0.1 Singular Integral Operators

In this section we show how Calderon-Zygmund decomposition is used in estimation of the convolution operator T, which defined on Schwartz space $\mathscr{S}$, $T(f) = K * f(x)$. The definition of singular integral operator given in book is different from others like that in Stein's *Singular Integrals and Differentiability Properties of Functions.*

Definition 0.1.0.1. *Given a tempered distribution K, the convolution operator*

$$Tf(x) = K * f(x) \quad (f \in \mathscr{S}(\mathbb{R}^n))$$

is called a singular integral operator if the following two conditions are satisfied:

1. $\hat{K} \in L^\infty(\mathbb{R}^n)$
2. *K coincides in $\mathbb{R}^n \setminus \{0\}$ with a locally integrable function $K(x)$ satisfying Hormander's condition:*

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \le B_k$$

Later we will see these two conditions guarantee the boundedness of $T(f)$.

0.1.1 Hormander's condition

If a locally integrable function $k(x)$ satisfying Hormander's condition and following two conditions:

$$\int_{r<|x|<2r} |k(x)|\, dx \le C_1$$

$$\begin{cases} \left| \int_{r<|x|<R} k(x)dx \right| \le C_2 \\ \lim_{r\to 0} \int_{r<|x|<1} k(x)dx \ \textit{exists} \end{cases}$$

Then $Tf(x) = \lim_{\epsilon\to 0} \int_{|y|>\epsilon} k(x-y)f(y)dy$ is a singular integrable operator (Proposition 5.5 in book). In other words, $\|\hat{k}\|_\infty < \infty$

If we let a locally integrable function $k(x)$ be $\Omega(x)\,|x|^{-n}$ with Ω homogeneous of degree 0 ($\Omega(rx) = \Omega(x)$ for $r \neq 0$). Then the following continuity condition on Ω ensures Hormander condition holds:

$$\int_0^1 w_1(\Omega;t)\frac{dt}{t} < \infty$$

where

$$w_1(\Omega;t) = \sup_{h\in\mathbb{R}^n, |h|\le t} \int_{|x'|=1} |\Omega(x'+h) - \Omega(x')|\, d\sigma(x')$$

Then $Tf(x) = \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \Omega(y) \, |y|^{-n} \, f(x-y) dy$ is a singular integrable operator. The condition $\|\hat{k}\|_\infty < \infty$ is ensured by explict formula for $\hat{k} = m(\xi)$ if Ω is odd:

$$m(\xi) = -\frac{\pi i}{2} \int_{|x'|=1} \Omega(x') \operatorname{sign}(x' \cdot \xi) d\sigma(x')$$

where $\operatorname{sign} x = \frac{x}{|x|}$ (Proposition 5.6 in book). If Ω is not necessarily odd then the formula for $m(\xi)$ is (refer section 4.2 in chapter 2 in Stein's *Singular Integrals and Differentiability Properties of Functions*):

$$m(\xi) = \frac{\pi i}{2} \int_{|x'|=1} (\operatorname{sign}(x' \cdot \xi) + \log(\frac{1}{|x' \cdot \xi|})) \Omega(x') d\sigma(x')$$

Remark 0.1.1.1. *Equation*

$$\frac{\pi}{2} c_n \int_{|x'|=1} (x' \cdot h) \operatorname{sign}(x' \cdot \xi) d\sigma(x') = \frac{\xi \cdot h}{|\xi|}$$

is by riesz representation for Hilbert space. If we fix ξ with $|\xi| = 1$, then left hand side is a linear function of h, say $\ell(h)$. Since

$$|\ell(h)| \leq \frac{\pi}{2} c_n \int_{|x'|=1} |x'| \, |h| \, d\sigma(x') = \frac{\pi}{2} c_n \int_{|x'|=1} |h| \, d\sigma(x') = |h|$$

, $\ell(h)$ is continuous linear functional with $\|\ell\| \leq 1$. Thus $\ell(h) = (h, g)$ for some $g \in \mathbb{R}^n$ with $|g| = \|\ell\| \leq 1$. Notice

$$\ell(\xi) = \frac{\pi}{2} c_n \int_{|x'|=1} (x' \cdot \xi) \operatorname{sign}(x' \cdot \xi) d\sigma(x') = \frac{\pi}{2} c_n \int_{|x'|=1} |x' \cdot \xi| \, d\sigma(x') = 1$$

Thus $\|\ell\| = 1$ with $|g| = 1$ and $1 = \ell(\xi) = (\xi, g) \leq |\xi| \, |g| = 1$. The inequality holds if and only if $g = k\xi$ for some $k \in \mathbb{R}$. Thus $g = \xi$ and $\ell(h) = (h, \xi)$

0.1.2 Estimation of singular integral operator

The following theorem is the main result concerning singular integral operators:

Theorem 0.1.2.1 (theorem 5.7 in book)**.** *Every singular integral operator satisfies the inequalities*

$$\|Tf\|_p \leq C_p \|f\|_p \quad (f \in L^2 \cap L^p; 1 < p < \infty) \tag{1}$$

$$|\{x : |Tf(x)| > t\}| \leq \frac{C_1}{t} \|f\|_1 \quad (f \in L^2 \cap L^1) \tag{2}$$

$$\|Tf\|_{\mathrm{BMO}} \leq C_\infty \|f\|_\infty \quad (f \in L^2 \cap L^\infty) \tag{3}$$

where C_p, $1 \leq p \leq \infty$, depends only on p, n and on the constants $\|\hat{K}\|_\infty$ and B_K of the kernel.

To prove this theorem, we only need to prove three case $p = 1$, $p = 2$, and $p = \infty$ and use Marcinkiewicz interpolation theorem in section 2 to derive $1 < p < 2$ case. And use another interpolation theorem 3.7 in section 3 to derive $2 < p < \infty$ case.

Remark 0.1.2.1 (notes on proof of inequality (2))**.** *The key idea to prove the case $p = 1$ is Calderon-Zygmund decomposition. First we can split the space into a family of non-overlapping cubes C_t and a set $\mathbb{R}^n \setminus \cup_{Q \in C_t} Q$. These sets satisfy:*

1. *For every $Q \in C_t$, $t < \frac{1}{|Q|} \int_{|Q|} |f(x)|\, dx \leq 2^n t$*
2. *For a.e. $X \notin \cup_{Q \in C_t} Q$, $|f(x)| \leq t$*

Let $\Omega = \cup_{Q \in C_t} Q$. For any function $f \in L^1$, the Calderon-Zygmund decomposition of $f(x) = g(x) + b(x)$ is:

$$g(x) = \sum_j (\frac{1}{|Q_j|} \int_{Q_j} f(t) dt) \chi_{Q_j}(x) + f(x) \chi_{\mathbb{R}^n \setminus \Omega}(x)$$

and

$$b(x) = f(x) - g(x) = \sum_j b_j(x) = \sum_j (f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(t) dt) \chi_{Q_j}(x)$$

Generally, to estimate the size of the set $\{x : |f(x)| > c\}$, we can use $(\frac{|f(x)|}{c})^p > 1$ in that set and integral the index function of that set on $\mathbb{R}^n$:

$$|\{x : |f(x)| > c\}| = \int_{|\{x:|f(x)|>c\}|} 1 dt \leq \int_{|\{x:|f(x)|>c\}|} \left(\frac{|f(x)|}{c}\right)^p dt \leq \frac{1}{c^p} \int |f(x)|^p\, dt$$

This inequality sometimes called Markov inequality.

Remark 0.1.2.2 (notes on proof of lemma 5.11 in book)**.** *In proof of theorem 5.20 in book, it says $f \in L_c^\infty$ was only imposed to ensure the $I_\epsilon = \int_{|y|>\epsilon} K(-y) f(y) dy$ exists. $M_p f(x)$ increases as p increases by Holder inequality. let $r < q$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, we have*

$$\begin{aligned}
(\frac{1}{|Q|} \int_Q |f(t)|^r\, dt)^{\frac{1}{r}} &\leq \frac{1}{|Q|^{\frac{1}{r}}} (\int_Q |f(t)|^p\, dt)^{\frac{1}{p}} (\int_Q 1^q dt)^{\frac{1}{q}} = \frac{|Q|^{\frac{1}{q}}}{|Q|^{\frac{1}{r}}} (\int_Q |f(t)|^p\, dt)^{\frac{1}{p}} \\
&= \frac{1}{|Q|^{\frac{1}{p}}} (\int_Q |f(t)|^p\, dt)^{\frac{1}{p}} = (\frac{1}{|Q|} \int_Q |f(t)|^p\, dt)^{\frac{1}{p}}
\end{aligned}$$

Remark 0.1.2.3 (notes on proof of inequality (3))**.** *We have $(Tf)^{\#}(0) = \sup_{0 \in Q} \frac{1}{|Q|} \int_Q |Tf(y) - (Tf)_Q|\, dy$ with $(Tf)_Q = \frac{1}{|Q|} \int_Q Tf(x) dx$*

We know $(Tf)^{\#}(x) \cong \sup_{x \in Q} \inf_{a \in \mathbb{R}} \frac{1}{|Q|} \int_Q |Tf(y) - a|\, dy$, where $\cong$ is used to indicate that each side is bounded by the other times an absolute constant (refer equation (3.1) in section 3, chapter 2 in book).

Observe that the cube $[-r,r]^n$ containing 0 is almost everywhere contained in the open ball $B(0,n^{\frac{1}{2}}r)$. Let d be the side length of Q. Let Q^2 be the cube with origin as center and with side length 2 times that of Q. $0\in Q$ implies $Q\subset Q^2$. Thus

$$\begin{aligned}(Tf)^{\#}(0) &\leq C\sup_{0\in Q}\inf_{a\in\mathbb{R}}\frac{1}{|Q|}\int_Q |Tf(y)-a|\,dy\\ &\leq C\sup_{r>0}\inf_{a\in\mathbb{R}}\frac{1}{2^n r^n}\int_{B(0,n^{\frac{1}{2}}r)}|Tf(y)-a|\,dy\\ &\leq C\sup_{r>0}\frac{1}{2^n r^n}\int_{|y|\leq n^{\frac{1}{2}}r}|Tf(y)-I_\epsilon|\,dy\\ &= C'\sup_{\epsilon>0}\epsilon^{-n}\int_{|y|\leq\frac{\epsilon}{2}}|Tf(y)-I_\epsilon|\,dy\end{aligned}$$

.

To estimate $\epsilon^{-n}\int_{|x|<\frac{\epsilon}{2}}|Tf(x)-I_\epsilon|\,dx$, we use lemma 7.11 in book and Hormander's condition:

$$\begin{aligned}&\epsilon^{-n}\int_{|x|<\frac{\epsilon}{2}}|Tf(x)-I_\epsilon|\,dx\\ \leq& C_pM_pf(0)+\epsilon^{-n}\iint_{2|x|<\epsilon<|y|}|K(x-y)-K(-y)|\,|f(y)|\,dxdy\\ \leq& C_p\|f\|_\infty+\|f\|_\infty\epsilon^{-n}\iint_{2|x|<\epsilon<|y|}|K(x-y)-K(-y)|\,dxdy\\ \leq& C_p\|f\|_\infty+\|f\|_\infty\epsilon^{-n}\int_{2|x|<\epsilon}\Big(\int_{2|x|<|y|}|K(x-y)-K(-y)|\,dy\Big)dx\\ =& C_p\|f\|_\infty+\|f\|_\infty\epsilon^{-n}\int_{2|x|<\epsilon}\Big(\int_{2|x|<|y|}|K(y-x)-K(y)|\,dy\Big)dx\\ \leq& C_p\|f\|_\infty+\|f\|_\infty\epsilon^{-n}\int_{2|x|<\epsilon}B_k dx\\ =& C_p\|f\|_\infty+B_k\|f\|_\infty\end{aligned}$$

We can change the sign of x and y in fifth line since the region is symmetric.

Remark 0.1.2.4. *I don't know why $\frac{1}{\pi}\log\left|\frac{x-a}{x-b}\right|$, with $a<b$ is in weak $-L^1\cap$* BMO.

0.1.3 Restriction and extension of singular integral operator

To understand the behavior of singular integral operators in L^1 and L^∞, we ask if there is a subspace of L^1, on which the singular integral operator is strong type $(1,1)$, and if the singular integral operator can "extend" from $L^2 \cap L^\infty$ to L^∞.

For the first question, we introduce the Banach space H^1_{at} (atomic H^1). More study of H^1_{at} is in Chapter 3. You can also refer section 5 and 6 in chapter 2 in Stein's *Functional Analysis*.

For the second question, notice $L^p \cap L^\infty$ is not dense in L^∞. The extension is not a trivial step. However, the function f and $f+C$ with constant C behavior the same in space BMO. Thus we introduce our definition of new T on L^∞.

Proposition 0.1.3.1 (proposition 5.15 in book)**.** *Given a singular integral operator T with kernel K, for each $f \in L^\infty$ we define:*

$$Tf(x) = \lim_{j\to\infty} (T(f\chi_{B_j})(x) - \int_{1<|y|<j} K(-y)f(y)dy)$$

The sequence to the right converges locally in L^1 and also pointwise a.e., and the extended operator T satisfies:

$$\|Tf\|_{\mathrm{BMO}} \leq C_\infty \|f\|_\infty \quad (f \in L^\infty) \tag{4}$$

Remark 0.1.3.1 (notes on proof of proposition 5.15 in book)**.** *It is not clear that $g_j(x)$ converges pointwise. $g_j(x)$ in $L^1(F)$ since $g_l(x) \in L^1(F)$ and the integral is bounded by $B_K\|f\|_\infty$ and F is compact.*

0.1.4 More precise estimation for more regular kernel

If the kernel is more regular, the estimation of operator T can be replaced by maximal function. More precisely:

Definition 0.1.4.1 (definition 5.17 in book)**.** *A singular integral operator $Tf = K * f$ is called regular if its kernel satisfies the following two conditions:*

1. $|K(x)| \leq B\,|x|^{-n}$ *for* $x \neq 0$
2. $|K(x-y) - K(x)| \leq B\,|y|\,|x|^{-n-1}$ *for* $|x| > 2\,|y| > 0$

Let $T_\epsilon f(x) = \int_{|y|>\epsilon} K(y)f(x-y)dy$ and $T^*f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|$. The estimation is the following:

Theorem 0.1.4.2 (theorem 5.20 in book)**.** *If T is a regular singular integral operator and $f \in L^p$, $1 \leq p < \infty$, then the following inequalities are verified:*

1. $(Tf)^\#(x) \leq C_q M_q f(x) \quad (q > 1)$
2. $T^*f(x) \leq C_q M_q f(x) + CM(Tf)(x) \quad (q > 1)$
3. $\|T^*f\|_p \leq C_p \|f\|_p \quad (1 < p < \infty)$
4. $|\{x : T^*f(x) > t\}| \leq Ct^{-1}\|f\|_1 \quad (t > 0)$

Remark 0.1.4.1 (notes on proof of theorem 5.20 in book)**.** *3 is a trivial consequence of 2 and the L^p inequalities for the operators M and T. The L^p inequality for operator M_q is similar with operator M. By 2, $\|T^*f\|_p \leq \|C_qM_qf\|_p + \|CM(Tf)\|_p$, and $\|CM(Tf)\|_p \leq C'\|Tf\|_p \leq C''\|f\|_p$.*

Now we prove the L^p inequality for operator M_q. We choose q with $1 < q < p$:

$$\begin{aligned}
\int_{\mathbb{R}^n} (M_qf(x))^p dx &= p\int_0^\infty t^{p-1} \left|\{x : M_qf(x) > t\}\right| dt \\
&= p\int_0^\infty t^{p-1} \left|\{x : \sup((\frac{1}{|Q|}\int_Q |f(t)|^q\, dt)^{\frac{1}{q}}) > t\}\right| dt \\
&= p\int_0^\infty t^{p-1} \left|\{x : \sup((\frac{1}{|Q|}\int_Q |f(t)|^q\, dt)) > t^q\}\right| dt \\
&\leq Cp\int_0^\infty t^{p-1} \int_{\{x:|f(x)|^q>\frac{t^q}{2}\}} \frac{|f(x)|^q}{t^q} dxdt \\
&= Cp\int_0^\infty \int_{\mathbb{R}^n} t^{p-1}\frac{|f(x)|^q}{t^q}\chi_{\{x:|f(x)|^q>\frac{t^q}{2}\}} dxdt \\
&= Cp\int_{\mathbb{R}^n}\int_0^{|f(x)|2^{\frac{1}{q}}} t^{p-1-q}\, |f(x)|^q\, dxdt \\
&= \frac{Cp}{p-q}\int_{\mathbb{R}^n} (|f(x)|\, 2^{\frac{1}{q}})^{p-q}\, |f(x)|^q\, dt \\
&= \frac{Cp2^{\frac{p-q}{q}}}{p-q}\int_{\mathbb{R}^n} |f(x)|^p\, dt
\end{aligned}$$

Thus $\|T^*f\|_p \leq \|C_qM_qf\|_p + \|CM(Tf)\|_p \leq C\|f\|_p$

Remark 0.1.4.2 (notes on corollary 5.22 in book)**.** *Notice (5.19) implies (b) in definition 5.1. By easy computation (5.18) implies (5.3). By proposition 5.5 in book, 5.1(b), (5.3) and (5.4) implies that $T = k * f$ is a singular integral operator. It is not clear how to pass $\mathscr{S}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.*

0.1.5 Proof of proposition 5.5 and 5.6 in book

Remark 0.1.5.1 (notes on proof of proposition 5.5 in book)**.** *Let $A = \{x : \epsilon < |x| < R\}$, $B = \{x : \epsilon < |x-y| < R\}$, $C = \{x : |x-y| \leq \epsilon\}$ and*

$B = \{x : |x-y| \geq R\}$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx \\
&= \int_{A\cap B} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx + \int_{\mathbb{R}^n\setminus(A\cap B)} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx \\
&= \int_{A\cap B} |k(x-y) - k(x)|\, dx + \int_{A^c\cup B^c} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx \\
&\leq \int_{A} |k(x-y) - k(x)|\, dx + \int_{A^c} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx + \int_{B^c} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx \\
&= \int_{A} |k(x-y) - k(x)|\, dx + \int_{A^c} \left|k_\epsilon^R(x-y)\right| dx + \int_{B^c} \left|k_\epsilon^R(x)\right| dx \\
&= \int_{A} |k(x-y) - k(x)|\, dx + \int_{A^c\cap B} |k(x-y)|\, dx + \int_{B^c\cap A} |k(x)|\, dx \\
&= \int_{A} |k(x-y) - k(x)|\, dx + \int_{A^c\cap B} |k(x-y)|\, dx + \int_{C\cap A} |k(x)|\, dx + \int_{D\cap A} |k(x)|\, dx
\end{aligned}$$

If $x \in C$, $|x| - |y| \leq |x-y| \leq \epsilon$, then $C \subset \{x : |x| \leq \epsilon + |y|\}$. If $x \in D$, $|x| + |y| \geq |x-y| \geq R$, then $D \subset \{x : R - |y| \leq |x| \leq R\}$. Since R is large enough and ϵ is small enough, we have $C \cap A \subset \{x : \epsilon < |x| \leq \epsilon + |y|\}$ and $D \cap A \subset \{x : R - |y| \leq |x| \leq R\}$. Thus

$$\int_{C\cap A} |k(x)|\, dx + \int_{D\cap A} |k(x)|\, dx \leq \int_{\epsilon<|x|\leq\epsilon+|y|} |k(x)|\, dx + \int_{R-|y|\leq|x|<R} |k(x)|\, dx$$

Let $x - y = t$,

$$\int_{A^c\cap B} |k(x-y)|\, dx = \int_{\epsilon<|t|<R \ and \ |t+y|\leq\epsilon} |k(t)|\, dt + \int_{\epsilon<|t|<R \ and \ |t+y|\geq R} |k(t)|\, dt$$

Use almost the same argument, we can show:

$$\int_{A^c\cap B} |k(x-y)|\, dx \leq \int_{\epsilon<|x|\leq\epsilon+|y|} |k(x)|\, dx + \int_{R-|y|\leq|x|<R} |k(x)|\, dx$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|k_\epsilon^R(x-y) - k_\epsilon^R(x)\right| dx \leq 2\int_{\epsilon<|x|\leq\epsilon+|y|} |k(x)|\, dx + 2\int_{R-|y|\leq|x|<R} |k(x)|\, dx \\
+ \int_A |k(x-y) - k(x)|\, dx
\end{aligned}$$

$\int_{r<|x|<2r} |k(x)|\, dx \leq C_1$ *implies* $\int_{|x|<r} |k(x)|\, |x|\, dx \leq 4C_1 r$*:*

$$\begin{aligned}
\int_{|x|<r} |k(x)|\, |x|\, dx &= \sum_{j=0}^{\infty} \int_{\frac{r}{2^{j+1}}|x|<\frac{r}{2^j}} |k(x)|\, |x|\, dx \leq \sum_{j=0}^{\infty} \int_{\frac{r}{2^{j+1}}|x|<\frac{r}{2^j}} |k(x)|\, \frac{r}{2^j} dx \\
&= \sum_{j=0}^{\infty} C_1 \frac{r}{2^j} = 2C_1 r
\end{aligned}$$

$\int_{|x|<r} |k(x)|\,|x|\,dx \le 4C_1 r$ implies $\int_{r<|x|<2r} |k(x)|\,dx \le C_1$:

$$\int_{r<|x|<2r} |k(x)|\,dx \le \int_{r<|x|<2r} |k(x)|\,\frac{|x|}{r}dx = \frac{1}{r}\int_{r<|x|<2r} |k(x)|\,|x|\,dx$$
$$\le \frac{1}{r}\int_{|x|<2r} |k(x)|\,|x|\,dx \le \frac{1}{r}8C_1 r = 8C_1$$

Thus $\int_{r<|x|<2r} |k(x)|\,dx \le C_1$ and $\int_{|x|<r} |k(x)|\,|x|\,dx \le 4C_1 r$ are equivalent.

Remark 0.1.5.2 (notes on proof of proposition 5.6 in book)**.** *Let $tx' = x - y$, we have:*

$$\begin{aligned}
\int_{|x|>2|y|} \frac{|\Omega(x-y)-\Omega(x)|}{|x-y|^n}dx &= \int_{|tx'+y|>2|y|} \frac{|\Omega(tx')-\Omega(tx'+y)|}{t^n}d(tx'+y) \\
&\le \int_{|tx'|+|y|>2|y|} \frac{|\Omega(tx')-\Omega(tx'+y)|}{t^n}d(tx') \\
&= \iint_{t>|y|} \frac{|\Omega(tx')-\Omega(tx'+y)|}{t^n}t^{n-1}dtd\sigma(x') \\
&= \int_{t>|y|}\int_{|x'|=1} |\Omega(tx')-\Omega(tx'+y)|\,d\sigma(x')\frac{dt}{t}
\end{aligned}$$

By mean value theorem, we have $|x-y|^{-n} - |x|^{-n} = n\frac{y\cdot\xi}{|\xi|^{n+2}}$ with $\xi = (1-c)(x-y)+cx = x-(1-c)y$. Notice $|\xi| \ge |x| - (1-c)\,|y| \ge |x| - |y|$. Since $|y| \le \frac{|x|}{2}$, $|\xi| \ge \frac{|x|}{2}$. Thus

$$\left||x-y|^{-n} - |x|^{-n}\right| \le n\frac{|y|\,|\xi|}{|\xi|^{n+2}} \le n2^{n+1}\frac{|y|}{|x|^{n+1}}$$