \frac{1}{2\pi}\int_{\delta<|t|\le\pi} d\,|\mu|\,(t) \\ &\quad + \left|(P_r(\theta-t)\cdot\frac{1}{2\pi}F(t))\Big|_{-\delta}^{\delta} + \int_{-\delta}^{\delta} P_r'(\theta-t)\frac{1}{2\pi}F(t)dt\right| \\ &\le (\sup_{\frac{3\delta}{4}<|t|<\pi+\frac{\delta}{4}} P_r(t)) \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} d\,|\mu|\,(t) \\ &\quad + \left|(P_r(\theta-t)\cdot\frac{1}{2\pi}F(t))\Big|_{-\delta}^{\delta}\right| + \left|\int_{-\delta}^{\delta} (P_r'(\theta-t)\frac{1}{2\pi}F(t))dt\right| \\ &\le (\sup_{|t|>\frac{3\delta}{4}} P_r(t)) \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} d\,|\mu|\,(t) \\ &\quad + (\sup_{|t|>\frac{3\delta}{4}} P_r(t)) \cdot \frac{1}{2\pi}\int_{-\delta}^{\delta} d\,|\mu|\,(t) + \left|\frac{1}{2\pi}\int_{-\delta}^{\delta} P_r'(\theta-t)F(t)dt\right|\end{aligned}$$

Lebesgue-Stieltjes integral is related to bounded increasing function. The integral has decomposition $d\mu(t) = f(t)dt + d\sigma(t)$, $f \in L^1$. $\int_0^\theta d\sigma(t)$ is a jump function. Thus $F'(\theta) = f(\theta)$ a.e.. Considering $P(f)$, $f \in L^p$, $1 \le p \le \infty$, or even $P(\mu)$, the integral $P(f)$ or $P(\mu)$ is L-S integral. Thus N.T. convergence holds a.e..

By section 1.1.1, Harmonic function with constraints (bounded in some sense) can be written as Poisson integral $P(f)$ or $P(\mu)$ (Corollary 1.10 in book). Thus the theorem of Fatou holds: *Any function holomorphic and bounded in D*

has non-tangential boundary values a.e..

The difference between $p = 1$ and $p > 1$ is the starting point of the theory of Hardy spaces.

1.1.4 Harmonic function in higher dimension

A continuous function is harmonic in region Ω if and only if it satisfies mean value property. For mean value property to harmonic, we need an approximate identity.

You can omit the following remark if you choose not to check the proof in book.

> **Remark 1.1.4.1** (Proof of Theorem 1.22)**.** In converse part, we choose region Ω_ϵ is to make integral $\int_{\mathbb{R}^n} u(x_0 + r\sigma - y)\phi_\epsilon(y)dy$ and $\int_{\mathbb{R}^n} u(x_0 - y)\phi_\epsilon(y)dy$ defined. In other words, if $x_0 + r\sigma - y$ or $x_0 - y$ out of Ω, then $\phi_\epsilon(y) = 0$.

A consequence of mean value property is maximum principle. The proof can be given by topological technic: Assumed u attains maximum value m in Ω. Let $A = \{x : u(x) = m\}$. Using mean value property, we show every x is interior point of A. Thus A is open. However, $\Omega \setminus A = \{x : u(x) < m\}$ is open since u is continuous. Ω is connected and A is not empty. $\Omega \setminus A$ is empty. Thus u is constant.

By maximum principle and minimum principle, we can show u is unique in Ω if u is harmonic in Ω and u is continuous on $\partial\Omega$.

We introduce the Poisson kernel $P(x, s) = \frac{1-|x|^2}{|x-s|^n}$ for Dirichlet problem in n dimension. The solution is $\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x, s)f(s)ds$.

There is a weaken form of mean value property (or called discrete mean value property). u is harmonic if mean value property is satisfied only on a sequence of $r_n \to 0$ (Theorem 1.30 in book).

You can omit the following remark if you choose not to check the proof in book.

> **Remark 1.1.4.2** (Proof of theorem 1.30)**.** The proof here is not well understand. The set $K = \{u(x) - v(x) = m : x \in \overline{B(x_0, R)}\}$ is compact in $\overline{B(x_0, R)}$ since K is closed in compact set $\overline{B(x_0, R)}$. We can use finite open cover to prove K is also compact in $B(x_0, R)$. Let $f : K \to \mathbb{R}$, $f(x) = d(x, x_0)$, f is continuous function on compact set K thus it attains maximum value of f. Let x_1 be a point of the maximum value. I just can image that for sphere $\partial B(x_1, r)$, only half of the sphere is in K otherwise $f(x_1)$ is not the maximum value. Here r need to be small enough to ensure $\partial B(x_1, r) \subset B(x_0, R)$. But then $u(x_1) - v(x_1) < m$ if we use integral with $r_j < r$. Thus it is a contradiction.
>
> I don't know how the sequence (r_n) plays in this proof.

Then we go to the reflection principle and Liouville theorem.

Finally we go to Dirichlet problem on unbounded domain, in particular $\mathbb{R}_+^{n+1}$. This problem cannot have a unique solution but can have a unique bounded solution. And we give Poisson kernel for $\mathbb{R}_+^{n+1}$ by Fourier transform method.

1.2 Subharmonic Functions

This section is about a new concept: subharmonic function. Subharmonic function can be considered as a generalization of harmonic function, as it preserves some important property of harmonic function such as maximum principle. On the other hand, we will see why we call it "sub" harmonic: subharmonic function can be controlled by harmonic function. Also, by some operations like composition and taking absolute value, subharmonic function can still be subharmonic, but harmonic function can not. Finally we will begin our study of zeroes of holomorphic function.

Now we give the definition of subharmonic function.

Definition 1.2.0.1. *A subharmonic function on an open set $\Omega \subset \mathbb{R}^n$ is a function v defined on Ω, with values $-\infty \leq v(x) < \infty$ and satisfying the following two conditions:*

1. *v is upper semicontinuous in Ω.*
2. *For every $x_0 \in \Omega$, there is a ball $B(x_0, r(x_0)) \subset \Omega$, $r(x_0) > 0$, such that for every r with $0 < r < r(x_0)$*

$$v(x_0) \leq \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} v(x_0 + r\sigma) d\sigma \tag{1.2.1}$$

1.2.1 Upper semicontinuous

There is two equivalence definition of v being upper semicontinuous in Ω:

1. For every $t \in \mathbb{R}$, the set $\{x \in \Omega : v(x) < t\}$ is open.
2. For every $x_0 \in \Omega$:

$$\limsup_{x \to x_0 \ in \ \Omega} v(x) \leq v(x_0) \tag{1.2.2}$$

This is equivalent to that for every $y > f(x_0)$, there exists a neighborhood U of x_0 such that $f(x) < y$ for all $x \in U$.

Remark 1.2.1.1 (Proof of equivalence)**.**
We prove by contradiction, Suppose that $\limsup v(x) > v(x_0)$, we can find x_k, $v(x_0) < v(x_k) < \limsup v(x)$. Since $v^{-1}([-\infty, v(x_k))$ is open and $v(x_0) < v(x_k)$, $v(x_0) \in v^{-1}([-\infty, v(x_k))$. Thus there is a neighborhood $U \in \mathscr{N}(x_0)$, $U \subset v^{-1}([-\infty, v(x_k))$. Now we can find another $x_n \in U$ s.t. $v(x_k) < v(x_n) < \limsup v(x)$. $v(x_n) > v(x_k)$ means $v(x_n) \notin v^{-1}([-\infty, v(x_k))$, contradicts to $v(x_n) \in U \subset v^{-1}([-\infty, v(x_k))$.

We prove the converse by contradiction. Suppose there is a number

$t_0 \in \mathbb{R}$, $v^{-1}([-\infty, t_0))$ is not open. So there is $x_0 \in v^{-1}([-\infty, t_0))$, such that $\forall U_k \in \mathscr{N}(x_0)$, there is $x_k \in U_k$, $x_k \notin v^{-1}([-\infty, t_0))$, which is equivalent to $v(x_k) \geq t$. This contradicts to $\limsup v(x_k) \leq v(x_0) < t_0$.

If v is subharmonic, inequality (1.2.1) implies another direction of inequality (1.2.2). Thus we actually have equality in (1.2.2).

An important and frequently used tool is following characterization of upper simicontinuity.

Proposition 1.2.1.1. *v is upper semicontinuous in Ω if and only if for every compact $K \subset \Omega$, v is the limit over K of a decreasing sequence of continuous function.*

This proposition is important tool in proof of some following theorems.

Remark 1.2.1.2 (Notes on proof of proposition 1.2.1.1)**.** The converse part, by using partition of the unity, we construct a sequence of decreasing function (u_k). We need to prove v is the limit of (u_k).

For any $x_0 \in K$, there is a sequence of balls $(B(x_{n,i}, \epsilon_n))$, $\epsilon_n \to 0$ s.t. $x_0 \in B(x_{n,i}, \epsilon_n)$ for all n. For each n, $B(x_{n,i}, \epsilon_n)$ is in finite open cover $(B(x_{n,i}, \epsilon_n))_i$ of K. Since $m_{n,i} = \sup_{B(x_{n,i},\epsilon_n)} v$, $u_n(x_0) \geq v(x_0)$ for all n. By definition of upper semicontinuous, for any $y > v(x_0)$, there is a neighborhood $U \in \mathscr{N}(x_0)$, $v(x) < y$ for all $x \in U$. Let $B(x_{n,i}, \epsilon_n) \subset U$, $m_{n,i} < y$. Thus $u_n(x_0) < y$ for all large enough n. Since y is any number larger than $v(x_0)$, $\limsup u(x) \leq v(x_0)$. This shows $u(x_0) = v(x_0)$.

1.2.2 Property of subharmonic function

First, subharmonic function satisfies maximum principle.

Remark 1.2.2.1 (Notes on proof of maximum principle)**.** Like proof of maximum principle for harmonic function, but we need to take care of semi-continuous. Assume $v(x_0) = M$, the maximum value. Choose r to satisfy inequality (1.2.1). If for some $x \in \partial B(x_0, r)$, $v(x) = m < M$, by semicontinuous, $\limsup v(x_k) \leq v(x) < m + \epsilon < M$. Thus there is a neighborhood $U \in \mathscr{N}(x)$, $\sup_{x_k \in U} v(x_k) < M$. Then $\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0 + r\sigma)d\sigma < M = v(x_0)$. This contradicts to inequality (1.2.1). Then the following is same as proof for maximum principle for harmonic function.

The best reason why we use name 'subharmonic' is following: v is subharmonic function if and only if when v less or equal to a harmonic function u on boundary of region, $v \leq u$ in entire region. We remind the reader that proposition 1.2.1.1 appears as an important step in proof.

There are two examples of using proposition 1.2.1.1 to detail with subharmonic function v. One is if v is not identically equal to $-\infty$, then

$$\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0 + r\sigma)d\sigma > -\infty$$

for every $\overline{B(x_0,r)} \subset \Omega$. In proof of this statement we also use Poisson representation of harmonic function and little topological trick. Another example is

$$m(r) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(r\sigma)d\sigma \tag{1.2.3}$$

is an increasing function.

There is another necessary and sufficient condition for v to be harmonic using Laplace operator. It says v is subharmonic if and only if $\Delta v \geq 0$.

> **Remark 1.2.2.2** (Proof of proposition 2.10 in book)**.** Author says we need to show that $v(x_0) \leq \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0 + r\sigma)d\sigma$. But I think this is obvious since we consider x_0 which $v(x_0) = 0$ and $v(x) \geq 0$ on Ω. And this inequality is not used in the following part of proof.

1.2.3 Estimation for zeroes of holomorphic function

We first state that if v is subharmonic, ϕ is increasing and convex function. Then $\phi \circ v$ is also subharmonic. This is useful when we need to connect holomorphic function with subharmonic function.

Now here comes our first theorem about zero points of holomorphic: Jensen's formula.

Theorem 1.2.3.1 (Jensen's formula)**.** *Let F be holomorphic in $D(0,R)$ and suppose that $F(0) \neq 0$. Let $0 < r < R$ and call $z_1, z_2, \cdots, z_n$ the zeroes of F in $\overline{D(0,r)}$ listed according to their multiplicities. Then:*

$$\log|F(0)| + \sum_{j=1}^{n}\log\frac{r}{|z_j|} = \frac{1}{\pi}\int_{-\pi}^{\pi}\log\left|F(re^{it})\right|dt. \tag{1.2.4}$$

The proof in book need a lemma: $\int_{-\pi}^{\pi}\log\left|1 - e^{it}\right|dt = 0$. You can also refer section 1 in Chapter 6 of Stein's *Complex Analysis*. The proof there is very different to the one in this book.

> **Remark 1.2.3.1** (Proof of lemma (Lemma 2.12 in book))**.** There is an inequality: for $|t| < \frac{\pi}{3}$, $\log\frac{1}{|\sin t|} \leq \frac{C_\alpha}{|t|^\alpha}$. I can prove it using elementary calculus, but I think it is an easy observation.

To continue our explorer of zeroes of holomorphic function, we show some connection between holomorphic function and subharmonic function. More precisely, If F is holomorphic, not identically 0, then $\log|F(z)|$, $\log^+|F(z)| =$

$\max(\log|F(z)|, 0)$ and $|F(z)|^a$ for any $0 < a < \infty$, are all subharmonic. Then we give definition of Hardy space on D. We define for $f \in H(D)$ (F is holomorphic in D) :

- $m_0(F, r) = \exp(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ \left|F(re^{it})\right| dt)$
- $m_p(F, r) = (\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F(re^{it})\right|^p dt)^{\frac{1}{p}}$
- $m_\infty(F, r) = \sup_t \left|F(re^{it})\right|$

This function is an increasing function of r in $[0, 1)$ (Hardy convex theorem), see equation (1.2.3) for case $0 \leq p < \infty$. $m_\infty(F, r)$ is also an increasing function but it uses a different method (Hadamard three-circle theorem).

Now we define Hardy space H^p:

Definition 1.2.3.2. *For $0 < p \leq \infty$, we define $H^p(D)$:*

$$H^p(D) = \{F \in H(D) : \|F\|_{H^p} = \sup_{0 \leq r < 1} m_p(F, r) < \infty\}$$

For $p = 0$, we have the Nevanlinna class N, defined by:

$$N = \{F \in H(D) : \sup_{0 \leq r < 1} m_0(F, r) < \infty\}$$

If $0 < p < q < \infty$, we have $H^\infty \subset H^q \subset H^p \subset N$

> **Remark 1.2.3.2.** The first two inclusions are as the same as the inclusion for L^p, the last inclusion is by:
>
> $$(m_0(r))^p = \exp(p\int_{-\pi}^{\pi} \log^+ |F| \frac{dt}{2\pi}) \leq \int_{-\pi}^{\pi} \exp(p \log^+ |F|) \frac{dt}{2\pi}$$
>
> Notice that:
>
> $$\int_{-\pi}^{\pi} \exp(p \log^+ |F|) \frac{dt}{2\pi} = \int_{\substack{t \in [-\pi, \pi] \\ |F| > 1}} |F|^p \frac{dt}{2\pi} + \int_{\substack{t \in [-\pi, \pi] \\ |F| \leq 1}} 1 \frac{dt}{2\pi}$$
>
> Thus:
>
> $$(m_0(r))^p \leq \int |F|^p \frac{dt}{2\pi} + 1$$

There left three theorems in this section. I interpret it shortly and informally. First one is for $F \in N$, the zeroes (z_j) of F cannot be too far from the boundary, or $\sum_j (1 - |z_j|) < \infty$. The second one is that if $\sum_j (1 - |z_j|) < \infty$ holds, the "Blaschke product"

$$B(z) = z^k \prod_{j=1}^{\infty} \frac{z_j - z}{1 - z\bar{z}_j} \frac{|z_j|}{z_j}$$

converges uniformly on each compact subset to a function H^∞ and they have the same zeroes.

Remark 1.2.3.3. If f is holomorphic in an open disc that vanishes on a sequence of distinct points with a limit point in the disc. Then f is identically 0 (Theorem 4.8 in chapter 2. Stein's Complex Analysis). However in Blaschke product case, there can be infinitely zeroes, since it can have limit points on boundary. Thus $B(z)$ can be not identically 0. However, for any $r < 1$, $B(z)$ can only have finite zeroes in $\overline{D(0,r)}$.

Here is a convention. If for some function F in D, the non-tangential boundary value of F is known to exist at e^{it}, we shall denote it by $F(e^{it})$.

The third theorem is the Blaschke product has properties: $\left|B(e^{it})\right| = 1$ for a.e. t and

$$\lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left|B(re^{it})\right| dt = 0.$$

If $F \in H^p$ with $p \geq 1$, which means F is holomorphic F and can be write as Poisson (or Poisson-Stieltjes) integral. By Fatou Theorem, we know that $F(e^{it})$ exists a.e.. In the next section we shall extend this result to any $p > 0$. The above three theorems play important roles in proving extension.

Remark 1.2.3.4 (Notes on proof of three theorems (theorem 2.19, 2.21 and 2.22))**.** In proof of theorem 2.19, we assume $F(0) = 0$, since we can use function $\frac{F(z)}{z^k}$ if $F(z)$ has zero of order k in $z = 0$. And this modification does not affect the sum $\sum_j (1 - |z_j|)$.

The step

$$\sum_1^n \log \frac{1}{|z_j|} \leq M - n \log r - \log |F(0)|$$

to

$$\sum_1^\infty \log \frac{1}{|z_j|} \leq M - \log |F(0)|$$

is not clear. I think we can not first let $r \to 1$ then $n \to \infty$. We can not control taking limit for which one first.

In proof of theorem 2.21, the final step is:

$$\begin{aligned}\left|1 - \frac{z_j - z}{1 - z\bar{z}_j} \frac{|z_j|}{z_j}\right| &= \left|1 - \frac{z_j\,|z_j| - z\,|z_j|}{z_j - z\,|z_j|^2}\right| = \left|\frac{z_j - z\,|z_j|^2 - z_j\,|z_j| + z\,|z_j|}{z_j - z\,|z_j|^2}\right| \\ &= (1 - |z_j|) \left|\frac{z_j + z\,|z_j|}{z_j - z\,|z_j|^2}\right| = (1 - |z_j|)\,|z_j| \left|\frac{e^{it} + z}{e^{it} - z\,|z_j|}\right| \\ &= (1 - |z_j|)\,|z_j| \left|\frac{z' + 1}{1 - z'\,|z_j|}\right|\end{aligned}$$

where $z' = z \cdot e^{-it}$. Since $|z_j| < 1$, $|z'| = |z| \leq r$, we have $|z' + 1| \leq$

$|z'| + 1 \le r + 1$, $|1 - z'|z_j|| \ge 1 - |z'|z_j|| = 1 - |z'||z_j| \ge 1 - r$. Thus

$$\left|1 - \frac{z_j - z}{1 - z\bar{z}_j}\frac{|z_j|}{z_j}\right| \le (1 - |z_j|)\frac{1 + r}{1 - r}$$

In proof of theorem 2.22, we know if $z\bar{w} \neq 1$, then Blaschke factors:

$$\left|\frac{w - z}{1 - \bar{w}z}\right| = 1 \ if \ \ |z| = 1 \ or \ \ |w| = 1$$

Since in $B_n(z)$ $\left|e^{it}\right| = 1$, I think $B_n(e^{it}) = 1$ everywhere, not a.e..

$\left|B_n(re^{it})\right| \to 1$ uniformly as $r \to 1$, since B_n is holomorphic in a neighborhood of $\bar{D}$. This is easy if we choose $D(0, 1 + \epsilon)$, s.t. $z\bar{z}_j \neq 1$ in $D(0, 1 + \epsilon)$.

1.3 F.Riesz Factorization Theorem

This section can be seen as a generalization of first section. In first section, we talk about norm convergence and pointwise convergence when boundary function f is in L^p, $1 < p \le \infty$ and f is a measure. This conclusion is for harmonic function. Harmonic function has series representation:

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}$$

and we can derive Poisson representation $u(re^{i\theta}) = P_r(f)$. Since holomorphic function also has series representation:

$$u(re^{i\theta}) = \sum_{k=0}^{\infty} a_k r^k e^{ik\theta}$$

, we can consider this representation as special case of harmonic function with $a_k = 0$ for $k < 0$. Poisson representation is also hold for holomorphic function, thus the converge result is hold also for holomorphic function. The following theorem is a summary of these results.

Theorem 1.3.0.1 (theorem 3.1 in book)**.** *Let $F \in H^p$ with $1 < p \le \infty$. Then:*

1. *For almost every t. the limit*

$$F(e^{it}) = \lim F(z) \ as \ z \to e^{it} \ N.T.$$

exists. The function $f(t) = F(e^{it})$ belongs to $L^p([-\pi, \pi])$ and $F = P(f)$

2. *If $p < \infty$:*

$$\int_{-\pi}^{\pi} \left|F(re^{it}) - F(e^{it})\right|^p dt \to 0 \ as \ r \to 1$$

If $p = \infty$, $F(re^{it}) \to F(e^{it})$ in the w^*-topology of L^∞ as $r \to 1$.

For each $1 < p \le \infty$: $\|F\|_{H^p} = \|f\|_p$.

3. F is the Cauchy integral of its boundary function, that is:

$$F(z) = \frac{1}{2\pi i}\int_{|\xi|=1}\frac{F(\xi)}{\xi - z}d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{F(e^{it})}{e^{it}-z}e^{it}dt$$

Remark 1.3.0.1. For first statement in theorem 1.3.0.1, N.T. limit holds for $p = 1$, but $P(f)$ may not be hold.

Remark 1.3.0.2. $u(re^{it}) = P_r(t)$ is neither in H^p nor N. $P_r(t)$ is harmonic but not holomorphic.

In this section we will extend this result to $p \le 1$. The main idea is to factorize $F(z)$ to a Blaschke product $B(z)$ and a non-vanish function $H(z)$.

1.3.1 Result of non-vanish case

Suppose that $F \in H^p$, $\frac{1}{2} \le p < 1$. If F does not vanish in D. Then $F(z) = e^{f(z)}$ for some holomorphic function f. Let $G(z) = e^{\frac{f(z)}{2}}$, we have $F(z) = G(z)^2$, $G(z) \in H^{2p}$ and $\|G\|^2_{H^{2p}} = \|F\|_{H^p}$. Since $2p \ge 1$, we have $G(e^{it}) = \lim G(z)$ a.e. as $z \to e^{it}$ N.T.. It follows that $F(e^{it}) = \lim F(z)$ a.e. as $z \to e^{it}$ N.T.

We know that $\int_{-\pi}^{\pi}\left|F(re^{it}) - F(e^{it})\right|^p dt \to 0$ as $r \to 1$ if $p > 1$. Suppose that $F \in H^p$, $\frac{1}{2} \le p < 1$ and we have $F(z) = G(z)^2$ as before, then:

$$\begin{aligned}
&\int_{-\pi}^{\pi}\left|F(re^{it}) - F(e^{it})\right|^p dt\\
=&\int_{-\pi}^{\pi}\left|G(re^{it})^2 - G(e^{it})^2\right|^p dt\\
=&\int_{-\pi}^{\pi}\left|G(re^{it}) + G(e^{it})\right|^p\left|G(re^{it}) - G(e^{it})\right|^p dt\\
\le&(\int_{-\pi}^{\pi}\left|G(re^{it}) + G(e^{it})\right|^{2p} dt)^{\frac{1}{2}}(\int_{-\pi}^{\pi}\left|G(re^{it}) - G(e^{it})\right|^{2p} dt)^{\frac{1}{2}}\\
\le&(\int_{-\pi}^{\pi}(2\left|G(e^{it})\right|)^{2p}dt)^{\frac{1}{2}}(\int_{-\pi}^{\pi}\left|G(re^{it}) - G(e^{it})\right|^{2p} dt)^{\frac{1}{2}}\\
\le&2^p\|G\|^p_{H^{2p}}(\int_{-\pi}^{\pi}\left|G(re^{it}) - G(e^{it})\right|^{2p} dt)^{\frac{1}{2}} \to 0 \; as \; r \to 1
\end{aligned}$$

We conclude that $F \in H^p$, $\frac{1}{2} \le p < 1$. If F does not vanish in D. Then there is a boundary function $F(e^{it})$, $F(z)$ converges to $F(e^{it})$ both in pointwise sense and norm sense.

> **Remark 1.3.1.1.** There is a basic inequality, used also in proving Minkowski inequality: $|a+b|^p \leq 2^p(|a|^p + |b|^p)$ for $p > 0$. To prove this we only need to consider two case: $|a| \geq |b|$ or $|a| \leq |b|$.

By induction, this conclusion can be extended to $0 < p < 1$. Thus two types of convergence holds for all $0 < p < \infty$.

> **Remark 1.3.1.2.** Author uses Fatou's lemma when $F(z)$ converges to $F(e^{it})$ N.T.. I think we can use this lemma even if it converges radially.

1.3.2 Result of H^p case

In the end of last section review, we state three theorems:

- For $F \in N$, the zeroes (z_j) of F satisfies $\sum_j (1 - |z_j|) < \infty$.
- If $\sum_j (1 - |z_j|) < \infty$ holds, the Blaschke product converges uniformly on each compact subset to a function $B(z) \in H^\infty$ and they have zeroes (z_j).
- $\left|B(e^{it})\right| = 1$ a.e.

If we let $H = \frac{F}{B}$, where Blaschke product is formed by zeroes of F, then H does not have any zeroes. Besides, if $F \in N$, then $H \in N$ and $\|H\|_N = \|F\|_N$. If $F \in H^p$, then $H \in H^p$ and $\|H\|_{H^p} = \|F\|_{H^p}$ (theorem 3.3 in book). Notice now we can use method in section 1.3.1 on H. We have following result:

Theorem 1.3.2.1 (theorem 3.6 in book)**.** *Let $F \in H^p$ with $0 < p \leq \infty$. Then:*

1. *For almost every t. the limit*

$$F(e^{it}) = \lim F(z) \text{ as } z \to e^{it} \text{ N.T.}$$

   exists. The function $f(t) = F(e^{it})$ belongs to $L^p([-\pi, \pi])$.

2. $\int_{-\pi}^{\pi} \left|F(re^{it}) - F(e^{it})\right|^p dt \to 0$ *as* $r \to 1$
3. $\|F\|_{H^p} = \lim_{r\to 1} (\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|F(re^{it})\right|^p dt)^{\frac{1}{p}} = (\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|F(e^{it})\right|^p dt)^{\frac{1}{p}}$

Another statement is that $F \in H^p$ can be improved to $F \in H^q$ if the boundary function $F(e^{it}) \in L^q$ (Corollary 3.7). The hard part of its proof is the case $p < q$ and $p \leq 1$. We factorize F as $F = BG^n$, where $np > 1$. Since $F(e^{it}) \in L^q$ and $\left|G(e^{it})\right|^n = \left|F(e^{it})\right|$, $G(e^{it}) \in L^{nq}$. Thus $G \in H^{nq}$ and $F \in H^q$.

1.3.3 H^1 function and its boundary

Recall in section 1, when u is a harmonic function in D and

$$\sup_{0\le r<1}\int_{-\pi}^{\pi}\left|F(re^{it})\right|dt<\infty$$

, we can only say u is $P(\mu)$ for some Borel measure and the result can not be improved (consider Poisson kernel). However, if $F\in H^1$, in other words $\sup_{0\le r<1}\int_{-\pi}^{\pi}\left|F(re^{it})\right|dt<\infty$ and F is holomorphic, then by $F(re^{it})\to F(e^{it})$ in L^1. Thus F can be written as the Poisson integral and the Cauchy integral of its boundary function $F(e^{it})$.

> **Remark 1.3.3.1** (notes on proof corollary 3.9 in book). Corollary 3.9 is Schwarz integral formula. The kernel $\frac{1}{2\pi}\frac{e^{it}+z}{e^{it}-z}$ is called Schwarz kernel.
>
> We can rewrite:
>
> $$\begin{aligned}G(z)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\operatorname{Re}F(e^{it})dt\\&=\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\operatorname{Re}F(e^{it})}{e^{it}-z}de^{it}+z\frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{\frac{\operatorname{Re}F(e^{it})}{e^{it}}}{e^{it}-z}de^{it}\end{aligned}$$
>
> Since $\operatorname{Re}F(e^{it})$ and $\frac{\operatorname{Re}F(e^{it})}{e^{it}}$ are continuous function on boundary of disk. Then $G(z)$ is holomorphic function.

An consequence of Poisson representation for H^1 functions is a famous theorem due to F. and M. Riesz. It says given a Borel measure μ, when negative frequencies of Fourier coefficients of μ is zero, then μ is absolutely continuous w.r.t. Lebesgue measure, i.e.: $d\mu(t)=f(t)dt$ for some $f\in L^1$. This theorem shows the difference between bounded holomorphic function $\sum_{k=0}^{\infty}a_kr^ke^{ik\theta}$ and bounded harmonic function $\sum_{k=-\infty}^{\infty}a_kr^ke^{ik\theta}$ (bounded as $\sup_{0\le r<1}\int_{-\pi}^{\pi}\left|F(re^{it})\right|dt<\infty$). The vanish of negative frequencies make bounded harmonic function (or Poisson integral of complex Borel measure) to bounded holomorphic function.

> **Remark 1.3.3.2** (notes on proof of corollary 3.11 in book). f is bounded variation, then f can be written as difference of two increasing bounded function. This is equivalent to f can be written as difference of two Borel measure. Thus $f(t)=c+\int_{-\pi}^{t}d\mu(s)$ where $c=f(-\pi)$.

The integration by parts:

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{ijt} d\mu(t) = & e^{ijt} \int_{-\pi}^{t} d\mu(s)\Big|_{-\pi}^{\pi} - ij \int_{-\pi}^{\pi} g(t) e^{ijt} dt \\
&= \left(e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s) - e^{-ij\pi} \int_{-\pi}^{-\pi} d\mu(s) \right) - ij \int_{-\pi}^{\pi} g(t) e^{ijt} dt \\
&= e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s) - ij \int_{-\pi}^{\pi} (F(e^{it}) - c) e^{ijt} dt \\
&= e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s) - \lim_{r\to 1} ij \int_{-\pi}^{\pi} F(re^{it}) e^{ijt} dt \\
&= e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s)
\end{aligned}$$

The limit in fourth equality is by $F \in H^1$, $F(re^{it}) \to F(e^{it})$ in L^1. This limit is 0 since $F \in H^1$, the negative frequencies are 0. $e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s) = e^{ij\pi} g(\pi)$ is 0 since $f(\pi) = f(-\pi) + g(\pi)$ and $f(\pi) = f(-\pi)$.

Corollary 3.11 in book shows a condition when bounded variation implies absolutely continuity. This Corollary emphases again 'holomorphic condition' or vanish of negative frequencies makes a Borel measure absolutely continuous. Theorem 3.12 in book says that $F' \in H^1$ is the necessary and sufficient condition of holomorphic $F \in H(D)$ is absolutely continuous on boundary.

Remark 1.3.3.3 (notes on proof of theorem 3.12 in book)**.** $F \in H^1$ implies $F' \in H(D)$. Since

$$\sup_{0\le r<1} \int_{-\pi}^{\pi} \left|F(re^{it})\right| dt = \sup_{0\le r<1} \int_{-\pi}^{\pi} \left|ire^{it} F(re^{it})\right| dt$$

, $F'(z) \in H^1$ if and only if $izF'(z) \in H^1$.

I don't know why the difference is harmonic and continuous at the origin, it has to be a constant c.

There is a corollary of Theorem 3.12 in book which is useful in next section.

Corollary 1.3.3.1 (corollary 3.13 in book)**.** *Let Γ be a Jordan curve and let F be a conformal map from D to interior domain bounded by a Jordan curve Γ. Then Γ is rectifiable if and only if $F' \in H^1$.*

1.4 Some Classical Inequalities

In this section we study two classical Inequalities: Hardy's inequality and Fejer-Riesz inequality. The first inequality is an example of why H^p is a natural replacement of L^p for $p \le 1$. The second inequality shows some geometry properties of conformal mappings.

1.4.1 Hardy's inequality

Theorem 1.4.1.1 (Hardy's inequality). *Let $F(z) = \sum_{j=0}^{\infty} a_j z^j$ be in H^1. Then:*

$$\sum_{j=0}^{\infty} \frac{|a_j|}{j+1} \le C\|F\|_{H^1}$$

with a constant C independent of F.

Remark 1.4.1.1 (notes on proof of theorem 1.4.1.1). We know the principal branch of the logarithm $\log z = \log r + i\theta$ where $z = re^{i\theta}$ with $|\theta| < \pi$. Thus $\operatorname{Im} \log 1 - z = \arg 1 - z$. It is easy to see $-\frac{\pi}{2} < \arg 1 - z < \frac{\pi}{2}$.

$$\begin{aligned} F(re^{it})u(re^{it}) =& (\sum_{j=0}^{\infty} a_j (re^{it})^j)(\frac{i}{2}\sum_{j\neq 0} j^{-1} r^{|j|} e^{ijt}) \\ =& (\sum_{j=0}^{\infty} a_j r^j e^{ijt})(\frac{i}{2}\sum_{k\neq 0} k^{-1} r^{|k|} e^{ikt}) \end{aligned}$$

After taking integral, only $j + k = 0$ term does not vanish, thus:

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} F(re^{it})u(re^{it})dt =& (\frac{i}{2}\sum_{j+k=0} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_j r^j e^{ijt} k^{-1} r^{|k|} e^{ikt} dt) \\ =& \frac{i}{2}\sum_{j+k=0} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_j r^{j+|k|} e^{i(j+k)t} k^{-1} dt \\ =& \frac{i}{2}\sum_{j=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_j r^{2j} (-j)^{-1} dt \\ =& \frac{i}{2}\sum_{j=1}^{\infty} a_j r^{2j} (-j)^{-1} \\ =& -\frac{i}{2}\sum_{j=1}^{\infty} a_j j^{-1} r^{2j} \end{aligned}$$

The corollary 4.2 in book shows that if $F(e^{it})$ is absolutely continuous (equivalent to $F' \in H^1$), then $(\hat{F}(n))_n \in \ell^1$. But the converse is not true. $(\hat{F}(n))_n \in \ell^1$ only implies F extends to a continuous function on $\bar{D}$

Remark 1.4.1.2 (Errata of $\operatorname{Re} H^1$). Let $g(t)$ be $\operatorname{Re} F(e^{it}) = \sum_{j\geq 0} a_j e^{ijt}$.

Then

$$
\begin{aligned}
g(t) =& \frac{F(e^{it}) + \overline{F(e^{it})}}{2} \\
=& \frac{a_0 + \bar{a}_0}{2} + \sum_{j\geq 0} \frac{a_j}{2} e^{ijt} + \sum_{j\geq 0} \frac{\bar{a}_j}{2} e^{-ijt} \\
=& \frac{a_0 + \bar{a}_0}{2} + \sum_{j>0} \frac{a_j}{2} e^{ijt} + \sum_{j<0} \frac{\overline{a_{-j}}}{2} e^{ijt} \\
=& \frac{a_0 + \bar{a}_0}{2} + \sum_{j\neq 0} \hat{g}(j) e^{ijt}
\end{aligned}
$$

where $\hat{g}(j) = \frac{a_j}{2}$ for $j > 0$, $\hat{g}(j) = \frac{\overline{a_{-j}}}{2}$ for $j > 0$ and $\hat{g}(j) = \operatorname{Re} a_0$. Thus $|a_j| = |\hat{g}(j)| + |\hat{g}(-j)|$. Substitute $|a_j|$ to $\sum_{j=1}^{\infty} \frac{|a_j|}{j} \leq \pi \|F\|_{H^1}$. We have $\sum_{j\neq 0} \left| \frac{\hat{f}(j)}{j} \right| \leq \pi \|f\|_{\operatorname{Re} H^1}$

We have $\operatorname{Re} H^1$ is a proper subspace of $\operatorname{Re} L^1$. And Hardy's inequality may be considered an extension to $p = 1$ of Paley's inequality which says that for $f \in L^p$ with $1 < p \leq 2$:

$$
\sum_{j=-\infty}^{\infty} \frac{\left|\hat{f}(j)\right|^p}{|j|^{p-2}} \leq C_p \|F\|_p^p
$$

Later we will see in $\mathbb{R}^n$ this inequality can be extended to H^p for $p < 1$. And H^p for $p \leq 1$ are natural substitutes of Lebesgue spaces L^p.

1.4.2 Fejer-Riesz inequality

Recall the final corollary in last section. Let F be a conformal map from D to interior domain bounded by a Jordan curve Γ. Then Γ is rectifiable if and only if $F' \in H^1$.

Theorem 1.4.2.1 (Fejer-Riesz inequality)**.** *Let $F \in H^p$, $0 < p < \infty$, then*

$$
\int_{-1}^{1} |F(x)|^p \, dx \leq \frac{1}{2} \int_{-\pi}^{\pi} \left|F(e^{it})\right|^p dt
$$

To prove this theorem, we first prove the $p = 2$ case. Then for $p \neq 2$ case, we factorize $F(z) = B(z)H(z)$ and let $|G(z)|^2 = |H(z)|^p$ to reduce this case to $p = 2$.

Here is a direct application of this inequality. Let F be the conformal map from D to interior domain bounded by a Jordan curve Γ. Then image of diameter of D has length at most half of length of Γ (corollary 4.6 in book).

Remark 1.4.2.1 (notes on proof of corollary 4.6 in book)**.** To prove that $\frac{1}{2}$ is the best constant in corollary 4.6 in book, we only need to show there is a conformal map from D to interior domain bounded by a rectifiable Jordan curve Γ, the constant $\frac{1}{2}$ can not be smaller. Let $F(z)$ is a conformal map from D to rectangle $\{x+iy : |x| < 1, |y| < \epsilon\}$ and F maps segment $(-1,1)$ in D to segment $(-1,1)$ in rectangle. It is easy to construct this map. The constant has to be at least $\frac{2}{4+4\epsilon}$. Let $\epsilon \to 0$ we conclude $\frac{1}{2}$ is the best constant.

Another usage of conformal mapping $F' \in H^1$ is following: F can be extended on $\bar{D}$ and F is still conformal. More precisely, Let F be a conformal mapping from D to interior domain bounded by a rectifiable Jordan curve Γ. F is also conformal at almost every boundary point (corollary 4.7 in book).

Remark 1.4.2.2 (notes on proof of corollary 4.7 in book)**.** The step:

$$\frac{F(e^{it_0}) - F(z)}{e^{it_0} - z} - F'(e^{it_0}) = \frac{1}{e^{it_0} - z} \int_z^{e^{it_0}} (F'(\xi) - F'(e^{it_0}))d\xi \to 0$$

as $z \to e^{it_0}$ N.T. is by mean value theorem of integration.

I don't know why the tangent to Γ at the point $F(e^{it_0})$ happens for a.e. boundary point e^{it_0}.

The angle between γ and boundary in D is $\lim \arg z - e^{it_0} - t_0 - \frac{\pi}{2}$ and The angle between $F(\gamma)$ and boundary in $F(D)$ is $\lim \arg F(z) - F(e^{it_0}) - t_0 - \arg(\frac{d}{dt}(F(e^{it}))|_{t=t_0})$. Since F is conformal in D, to prove F is conformal in $\bar{D}$, we only need to prove the conformal map preserves angle on boundary. That is:

$$\lim_{z\to e^{it_0}} \arg(z - e^{it_0}) - t_0 - \frac{\pi}{2} = \lim_{z\to e^{it_0}} \arg(F(z) - F(e^{it_0})) - \arg(\frac{d}{dt}(F(e^{it}))|_{t=t_0})$$

We have $\frac{d}{dt}F(e^{it})|_{t=t_0} = ie^{it_0}F'(e^{it_0})$. Thus $\arg(\frac{d}{dt}(F(e^{it}))|_{t=t_0}) = \frac{\pi}{2} + t_0 + \arg F'(e^{it_0})$. So the equality is the same as:

$$\lim_{z\to e^{it_0}} \arg(z - e^{it_0}) = \lim_{z\to e^{it_0}} \arg(F(z) - F(e^{it_0})) - \arg F'(e^{it_0})$$

which is clearly if we take arg in both sides in $\lim_{z\to e^{it_0}} \frac{F(e^{it_0}) - F(z)}{e^{it_0} - z} = F'(e^{it_0})$. We use $t_0 + \frac{\pi}{2}$ instead of $t_0 - \frac{\pi}{2}$ match to $\arg(\frac{d}{dt}(F(e^{it}))|_{t=t_0})$ since they are in the same direction.

1.5 The Conjugate Function

In section 1, given a integrable function f, $f \in L^1$, we know the harmonic function $u = P(f)$ determines a holomorphic function F up to a constant c if we consider u as the real part of F. Let $v(re^{it}) = F(re^{it})$ with $v(0) = 0$. Then we can define the conjugate function of f to be:

$$\tilde{f}(t) = \lim_{t \to 1} v(re^{it})$$

The next theorem ensures the above limit exists for a.e. t:

Theorem 1.5.0.1 (theorem 5.2 in book)**.** *Let $F \in H(D)$ be such that $\operatorname{Re} F(z) \geq 0$ for every $z \in D$. Then F has N.T. limits at almost every boundary point.*

> **Remark 1.5.0.1** (notes on proof of theorem 5.2 in book)**.** I don't know why $\lim G(z) = \lim \frac{1}{1+F(z)}$ as $z \to e^{it}$ N.T. is different from 0 a.e. t.

1.5.1 Estimate conjugate function $\tilde{f}$ for $f \in L^p$, $1 < p < \infty$

The following theorem is key to estimate $\|\tilde{f}\|_p$ for $1 < p < \infty$:

Theorem 1.5.1.1 ((theorem 5.3 in book))**.** *For every p with $1 < p \leq 2$, there is a constant C_p, s.t. for every $F(z) = u(z) + iv(z)$, holomorphic in D, with $u(z) > 0$ on D, v real valued and $v(0) = 0$, the inequality:*

$$\int_{-\pi}^{\pi} \left|v(re^{it})\right|^p dt \leq C_p \int_{-\pi}^{\pi} \left|u(re^{it})\right|^p dt$$

holds for every $0 < r < 1$.

> **Remark 1.5.1.1** (notes on proof of theorem 1.5.1.1)**.** We need inequality $|\sin\theta|^p \leq C_p |\cos\theta|^p - D_p \cos(p\theta)$ for $|\theta| < \frac{\pi}{2}$. To use this inequality, we need $\psi(z) < \frac{\pi}{2}$ where $\psi(z) = \arg F(z)$. $u(z) > 0$ ensures this inequality hold.
>
> However, this inequality actually holds if $u \geq 0$ since $u \geq 0$ implies $u > 0$ by maximum principle.

Now we give the estimation of $\|\tilde{f}\|_p$ for $1 < p < \infty$:

Corollary 1.5.1.2 (Marcel Riesz inequality (corollary 5.5 in book))**.** *For every p with $1 < p < \infty$, there is a constant C_p, s.t. for each $f \in L^p$:*

$$\int_{-\pi}^{\pi} |\tilde{f}(t)|^p dt \leq B_p \int_{-\pi}^{\pi} |f(t)|^p \, dt \tag{1.5.1}$$

> **Remark 1.5.1.2** (notes on proof of corollary 1.5.1.2)**.** First consider $1 < p \leq 2$ case. Let $f^+ = \max(f, 0)$, $v_1 = \tilde{f^+}$, $f^- = \max(-f, 0)$ and $v_2 = \tilde{f^-}$. Notice $u_1 = P(f^+) = 0$ and $v_1 = 0$ if $f^+ = 0$, $u_2 = P(f^-) = 0$ and $v_2 = 0$

if $f^{-} = 0$. Thus we can write:

$$\int_{-\pi}^{\pi} |v(re^{it})|^p dt = \int_{-\pi}^{\pi} \left|v_1(re^{it})\right|^p + \left|v_2(re^{it})\right|^p dt$$

$$\int_{-\pi}^{\pi} |u(re^{it})|^p dt = \int_{-\pi}^{\pi} \left|u_1(re^{it})\right|^p + \left|u_2(re^{it})\right|^p dt$$

Since $\int \left|v_1(re^{it})\right| \leq C_p \int \left|u_1(re^{it})\right|$ and $\int \left|v_2(re^{it})\right| \leq C_p \int \left|u_2(re^{it})\right|$ by theorem 1.5.1.1, we have $\int \left|v(re^{it})\right| \leq C_p \int \left|u(re^{it})\right|$. Fatou lemma yields inequality (1.5.1).

For $2 < p < \infty$ case, we use $\|v(re^{i\cdot})\|_p = \sup_{\|g\|_{p'} \leq 1} \int v(re^{it}) g(t) dt$ by Hahn-Banach theorem.

Let $h = P(g)$ and $w = \tilde{h}$ with $w(0) = 0$. Since $h + iw \in H^{p'}$, $h + iw$ can be written as Poisson integral of boundary function, $h(re^{it}) + iw(re^{it}) = P(g + i\tilde{g})$. Thus we have $w = P(\tilde{g})$.

By Holder inequality, We have:

$$\begin{aligned}
&\int \left|(u(rz) + iv(rz))(h(z) + iw(z)) - (u(re^{it}) + iv(re^{it}))(g(t) + i\tilde{g}(t))\right| \\
\leq &\int |(u(rz) + iv(rz))(h(z) + iw(z)) - (u(rz) + iv(rz))(g(t) + i\tilde{g}(t)) \\
&+ (u(rz) + iv(rz))(g(t) + i\tilde{g}(t))(u(re^{it}) + iv(re^{it}))(g(t) + i\tilde{g}(t))| \\
\leq &\int |u(rz) + iv(rz)| \, |((h(z) + iw(z)) - (g(t) + i\tilde{g}(t))| \\
&+ \int \left|((u(rz) + iv(rz)) - (u(re^{it}) + iv(re^{it})))\right| |(g(t) + i\tilde{g}(t))| \\
\leq &(\int |u(rz) + iv(rz)|^p)^{\frac{1}{p}} (\int |((h(z) + iw(z)) - (g(t) + i\tilde{g}(t)))|^{p'})^{\frac{1}{p'}} \\
&+ (\int \left|((u(rz) + iv(rz)) - (u(re^{it}) + iv(re^{it})))\right|^p)^{\frac{1}{p}} (\int |(g(t) + i\tilde{g}(t))|^{p'})^{\frac{1}{p'}}
\end{aligned}$$

We have $(\int |((h(z) + iw(z)) - (g(t) + i\tilde{g}(t)))|^{p'})^{\frac{1}{p'}} \to 0$ since $h + iw \in H^{p'}$.
We have$(\int \left|(u(rz) + iv(rz)) - (u(re^{it}) + iv(re^{it}))\right|^p)^{\frac{1}{p}} \to 0$ since $\left|re^{it}\right| < 1$.

Thus $(u(rz) + iv(rz))(h(z) + iw(z)) \to (u(re^{it}) + iv(re^{it}))(g(t) + i\tilde{g}(t))$ in L^1.

Our final result is estimation of imaginary part of holomorphic function:

Corollary 1.5.1.3 (corollary 5.8 in book)**.** *If $F \in H(D)$, then for every $1 < p < \infty$ and every $0 \leq r < 1$:*

$$(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\operatorname{Im} F(re^{it})\right|^p dt)^{\frac{1}{p}} \leq B_p^{\frac{1}{p}} (\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\operatorname{Re} F(re^{it})\right|^p dt)^{\frac{1}{p}} + |\operatorname{Im} F(0)|$$

1.5.2 Estimate conjugate function $\tilde{f}$ for $f \in L^1$

The conjugate or harmonic function is always exists, but the behavior of boundary is not simple. The corollary 1.5.1.2 does not hold for $p = 1$. For example, Poisson kernel $P_r(t)$ is in L^1, $\|P_r(t)\|_1 = 2\pi$ but conjugate Poisson kernel $Q_r(t)$ is not, $\int |Q_r(t)| = 4 \log \frac{1+r}{1-r}$.

> **Remark 1.5.2.1.** By Hahn-Banach theorem and duality of L^p, for $v \in L^1$, $\int |v| = \sup_{\|g\|_\infty \leq 1} \left|\int vg\right|$. I don't know how the author concludes conjugate function operator is not bounded in L^∞.

For $f \in L^1$, conjugate operator is of weak type $(1,1)$ (theorem 5.9 in book) and type $(1,p)$ for $0 < p < 1$ (corollary 5.10 in book).

> **Remark 1.5.2.2** (notes on proof of theorem 5.9 in book)**.** $v(0) = 0$ is a necessary condition for conjugate operator is linear operator.
>
> The function $f(z) = \frac{z-i\lambda}{z+i\lambda} = \frac{|z|^2-\lambda^2-2i\lambda \operatorname{Re} z}{|z|^2+\lambda^2+2\lambda \operatorname{Im} z}$ maps $\operatorname{Re} z > 0$ to $\operatorname{Im} z < 0$ since $1 \mapsto \frac{1-\lambda^2-2i\lambda}{1+\lambda^2}$.
>
> If $|z| = \lambda$, $f(z) = \frac{-i \operatorname{Re} z}{\lambda + \operatorname{Im} z}$ and $\arg f(z) = -\frac{\pi}{2}$. Thus $h_\lambda(z) = \frac{1}{2}$. If $k \neq \frac{1}{2}$, the level lines $h_\lambda(z) = k$ when $\tan \arg f(z) = \frac{-2\lambda \operatorname{Re} z}{|z|^2-\lambda^2} = \tan \pi(k-1)$. Which is $-2\lambda x = c(k)(x^2+y^2-\lambda^2)$. This is a circle passing through $i\lambda$ and $-i\lambda$.
>
> Let $f(x) = \frac{1}{x} + \arg\tan x - \frac{\pi}{2}$. $f'(x) = -\frac{1}{x^2} + \frac{1}{1+x^2} < 0$ and $\lim_{x\to\infty} f(x) = 0$. Thus $f(x) > 0$. The inequality $\frac{\pi}{2} - \arg\tan\lambda \leq \frac{1}{\lambda}$ holds.
>
> u is harmonic and F is holomorphic, then $u \circ F$ is holomorphic. u is $\operatorname{Re} G$ with G holomorphic. $G \circ F$ is holomorphic. Thus $u \circ F = \operatorname{Re} G \circ F$ is harmonic.
>
> We should pay attention to this statement: *Since* $\tilde{f}(t) = \lim_{r\to 1} v(re^{it})$, *then*
>
> $$\begin{aligned} |E_\lambda| &= \left| \bigcup_{n=1}^\infty \bigcap_{j=n}^\infty \{t : \left|v(r_j e^{it})\right| > \lambda\} \right| \\ &= \left| \lim_{n\to\infty} \bigcap_{j=n}^\infty \{t : \left|v(r_j e^{it})\right| > \lambda\} \right| \\ &= \lim_{n\to\infty} \left| \bigcap_{j=n}^\infty \{t : \left|v(r_j e^{it})\right| > \lambda\} \right| \end{aligned}$$
>
> The last equality is by continuity of measure.
>
> I don't know why author choose $C = \frac{64}{\pi}$ finally.

1.5.3 Conjugate function and H^p space

We now describe H^p for $1 \leq q \leq \infty$. Suppose $F \in H^p$, we know $F = P(f)$ for some $f \in L^1$. $f \in L^1$ implies $\tilde{f}$ is well defined. We can write $F = f + \tilde{f} + i \operatorname{Im} F(0)$ for boundary F. $F(z) \in H^p$ implies $f \in \operatorname{Re} L^p$ (theorem 3.6 in book). Thus:

$$H^p \subset \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^p, c \in \mathbb{R}\}$$

By writing $\tilde{f} = F - f - \operatorname{Im} F(0)$, we know $\tilde{f} \in L^p$ for $1 \leq p \leq \infty$ (conclusion in book).

When $1 < p < \infty$, $f \in L^p$ guarantees $\tilde{f} \in L^p$. $(f + i\tilde{f} + ic) \in L^p$ implies $P(f + i\tilde{f} + ic) \in H^p$. Thus for $1 < p < \infty$:

$$H^p \supset \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^p, c \in \mathbb{R}\}$$

Here we take $f \in \operatorname{Re} L^p$ to make $\tilde{f}$ meaningful. We also have $\operatorname{Re} H^p = \operatorname{Re} L^p$ in this case.

For $p = 1$, $f \in L^p$ no longer guarantees $\tilde{f} \in L^p$. But we know if $f, \tilde{f} \in L^1$, then $F = P(f + i\tilde{f} + ic) \in H^1$. Thus:

$$H^1 \supset \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^1, \tilde{f} \in L^1, c \in \mathbb{R}\}$$

Here we take $f \in \operatorname{Re} L^p$ to make $\tilde{f}$ meaningful. We also have $\operatorname{Re} H^1 = \{f \in \operatorname{Re} L^1 : \tilde{f} \in L^1\} \subsetneqq \operatorname{Re} L^1$ by counterexample Poisson kernel.

Similarly we have:

$$H^\infty \supset \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^\infty, \tilde{f} \in L^\infty, c \in \mathbb{R}\}$$

and

$$\operatorname{Re} H^\infty = \{f \in \operatorname{Re} L^\infty : \tilde{f} \in L^\infty\} \subsetneqq \operatorname{Re} L^\infty$$

> **Remark 1.5.3.1.** Author suppose $F(e^{it}) \in H^p$, but if $F(re^{it}) \in H^p$ we only know $F(e^{it}) \in L^p$. In other words, $F(e^{it}) \in H^p$ is not clear.
>
> Conclusion for H^∞ is from remark 1.5.2.1.

1.5.4 Conjugate operator

> **Remark 1.5.4.1.**
> $$\frac{r \sin t}{1 + r^2 - 2r \cos t} \to \frac{1}{2 \tan \frac{t}{2}}$$

as $r \to 1$ fails if $t = 0$.

$$\begin{aligned}
\left|\frac{r\sin t}{1+r^2-2r\cos t}\right| &= \left|\frac{2r\sin\frac{t}{2}\cos\frac{t}{2}}{1+r^2-2r(2(\cos\frac{t}{2})^2-1)}\right| \\
&= \left|\frac{2r\sin\frac{t}{2}\cos\frac{t}{2}}{(1+r)^2-4r(\cos\frac{t}{2})^2}\right| \\
&= \left|\frac{2r\sin\frac{t}{2}\cos\frac{t}{2}}{(1-r)^2+4r(\sin\frac{t}{2})^2}\right| \\
&\le \left|\frac{2r\sin\frac{t}{2}\cos\frac{t}{2}}{4r(\sin\frac{t}{2})^2}\right| \\
&= \left|\frac{1}{2\tan\frac{t}{2}}\right| \\
&= \frac{1}{2\tan\left|\frac{t}{2}\right|} \\
&\le \frac{1}{|t|}
\end{aligned}$$

The following theorem shows we can define conjugate function without getting inside the disk, by singular integral:

Theorem 1.5.4.1 (theorem 5.14 in book)**.** *If $f \in L^1$, then*

$$\tilde{f}(\theta) = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{0<\epsilon<|t|<\pi}\frac{1}{2\tan\frac{t}{2}}f(\theta-t)dt$$

for every θ in the Lebesgue set of f and, consequently for a.e. θ.

Remark 1.5.4.2. I don't know why

$$\frac{1}{\pi}\int_{1-r<|t|<\pi}\left(\frac{r\sin t}{1+r^2-2r\cos t}-\frac{1}{2\tan\frac{t}{2}}\right)f(\theta-t)dt$$

is bounded by

$$C(1-r)^2\int_{1-r<|t|<\pi}\frac{|f(\theta-t)-f(\theta)|}{|t|^3}dt$$

as $r \to 1$.

For upper half plane, we know $u(x,t) = P_t * f(x)$ is harmonic function. We have Hilbert transform as counterpart of conjugate function:

$$Hf(x) = \lim_{\epsilon\to 0}\int_{|y|>\epsilon}\frac{f(x-y)}{y}dy$$

for a.e. x.

1.6 H^p as a Linear Space

In this section we look at H^p as a topological vector space. By considering distance $d(F,G) = \|F - G\|_{H^p}$ for $p \geq 1$ and $d(F,G) = \|F - G\|_{H^p}^p$ for $p < 1$, H^p is a metric space. By considering mapping $F(z) \mapsto F(e^{it})$, H^p is isometric to subspace of L^p. The main topic in this section is dual of H^p.

1.6.1 H^p is not Locally convex for $0 < p < 1$

If a space is locally convex, there is a convex neighborhood V contained in ball $B(0,1)$. Since V is a neighborhood, there is a ball $B(0,\epsilon)$ contained in V. Thus by contrapositive, If for all $\epsilon > 0$, there is a convex combination of F in ball $B(0,\epsilon)$ is out of $B(0,1)$, then the space is not locally convex.

It is easy to prove for $0 < p < 1$, L^p is not locally convex by using triangle wave function. To prove the same fact for H^p, we use trigonometric polynomials to approximate triangle wave function. And conclude these polynomials are in ball $B(0,\epsilon)$ and their convex combination is out of ball $B(0,1)$.

> **Remark 1.6.1.1** (notes on proof of theorem 6.2 in book)**.** For $1 < p < 1$, $(a+b)^p \leq a^p + b^p$ by $(a+b)^p \leq \frac{(2a)^p + (2b)^p}{2} = 2^{p-1}a^p + 2^{p-1}b^p \leq a^p + b^p$

> **Remark 1.6.1.2** (Algebric dual space and topological dual space from wikipedia: dual space)**.** Given any vector space V over a field $\mathbb{F}$, the algebraic dual space V^* is defined as the set of all linear functionals $\phi : V \to \mathbb{F}$.
>
> When dealing with topological vector spaces, one is typically only interested in the continuous linear functionals $\phi : V \to \mathbb{F}$. This gives rise to the notion of the "continuous dual space" or "topological dual" which is a linear subspace of the algebraic dual space. For any finite-dimensional normed vector space or topological vector space, such as Euclidean n-space, the continuous dual and the algebraic dual coincide. This is however false for any infinite-dimensional normed space.

Being non locally convex has a great deal of continuous linear functionals. The topological dual or continuous linear functional on H^p is zero. First we prove the only convex neighborhood of 0 is the whole space. By using the proof of non locally convex reversely, Given a convex and open set $V \subset H^p$ and $0 \in V$, we can show for any $F \in H^p$, there is a combination $\sum_j \lambda_j F_j = F$, s.t. $\sum_j \lambda_j = 1$ and $F_j \in V$. Thus $F \in V$ by V convex and we have $V = H^p$.

Then we consider the continuous linear functionals on H^p. Assume $\phi : H^p \to \mathbb{F}$ is a continuous linear functional. Let $\mathscr{B}$ be a locally convex base for $\mathbb{F}$. For any $W \in \mathscr{B}$, we have $\phi^{-1}(W)$ is convex and open hence is H^p. $\phi(H^p) \subset W$ for all $W \in \mathscr{B}$. We conclude that $\phi(F) = 0$ for all $F \in H^p$. Thus all continuous linear functionals on H^p are zero (This part is following the section 1.47 in Rudin, 1991).

Using inequality:

$$|F(z)| \leq \frac{1}{(1-|z|)^{\frac{1}{p}}}\|F\|_{H^p}$$

for $F \in H^p$ with $0 < p < \infty$, we can prove H^p is a complete space. Thus H^p is closed subspace of L^p in isometry sense.

> **Remark 1.6.1.3.** I don't know why H^p is the minimal closed subspace which contains $\{e^{ijt} : j = 0, 1, \cdots\}$. The author give the reason as follows: If $F(z) = \sum_0^\infty a_j z^j$ is in H^p, $F(re^{it}) \to F(e^{it})$ in L^p as $r \to 1$. And for r fixed, $\sum_0^n a_j r^j e^{ijt} \to F(re^{it})$ uniformly as $n \to \infty$.

1.6.2 Dual of H^p

In subsection 1.6.1, we show for $0 < p < 1$, the dual of H^p is zero. We investigate case $1 \leq p \leq \infty$ in this subsection.

In section 5, we show for $1 < p < \infty$, $H^p = \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^p, c \in \mathbb{R}\}$ and for $p = 1$, $H^1 = \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^1, \tilde{f} \in L^1, c \in \mathbb{R}\}$. Thus H^p is a proper subspace of L^p.

By dual of L^p, any continuous linear functional $\phi(g)$ for $g \in L^p$ can be written as $\phi_f(g) = \int gf$ with $f \in L^{p'}$. We consider the restriction of ϕ to H^p. This is the continuous linear functional $\phi_f(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})f(t)dt$ on H^p. If we consider this mapping is as from $L^{p'} \to (H^p)^*$, we have:

$$\begin{aligned}
\|\phi_f\| &= \sup_{\|f\|_{p'}=1} \frac{\|\phi_f\|_{(H^p)^*}}{\|f\|_{p'}} \\
&= \sup_{\|f\|_{p'}=1} \sup_{\|F\|_{H^p}=1} \frac{\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})f(t)dt\right|}{\|F\|_{H^p}} \\
&\leq \sup_{\|f\|_{p'}=1} \sup_{\|F\|_{H^p}=1} \frac{\frac{1}{2\pi}(\int_{-\pi}^{\pi} \left|F(e^{it})\right|^p dt)^p (\int_{-\pi}^{\pi} |f(t)|^p dt)^{p'}}{\|F\|_{H^p}} \\
&= \sup_{\|f\|_{p'}=1} \sup_{\|F\|_{H^p}=1} \frac{\|F\|_{H^p}\|f\|_{p'}}{\|F\|_{H^p}} \\
&= 1
\end{aligned}$$

Thus the mapping ϕ from $L^{p'} \to (H^p)^*$, $f \mapsto \phi_f$ is a continuous linear mapping. The Hahn-Banach theorem tells us that every $\Lambda \in (H^p)^*$ is of the form $\Lambda = \phi_f$ for some f with $\|f\|_{p'} \leq \|\Lambda\|$. More precisely, any continuous linear functional $\Lambda \in (H^p)^*$ can be extended to all of L^p. Thus we get $f \in L^{p'}$ with $\phi_f(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})f(t)dt$ restricted back to H^p.

The kernel of mapping ϕ is $f \in L^{p'}$ for which $\phi_f = 0$. This is equivalent $\phi_f(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})f(t)dt = 0$ for all $F \in H^p$, clearly,

$$\ker\phi = \{f \in L^{p'} : \hat{f}(-j) = \int_{-\pi}^{\pi} e^{ijt} f(t)\frac{dt}{2\pi} = 0, j = 0, 1, \cdots\}$$

$\hat{f}(j)$ is zero for non-positive frequency j is equivalent to $f \in H^p$ and $\hat{f}(0) = 0$. Thus

$$\begin{aligned}
&\{f \in L^{p'} : \hat{f}(-j) = \int_{-\pi}^{\pi} e^{ijt} f(t) \frac{dt}{2\pi} = 0, j = 0, 1, \cdots\} \\
=&\{f \in H^{p'} : \int_{-\pi}^{\pi} f(t) dt = 0\}
\end{aligned}$$

We denote this space by $H^{p'}(0)$ and obtain an isometry

$$L^{p'}/H^{p'}(0) \cong (H^p)^* \tag{1.6.1}$$

Now we consider the continuous linear functionals on $H^p(0)$, We consider the kernel of mapping $L^{p'} \to (H^p(0))^*$:

$$\{f \in L^{p'} : \hat{f}(-j) = \int_{-\pi}^{\pi} e^{ijt} f(t) \frac{dt}{2\pi} = 0, j = 1, 2, \cdots\} = H^{p'}$$

Thus we obtain an isometry

$$(H^{p'}(0))^* \cong L^{p'}/H^{p'} \tag{1.6.2}$$

Remark 1.6.2.1 (Topological complement)**.** Two vector subspace X and Y are algebraic complement of each other if $X + Y = E$ and $X \cap Y = \{0\}$. We can write $X \oplus Y = E$.

Two vector subspace X and Y are topological complement of each other if they are algebraic complement of each other and P_X (Projection from E to X) is continuous. If E is Banach space, another equivalent condition for topological complement is they are algebraic complement of each other and X and Y are closed.

X and Y are topological complement of each other means $E = X \oplus Y$ and $E \cong X \oplus Y$ in isometry sense.

For $1 < p < \infty$, $H^{p'}(0)$ has a topological complement in $L^{p'}$. Let us see how to construct this. Consider $f \in L^{p'}$, $f = \sum_{-\infty}^{\infty} a_j e^{ijt}$. Set

$$A(f) = \frac{1}{2}(f + \tilde{f} - \hat{f}(0)) = \sum_{j>0} a_j e^{ijt}$$

Then A is the projection of $L^{p'}$ onto $H^{p'}(0)$. $f - A(f) = \sum_{j \leq 0} a_j e^{ijt} = \sum_{j \geq 0} a_{-j} e^{-ijt}$. If we write $F(z) = \sum_{j \geq 0} a_{-j} z^j$, we have $F \in H^{p'}$ and $f(t) = Af(t) + F(e^{-it})$. If we write $G(z) = \sum_{j \geq 0} \overline{a_{-j}} z^j$. $G(e^{it}) = F(e^{-it}) \in H^{p'}$. Thus we have:

$$f(t) = Af(t) + \overline{G(e^{it})}$$

Using notation $\overline{H^{p'}} = \overline{h(t)} : h \in H^{p'}$ and $(H^{p'})^- = h(-t) : h \in H^{p'}$. We have:

$$L^{p'} = H^{p'}(0) \oplus \overline{H^{p'}} = H^{p'}(0) \oplus (H^{p'})^-$$

If we consider $B(f) = \frac{1}{2}(f + \tilde{f} + \hat{f}(0)) = \sum_{j \geq 0} a_j e^{ijt}$. We have:

$$L^{p'} = H^{p'} \oplus \overline{H^{p'}(0)} = H^{p'} \oplus (H^{p'}(0))^-$$

By topological direct sum we have topological isomorphisms:

$$L^{p'}/H^{p'}(0) \cong H^{p'}$$

$$L^{p'}/H^{p'} \cong H^{p'}(0)$$

By equation (1.6.1) and (1.6.2), we derive a conclusion for H^p similar with dual of L^p.

$$(H^p)^* \cong H^{p'}$$

with the pairing

$$< G, F >= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{it})G(e^{-it})dt$$

where $F \in H^p$, $G \in H^{p'}$, or

$$< G, F >= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{it})\overline{G}(e^{it})dt$$

where $F \in H^p$, $G \in H^{p'}$. Besides, we also have $(H^p(0))^* \cong H^{p'}(0)$. We can not use same argument for $p = 1$ case since for $f \in L^{1'} = L^\infty$, $\hat{f}$ no longer in L^∞. We will study $(H^1)^*$ in section 9. Now we only have $(H^1)^* \cong L^\infty/H^\infty(0)$ by equation (1.6.1).

1.7 Canonical Factorization Theorem

In section 3, we show a holomorphic function $F \in H^p$, $0 < p \leq \infty$ can be written as $F = BH$, where B is the Blaschke product, H is never zero and $\|H\|_{H^p} = \|F\|_{H^p}$. We will get a finer factorization: canonical factorization. We can factorize F as product of inner and outer function:

$$F(z) = I_F(z)E_F(z)$$

where:

$$I_F(z) = e^{ic}B(z) \exp\left(-\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\sigma(t)\right)$$

and

$$E_F(z) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log\left|F(e^{it})\right| dt\right)$$

1.7.1 Canonical factorization

For $F \in H^p$, $0 < p \leq \infty$, we use Riesz factorization $F = BH$. We want to keep factorizing non zero function H. Write $\log\left|H(r_j e^{it})\right| = \log^+\left|H(r_j e^{it})\right| - \log^-\left|H(r_j e^{it})\right|$. We know $\log^-\left|H(r_j e^{it})\right|$ converges to a positive measure μ_2. And we observe $\log^+\left|H(r_j e^{it})\right| \to \log^+\left|H(e^{it})\right|$ in H^1 norm.

Since $\log H(r_j z) = \log|H(r_j z)| + i\theta$, $|\log H(r_j z)| \leq |\log|H(r_j z)|| + |\theta|$. By theorem 3.2 in book, we show that $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\log\left|H(re^{it})\right|\right| dt$ is finite. Thus $\log H(r_j z) \in H^1$.

Now we factorize the non zero function H. by corollary 3.9 in book:

$$\log H(r_j z) = i \arg H(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log\left|H(r_j e^{it})\right| dt$$

Using $\log = \log^+ - \log^-$, and let $r_j \to 1$, We have:

$$\log H(z) = ic + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log^+\left|H(e^{it})\right| dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu_2(t)$$

Write $d\mu_2(t) = g(t)dt + d\sigma(t)$, $k(t) = \log^+\left|H(e^{it})\right| - g(t)$. We finally get canonical theorem:

$$F(z) = I_F(z)E_F(z)$$

where:

$$I_F(z) = e^{ic}B(z)\exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\sigma(t)\right)$$

and

$$E_F(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log\left|F(e^{it})\right| dt\right)$$

Remark 1.7.1.1. If $p < \infty$, we have $|E_F(z)|^p \leq P(\left|F(e^{it})\right|^p)$, or $\left|E_F(re^{i\theta})\right|^p \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\left|F(e^{it})\right|^p dt$. Integrate both side by θ we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|E_F(re^{i\theta})\right|^p d\theta &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\left|F(e^{it})\right|^p dt d\theta \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|F(e^{it})\right|^p dt\end{aligned}$$

Thus $F \in H^p$ implies $E_F \in H^p$. The $p = \infty$ case is trivial.

$\left|E_F(e^{it})\right| = \left|F(e^{it})\right|$ a.e.t since $I_F(e^i t)$ has finite non zero point.

We say $F \in H^p$ is an inner function if and only if $E_F = 1$, and say $F \in H^p$ is an outer function if and only if I_F is constant.

1.7.2 Outer function

We list some conclusions about outer functions. Most of them are criterions of outer function.

Corollary 1.7.2.1. *If $F \in H^p$, $0 < p \leq \infty$, and is not identically zero, then:*

$$\log|F(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log\left|F(e^{it})\right| dt$$

and equality holds if and only if F is an outer functions.

The following theorem is an easy consequence of the above corollary.

Theorem 1.7.2.2 (criterion of outer function. theorem 7.5 in book)**.** *Suppose that $F \in H^p$ and $F^{-1} \in H^p$ for some $0 < p \leq \infty$. Then F is an outer function.*

The next theorem states that the limit of sequence of decreasing outer functions is outer function.

Theorem 1.7.2.3. *Let $F_j \in H^p$ be outer functions for $j = 1, 2, \cdots$. Suppose that $|F_1(z)| \geq |F_2(z)| \geq \cdots$ for every $z \in D$ and $F_j(z) \to F(z)$ uniformly over compact subsets of D, as $j \to \infty$. Then, if F is not identically zero, F is an outer function.*

> **Remark 1.7.2.1** (notes on proof of theorem 7.6 in book)**.**
>
> $$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^{-}\left|F_j(e^{it})\right| dt \to \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^{-}\left|F(e^{it})\right| dt$$
>
> by monotone convergence. But monotone convergence does not ensure limit is finite. I wonder if it can be infinite.

The following theorem is an easy consequence of the above theorem.

Theorem 1.7.2.4 (theorem 7.7 in book)**.** *Let $F \in H^p$, $0 < p \leq \infty$, not identically zero, and such that $\operatorname{Re} F(z) \geq 0$ for every $z \in D$. Then F is an outer function.*

> **Remark 1.7.2.2** (notes on proof of theorem 7.8 in book)**.**
> Proof of $|K(z)|^p = \exp P(\log(\left|K(e^{it})\right|^p)) \leq P(\left|K(e^{it})\right|^p)$ implies $K \in H^p$ and $p = \infty$ case is similar with remark 1.7.1.1.

1.7.3 Inner function

We can prove the following corollary, although sometimes it is considered as definition of inner functions.

Corollary 1.7.3.1 (corollary 7.2 in book)**.** *The inner functions are precisely those functions $F \in H^\infty$ for which $\left|F(e^{it})\right| = 1$ almost everywhere.*

Remark 1.7.3.1 (notes on proof of corollary 7.2 in book)**.** Since $|E_F(z)| = \exp P(\log \left|F(e^{it})\right|)$. One direction is

$$E_F = 1 \implies |E_F| = 1 \implies \log \left|F(e^{it})\right| = 0 \implies \left|F(e^{it})\right| = 1$$

Another direction is

$$\left|F(e^{it})\right| = 1 \implies \log \left|F(e^{it})\right| = 0 \implies E_F = 1$$

Consider space $F \cdot \mathscr{P}$, where $\mathscr{P}$ is the space of polynomials. Theorem 7.9 in book shows if F is an outer function, then $F \cdot \mathscr{P}$ is dense in H^p. Corollary 7.11 in book shows if F is just in H^p, then closure of $F \cdot \mathscr{P}$ is $I_F \cdot H^p$. These conclusions show how the inner factor plays in some approximation problems.

Remark 1.7.3.2 (notes on theorem 7.9 in book)**.** Even if $\mathscr{P}$ is dense in H^p, $F \cdot \mathscr{P}$ may not be dense in H^p. For example if $F(0) = 0$ in a positive measure set.

By Hahn-Banach theorem, if E is locally convex and F is subspace of E, then F is dense in E if and only if for any $f \in E^*$, $f|_F = 0 \implies f = 0$. Thus for $p \geq 1$, $E_F \cdot \mathscr{P}$ is dense in H^p is equivalent to for any $k(e^{it}) \in L^{p'}$, $\int_{-\pi}^{\pi} E_F(e^{it})e^{ijt}k(e^{it})dt = 0 \implies \int_{-\pi}^{\pi} G(e^{it})k(e^{it})dt = 0$ for each $G \in H^p$. But $\int_{-\pi}^{\pi} E_F(e^{it})e^{ijt}k(e^{it})dt = 0$ also implies the non positive frequencies of $E_F(e^{it})k(e^{it})$ is 0. This means $E_F(e^{it})k(e^{it}) = \sum_{j=1}^{\infty} a_n e^{ijt}$. Thus $E_F(e^{it})k(e^{it}) = e^{it}H(e^{it})$. By Holder inequality, we see $H \in H^1$.

I wonder in which sense $E_F \cdot \mathscr{P}$ is not dense in H^p and how the proof excludes this case.

By Holder inequality, $\int |R - E_K \cdot Q|^p \leq (\int |R - E_K \cdot Q|^{2p})^{\frac{1}{2}}(\int 1)^{\frac{1}{2}}$. Thus $\|R - E_K \cdot Q\|_{H^{2p}} < \epsilon$ implies $\|R - E_K \cdot Q\|_{H^p} < \epsilon$

1.8 The Helson-Szego Theorem

In this section, we no longer focus on Lebesgue measure. We will reexamine some results for boundedness of conjugate function on Borel measure. The main result we give is the Helson-Szego theorem. The Helson-Szego theorem is a characterization of those positive (Borel) measure μ on $[-\pi, \pi]$ for which the conjugate function operator is bounded in $L^2(\mu)$. More precisely, we answer the problem for what positive measure the following inequality holds:

$$\int_{-\pi}^{\pi} |\tilde{f}(t)|^2 d\mu(t) \leq C \int_{-\pi}^{\pi} |f(t)|^2 \, d\mu(t)$$

where $f(t) = \sum_{j=-n}^{n} a_j e^{ijt}$ and $\tilde{f}(t) = -i \sum_{j=-n}^{n} (\operatorname{sgn} j) a_j e^{ijt}$. $f(t) + i\tilde{f}(t)$ only left non negative frequency part. We will call such measures Helson-Szego measures.

1.8.1 Distance from constant function 1 to space $\mathscr{P}(0)$

Remark 1.8.1.1 (notes on proof of theorem 8.2 in book)**.** To proof $1 \in \overline{\mathscr{P}(0)}$, we can use Hahn-Banach theorem: if E is locally convex and F is subspace of E and $x_0 \in E$, then $x_0 \in \bar{F}$ if and only if for any $f \in E^*$, $f|_F = 0 \implies f(x_0) = 0$

μ and ν are mutually singular if there are disjoint subsets A and B in $\mathscr{M}$ s.t. $\nu(E) = \nu(A \cap E)$ and $\mu(E) = \mu(B \cap E)$ for all $E \in \mathscr{M}$. ν is absolutely continuous w.r.t. μ if $\nu(E) = 0$ whenever $E \in \mathscr{M}$ and $\mu(E) = 0$ (section 4.2 in Chapter 6 of Stein's Real Analysis).

Thus if μ and ν are mutually singular and ν is absolutely continuous w.r.t. μ, then for all $E \in \mathscr{M}$, $\mu(A \cap E) = \mu(A \cap B \cap E) = 0$ and hence $\nu(E) = \nu(A \cap E) = 0$. Therefore ν is identically zero.

The following lemma is useful in specifying Helson-Szego measure. It states there is a holomorphic function continuous on boundary can separate closed Lebesgue measure 0 set E and set $\bar{D} \setminus E$. Using this lemma, we can narrow our candidates non negative measure to absolutely continuous measure. All Helson-Szego measure $d\mu$ can be written as $d\mu = w(t)dt$.

Lemma 1.8.1.1 (lemma 8.3 in book)**.** *Let E be a closed subset of T having Lebesgue measure 0. Then, there exists a function $F \in \mathscr{A}$ (that is: F is holomorphic in D and continuous on $\bar{D}$) such that $F(z) = 1$ for every $z \in E$ and $|F(z)| < 1$ for every $z \in \bar{D} \setminus E$.*

Remark 1.8.1.2 (notes on proof of theorem 8.4 in book)**.** In theorem 8.4 in book we assumes $w \geq 0$ but in proof this condition is never used. Suppose μ is a Borel measure and is finite on all finite radius balls. Then for any Borel set E and any $\epsilon > 0$, there are an open set O and a closed set F such that $E \subset O$ and $\mu(O - E) \leq \epsilon$ while $F \subset E$ and $\mu(E - F) \leq \epsilon$ (proposition 1.3 in Chapter 6 of Stein's Real Analysis).

By above statement, inequality (8.6) in book can be hold.

The following theorem gives a precise formula for the distance we are looking for:

Theorem 1.8.1.2 (theorem 8.7 in book)**.** *For $w \geq 0$, $w \in L^1$:*

$$\inf_{P \in \mathscr{P}(0)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|1 - P(e^{it})\right|^p w(t)dt = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log w(t)dt\right), 1 \leq p < \infty$$

1.8.2 The Helson-Szego theorem

By theorem 1.8.1.2, if $\log w(t) = -\infty$, there is a sequence of $(P_j) \subset \mathscr{P}(0)$, s.t. $\int_{-\pi}^{\pi} \left|1 - P_j(e^{it})\right|^2 w(t)dt \to 0$. Since the conjugate function of $1 - P_j(e^{it})$

is $iP_j(e^{it})$ and μ is Helson-Szego measure, we have $\int_{-\pi}^{\pi} \left|P_j(e^{it})\right|^2 w(t)dt \to 0$. By $1 \leq \left|P_j(e^{it})\right|^2 + \left|1 - P_j(e^{it})\right|^2$, we show $\int_{-\pi}^{\pi} 1w(t)dt = 0$. We conclude $\log w(t) = -\infty$ implies μ is a trivial measure. Thus we can only focus on Helson-Szego measure μ which $d\mu = w(t)dt$, $w(t) \in L^1$ and $\log w(t) \in L^1$.

Given $f(t) = \sum_j a_j e^{ijt}$, operator A sends $f(t)$ to $Af(t) = \sum_{j>0} a_j e^{ijt}$. By easy calculation we have following identities:

$$\tilde{f} = -i(Af - \overline{A\bar{f}}) \; and \; Af = \frac{-\tilde{\tilde{f}} + i\tilde{f}}{2}$$

> **Remark 1.8.2.1** (proof of lemma 8.10)**.** This proof is from P165 in Paul Koosis' *Introduction to Hp spaces*.
>
> By $\tilde{f} = -i(Af - \overline{A\bar{f}})$, if $\|Af\|_w \leq C\|f\|_w$, then
>
> $$\|\tilde{f}\|_w \leq \|Af\|_w + \|\overline{A\bar{f}}\|_w = \|Af\|_w + \|A\bar{f}\|_w \leq C\|f\|_w + C\|\bar{f}\|_w \leq 2C\|f\|_w$$
>
> By $Af = \frac{-\tilde{\tilde{f}}+i\tilde{f}}{2}$, if $\|\tilde{f}\|_w \leq C\|f\|_w$, then
>
> $$\|Af\|_w \leq \frac{1}{2}\|\tilde{\tilde{f}}\|_w + \frac{1}{2}\|\tilde{f}\|_w \leq \frac{C}{2}\|\tilde{f}\|_w + \frac{C}{2}\|f\|_w \leq \frac{C^2}{2}\|f\|_w + \frac{C}{2}\|f\|_w \leq \frac{C^2 + C}{2}\|f\|_w$$
>
> Thus conjugate operator bounded in L^2 is equivalent to operator A bounded in L^2.

We now talk about how Helson-Szego theorem reached. Every lemmas or theorems contain different ideas in the path to Helson-Szego theorem. Lemma 8.10 in book shows we can study boundedness of operator A instead of boundedness of conjugate operator. In theorem 8.11 in book, if we write $f(t) = P(e^{it}) + e^{it}\overline{Q(e^{it})}$ for some $P, Q \in \mathscr{P}(0)$, we can study the boundedness in space $\mathscr{P}(0)$ but we need an additional restriction on some constants. Later we state theorem 8.12 in book, we further shows the restriction on the constant in theorem 8.11 in book becomes the restriction on the norm in quotient space $(H^1(0))^* = L^\infty/H^\infty$. The norm on quotient space is naturally the distance from represent element to space H^∞.

Now we state the Helson-Szego theorem:

Theorem 1.8.2.1 (Helson-Szego Theorem, theorem 8.14 in book)**.** *w is a Helson-Szego weight if and only if $w(t) = e^{u(t)+\tilde{v}(t)}$ with u and v real, bounded and $\|v\|_\infty < \frac{\pi}{2}$*

> **Remark 1.8.2.2** (notes on proof of Helson-Szego theorem)**.** I also refer another proof of Helson-Szego theorem in P167 in Paul Koosis' *Introduction to Hp spaces*.
>
> Since u is bounded, we have $0 < C_1 < e^u < C_2 < \infty$. If $w(t) = e^{\tilde{v}(t)}$ is

a Helson-Szego measure, then:

$$\int_{-\pi}^{\pi} |\tilde{f}(t)|^2 e^{u+\tilde{v}} dt \le \int_{-\pi}^{\pi} |\tilde{f}(t)|^2 C_2 e^{\tilde{v}} dt \le C \int_{-\pi}^{\pi} |f(t)|^2 C_2 e^{\tilde{v}} dt \le C \int_{-\pi}^{\pi} |f(t)|^2 \frac{C_2}{C_1} e^{u+\tilde{v}} dt$$

Thus we just need to consider the case $w(t) = e^{\tilde{v}(t)}$.

Recall conjugate function is real. Suppose $\Phi(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\frac{1}{2}\tilde{v}(t)dt\right)$. We have $|\Phi(z)|^2 = \exp(P(\tilde{v}))$ and $\Phi(z)^2 = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\tilde{v}(t)dt\right)$. Now we compute the angle $\frac{\Phi^2}{|\Phi|^2}$:

$$\begin{aligned}
\frac{\Phi(z)^2}{|\Phi(z)|^2} &= \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{e^{it}+z}{e^{it}-z} - P_r(\theta-t)\right)\tilde{v}(t)dt\right) \\
&= \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} iQ_r(\theta-t)\tilde{v}(t)dt\right) \\
&= \exp\left(i\tilde{\tilde{v}}(re^{i\theta})\right)
\end{aligned}$$

If $v(t) = \sum_j a_j e^{ijt}$, then $\tilde{v}(t) = -i\sum_j (\operatorname{sgn} j) a_j e^{ijt}$ and $\tilde{\tilde{v}}(t) = -\sum_{j\neq 0} a_j e^{ijt} = -v(t) + a_0 = -v(t) + v(0)$. Thus $\frac{\Phi^2}{|\Phi|^2} = \exp(-iP(v) + iv(0))$.

Now we have

$$\begin{aligned}
\Phi(z)^2 &= |\Phi(z)|^2 \frac{\Phi(z)^2}{|\Phi(z)|^2} \\
&= \exp(P(\tilde{v}))\exp(-iP(v) + iv(0)) \\
&= \exp(P(\tilde{v} - iv))\exp iv(0))
\end{aligned}$$

Thus we can see the real number τ in book is $v(0)$. τ is irrelevant since $\inf_{g\in H^\infty} \|e^{-i\tau}e^{iv(t)} - g\|_\infty = \inf_{g\in H^\infty} \|e^{iv(t)} - g\|_\infty$. Notice that constants are in H^∞. Thus $\left|e^{iv(t)} - sin\epsilon\right| < 1$ implies $d(e^{iv}, H^\infty) < 1$.

The inequality $\left|e^{i\phi(t)} - H(e^{it})\right| \le \alpha$ in book should be $\left|e^{i\phi(t)} - H(e^{it})\right| \le \alpha_1$ for some α_1 with $\inf_{g\in H^\infty} \|e^{iv(t)} - g\|_\infty = \alpha < \alpha_1 < 1$.

An outer function F has representation:

$$F(z) = C\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}w(t)dt\right)$$

where $C = I_F(z)$ is constant. Decompose $\frac{e^{it}+z}{e^{it}-z} = P_r(\theta-t) + iQ_r(\theta-t)$, we have:

$$F(z) = C\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(P_r(\theta-t) + iQ_r(\theta-t))w(t)dt\right) = C\exp\left(u(re^{i\theta}) + iv(re^{i\theta})\right)$$

where u is a real harmonic function and v its conjugate. Let $r \to 1$ and notice $|C| \to 1$. Thus we have $F(e^{it}) = e^{i\tau}e^{u+iv}$.

We conclude that an outer function has representation $e^{i\tau}e^{u+iv}$ where τ is a real number. u is a real harmonic function and v its conjugate.

Chapter 2

Calderon-Zygmund Theory

2.1 Singular Integral Operators

In this section we show how Calderon-Zygmund decomposition is used in estimation of the convolution operator T, which defined on Schwartz space $\mathscr{S}$, $T(f) = K * f(x)$. The definition of singular integral operator given in book is different from others like that in Stein's *Singular Integrals and Differentiability Properties of Functions.*

Definition 2.1.0.1. *Given a tempered distribution K, the convolution operator*

$$Tf(x) = K * f(x) \quad (f \in \mathscr{S}(\mathbb{R}^n))$$

is called a singular integral operator if the following two conditions are satisfied:

1. $\hat{K} \in L^\infty(\mathbb{R}^n)$
2. *K coincides in $\mathbb{R}^n \setminus \{0\}$ with a locally integrable function $K(x)$ satisfying Hormander's condition:*

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \le B_k$$

Later we will see these two conditions guarantee the boundedness of $T(f)$.

2.1.1 Hormander's condition

If a locally integrable function $k(x)$ satisfying Hormander's condition and following two conditions:

$$\int_{r<|x|<2r} |k(x)|\, dx \le C_1$$

$$\begin{cases} \left| \int_{r<|x|<R} k(x)dx \right| \leq C_2 \\ \lim_{r\to 0} \int_{r<|x|<1} k(x)dx \ exists \end{cases}$$

Then $Tf(x) = \lim_{\epsilon\to 0} \int_{|y|>\epsilon} k(x-y)f(y)dy$ is a singular integrable operator (Proposition 5.5 in book). In other words, $\|\hat{k}\|_\infty < \infty$

If we let a locally integrable function $k(x)$ be $\Omega(x)\,|x|^{-n}$ with Ω homogeneous of degree 0 ($\Omega(rx) = \Omega(x)$ for $r \neq 0$). Then the following continuity condition on Ω ensures Hormander condition holds:

$$\int_0^1 w_1(\Omega; t)\frac{dt}{t} < \infty$$

where

$$w_1(\Omega;t) = \sup_{h\in\mathbb{R}^n, |h|\leq t} \int_{|x'|=1} |\Omega(x'+h) - \Omega(x')|\, d\sigma(x')$$

Then $Tf(x) = \lim_{\epsilon\to 0} \int_{|y|>\epsilon} \Omega(y)\,|y|^{-n}\, f(x-y)dy$ is a singular integrable operator. The condition $\|\hat{k}\|_\infty < \infty$ is ensured by explict formula for $\hat{k} = m(\xi)$ if Ω is odd:

$$m(\xi) = -\frac{\pi i}{2}\int_{|x'|=1} \Omega(x')\operatorname{sign}(x'\cdot\xi)d\sigma(x')$$

where $\operatorname{sign} x = \frac{x}{|x|}$ (Proposition 5.6 in book). If Ω is not necessarily odd then the formula for $m(\xi)$ is (refer section 4.2 in chapter 2 in Stein's *Singular Integrals and Differentiability Properties of Functions*):

$$m(\xi) = \frac{\pi i}{2}\int_{|x'|=1} (\operatorname{sign}(x'\cdot\xi) + \log(\frac{1}{|x'\cdot\xi|}))\Omega(x')d\sigma(x')$$

Remark 2.1.1.1. Equation

$$\frac{\pi}{2}c_n \int_{|x'|=1} (x'\cdot h)\operatorname{sign}(x'\cdot\xi)d\sigma(x') = \frac{\xi\cdot h}{|\xi|}$$

is by riesz representation for Hilbert space. If we fix ξ with $|\xi| = 1$, then left hand side is a linear function of h, say $\ell(h)$. Since

$$|\ell(h)| \leq \frac{\pi}{2}c_n \int_{|x'|=1} |x'|\,|h|\,d\sigma(x') = \frac{\pi}{2}c_n\int_{|x'|=1} |h|\,d\sigma(x') = |h|$$

, $\ell(h)$ is continuous linear functional with $\|\ell\| \leq 1$. Thus $\ell(h) = (h, g)$ for some $g \in \mathbb{R}^n$ with $|g| = \|\ell\| \leq 1$. Notice

$$\ell(\xi) = \frac{\pi}{2}c_n \int_{|x'|=1} (x'\cdot\xi)\operatorname{sign}(x'\cdot\xi)d\sigma(x') = \frac{\pi}{2}c_n\int_{|x'|=1} |x'\cdot\xi|\,d\sigma(x') = 1$$

Thus $\|\ell\| = 1$ with $|g| = 1$ and $1 = \ell(\xi) = (\xi, g) \leq |\xi|\,|g| = 1$. The inequality holds if and only if $g = k\xi$ for some $k \in \mathbb{R}$. Thus $g = \xi$ and $\ell(h) = (h, \xi)$

2.1.2 Estimation of singular integral operator

The following theorem is the main result concerning singular integral operators:

Theorem 2.1.2.1 (theorem 5.7 in book)**.** *Every singular integral operator satisfies the inequalities*

$$\|Tf\|_p \leq C_p \|f\|_p \quad (f \in L^2 \cap L^p; 1 < p < \infty) \tag{2.1.1}$$

$$|\{x : |Tf(x)| > t\}| \leq \frac{C_1}{t}\|f\|_1 \quad (f \in L^2 \cap L^1) \tag{2.1.2}$$

$$\|Tf\|_{\text{BMO}} \leq C_\infty \|f\|_\infty \quad (f \in L^2 \cap L^\infty) \tag{2.1.3}$$

where C_p, $1 \leq p \leq \infty$, depends only on p, n and on the constants $\|\hat{K}\|_\infty$ and B_K of the kernel.

To prove this theorem, we only need to prove three case $p = 1$, $p = 2$, and $p = \infty$ and use Marcinkiewicz interpolation theorem in section 2 to derive $1 < p < 2$ case. And use another interpolation theorem 3.7 in section 3 to derive $2 < p < \infty$ case.

Remark 2.1.2.1 (notes on proof of inequality (2.1.2))**.** The key idea to prove the case $p = 1$ is Calderon-Zygmund decomposition. First we can split the space into a family of non-overlapping cubes C_t and a set $\mathbb{R}^n \setminus \cup_{Q \in C_t} Q$. These sets satisfy:

1. For every $Q \in C_t$, $t < \frac{1}{|Q|}\int_{|Q|} |f(x)|\,dx \leq 2^n t$
2. For a.e. $X \notin \cup_{Q \in C_t} Q$, $|f(x)| \leq t$

Let $\Omega = \cup_{Q \in C_t} Q$. For any function $f \in L^1$, the Calderon-Zygmund decomposition of $f(x) = g(x) + b(x)$ is:

$$g(x) = \sum_j \left(\frac{1}{|Q_j|}\int_{Q_j} f(t)dt\right)\chi_{Q_j}(x) + f(x)\chi_{\mathbb{R}^n \setminus \Omega}(x)$$

and

$$b(x) = f(x) - g(x) = \sum_j b_j(x) = \sum_j \left(f(x) - \frac{1}{|Q_j|}\int_{Q_j} f(t)dt\right)\chi_{Q_j}(x)$$

Generally, to estimate the size of the set $\{x : |f(x)| > c\}$, we can use

$(\frac{|f(x)|}{c})^p > 1$ in that set and integral the index function of that set on $\mathbb{R}^n$:

$$|\{x : |f(x)| > c\}| = \int_{|\{x:|f(x)|>c\}|} 1dt \le \int_{|\{x:|f(x)|>c\}|} \left(\frac{|f(x)|}{c}\right)^p dt \le \frac{1}{c^p} \int |f(x)|^p \, dt$$

This inequality sometimes called Markov inequality.

Remark 2.1.2.2 (notes on proof of lemma 5.11 in book)**.** In proof of theorem 5.20 in book, it says $f \in L_c^\infty$ was only imposed to ensure the $I_\epsilon = \int_{|y|>\epsilon} K(-y)f(y)dy$ exists. $M_p f(x)$ increases as p increases by Holder inequality. let $r < q$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, we have

$$\begin{aligned}(\frac{1}{|Q|} \int_Q |f(t)|^r \, dt)^{\frac{1}{r}} \le \frac{1}{|Q|^{\frac{1}{r}}} (\int_Q |f(t)|^p \, dt)^{\frac{1}{p}} (\int_Q 1^q dt)^{\frac{1}{q}} = \frac{|Q|^{\frac{1}{q}}}{|Q|^{\frac{1}{r}}} (\int_Q |f(t)|^p \, dt)^{\frac{1}{p}} \\ = \frac{1}{|Q|^{\frac{1}{p}}} (\int_Q |f(t)|^p \, dt)^{\frac{1}{p}} = (\frac{1}{|Q|} \int_Q |f(t)|^p \, dt)^{\frac{1}{p}}\end{aligned}$$

Remark 2.1.2.3 (notes on proof of inequality (2.1.3))**.** We have $(Tf)^{\#}(0) = \sup_{0 \in Q} \frac{1}{|Q|} \int_Q |Tf(y) - (Tf)_Q| \, dy$ with $(Tf)_Q = \frac{1}{|Q|} \int_Q Tf(x)dx$

We know $(Tf)^{\#}(x) \cong \sup_{x \in Q} \inf_{a \in \mathbb{R}} \frac{1}{|Q|} \int_Q |Tf(y) - a| \, dy$, where $\cong$ is used to indicate that each side is bounded by the other times an absolute constant (refer equation (3.1) in section 3, chapter 2 in book).

Observe that the cube $[-r, r]^n$ containing 0 is almost everywhere contained in the open ball $B(0, n^{\frac{1}{2}} r)$. Let d be the side length of Q. Let Q^2 be the cube with origin as center and with side length 2 times that of Q. $0 \in Q$ implies $Q \subset Q^2$. Thus

$$\begin{aligned}(Tf)^{\#}(0) &\le C \sup_{0 \in Q} \inf_{a \in \mathbb{R}} \frac{1}{|Q|} \int_Q |Tf(y) - a| \, dy \\ &\le C \sup_{r>0} \inf_{a \in \mathbb{R}} \frac{1}{2^n r^n} \int_{B(0, n^{\frac{1}{2}} r)} |Tf(y) - a| \, dy \\ &\le C \sup_{r>0} \frac{1}{2^n r^n} \int_{|y| \le n^{\frac{1}{2}} r} |Tf(y) - I_\epsilon| \, dy \\ &= C' \sup_{\epsilon>0} \epsilon^{-n} \int_{|y| \le \frac{\epsilon}{2}} |Tf(y) - I_\epsilon| \, dy\end{aligned}$$

.

To estimate $\epsilon^{-n} \int_{|x|<\frac{\epsilon}{2}} |Tf(x) - I_\epsilon| \, dx$, we use lemma 7.11 in book and

Hormander's condition:

$$
\begin{aligned}
&\epsilon^{-n}\int_{|x|<\frac{\epsilon}{2}}|Tf(x)-I_\epsilon|\,dx\\
\leq&C_pM_pf(0)+\epsilon^{-n}\iint_{2|x|<\epsilon<|y|}|K(x-y)-K(-y)|\,|f(y)|\,dxdy\\
\leq&C_p\|f\|_\infty+\|f\|_\infty\epsilon^{-n}\iint_{2|x|<\epsilon<|y|}|K(x-y)-K(-y)|\,dxdy\\
\leq&C_p\|f\|_\infty+\|f\|_\infty\epsilon^{-n}\int_{2|x|<\epsilon}(\int_{2|x|<|y|}|K(x-y)-K(-y)|\,dy)dx\\
=&C_p\|f\|_\infty+\|f\|_\infty\epsilon^{-n}\int_{2|x|<\epsilon}(\int_{2|x|<|y|}|K(y-x)-K(y)|\,dy)dx\\
\leq&C_p\|f\|_\infty+\|f\|_\infty\epsilon^{-n}\int_{2|x|<\epsilon}B_kdx\\
=&C_p\|f\|_\infty+B_k\|f\|_\infty
\end{aligned}
$$

We can change the sign of x and y in fifth line since the region is symmetric.

Remark 2.1.2.4. I don't know why $\frac{1}{\pi}\log\left|\frac{x-a}{x-b}\right|$, with $a<b$ is in $weak-L^1\cap\mathrm{BMO}$.

2.1.3 Restriction and extension of singular integral operator

To understand the behavior of singular integral operators in L^1 and L^∞, we ask if there is a subspace of L^1, on which the singular integral operator is strong type $(1,1)$, and if the singular integral operator can "extend" from $L^2\cap L^\infty$ to L^∞.

For the first question, we introduce the Banach space H^1_{at} (atomic H^1). More study of H^1_{at} is in Chapter 3. You can also refer section 5 and 6 in chapter 2 in Stein's *Functional Analysis*.

For the second question, notice $L^p\cap L^\infty$ is not dense in L^∞. The extension is not a trivial step. However, the function f and $f+C$ with constant C behavior the same in space BMO. Thus we introduce our definition of new T on L^∞.

Proposition 2.1.3.1 (proposition 5.15 in book)**.** *Given a singular integral operator T with kernel K, for each $f\in L^\infty$ we define:*

$$Tf(x)=\lim_{j\to\infty}(T(f\chi_{B_j})(x)-\int_{1<|y|<j}K(-y)f(y)dy)$$

The sequence to the right converges locally in L^1 and also pointwise a.e., and the extended operator T satisfies:

$$\|Tf\|_{\mathrm{BMO}}\leq C_\infty\|f\|_\infty\quad(f\in L^\infty)\tag{2.1.4}$$

Remark 2.1.3.1 (notes on proof of proposition 5.15 in book)**.** It is not clear that $g_j(x)$ converges pointwise. $g_j(x)$ in $L^1(F)$ since $g_l(x) \in L^1(F)$ and the integral is bounded by $B_K \|f\|_\infty$ and F is compact.

2.1.4 More precise estimation for more regular kernel

If the kernel is more regular, the estimation of operator T can be replaced by maximal function. More precisely:

Definition 2.1.4.1 (definition 5.17 in book)**.** *A singular integral operator $Tf = K * f$ is called regular if its kernel satisfies the following two conditions:*

1. $|K(x)| \leq B\,|x|^{-n}$ *for* $x \neq 0$
2. $|K(x-y) - K(x)| \leq B\,|y|\,|x|^{-n-1}$ *for* $|x| > 2\,|y| > 0$

Let $T_\epsilon f(x) = \int_{|y|>\epsilon} K(y) f(x-y)dy$ and $T^* f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|$. The estimation is the following:

Theorem 2.1.4.2 (theorem 5.20 in book)**.** *If T is a regular singular integral operator and $f \in L^p$, $1 \leq p < \infty$, then the following inequalities are verified:*

1. $(Tf)^\#(x) \leq C_q M_q f(x) \quad (q > 1)$
2. $T^* f(x) \leq C_q M_q f(x) + CM(Tf)(x) \quad (q > 1)$
3. $\|T^* f\|_p \leq C_p \|f\|_p \quad (1 < p < \infty)$
4. $|\{x : T^* f(x) > t\}| \leq C t^{-1} \|f\|_1 \quad (t > 0)$

Remark 2.1.4.1 (notes on proof of theorem 5.20 in book)**.** 3 is a trivial consequence of 2 and the L^p inequalities for the operators M and T. The L^p inequality for operator M_q is similar with operator M. By 2, $\|T^* f\|_p \leq \|C_q M_q f\|_p + \|CM(Tf)\|_p$, and $\|CM(Tf)\|_p \leq C' \|Tf\|_p \leq C'' \|f\|_p$.

Now we prove the L^p inequality for operator M_q. We choose q with

$1 < q < p$:

$$\begin{aligned}
\int_{\mathbb{R}^n} (M_q f(x))^p dx &= p \int_0^\infty t^{p-1} \left|\{x : M_q f(x) > t\}\right| dt \\
&= p \int_0^\infty t^{p-1} \left|\{x : \sup((\frac{1}{|Q|}\int_Q |f(t)|^q\, dt)^{\frac{1}{q}}) > t\}\right| dt \\
&= p \int_0^\infty t^{p-1} \left|\{x : \sup((\frac{1}{|Q|}\int_Q |f(t)|^q\, dt)) > t^q\}\right| dt \\
&\leq Cp \int_0^\infty t^{p-1} \int_{\{x:|f(x)|^q > \frac{t^q}{2}\}} \frac{|f(x)|^q}{t^q} dx dt \\
&= Cp \int_0^\infty \int_{\mathbb{R}^n} t^{p-1} \frac{|f(x)|^q}{t^q} \chi_{\{x:|f(x)|^q > \frac{t^q}{2}\}} dx dt \\
&= Cp \int_{\mathbb{R}^n} \int_0^{|f(x)| 2^{\frac{1}{q}}} t^{p-1-q} |f(x)|^q\, dx dt \\
&= \frac{Cp}{p-q} \int_{\mathbb{R}^n} (|f(x)|\, 2^{\frac{1}{q}})^{p-q} |f(x)|^q\, dt \\
&= \frac{Cp 2^{\frac{p-q}{q}}}{p-q} \int_{\mathbb{R}^n} |f(x)|^p\, dt
\end{aligned}$$

Thus $\|T^* f\|_p \leq \|C_q M_q f\|_p + \|CM(Tf)\|_p \leq C\|f\|_p$

Remark 2.1.4.2 (notes on corollary 5.22 in book)**.** Notice (5.19) implies (b) in definition 5.1. By easy computation (5.18) implies (5.3). By proposition 5.5 in book, 5.1(b), (5.3) and (5.4) implies that $T = k * f$ is a singular integral operator. It is not clear how to pass $\mathscr{S}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

2.1.5 Proof of proposition 5.5 and 5.6 in book

Remark 2.1.5.1 (notes on proof of proposition 5.5 in book)**.** Let $A = \{x : \epsilon < |x| < R\}$, $B = \{x : \epsilon < |x - y| < R\}$, $C = \{x : |x - y| \leq \epsilon\}$ and

$B = \{x : |x - y| \geq R\}$

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx \\
&= \int_{A\cap B} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx + \int_{\mathbb{R}^n\setminus(A\cap B)} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx \\
&= \int_{A\cap B} |k(x-y) - k(x)|\, dx + \int_{A^c\cup B^c} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx \\
&\leq \int_{A} |k(x-y) - k(x)|\, dx + \int_{A^c} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx + \int_{B^c} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx \\
&= \int_{A} |k(x-y) - k(x)|\, dx + \int_{A^c} \left|k_\epsilon^R(x-y)\right| dx + \int_{B^c} \left|k_\epsilon^R(x)\right| dx \\
&= \int_{A} |k(x-y) - k(x)|\, dx + \int_{A^c\cap B} |k(x-y)|\, dx + \int_{B^c\cap A} |k(x)|\, dx \\
&= \int_{A} |k(x-y) - k(x)|\, dx + \int_{A^c\cap B} |k(x-y)|\, dx + \int_{C\cap A} |k(x)|\, dx + \int_{D\cap A} |k(x)|\, dx
\end{aligned}
$$

If $x \in C$, $|x| - |y| \leq |x - y| \leq \epsilon$, then $C \subset \{x : |x| \leq \epsilon + |y|\}$. If $x \in D$, $|x| + |y| \geq |x - y| \geq R$, then $D \subset \{x : R - |y| \leq |x| \leq R\}$. Since R is large enough and ϵ is small enough, we have $C \cap A \subset \{x : \epsilon < |x| \leq \epsilon + |y|\}$ and $D \cap A \subset \{x : R - |y| \leq |x| \leq R\}$. Thus

$$\int_{C\cap A} |k(x)|\, dx + \int_{D\cap A} |k(x)|\, dx \leq \int_{\epsilon<|x|\leq\epsilon+|y|} |k(x)|\, dx + \int_{R-|y|\leq|x|<R} |k(x)|\, dx$$

Let $x - y = t$,

$$\int_{A^c\cap B} |k(x-y)|\, dx = \int_{\epsilon<|t|<R \text{ } and \text{ } |t+y|\leq\epsilon} |k(t)|\, dt + \int_{\epsilon<|t|<R \text{ } and \text{ } |t+y|\geq R} |k(t)|\, dt$$

Use almost the same argument, we can show:

$$\int_{A^c\cap B} |k(x-y)|\, dx \leq \int_{\epsilon<|x|\leq\epsilon+|y|} |k(x)|\, dx + \int_{R-|y|\leq|x|<R} |k(x)|\, dx$$

Thus

$$
\begin{aligned}
\int_{\mathbb{R}^n} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx \leq\ & 2\int_{\epsilon<|x|\leq\epsilon+|y|} |k(x)|\, dx + 2\int_{R-|y|\leq|x|<R} |k(x)|\, dx \\
&+ \int_A |k(x-y) - k(x)|\, dx
\end{aligned}
$$

$\int_{r<|x|<2r} |k(x)|\, dx \le C_1$ implies $\int_{|x|<r} |k(x)|\, |x|\, dx \le 4C_1 r$:

$$\begin{aligned}\int_{|x|<r} |k(x)|\, |x|\, dx =& \sum_{j=0}^{\infty} \int_{\frac{r}{2^{j+1}}\,|x|<\frac{r}{2^j}} |k(x)|\, |x|\, dx \le \sum_{j=0}^{\infty} \int_{\frac{r}{2^{j+1}}\,|x|<\frac{r}{2^j}} |k(x)|\, \frac{r}{2^j} dx \\ =& \sum_{j=0}^{\infty} C_1 \frac{r}{2^j} = 2C_1 r\end{aligned}$$

$\int_{|x|<r} |k(x)|\, |x|\, dx \le 4C_1 r$ implies $\int_{r<|x|<2r} |k(x)|\, dx \le C_1$:

$$\begin{aligned}\int_{r<|x|<2r} |k(x)|\, dx \le& \int_{r<|x|<2r} |k(x)|\, \frac{|x|}{r} dx = \frac{1}{r} \int_{r<|x|<2r} |k(x)|\, |x|\, dx \\ \le& \frac{1}{r} \int_{|x|<2r} |k(x)|\, |x|\, dx \le \frac{1}{r} 8C_1 r = 8C_1\end{aligned}$$

Thus $\int_{r<|x|<2r} |k(x)|\, dx \le C_1$ and $\int_{|x|<r} |k(x)|\, |x|\, dx \le 4C_1 r$ are equivalent.

Remark 2.1.5.2 (notes on proof of proposition 5.6 in book)**.** Let $tx' = x - y$, we have:

$$\begin{aligned}\int_{|x|>2|y|} \frac{|\Omega(x-y) - \Omega(x)|}{|x-y|^n} dx =& \int_{|tx'+y|>2|y|} \frac{|\Omega(tx') - \Omega(tx'+y)|}{t^n} d(tx'+y) \\ \le& \int_{|tx'|+|y|>2|y|} \frac{|\Omega(tx') - \Omega(tx'+y)|}{t^n} d(tx') \\ =& \iint_{t>|y|} \frac{|\Omega(tx') - \Omega(tx'+y)|}{t^n} t^{n-1} dt d\sigma(x') \\ =& \int_{t>|y|} \int_{|x'|=1} |\Omega(tx') - \Omega(tx'+y)|\, d\sigma(x') \frac{dt}{t}\end{aligned}$$

By mean value theorem, we have $|x-y|^{-n} - |x|^{-n} = n\frac{y\cdot\xi}{|\xi|^{n+2}}$ with $\xi = (1-c)(x-y) + cx = x - (1-c)y$. Notice $|\xi| \ge |x| - (1-c)\, |y| \ge |x| - |y|$. Since $|y| \le \frac{|x|}{2}$, $|\xi| \ge \frac{|x|}{2}$. Thus

$$\left| |x-y|^{-n} - |x|^{-n} \right| \le n \frac{|y|\, |\xi|}{|\xi|^{n+2}} \le n 2^{n+1} \frac{|y|}{|x|^{n+1}}$$

2.2 Multipliers

In this section, we keep talk about the convergence property and norm estimate of convolution operator. In last section, we know some convolution operator can not be defined as usual Lebesgue integral but a limit process called singular integral. The stimulation of studying the convolution operator is that the convolution operator is equivalent to translation invariant operator.

Suppose $Tg = \int f(x-y)g(y)dy$ is a convolution operator. Then by Fourier transform, we have $(Tf)^\wedge = \hat{f}\hat{g}$. The concept of multipliers give us another approach to study the convolution operator T:

Definition 2.2.0.1. *Let $1 \le p < \infty$. Given $m \in L^\infty$, we say that m is a (Fourier) multiplier for L^p if the operator T_m, initially defined in L^2 by the relation:*

$$(T_m f)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$$

satisfies the inequality

$$\|T_m f\|_p \le C\|f\|_p \quad (f \in L^2 \cap L^p)$$

> **Remark 2.2.0.1.** Given a sequence of kernels $(k_N)_N$ in L^1, if $T_N f = k_N * f$ are uniformly bounded from $\mathscr{S}$ to L^p, it is easy to see $(T_N)_N$ is a Cauchy sequence of in bounded linear functional space $\mathscr{B}(\mathscr{S}, L^p)$ ($\mathscr{B}(\mathscr{S}, L^p)$ is complete by assigning L^p norm on $\mathscr{S}$ and completeness of L^p), then the limit of T_N exists. Indeed, under hypothesis, proposition 5.5 in book gives $T_N f$ is a singular integral for $f \in \mathscr{S}(\mathbb{R}^n)$ and T_N is uniformly bounded. Since $\mathscr{S}(\mathbb{R}^n)$ is dense in L^p, T can be uniquely extends to L^p.
>
> Notice that the limit process agrees the definition of singular integral.

2.2.1 Hormander-Mihlin multiplier theorem

We know every $m \in L^\infty$ is a multiplier for L^2 by Plancherel's theorem. The Hormander-Mihlin multiplier theorem gives a sufficient condition for $m \in L^\infty$ is a multiplier for L^p.

Theorem 2.2.1.1 (Hormander-Mihlin multiplier theorem)**.** *Let $a = [\frac{n}{2}] + 1$ be the first integer greater than $\frac{n}{2}$. If $m \in L^\infty(\mathbb{R}^n)$ is of class C^a outside the origin and satisfies:*

$$(R^{-n}\int_{R<|x|<2R} |D^\alpha m(\xi)|^2 \, d\xi)^{\frac{1}{2}} \le CR^{-|\alpha|} \quad (0 < R < \infty) \tag{2.2.1}$$

for every multi-index α such that $|\alpha| \le a$, then m is a multiplier for L^p, $1 < p < \infty$.

This theorem is an improvement of Mihlin multiplier theorem. Mihlin multiplier theorem supposes a stronger condition:

$$|D^\alpha m(\xi)| \le CR^{-|\alpha|} \quad (|\alpha| \le a) \tag{2.2.2}$$

In other words, Hormander weaken the decreasing speed as α increasing. The decreasing of uniform boundedness of $|D^\alpha m(\xi)|$ is weakened to the decreasing of L^2 norm of $|D^\alpha m(\xi)|$.

Inequality (2.2.2) is satisfied by every function $m(\xi)$ of class C^a outside the origin of degree ib. Indeed by chain rule: $\frac{\partial}{\partial x_i}(f_i(tx)) = f_i(tx)t$. And $\frac{\partial}{\partial x_i}(t^p f_i(x)) = t^p f_i(x) \leq C\,|\xi|^{-1}$. We have $f_i(tx)t = t^p f_i(x)$. Thus $|m_i(\xi)| = \left||\xi|^{ib-1}\, m_i(x')\right| = \frac{|m_i(x')|}{|\xi|}$ with $|x'| = 1$. By induction, $|D^\alpha m(\xi)| \leq \frac{|D^\alpha m(x')|}{\xi^{-\alpha}}$. Let

$$C = \max_{|\alpha|\leq a} \sup_{|x'|=1} |D^\alpha m(x')|$$

. We have $|D^\alpha m(\xi)| \leq C\,|\xi|^{-\alpha}$

There are two standard techniques in proof of the Hormander-Mihlin multiplier theorem. The first one is the smooth cutting of the multiplier into dyadic pieces.

Lemma 2.2.1.2 (Lemma 6.5 in book)**.** *There is a non negative function $\phi \in C^\infty$ supported in the spherical shell $\{\xi : \frac{1}{2}\,|\xi| < 2\}$ such that*

$$\sum_{j\in\mathbb{Z}} \phi(2^{-j}\xi) = 1 \quad (\xi \neq 0)$$

The second is an explicit formulation of the well known fact that the regularity of the multiplier is translated into control of the size of kernel.

Lemma 2.2.1.3 (Lemma 6.6 in book)**.** *Let $a = [\frac{n}{2}] + 1$, and let s be such that $a = \frac{n}{s} + \frac{1}{2}$ (so that $s \leq 2$). If $k \in L^2$ is such that $\hat{k}$ is of class C^a, then,*

$$\int_{|x|>t} |k(x)|\,dx \leq C_n t^{-\frac{1}{2}} \max_{|\alpha|=a} \|D^\alpha \hat{k}\|_s \quad (0 < t < \infty)$$

The details of proof of two lemmas and Hormander-Mihlin multiplier theorem is in book and the subsection 2.2.4.

2.2.2 More precise estimation for more regular multiplier

If we consider Mihlin multiplier theorem, the stronger condition (2.2.2) gives the smoothness of multipliers. And we have more precise estimation like Theorem 5.20 (i) in book.

Theorem 2.2.2.1 (theorem 6.10 in book)**.** *Let a be an integer such that $\frac{n}{2} < a \leq n$, and suppose that the multiplier $m(\xi)$ satisfies* (2.2.2) *for all $|\alpha| \leq a$. Then, for every $q > \frac{n}{a}$, the operator T_m satisfies*

$$(T_m f)^{\#}(x) \leq C_q M_q f(x) \quad (f \in \cup_{1<p<\infty} L^p)$$

2.2.3 Some properties of multipliers

Theorem 2.2.3.1. *m is a multiplier for L^p if and only if it is a multiplier for $L^{p'}$. And the norm of operator are identical.*

Proof. I want to use:

$$\begin{aligned}
\int T_m f(x)\overline{g(x)}dx &= \int (\int m(y)\hat{f}(y)e^{2\pi i x\cdot y}dy)\overline{g(x)}dx \\
&= \iint m(y)\hat{f}(y)e^{2\pi i x\cdot y}\overline{g(x)}dydx \\
&= \iint m(y)(\int f(z)e^{-2\pi i y\cdot z}dz)e^{2\pi i x\cdot y}\overline{g(x)}dydx \\
&= \iiint m(y)f(z)e^{-2\pi i y\cdot z}e^{2\pi i x\cdot y}\overline{g(x)}dzdydx \\
&= \iint m(y)f(z)e^{-2\pi i y\cdot z}\overline{(\int e^{-2\pi i x\cdot y}g(x)dx)}dzdy \\
&= \iint m(y)f(z)e^{-2\pi i y\cdot z}\overline{\hat{g}(y)}dzdy \\
&= \int f(z)\overline{(\int \overline{m(y)}\hat{g}(y)e^{2\pi i y\cdot z}dy)}dz \\
&= \int f(z)\overline{T_{\bar{m}}g(z)}dz
\end{aligned}$$

By dual of L^p and Hahn-Banach theorem,

$$\begin{aligned}
\|T_m\| &= \sup_{\|f\|_p\leq 1}\|T_m f\|_p \\
&= \sup_{\|f\|_p\leq 1}\sup_{\|g\|_q\leq 1}\int T_m f\bar{g} = \sup_{\|f\|_p\leq 1}\sup_{\|g\|_q\leq 1}\int f\overline{T_{\bar{m}}g} = \|T_{\bar{m}}\|
\end{aligned}$$

Thus m is a multiplier for L^p if and only if $\bar{m}$ is a multiplier for $L^{p'}$. And the norm of operator are identical. And $\bar{m}$ is a multiplier for $L^{p'}$ implies m is a multiplier for $L^{p'}$. And their norms are equal. $\square$

Theorem 2.2.3.2. *If m is a multiplier for L^p, then m is a multiplier for L^q with $p' < q < p$ or $p < q < p'$.*

This theorem is by theorem 2.2.3.1 and Marcinkiewicz' interpolation theorem. If we use Riesz-Thorin interpolation theorem, we can get estimation $\|T_m\|_{p,p} \geq \|m\|_\infty$.

Theorem 2.2.3.3. *The multipliers for L^p form a subalgebra of $L^\infty(\mathbb{R}^n)$ which is invariant under translations, rotations and dilations.*

2.2.4 Details of proof and errata

Note 1 (proof of lemma 6.1)**.** Since the constant C_p depends only on p, n and on the constants $\|\hat{K}\|_\infty$ and B_K of the kernel, and $\|\hat{K}\|_\infty$ and B_K is bounded by hypothesis a) and b), $T_N f = k_N * f$ are uniformly bounded.

Errata 1 (P209). $k_N = k\chi_{\{x:\frac{1}{N}\leq|x|\leq N\}}$

Note 2 (proof of norm of T_m as an operator from L^2 to L^2 is $\|m\|_\infty$). $T_m \leq \|m\|_\infty$ is easy. To prove $T_m \geq \|m\|_\infty$, by Plancherel's theorem, this is equivalent to $\|\hat{Tf}\|_2 \geq \|m\|_\infty\|\hat{f}\|_2$ for some $\hat{f}$. For any $\epsilon > 0$, let $A_\epsilon = \{x \in [0,1] : m(x) > \|m\|_\infty - \epsilon\}$, $\hat{f}(\xi) = \chi_{A_\epsilon}$. $\hat{Tf} = m\hat{f} > (\|m\|_\infty - \epsilon)\hat{f}$ hence $\|\hat{Tf}\|_2 > (\|m\|_\infty - \epsilon)\|\hat{f}\|_2$. Thus $\|\hat{Tf}\|_2 \geq \|m\|_\infty\|\hat{f}\|_2$.

Note 3 (proof of lemma 6.6 in book). The proof of lemma 6.6 is bad formatted. We give the complete proof of lemma 6.6.

First we have $|x|^a \leq (\sqrt{n}\max_i {x_i}^2)^a \leq n^{\frac{a}{2}} max_i |x_i|^a \leq n^{\frac{a}{2}}(\sum_i |x_i|^a)$.

By $|x|^a \leq n^{\frac{a}{2}}(\sum_i |x_i|^a)$, Holder inequality and Minkowski inequality for $s' \geq 2$:

$$\begin{aligned}
\int_{|x|>t} |k(x)|\,dx &= \int_{|x|>t} \frac{\sum_{j=1}^n (|x_j|^a)\,|k(x)|}{\sum_{j=1}^n (|x_j|^a)} dx \\
&\leq \int_{|x|>t} \frac{C\sum_{j=1}^n (|x_j|^a)\,|k(x)|}{|x|^a} dx \\
&\leq C\Big(\int (\sum_{j=1}^n (\big|x_j^a k(x)\big|)^{s'})dx\Big)^{\frac{1}{s'}}\Big(\int_{|x|>t} \frac{1}{|x|^{as}} dx\Big)^{\frac{1}{s}} \\
&\leq C\sum_{j=1}^n\Big(\int \big|x_j^a k(x)\big|^{s'}\,dx\Big)^{\frac{1}{s'}}\Big(\int_{|x|>t} \frac{1}{|x|^{as}} dx\Big)^{\frac{1}{s}}
\end{aligned}$$

Notice $\frac{2}{3} \leq s \leq 2$, $n + \frac{1}{3} \leq as \leq n+1$. Thus $\int_{|x|>t} \frac{1}{|x|^{as}} dx$ converges and $(\int_{|x|>t} \frac{1}{|x|^{as}} dx)^{\frac{1}{s}} = Ct^{-\frac{1}{2}}$.

Let $1 \leq p \leq 2$ and q is the dual exponent if p, the Hausdorff-Young inequality is:

$$(\sum |a_n|^q)^{\frac{1}{q}} \leq (\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^p\,d\theta)^{\frac{1}{p}}$$

and its dual:

$$(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^q\,d\theta)^{\frac{1}{q}} \leq (\sum |a_n|^p)^{\frac{1}{p}}$$

But here it uses the analog for the Fourier transform (Corollary 2.6 in Chapter 2 in stein's *Functional Analysis*):

Theorem 2.2.4.1. *If $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the Fourier transform T has a unique extension to a bounded map from L^p to L^q, with $\|T(f)\|_q \leq \|f\|_p$*

By Theorem 2.2.4.1 and property of Fourier transform.

$$\left(\int \left|(x_j^a k(x))\right|^{s'} dx\right)^{\frac{1}{s'}} \leq \left(\int \left|(x_j^a k(x))^\wedge\right|^{s} dx\right)^{\frac{1}{s}}$$
$$= \left(\int \left|(-2\pi i)^{-a} D_j^a \hat{k}(\xi)\right|^{s} dx\right)^{\frac{1}{s}} = C\|D_j^a \hat{k}\|_s$$

Thus

$$\int_{|x|>t} |k(x)|\, dx \leq C t^{-\frac{1}{2}} \sum_{j=1}^{n} \|D_j^a \hat{k}\|_s$$

Observe that $\sum_{j=1}^n \|D_j^a \hat{k}\|_s \leq n \max_j \|D_j^a \hat{k}\|_s \leq n \max_{|\alpha|=a} \|D^\alpha \hat{k}\|_s$. We finally prove that:

$$\int_{|x|>t} |k(x)|\, dx \leq C t^{-\frac{1}{2}} \max_{|\alpha|=a} \|D^\alpha \hat{k}\|_s$$

Errata 2 (P212)**.** By Leibnitz rule

$$D^\alpha m_j(\xi) = \sum_{\alpha=\beta+\gamma} \binom{|\alpha|}{|\beta|} D^\beta m(\xi) 2^{-j|\gamma|} D^\gamma \phi(2^{-j}\xi)$$

with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$

Errata 3 (P212)**.** $m = \sum_j m_j$ and at most two m_j can be non zero at any point. We have $\sum_j \left|\hat{k_j}(\xi)\right| = \sum_j |m_j(\xi)| \leq 2\|m\|_\infty$

Errata 4 (P214)**.** $|(2\pi y)^\gamma| \leq C\, |y|^{|\gamma|}$

Note 4 (proof of theorem 6.3 in book)**.** By Leibnitz rule we have

$$D^\alpha m_j(\xi) = \sum_{\alpha=\beta+\gamma} \binom{|\alpha|}{|\beta|} D^\beta m(\xi) 2^{-j|\gamma|} D^\gamma \phi(2^{-j}\xi)$$

Since for each γ, $\left|D^\gamma \phi(2^{-j}\xi)\right|$ is uniformly bounded and there are finite choice of γ, there is a constant C with $\left|D^\gamma \phi(2^{-j}\xi)\right| \leq C$ for all ξ and γ. Notice $\binom{|\alpha|}{|\beta|}$ is also bounded. Thus

$$|D^\alpha m_j(\xi)| \leq C' \sum_{|\beta|\leq|\alpha|} \left|D^\beta m(\xi) 2^{-j|\gamma|}\right| = C' 2^{-j|\alpha|} \sum_{|\beta|\leq|\alpha|} \left|D^\beta m(\xi) 2^{j|\beta|}\right|$$

Now we estimate the norm of $D^\alpha m_j(\xi)$. Notice $\mathrm{supp}(m_j) \subset \{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. Thus the support of $D^\alpha m_j(\xi)$ is also in $\{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. Thus only need to integrate each $D^\beta m(\xi)$ on $\{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. By separate the region of integral:

$$
\begin{aligned}
&(\int_{2^{j-1}\leq|\xi|\leq 2^{j+1}} \left|D^\beta m(\xi)\right|^s d\xi)^{\frac{1}{s}} \\
\leq&(\int_{2^{j-1}\leq|\xi|\leq 2^{j}} \left|D^\beta m(\xi)\right|^s d\xi)^{\frac{1}{s}} + (\int_{2^{j}\leq|\xi|\leq 2^{j+1}} \left|D^\beta m(\xi)\right|^s d\xi)^{\frac{1}{s}}
\end{aligned}
$$

By Holder inequality and hypothesis (2.2.1), the first integral on right hand side is:

$$
\begin{aligned}
&(\int_{2^{j-1}\leq|\xi|\leq 2^{j}} \left|D^\beta m(\xi)\right|^s d\xi)^{\frac{1}{s}} \\
\leq&(\int_{2^{j-1}\leq|\xi|\leq 2^{j}} \left|D^\beta m(\xi)\right|^2 d\xi)^{\frac{1}{2}}(\int_{2^{j-1}\leq|\xi|\leq 2^{j}} 1 d\xi)^{\frac{1}{s}-\frac{1}{2}} \\
\leq&C_1(2^{j-1})^{-|\beta|}(2^{j-1})^{\frac{n}{2}}C_2(2^{jn}-2^{(j-1)n})^{\frac{1}{s}-\frac{1}{2}} \\
=&C_1C_2(2^{j-1})^{\frac{n}{2}-|\beta|}(2^{jn}(1-2^{-n}))^{\frac{1}{s}-\frac{1}{2}} \\
=&C(1-2^{-n})^{\frac{1}{s}-\frac{1}{2}}(2^{-1})^{\frac{n}{2}-|\beta|}(2^{j})^{\frac{n}{2}-|\beta|}(2^{jn})^{\frac{1}{s}-\frac{1}{2}} \\
\leq&C'(2^{j})^{\frac{n}{2}-|\beta|}(2^{jn})^{\frac{1}{s}-\frac{1}{2}}
\end{aligned}
$$

Using the same argument for the second integral on right hand side:

$$
\begin{aligned}
(\int_{2^{j}\leq|\xi|\leq 2^{j+1}} \left|D^\beta m(\xi)\right|^s d\xi)^{\frac{1}{s}} &\leq C_1(2^{j})^{-|\beta|}(2^{j})^{\frac{n}{2}}C_2(2^{(j+1)n}-2^{jn})^{\frac{1}{s}-\frac{1}{2}} \\
&= C(2^n-1)^{\frac{1}{s}-\frac{1}{2}}(2^{j})^{\frac{n}{2}-|\beta|}(2^{jn})^{\frac{1}{s}-\frac{1}{2}} \\
&\leq C'(2^{j})^{\frac{n}{2}-|\beta|}(2^{jn})^{\frac{1}{s}-\frac{1}{2}}
\end{aligned}
$$

Thus

$$
(\int_{2^{j-1}\leq|\xi|\leq 2^{j+1}} \left|D^\beta m(\xi)\right|^s d\xi)^{\frac{1}{s}} \leq C'(2^{j})^{\frac{n}{2}-|\beta|}(2^{jn})^{\frac{1}{s}-\frac{1}{2}} = C'(2^j)^{-|\beta|+\frac{n}{s}}
$$

Note 5 (proof of theorem 6.3 in book)**.** By Plancherel's theorem:

$$
\|T^N f - T_m f\|_2 = \|(m-m^N)\hat{f}\|_2 = \|(m-m^N)\|_2\|\hat{f}\|_2
$$

Thus $T^N f \to T_m f$ in L^2 for every $f \in L^2$.

Generally, L^p convergence does not implies pointwise convergence. But there is a subsequence converges pointwise. Thus we have:

$$
\liminf_{N\to\infty} \left|T^N f(x) - T_m f(x)\right| = 0 \quad a.e.
$$

By triangular inequality:

$$
\begin{aligned}
(\int |T_m f(x)|^p)^{\frac{1}{p}} \leq&(\int \left|T^N f(x) - T_m f(x)\right|^p)^{\frac{1}{p}} + \int \left|T^N f(x)\right|^p \\
\leq&(\int \left|T^N f(x) - T_m f(x)\right|^p)^{\frac{1}{p}} + C_p\|f\|_p
\end{aligned}
$$

Take lim inf on both side and use Fatou's Lemma:

$$
\begin{aligned}
(\int |T_m f(x)|^p)^{\frac{1}{p}} \leq & \liminf(\int \left|T^N f(x) - T_m f(x)\right|^p)^{\frac{1}{p}} + C_p\|f\|_p \\
\leq & (\int \liminf \left|T^N f(x) - T_m f(x)\right|^p)^{\frac{1}{p}} + C_p\|f\|_p \\
= & C_p\|f\|_p
\end{aligned}
$$

Note 6 (proof of theorem 6.3 in book)**.** By Leibnitz rule:

$$
\begin{aligned}
|D^\alpha \tilde{m_j}(\xi)| &= \left|\sum_{\alpha=\beta+\gamma} \binom{|\alpha|}{|\beta|} D^\beta m_j(\xi) D^\gamma (e^{-2\pi i y\cdot\xi} - 1)\right| \\
&\leq C \sum_{\alpha=\beta+\gamma} \left|D^\beta m_j(\xi)\right| \left|D^\gamma (e^{-2\pi i y\cdot\xi} - 1)\right| \\
&\leq C' \sum_{\alpha=\beta+\gamma} |y|\, 2^{j(1-|\gamma|)} \left|D^\beta m_j(\xi)\right| \\
&\leq C' \sum_{\alpha=\beta+\gamma} |y|\, 2^{j(1-|\gamma|)} C'' 2^{-j|\beta|} \sum_{|\beta'|\leq|\beta|} \left|D^{\beta'} m(\xi) 2^{j\left|\beta'\right|}\right| \\
&\leq C \sum_{\alpha=\beta+\gamma} |y|\, 2^{j(1-|\alpha|)} \sum_{|\beta'|\leq|\beta|} \left|D^{\beta'} m(\xi) 2^{j\left|\beta'\right|}\right| \\
&\leq C \sum_{|\beta|\leq a} |y|\, 2^{j(1-a)} \sum_{|\beta'|\leq|\beta|} \left|D^{\beta'} m(\xi) 2^{j\left|\beta'\right|}\right| \\
&\leq C \sum_{|\beta|\leq a} |y|\, 2^{j(1-a)} \sum_{|\beta'|\leq a} \left|D^{\beta'} m(\xi) 2^{j\left|\beta'\right|}\right| \\
&= Ca\, |y|\, 2^{j(1-a)} \sum_{|\beta'|\leq a} \left|D^{\beta'} m(\xi) 2^{j\left|\beta'\right|}\right|
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sup_{|\alpha|=a} \|D^\alpha \tilde{m_j}\|_s &\leq C\, |y|\, 2^{j(1-a)} \sum_{|\beta'|\leq a} 2^{j\left|\beta'\right|} \|D^{\beta'} m(\xi)\|_s \\
&\leq C\, |y|\, 2^{j(1-a)} \sum_{|\beta'|\leq a} 2^{j\left|\beta'\right|} C 2^{j(-\left|\beta'\right| + \frac{n}{s})} \\
&\leq C'\, |y|\, 2^{j(1-a)} \sum_{|\beta'|\leq a} 2^{j\frac{n}{s}} \\
&\leq C' a\, |y|\, 2^{j(1-a+\frac{n}{s})}
\end{aligned}
$$

Note 7 (proof of theorem 6.10 in book)**.** We denote $t_j = 2^j |y|$

$$
\begin{aligned}
&\int_{|x|>2|y|} \left|k^N(x-y) - k^N(x)\right| |f(x)|\, dx \\
=&\sum_{j=1}^{\infty} \int_{t_j<|x|\le 2t_j} \left|k^N(x-y) - k^N(x)\right| |f(x)|\, dx \\
\le&\sum_{j=1}^{\infty} (\int_{t_j<|x|\le 2t_j} \left|k^N(x-y) - k^N(x)\right|^{q'} dx)^{\frac{1}{q'}} (\int_{t_j<|x|\le 2t_j} |f(x)|^q\, dx)^{\frac{1}{q}} \\
\le&\sum_{j=1}^{\infty} (\int_{t_j<|x|} \left|k^N(x-y) - k^N(x)\right|^{q'} dx)^{\frac{1}{q'}} (\int_{t_j<|x|\le 2t_j} |f(x)|^q\, dx)^{\frac{1}{q}} \\
\le&(\sum_{j=1}^{\infty} C t_j^{-\epsilon-\frac{n}{q}} |y|^{\epsilon} |B(0,2t_j)|^{\frac{1}{q}}) \sup_j (\frac{1}{|B(0,2t_j)|} \int_{|x|\le 2t_j} |f(x)|^q\, dx)^{\frac{1}{q}} \\
\le&C(\sum_{j=1}^{\infty} t_j^{-\epsilon-\frac{n}{q}} |y|^{\epsilon} t_j^{\frac{n}{q}}) M_q f(0) \\
=&C|y|^{\epsilon} (\sum_{j=1}^{\infty} t_j^{-\epsilon}) M_q f(0) \\
=&C|y|^{\epsilon} |y|^{-\epsilon} (\sum_{j=1}^{\infty} 2^{-j\epsilon}) M_q f(0) \\
=&C' M_q f(0)
\end{aligned}
$$

Since we prove that $(T_m f)^{\#}(0) \le A_n \sup_{\epsilon>0} \epsilon^{-n} \int_{|x|\le\frac{\epsilon}{2}} |f(y) - I_\epsilon|\, dy$ and using lemma 5.11 in book:

$$
\begin{aligned}
&\epsilon^{-n} \int_{|x|<\frac{\epsilon}{2}} |Tf(x) - I_\epsilon|\, dx \\
\le&C_q M_q f(0) + \epsilon^{-n} \iint_{2|x|<\epsilon<|y|} |K(x-y) - K(-y)|\, |f(y)|\, dxdy \\
\le&C_q M_q f(0) + \epsilon^{-n} \int_{2|x|<\epsilon} (\int_{2|x|<|y|} |K(y-x) - K(y)|\, |f(y)|\, dy) dx \\
\le&C_q M_q f(0) + \epsilon^{-n} \int_{2|x|<\epsilon} (C' M_q f(0)) dx \\
\le&C_q M_q f(0) + C'' M_q f(0) \\
\le&C M_q f(0)
\end{aligned}
$$

Thus $(T^N f)^{\#}(0) \le C M_q f(0)$.

Note 8 (proof of (6.11) in book)**.** First we prove a variant of Lemma 6.6:

$$(\int_{|x|>t} |k(x)|^{q'}\, dx)^{\frac{1}{q'}} \le Ct^{\epsilon-\frac{n}{q}} \max_{|\alpha|=a} \|D^{\alpha}\hat{k}\|_s \quad (0<t<\infty) \tag{2.2.3}$$

The proof is entirely similar with lemma 6.6, but Holder's inequality is applied in the form: $\|\cdot\|_{q'} \le \|\cdot\|_r \|\cdot\|_{s'}$, with $\frac{1}{r}=\frac{1}{s}-\frac{1}{q}$. Notice $-ar+n-1<-1$ by $a-\frac{n}{s}=\epsilon>-\frac{1}{q}$, the second integral on right hand side converges.

$$\begin{aligned}(\int_{|x|>t} |k(x)|^{q'}\, dx)^{\frac{1}{q'}} &\le C(\int (\sum_{j=1}^{n} (\left|x_j^a k(x)\right|)^{s'})dx)^{\frac{1}{s'}} (\int_{|x|>t} \frac{1}{|x|^{ar}} dx)^{\frac{1}{r}} \\ &\le C\sum_{j=1}^{n} (\int \left|x_j^a k(x)\right|^{s'} dx)^{\frac{1}{s'}} (t^{n-ar})^{\frac{1}{r}} \\ &\le C \max_{|\alpha|=a} \|D^{\alpha}\hat{k}\|_s (t^{\frac{n}{r}-a})\end{aligned}$$

By $\frac{n}{r}-a=\frac{n}{s}-\frac{n}{q}-a=-\epsilon-\frac{n}{q}$, we have proof inequality (2.2.3).

Now we prove:

$$(\int_{|x|>2t} |k_j(x-y)-k_j(x)|^{q'} dx)^{\frac{1}{q'}} \le \begin{cases} Ct^{-\epsilon-\frac{n}{q}} 2^{-j\epsilon} & (2^j\,|y|\ge 1) \\ Ct^{-\epsilon-\frac{n}{q}}\,|y|\,2^{j(1-\epsilon)} & (2^j\,|y|<1)\end{cases}$$

The first inequality is by inequality (2.2.3) and inequality (6.7) in book:

$$\|D^{\alpha} m_j\|_s \le C2^{j(-|\alpha|+\frac{n}{s})} \quad (|\alpha|\le a; 1\le s\le 2)$$

we have

$$(\int_{|x|>t} |k_j(x)|^{q'}\, dx)^{\frac{1}{q'}} \le Ct^{-\epsilon-\frac{n}{q}} 2^{-j\epsilon} \tag{2.2.4}$$

The proof of the second inequality is similar with (6.9) in book. By the same argument as above, we have

$$(\int_{|x|>t} |\tilde{k_j}(x)|^{q'} dx)^{\frac{1}{q'}} \le Ct^{-\epsilon-\frac{n}{q}} \max_{|\alpha|=a} \|D^{\alpha}\hat{\tilde{k}}_j\|_s$$

By $\hat{\tilde{k}}=\tilde{m}$ and inequality (6.9) in book

$$\sup_{|\alpha|=a} \|D^{\alpha}\tilde{m}_j\|_s \le C\,|y|\,2^{j(1-a+\frac{n}{s})} = C\,|y|\,2^{j(1-\epsilon)} \quad (1\le s\le 2; 2^j\,|y|<1)$$

, we have:

$$(\int_{|x|>t} |\tilde{k_j}(x)|^{q'} dx)^{\frac{1}{q'}} \le Ct^{-\epsilon-\frac{n}{q}}\,|y|\,2^{j(1-\epsilon)} \quad (1\le s\le 2; 2^j\,|y|<1) \tag{2.2.5}$$

Now by inequality (2.2.4) and inequality (2.2.5) we have:

$$(\int_{|x|>2t} |k_j(x-y) - k_j(x)|^{q'} dx)^{\frac{1}{q'}} \leq \begin{cases} Ct^{-\epsilon-\frac{n}{q}} 2^{-j\epsilon} & (2^j\,|y| \geq 1) \\ Ct^{-\epsilon-\frac{n}{q}}\,|y|\,2^{j(1-\epsilon)} & (2^j\,|y| < 1) \end{cases}$$

Finally, by summing with different part of index, (6.11) is an easy consequence of above inequality.

Note 9 (proof of (6.14) in book)**.** By definition of subalgebra, we only need to check that $T_{am_1+bm_2}$, $T_{m_1m_2}$ are bounded. These are easy since T_m is linear on m and $T_{m_1m_2} = T_{m_1}T_{m_2}$. Suppose the affine transformations $A(x) = h + L(x)$ where L is linear transformation. I don't know why the following holds:

$$(f \circ A)^\wedge(\xi) = e^{2\pi i h\cdot\tilde{L}(\xi)}\,|\det L|^{-1}\,\hat{f}(\tilde{L}(\xi))$$

with $\tilde{L} = (L^*)^{-1}$ and how this implies multipliers are invariant under transformation A.