0.1 F.Riesz Factorization Theorem

This section can be seen as a generalization of first section. In first section, we talk about norm convergence and pointwise convergence when boundary function f is in L^p, $1 < p \leq \infty$ and f is a measure. This conclusion is for harmonic function. Harmonic function has series representation:

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}$$

and we can derive Poisson representation $u(re^{i\theta}) = P_r(f)$. Since holomorphic function also has series representation:

$$u(re^{i\theta}) = \sum_{k=0}^{\infty} a_k r^k e^{ik\theta}$$

, we can consider this representation as special case of harmonic function with $a_k = 0$ for $k < 0$. Poisson representation is also hold for holomorphic function, thus the converge result is hold also for holomorphic function. The following theorem is a summary of these results.

Theorem 0.1.0.1 (theorem 3.1 in book)**.** *Let $F \in H^p$ with $1 < p \leq \infty$. Then:*

1. *For almost every t. the limit*

   $$F(e^{it}) = \lim F(z) \text{ as } z \to e^{it} \text{ N.T.}$$

   exists. The function $f(t) = F(e^{it})$ belongs to $L^p([-\pi, \pi])$ and $F = P(f)$

2. *If $p < \infty$:*

   $$\int_{-\pi}^{\pi} \left|F(re^{it}) - F(e^{it})\right|^p dt \to 0 \text{ as } r \to 1$$

   If $p = \infty$, $F(re^{it}) \to F(e^{it})$ in the w^*-topology of L^∞ as $r \to 1$.

   For each $1 < p \leq \infty$: $\|F\|_{H^p} = \|f\|_p$.

3. *F is the Cauchy integral of its boundary function, that is:*

   $$F(z) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{F(\xi)}{\xi - z} d\xi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F(e^{it})}{e^{it} - z} e^{it} dt$$

Remark. *For first statement in theorem 0.1.0.1, N.T. limit holds for $p = 1$, but $P(f)$ may not be hold.*

Remark. *$u(re^{it}) = P_r(t)$ is neither in H^p nor N. $P_r(t)$ is harmonic but not holomorphic.*

In this section we will extend this result to $p \leq 1$. The main idea is to factorize $F(z)$ to a Blaschke product $B(z)$ and a non-vanish function $H(z)$.

0.1.1 Result of non-vanish case

Suppose that $F \in H^p$, $\frac{1}{2} \leq p < 1$. If F does not vanish in D. Then $F(z) = e^{f(z)}$ for some holomorphic function f. Let $G(z) = e^{\frac{f(z)}{2}}$, we have $F(z) = G(z)^2$, $G(z) \in H^{2p}$ and $\|G\|_{H^{2p}}^2 = \|F\|_{H^p}$. Since $2p \geq 1$, we have $G(e^{it}) = \lim G(z)$ a.e. as $z \to e^{it}$ N.T.. It follows that $F(e^{it}) = \lim F(z)$ a.e. as $z \to e^{it}$ N.T.

We know that $\int_{-\pi}^{\pi} \left|F(re^{it}) - F(e^{it})\right|^p dt \to 0$ as $r \to 1$ if $p > 1$. Suppose that $F \in H^p$, $\frac{1}{2} \leq p < 1$ and we have $F(z) = G(z)^2$ as before, then:

$$
\begin{aligned}
&\int_{-\pi}^{\pi} \left|F(re^{it}) - F(e^{it})\right|^p dt \\
=&\int_{-\pi}^{\pi} \left|G(re^{it})^2 - G(e^{it})^2\right|^p dt \\
=&\int_{-\pi}^{\pi} \left|G(re^{it}) + G(e^{it})\right|^p \left|G(re^{it}) - G(e^{it})\right|^p dt \\
\leq&(\int_{-\pi}^{\pi} \left|G(re^{it}) + G(e^{it})\right|^{2p} dt)^{\frac{1}{2}}(\int_{-\pi}^{\pi} \left|G(re^{it}) - G(e^{it})\right|^{2p} dt)^{\frac{1}{2}} \\
\leq&(\int_{-\pi}^{\pi} (2\left|G(e^{it})\right|)^{2p} dt)^{\frac{1}{2}}(\int_{-\pi}^{\pi} \left|G(re^{it}) - G(e^{it})\right|^{2p} dt)^{\frac{1}{2}} \\
\leq&2^p \|G\|_{H^{2p}}^p (\int_{-\pi}^{\pi} \left|G(re^{it}) - G(e^{it})\right|^{2p} dt)^{\frac{1}{2}} \to 0 \; as \; r \to 1
\end{aligned}
$$

We conclude that $F \in H^p$, $\frac{1}{2} \leq p < 1$. If F does not vanish in D. Then there is a boundary function $F(e^{it})$, $F(z)$ converges to $F(e^{it})$ both in pointwise sense and norm sense.

Remark. *There is a basic inequality, used also in proving Minkowski inequality:* $|a+b|^p \leq 2^p(|a|^p + |b|^p)$ *for* $p > 0$*. To prove this we only need to consider two case:* $|a| \geq |b|$ *or* $|a| \leq |b|$.

By induction, this conclusion can be extended to $0 < p < 1$. Thus two types of convergence holds for all $0 < p < \infty$.

Remark. *Author uses Fatou's lemma when* $F(z)$ *converges to* $F(e^{it})$ *N.T.. I think we can use this lemma even if it converges radially.*

0.1.2 Result of H^p case

In the end of last section review, we state three theorems:

- For $F \in N$, the zeroes (z_j) of F satisfies $\sum_j (1 - |z_j|) < \infty$.
- If $\sum_j (1 - |z_j|) < \infty$ holds, the Blaschke product converges uniformly on each compact subset to a function $B(z) \in H^\infty$ and they have zeroes (z_j).
- $\left|B(e^{it})\right| = 1$ a.e.

If we let $H = \frac{F}{B}$, where Blaschke product is formed by zeroes of F, then H does not have any zeroes. Besides, if $F \in N$, then $H \in N$ and $\|H\|_N = \|F\|_N$. If $F \in H^p$, then $H \in H^p$ and $\|H\|_{H^p} = \|F\|_{H^p}$ (theorem 3.3 in book). Notice now we can use method in section 0.1.1 on H. We have following result:

Theorem 0.1.2.1 (theorem 3.6 in book)**.** *Let $F \in H^p$ with $0 < p \leq \infty$. Then:*

1. *For almost every t. the limit*

$$F(e^{it}) = \lim F(z) \text{ as } z \to e^{it} \text{ N.T.}$$

exists. The function $f(t) = F(e^{it})$ belongs to $L^p([-\pi, \pi])$.

2. $\int_{-\pi}^{\pi} \left|F(re^{it}) - F(e^{it})\right|^p dt \to 0$ *as* $r \to 1$

3. $\|F\|_{H^p} = \lim_{r\to 1} (\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F(re^{it})\right|^p dt)^{\frac{1}{p}} = (\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F(e^{it})\right|^p dt)^{\frac{1}{p}}$

Another statement is that $F \in H^p$ can be improved to $F \in H^q$ if the boundary function $F(e^{it}) \in L^q$ (Corollary 3.7). The hard part of its proof is the case $p < q$ and $p \leq 1$. We factorize F as $F = BG^n$, where $np > 1$. Since $F(e^{it}) \in L^q$ and $\left|G(e^{it})\right|^n = \left|F(e^{it})\right|$, $G(e^{it}) \in L^{nq}$. Thus $G \in H^{nq}$ and $F \in H^q$.

0.1.3 H^1 function and its boundary

Recall in section 1, when u is a harmonic function in D and

$$\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} \left|F(re^{it})\right| dt < \infty$$

, we can only say u is $P(\mu)$ for some Borel measure and the result can not be improved (consider Poisson kernel). However, if $F \in H^1$, in other words $\sup_{0\leq r<1} \int_{-\pi}^{\pi} \left|F(re^{it})\right| dt < \infty$ and F is holomorphic, then by $F(re^{it}) \to F(e^{it})$ in L^1. Thus F can be written as the Poisson integral and the Cauchy integral of its boundary function $F(e^{it})$.

Remark (notes on proof corollary 3.9 in book)**.** *Corollary 3.9 is Schwarz integral formula. The kernel $\frac{1}{2\pi}\frac{e^{it}+z}{e^{it}-z}$ is called Schwarz kernel.*

We can rewrite:

$$\begin{aligned} G(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \operatorname{Re} F(e^{it}) dt \\ &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{\operatorname{Re} F(e^{it})}{e^{it}-z} de^{it} + z\frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{\frac{\operatorname{Re} F(e^{it})}{e^{it}}}{e^{it}-z} de^{it} \end{aligned}$$

Since $\operatorname{Re} F(e^{it})$ and $\frac{\operatorname{Re} F(e^{it})}{e^{it}}$ are continuous function on boundary of disk. Then $G(z)$ is holomorphic function.

An consequence of Poisson representation for H^1 functions is a famous theorem due to F. and M. Riesz. It says given a Borel measure μ, when negative frequencies of Fourier coefficients of μ is zero, then μ is absolutely continuous w.r.t. Lebesgue measure, i.e.: $d\mu(t) = f(t)dt$ for some $f \in L^1$. This theorem shows the difference between bounded holomorphic function $\sum_{k=0}^{\infty} a_k r^k e^{ik\theta}$ and bounded harmonic function $\sum_{k=-\infty}^{\infty} a_k r^k e^{ik\theta}$ (bounded as $\sup_{0\leq r<1} \int_{-\pi}^{\pi} \left|F(re^{it})\right| dt < \infty$). The vanish of negative frequencies make bounded harmonic function (or Poisson integral of complex Borel measure) to bounded holomorphic function.

Remark (notes on proof of corollary 3.11 in book)**.** *f is bounded variation, then f can be written as difference of two increasing bounded function. This is equivalent to f can be written as difference of two Borel measure. Thus $f(t) = c + \int_{-\pi}^{t} d\mu(s)$ where $c = f(-\pi)$.*

The integration by parts:

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{ijt} d\mu(t) &= e^{ijt} \int_{-\pi}^{t} d\mu(s)\Big|_{-\pi}^{\pi} - ij \int_{-\pi}^{\pi} g(t) e^{ijt} dt \\
&= \left(e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s) - e^{-ij\pi} \int_{-\pi}^{-\pi} d\mu(s) \right) - ij \int_{-\pi}^{\pi} g(t) e^{ijt} dt \\
&= e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s) - ij \int_{-\pi}^{\pi} (F(e^{it}) - c) e^{ijt} dt \\
&= e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s) - \lim_{r\to 1} ij \int_{-\pi}^{\pi} F(re^{it}) e^{ijt} dt \\
&= e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s)
\end{aligned}$$

The limit in fourth equality is by $F \in H^1$, $F(re^{it}) \to F(e^{it})$ in L^1. This limit is 0 since $F \in H^1$, the negative frequencies are 0. $e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s) = e^{ij\pi} g(\pi)$ is 0 since $f(\pi) = f(-\pi) + g(\pi)$ and $f(\pi) = f(-\pi)$.

Corollary 3.11 in book shows a condition when bounded variation implies absolutely continuity. This Corollary emphases again 'holomorphic condition' or vanish of negative frequencies makes a Borel measure absolutely continuous. Theorem 3.12 in book says that $F' \in H^1$ is the necessary and sufficient condition of holomorphic $F \in H(D)$ is absolutely continuous on boundary.

Remark (notes on proof of theorem 3.12 in book)**.** *$F \in H^1$ implies $F' \in H(D)$. Since*

$$\sup_{0\leq r<1} \int_{-\pi}^{\pi} \left|F(re^{it})\right| dt = \sup_{0\leq r<1} \int_{-\pi}^{\pi} \left|ire^{it} F(re^{it})\right| dt$$

, $F'(z) \in H^1$ if and only if $izF'(z) \in H^1$.

I don't know why the difference is harmonic and continuous at the origin, it has to be a constant c.

There is a corollary of Theorem 3.12 in book which is useful in next section.

Corollary 0.1.3.1 (corollary 3.13 in book). *Let* Γ *be a Jordan curve and let* F *be a conformal map from* D *to interior domain bounded by a Jordan curve* Γ. *Then* Γ *is rectifiable if and only if* $F' \in H^1$.