Chapter 1

Classical Theory of Hardy Space

1.1 Harmonic Functions, Poisson Representation

In plane case, we can rewrite harmonic condition: $\Delta F = 0$ as $\Delta F = (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})F = 0$. Notices the equation $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 0$ is equivalent to the Cauchy-Riemann equations for function $F = u + iv$.

Remark (holomorphic indicates harmonic)**.** *If function F satisfies C-R equations, then F satisfies Laplace equation for $\mathbb{C}$. And if F satisfies $\Delta = 0$, so does $\bar{F}$. Thus F and $\bar{F}$ are all harmonic functions. Besides, the real and imaginary part of F, u and v also satisfy $\Delta = 0$, means u and v are harmonic functions.*

Remark (harmonic indicates holomorphic)**.** *Assume that u satisfies $\Delta = 0$. If we let $v_x = -u_y$ and $v_y = u_x$, then we have $v_{xy} = v_{yx}$, which indicates exists of v (equivalent of Fubini's Theorem and the equality of the mixed partial derivatives). And $v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$ indicates v is also harmonic. v is determined up to an additive constant, and $F = u + iv$ is holomorphic.*

Remark. *Later we will see holomorphic function is the special case of harmonic function.*

If u is a real harmonic function, by above remark, we know $u = \operatorname{Re} F$ for some holomorphic function $F(z) = \sum_{k=0}^{\infty} c_k z^k$. Then we can derive the series representation of u:

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta} \tag{1.1}$$

and it converges uniformly on compact subsets of $D(0, R)$.

Remark (uniformly converge on compact subsets)**.** *Using mean value property, sequence of holomorphic functions which uniformly converge on compact subsets, the limit is a holomorphic function.*

The partial sum of $u_n(re^{i\theta}) = \sum_{k=-n}^{n} a_k r^{|k|} e^{ik\theta}$ is obviously harmonic, and it converges uniformly on compact subsets of $D(0,R)$. Thus $u(re^{i\theta})$ is harmonic.

1.1.1 Harmonic function to Poisson integral (or Poisson representation of harmonic function)

Assumed u is harmonic in $D(0,R)$.If $R > 1$, we can represent a_k in equation (1.1) by $u(e^{it})$ using Fourier analysis. Finally we derive the Poisson kernel and the Poisson representation:

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) u(e^{it}) dt \tag{1.2}$$

If u is only harmonic on $D(0,1)$, the equation (1.2) still can be valid in some sense once we add some restriction on u.

Theorem 1.1.1.1 (theorem 1.3 in book)**.** *Let u be a harmonic function in D such that*

$$\sup_{0 \le r < 1} \int_{-\pi}^{\pi} \left| u(re^{it}) \right|^p dt < \infty \tag{1.3}$$

for some $p > 1$. Then there is a function $f \in L^p$ such that

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(t) dt \tag{1.4}$$

In later section, we will see left side of inequality (1.3) is H^p norm. The proof of Theorem 1.1.1.1 shows that how dual space, w*-topology and Banach-Alaoglu theorem perform in integral representation. The tricky part is using representation $u(r_n re^{i\theta})$. In the proof we also need $P_r(\theta) \in L^{p'}$ which is easy since $\|P_r(\theta)\|_\infty = \frac{1+r}{1-r}$ and $L^\infty([-\pi,\pi]) \subset L^{p'}([-\pi,\pi])$.

Remark (some details of proof of theorem 1.1.1.1)**.** *In metrizable space, compact and sequentially compact are equivalent.*

Let $f_n(t) = u(r_n e^{it})$, $(f_n) \subset L^{p'*}$ is in a close ball w.r.t p-norm and Banach-Alaoglu theorem indicates this ball is w^* compact. Since w^*-compact is equivalent to w^*-sequentially compact if this close ball is metrizable in w^*-topology, we need to prove this close ball is metrizable in w^*-topology.

Here is a direct proof. Since $L^{p'}$ is separable there is a countable dense set $(g_n) \in L^{p'}$. For every $g \in L^{p'}$, let $\hat{g}(f) = f(g)$ where $f \in L^{p'*}$. Any pair of points $f_1, f_2 \in L^{p'*}$ can be separate by some $g_i \in L^{p'}$ (since every $\hat{g_n}$ is w^*-continuous, density of (g_n) ensures f_1, f_2 can be separate), Thus $(\hat{g_n})$ is a countable family of continuous functions that separates points in $L^{p'*}$. Then we can use metric: $d(f_1, f_2) = \sum_{n=0}^{\infty} 2^{-n} |f_1(g_n) - f_2(g_n)|$.

We have proved that $L^{p'*}$ is metrizable in w^*-topology. Here is a topological thought. By Nagata-Smirnov metrization theorem, $L^{p'*}$ is regular. we don't know how to shown this property directly. Also, $L^{p'*}$ a basis countably locally *finite (by Nagata-Smirnov metrization theorem), we also want to show this property directly. Besides, consider Urysohn metrization theorem, if we have shown $L^{p'*}$ is regular, $L^{p'*}$ is metrizable once we show there is a countable basis for $L^{p'*}$. We don't know if it is possible.*

Now (f_n) is w^* sequentially compact in $L^{p'*}$. This means for any $g \in L^{p'}$, there is a subsequence (f_{n_k}) and $f \in L^{p'*}$ such that $\int gf_{n_k} \to \int gf$.

Remark. *$L^{p'*} \cong L^p$. Thus $L^{p'*}$ is a normed space and $L^{p'*}$ is metrizable.*

The $p = 1$ case is failed since we can not use w*-topology. L^1 is not a separate dual space of some space (Assume $L^1 = X^*$. L^1 is separable in w*-topology implies X is separable). One method showing L^1 is not a separate dual space of some space by showing L^1 does not have the Radon-Nikodym property (Radon-Nikodym theorem is valid).

Remark. *If X is a Banach space, then X^* has the Radon-Nikodym property (RNP) if (and only if) every separable, linear subspace of X has a separable dual (Charles Stegall: The Radon-Nikodym property in Conjugate Banach Space. II).*

Thus for $p = 1$ case, we can not use same argument as Theorem 1.1.1.1. However there is a relative result. L^1 can be isometrically imbedded in to M, the space of Borel measures with bounded variation, which is dual of continuous function with compact support space C. Thus we can use same argument to M and C and introduce Poisson-Stieltjes integral.

When we use Borel measures case in Theorem 1.1.1.1, we get *Poisson integral of positive measure for harmonic functions.*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{\pi}^{\pi} P_r(\theta - t)d\mu(t) \tag{1.5}$$

Here $d\mu(t)$ is the w*-limit of $u(r_n e^{it})dt$. The difference between (1.2) and (1.5) is that in (1.2), u needs to be harmonic in a little larger disk $D(0, R), R > 1$, but in (1.5) u needs not. However, this is not hold for all harmonic functions since we need constraints.

1.1.2 Poisson integral indicates harmonic

Using Fourier series, we show that given a function $f \in L^p$, $1 \leq p \leq \infty$ (or a complex Borel measure μ), Poisson integral u=$P(f)$ (or u=$P(\mu)$) is real part of a holomorphic function. Thus it is harmonic (Theorem 1.11 (or 1.14) in book). And we give bound of norm of u by norm of f (or measure μ):

- $\int_{-\pi}^{\pi} \left|u(re^{it})\right|^p dt \leq \int_{-\pi}^{\pi} |f(t)|^p dt \; for \; p < \infty$
- $|u(z)| \leq \|f(t)\|_\infty \; for \; p = \infty$
- $\int_{-\pi}^{\pi} \left|u(re^{it})\right| dt \leq \int_{-\pi}^{\pi} d|\mu|(t)$

For $f \in L^p$, $p \leq \infty$ case, the equality holds when $r \to 1$ or take sup on right hand side. We prove this in section 3.

1.1.3 Boundary behavior of Poisson integral (or norm convergence and pointwise convergence)

Definition 1.1.3.1 (Dirichlet problem)**.** *Given a continuous function f on ∂D, we want to find a continuous function on $\bar{D}$, which is harmonic in D and coincides with f on ∂D.*

From section 1.1.2, we can see $u(re^{it}) = P(f)$ is harmonic function. Roughly speaking, if $u(e^{it}) = f$ (Actually the domain of u is D, so $u(e^{it})$ is not defined), the classical Dirichlet problem is solved. Thus we need to study the boundary behavior of Poisson integral. *The key idea is approximate identity.* Using this idea, we prove the following theorem.

Theorem 1.1.3.2 (Theorem 1.16 in book)**.** *Let ϕ_α be an approximate identity on the torus T. Then:*

1. *If $f \in L^p([-\pi,\pi])$ with $1 \leq p < \infty$ and f_α stands for convolution*

$$f_\alpha(\theta) = (f * \phi_\alpha)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - t)\phi_\alpha(t)dt$$

   it follows that $f_\alpha \to f$ in L^p, i.e.:

$$\int_{-\pi}^{\pi} |f_\alpha(t) - f(t)|^p \, dt \to 0$$

2. *If f is a continuous 2π-periodic function, we have $f_\alpha \to f$ uniformly on T.*

Theorem 1.1.3.2 shows that for $f \in L^p$, $1 \leq p < \infty$ or $f \in C$, Poisson integral converges to boundary in norm. Following the proof of theorem 1.1.3.2, we can show another two case: $p = \infty$ and $f \in M$. The convergence becomes w* convergence (Corollary 1.19 in book).

Here comes another topic: What about pointwise convergence for $P(\mu)$ on boundary if $P(\mu)$ is Poisson-Stieltjes integral? We need a new concept: non-tangentially converge. We show that $P(\mu)(z) \to F'(\theta_1)$ as $z \to e^{i\theta_1}$ N.T., where $F(\theta) = \int_0^\theta d\mu(t)$ and $F'(\theta_1)$ exists and finite (Theorem 1.20 in book).

You can omit the following remark if you choose not to check the proof in book.

Remark (Proof of Theorem 1.20)**.** *First we take $c > 0$, since we need proof for any $c > 0$. This c decides the approach region.*
Then given $\epsilon > 0$, we take δ small enough s.t. $|F(t)| < \epsilon|t|$ when $|t| < \delta$. This δ can be taken since $F(0) = 0$ and $F'(0) = 0$.
If we take r large enough and $re^{i\theta}$ in the region. then $|\theta|$ can be less than $\frac{\delta}{4}$.

The estimation of $u(re^{i\theta})$ is (we use $F(t)=\int_{-\pi}^{\pi} d\mu(t)$ in the third line):

$$\begin{aligned}
\left|u(re^{i\theta})\right| &= \left|\frac{1}{2\pi}\int_{\delta<|t|\le\pi} P_r(\theta-t)d\mu(t) + \frac{1}{2\pi}\int_{-\delta}^{\delta} P_r(\theta-t)d\mu(t)\right| \\
&\le \left|\frac{1}{2\pi}\int_{\delta<|t|\le\pi} P_r(\theta-t)d\mu(t)\right| + \left|\frac{1}{2\pi}\int_{-\delta}^{\delta} P_r(\theta-t)d\mu(t)\right| \\
&\le (\sup_{|t|>\delta} P_r(\theta-t))\cdot\frac{1}{2\pi}\int_{\delta<|t|\le\pi} d\left|\mu\right|(t) \\
&\quad + \left|(P_r(\theta-t)\cdot\frac{1}{2\pi}F(t))\Big|_{-\delta}^{\delta} + \int_{-\delta}^{\delta} P_r'(\theta-t)\frac{1}{2\pi}F(t)dt\right| \\
&\le (\sup_{\frac{3\delta}{4}<|t|<\pi+\frac{\delta}{4}} P_r(t))\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} d\left|\mu\right|(t) \\
&\quad + \left|(P_r(\theta-t)\cdot\frac{1}{2\pi}F(t))\Big|_{-\delta}^{\delta}\right| + \left|\int_{-\delta}^{\delta}(P_r'(\theta-t)\frac{1}{2\pi}F(t))dt\right| \\
&\le (\sup_{|t|>\frac{3\delta}{4}} P_r(t))\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} d\left|\mu\right|(t) \\
&\quad + (\sup_{|t|>\frac{3\delta}{4}} P_r(t))\cdot\frac{1}{2\pi}\int_{-\delta}^{\delta} d\left|\mu\right|(t) + \left|\frac{1}{2\pi}\int_{-\delta}^{\delta} P_r'(\theta-t)F(t)dt\right|
\end{aligned}$$

Lebesgue-Stieltjes integral is related to bounded increasing function. The integral has decomposition $d\mu(t) = f(t)dt + d\sigma(t)$, $f \in L^1$. $\int_0^\theta d\sigma(t)$ is a jump function. Thus $F'(\theta) = f(\theta)$ a.e.. Considering $P(f)$, $f \in L^p$, $1 \le p \le \infty$, or even $P(\mu)$, the integral $P(f)$ or $P(\mu)$ is L-S integral. Thus N.T. convergence holds a.e..

By section 1.1.1, Harmonic function with constraints (bounded in some sense) can be written as Poisson integral $P(f)$ or $P(\mu)$ (Corollary 1.10 in book). Thus the theorem of Fatou holds: *Any function holomorphic and bounded in D has non-tangential boundary values a.e..*

The difference between $p = 1$ and $p > 1$ is the starting point of the theory of Hardy spaces.

1.1.4 Harmonic function in higher dimension

A continuous function is harmonic in region Ω if and only if it satisfies mean value property. For mean value property to harmonic, we need an approximate identity.

You can omit the following remark if you choose not to check the proof in book.

Remark (Proof of Theorem 1.22)**.** *In converse part, we choose region Ω_ϵ is to make integral $\int_{\mathbb{R}^n} u(x_0 + r\sigma - y)\phi_\epsilon(y)dy$ and $\int_{\mathbb{R}^n} u(x_0 - y)\phi_\epsilon(y)dy$ defined. In other words, if $x_0 + r\sigma - y$ or $x_0 - y$ out of Ω, then $\phi_\epsilon(y) = 0$.*

A consequence of mean value property is maximum principle. The proof can be given by topological technic: Assumed u attains maximum value m in Ω. Let $A = \{x : u(x) = m\}$. Using mean value property, we show every x is interior point of A. Thus A is open. However, $\Omega \setminus A = \{x : u(x) < m\}$ is open since u is continuous. Ω is connected and A is not empty. $\Omega \setminus A$ is empty. Thus u is constant.

By maximum principle and minimum principle, we can show u is unique in Ω if u is harmonic in Ω and u is continuous on $\partial\Omega$.

We introduce the Poisson kernel $P(x, s) = \frac{1-|x|^2}{|x-s|^n}$ for Dirichlet problem in n dimension. The solution is $\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x, s)f(s)ds$.

There is a weaken form of mean value property (or called discrete mean value property). u is harmonic if mean value property is satisfied only on a sequence of $r_n \to 0$ (Theorem 1.30 in book).

You can omit the following remark if you choose not to check the proof in book.

Remark (Proof of theorem 1.30)**.** *The proof here is not well understand. The set $K = \{u(x) - v(x) = m : x \in \overline{B(x_0, R)}\}$ is compact in $\overline{B(x_0, R)}$ since K is closed in compact set $\overline{B(x_0, R)}$. We can use finite open cover to prove K is also compact in $B(x_0, R)$. Let $f : K \to \mathbb{R}$, $f(x) = d(x, x_0)$, f is continuous function on compact set K thus it attains maximum value of f. Let x_1 be a point of the maximum value. I just can image that for sphere $\partial B(x_1, r)$, only half of the sphere is in K otherwise $f(x_1)$ is not the maximum value. Here r need to be small enough to ensure $\partial B(x_1, r) \subset B(x_0, R)$. But then $u(x_1) - v(x_1) < m$ if we use integral with $r_j < r$. Thus it is a contradiction.*

I don't know how the sequence (r_n) plays in this proof.

Then we go to the reflection principle and Liouville theorem.

Finally we go to Dirichlet problem on unbounded domain, in particular $\mathbb{R}^{n+1}_+$. This problem cannot have a unique solution but can have a unique bounded solution. And we give Poisson kernel for $\mathbb{R}^{n+1}_+$ by Fourier transform method.

1.2 Subharmonic Functions

This section is about a new concept: subharmonic function. Subharmonic function can be considered as a generalization of harmonic function, as it preserves some important property of harmonic function such as maximum principle. On the other hand, we will see why we call it "sub" harmonic: subharmonic function can be controlled by harmonic function. Also, by some operations like composition and taking absolute value, subharmonic function can still be subharmonic, but harmonic function can not. Finally we will begin our study of zeroes of holomorphic function.

Now we give the definition of subharmonic function.

Definition 1.2.0.1. *A subharmonic function on an open set $\Omega \subset \mathbb{R}^n$ is a function v defined on Ω, with values $-\infty \le v(x) < \infty$ and satisfying the following two conditions:*

1. *v is upper semicontinuous in Ω.*

2. *For every $x_0 \in \Omega$, there is a ball $B(x_0, r(x_0)) \subset \Omega$, $r(x_0) > 0$, such that for every r with $0 < r < r(x_0)$*

$$v(x_0) \leq \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} v(x_0 + r\sigma) d\sigma \tag{1.6}$$

1.2.1 Upper semicontinuous

There is two equivalence definition of v being upper semicontinuous in Ω:

1. For every $t \in \mathbb{R}$, the set $\{x \in \Omega : v(x) < t\}$ is open.

2. For every $x_0 \in \Omega$:
$$\limsup_{x \to x_0 \ in \ \Omega} v(x) \leq v(x_0) \tag{1.7}$$
This is equivalent to that for every $y > f(x_0)$, there exists a neighborhood U of x_0 such that $f(x) < y$ for all $x \in U$.

Remark (Proof of equivalence)**.**
We prove by contradiction, Suppose that $\limsup v(x) > v(x_0)$, we can find x_k, $v(x_0) < v(x_k) < \limsup v(x)$. Since $v^{-1}([-\infty, v(x_k))$ is open and $v(x_0) < v(x_k)$, $v(x_0) \in v^{-1}([-\infty, v(x_k))$. Thus there is a neighborhood $U \in \mathscr{N}(x_0)$, $U \subset v^{-1}([-\infty, v(x_k))$. Now we can find another $x_n \in U$ s.t. $v(x_k) < v(x_n) < \limsup v(x)$. $v(x_n) > v(x_k)$ means $v(x_n) \notin v^{-1}([-\infty, v(x_k))$, contradicts to $v(x_n) \in U \subset v^{-1}([-\infty, v(x_k))$.

We prove the converse by contradiction. Suppose there is a number $t_0 \in \mathbb{R}$, $v^{-1}([-\infty, t_0))$ is not open. So there is $x_0 \in v^{-1}([-\infty, t_0))$, such that $\forall U_k \in \mathscr{N}(x_0)$, there is $x_k \in U_k$, $x_k \notin v^{-1}([-\infty, t_0))$, which is equivalent to $v(x_k) \geq t$. This contradicts to $\limsup v(x_k) \leq v(x_0) < t_0$.

If v is subharmonic, inequality (1.6) implies another direction of inequality (1.7). Thus we actually have equality in (1.7).

An important and frequently used tool is following characterization of upper simicontinuity.

Proposition 1.2.1.1. *v is upper semicontinuous in Ω if and only if for every compact $K \subset \Omega$, v is the limit over K of a decreasing sequence of continuous function.*

This proposition is important tool in proof of some following theorems.

Remark (Notes on proof of proposition 1.2.1.1)**.** *The converse part, by using partition of the unity, we construct a sequence of decreasing function (u_k). We need to prove v is the limit of (u_k).*

For any $x_0 \in K$, there is a sequence of balls $(B(x_{n,i}, \epsilon_n))$, $\epsilon_n \to 0$ s.t. $x_0 \in B(x_{n,i}, \epsilon_n)$ for all n. For each n, $B(x_{n,i}, \epsilon_n)$ is in finite open cover $(B(x_{n,i}, \epsilon_n))_i$ of K. Since $m_{n,i} = \sup_{B(x_{n,i}, \epsilon_n)} v$, $u_n(x_0) \geq v(x_0)$ for all n. By definition of *upper semicontinuous, for any $y > v(x_0)$, there is a neighborhood $U \in \mathcal{N}(x_0)$, $v(x) < y$ for all $x \in U$. Let $B(x_{n,i}, \epsilon_n) \subset U$, $m_{n,i} < y$. Thus $u_n(x_0) < y$ for all large enough n. Since y is any number larger than $v(x_0)$, $\limsup u(x) \leq v(x_0)$. This shows $u(x_0) = v(x_0)$.*

1.2.2 Property of subharmonic function

First, subharmonic function satisfies maximum principle.

Remark (Notes on proof of maximum principle)**.** *Like proof of maximum principle for harmonic function, but we need to take care of semicontinuous. Assume $v(x_0) = M$, the maximum value. Choose r to satisfy inequality* (1.6)*. If for some $x \in \partial B(x_0, r)$, $v(x) = m < M$, by semicontinuous, $\limsup v(x_k) \leq v(x) < m + \epsilon < M$. Thus there is a neighborhood $U \in \mathcal{N}(x)$, $\sup_{x_k \in U} v(x_k) < M$. Then $\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0 + r\sigma)d\sigma < M = v(x_0)$. This contradicts to inequality* (1.6)*. Then the following is same as proof for maximum principle for harmonic function.*

The best reason why we use name 'subharmonic' is following: v is subharmonic function if and only if when v less or equal to a harmonic function u on boundary of region, $v \leq u$ in entire region. We remind the reader that proposition 1.2.1.1 appears as an important step in proof.

There are two examples of using proposition 1.2.1.1 to detail with subharmonic function v. One is if v is not identically equal to $-\infty$, then

$$\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0 + r\sigma)d\sigma > -\infty$$

for every $\overline{B(x_0, r)} \subset \Omega$. In proof of this statement we also use Poisson representation of harmonic function and little topological trick. Another example is

$$m(r) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(r\sigma)d\sigma \tag{1.8}$$

is an increasing function.

There is another necessary and sufficient condition for v to be harmonic using Laplace operator. It says v is subharmonic if and only if $\Delta v \geq 0$.

Remark (Proof of proposition 2.10 in book)**.** *Author says we need to show that $v(x_0) \leq \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0 + r\sigma)d\sigma$. But I think this is obvious since we consider x_0 which $v(x_0) = 0$ and $v(x) \geq 0$ on Ω. And this inequality is not used in the following part of proof.*

1.2.3 Estimation for zeroes of holomorphic function

We first state that if v is subharmonic, ϕ is increasing and convex function. Then $\phi \circ v$ is also subharmonic. This is useful when we need to connect holomorphic function with subharmonic function.

Now here comes our first theorem about zero points of holomorphic: Jensen's formula.

Theorem 1.2.3.1 (Jensen's formula)**.** *Let F be holomorphic in $D(0,R)$ and suppose that $F(0) \neq 0$. Let $0 < r < R$ and call $z_1, z_2, \cdots, z_n$ the zeroes of F in $\overline{D(0,r)}$ listed according to their multiplicities. Then:*

$$\log|F(0)| + \sum_{j=1}^{n} \log \frac{r}{|z_j|} = \frac{1}{\pi} \int_{-\pi}^{\pi} \log \left|F(re^{it})\right| dt. \tag{1.9}$$

The proof in book need a lemma: $\int_{-\pi}^{\pi} \log \left|1 - e^{it}\right| dt = 0$. You can also refer section 1 in Chapter 6 of Stein's *Complex Analysis*. The proof there is very different to the one in this book.

Remark (Proof of lemma (Lemma 2.12 in book))**.** *There is an inequality: for $|t| < \frac{\pi}{3}$, $\log \frac{1}{|\sin t|} \leq \frac{C_\alpha}{|t|^\alpha}$. I can prove it using elementary calculus, but I think it is an easy observation.*

To continue our explorer of zeroes of holomorphic function, we show some connection between holomorphic function and subharmonic function. More precisely, If F is holomorphic, not identically 0, then $\log|F(z)|$, $\log^+|F(z)| = \max(\log|F(z)|, 0)$ and $|F(z)|^a$ for any $0 < a < \infty$, are all subharmonic. Then we give definition of Hardy space on D. We define for $f \in H(D)$ (F is holomorphic in D) :

- $m_0(F, r) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ \left|F(re^{it})\right| dt\right)$
- $m_p(F, r) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|F(re^{it})\right|^p dt\right)^{\frac{1}{p}}$
- $m_\infty(F, r) = \sup_t \left|F(re^{it})\right|$

This function is an increasing function of r in $[0, 1)$ (Hardy convex theorem), see equation (1.8) for case $0 \leq p < \infty$. $m_\infty(F, r)$ is also an increasing function but it uses a different method (Hadamard three-circle theorem).

Now we define Hardy space H^p:

Definition 1.2.3.2. *For $0 < p \leq \infty$, we define $H^p(D)$:*

$$H^p(D) = \{F \in H(D) : \|F\|_{H^p} = \sup_{0 \leq r < 1} m_p(F, r) < \infty\}$$

For $p = 0$, we have the Nevanlinna class N, defined by:

$$N = \{F \in H(D) : \sup_{0 \leq r < 1} m_0(F, r) < \infty\}$$

If $0 < p < q < \infty$, we have $H^\infty \subset H^q \subset H^p \subset N$

Remark. *The first two inclusions are as the same as the inclusion for L^p, the last inclusion is by:*

$$(m_0(r))^p = \exp\left(p \int_{-\pi}^{\pi} \log^+ |F| \frac{dt}{2\pi}\right) \leq \int_{-\pi}^{\pi} \exp\left(p \log^+ |F|\right) \frac{dt}{2\pi}$$

Notice that:

$$\int_{-\pi}^{\pi} \exp\left(p \log^+ |F|\right) \frac{dt}{2\pi} = \int_{\substack{t\in[-\pi,\pi] \\ |F|>1}} |F|^p \frac{dt}{2\pi} + \int_{\substack{t\in[-\pi,\pi] \\ |F|\leq 1}} 1 \frac{dt}{2\pi}$$

Thus:

$$(m_0(r))^p \leq \int |F|^p \frac{dt}{2\pi} + 1$$

There left three theorems in this section. I interpret it shortly and informally. First one is for $F \in N$, the zeroes (z_j) of F cannot be too far from the boundary, or $\sum_j (1 - |z_j|) < \infty$. The second one is that if $\sum_j (1 - |z_j|) < \infty$ holds, the "Blaschke product"

$$B(z) = z^k \prod_{j=1}^{\infty} \frac{z_j - z}{1 - z\bar{z}_j} \frac{|z_j|}{z_j}$$

converges uniformly on each compact subset to a function H^∞ and they have the same zeroes.

Remark. *If f is holomorphic in an open disc that vanishes on a sequence of distinct points with a limit point in the disc. Then f is identically 0 (Theorem 4.8 in chapter 2. Stein's Complex Analysis). However in Blaschke product case, there can be infinitely zeroes, since it can have limit points on boundary. Thus $B(z)$ can be not identically 0. However, for any $r < 1$, $B(z)$ can only have finite zeroes in $\overline{D(0,r)}$.*

Here is a convention. If for some function F in D, the non-tangential boundary value of F is known to exist at e^{it}, we shall denote it by $F(e^{it})$.

The third theorem is the Blaschke product has properties: $\left|B(e^{it})\right| = 1$ for a.e. t and

$$\lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left|B(re^{it})\right| dt = 0.$$

If $F \in H^p$ with $p \geq 1$, which means F is holomorphic F and can be write as Poisson (or Poisson-Stieltjes) integral. By Fatou Theorem, we know that $F(e^{it})$ exists a.e.. In the next section we shall extend this result to any $p > 0$. The above three theorems play important roles in proving extension.

Remark (Notes on proof of three theorems (theorem 2.19, 2.21 and 2.22))**.** *In proof of theorem 2.19, we assume $F(0) = 0$, since we can use function $\frac{F(z)}{z^k}$ if $F(z)$ has zero of order k in $z = 0$. And this modification does not affect the sum $\sum_j (1 - |z_j|)$.*

The step

$$\sum_1^n \log \frac{1}{|z_j|} \leq M - n\log r - \log |F(0)|$$

to

$$\sum_1^\infty \log \frac{1}{|z_j|} \leq M - \log |F(0)|$$

is not clear. I think we can not first let $r \to 1$ then $n \to \infty$. We can not control taking limit for which one first.

In proof of theorem 2.21, the final step is:

$$\begin{aligned}\left|1 - \frac{z_j - z}{1 - z\bar{z}_j}\frac{|z_j|}{z_j}\right| &= \left|1 - \frac{z_j|z_j| - z|z_j|}{z_j - z|z_j|^2}\right| = \left|\frac{z_j - z|z_j|^2 - z_j|z_j| + z|z_j|}{z_j - z|z_j|^2}\right| \\ &= (1 - |z_j|)\left|\frac{z_j + z|z_j|}{z_j - z|z_j|^2}\right| = (1 - |z_j|)|z_j|\left|\frac{e^{it} + z}{e^{it} - z|z_j|}\right| \\ &= (1 - |z_j|)|z_j|\left|\frac{z' + 1}{1 - z'|z_j|}\right|\end{aligned}$$

where $z' = z \cdot e^{-it}$. Since $|z_j| < 1$, $|z'| = |z| \leq r$, we have $|z' + 1| \leq |z'| + 1 \leq r + 1$, $|1 - z'|z_j|| \geq 1 - |z'|z_j|| = 1 - |z'||z_j| \geq 1 - r$. Thus

$$\left|1 - \frac{z_j - z}{1 - z\bar{z}_j}\frac{|z_j|}{z_j}\right| \leq (1 - |z_j|)\frac{1 + r}{1 - r}$$

In proof of theorem 2.22, we know if $z\bar{w} \neq 1$, then Blaschke factors:

$$\left|\frac{w - z}{1 - \bar{w}z}\right| = 1 \ if \ \ |z| = 1 \ or \ \ |w| = 1$$

Since in $B_n(z)$ $\left|e^{it}\right| = 1$, I think $B_n(e^{it}) = 1$ everywhere, not a.e..

$\left|B_n(re^{it})\right| \to 1$ uniformly as $r \to 1$, since B_n is holomorphic in a neighborhood of $\bar{D}$. This is easy if we choose $D(0, 1 + \epsilon)$, s.t. $z\bar{z}_j \neq 1$ in $D(0, 1 + \epsilon)$.

1.3 F.Riesz Factorization Theorem

This section can be seen as a generalization of first section. In first section, we talk about norm convergence and pointwise convergence when boundary function f is in L^p, $1 < p \leq \infty$ and f is a measure. This conclusion is for harmonic function. Harmonic function has series representation:

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}$$

and we can derive Poisson representation $u(re^{i\theta}) = P_r(f)$. Since holomorphic function also has series representation:

$$u(re^{i\theta}) = \sum_{k=0}^{\infty} a_k r^k e^{ik\theta}$$

, we can consider this representation as special case of harmonic function with $a_k = 0$ for $k < 0$. Poisson representation is also hold for holomorphic function, thus the converge result is hold also for holomorphic function. The following theorem is a summary of these results.

Theorem 1.3.0.1 (theorem 3.1 in book)**.** *Let $F \in H^p$ with $1 < p \leq \infty$. Then:*

1. *For almost every t. the limit*

$$F(e^{it}) = \lim F(z) \text{ as } z \to e^{it} \text{ N.T.}$$

   exists. The function $f(t) = F(e^{it})$ belongs to $L^p([-\pi, \pi])$ and $F = P(f)$

2. *If $p < \infty$:*

$$\int_{-\pi}^{\pi} \left|F(re^{it}) - F(e^{it})\right|^p dt \to 0 \text{ as } r \to 1$$

   If $p = \infty$, $F(re^{it}) \to F(e^{it})$ in the w^*-topology of L^∞ as $r \to 1$.

   For each $1 < p \leq \infty$: $\|F\|_{H^p} = \|f\|_p$.

3. *F is the Cauchy integral of its boundary function, that is:*

$$F(z) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{F(\xi)}{\xi - z} d\xi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F(e^{it})}{e^{it} - z} e^{it} dt$$

Remark. *For first statement in theorem 1.3.0.1, N.T. limit holds for $p = 1$, but $P(f)$ may not be hold.*

Remark. *$u(re^{it}) = P_r(t)$ is neither in H^p nor N. $P_r(t)$ is harmonic but not holomorphic.*

In this section we will extend this result to $p \leq 1$. The main idea is to factorize $F(z)$ to a Blaschke product $B(z)$ and a non-vanish function $H(z)$.

1.3.1 Result of non-vanish case

Suppose that $F \in H^p$, $\frac{1}{2} \leq p < 1$. If F does not vanish in D. Then $F(z) = e^{f(z)}$ for some holomorphic function f. Let $G(z) = e^{\frac{f(z)}{2}}$, we have $F(z) = G(z)^2$, $G(z) \in H^{2p}$ and $\|G\|_{H^{2p}}^2 = \|F\|_{H^p}$. Since $2p \geq 1$, we have $G(e^{it}) = \lim G(z)$ a.e. as $z \to e^{it}$ N.T.. It follows that $F(e^{it}) = \lim F(z)$ a.e. as $z \to e^{it}$ N.T.

We know that $\int_{-\pi}^{\pi} \left|F(re^{it}) - F(e^{it})\right|^p dt \to 0$ as $r \to 1$ if $p > 1$. Suppose that $F \in H^p$, $\frac{1}{2} \le p < 1$ and we have $F(z) = G(z)^2$ as before, then:

$$
\begin{aligned}
&\int_{-\pi}^{\pi} \left|F(re^{it}) - F(e^{it})\right|^p dt \\
=&\int_{-\pi}^{\pi} \left|G(re^{it})^2 - G(e^{it})^2\right|^p dt \\
=&\int_{-\pi}^{\pi} \left|G(re^{it}) + G(e^{it})\right|^p \left|G(re^{it}) - G(e^{it})\right|^p dt \\
\le&(\int_{-\pi}^{\pi} \left|G(re^{it}) + G(e^{it})\right|^{2p} dt)^{\frac{1}{2}}(\int_{-\pi}^{\pi} \left|G(re^{it}) - G(e^{it})\right|^{2p} dt)^{\frac{1}{2}} \\
\le&(\int_{-\pi}^{\pi} (2\left|G(e^{it})\right|)^{2p} dt)^{\frac{1}{2}}(\int_{-\pi}^{\pi} \left|G(re^{it}) - G(e^{it})\right|^{2p} dt)^{\frac{1}{2}} \\
\le&2^p \left\|G\right\|_{H^{2p}}^p (\int_{-\pi}^{\pi} \left|G(re^{it}) - G(e^{it})\right|^{2p} dt)^{\frac{1}{2}} \to 0 \ as \ r \to 1
\end{aligned}
$$

We conclude that $F \in H^p$, $\frac{1}{2} \le p < 1$. If F does not vanish in D. Then there is a boundary function $F(e^{it})$, $F(z)$ converges to $F(e^{it})$ both in pointwise sense and norm sense.

Remark. *There is a basic inequality, used also in proving Minkowski inequality:* $|a+b|^p \le 2^p(|a|^p + |b|^p)$ *for* $p > 0$. *To prove this we only need to consider two case:* $|a| \ge |b|$ *or* $|a| \le |b|$.

By induction, this conclusion can be extended to $0 < p < 1$. Thus two types of convergence holds for all $0 < p < \infty$.

Remark. *Author uses Fatou's lemma when $F(z)$ converges to $F(e^{it})$ N.T.. I think we can use this lemma even if it converges radially.*

1.3.2 Result of H^p case

In the end of last section review, we state three theorems:

- For $F \in N$, the zeroes (z_j) of F satisfies $\sum_j (1 - |z_j|) < \infty$.
- If $\sum_j (1 - |z_j|) < \infty$ holds, the Blaschke product converges uniformly on each compact subset to a function $B(z) \in H^\infty$ and they have zeroes (z_j).
- $\left|B(e^{it})\right| = 1$ a.e.

If we let $H = \frac{F}{B}$, where Blaschke product is formed by zeroes of F, then H does not have any zeroes. Besides, if $F \in N$, then $H \in N$ and $\left\|H\right\|_N = \left\|F\right\|_N$. If $F \in H^p$, then $H \in H^p$ and $\left\|H\right\|_{H^p} = \left\|F\right\|_{H^p}$ (theorem 3.3 in book). Notice now we can use method in section 1.3.1 on H. We have following result:

Theorem 1.3.2.1 (theorem 3.6 in book)**.** *Let $F \in H^p$ with $0 < p \le \infty$. Then:*

1. *For almost every t. the limit*

$$F(e^{it}) = \lim F(z) \text{ as } z \to e^{it} \text{ N.T.}$$

exists. The function $f(t) = F(e^{it})$ belongs to $L^p([-\pi,\pi])$.

2. $\int_{-\pi}^{\pi} \left|F(re^{it}) - F(e^{it})\right|^p dt \to 0$ *as* $r \to 1$

3. $\|F\|_{H^p} = \lim_{r\to 1} (\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F(re^{it})\right|^p dt)^{\frac{1}{p}} = (\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F(e^{it})\right|^p dt)^{\frac{1}{p}}$

Another statement is that $F \in H^p$ can be improved to $F \in H^q$ if the boundary function $F(e^{it}) \in L^q$ (Corollary 3.7). The hard part of its proof is the case $p < q$ and $p \leq 1$. We factorize F as $F = BG^n$, where $np > 1$. Since $F(e^{it}) \in L^q$ and $\left|G(e^{it})\right|^n = \left|F(e^{it})\right|$, $G(e^{it}) \in L^{nq}$. Thus $G \in H^{nq}$ and $F \in H^q$.

1.3.3 H^1 function and its boundary

Recall in section 1, when u is a harmonic function in D and

$$\sup_{0\leq r<1} \int_{-\pi}^{\pi} \left|F(re^{it})\right| dt < \infty$$

, we can only say u is $P(\mu)$ for some Borel measure and the result can not be improved (consider Poisson kernel). However, if $F \in H^1$, in other words $\sup_{0\leq r<1}\int_{-\pi}^{\pi} \left|F(re^{it})\right| dt < \infty$ and F is holomorphic, then by $F(re^{it}) \to F(e^{it})$ in L^1. Thus F can be written as the Poisson integral and the Cauchy integral of its boundary function $F(e^{it})$.

Remark (notes on proof corollary 3.9 in book)**.** *I don't know why*

$$G(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \operatorname{Re} F(e^{it}) dt$$

is holomorphic function.

An consequence of Poisson representation for H^1 functions is a famous theorem due to F. and M. Riesz. It says given a Borel measure μ, when negative frequencies of Fourier coefficients of μ is zero, then μ is absolutely continuous w.r.t. Lebesgue measure, i.e.: $d\mu(t) = f(t)dt$ for some $f \in L^1$. This theorem shows the difference between bounded holomorphic function $\sum_{k=0}^{\infty} a_k r^k e^{ik\theta}$ and bounded harmonic function $\sum_{k=-\infty}^{\infty} a_k r^k e^{ik\theta}$ (bounded as $\sup_{0\leq r<1}\int_{-\pi}^{\pi} \left|F(re^{it})\right| dt < \infty$). The vanish of negative frequencies make bounded harmonic function (or Poisson integral of complex Borel measure) to bounded holomorphic function.

Remark (notes on proof of corollary 3.11 in book)**.** *f is bounded variation, then f can be written as difference of two increasing bounded function. This is equivalent to f can be written as difference of two Borel measure. Thus $f(t) = c + \int_{-\pi}^{t} d\mu(s)$ where $c = f(-\pi)$.*

The integration by parts:

$$\begin{aligned}\int_{-\pi}^{\pi} e^{ijt} d\mu(t) &= e^{ijt} \int_{-\pi}^{t} d\mu(s)\Big|_{-\pi}^{\pi} - ij \int_{-\pi}^{\pi} g(t) e^{ijt} dt \\ &= \left(e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s) - e^{-ij\pi} \int_{-\pi}^{-\pi} d\mu(s) \right) - ij \int_{-\pi}^{\pi} g(t) e^{ijt} dt \\ &= e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s) - ij \int_{-\pi}^{\pi} (F(e^{it}) - c) e^{ijt} dt \\ &= e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s) - \lim_{r\to 1} ij \int_{-\pi}^{\pi} F(re^{it}) e^{ijt} dt \\ &= e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s)\end{aligned}$$

The limit in fourth equality is by $F \in H^1$, $F(re^{it}) \to F(e^{it})$ in L^1. This limit is 0 since $F \in H^1$, the negative frequencies are 0. $e^{ij\pi} \int_{-\pi}^{\pi} d\mu(s) = e^{ij\pi} g(\pi)$ is 0 since $f(\pi) = f(-\pi) + g(\pi)$ and $f(\pi) = f(-\pi)$.

Corollary 3.11 in book shows a condition when bounded variation implies absolutely continuity. This Corollary emphases again 'holomorphic condition' or vanish of negative frequencies makes a Borel measure absolutely continuous. Theorem 3.12 in book says that $F' \in H^1$ is the necessary and sufficient condition of holomorphic $F \in H(D)$ is absolutely continuous on boundary.

Remark (notes on proof of theorem 3.12 in book)**.** *$F \in H^1$ implies $F' \in H(D)$. Since*

$$\sup_{0\le r<1} \int_{-\pi}^{\pi} \left|F(re^{it})\right| dt = \sup_{0\le r<1} \int_{-\pi}^{\pi} \left|ire^{it} F(re^{it})\right| dt$$

, $F'(z) \in H^1$ if and only if $izF'(z) \in H^1$.

I don't know why the difference is harmonic and continuous at the origin, it has to be a constant c.

There is a corollary of Theorem 3.12 in book which is useful in next section.

Corollary 1.3.3.1 (corollary 3.13 in book)**.** *Let Γ be a Jordan curve and let F be a conformal map from D to interior domain bounded by a Jordan curve Γ. Then Γ is rectifiable if and only if $F' \in H^1$.*

1.4 Some Classical Inequalities

In this section we study two classical Inequalities: Hardy's inequality and Fejer-Riesz inequality. The first inequality is an example of why H^p is a natural replacement of L^p for $p \le 1$. The second inequality shows some geometry properties of conformal mappings.

1.4.1 Hardy's inequality

Theorem 1.4.1.1 (Hardy's inequality)**.** *Let* $F(z) = \sum_{j=0}^{\infty} a_j z^j$ *be in* H^1*. Then:*

$$\sum_{j=0}^{\infty} \frac{|a_j|}{j+1} \leq C \|F\|_{H^1}$$

with a constant C *independent of* F*.*

Remark (notes on proof of theorem 1.4.1.1)**.** *We know the principal branch of the logarithm* $\log z = \log r + i\theta$ *where* $z = re^{i\theta}$ *with* $|\theta| < \pi$*. Thus* $\operatorname{Im} \log 1 - z = \arg 1 - z$*. It is easy to see* $-\frac{\pi}{2} < \arg 1 - z < \frac{\pi}{2}$*.*

$$\begin{aligned}
F(re^{it})u(re^{it}) =& (\sum_{j=0}^{\infty} a_j (re^{it})^j)(\frac{i}{2}\sum_{j\neq 0} j^{-1} r^{|j|} e^{ijt}) \\
=& (\sum_{j=0}^{\infty} a_j r^j e^{ijt})(\frac{i}{2}\sum_{k\neq 0} k^{-1} r^{|k|} e^{ikt})
\end{aligned}$$

After taking integral, only $j + k = 0$ *term does not vanish, thus:*

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} F(re^{it})u(re^{it})dt =& (\frac{i}{2}\sum_{j+k=0} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_j r^j e^{ijt} k^{-1} r^{|k|} e^{ikt} dt) \\
=& \frac{i}{2}\sum_{j+k=0} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_j r^{j+|k|} e^{i(j+k)t} k^{-1} dt \\
=& \frac{i}{2}\sum_{j=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_j r^{2j} (-j)^{-1} dt \\
=& \frac{i}{2}\sum_{j=1}^{\infty} a_j r^{2j} (-j)^{-1} \\
=& -\frac{i}{2}\sum_{j=1}^{\infty} a_j j^{-1} r^{2j}
\end{aligned}$$

The corollary 4.2 in book shows that if $F(e^{it})$ is absolutely continuous (equivalent to $F' \in H^1$), then $(\hat{F}(n))_n \in \ell^1$. But the converse is not true. $(\hat{F}(n))_n \in \ell^1$ only implies F extends to a continuous function on $\bar{D}$

Remark (Errata of $\operatorname{Re} H^1$)**.** *Let $g(t)$ be $\operatorname{Re} F(e^{it}) = \sum_{j\geq 0} a_j e^{ijt}$. Then*

$$\begin{aligned} g(t) =& \frac{F(e^{it}) + \overline{F(e^{it})}}{2} \\ =& \frac{a_0 + \bar{a}_0}{2} + \sum_{j\geq 0} \frac{a_j}{2} e^{ijt} + \sum_{j\geq 0} \frac{\bar{a}_j}{2} e^{-ijt} \\ =& \frac{a_0 + \bar{a}_0}{2} + \sum_{j>0} \frac{a_j}{2} e^{ijt} + \sum_{j<0} \frac{\overline{a_{-j}}}{2} e^{ijt} \\ =& \frac{a_0 + \bar{a}_0}{2} + \sum_{j\neq 0} \hat{g}(j) e^{ijt} \end{aligned}$$

where $\hat{g}(j) = \frac{a_j}{2}$ for $j > 0$, $\hat{g}(j) = \frac{\overline{a_{-j}}}{2}$ for $j > 0$ and $\hat{g}(j) = \operatorname{Re} a_0$. Thus $|a_j| = |\hat{g}(j)| + |\hat{g}(-j)|$. Substitute $|a_j|$ to $\sum_{j=1}^{\infty} \frac{|a_j|}{j} \leq \pi \|F\|_{H^1}$. We have $\sum_{j\neq 0} \left|\frac{\hat{f}(j)}{j}\right| \leq \pi \|f\|_{\operatorname{Re} H^1}$

We have $\operatorname{Re} H^1$ is a proper subspace of $\operatorname{Re} L^1$. And Hardy's inequality may be considered an extension to $p = 1$ of Paley's inequality which says that for $f \in L^p$ with $1 < p \leq 2$:

$$\sum_{j=-\infty}^{\infty} \frac{\left|\hat{f}(j)\right|^p}{|j|^{p-2}} \leq C_p \|F\|_p^p$$

Later we will see in $\mathbb{R}^n$ this inequality can be extended to H^p for $p < 1$. And H^p for $p \leq 1$ are natural substitutes of Lebesgue spaces L^p.

1.4.2 Fejer-Riesz inequality

Recall the final corollary in last section. Let F be a conformal map from D to interior domain bounded by a Jordan curve Γ. Then Γ is rectifiable if and only if $F' \in H^1$.

Theorem 1.4.2.1 (Fejer-Riesz inequality)**.** *Let $F \in H^p$, $0 < p < \infty$, then*

$$\int_{-1}^{1} |F(x)|^p \, dx \leq \frac{1}{2} \int_{-\pi}^{\pi} \left|F(e^{it})\right|^p dt$$

To prove this theorem, we first prove the $p = 2$ case. Then for $p \neq 2$ case, we factorize $F(z) = B(z)H(z)$ and let $|G(z)|^2 = |H(z)|^p$ to reduce this case to $p = 2$.

Here is a direct application of this inequality. Let F be the conformal map from D to interior domain bounded by a Jordan curve Γ. Then image of diameter of D has length at most half of length of Γ (corollary 4.6 in book).

Remark (notes on proof of corollary 4.6 in book)**.** *To prove that* $\frac{1}{2}$ *is the best constant in corollary 4.6 in book, we only need to show there is a conformal map from* D *to interior domain bounded by a rectifiable Jordan curve* Γ*, the constant* $\frac{1}{2}$ *can not be smaller. Let* $F(z)$ *is a conformal map from* D *to rectangle* $\{x+iy : |x| < 1, |y| < \epsilon\}$ *and* F *maps segment* $(-1,1)$ *in* D *to segment* $(-1,1)$ *in rectangle. It is easy to construct this map. The constant has to be at least* $\frac{2}{4+4\epsilon}$*. Let* $\epsilon \to 0$ *we conclude* $\frac{1}{2}$ *is the best constant.*

Another usage of conformal mapping $F' \in H^1$ is following: F can be extended on $\bar{D}$ and F is still conformal. More precisely, Let F be a conformal mapping from D to interior domain bounded by a rectifiable Jordan curve Γ. F is also conformal at almost every boundary point (corollary 4.7 in book).

Remark (notes on proof of corollary 4.7 in book)**.** *The step:*

$$\frac{F(e^{it_0}) - F(z)}{e^{it_0} - z} - F'(e^{it_0}) = \frac{1}{e^{it_0} - z} \int_z^{e^{it_0}} (F'(\xi) - F'(e^{it_0}))d\xi \to 0$$

as $z \to e^{it_0}$ *N.T. is by mean value theorem of integration.*

I don't know why the tangent to Γ *at the point* $F(e^{it_0})$ *happens for a.e. boundary point* e^{it_0}*.*

The angle between γ *and boundary in* D *is* $\lim \arg z - e^{it_0} - t_0 - \frac{\pi}{2}$ *and The angle between* $F(\gamma)$ *and boundary in* $F(D)$ *is* $\lim \arg F(z) - F(e^{it_0}) - t_0 - \arg(\frac{d}{dt}(F(e^{it}))|_{t=t_0})$*. Since* F *is conformal in* D*, to prove* F *is conformal in* $\bar{D}$*, we only need to prove the conformal map preserves angle on boundary. That is:*

$$\lim_{z \to e^{it_0}} \arg(z - e^{it_0}) - t_0 - \frac{\pi}{2} = \lim_{z \to e^{it_0}} \arg(F(z) - F(e^{it_0})) - \arg(\frac{d}{dt}(F(e^{it}))|_{t=t_0})$$

We have $\frac{d}{dt}F(e^{it})|_{t=t_0} = ie^{it_0}F'(e^{it_0})$*. Thus* $\arg(\frac{d}{dt}(F(e^{it}))|_{t=t_0}) = \frac{\pi}{2} + t_0 + \arg F'(e^{it_0})$*. So the equality is the same as:*

$$\lim_{z \to e^{it_0}} \arg(z - e^{it_0}) = \lim_{z \to e^{it_0}} \arg(F(z) - F(e^{it_0})) - \arg F'(e^{it_0})$$

which is clearly if we take arg *in both sides in* $\lim_{z \to e^{it_0}} \frac{F(e^{it_0}) - F(z)}{e^{it_0} - z} = F'(e^{it_0})$*. We use* $t_0 + \frac{\pi}{2}$ *instead of* $t_0 - \frac{\pi}{2}$ *match to* $\arg(\frac{d}{dt}(F(e^{it}))|_{t=t_0})$ *since they are in the same direction.*