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Weighted Norm Inequalities and Related Topics

J. GARCÍA-CUERVA and J.L. RUBIO DE FRANCIA

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AND RELATED TOPICS

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WEIGHTED NORM INEQUALITIES AND RELATED TOPICS

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This book is dedicated to

Miguel de Guzmán

for his constant efforts to

create a stimulating mathe-

matical atmosphere in Spain.

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PREFACE

This book presents some of the basic aspects of weighted norm inequalities for the classical operators in Fourier Analysis but, as the title tries to indicate, this is not the only subject to be discussed. Rather, we take weighted inequalities as the central theme around which other closely related topics appear. Among them, a great deal of space is devoted to the theory of Hardy spaces.

After the classical monograph of A. Zygmund [1], the standard references for the many important developments that occurred in Fourier Analysis during the second half of this century are E.M.Stein [1] and Stein-Weiss [2], both published around 1970. However, both Hardy spaces and weighted norm inequalities, though studied for many years before 1970, have undergone a most spectacular development since that date. In the preface of the book by M. de Guzmán [2], both topics were described as being in a very fluid state, making their exposition rather difficult. We felt the time had come to fill such a gap in the current mathematical literature by writing such an exposition.

We do not intend to cover everything that is known in this area, nor to give the most general form of each result. We have tried to present the material in an easily readable form, with the main ideas appearing as naturally as possible, showing first how these work in the simplest contexts, and bringing to light the connections among the different topics. Thus, Hardy spaces are first considered, in chapter I, from the classical point of view of Complex Analysis, while the more recent real variable approach is presented in chapter III. Also, in chapter II, we give a detailed account of the theory of maximal and singular integral operators in the spaces $L^p(\mathbb{R}^n)$; the extensions to the weighted setting or to vector valued functions are studied later, in chapters IV and V respectively. As an outcome of our emphasis on the mutual implications between different subjects, the central results of the theory are obtained from a variety of approaches. In particular, we give three different proofs (in chapters I, III and IV) of C. Fefferman's H^1-BMO duality theorem, four proofs of Muckenhoupt's characterization of the weighted inequalities for the maximal function and one and a half proofs of the Helson-Szegö theorem. Likewise, chapter VI tries to set up the basis for an (abstract) alternative approach to the two-weights problem for general operators, some of whose concrete aspects are described, by more classical means, in chapter IV.

We wanted the book to be accessible to anyone who is familiar with the Lebesgue integral and the basic facts of Functional Analysis and Complex Function Theory. We hope that the rather leisurely exposition will make the book more useful for those readers who want an introduction to any of the areas considered. This, unfortunately, also made the book grow beyond the previously conceived size. It also forced us to abandon the original idea of indluding further developments in the main subjects, such as weighted Hardy spaces, martingales, quasi-conformal group and Calderon's theorem on the Cauchy integral, harmonic measure and Hardy spaces in nonsmooth domains, etc. In an attempt to conciliate this very ambitious plan with the finiteness of our strength (or lifetime) and of the editor's patience, a last section of "Notes and Further Results" was added to each chapter. These sections include some historical notes, credit for the authorship of the more recent important theorems and further results (with references and/or a sketch of the proof) complementing the material covered in the main text. Our personal taste and, in many cases, our ignorance, are responsible for the selection of the results mentioned, but we are aware of the fact that many other contributions to this field in recent years deserve to appear in these sections. To the authors of these fine contributions (whose number we estimate to be 10^N for some large N) and to those who could complain for our giving inaccurate credit to some result, we offer our apologies and plead guilty in advance.

This work would have never been started without the encouragement of M. de Guzmán. We are most grateful to him and to G. Weiss for their continuous stimulus. Both of us gave graduate courses based on the material covered by this book in the academic year 1981/82, and have benefited from the criticism and encouragement of our colleagues at the Universidad Autónoma during and after that course. We must particularly mention the invaluable help of M. Walias and J.L. Torrea, who revised the whole manuscript and made many pertinent suggestions. Thanks finally to Caroline, Paloma and Soledad who typed the manuscript.

Spring 1985

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J.L. RUBIO DE FRANCIA

CONTENTS

CHAPTER I

CLASSICAL THEORY OF HARDY SPACES

The theory of Fourier series of one variable is described by A. Zygmund in his famous monograph "Trigonometric Series" as "the meeting ground of real and complex variables". Most of the topics covered by this book have their origin in this connection between properties of harmonic or analytic functions in the unit disc D of the complex plane and the Fourier Analysis of their boundary values in the torus $T = \partial D$. No doubt, the problems that arose in this context were initially attacked with the help of the powerful methods of Complex Analysis, and this is the approach followed in this chapter. Even though such methods are of no use for dealing with the similar, but more general, situations in $\mathbb{R}^n$, we hope to gain some perspective for our future work by first discussing the problems in their simplest formulation.

Thus, the theory of Hardy spaces in the unit disc, which was developed in its essential aspects during the first half of this century, occupies a central part in the chapter, and is based on factorizations of analytic functions due to F. Riesz and Smirnov, of which no satisfactory analogue is available for harmonic functions in $\mathbb{R}^n$. In spite of this, the theory has a natural continuation, by real variable methods, in Chapter III. One of the basic elements of the theory of H^p spaces is the conjugate function, the simplest example of a singular integral operator, whose n-dimensional analogues will be studied within the Calderón-Zygmund theory described in Chapter II. The same can be said of the maximal function of Hardy and Littlewood.

The material contained in the last three sections is not so classical: The Helson-Szegö theorem, our first example of a weighted norm inequality, must be considered in connection with the inequalities of the same type (though quite different in the spirit of the methods) which we shall obtain in Chapters II and IV, while C. Fefferman's celebrated duality theorem will be revisited in sections III.5 and IV.5 with different proofs (one of them, by the way, related to the Helson-Szego theorem).

Although the relevant problems studied in this chapter are one-dimensional, we do obtain some basic properties of harmonic and subharmonic functions of n real variables. This requires only slightly more effort than when dealing with functions of one complex variable, and paves the way for later developments.

1. HARMONIC FUNCTIONS, POISSON REPRESENTATION

Recall that a harmonic function on a domain $\Omega \subset \mathbb{R}^n$ is a function $u \in C^2(\Omega)$ (i.e. continuously differentiable up to order 2 in Ω) and satisfying Laplace's equation $\Delta u = 0$ in Ω, where, of course, Δ is the Laplacian differential operator $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$

In this chapter we shall be concerned, primarily, with harmonic functions on a plane domain. For the plane we shall use the complex coordinate $z = x + iy$, x and y real. There is a natural relation between harmonic and holomorphic functions in the plane. Indeed, the Laplacian operator can be factored as

$$\Delta = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

The equation $\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = 0$ for a complex valued function $F = u + iv$ with u and v real, is equivalent to:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$$

which is the Cauchy-Riemann system, whose solutions $F = u + iv$ are precisely the holomorphic functions. Any holomorphic function is, therefore, harmonic. Besides if $F = u + iv$ (with u and v real) is holomorphic, taking complex conjugates in the identity $\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = 0$, we see that $\overline{F} = u - iv$ satisfies the equation $\frac{\partial \overline{F}}{\partial x} - i\frac{\partial \overline{F}}{\partial y} = 0$ and, consequently, is also harmonic. It follows that both $u = (F+\overline{F})/2$ and $v = (F-\overline{F})/2i$ are harmonic functions; that is: the real and imaginary parts of a holomorphic function are harmonic.

On the other hand, if u is a real harmonic function on a simply connected domain $\Omega \subset \mathbb{C}$ (the complex plane), the condition $\Delta u = (u_x)_x - (-u_y)_y = 0$ guarantees the existence of v such that $dv = -u_y\, dx + u_x\, dy$, that is: $v_x = -u_y$ and $v_y = u_x$. Then the function $F = u + iv$ is holomorphic and the function v(which is harmonic since $v_{xx} = -u_{yx}$ and $v_{yy} = u_{xy}$) is said to be a harmonic conjugate of u . Clearly v is determined up to an additive constant.

So, we have seen that a real function u defined on a simply connected domain of $\mathbb{C}$ is harmonic if and only if it is the real part of some holomorphic function.

Suppose u is a real harmonic function on the disk $D(0,R) = \{z \in \mathbb{C} : |z| < R\}$. Then we know that $u(z) = \operatorname{Re} F(z)$ for some holomorphic function F. Let $F(z) = \sum_{k=0}^{\infty} c_k z^k$ be the power series representarion of F.

We can write $u(z) = (F(z) + \overline{F(z)})2$ and get a series representation for u. Let us do that using the polar form of $z = re^{i\theta}$ with $r = |z|$ and $-\pi \le \theta \le \pi$, We obtain:

$$u(re^{i\theta}) = \tfrac{1}{2}\Big(\sum_{k=0}^{\infty} c_k r^k e^{ik\theta} + \sum_{k=0}^{\infty} \bar{c}_k r^k e^{-ik\theta}\Big) = \sum_{-\infty}^{\infty} a_k r^{|k|} e^{ik\theta} \qquad \text{with}$$

$a_k = c_k/2$ for $k > 0$, $a_0 = \operatorname{Re} c_0$ and $a_k = \bar{c}_{-k}/2$ for $k < 0$.

We conclude that *any* u *harmonic in* $D(0,R)$ *has a series representation*

$$(1.1) \qquad u(re^{i\theta}) = \sum_{-\infty}^{\infty} a_k \, r^{|k|} e^{ik\theta}$$

converging uniformly on compact subsets of $D(0,R)$.

Suppose $R > 1$. Since (1.1) converges uniformly for $r = 1$, we see that a_k is the Fourier coefficient, corresponding to the frequency k , of the function $t \longmapsto u(e^{it})$, that is

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it})\, e^{-ikt}\, dt$$

Substituting this integral for a_k in (1.1) we get:

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it}) \sum_{-\infty}^{\infty} r^{|k|} e^{ik(\theta-t)}\, dt$$

For $0 \le r < 1$ the series converges uniformly, its sum being

$$\sum_{-\infty}^{\infty} r^{|k|} e^{ikt} = 1 + 2\operatorname{Re} \sum_{k=1}^{\infty} (re^{it})^k = \frac{1 - r^2}{1 + r^2 - 2r\cos t}$$

This function is the Poisson kernel for the unit disk and will be denoted by $P_r(t)$.

For a function u harmonic in $D(0,R)$, $R > 1$, we have obtained the

Poisson representation:

$$(1.2)\quad u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1+r^2-2r\cos(\theta-t)} u(e^{it})dt =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, u(e^{it})\, dt \qquad (0 \le r < 1,\ -\pi \le \theta \le \pi)$$

which exhibits the function $u_r(e^{it}) = u(re^{it})$ as the convolution (on the torus group $\mathbb{T} = \{e^{it}: t \in \mathbb{R}\}$ which we identify with the interval $[-\pi,\pi]$) of the functions $u(e^{it})$ and P_r.

Formula (1.2) provides the key to the solution of the Dirichlet problem for the disk. This basic problem consists in finding a function u continuous on $\overline{D} = \{z \in \mathbb{C}: |z| \le 1\}$ and harmonic in D, whose restriction to the boundary of D, $u(e^{it})$, coincides with a previously given continuous function $f(t)$ on $[-\pi,\pi]$ such that $f(-\pi) = f(\pi)$. The natural candidate for the solution will be the integral in (1.2) with $f(t)$ in place of $u(e^{it})$, that is, the function $t \longmapsto u(re^{it})$ for $0 \le r < 1$, is the convolution of f and the Poisson kernel P_r. We write $u(re^{i\theta}) = P_r * f(\theta)$ or $u = P(f)$ and say that u is the Poisson integral of f. That this function u is indeed a solution will be seen shortly. First, we shall show that the Poisson representation (1.2) remains valid for a much wider class of harmonic functions in the unit disk. One consequence will be the uniqueness of the solution to the Dirichlet problem. We start with the following

THEOREM 1.3. *Let* u *be a harmonic function in* D *such that*

$$(1.4)\qquad \sup_{0\le r<1} \int_{-\pi}^{\pi} |u(re^{it})|^p\, dt < \infty$$

for some $p > 1$. *Then there is a function* $f \in L^p([-\pi,\pi])$ *such that*

$$(1.5)\qquad u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, f(t)\, dt$$

that is: u *is the Poisson integral of some* L^p *function* f.

Proof: Let $r_n \uparrow 1$ (i.e. r_n is an increasing sequence converging to 1). Consider the functions $f_n(t) = u(r_n e^{it})$. From condition (1.4) (f_n) is a bounded sequence in $L^p([-\pi,\pi]) = L^{p'}([-\pi,\pi])^*$, where p'

denotes the exponent conjugate to p (that is: $(1/p) + (1/p') = 1$) and the star $*$ is used to denote the topological dual. Thus the sequence (f_n) is in a closed ball of the normed dual to $L^{p'}([-\pi,\pi])$. The Banach-Alaoglu theorem asserts that such ball is weak-$*$ compact and therefore, since $L^{p'}([-\pi,\pi])$ is separable, also metrizable (see Rudin [2] pp. 66-68). It follows that (f_n) has a subsequence converging in the weak-$*$ topology to a certain $f \in L^p([-\pi,\pi])$, that is: for every $g \in L^{p'}([-\pi,\pi])$

$$(1.6) \qquad \int_{-\pi}^{\pi} g(t)\, f_n(t)\, dt \longrightarrow \int_{-\pi}^{\pi} g(t)\, f(t)\, dt \quad \text{as} \quad n \to \infty$$

For each n, the function $z \mapsto u(r_n z)$ is harmonic in $D(0,r_n^{-1})$, a disk of radius bigger than one. Therefore, we have the Poisson representation:

$$u(r_n re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1+r^2-2r\cos(\theta-t)}\, u(r_n e^{it})dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta-t) f_n(t)dt$$

Letting n go to ∞, the left hand side tends to $u(re^{i\theta})$ while the right hand side tends to $\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta-t)\, f(t)\, dt$, according to (1.6). This yields the Poisson representation (1.5). $\square$

Let us observe that the theorem remains valid for $p = \infty$ substituting for (1.4) the condition:

$$(1.7) \qquad \sup_{0 \le r < 1} \|u_r\|_\infty < \infty, \text{ where } u_r \text{ is the function } t \mapsto u(re^{it});$$

or, in other words, the hypothesis that u is bounded in D. All one needs to realize is that still $L^\infty([-\pi,\pi]) = L^1([-\pi,\pi])^*$.

A relevant question at this point in whether condition (1.4) with $p = 1$ will imply a Poisson representation. The proof of theorem 1.3 does not extend to this case because $L^1([-\pi,\pi])$ is not a dual space (any separable dual has the Radon-Nikodym property and $L^1([-\pi,\pi])$ does not have it. See Diestel-Uhl [1]). However $L^1([-\pi,\pi])$ can be isometrically imbedded into the space $M([-\pi,\pi])$ of Borel measures on $[-\pi,\pi]$ by assigning to each $f \in L^1([-\pi,\pi])$, the measure $d\mu(t) = f(t)\,dt$. The space $M([-\pi,\pi])$ is the dual of the space $C([-\pi,\pi])$ of continuous functions on $[-\pi,\pi]$ with the supremum norm. Then we can repeat the argument used in the proof of theorem 1.3 and obtain:

THEOREM 1.8. Let u *be a harmonic function in* D, *such that:*

$$\sup_{0 \le r < 1} \int_{-\pi}^{\pi} |u(re^{it})| dt < \infty \tag{1.9}$$

Then, there is a Borel measure μ *on* $[-\pi, \pi]$, *such that:*

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\, d\mu(t)$$

That is: u *is the Poisson integral of the measure* μ *(we shall write* $u = P(\mu)$*). These functions* u *are often called Poisson-Stieltjes integrals.*

Let us observe that condition (1.9) is automatically satisfied if the harmonic function u is ≥ 0. Indeed, in that case:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{it})|\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{it})\, dt = u(0)$$

The last identiy is an instance of the mean value property of harmonic functions, which follows immediately from Poisson's representation. We obtain, therefore, the following

COROLLARY 1.10. Any positive harmonic function in D *is the Poisson integral of some positive measure on* T. □

The measure is positive because it is obtained as a weak-* limit of positive measures.

For $p = \infty$, theorem 1.3 and its proof imply that the solution u of the Dirichlet problem on D with boundary function f is, if any, $P(f)$, the Poisson integral of f. We shall presently see that $P(f)$ is indeed a solution. This will be based upon the fact that the Poisson kernel gives rise to an approximate identity. To see what this means, we examine closely the Poisson kernel

$$P_r(t) = \frac{1 - r^2}{1+r^2-2r \cos t} = \sum_{-\infty}^{\infty} r^{|k|} e^{ikt} \qquad 0 < r < 1,\ t \in \mathbb{R}.$$

It is obviously a 2π-periodic continuous function of t. It is also positive. Besides

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{-\infty}^{\infty} r^{|k|} e^{ikt}\, dt = 1$$

And finally, for any $\delta > 0$ is $\sup_{\delta<|t|\leq\pi} P_r(t) \to 0$ as $r \to 1$.

Indeed for $\delta \leq |t| \leq \pi$ is

$$P_r(t) \leq \frac{1-r^2}{1+r^2-2r\cos\delta} \leq \frac{1-r^2}{1-\cos^2\delta} \to 0 \qquad \text{as } r \to 1.$$

The last inequality is due to the fact that the denominator is minimal for $r = \cos\delta$.

In general, an approximate identity on the torus T will be a family ϕ_α of 2π-periodic functions in $L^1([-\pi,\pi])$, with indices α ranging over a directed set and satisfying the following three conditions:

i) $\sup_\alpha \frac{1}{2\pi}\int_{-\pi}^{\pi}|\phi_\alpha(t)|\,dt = k < \infty$

ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_\alpha(t)\,dt = 1$ for every α

iii) $\int_{\delta<|t|\leq\pi}|\phi_\alpha(t)|dt \to 0$ for every $\delta > 0$

Clearly, the Poisson kernel gives an approximate identity P_r. In this case $k = 1$ in i) and an even stronger version of iii) holds as we have seen

The fact that the Poisson kernel is positive and satisfies ii) gives converses to theorems 1.3 and 1.8.

THEOREM 1.11. Let $f \in L^p([-\pi,\pi])$, $1 \leq p \leq \infty$, *and let* $u = P(f)$ *be its Poisson integral; that is:*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, f(t)\,dt, \qquad 0 \leq r < 1,\ -\pi \leq \theta \leq \pi.$$

Then $u(z)$ *is harmonic in* D. *Besides, if* $p < \infty$, *we have:*

$$(1.12) \qquad \int_{-\pi}^{\pi}|u(re^{it})|^p\,dt \leq \int_{-\pi}^{\pi}|f(t)|^p\,dt \quad \textit{for every} \quad r < 1$$

and if $p = \infty$

$$(1.13) \qquad |u(z)| \leq \|f\|_\infty \quad \textit{for every} \quad z \in D$$

Proof: If the Fourier series of f is $\sum_{-\infty}^{\infty} a_k\, e^{ik\theta}$, then

$$u(re^{i\theta}) = \sum_{-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}$$

If f is real-valued, so is u, and, clearly

$$u(z) = \mathrm{Re}(a_0 + 2\sum_{1}^{\infty} a_k z^k)$$

the real part of a holomorphic function. Consequently, u is harmonic in D.

(1.12) and (1.13) are particular instances of Young's inequality. They can be obtained very easily by writing

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\, f\,(\theta-t)\, dt,$$

and applying Minkowski's inequality for integrals (see appendix A.1 in Stein [1]):

$$\|u(re^{i\cdot})\|_p \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\, \|f(.-t)\|_p\, dt = \|f\|_p$$

(the dot . stands for the variable with respect to which norms are taken). $\square$

THEOREM 1.14. Let μ *be a complex Borel measure on* $[-\pi,\pi]$ *and* $u = P(\mu)$ *its Poisson integral.*

Then $u(z)$ *is harmonic in* D *and*

$$(1.15) \qquad \int_{-\pi}^{\pi} |u(re^{it})|\, dt \leq \int_{-\pi}^{\pi} d|\mu|\,(t)$$

(*the last integral denotes the total variation of* μ)

Proof: If the Fourier series of μ is $\sum a_k e^{ik\theta}$, that is, if

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\, d\mu(t)$$

then

$$u(re^{i\theta}) = \sum_{-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}$$

As before, u is clearly harmonic. Besides

$$\int_{-\pi}^{\pi} |u(re^{i\theta})|\, d\theta \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P_r(\theta - t)\, d|\mu|(t)\, d\theta = \int_{-\pi}^{\pi} d|\mu|(t)$$

by Fubini's theorem. $\square$

Thus, we have seen that, in the class of harmonic functions in D, condition (1.4) for $p > 1$ characterizes those which are Poisson integrals of L^p functions, and condition (1.4) for $p = 1$ characterizes those which are Poisson integrals of Borel measures of finite total variation.

We shall study next the boundary behaviour of Poisson integrals. This will allow us to solve the Dirichlet problem and several variants of it by means of Poisson integrals and will also give us a better understanding of how u determines f in theorem 1.3 or μ in theorem 1.8. First we study the convergence in the L^p norm. We can state a general result valid for every approximate identity ϕ_α.

THEOREM 1.16. Let ϕ_α *be an approximate identity on the torus* T *Then*:

a) If $f \in L^p([-\pi,\pi])$ *with* $1 \le p < \infty$ *and* f_α *stands for the convolution*

$$f_\alpha(\theta) = (f * \phi_\alpha)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta - t)\, \phi_\alpha(t)\, dt$$

it follows that $f_\alpha \to f$ *in* L^p, *i.e.*:

$$\int_{-\pi}^{\pi} |f_\alpha(t) - f(t)|^p\, dt \to 0$$

b) If f *is a continuous* 2π-*periodic function*, *we have* $f_\alpha \to f$ *uniformly on* T

Proof: $f_\alpha(\theta) - f(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(\theta - t) - f(\theta))\, \phi_\alpha(t)\, dt$

because of property ii) of the approximate identity. Then, Minkowski's inequality for integrals implies that:

$$\|f_\alpha - f\|_p \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(\cdot - t) - f\|_p |\phi_\alpha(t)|\, dt = \frac{1}{2\pi} \int_{-\delta}^{\delta} + \frac{1}{2\pi} \int_{\delta < |t| \le \pi}$$

for an arbitrary $\delta > 0$. The second term in this sum is bounded by $\frac{1}{\pi}\|f\|_p \int_{\delta<|t|\leq\pi} |\phi_\alpha(t)|\,dt$, which, according to property iii) of the approximate identity, tends to zero as α moves in the directed set of indices, no matter how small δ is. The first term in the sum is bounded by

$$\left(\sup_{|t|<\delta} \|f(\cdot-t)-f\|_p\right)\left(\sup_{\alpha} \frac{1}{2\pi}\int_{-\pi}^{\pi}|\phi_\alpha(t)|\,dt\right) = K \sup_{|t|<\delta}\|f(\cdot-t)-f\|_p$$

with $K < \infty$ (property i) of the approximate identity). But $\sup_{|t|<\delta}\|f(\cdot-t)-f\|_p \equiv \omega_p(f,\delta)$, the L^p-modulus of continuity of f, can be made small by taking δ small. Indeed, it is clear that $\|f(\cdot-t)-f\|_p \to 0$ as $t \to 0$ (note that we are taking f continuous when $p = \infty$. For $p < \infty$, we just need to approximate f in the L^p norm by continuous functions in order to justify the claim) Now, to conclude the proof of the theorem, given $\varepsilon > 0$, we first choose $\delta > 0$ small enough to have

$$\frac{1}{2\pi}\int_{-\delta}^{\delta}\|f(\cdot-t)-f\|_p\ |\phi_\alpha(t)|\ dt < \frac{\varepsilon}{2}$$

independently of α. Then, with this δ fixed, all we have to do is to take α far enough in the order of the directed set of indices to render

$$\frac{1}{2\pi}\int_{\delta<|t|\leq\pi}\|f(\cdot-t)-f\|_p\ |\phi_\alpha(t)|\ dt < \frac{\varepsilon}{2}$$

This will suffice to have $\|f_\alpha - f\|_p < \varepsilon$. □

Taking as approximate identity the Poisson kernel P_r, we obtain

COROLLARY 1.17 *Let* f *be a* 2π-*periodic function on* $\mathbb{R}$, *and let* $u = P(f)$. *Then:*

a) *If* $f \in L^p([-\pi\ \pi])$ *with* $1 \leq p < \infty$ *we have:*

$$\int_{-\pi}^{\pi}|u(re^{it})-f(t)|^p\ dt \to 0 \quad \textit{as} \quad r \to 1$$

b) *If* f *is continuous* $u(re^{it}) \to f(t)$ *uniformly in* t *as* $r \to 1$.

Thus, in theorem 1.3, the function f is the limit in $L^p(T)$ of the functions $u_r(t) = u(re^{it})$, $1 < p < \infty$.

Part b) of the corollary implies that, for f continuous on T, $u = P(f)$ is, indeed, the solution of the classical Dirichlet problem. Of course, part a) implies that for $1 \le p < \infty$, $u = P(f)$ is the solution of an L^p version of the Dirichlet problem.

By duality, theorem 1.16 implies

THEOREM 1.18. Let ϕ_α *be an approximate identity on the torus* T. *Then*:

a) *If* $f \in L^\infty([-\pi,\pi])$ *and* $f_\alpha = f * \phi_\alpha$, *it follows taht* $f_\alpha \to f$ *in the weak-* topology of* L^∞.

b) *if* $\mu \in M(T)$ *and* $f_\alpha = \mu * \phi_\alpha$, *it follows that* $f_\alpha \to \mu$ *in the weak-* topology of* $M(T)$.

Proof: a) We have to see that $\int f_\alpha(\theta)\,\psi(\theta)\,d\theta$ converges to $\int f(\theta)\,\psi(\theta)\,d\theta$ for every $\psi \in L^1([-\pi,\pi])$. But

$$\int_{-\pi}^{\pi} f_\alpha(\theta)\psi(\theta)d\theta = \int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\phi_\alpha(\theta-t)dt\ \psi(\theta)d\theta = \int_{-\pi}^{\pi}(\psi*\phi_\alpha(-\cdot))(t)f(t)dt \to$$

$$\to \int_{-\pi}^{\pi}\psi(t)f(t)dt \quad \text{since } \psi*\phi_\alpha(-\cdot)\to\psi \text{ in } L^1 \text{ and } f \in L^\infty.$$

The proof of b) is entirely similar, only this time we take ψ continuous and apply part b) of the theorem. □

COROLLARY 1.19 a) *If* $f \in L^\infty([-\pi,\pi])$ *and* $u = P(f)$, *we have* $u_r(t) = u(re^{it}) \to f(t)$ *in the weak-* topology of* L^∞

b) *If* $\mu \in M(T)$ *and* $u = P(\mu)$, *we have* $u_r(t)dt \to d\mu(t)$ *in the weak-* topology of* M. □

It should be noted that if an integrable function f or a complex Borel measure μ has Fourier series $\sum a_k e^{ik\theta}$, then, the Poisson integral of f or μ is the function $u(re^{i\theta}) = \sum a_k r^{|k|} e^{ik\theta}$ which, for each fixed r, can be viewed as an average of the partial sums

$$S_n(\theta) = \sum_{-n}^{n} a_k e^{ik\theta}$$

Indeed

$$u(re^{i\theta}) = (1-r) \sum_0^\infty r^n S_n(\theta).$$

The functions $u_r(\theta) = u(re^{i\theta})$ are called the Abel means of (the Fourier series of) f or μ. Thus, every theorem about the boundary behaviour of the function u can be read as a theorem on the Abel summability of the Fourier series of f or μ. So far, we have established the Abel summability in the L^p norm. Now we shall analize the problem of pointwise summability or, in other words, we shall study the pointwise behaviour of a Poisson-Stieltjes integral. We shall no longer obtain results for a general approximate identity. Now, the particular structure of the Poisson kernel (more specifically the fact that $P_r(t)$ is decreasing as a function of $|t|$), will be decisive.

THEOREM 1.20. Let μ *be a Borel measure on* T, *and call*

$$F(\theta) = \int_0^\theta d\mu(t).$$

We know that F *is a function of bounded variation and, hence, it has a (finite) derivative at almost every point* θ. *Let* θ_1 *be one of those points at which* $F'(\theta_1)$ *exists and is finite. Let* $u = P(\mu)$. *Then* $u(z)$ *converges to* $F'(\theta_1)$ *as* z *tends to* $e^{i\theta_1}$ *non-tangentially. By this we mean that for every* $c > 0$ $u(z) \to F'(\theta_1)$ *as* $z = re^{i\theta}$ *tends to* $e^{i\theta_1}$ *remaining in the region* $\{re^{i\theta}: |\theta-\theta_1| < c(1-r)\}$. *We shall indicate this type of convergence by writing* $u(z) \to F'(\theta_1)$ *as* $z \xrightarrow{N.T.} e^{i\theta_1}$.

Proof: Note that $F'(\theta_1)$ is, by definition, a limit of averages of the measure μ. So, it should not come as a surprise that the functions u_r which are just another type of averages, converge to $F'(\theta_1)$ too. The proof will be based on integration by parts. First, just to simplify the writing, we may clearly assume that $\theta_1 = 0$, and also that $F'(0) = 0$ (otherwise, we consider the measure $d\mu(t) - F'(0)dt$). Take $c > 0$. We shall show that $u(re^{i\theta})$ can be made small uniformly in θ for $|\theta| < c(1-r)$, by just taking r close enough to 1. Let $\varepsilon > 0$. Take $\delta > 0$ such that $|F(t)| < \varepsilon|t|$ every time that $|t| < \delta$. Look only at r's so close to 1 that if $re^{i\theta}$ is in our region of approach, then $|\theta| < \delta/4$, in other words, let $c(1-r) < \delta/4$. Then, for $re^{i\theta}$ in our region

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\, d\mu(t) = \frac{1}{2\pi}\int_{\delta<|t|\le\pi} + \frac{1}{2\pi}\int_{-\delta}^{\delta}$$

The first term in this sum is majorized by

$$\sup_{|t|>\delta/2} P_r(t)\cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} d|\mu|(t)$$

Clearly, this tends to 0 as $r \to 1$, and we just need to worry about the other term, which we can write, after integrating by parts as something bounded by a constant times $\sup_{|t|>\delta/2} P_r(t)$ plus:

$$\frac{1}{\pi}\int_{-\delta}^{\delta} \frac{(1-r^2)r\sin(t-\theta)}{|1+r^2-2r\cos(\theta-t)|^2}\, F(t)\, dt$$

Again we just need to study this last term. Suppose, just for definiteness, that $\theta > 0$, and decompose the integral as $\int_{-\delta}^{0} + \int_{0}^{2\theta} + \int_{2\theta}^{\delta}$. We look at each of the terms in this sum.

On the first one we use $|F(t)| < \varepsilon|t| = \varepsilon(-t) \le \varepsilon(\theta-t)$ in such a way that, after changing $\theta-t$ to t, we obtain as a bound for the absolute value of the first term:

$$\frac{\varepsilon}{\pi}\int_{\theta}^{\theta+\delta} \frac{(1-r^2)r\sin t}{(1+r^2-2r\cos t)^2}\, t\, dt \le \frac{\varepsilon}{\pi}\int_{0}^{\pi} \frac{(1-r^2)r\sin t}{(1+r^2-2r\cos t)^2}\, t\, dt$$

Integrating by parts again we see that this quantity is bounded by

$$\frac{\varepsilon}{2\pi}\int_{0}^{\pi} \frac{1-r^2}{1+r^2-2r\cos t}\, dt = \frac{\varepsilon}{2}$$

For the second term in the sum we use $|F(t)| < \varepsilon t$ and $|\theta-t| < \theta$. In this way, the second term is bounded by

$$\frac{1}{\pi}\int_{0}^{2\theta} \frac{(1-r)(1+r)r\theta}{(1-r)^4}\, \varepsilon t\, dt \le \frac{8\,\theta^3\,\varepsilon}{\pi(1-r)^3} \le \frac{8C^3}{\pi}\varepsilon$$

As for the third and last term in the sum, we use $|F(t)| < \varepsilon\, t \le 2\varepsilon(t-\theta)$ and proceed exactly as with the first term. $\square$

When $d\mu(t) = f(t)dt$ with $f \in L^1([-\pi,\pi])$, we know that for almost every θ is $\frac{1}{t}\int_0^t |f(\theta+s) - f(\theta)|ds \to 0$ as $t \to 0$. (Actually, this fact, which is equivalent to Lebesgue's differentiation theorem, will be proved in chapter II. See theorem 1.9 there). Those θ for which this holds are called Lebesgue points for f. In such a point θ, clearly $F'(\theta) = f(\theta)$. Therefore, we get

COROLLARY 1.21. Let $f \in L^1([-\pi,\pi])$, *and let* $u = P(f)$. *Then, for every Lebesgue point* θ

$$u(z) \to f(\theta) \qquad \text{as} \qquad z \xrightarrow{N.T.} e^{i\theta}$$

In particular, this is true almost everywhere. □

When $d\mu(t) = f(t)dt + d\sigma(t)$, where $f \in L^1([-\pi,\pi])$ and σ is singular, it is known that $F'(\theta) = f(\theta)$ for a.e.θ (see Rudin [1]), so that every Lebesgue-Stieltjes integral has non-tangential boundary values a.e. This applies in particular to the harmonic functions in D satisfying any of the conditions (1.4),(1.7) or (1.9). This result contains the classical *theorem of Fatou* stating that

"Any function holomorphic and bounded in D *has non-tangential boundary values a.e."*.

When a harmonic function u satisfies condition (1.4) for some $p > 1$, then u can be recovered from its boundary function f. Indeed, we know that $u = P(f)$. However if u just satisfies (1.9) (i.e., if $p = 1$), then it is no longer true that u is the Poisson integral of its boundary function. For example, let

$$u(re^{it}) = P_r(t) = \sum_{-\infty}^{\infty} r^{|k|}e^{ikt}$$

This is, of course, a harmonic function, and it clearly satisfies (1.9). Its boundary function is 0. Indeed $P_r(t) \to 0$ as $r \to 1$ for every $t \neq 0$ in $[-\pi,\pi]$. However $u > 0$, and it cannot be the Poisson integral of 0, which is identically 0. Actually, in this case $u = P(\delta)$ where δ is the Dirac delta or, in other words, the unit mass concentrated at 0 in $[-\pi,\pi]$. This difference in the behaviour of an L^p-bounded harmonic function for $p = 1$ or $p > 1$ is a basic fact and, as we shall see, is the natural starting point for the theory of Hardy spaces, which is the main topic in this chapter.

Even though in this chapter we shall only deal with harmonic functions in a plane domain, which almost always will be the unit disk D, it seems natural to try to see to which extent our discussion of harmonic functions can be carried over to higher real dimensions or plane domains different from D. We shall start by showing that harmonic functions are characterized by the mean value property.

THEOREM 1.22. Let u *be a continuous function on a domain* $\Omega \subset \mathbb{R}^n$. *Then* u *is harmonic in* Ω *if and only if* u *satisfies the following property (known as mean value property) : For every* $x_0 \in \Omega$ *and for every* $r > 0$ *such that* $\overline{B(x_0,r)} = \{x \in \mathbb{R}^n : |x-x_0| \le r\} \subset \Omega$

$$(1.23) \qquad u(x_0) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} u(x_0+r\sigma)d\sigma$$

where $\Sigma_{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ *is the unit sphere in* $\mathbb{R}^n$, $d\sigma$ *is Lebesgue measure on* Σ_{n-1} *and* $|\Sigma_{n-1}| = \int_{\Sigma_{n-1}} d\sigma$

Proof: Suppose that u is harmonic in Ω, so that $\Delta u = 0$ in Ω. Let $x_0 \in \Omega$ and $r > 0$ be such that $\overline{B(x_0,r)} \subset \Omega$. For $0 < s \le r$ let $f(s)$ stand for the average of u over the sphere of center x_0 and radius s, that is

$$f(s) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} u(x_0+s\sigma)\, d\sigma$$

We find that the function f defined in the interval $(0,r]$ is differentiable and its derivative is

$$f'(s) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} \sum_{j=1}^{n} u_{x_j}(x_0+s\sigma)\,\sigma_j\, d\sigma$$

The integrand equals $D_\sigma u(x_0+s\sigma)$, which is the derivative of u in the direction of the outer normal in the point $x_0+s\sigma$. Then

$$f'(s) = \frac{1}{s^{n-1}|\Sigma_{n-1}|}\int_{\partial B(x_0,s)} D_\sigma u(x)\, d\sigma_s(x),$$

where $\partial B(x_0,s)$ is the boundary of the ball, that is, $\partial B(x_0,s) = \Sigma(x_0,s)$, the sphere of center x_0 and radius s; and $d\sigma_s$ is the natural Lebesgue measure on $\partial B(x_0,s)$. By applying Green's theorem we get

$$f'(s) = \frac{1}{s^{n-1}|\Sigma_{n-1}|}\int_{B(x_0,s)} \Delta u = 0.$$

Thus $f(s)$ is constant for $0 < s \leq r$. But, clearly, $f(s) \to u(x_0)$ for $s \to 0$ and $f(s) \to f(r)$ for $s \to r$. Therefore $f(r) = u(x_0)$ and (1.23) is proved.

The converse is equally easy if we assume, to start with, that u is twice differentiable. If this is the case and we assume that the mean value property holds, then, with the same notation used above we have $f(s) = u(x_0)$, constant on $[0,r]$.

Consequently $f'(s) = 0$ and from here

$$\frac{1}{|B(x_0,s)|}\int_{B(x_0,s)} \Delta u(x)dx = \frac{n}{s^n|\Sigma_{n-1}|}\int_{B(x_0,s)} \Delta u(x)dx = 0$$

But since Δu is continuous,

$$\frac{1}{|B(x_0,s)|}\int_{B(x_0,s)} \Delta u(x)dx \to \Delta u(x_0) \quad \text{as} \quad s \to 0$$

Thus $\Delta u(x_0) = 0$. Since this is true for every $x_0 \in \Omega$, we get that u is harmonic.

In case u is just a continuous function in Ω satisfying the mean value property, we shall show that u is harmonic by reducing the problem to the case of a smooth function considered previously

Since the problem is local we can assume that Ω is bounded and u is bounded. We shall use a C^∞ function ϕ with support in $B(0,1)$ and having $\int_{\mathbb{R}^n} \phi = 1$ and the approximate identity in $\mathbb{R}^n$, ϕ_ε, to which it gives rise: $\phi_\varepsilon(x) = \varepsilon^{-n}\phi\,(\varepsilon^{-1}x)$ for $\varepsilon > 0$. It can be checked quite easily that ϕ_ε is indeed an approximate identity in $\mathbb{R}^n$. The definition is entirely similar to the one given for the torus T (right before theorem 1.11). We shall also require ϕ to be radial, that is: $\phi(x) = \psi(|x|)$.

Then we approximate u by the functions

$$u_\varepsilon(x) = u*\phi_\varepsilon(x) = \int_{\mathbb{R}^n} u(x-y)\ \phi_\varepsilon(y)\ dy = \int_{\mathbb{R}^n} u(y)\phi_\varepsilon(x-y)\ dy$$

(it is understood that u is extended to $\mathbb{R}^n$ by making it equal to 0 outside of Ω). We see in the last integral that the smoothness of ϕ implies that u_ε is smooth too. Now observe that u_ε

satisfies the mean value property in $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. Indeed if $x_0 \in \Omega_\varepsilon$ and $\overline{B(x_0,r)} \subset \Omega_\varepsilon$, we have

$$\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} u_\varepsilon(x_0+r\sigma)\,d\sigma = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}}\int_{\mathbb{R}^n} u(x_0+r\sigma-y)\phi_\varepsilon(y)\,dy\,d\sigma =$$

$$= \int_{\mathbb{R}^n}\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} u(x_0-y+r\sigma)\,d\sigma\,\phi_\varepsilon(y)dy = \int_{\mathbb{R}^n} u(x_0-y)\phi_\varepsilon(y)dy = u_\varepsilon(x_0)$$

since u satisfies the mean value property in Ω and $\overline{B(x_0,r+\varepsilon)} \subset \Omega$. Consequently u_ε is harmonic in Ω_ε. But on the other hand for $x \in \Omega_\varepsilon$ is:

$$u_\varepsilon(x) = \int_{\mathbb{R}^n} u(x-y)\phi_\varepsilon(y)dy = \int_0^\infty r^{n-1}\int_{\Sigma_{n-1}} u(x-r\sigma)\phi_\varepsilon(r\sigma)d\sigma\,dr =$$

$$= \varepsilon^{-n}\int_0^\infty r^{n-1}\int_{\Sigma_{n-1}} u(x-r\sigma)\,d\sigma\,\psi(\varepsilon^{-1}r)dr = \varepsilon^{-n}\int_0^\infty r^{n-1}|\Sigma_{n-1}|\,u(x)\psi(\varepsilon^{-1}r)dr$$

$$= u(x)\int_{\mathbb{R}^n}\phi_\varepsilon(x)dx = u(x)$$

This ends the proof that u is harmonic because in a neighbourhood of a given $x \in \Omega$, u coincides with u_ε for ε small enough and we already know that u_ε is harmonic. $\square$

It has to be observed that the mean value property in theorem 1.22 is equivalent to the fact that for every $x_0 \in \Omega$ and every $r > 0$ such that $\overline{B(x_0,r)} \subset \Omega$

$$(1.24) \qquad u(x_0) = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} u(x)\,dx$$

Indeed if the first mean value property holds and $\overline{B(x_0,r)} \subset \Omega$, we have:

$$u(x_0) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} u(x_0+s\sigma)\,d\sigma$$

for all s with $0 < s \leq r$. Integrating both sides against s^{n-1} from 0 to r, we get:

$$\frac{r^n}{n}u(x_0) = \frac{1}{|\Sigma_{n-1}|}\int_0^r s^{n-1}\int_{\Sigma_{n-1}} u(x_0+s\sigma)d\sigma\,ds = \frac{1}{|\Sigma_{n-1}|}\int_{B(x_0,r)} u(x)dx$$

Thus

$$u(x_0) = \frac{n}{r^n|\Sigma_{n-1}|}\int_{B(x_0,r)} u(x)dx = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} u(x)\,dx.$$

Which is (1.24). Conversely, if we assume the second mean value property and $x_0 \in \Omega$, we shall have, for all r in an interval to the right of 0:

$$u(x_0) = \frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} u(x)dx = \frac{n}{r^n|\Sigma_{n-1}|}\int_0^r s^{n-1}\int_{\Sigma_{n-1}} u(x_0+s\sigma)d\sigma\,ds$$

The right hand side, as a function of r, will have derivative 0:

$$0 = \frac{n}{r^n|\Sigma_{n-1}|}\, r^{n-1}\int_{\Sigma_{n-1}} u(x_0+r\sigma)d\sigma - \frac{n}{r}\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} u(x)\,dx$$

or equivalently

$$u(x_0) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} u(x_0+r\sigma)\,d\sigma\ , \text{ which is (1.23).}$$

A consequence of the mean value property is the so called *maximum principle* for harmonic functions, which can be stated as follows:

COROLLARY 1.25 Let u *be a real-valued harmonic function in a domain* $\Omega \subset \mathbb{R}^n$. *Then* u *cannot attain a maximum value unless it is constant.*

Proof: Suppose that u does attain a maximu value, that is, there exists $x_0 \in \Omega$ such that $u(x) \leq u(x_0) = m$ for every $x \in \Omega$. Take $r > 0$ such that $\overline{B(x_0,r)} \subset \Omega$. Then

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} u(x)\,dx = u(x_0) = m.$$

Since $u(x) \leq m$ for every x and u is continuous, if we had $u(x) < m$ for some $x \in B(x_0,r)$, then the average of u over $B(x_0,r)$ would have to be $< m$. Thus $u(x) = m$ for every $x \in B(x_0,r)$ This shows that the set A of points of Ω where $u(x) = m$, is an open set. But $B = \Omega \setminus A = \{x \in \Omega: u(x) < m\}$ is also open because u is continuous. Since A is not empty and Ω is connected, B has to be necessarily empty. Consequently $u(x) = m$ for every $x \in \Omega$. $\square$

Observe that the same result is true with minimum instead of maximum.

Here is an equivalent formulation of the maximum and minimum principles:

COROLLARY 1.26. Let u *be a real-valued function, continuous on the closure* $\bar{\Omega}$ *of a bounded domain* $\Omega \subset \mathbb{R}^n$, *and harmonic in* Ω. *Then* u *attains its maximum and its minimum at the boundary of* Ω *(only at the boundary if* u *is not a constant).*

From this we can derive a uniqueness result for the solution of the Dirichlet problem for a bounded domain. Namely:

COROLLARY 1.27. Let u_1 *and* u_2 *be two functions continuous on the closure* $\bar{\Omega}$ *of a bounded domain* Ω, *harmonic in* Ω *and such that* $u_1(x) = u_2(x)$ *for every* $x \in \partial\Omega$, *the boundary of* Ω. *Then* $u_1(x) = u_2(x)$ *for every* $x \in \bar{\Omega}$.

Proof: We may assume that u_1 and u_2 are real-valued. Then, we just need to observe that $u_1 - u_2$ attains its maximum and its minimun at the boundary. But in the boundary it is always zero. Hence $u_1 - u_2$ is zero all over Ω. $\square$

Now we shall briefly discuss the Dirichlet problem for a ball, say the unit ball B_n of $\mathbb{R}^n$, that is: $B_n = B(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$. We have already solved this problem for $n = 2$. The solution u was the Poisson integral of the boundary function f:

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-t)}\,f(t)\,dt =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-|re^{i\theta}|^2}{|re^{i\theta}-e^{it}|^2}\,f(t)\,dt.$$

We shall see that for general n, the solution is given by:

$$u(x) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}}\frac{1-|x|^2}{|x-s|^n}\,f(s)\,ds \tag{1.28}$$

We shall write $(1-|x|^2)\,|x-s|^{-n} = P(x,s)$, the Poisson kernel for the ball. The fact that (1.28) is indeed the solution of the Dirichlet problem for B_n depends upon the following properties of the Poisson kernel:

a) $P(x,s)$ is harmonic in $x \in B_n$ for each fixed $s \in \Sigma_{n-1}$.

b) $P(x,s) \geq 0$ and $\frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} P(x,s)\, ds = 1$ for every $x \in B_n$

c) For every $\delta > 0$

$$\lim_{r \to 1} \int_{|s-x'| > \delta} P(rx',s)\, ds = 0$$

uniformly in $x' \in \Sigma_{n-1}$.

Let us check these three properties. Let $n > 2$. For a), observe that

$$P(x,s) = \frac{1-|x|^2}{|x-s|^n} = -|x-s|^{2-n} - (2/(n-2)) \sum_j s_j \frac{\partial}{\partial s_j} (|x-s|^{2-n})$$

and, therefore, the harmonicity of $x \mapsto P(x,s)$ follows from the well known fact that $|x|^{2-n}$ is harmonic in $\mathbb{R}^n \setminus \{0\}$ (A radial function in $\mathbb{R}^n$ of the form $\phi(|x|)$ has Laplacian equal to $\phi''(|x|) + \frac{n-1}{|x|} \phi'(|x|)$. Thus if we want $\phi(|x|)$ to be harmonic in $\mathbb{R}^n \setminus \{0\}$, all we have to do is to solve the ordinary differential equation $\phi''(r) + \frac{n-1}{r} \phi'(r) = 0$ Clearly r^{2-n} is a solution)

In b) $P(x,s) \geq 0$ is obvious and, by a)

$$1 = P(0,s) = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} P(rx',s) dx'$$

for every r with $0 < r < 1$.

But, clearly $|rx'-s| = |rs-x'|$ and, consequently:

$$P(rx',s) = P(rs,x').$$

Thus

$$1 = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} P(rs,x')\, dx'$$

which is the equality in b).

Finally c) is very simple, since for each $s \in \Sigma_{n-1}$ with $|s-x'| > \delta > 0$ is $|rx'-s| \geq c_\delta > 0$ for every $r \in [0,1]$, with c_δ independent of $x' \in \Sigma_{n-1}$.

With these three properties we can prove the following

THEOREM 1.29. Let f *be a continuous function on* Σ_{n-1}. *Then the function* u *defined in* $\overline{B(0,1)}$ *as:*

$$u(x) = \begin{cases} \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} P(x,s)\, f(s)\, ds & \textit{if} \quad |x| < 1 \\ \\ f(x) \quad \textit{if} \quad |x| = 1 \end{cases}$$

is continuous in $\overline{B(0,1)}$ *and harmonic in* $B(0,1)$. *It is, therefore, the solution of the Dirichlet problem in* B_n *with boundary function* f

Proof: The harmonicity of u in $B(0,1)$ follows from the harmonicity of $P(x,s)$ as a function of x (either by interchanging the Laplacian with the integral sign or by checking the mean value property directly by using Fubini's theorem).

For the continuity we just need to show that $u(rx') \to f(x')$ as $r \to 1$ uniformly in $x' \in \Sigma_{n-1}$. We write

$$u(rx')-f(x') = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} P(rx',s)(f(s)-f(x'))\, ds$$

by using property b) of the Poisson kernel. Then

$$|u(rx')-f(x')| \leq \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} P(rx',s)|f(s)-f(x')|\, ds =$$
$$= \frac{1}{|\Sigma_{n-1}|} \left(\int_{|s-x'| < \delta} + \int_{|s-x'| \geq \delta}\right)$$

Since f is continuous, given $\varepsilon > 0$, δ can be chosen in such a way that the first term in the sum is $< \varepsilon/2$ independently of x'. Then, with this choice of $\delta > 0$, r can be chosen so close to 1 that the second term in the sum is also smaller than $\varepsilon/2$ independently of x' (by property c)). $\square$

If we want to solve the Dirichlet problem for a ball centered at $x_0 \in \mathbb{R}^n$ and having radius R, all we have to do is to reduce the problem to the unit ball by translation and dilation. Let f be the boundary function. Then consider for $x' \in \Sigma_{n-1}$ the function $g(x') = f(x_0+Rx')$ and solve the Dirichlet problem in the unit ball with boundary function g. The solution will be

$$v(x) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x,s)\, g(s)\, ds$$

Then $u(x) = v\left(\frac{x-x_0}{R}\right)$ will be the solution of the original Dirichlet problem for $B(x_0,R)$. We can write

$$u(x) = v\left(\frac{x-x_0}{R}\right) = \frac{R^{n-2}}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} \frac{R^2-|x-x_0|^2}{|x-x_0-Rs|^n}\, f(x_0+Rs)\, ds$$

The existence of solution for the Dirichlet problem in a ball allows us to make the following useful observation

THEOREM 1.30. Suppose that u *is continuous in an open set* Ω *and satisfies the following seemingly weaker form of the mean value property: for each* $x_0 \in \Omega$*, there is a sequence of positive numbers* $r_j \downarrow 0$ *(the* r_j's *depending possibly on the particular* x_0*) such that for each* r_j:

$$u(x_0) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} u(x_0+r_j x')\, dx'$$

Then u *is harmonic in* Ω

Proof: Let $x_0 \in \Omega$ and suppose that $\overline{B(x_0,R)} \subset \Omega$. Let v be the solution of the Dirichlet problem for $B(x_0,R)$ with boundary function coinciding with u. We shall show that $u = v$ in $B(x_0,R)$. Since x_0 is arbitrary, this will end the proof of the theorem. Suppose that $u-v > 0$ for some point of $B(x_0,R)$, and let $m = \max\{u(x)-v(x): x \in \overline{B(x_0,R)}\} > 0$. Since $u-v = 0$ on the boundary of $B(x_0,R)$, the set of points of $\overline{B(x_0,R)}$ where $u-v$ attains the value m, will be a compact subset K of $B(x_0,R)$. Let x_1 be a point of this compact K having maximal distance to x_0. For each $r > 0$ small enough, at least half of the sphere $\partial B(x_1,r)$ is not in K. But then $u(x_1)-v(x_1)$, which is the average of $u-v$ over each of the spheres of center x_1 and radius r_j (each of the radii corresponding to x_1) will have to be $<m$, which is a contradiction. If we had $u-v < 0$ for some point in $B(x_0,R)$ we would proceed exactly in the same way but using the minimum instead of the maximum. $\square$

Theorem 1.30 allows us to give a very simple proof of the following *reflection principle*

THEOREM 1.31. Let Ω *be a domain in* $\mathbb{R}^n$, *symmetric with respect to the hyperplane* $x_n = 0$. *Suppose that* u *is a function continuous in* Ω, *harmonic in* $\Omega^+ = \{x \in \Omega: x_n > 0\}$ *and odd in the variable* x_n, *that is:* $u(x_1, \ldots, x_{n-1}, -x_n) = -u(x_1, \ldots, x_{n-1}, x_n)$ *for every* $x = (x_1, \ldots, x_n) \in \Omega$. *Then,* u *is harmonic in* Ω.

Proof: We just need to show that u satisfies the mean value property in the form appearing in 1.30. But this is obvious, because: 1) If $x_0 \in \Omega^+$ we know that u is harmonic in Ω^+ and, consequently for those r's such that $\overline{B(x_0,r)} \subset \Omega^+$, $u(x_0)$ coincides with the average of u over the sphere $\partial B(x_0,r)$. 2) If $x_0 \in \Omega^- = \{x \in \Omega: x_n < 0\}$, we have the same situation because for $x = (x_1, \ldots, x_n) \in \Omega^-$, is $u(x_1, \ldots, x_{n-1}, -x_n) = -u(x_1, \ldots, x_{n-1}, -x_n)$ and this implies that $\Delta u(x) = 0$ also in Ω^-. 3) If $x_0 = (x_1, \ldots$ $\ldots, x_{n-1}, 0)$, we have $u(x_0) = 0$ and also, the average of u over each sphere centered at x_0 is necessarily o because u takes opposite values at a point and its symmetric in the other halfspace. $\square$

As a last application of the mean value property we shall prove the following extension of a classical theorem of Liouville:

THEOREM 1.32. The only bounded harmonic functions in $\mathbb{R}^n$ *are the constants.*

Proof: Suppose u is harmonic in $\mathbb{R}^n$ and also $|u(x)| \leq M < \infty$ for every $x \in \mathbb{R}^n$. Let x_1 and x_2 be two arbitrary points chosen in $\mathbb{R}^n$. Then, using the mean value property we have

$$u(x_1) - u(x_2) = \frac{1}{|B(x_1,r)|}\int_{B(x_1,r)} u(x)dx - \frac{1}{|B(x_2,r)|}\int_{B(x_2,r)} u(x)dx$$

Take r much bigger than $|x_1 - x_2| = d$. Then

$$|u(x_1)-u(x_2)| \leq \frac{Mn}{|\Sigma_{n-1}|r^n} |B(x_1,r)\Delta B(x_2,r)| \text{ , where } B(x_1,r)\Delta B(x_2,r)$$

is the symmetric difference of the two balls. Then, since

$$|B(x_1,r)\setminus B(x_2,r)| \leq |B(0,1)| \; (r^n-(r-d)^n) \leq \frac{|\Sigma_{n-1}|}{n} nr^{n-1}d = |\Sigma_{n-1}| \, r^{n-1}d$$

we get $|u(x_1) - u(x_2)| \leq \frac{2Mnd}{r}$. Now just let $r \to \infty$. $\square$

Of course, the Dirichlet problem can also be posed for an unbounded domain. As a preparation for later chapters we shall have to deal with the Dirichlet problem for a half-space. To set up the problem, we consider the euclidean space $\mathbb{R}^{n+1}$ whose points we denote by (x,t) with $x = (x_1, \dots x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. That is, we single out the last coordinate. Our domain will be $\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1}:$ $: t > 0\}$ whose boundary $\partial\mathbb{R}^{n+1}_+ = \{(x,0): x \in \mathbb{R}^n\}$ we shall identify with $\mathbb{R}^n$ in the natural way. The Dirichlet problem for $\mathbb{R}^{n+1}_+$ is the following: Given f, a continuous function in $\mathbb{R}^n$, we have to find $u(x,t)$, continuous in $\overline{\mathbb{R}^{n+1}_+} = \{(x,t): x \in \mathbb{R}^n, t \geq 0\}$, harmonic in $\mathbb{R}^{n+1}_+$ and such that $u(x,0) = f(x)$ for every $x \in \mathbb{R}^n$. We observe that this problem cannot have a unique solution. Indeed if $u(x,t)$ is a solution, $v(x,t) = u(x,t) + t$ is a solution too. However, we can see that it can have at most one bounded solution. We just need to prove the following

THEOREM 1.33. Let u *be a function continuous and bounded in* $\overline{\mathbb{R}^{n+1}_+}$ *and harmonic in* $\mathbb{R}^{n+1}$*, such that* $u(x,0) = 0$ *for every* $x \in \mathbb{R}^n$*. Then* u *is identically zero.*

Proof: Define a function $v(x,t)$ in $\mathbb{R}^{n+1}$ by letting

$$v(x,t) = \begin{cases} u(x,t) & \text{if } t \geq 0 \\ -u(x,-t) & \text{if } t \leq 0 \end{cases}$$

According to theorem 1.31, v is harmonic in $\mathbb{R}^{n+1}$.

It is also, obviously, bounded. By theorem 1.32, it has to be a constant. Since $v(x,0) = u(x,0) = 0$, we have, necessarily $v \equiv 0$ and, therefore $u \equiv 0$. $\square$

Next we shall seek a solution of the Dirichlet problem in $\mathbb{R}^{n+1}_+$ for nice boundary functions, say $f \in C^\infty_c(\mathbb{R}^n)$. Our tool will be now the Fourier transform $\mathcal{F}$ in $\mathbb{R}^n$ (see Stein-Weiss [2], Chapter I), just as the Fourier series led us to the solution for the disk. If $u(x,t)$ is a solution, we shall have

$$\begin{cases} \Delta u(x,t) = \Delta_x u(x,t) + \dfrac{\partial^2 u}{\partial t^2}(x,t) = 0 & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = f(x), \quad x \in \mathbb{R}^n. \end{cases}$$

We take Fourier transforms with respect to the variable x. If we call $\mathcal{F}(u(\cdot,t))(\xi) = h(\xi,t)$ and $\mathcal{F} f(\xi) = \hat{f}(\xi)$, we shall have

$$\begin{cases} -4\pi^2 \, |\xi|^2 \, h(\xi,t) + \dfrac{\partial^2 h}{\partial t^2} (\xi,t) = 0 \\ h(\xi,0) = \hat{f}(\xi) \end{cases}$$

For each ξ, this is a very simple linear ordinary differential equation with constant coefficients. The only solution that remains bounded for $t \to \infty$ is $h(\xi,t) = \hat{f}(\xi)e^{-2\pi|\xi|t}$. The inverse Fourier transform of this function in the variable ξ will be, hopefully, the solution $u(x,t)$ of our problem

$$u(x,t) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{-2\pi|\xi|t}e^{2\pi i\xi\cdot x}d\xi$$

If we call $P(x) = \mathcal{F}(e^{-2\pi|\cdot|})(x)$, then the Fourier transform of $\xi \mapsto e^{-2\pi|\xi|t}e^{2\pi i\xi\cdot x}$ will be

$$P_t(y-x) = P_t(x-y) \qquad \text{where} \qquad P_t(x) = t^{-n}P(t^{-1}x)$$

Thus, our candidate for a solution is

$$(1.34) \qquad u(x,t) = \int_{\mathbb{R}^n} P_t(x-y)\, f(y)\, dy$$

P_t is called the Poisson kernel for $\mathbb{R}^{n+1}_+$ (or for $\mathbb{R}^n$). For $n = 1$ it is very simple to obtain $P(x)$

$$P(x) = \int_{-\infty}^{\infty} e^{-2\pi|\xi|}e^{-2\pi i\xi x}d\xi = \int_0^{\infty} e^{-2\pi\xi(1+ix)}d\xi \; +$$

$$+ \int_0^{\infty} e^{-2\pi\xi(1-ix)}d\xi = \frac{1}{2\pi}\left(\frac{1}{1+ix} + \frac{1}{1-ix} \right) = \frac{1}{\pi}\,\frac{1}{1+x^2}$$

Then $P_t(x) = \frac{1}{\pi}\frac{t}{t^2+x^2}$ and the proposed solution is in this case:

$$(1.35) \qquad u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{t}{t^2+(x-y)^2}\, f(y)\, dy$$

For $n > 1$, the computation of $P(x)$ is not so simple and we refer the reader to Stein-Weiss [2] Ch. I, where it is proved that

$$P(x) = c_n \frac{1}{(1+|x|^2)^{(n+1)/2}}$$

with $\quad c_n = \dfrac{2}{|\Sigma n\ |} = \Gamma((n+1)/2)/(\pi^{(n+1)/2})$

Thus, in $\mathbb{R}^n$ $\quad P_t(x) = c_n t(t^2 + |x|^2)^{-(n+1)/2}$ and

$$(1.36) \qquad u(x,t) = c_n \int_{\mathbb{R}^n} \frac{t}{(t^2 + |x-y|^2)^{(n+1)/2}} f(y)\, dy.$$

The integral (1.34) is denoted as $P(f)(x,t)$ and it is called the Poisson integral of f. We shall see that $P(f)$ is the solution of the Dirichlet problem in the 4^{th} section of chapter II and we shall try to extend to $\mathbb{R}^{n+1}$ the Fatou-type theorems which we have seen hold in the disk.

2. SUBHARMONIC FUNCTIONS

DEFINITION 2.1. A subharmonic function on an open set $\Omega \subset \mathbb{R}^n$ *is a function* v *defined on* Ω, *with values* $-\infty \leq v(x) < \infty$ *and satisfying the following two conditions:*

i) v *is upper semicontinuous in* Ω.

ii) For every $x_0 \in \Omega$, *there is a ball* $B(x_0, r(x_0)) \subset \Omega$, $r(x_0) > 0$, *such that for every* r *with* $0 < r < r(x_0)$

$$(2.2) \qquad v(x_0) \leq \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} v(x_0 + r\sigma)\, d\sigma$$

To say that v is upper semicontinuous in Ω means that for every $t \in \mathbb{R}$, the set $\{x \in \Omega : v(x) < t\}$ is open. This is, in turn, clearly equivalent to the following:

$$(2.3) \qquad \text{For every } x_0 \in \Omega : \limsup_{x \to x_0 \text{ in } \Omega} v(x) \leq v(x_0)$$

Like continuity, upper semicontinuity is a pointwise property. When the inequality in (2.3) holds for a given $x_0 \in \Omega$, it is said that v is upper semicontinuous at x_0.

Observe that the upper semicontinuity of v, together with the fact that $v(x) < \infty$ for every $x \in \Omega$, imply that v is bounded above on every compact $K \subset \Omega$. Indeed, let $K_j = \{x \in K : v(x) \geq j\}$ for $j = 1, 2, \ldots$ These K_j's are compact sets and $K \quad K_1 \quad K_2 \quad \ldots$ Since $\bigcap_{j=1} K_j = \{x \in K : v(x) = \infty\} = \emptyset$ by assumption, we must have

$K_j = \emptyset$ for some j, that is: $v(x) < j$ for every $x \in K$.

This is what allows us to assign a meaning to the integrals appearing in (2.2). Each of these integrals is defined as the integral of the positive part of the function $v(x_0+r\sigma)$ minus the integral of its negative part. This difference makes sense because the integral of the positive part is finite, being the integral of a bounded function. A priori, the integrals in (2.2) can be either a real number or $-\infty$. Actually, we shall eventually show that, unless v is identically equal to $-\infty$, none of these integrals can be $-\infty$, in such a way that v is integrable over the corresponding spheres.

It has to be noted that, for v subharmonic, (2.2) implies that $v(x_0) \leq \limsup_{x \to x_0 \text{ in } \Omega} v(x)$ and, consequently, we actually have equality in (2.3).

Here is a useful characterization of upper semicontinuity.

PROPOSITION 2.4. v is upper semicontinuous in Ω if and only if, for every compact $K \subset \Omega$, v is the limit over K of a decreasing sequence of continuous functions.

Proof: First of all, we see that the infimum v of a family v_α of upper semicontinuous functions, is itself upper semicontinuous. Indeed $\{x: v(x) < t\} = \bigcup_\alpha \{x: v_\alpha(x) < t\}$, is an open set, since it is union of open sets. In particular, if v is the limit of a decreasing sequence of continuous functions in K, v will be upper semicontinuous in K.

For the converse, if v is upper semicontinuous in Ω, K is a compact subset of Ω and $\varepsilon > 0$ is small enough, we find a finite covering of K by balls $B(x_j,\varepsilon) \subset \Omega$ with $x_j \in K$. Then let ϕ_j be continuous ≥ 0, with support contained in $B(x_j,\varepsilon)$ and such that $\sum_j \phi_j(x) = 1$ for every $x \in K$. (The ϕ_j's form what is known as a partition of the unity in K subordinated to the covering. See Rudin [1])

Let $m_j = \sup_{B(x_j,\varepsilon)} v$ and consider the continuous function

$\psi(x) = \sum_j m_j \phi_j(x)$. We can do this for a decreasing sequence $\varepsilon_1 > \varepsilon_2 > \ldots \to 0$, obtaining corresponding functions $\psi_1, \psi_2, \ldots$. Then let $u_1 = \psi_1$, $u_2 = \min(\psi_2, u_1)$ ..., $u_j = \min(\psi_j, u_{j-1})$, ... In this way, we obtain continuous functions $u_1 \geq u_2 \geq \ldots$, and this sequence can be easily seen to converge to v by using the upper semicontinuity of v. □

Of course, any real-valued harmonic function is subharmonic, since it is continuous and has the mean value property (1.23), whcih is stronger than (2.2). However, subharmonicity is all that is needed for the maximum principle. We can state

THEOREM 2.5. Let v be a subharmonic function in a domain $\Omega \subset \mathbb{R}^n$. Then v cannot attain a maximum value unless it is constant.

Proof: The proof of corollary 1.25 applies to this case with minor changes. □

The maximum principle can also be given in this form

COROLLARY 2.6. Let Ω be a bounded domain in $\mathbb{R}^n$ and let $v: \overline{\Omega} \longrightarrow [-\infty, \infty)$ be upper semicontinuous in $\overline{\Omega}$ and subharmonic in Ω. Then v has a maximum in $\overline{\Omega}$ and attains it at the boundary (only at the boundary if v is not a constant).

Proof: The fact that v has a maximum in $\overline{\Omega}$ follows simply from the upper semicontinuity (exactly in the same way that the fact that it is bounded above). Then, we just need to apply theorem 2.5. □

The maximum principle can be used to establish the following characterization of subharmonic functions, which is the best justification for the name "subharmonic".

THEOREM 2.7. Let $v: \Omega \longrightarrow [-\infty, \infty)$ be upper semicontinuous in the open set Ω. Then, the following conditions are equivalent:

a) v *is subharmonic in* Ω

b) *Whenever* u *is a real-valued continuous function in* $\overline{G}$, *harmonic in* G, G *being an open and bounded set with* $\overline{G} \subset \Omega$, *and* u *satisfies* $v(x) \leq u(x)$ *for every* $x \in \partial G$, *then*

$v(x) \leq u(x)$ *for every* $x \in G$.

Proof: Suppose a) holds and u satisfies the assumption made in b) We can assume that G is connected. Then the function $v-u$ is upper semicontinuous in $\overline{G}$, subharmonic in G and $v-u \leq 0$ in ∂G. It follows from corollary 2.6 that $v-u \leq 0$ also in G.

Conversely, assuming that b) holds, let us prove that v is subharmonic. Let $x_0 \in \Omega$ and $\overline{B(x_0,r)} \subset \Omega$. We shall prove that (2.2) holds. Since v is upper semicontinuous, we know from Proposition 2.4 that there is a decreasing sequence of functions $u_1 \geq u_2 \geq \ldots$ $\ldots \geq u_j \geq \ldots$ continuous in $\partial B(x_0,r)$, converging to v in $\partial B(x_0,r)$. Abusing the notation a little, let us denote also by u_j the function continuous in $\overline{B(x_0,r)}$ and harmonic in $B(x_0,r)$ which coincides with u_j in $\partial B(x_0,r)$, that is, the solution of the Dirichlet problem in $B(x_0,r)$ with boundary function u_j. It follows from b) that $v(x) \leq u_j(x)$ for every j and every $x \in B(x_0,r)$. Also, the subharmonicity of u_{j+1} and the already proved fact that a) implies b) yield $u_j(x) \geq u_{j+1}(x)$ for every j and every $x \in B(x_0,r)$. We can write:

$$v(x_0) \leq \lim_{j\to\infty} u_j(x_0) = \lim_{j\to\infty} \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} u_j(x_0+r\sigma)\, d\sigma =$$

$$= \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} \lim_{j\to\infty} u_j(x_0+r\sigma)\, d\sigma = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} v(x_0+r\sigma)\, d\sigma.$$

and (2.2) is proved. The interchanging of the limit and the integral sign is justified by the monotone convergence theorem. $\square$

Observe that in the course of proving theorem 2.7 we have also proved that if v is subharmonic in Ω, then (2.2) holds whenever $\overline{B(x_0,r)} \subset \Omega$. This is stronger than what we assumed in the definition of subharmonicity, namely, that (2.2) holds for small enough radii, $r < r(x_0)$ depending on the point $x_0 \in \Omega$.

Proposition 2.4. provides a very useful technique for dealing with subharmonic functions. Two more examples of its use follow

PROPOSITION 2.8. Let v *be a subharmonic function in a domain* $\Omega \subset \mathbb{R}^n$, *and suppose that* v *is not identically equal to* $-\infty$. *Then*, *whenever* $\overline{B(x_0,r)} \subset \Omega$, *we have.*

$$\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0+r\sigma)\,d\sigma > -\infty$$

Proof: Let $\overline{B(x_0,r)} \subset \Omega$, and let $u_1 \geq u_2 \geq \ldots \geq u_j \geq \ldots$ be a sequence of continuous functions in $\partial B(x_0,r)$ converging to v in $\partial B(x_0,r)$. As before, consider u_j extended as a harmonic function in $B(x_0,r)$, continuous in $\overline{B(x_0,r)}$. Then, for every $x \in \mathbb{R}^n$ with $|x| < 1$ is:

$$v(x_0+rx) \leq \lim_{j\to\infty} u_j(x_0+rx) = \lim_{j\to\infty} \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x,s)\,u_j(x_0+rs)\,ds =$$

$$= \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x,s) \lim_{j\to\infty} u_j(x_0+rs)\,ds = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} P(x,s)v(x_0+rs)ds,$$

where we have used the monotone convergence theorem (note that $P(x,s) \geq 0$). Now, if

$$\int_{\Sigma_{n-1}} v(x_0+r\sigma)\,d\sigma = -\infty,$$

then also

$$\int_{\Sigma_{n-1}} P(x,s)\,v(x_0+rs)\,ds = -\infty,$$

because $P(x,s)$ is positive and bounded as a function of s. It follows that $v(x) = -\infty$ for every $x \in B(x_0,r)$.

What we have shown above implies that the set

$$A = \{x \in \Omega : \int_{\Sigma_{n-1}} v(x+r\sigma)\,d\sigma = -\infty \ \text{ for some } r > 0 \text{ with } \overline{B(x,r)} \subset \Omega\}$$

is open. But its complement $B = \Omega \setminus A$ is also open. Indeed, if $x_0 \in B$ and $\overline{B(x_0,r)} \subset \Omega$, no point of A can belong to $B(x_0,r)$ because this would imply that

$$\int_{\Sigma_{n-1}} v(x_0+r'\sigma)\,d\sigma = -\infty$$

for some $r' < r$, since v would be equal to $-\infty$ in a whole open subset of the sphere $\Sigma(x_0,r')$.

Therefore, either $A = \emptyset$ or $A = \Omega$, and in this latter case v is identically $-\infty$. $\square$

PROPOSITION 2.9. Let v *be a subharmonic function in* $B(0,R) \subset \mathbb{R}^n$. *Then,*

$$m(r) = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} v(r\sigma)\, d\sigma$$

in an increasing function in the interval $[0,R)$.

Proof: Let $r_1 < r_2 < R$. Let $u_1 \geq u_2 \geq \dots \geq u_j \geq \dots$ be a sequence of functions continuous in $\partial B(0,r_2)$ converging to v in $\partial B(0,r_2)$. For each j, we also denote by u_j the function continuous in $\overline{B(0,r_2)}$ and harmonic in $B(0,r_2)$ coinciding with our original u_j in $\partial B(0,r_2)$. Then

$$m(r_1) = \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} v(r_1\sigma) d\sigma \leq \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} u_j(r_1\sigma) d\sigma = u_j(0) =$$

$$= \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} u_j(r_2\sigma) d\sigma \xrightarrow[j\to\infty]{} \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} v(r_2\sigma) d\sigma = m(r_2)$$

Thus $m(r_1) \leq m(r_2)$ and our statement is proved. $\square$

It has to be observed that in the definition 2.1 of a subharmonic function, condition ii) can be replaced by:

ii)' *For every* $x_0 \in \Omega$ *and every* $r > 0$ *such that* $\overline{B(x_0,r)} \subset \Omega$

$$v(x_0) \leq \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} v(x)\, dx$$

We do not even need to require this for every $r > 0$ with $\overline{B(x_0,r)} \subset \Omega$, but just for those satisfying $0 < r < r(x_0)$ for some $r(x_0)$ with $B(x_0,r(x_0)) \subset \Omega$.

Indeed, if we have (2.2) with s in place of r, for every $s \in (0,r]$, we get condition ii)' just by integrating both sides of (2.2) against s^{n-1} between o and r, exactly as we did in the remark following theorem 1.22. Conversely if ii)' holds even with the restriction $r < r(x_0)$, then it can be seen, just as in theorem 2.5 or corollary 1.25, that v satisfies the maximum principle; and once this is done, we see that v satisfies property b) in theorem 2.7 and, consequently, v is subharmonic.

For $v \in C^2(\Omega)$, the method of proof of theorem 1.22 gives a necessary and sufficient condition for v to be subharmonic in terms of Δv, namely, we must have $\Delta v(x) \geq 0$ for every $x \in \Omega$. The proof of the necessity uses also proposition 2.9. The criterion can be extended to arbitrary functions by interpreting Δv in the distributions sense (see Stein-Weiss [2], Ch. II), but we shall not make use of this. We shall however need the following

PROPOSITION 2.10. Suppose that v *is a non-negative continuous function on an open set* Ω *such that* v *is of class* C^2 *on the open set* $\Omega_0 = \{x \in \Omega: v(x) > 0\}$ *and satisfies* $\Delta v \geq 0$ *on* Ω_0. *Then* v *is subharmonic in the whole set* Ω.

Proof: We already know that v is subharmonic in Ω_0, so that we only need to consider points $x_0 \in \Omega$ such that $v(x_0) = 0$. Let x_0 be one of such points, and suppose $\overline{B(x_0,r)} \subset \Omega$. We must show that

$$v(x_0) \leq \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} v(x_0 + r\sigma)\, d\sigma$$

Let u be harmonic in $B(x_0,r)$ coinciding with v in the boundary of $B(x_0,r)$. It is enough to show that $v(x) \leq u(x)$ for every $x \in B(x_0,r)$. Suppose that $v(x) - u(x) > 0$ for some $x \in B(x_0,r)$, and define then

$$\max_{x \in \overline{B(x_0,r)}} (v(x) - u(x)) = \delta > 0$$

Let $A = \{x \in B(x_0,r): v(x) - u(x) = \delta\}$. If $x \in A$, we have $v(x) = u(x) + \delta \geq \delta > 0$, so that $A \subset \Omega_0$. On the other hand, A is a non-empty closed subset of $B(x_0,r)$. It turns out that A is also open. Indeed, let $x \in A$, and let $B(x,r') \subset B(x_0,r) \cap \Omega_0$. Then, since $v-u$ is subharmonic in Ω_0, we have

$$\delta = v(x) - u(x) \leq \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} (v(x+r'\sigma) - u(x+r'\sigma))\, d\sigma \leq \delta$$

It follows that $v-u = \delta$ all over $B(x,r')$. Since A is both open and closed, it must be $A = B(x_0,r)$, but this is impossible because $v(x_0) - u(x_0) = -u(x_0) \leq 0$. $\square$

THEOREM 2.11. Let v *be a subharmonic function in* Ω, *and suppose* Φ *is a function increasing and convex in* $\mathbb{R}$. *Then the composite* *function* $\phi o v$ *is also subharmonic (define* $\phi(-\infty)$ *so that* ϕ *becomes continuous at* $-\infty$ *. That way, the composition will always make sense)*

Proof: First we have to see that $\phi o v$ is upper semicontinuous, i.e. that $(\phi o v)^{-1}([-\infty,t))$ is always open. But $(\phi o v)^{-1}([-\infty,t)) = v^{-1}(\phi^{-1}([-\infty,t))$ and, since ϕ is increasing and continuous, $\phi^{-1}([-\infty,t)) = [-\infty,s)$ unless it is empty. Since v is upper semicontinuous $v^{-1}([-\infty,s))$ is open. Thus, $(\phi o v)^{-1}([-\infty,t))$ is always open. Next, let $\overline{B(x_0,r)} \subset \Omega$. Then

$$\phi(v(x_0)) \leq \phi\left(\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0+r\sigma)d\sigma\right) \leq \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} \phi(v(x_0+r\sigma))d\sigma,$$

the last inequality being a consequence of convexity (Jensen's inequality). $\square$

We shall give now our main example of a subharmonic function, namely, $\log|F(z)|$ for F holomorphic not identically 0 in a plane domain. The subharmonicity of this function will be one of our main tools. If F is never 0, then $\log|F(z)| = \operatorname{Re}(\log F(z))$, where $\log F(z)$ can be defined locally as a holomorphic function. It follows that for F holomorphic without zeroes $\log|F(z)|$ is actually a harmonic function. In the general case, some work will be needed to get rid of the zeroes.

LEMMA 2.12. $$\int_{-\pi}^{\pi} \log|1-e^{it}|\,dt = 0$$

Proof: Since $1-z$ is holomorphic in D and has no zeroes, we see that $\log|1-z|$ is a harmonic function, and the mean value property yields:

$$(2.13)\quad \int_{-\pi}^{\pi} \log|1-re^{it}|\,dt = 0 \quad \text{for every} \quad r \in [0,1)$$

Now we can let r tend to 1 and use Lebesgue dominated convergence theorem. Indeed, for $|t| < \pi/3$, we have:

$$|\log|1-re^{it}|| = \log\frac{1}{|1-re^{it}|} = \log\frac{1}{\sqrt{1+r^2-2r\cos t}} \leq \log\frac{1}{|\sin t|} \leq \frac{C_\alpha}{|t|^\alpha}$$

for any $0 < \alpha < 1$; and if $|t| \geq \pi/3$, then $(\sqrt{3}/2) \leq |1-re^{it}| \leq 2$

and $|\log|1-re^{it}|| \leq \log 2$. Thus, the integrands in (2.13) are bounded in absolute value by the same integrable function. $\square$

THEOREM 2.14. (*Jensen's formula*) *Let* F *be holomorphic in* $D(0,R)$ *and suppose that* $F(0) \neq 0$. *Let* $0 < r < R$ *and call* $z_1, z_2, \ldots, z_n$ *the zeroes of* F *in* $\overline{D(0,r)}$ *listed according to their multiplicities. Then*:

$$\log|F(0)| + \sum_{j=1}^{n} \log \frac{r}{|z_j|} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(re^{it})|\, dt.$$

Proof: Suppose $z_1, \ldots, z_m \in D(0,r)$ and $|z_{m+1}| = \ldots = |z_n| = r$ Define

$$G(z) = F(z) \prod_{j=1}^{m} \frac{r^2 - z\overline{z_j}}{r(z-z_j)} \prod_{j=m+1}^{n} \frac{z_j}{z-z_j}$$

G is holomorphic and nowhere zero in $D(0,r+\varepsilon)$ for some $\varepsilon > 0$, therefore $\log|G(z)|$ will be harmonic in $D(0,r+\varepsilon)$ and we shall have:

$$\log|G(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|G(re^{it})|\, dt$$

But

$$\log|G(0)| = \log|F(0)| + \sum_{j=1}^{m} \log\frac{r}{|z_j|} = \log|F(0)| + \sum_{j=1}^{n} \log\frac{r}{|z_j|}$$

Since

$$\left|\frac{r^2 - re^{it}\overline{z_j}}{r(re^{it}-z_j)}\right| = \left|\frac{r-e^{it}\overline{z_j}}{re^{it}-z_j}\right| = 1 \text{ and } \left|\frac{z_j}{re^{it}-z_j}\right| = \left|\frac{re^{it_j}}{r(e^{it}-e^{it_j})}\right| =$$

$$= \frac{1}{|1-e^{i(t-t_j)}|}$$

we finally obtain:

$$\log|F(0)| + \sum_{j=1}^{n} \log\frac{r}{|z_j|} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(re^{it})|\, dt +$$

$$+ \sum_{j=m+1}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\frac{1}{|1-e^{i(t-t_j)}|}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(re^{it})|\, dt$$

by the previous lemma. $\square$

COROLLARY 2.15. Let F *be holomorphic, not identically* 0 *on an open set* $\Omega \subset \mathbb{C}$*. Then, the functions* $\log |F(z)|$, $\log^+ |F(z)| = \max(\log |F(z)|, 0)$ *and* $|F(z)|^a$ *for any* $0 < a < \infty$*, are all subharmonic in* Ω*.*

Proof: First of all, let us see that the function $\log |F(z)|$ is subharmonic. It is a continuous function with values in $[-\infty, \infty)$. Besides, if $\overline{D(z_0,r)} \subset \Omega$, we have:

$$\log |F(z_0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(z_0+re^{it})| \, dt$$

This is clear if $F(z_0) = 0$. If this is not the case, it follows from theorem 2.14 applied to the function $z \longmapsto F(z_0+z)$ which is holomorphic in $D(0,r+\varepsilon)$ for some $\varepsilon > 0$ and does not vanish at 0.

As for the functions $\log^+ |F(z)|$ and $|F(z)|^a$, $a > 0$, they result from composing $\log |F(z)|$ with the functions $\phi(t) = \max(t,0)$ or $\phi(t) = e^{at}$ respectively, which are increasing and convex. The subharmonicity follows from theorem 2.11. $\square$

THEOREM 2.16. For $F \in H(D)$ *(that is:* F *is holomorphic in* D*), and* $0 \leq r < 1$*, we define:*

$$m_0(F,r) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |F(re^{it})| \, dt\right)$$

$$m_p(F,r) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{it})|^p \, dt\right)^{1/p}, \quad 0 < p < \infty.$$

$$m_\infty(F,r) = \sup_t |F(re^{it})|$$

Then, for each $F \in H(D)$ *and each* $0 \leq p \leq \infty$*,* $m_p(F,r)$ *is an increasing function of* r *in* $[0,1)$.

Proof: This is just a consequence of corollary 2.15 and proposition 2.9. $\square$

Theorem 2.16. (due to Hardy [1]) is the starting point of the theory of Hardy spaces (H^p) introduced by F. Riesz in [1].

DEFINITION 2.17. For $0 < p \leq \infty$*, we shall define* $H^p(D)$ *(also denoted simply by* H^p *when no confusion is likely to arise) to be the* *following class of functions:*

$$H^p(D) = \{F \in H(D) : \|F\|_{H^p} \equiv \sup_{0 \leq r < 1} m_p(F,r) < \infty\}$$

For $p = 0$, *we have the Nevanlinna class* N, *defined by:*

$$N = \{F \in H(D) : \sup_{0 \leq r < 1} m_0(F,r) < \infty\}$$

If $0 < p < q < \infty$, *we clearly have* $H^\infty \subset H^q \subset H^p \subset N$.

PROPOSITION 2.18. Let $F \in N$ *be such that* $F(0) \neq 0$. *Then*

$$\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\log |F(re^{it})|| \, dt < \infty.$$

Proof:

$$-\infty < \log |F(0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(re^{it})| dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |F(re^{it})| dt -$$
$$- \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |F(re^{it})| \, dt$$

Thus

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |F(re^{it})| dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |F(re^{it})| \, dt - \log |F(0)|$$

and, consequently

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\log |F(re^{it})|| \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |F(re^{it})| dt +$$
$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |F(re^{it})| \, dt \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \log^+ |F(re^{it})| \, dt - \log |F(0)|$$

From this inequality, our claim follows immediately. $\square$

Next, we shall use Jensen's formula to derive a basic fact, namely: that the zeroes of $F \in N$ cannot be too far from the boundary.

THEOREM 2.19. Suppose $F \in N$ *and* F *is not identically* 0. *Let* z_j *be the zeroes of* F *listed according to their multiplicities. Then:*

$$\sum_j (1 - |z_j|) < \infty \tag{2.20}$$

Proof: We may assume $|z_1| \leq |z_2| \leq \dots$ and $F(0) \neq 0$.

Applying Jensen's formula, we get, for every $0 < r < 1$

$$\log |F(0)| + \sum_{|z_j| \leq r} \log \frac{r}{|z_j|} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(re^{it})| \, dt \leq$$
$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |F(re^{it})| \, dt \leq M < \infty$$

with M independent of r. Given n, we just need to take $r < 1$ close enough to 1, to have $|z_n| \leq r$ and, consequently $|z_j| \leq r$ for $j = 1, 2, \dots, n$. Thus, for n fixed

$$\sum_1^n \log \frac{1}{|z_j|} \leq M - n \log r - \log |F(0)|$$

is true for all r bigger than certain $r(n)$ which depends on n. Letting r tend to 1 we obtain

$$\sum_1^n \log \frac{1}{|z_j|} \leq M - \log |F(0)|$$

Since this is true for every n, we have

$$\sum_1^\infty \log \frac{1}{|z_j|} \leq M - \log |F(0)| < \infty .$$

Now, observing that $1 - |z_j| \leq \log \frac{1}{|z_j|}$, we get (2.20). $\square$

The importance of condition (2.20) comes to light in the following result.

THEOREM 2.21. Let (z_j) *be a sequence of complex numbers* $0 < |z_j| < 1$, *such that (2.20) holds. Let* k *be a non-negative integer. Then the "Blaschke product".*

$$B(z) = z^k \prod_{j=1}^{\infty} \frac{z_j - z}{1 - z\bar{z}_j} \frac{|z_j|}{z_j}$$

converges uniformly on each compact subset of the unit disk, to a function $B \in H^\infty$ *whose zeroes are precisely the* z_j's *plus a zero of order* k *at* 0 *if* $k > 0$.

Proof: All we need to prove is that the series

$$\sum_{j=1}^{\infty} \left| 1 - \frac{z_j - z}{1 - z\bar{z}_j} \frac{|z_j|}{z_j} \right|$$

converges uniformly on compact subsets of D. But, for $|z| \leq r < 1$, we have the estimate :

$$\left|1-\frac{z_j - z}{1-z\,\bar{z}_j}\,\frac{|z_j|}{z_j}\right| = (1-|z_j|)\left|\frac{z_j+z\;|z_j|}{z_j-z\;|z_j|^2}\right| \le (1-|z_j|)\,\frac{1+r}{1-r}$$

so that (2.20) is sufficient for our purposes. $\square$

Since $B \in H^\infty$, Fatou's theorem implies that B has non-tangential limits at almost every boundary point. We shall write $B(e^{it}) = \lim B(z)$ (as $z \xrightarrow{N.T.} e^{it}$).

In general, if for some function F in D, the non-tangential boundary value of F is known to exist at e^{it}, we shall denote it by $F(e^{it})$.

If $F \in H^p$ with $p \ge 1$, we know that $F(e^{it})$ exists a.e., since F is a Poisson (or Poisson-Stieltjes) integral according to theorems 1.3 and 1.8. In the next section we shall extend this result to any $p > 0$.

THEOREM 2.22. Let B *be the Blaschke product appearing in theorem 2.21. Then* $|B(e^{it})| = 1$ *for a.e.* t *and*

$$\lim_{r\to1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|B(re^{it})|\,dt = 0.$$

Proof: We know that the limit exists because the integral is an increasing function of r. Also $|B(z)| \le 1$, so that $\log(1/|B(re^{it})|) \ge 0$ and Fatou's lemma can be used to obtain:

$$\int_{-\pi}^{\pi}\log(1/|B(re^{it})|)\,dt \le \lim_{r\to1}\int_{-\pi}^{\pi}\log(1/|B(re^{it})|)\,dt$$

or equivalently:

$$\lim_{r\to1}\int_{-\pi}^{\pi}\log|B(re^{it})|\,dt \le \int_{-\pi}^{\pi}\log|B(e^{it})|\,dt \le 0.$$

If we prove that this limit is also ≥ 0, we shall be done. Let us call

$$B_n(z) = z^k\prod_{j=1}^{n}\frac{z_j - z}{1-z\,\bar{z}_j}\,\frac{|z_j|}{z_j}.$$

Note that $|B_n(e^{it})| = 1$ a.e. and also $|B_n(re^{it})| \to 1$ uniformly as $r \to 1$, since B_n is holomorphic in a neighbourhood of $\overline{D}$. Then:

$$\lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log |B(re^{it})|\, dt = \lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \left| \frac{B(re^{it})}{B_n(re^{it})} \right| dt.$$ But

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \left| \frac{B(re^{it})}{B_n(re^{it})} \right| dt \geq \log \left| \frac{B(0)}{B_n(0)} \right| = \sum_{n+1}^{\infty} \log |z_j|$$

Thus, for every n:

$$\sum_{n+1}^{\infty} \log |z_j| \leq \lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log |B(re^{it})|\, dt.$$

Now we realize that

$$\sum_{1}^{\infty} \log \frac{1}{|z_j|} \leq \sum_{1}^{\infty} C(1-|z_j|) < \infty$$

for a C depending on how small is the smallest z_j. It follows that

$$\sum_{n+1}^{\infty} \log |z_j| \to 0 \quad \text{as} \quad n \to \infty$$

and, consequently:

$$0 \leq \lim_{r\to 1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log |B(re^{it})|\, dt. \square$$

3. F. RIESZ FACTORIZATION THEOREM

For $F \in H(D)$ and $0 < p < \infty$, we have denoted by $\|F\|_{H^p}$ the quantity

$$\sup_{0\leq r<1} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{it})|^p\, dt\right)^{1/p} = \lim_{r\to 1} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(re^{it})|^p\, dt\right)^{1/p}$$

with the obvious modification for $p = \infty$. Then, we have introduced the space H^p consisting of those $F \in H(D)$ for which $\|F\|_{H^p} < \infty$.

If $1 \leq p \leq \infty$ and $F, G \in H(D)$, Minkowski's inequality gives: $m_p(F+G,r) \leq m_p(F,r)+m_p(G,r)$ for $0 \leq r < 1$, so, letting $r \to 1$, we get

$$\|F + G\|_{H^p} \leq \|F\|_{H^p} + \|G\|_{H^p}$$

Since, clearly, $\|\ \|_{H^p}$ is homogeneous, and $\|F\|_{H^p} = 0$ implies $F \equiv 0$, it follows that, for $1 \le p \le \infty$, H^p is a linear space and $\|\ \|_{H^p}$ is a norm on it.

If $0 < p < 1$, $\|\ \|_{H^p}$ no longer satisfies the triangle inequality. Instead, in this case, we have, for $F, G \in H(D)$:

$$\|F + G\|^p_{H^p} \le \|F\|^p_{H^p} + \|G\|^p_{H^p}$$

so that $(F,G) \mapsto \|F - G\|^p_{H^p}$ is an invariant metric on H^p. Of course $\|\cdot\|^p_{H^p}$ is p-homogeneous. It follows that H^p is still a linear space and $\|\ \|^p_{H^p}$ is a p-norm on it.

In this section we shall look at properties of the individual functions in H^p. In later sections we shall study H^p as a metric linear space.

Among all the H^p's, H^2 is the simplest to characterize. Indeed if $F \in H(D)$ has the power series expansion $F(z) = \sum_{j\ge0} a_j z^j$, the function $t \mapsto F(re^{it})$ will have the Fourier series $\sum_{j\ge0} a_j\, r^j\, e^{ijt}$ and, by Plancherel's theorem

$$m_2(F,r)^2 = \sum_{j=0}^{\infty} |a_j|^2\, r^{2j}$$

so that

$$\|F\|^2_{H^2} = \lim_{r\to1} m_2(F,r)^2 = \sum_{j=0}^{\infty} |a_j|^2$$

Thus $F \in H^2$ if and only if $\sum_{j\ge0} |a_j|^2 < \infty$. But, if this is the case, the Riesz-Fisher theorem tells us that there is some function $f \in L^2([-\pi,\pi])$ with Fourier series $\sum_{j\ge0} a_j\, e^{ijt}$. Then, the Poisson integral of f will be $\sum_{j\ge0} a_j\, r^j e^{ijt} = F(re^{it})$. Thus, the functions in H^2 are precisely the Poisson integrals of functions in L^2 whose Fourier coefficients a_j are 0 for each $j < 0$.

It follows from theorem 1.3. that this characterization remains true at least for $1 < p \le \infty$. Indeed, if $F \in H^p$, $1 < p \le \infty$ and $F(z) = \sum_{j\ge0} a_j\, z^j$, we know that $F = P(f)$ with $f(t) = F(e^{it})$

belonging to L^p, and also

$$\int_{-\pi}^{\pi} |F(re^{it}) - F(e^{it})|^p \, dt \to 0, \quad \text{as} \quad r \to 1 \qquad (1 < p < \infty)$$

and $F(re^{it}) \to F(e^{it})$ $\star$-weakly in L^∞ as $r \to 1$. This implies that the Fourier series of $f(t) = F(e^{it})$ is $\sum_{j=0}^{\infty} a_j e^{ijt}$.

All we can say at this point about boundary bahaviour and representation in terms of the boundary function is collected in the following

THEOREM 3.1. Let $F \in H^p$ *with* $1 < p \leq \infty$. *Then:*

a) *For almost every* t, *the limit*

$$F(e^{it}) = \lim F(z) \qquad (\text{as} \quad z \xrightarrow{\text{N.T.}} e^{it})$$

exists, the function $f(t) = F(e^{it})$ *belongs to* $L^p([-\pi,\pi])$ *and* $F = P(f)$

b) *If* $p < \infty$:

$$\int_{-\pi}^{\pi} |F(re^{it}) - F(e^{it})|^p \, dt \to 0 \quad \text{as} \quad r \to 1$$

If $p = \infty$, $F(re^{it}) \to F(e^{it})$ *in the weak-* topology of* L^∞ *as* $r \to 1$.

For each $1 < p \leq \infty$: $\|F\|_{H^p} = \|f\|_p$.

c) F *is the Cauchy integral of its boundary function; that is:*

$$F(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{F(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F(e^{it})}{e^{it} - z} e^{it} \, dt$$

Proof: Theorem 1.3 implies that $F = P(f)$ for some $f \in L^p([-\pi,\pi])$. Then, corollary 1.21. gives $F(e^{it}) = f(t)$ for a.e.t. The convergence in b) follows from corollary 1.17. (a) for $p < \infty$ and from theorem 1.18. (a) for $p = \infty$.

Fatou's lemma gives $\|f\|_p \leq \|F\|_p$ and the converse inequality follows from theorem 1.11.

As for c), we know that for $r < 1$, Cauchy's formula holds:

$$F(rz) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{F(r\zeta)}{\zeta - z}\, d\zeta$$

Then all we have to do is to let $r \to 1$ and use b). $\square$

THEOREM 3.2. Let F *be a function in* N, *not identically* 0. *Then:*

$$\limsup_{r \to 1} \int_{-\pi}^{\pi} |\log|F(re^{it})||\, dt < \infty$$

Hence, if $F(e^{it}) = \lim_{r\to 1} F(re^{it})$ *exists a.e. (It always does, but we have proved this only for* $F \in H^p$, $p \geq 1$*), then*

$$\int_{-\pi}^{\pi} |\log|F(e^{it})||\, dt < \infty$$

and, consequently, $F(e^{it})$ *can vanish only on a set of measure* 0.

Proof: All we need to prove is the finiteness of the lim sup, because then the integrability of $\log|F(e^{it})|$ follows by using Fatou's lemma and, since $F(e^{it}) = 0$ precisely where $|\log|F(e^{it})|| = \infty$, and this can only happen on a set of measure 0, everything will be proved.

Suppose first that $F(0) \neq 0$. Then

$$-\infty < \log|F(0)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(re^{it})|\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|F(re^{it})|\, dt -$$

$$- \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^-|F(re^{it})|\, dt.$$

Thus,

$$0 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^-|F(re^{it})|\, dt \leq \sup_{r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|F(re^{it})|\, dt - \log|F(0)| <$$

$$< \infty$$

and, consequently

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\log|F(re^{it})||\, dt \leq 2 \sup_{r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|F(re^{it})|\, dt - \log|F(0)| =$$

$$= C < \infty.$$

which completes the proof in this case.

If $F(0) = 0$, let $F(z) = z^k H(z)$ with $H(0) \neq 0$. Then

$$-\infty < \log |H(0)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log |H(re^{it})|\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log |F(re^{it})|\, dt - k \log r$$

and we get, exactly as before,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\log |F(re^{it})||\, dt \leq 2 \sup_{r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |F(re^{it})|\, dt - \log |H(0)| - k \log r$$

The right hand side is uniformly bounded for $r > 1/2$, say, and this ends the proof. $\square$

Now we wish to study the boundary behaviour of an $F \in H^p$ with $0 < p \leq 1$. For $p = 1$ we know the existence of the boundary function f. But we do not know whether F has to be necessarily the Poisson integral of f. We shall see later that, indeed, $F = P(f)$. This is very interesting because if we merely knew that F is harmonic, the functions $t \mapsto F(re^{it})$ being still uniformly in L^1, all we could say is that F is the Poisson integral of some measure, not necessarily a function. For $p < 1$, we do not even know whether $F(e^{it})$ exists.

Suppose that $F \in H^p$, $0 < p < \infty$, and F does not have any zeroes in D. Then $F(z) = G(z)^2$ for some $G \in H(D)$. Since $|G(z)|^{2p} = |F(z)|^p$, it follows that $G \in H^{2p}$ with $\|G\|^2_{H^{2p}} = \|F\|_{H^p}$. If $1/2 \leq p < 1$, we know that

$$G(e^{it}) = \lim G(z) \qquad (\text{as } z \xrightarrow{\text{N.T.}} e^{it})$$

exists a.e. and belongs to L^{2p}, because $2p \geq 1$. It follows that

$$F(e^{it}) = \lim F(z) \qquad (\text{as } z \xrightarrow{\text{N.T.}} e^{it}) = G(e^{it})^2$$

also exists a.e. and belongs to L^p.

Then we can go to $1/4 \leq p < 1/2$ and so on. In other words, we can prove by induction, that if $F \in H^p$, $0 < p \leq \infty$, and F is nowhere 0 in D, then F has a non-tangential boundary function $F(e^{it})$ belonging to L^p. Of course we know that the hypothesis "F nowhere 0 in D" can be removed when $p \geq 1$. We shall eventually see that it can also be removed for $0 < p < 1$.

We shall prove next that, if $F \in H^p$, $0 < p < \infty$ and F is nowhere 0 in D, then

$$\int_{-\pi}^{\pi} |F(re^{it}) - F(e^{it})|^p \, dt \to 0 \quad \text{as} \quad r \to 1$$

We know that this holds for $p > 1$ (even for an F with zeroes). Now we shall see that if it holds with $2p$ in place of p, so it does with p. Indeed, let $F \in H^p$ have no zeroes in D. Write, as before, $F(z) = G(z)^2$ with $\|G\|^2_{H^{2p}} = \|F\|_{H^p}$. Then:

$$\int_{-\pi}^{\pi} |F(re^{it}) - F(e^{it})|^p \, dt = \int_{-\pi}^{\pi} |G(re^{it})^2 - G(e^{it})^2|^p \, dt =$$

$$= \int_{-\pi}^{\pi} |G(re^{it}) + G(e^{it})|^p |G(e^{it}) - G(re^{it})|^p \, dt \leq$$

$$\leq \left(\int_{-\pi}^{\pi} |G(re^{it}) + G(e^{it})|^{2p} \, dt\right)^{1/2} \left(\int_{-\pi}^{\pi} |G(re^{it}) - G(e^{it})|^{2p} \, dt\right)^{1/2} \leq$$

$$\leq C \, \|G\|^p_{H^{2p}} \left(\int_{-\pi}^{\pi} |G(re^{it}) - G(e^{it})|^{2p} \, dt\right)^{1/2} \to 0 \quad \text{as} \quad r \to 1.$$

Again, by induction, we prove the convergence for any $p > 0$. Observe that, for $p = 1$, this convergence implies the Poisson and Cauchy representations in terms of the boundary function for an $F \in H^1$ without zeroes.

Now the problem is to see what can be done for functions with zeroes. F. Riesz made the fundamental observation that the zeroes do not matter because we can factor them out. Indeed, if $F \in H^p$, $0 < p \leq \infty$ (or, even more generally, if $f \in N$), and is not identically 0, we have seen (theorem 2.19). that the zeroes $\{z_j\}$ of F, listed according to their multiplicities, satisfy

$$\sum_{j=1}^{\infty} (1 - |z_j|) < \infty,$$

and this is precisely the condition that guarantees the convergence of the Blaschke product B formed with the sequence (z_j) (theorem 2.21). Then we have a function $B \in H^\infty$ with exactly the same zeroes as F and with $|B(e^{it})| = 1$ a.e.. We can write $F = B \cdot H$, where $H \in H(D)$ does not have any zeroes.

Here is a more precise statement:

THEOREM 3.3. Let $F \in N$ *be not identically* 0, *and let* B *be the Blaschke product formed with the zeroes of* F. *Then* $F(z) = B(z)H(z)$ *with* $H \in N$ *vanishing nowhere. Besides* $\|H\|_N = \|F\|_N$ *and if* $F \in H^p$, *then also* $H \in H^p$ *and* $\|H\|_{H^p} = \|F\|_{H^p}$.

Proof: From the simple inequality $\log^+(a \cdot b) \leq \log^+ a + \log^+ b$ we obtain:

$$\int_{-\pi}^{\pi} \log^+|H(re^{it})| \, dt \leq \int_{-\pi}^{\pi} \log^+|F(re^{it})| \, dt + \int_{-\pi}^{\pi} \log \frac{1}{|B(re^{it})|} \, dt$$

(note $|B(re^{it})| \leq 1$). The second integral in the right hand side tends to 0 as r tends to 1. (theorem 2.22).

Thus, $\|H\|_N \leq \|F\|_N$. But $|B(z)| \leq 1$, so that $|F(z)| \leq |H(z)|$ and $\|F\|_N \leq \|H\|_N$. We have proved $\|H\|_N = \|F\|_N$.

Suppose now that $F \in H^p$. Exactly as in the proof of theorem 2.22, denote by $B_n(z)$ the finite partial products of $B(z)$ and let $H_n(z) = F(z)/B_n(z)$.

For each n, $|B_n(re^{it})| \to 1$ uniformly in t, as $r \to 1$. Then, if $p < \infty$

$$\|H_n\|_{H^p}^p = \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{it})/B_n(re^{it})|^p \, dt =$$

$$= \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} (|F(re^{it})/B_n(re^{it})|^p - |F(re^{it})|^p) \, dt + \|F\|_{H^p}^p = \|F\|_{H^p}^p.$$

If $p = \infty$

$$\|H_n\|_{H^\infty} = \lim_{r \to 1} m_\infty(F/B_n, r) \leq$$

$$\leq \lim_{r \to 1} \sup_t (|F(re^{it})/B_n(re^{it})| - |F(re^{it})|) + \lim_{r \to 1} m_\infty(F, r) = \|F\|_{H^\infty}$$

But $|F(z)| \leq |H_n(z)|$, so that $\|H_n\|_{H^\infty} = \|F\|_{H^\infty}$. Thus, for any p: $\|H_n\|_{H^p} = \|F\|_{H^p}$ for all n. Now, for r fixed and $n \to \infty$

$|H_n(re^{it})| \uparrow |H(re^{it})|$. If $p < \infty$, we apply Lebesgue's monotone convergence theorem, to get:

$$m_p(H,r) = \lim_{n\to\infty} m_p(H_n,r) \leq \|F\|_{H^p}$$

and this implies $\|H\|_{H^p} \leq \|F\|_{H^p}$. But $|F(z)| \leq |H(z)|$, so that $\|F\|_{H^p} \leq \|H\|_{H^p}$ and we have finally $\|H\|_{H^p} = \|F\|_{H^p}$. The case $p = \infty$ is even simpler. □

COROLLARY 3.4. Let $F \in H^p$, $0 < p \leq \infty$. *Then* $F = F_1 \cdot F_2$ *with*

$$\|F_1\|_{H^{2p}} = \|F_2\|_{H^{2p}} = \|F\|_{H^p}^{1/2}$$

Proof: Let $F(z) = B(z)\,H(z)$ be the factorization given by theorem 3.3. Since H has no zeroes, it can be written as $H(z) = G(z)^2$ with $\|G\|_{H^{2p}}^{2p} = \|H\|_{H^p}^p = \|F\|_{H^p}^p$. Then, it is enough to write $F_1 = BG$ and $F_2 = G$. □

COROLLARY 3.5. Let $F \in H^p$, $0 < p \leq \infty$. *Then* $F = F_1 - F_2$ *where* F_1 *and* F_2 *are non-vanishing* H^p *functions satisfying* $\|F_j\|_{H^p} \leq \|F\|_{H^p}$, $j = 1,2$.

Proof: Write, as above

$$F = B \cdot H = ((1+B)/2)H - ((1-B)/2)H = F_1 - F_2$$

if we let $F_1 = (1+B)/2$, $F_2 = (1-B)/2$.

Since $|B(z)| < 1$ for every $z \in D$, it is clear that the functions F_j, $j = 1, 2$, do not vanish anywhere and satisfy $\|F_j\|_{H^p} \leq \|F\|_{H^p}$. □

Now we can extend the results on boundary behaviour, obtained for non-vanishing H^p functions, to arbitrary H^p functions.

THEOREM 3.6. Let $F \in H^p$ *with* $0 < p < \infty$. *Then*:

a) *The non-tangential limit*

$$F(e^{it}) = \lim F(z) \quad \text{as} \quad z \xrightarrow{N.T.} e^{it}$$

exists for almost every t, *and the function:* $t \mapsto F(e^{it})$

belongs to $L^p([-\pi,\pi])$.

b) $\int_{-\pi}^{\pi} |F(re^{it}) - F(e^{it})|^p \, dt \to 0$ *as* $r \to 1$.

c) $\|F\|_{H^p} = \lim_{r\to 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(re^{it})|^p dt\right)^{1/p} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{it})|^p dt\right)^{1/p}$

Proof: Suppose F is not identically 0. Let B be the Blaschke product formed with the zeroes of F. Write $F(z) = B(z)\, H(z)$. According to theorem 3.3: $\|H\|_{H^p} = \|F\|_{H^p}$ and H does not vanish. Since both B and H have non-tangential limits at a.e. boundary point, the same holds for F; that is, for a.e. t,

$$F(e^{it}) = B(e^{it})\, H(e^{it}) = \lim F(z) \quad \text{as} \quad z \xrightarrow{\text{N.T.}} e^{it}$$

Besides, since $|B(e^{it})| = 1$ a.e., it follows that $|F(e^{it})| = |H(e^{it})|$ a.e., and consequently $F(e^{it})$ belongs to $L^p([-\pi,\pi])$. This finishes the proof of a).

To prove b), observe that

$$\int_{-\pi}^{\pi} |F(re^{it}) - F(e^{it})|^p \, dt = \int_{-\pi}^{\pi} |B(re^{it})\, H(re^{it}) - B(e^{it})\, H(e^{it})|^p dt$$

$$\le C\Big(\int_{-\pi}^{\pi} |B(re^{it})\, H(re^{it}) - B(re^{it})\, H(e^{it})|^p \, dt \, +$$

$$+ \int_{-\pi}^{\pi} |B(re^{it})\, H(e^{it}) - B(e^{it})\, H(e^{it})|^p \, dt\Big) \le$$

$$\le C\Big(\int_{-\pi}^{\pi} |H(re^{it}) - H(e^{it})|^p \, dt + \int_{-\pi}^{\pi} |B(re^{it}) - B(e^{it})|^p |H(e^{it})|^p dt\Big) \to$$

$$\to 0 \quad \text{as} \quad r \to 1.$$

Indeed, the first integral in the sum tends to 0 because H does not vanish and the second integral tends to 0 simply by dominated convergence.

Finally c) follows from b): Fatou's lemma gives

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{it})|^p \, dt\right)^{1/p} \le \|F\|_{H^p}$$

and the opposite inequality for $p > 1$ was already obtained from

Poisson representation. If $p \leqq 1$, we have:

$$\int_{-\pi}^{\pi} |F(re^{it})|^p \, dt \leq \int_{-\pi}^{\pi} |F(re^{it}) - F(e^{it})|^p \, dt + \int_{-\pi}^{\pi} |F(e^{it})|^p \, dt$$

Letting $r \to 1$ and using b) we get:

$$\|F\|_{H^p} \leqq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{it})|^p \, dt\right)^{1/p}$$

This completes the proof.

COROLLARY 3.7. Let $F \in H^p$ *for some* $0 < p < \infty$, *and suppose that the boundary function* $F(e^{it})$ *belongs to* $L^q([-\pi,\pi])$. *Then* $F \in H^q$.

Proof: If $q \leqq p$, nothing has to be proved, since, in this case $H^p \subset H^q$. Let, therefore $p < q$. If $1 < p$, it is clear that $F \in H^q$, because F is the Poisson integral of its boundary function and this is supposed to belong to L^q. For the case $p \leq 1$, we use the factorization theorem. We write $F = B \cdot H$ where B is the Blaschke product formed with the zeroes of F and H is a non-vanishing holomorphic function such that $\|H\|_{H^p} = \|F\|_{H^p}$. Then we take an integer n such that $1 < np$ and write $H = G^n$. Since $|H|^p = |G|^{np}$, it follows that $G \in H^{np}$ with $\|G\|_{H^{np}}^{np} = \|H\|_{H^p}^p = \|F\|_{H^p}^p$. The boundary function satisfies $|G(e^{it})|^n = |H(e^{it})| = |F(e^{it})|$. The fact that $F(e^{it})$ belongs to L^q implies that $G(e^{it})$ belongs to L^{nq}. Therefore $G \in H^{nq}$ and, consequently $F \in H^q$. $\square$

COROLLAY 3.8. Every $F \in H^1$ *is the Poisson integral and the Cauchy integral of its boundary function* $F(e^{it})$.

Proof: Let $F \in H^1$ and take $0 < s < 1$. We know that, for $z = re^{i\theta} \in D$

$$F(sre^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1+r^2-2r\cos(\theta-t)} F(se^{it}) \, dt$$

Now, letting $s \to 1$ and using the fact, contained in theorem 3.6., that the functions $F(se^{it})$ tend in L^1 to the function $F(e^{it})$ as $s \to 1$, we get:

$$F(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1+r^2-2r\cos(\theta-t)} F(e^{it}) \, dt$$

which is the Poisson representation.

The Cauchy representation is obtained in a similar way starting from:

$$F(sz) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{F(s\zeta)}{\zeta - z}\, d\zeta . \quad \square$$

COROLLARY 3.9. If $F \in H^1$, *then for every* $z \in D$

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \operatorname{Re} F(e^{it})\, dt + i \operatorname{Im} F(0)$$

Proof: We just need to observe that, if $z = re^{i\theta}$, then

$$\operatorname{Re}\left(\frac{e^{it}+z}{e^{it}-z} \right) = \frac{1 - |z|^2}{|e^{it}-z|^2} = P_r(\theta - t)$$

and therefore,

$$G(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \operatorname{Re} F(e^{it})\, dt$$

is a holomorphic function, whose real part coincides with that of F. Hence $F(z) - G(z)$ is constant. Since $G(0) = \operatorname{Re} F(0)$, it follows that $F(z) = G(z) + i \operatorname{Im} F(0)$. $\square$

As a consequence of the Poisson representation for H^1 functions, we can prove a very famous theorem due to F. and M. Riesz.

THEOREM 3.10. Let μ *be a Borel measure on* T *whose Fourier coefficients vanish for all negative frequencies, that is:*

$$\int_{-\pi}^{\pi} e^{ijt}\, d\mu(t) = 0 \quad \textit{for} \quad j = 1, 2, \ldots$$

Then μ *is absolutely continuous with respect to Lebesgue measure, i.e.:* $d\mu(t) = f(t)\, dt$ *for some* $f \in L^1$.

Proof: Let F be the Poisson integral of μ

$$F(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1+r^2-2r\cos(\theta-t)}\, d\mu(t) = \sum_{-\infty}^{\infty} a_j r^{|j|} e^{ij\theta}$$

where

where

$$a_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ijt}d\mu(t), \qquad j = 0, \pm 1, \pm 2, \ldots$$

Since $a_j = 0$ for each $j < 0$, we have:

$F(z) = \sum_0^\infty a_j z^j$, a holomorphic function

On the other hand, (1.15) gives:

$$\int_{-\pi}^{\pi} |F(re^{it})|\, dt \leq \int_{-\pi}^{\pi} d|\mu|\,(t) \qquad \text{for every} \quad 0 < r < 1$$

so that $F \in H^1$. Call $f(t) = F(e^{it})$. Then $F = P(f)$ and, consequently, the measures $f(t)\,dt$ and $d\mu(t)$ have the same Fourier coefficients. Hence $d\mu(t) = f(t)\,dt$ as was to be proved. □

COROLLARY 3.11. Let $F \in H^1$ *and suppose that the boundary function* $F(e^{it})$ *coincides almost everywhere with a function of bounded variation. Then* $F(z)$ *can be extended to a function coninuous on* $\overline{D}$ *and* $F(e^{it})$ *is an absolutely continuous function.*

Proof: If we are able to see that $F(e^{it})$ coincides a.e. with an absolutely continuous function $h(t)$, then, since $F(z)$ is the Poisson integral of h, corollary 1.17 (b) will yield the continuity of F up to the boundary.

We are assuming that $F(e^{it}) = f(t)$ for a.e.t, where $f(t)$ is a function of bounded variation. We can write $f(t) = c + g(t)$ with $g(t) = \int_{-\pi}^{t} d\mu(s)$ for a.e.t,μ being a certain Borel measure. For $j = 1, 2, \ldots$, we get, by integrating by parts:

$$\int_{-\pi}^{\pi} e^{ijt}d\mu(t) = -ij \int_{-\pi}^{\pi} g(t)e^{ijt}dt = -ij\int_{-\pi}^{\pi} F(e^{it})\, e^{ijt}dt =$$

$$= \lim_{r\to 1} \left(-ij \int_{-\pi}^{\pi} F(re^{it})\, e^{ijt}dt\right) = 0.$$

So, by the F. and M. Riesz theorem: $d\mu(t) = k(t)\,dt$ with $k(t)$ integrable and, consequently, g is an absolutely continuous function. □

THEOREM 3.12. Let $F \in H(D)$. *Then* F *is cotinuous up to the boundary* *with an absolutely continuous boundary function if and only if* $F' \in H^1$. *Besides, when* $F' \in H^1$, *we have:*

$$\frac{d}{dt}\left(F(e^{it})\right) = ie^{it}F'(e^{it}) \qquad \textit{a.e.}$$

where, of course, $F'(e^{it})$ *is the boundary function of* F', *i.e.*

$$F'(e^{it}) = \lim F'(z) \quad \textit{as} \quad z \xrightarrow{\text{N.T.}} e^{it}$$

Proof: Suppose that $F \in H(D)$ is continuous up to the boundary and its boundary function $f(t) = F(e^{it})$ is absolutely continuous. Then, of course, F is the Poisson integral of f:

$$F(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, f(t)\, dt$$

Differentiating with respect to θ and integrating by parts we get:

$$ire^{i\theta}\, F'(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r'(\theta-t)\, f(t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, f'(t)\, dt$$

Thus, the holomorphic function $izF'(z)$ is the Poisson integral of f', an integrable function. It follows that $izF'(z)$ is in H^1 and consequently, $F' \in H^1$.

To prove the converse, let $F \in H(D)$ be such that $F' \in H^1$. Then, $izF'(z)$ is also in H^1 and, of course, it will be the Poisson integral of its boundary function:

$$ire^{i\theta}F'(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, ie^{it}F'(e^{it})\, dt$$

Call $g(\theta) = \int_{-\pi}^{\theta} ie^{it}F'(e^{it})\, dt$. Then, integrating by parts:

$$ire^{i\theta}F'(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r'(\theta-t)\, g(t)\, dt = \frac{\partial}{\partial\theta}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\, g(t)\, dt\right)$$

We have seen this way that the functions $F(re^{i\theta})$ and $P(g)(re^{i\theta})$ have the same derivative with respect to θ. It follows that their difference depends only on r. But, since this difference is harmonic and continuous at the origin, it has to be a constant c. Thus, finally: $F = P(g+c)$, the Poisson integral of an absolutely continuous function. Besides:

$$\frac{d}{dt}(F(e^{it})) = \frac{d}{dt}(g(t) + c) = ie^{it}F'(e^{it})$$

by the definition of g. $\square$

COROLLARY 3.13. Let Γ *be a Jordan curve and let* F *be a conformal equivalence from* D *onto the interior domain limited by* Γ. *Then* Γ *is rectifiable if and only if* $F' \in H^1$.

Proof: A classical result of Caratheodory (see Koosis [1] for its proof) says that for any Jordan curve Γ, the conformal equivalence F can be extended continuously to $\overline{D}$ and the extension is a homeomorphism between the two closures. In particular, $F(e^{it})$ is a parametrization of Γ. If Γ is rectifiable, $F(e^{it})$ will be of bounded variation and, by corollary 3.11, absolutely continuous. It follows from theorem 3.12 that $F' \in H^1$. Theorem 3.12 gives also the converse and implies that arc length on Γ is given by the measure $|F'(e^{it})|dt$. $\square$

Note that, from the fact that $t \mapsto F(e^{it})$ is absolutely continuous, it follows that it carries sets of measure 0 in T to sets of measure (arc-length) 0 in Γ.

We also have the existence a.e. of the derivative $\frac{d}{dt}(F(e^{it})) = ie^{it}F'(e^{it}) \neq 0$ (theorem 3.2), which provides a tangent to Γ at almost every point. (with respect to arc-length).

4. SOME CLASSICAL INEQUALITIES

THEOREM 4.1. (Hardy's inequality) Let $F(z) = \sum_{j=0}^{\infty} a_j z^j$ *be in* H^1. *Then:*

$$\sum_{j=0}^{\infty} \frac{|a_j|}{j+1} \leq C \, \|F\|_{H^1}$$

with a constant C *independent of* F.

Proof: We shall see that one can find $G(z) = \sum_{j=0}^{\infty} A_j z^j$ such that $\|G\|_{H^1} \leq \|F\|_{H^1}$ and $|a_j| \leq A_j$ for every $j = 0, 1, \ldots$.

Once this is done, if the inequality holds for G, which has non-negative coefficients, it will also hold for F with the same constant. Thus, we shall be reduced to proving Hardy's inequality with the additional hypothesis that $0 \le a_j$ for every j. Let us see how to obtain G. We apply corollary 3.4 and write $F = F_1 \cdot F_2$ with $\|F_1\|_{H^2} = \|F_2\|_{H^2} = \|F\|_{H^1}^{1/2}$. Let

$$F_1(z) = \sum_{j=0}^{\infty} b_j z^j \quad , \quad F_2(z) = \sum_{j=0}^{\infty} c_j z^j$$

Then we consider

$$G_1(z) = \sum_{j=0}^{\infty} |b_j| z^j \quad , \quad G_2(z) = \sum_{j=0}^{\infty} |c_j| z^j$$

and the Plancherel theorem characterization of H^2 tells us that $\|G_1\|_{H^2} = \|F_1\|_{H^2}$ and $\|G_2\|_{H^2} = \|F_2\|_{H^2}$. Now, call $G = G_1 \cdot G_2$. Then $\|G\|_{H^1} \le \|G_1\|_{H^2} \cdot \|G_2\|_{H^2} = \|F\|_{H^1}$ and $G(z) = \sum_{j=0}^{\infty} A_j z^j$ with

$$A_j = \sum_{k=0}^{j} |b_k||c_{j-k}| \quad , \quad a_j = \sum_{k=0}^{j} b_k c_{j-k}$$

so that, clearly, $|a_j| \le A_j$. Thus, the function G satisfies our requirements and the claim that we can assume $0 \le a_j$ is justified.

Let us suppose, therefore, that $0 \le a_j$ for $j = 0, 1, \ldots$ Let $u(z) = \operatorname{Im} \log(1-z)$ for $|z| < 1$, where we have chosen the principal branch of the logarithm. The function u is harmonic in D and satisfies $-\pi/2 < u(z) < \pi/2$. From the Taylor expansion of $\log(1-z)$ we get

$$u(re^{it}) = (i/2) \sum_{j \neq 0} j^{-1} r^{|j|} e^{ijt}$$

and can write, for every $r < 1$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F(re^{it}) u(re^{it})\, dt = -\frac{i}{2} \sum_{j=1}^{\infty} a_j j^{-1} r^j$$

and, since $a_j \ge 0$

$$\sum_{j=1}^{\infty} a_j j^{-1} r^j = 2 \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} F(re^{it}) u(re^{it})\, dt \right| \le \pi \|F\|_{H^1}$$

Making $r \to 1$ yields

$$\sum_{j=1}^{\infty} j^{-1} a_j \le \pi \|F\|_{H^1}.$$

Taking into account that $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})\, dt$, we finally arrive at:

$$\sum_{j=0}^{\infty} \frac{a_j}{j+1} \le (1+\pi) \|F\|_{H^1}.$$

This proves Hardy's inequality with $C = 1 + \pi$. $\square$

<u>*COROLLARY 4.2. Let*</u> $F(z) = \sum_{j=0}^{\infty} a_j z^j$ <u>*be a conformal equivalence from*</u> D <u>*onto the interior domain limited by a rectifiable Jordan curve*</u> Γ. <u>*Then*</u> $\sum_{j=0}^{\infty} |a_j| < \infty$.

<u>Proof</u>: From the fact that Γ is rectifiable , it follows that

$F'(z) = \sum_{j=0}^{\infty} (j+1) a_{j+1} z^j$ is in H^1 (corollary 3.13).

Then, Hardy's inequality applied to F' gives $\sum_{j=0}^{\infty} |a_{j+1}| < \infty$. $\square$

The corollary we have just proved is an strengthening, for Γ rectifiable of the fact that F extends to the boundary of D with continuity. In general, for $F(z) = \sum_{j=0}^{\infty} a_j z^j$ in $H(D)$, the following implications hold:

$$F' \in H^1 \Longrightarrow \sum_{j=0}^{\infty} |a_j| < \infty.$$

$\sum_{j=0}^{\infty} |a_j| < \infty \Longrightarrow F$ extends to a continuous function on $\overline{D}$.

However, examples can be given to show that the opposite implications do not hold.

There is another way to state Hardy's inequality. We have seen that the mapping

$$H^1 \longrightarrow L^1(T)$$

$$F(z) \longmapsto f(t) = F(e^{it})$$

is a linear isometry. By means of this isometry we may identify H^1 with a certain subspace of L^1 which we may also call H^1. The ambiguity disappears with the habit we have developed, of using capital letters for holomorphic functions on D and small type letters for functions defined on T or, what is the same, on $[-\pi,\pi]$.

If $F(z) = \sum_{j\geq 0} a_j z^j$ is in H^1, the Fourier coefficients of $f(t) = F(e^{it})$ are $\hat{f}(j) = a_j$ if $j = 0, 1, 2, \ldots$ and $\hat{f}(j) = 0$ if $j < 0$. Thus, we can write Hardy's inequality as:

$$\sum_{j=0}^{\infty} \frac{|\hat{f}(j)|}{j+1} \leq C \|f\|_1$$

We can also formulate Hardy's inequality in terms of the space $\mathrm{Re}\, H^1$ formed by those functions on the boundary which are the real parts of the functions in H^1. After all, any $f \in H^1$ is determined by its real part except for an additive purely imaginary constant. If we assign to every $g \in \mathrm{Re}\, H^1$ the function $F(z)$ in H^1 such that $\mathrm{Re}\, F(e^{it}) = g(t)$ and $F(0)$ is real, we have a one to one correspondence and we can define $\|g\|_{\mathrm{Re}\, H^1} = \|F\|_{H^1}$. If $F(z) = \sum_{j\geq 0} a_j z^j$, then, $\hat{g}(j) = -\frac{i}{2} a_j$ for $j > 0$, $\hat{g}(j) = \frac{i}{2} a_{-j}$ for $j < 0$ and $\hat{g}(0) = \mathrm{Im}\, a_0$. Therefore Hardy's inequality can be viewed as an inequality for the Fourier coefficients of any $f \in \mathrm{Re}\, H^1$, namely:

$$(4.3) \qquad \sum_{-\infty}^{\infty}{}' \left|\frac{\hat{f}(j)}{j}\right| \leq C \|f\|_{\mathrm{Re}\, H^1}$$

where the prime is used to indicate that the sum is taken over those $j \neq 0$.

The function

$$f(t) = \sum_{j=2}^{\infty} \frac{\cos(jt)}{\log j}$$

is in $\mathrm{Re}\, L^1$ (see Zygmund [1] Ch.V, 1.) and for it, the sum in 4.3 is ∞. This shows that $\mathrm{Re}\, H^1$ is a proper subspace of $\mathrm{Re}\, L^1$. We shall see later that, for $1 < p < \infty$, $\mathrm{Re}\, H^p = \mathrm{Re}\, L^p$. After this, Hardy's inequality may be considered an extension to $p = 1$ of Paley's inequality which says that for $f \in L^p$ with $1 < p \leq 2$.

$$\sum_{-\infty}^{\infty} |\hat{f}(j)|^p \, |j|^{p-2} \leq C_p \|f\|_p^p$$

Of course, for $p = 2$, this is just a consequence of Plancherel theorem. In the next two chapters we shall see how Paley's inequality can be obtained by interpolation between the case $p = 2$ and either Hardy's inequality or, more simply, the estimate $|\hat{f}(j)| \leq \|f\|_1$. We shall also see, in the context of $\mathbb{R}^n$, how to extend this inequality to H^p for $p < 1$. It will then become clear that the Hardy spaces H^p, for $p \leq 1$, are natural substitutes of the Lebesgue spaces L^p which are only "reasonable" for $p > 1$.

The name of Paley is also associated to the following result:

THEOREM 4.4 Let $F(z) = \sum_{j=0}^{\infty} a_j z^j$ *be in* H^1. *Then:*

$$\sum_{j=0}^{\infty} |a_{2^j}|^2 \leq C \|F\|_{H^1}^2 .$$

Proof: Write $F(z) = F_1(z) \cdot F_2(z)$ with $\|F_1\|_{H^2} = \|F_2\|_{H^2} = \|F\|_{H^1}^{1/2}$.

Let $F_1(z) = \sum_{j\geq 0} b_j z^j$ and $F_2(z) = \sum_{j\geq 0} c_j z^j$. Then

$$a_j = \sum_{k=0}^{j} b_k c_{j-k} \quad , \text{ so that:}$$

$$|a_{2^j}| \leq \sum_{k=0}^{2^{j-1}} |b_k||c_{2^j-k}| + \sum_{k=2^{j-1}+1}^{2^j} |b_k||c_{2^j-k}| \leq$$

$$\leq \Big(\sum_{k=2^{j-1}}^{2^j} |c_k|^2\Big)^{1/2} \cdot \|F_1\|_{H^2} + \Big(\sum_{k=2^{j-1}}^{2^j} |b_k|^2\Big)^{1/2} \cdot \|F_2\|_{H^2} =$$

$$= \|F\|_{H^1}^{1/2} \Big\{\Big(\sum_{2^{j-1}}^{2^j} |c_k|^2\Big)^{1/2} + \Big(\sum_{2^{j-1}}^{2^j} |b_k|^2\Big)^{1/2}\Big\}$$

It follows that:

$$|a_{2^j}|^2 \leq 2 \|F\|_{H^1} \Big(\sum_{2^{j-1}}^{2^j} |c_k|^2 + \sum_{2^{j-1}}^{2^j} |b_k|^2\Big)$$

and, summing in j

$$\sum_{j=0}^{\infty} |a_{2^j}|^2 \leq 2 \|F\|_{H^1} (\|F_1\|_{H^2}^2 + \|F_2\|_{H^2}^2) = 4 \|F\|_{H^1}^2 ,$$

as we wanted to show. □

Finally we shall present the so called Féjer-Riesz inequality:

THEOREM 4.5. Let $F \in H^p$, $0 < p < \infty$. *Then*

$$\int_{-1}^{1} |F(x)|^p \, dx \le \frac{1}{2} \int_{-\pi}^{\pi} |F(e^{it})|^p \, dt$$

Proof: Suppose first that $p = 2$. We take $0 < r < 1$ and apply Cauchy's theorem to the function $F(z)\,\overline{F(\bar{z})}$, which is holomorphic in D, and the path formed by the segment $[-r, r]$ followed by the semicircle $\{re^{i\theta}: 0 \le \theta \le \pi\}$. We get:

$$\int_{-r}^{r} |F(x)|^2 \, dx + ir\int_{0}^{\pi} F(re^{it})\overline{F(re^{-it})}e^{it}dt = 0$$

and, from that

$$\int_{-r}^{r} |F(x)|^2 \, dx \le \int_{0}^{\pi} |F(re^{it})| \; |F(re^{-it})| \, dt \le$$

$$\le \left(\int_{0}^{\pi} |F(re^{it})|^2 \, dt\right)^{1/2} \left(\int_{0}^{\pi} |F(re^{-it})|^2 \, dt\right)^{1/2} =$$

$$= \left(\int_{0}^{\pi} |F(re^{it})|^2 \, dt\right)^{1/2} \left(\int_{-\pi}^{0} |F(re^{it})|^2 \, dt\right)^{1/2} \le$$

$$\le \frac{1}{2} \left(\int_{0}^{\pi} |F(re^{it})|^2 \, dt + \int_{-\pi}^{0} |F(re^{it})|^2 \, dt\right) =$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} |F(re^{it})|^2 \, dt \le \frac{1}{2} \int_{-\pi}^{\pi} |F(e^{it})|^2 \, dt.$$

Letting $r \to 1$, we get the desired inequality for $p = 2$. If $p \neq 2$, we write $F(z) = B(z)\, H(z)$ where B is the Blaschke product formed with the zeroes of F, and H is a non-vanishing holomorphic function in D. We know that $\|H\|_{H^p} = \|F\|_{H^p}$. It will be $H(z) =$ $= \exp A(z)$ for some holomorphic function A. Writing $G(z) =$ $= \exp((p/2)\, A(z))$ we have $|G(z)|^2 = |H(z)|^p$, so that

$$\|G\|_{H^2}^2 = \|H\|_{H^p}^p = \|F\|_{H^p}^p \qquad \text{and}$$

$$\int_{-1}^{1} |F(x)|^p \, dx \le \int_{-1}^{1} |H(x)|^p \, dx = \int_{-1}^{1} |G(x)|^2 \, dx \le$$

$$\le \frac{1}{2} \int_{-\pi}^{\pi} |G(e^{it})|^2 \, dt = \frac{1}{2} \|G\|_{H^2}^2 = \frac{1}{2} \|F\|_{H^p}^p = \frac{1}{2} \int_{-\pi}^{\pi} |F(e^{it})|^p \, dt. \; \square$$

COROLLARY 4.6. Let F *be a conformal equivalence from* D *onto the interior domain bounded by a rectifiable Jordan curve* Γ. *Then, the image of each diameter is a rectifiable curve whose length is at* *most half the length of* Γ.

Proof: After a rotation, apply Fejer-Riesz inequality to F', which is an H^1 function. □

Observe that 1/2 is the best possible constant in 4.6. Indeed, let F map D conformally onto the rectangle $\{x + iy: |x| < 1, |y| < \varepsilon\}$. We may assume that $F(t) = t$ for $-1 < t < 1$. Since the perimeter of the rectangle is $4 + 4\varepsilon$, the constant in 4.6. has to be at least $2/(4 + 4\varepsilon)$. Letting $\varepsilon \to 0$, we conclude that the constant is $\geq 1/2$.

Next, we shall use the integrability along radii of H^1 functions to continue the study of the boundary behaviour of conformal mappings.

COROLLARY 4.7. *Let* F *be as in corollary 4.6. and let* e^{it_0} *be such that the non-tangential boundary value* $F'(e^{it_0})$ *exists. Then*

$$(4.8) \qquad \frac{F(z) - F(e^{it_0})}{z - e^{it_0}} \to F'(e^{it_0}) \quad \textit{as} \quad z \xrightarrow{\text{N.T.}} e^{it_0}$$

It follows that F *is also conformal at almost every boundary point.*

Proof: $F(e^{it_0}) - F(z) = \int_z^{e^{it_0}} F'(\zeta)\, d\zeta$

where the integral is taken over the segment joining z and e^{it_0}. Thus, if we remain in the region $|\arg(z) - t_0| < C(1 - |z|)$, then

$$\frac{F(e^{it_0}) - F(z)}{e^{it_0} - z} - F'(e^{it_0}) = \frac{1}{e^{it_0} - z} \int_z^{e^{it_0}} (F'(\zeta) - F'(e^{it_0}))\, d\zeta \to 0$$

as $z \xrightarrow{\text{N.T.}} e^{it_0}$

Now, suppose that e^{it_0} is such that

$$\frac{d}{dt}(F(e^{it}))\Big|_{t=t_0} = ie^{it_0} F'(e^{it_0}) \neq 0$$

Then, we can speak about the tangent to Γ at the point $F(e^{it_0})$. We know that this happens for a.e. boundary point e^{it_0}. Let γ be a curve in D which ends at e^{it_0} and meets the boundary non-tangentially, that is: $\arg(z-e^{it_0})$ has a limit as $z \to e^{it_0}$ along γ and this limit is not $t_0 \pm (\pi/2)$. Consider the curve $F(\gamma)$. We are going to see that $F(\gamma)$ and Γ meet at $F(e^{it_0})$ with the same angle as the one with which γ and ∂D meet at e^{it_0}. We have to see that

$$\lim_{\substack{z\to e^{it_0}\\ \gamma}} \arg(z-e^{it_0}) - t_0 - \frac{\pi}{2} = \lim_{\substack{z\to e^{it_0}\\ \gamma}} \arg(F(z) - F(e^{it_0})) - $$
$$- \arg\left(\frac{d}{dt} (F(e^{it}))\Big|_{t=t_0} \right)$$

or, what is the same, that:

$$\lim_{\substack{z\to e^{it_0}\\ \gamma}} \arg(z-e^{it_0}) = \lim_{\substack{z\to e^{it_0}\\ \gamma}} \arg(F(z) - F(e^{it_0})) - \arg F'(e^{it_0})$$

But this is clearly a consequence of 4.8.

5. THE CONJUGATE FUNCTION

DEFINITION 5.1. *Let* $f(t)$ *be a* 2π*-periodic function, integrable on* $[-\pi,\pi]$. *Let* $u(re^{it})$ *be its Poisson integral and let* $v(re^{it})$ *be the harmonic conjugate of* u *uniquely determined by the condition* $v(0) = 0$. *Then we define the conjugate function of* f *to be the function:*

$$\tilde{f}(t) = \lim_{r\to 1} v(re^{it})$$

For this definition to make sense, we need to show that the limit exists for a.e.t. Of course, by writing $f = f^+ - f^-$, we may restrict our attention to $f \geqq 0$, which implies also $u \geqq 0$. Then the existence of the limit follows from

THEOREM 5.2. *Let* $F \in H(D)$ *be such that* $\operatorname{Re} F(z) \geqq 0$ *for every* $z \in D$. *Then* F *has non-tangential limits at almost every boundary point.*

Proof: Let $G(z) = 1/(1+F(z))$. Then $G \in H^\infty(D)$ since $|1+F(z)| \geqq \operatorname{Re}(1+F(z)) \geqq 1$, so that $|G(z)| \leqq 1$. It follows from

Fatou's theorem that $G(e^{it}) = \lim G(z)$ as $z \xrightarrow{N.T.} e^{it}$, exists and is different from 0 for a.e.t. But $F(z) = G(z)^{-1} - 1$. Hence $F(e^{it}) = \lim F(z)$ as $z \xrightarrow{N.T.} e^{it}$ exists a.e. and equals $G(e^{it})^{-1} - 1$. $\square$

We have established the existence of $\tilde{f}$. We even know that $\tilde{f}(t) = \lim v(z)$ as $z \xrightarrow{N.T.} e^{it}$. Next, we shall investigate the size of $\tilde{f}$. First of all, we shall prove the theorem of Marcel Riesz saying that the conjugate function operator $f \longrightarrow \tilde{f}$ is bounded in L^p for $1 < p < \infty$. Here is the key to this boundedness:

THEOREM 5.3. For every p *with* $1 < p \leq 2$, *there is a constant* C_p, *such that for every* $F(z) = u(z) + iv(z)$, *holomorphic in* D, *with* $u(z) > 0$ *on* D, v *real valued and* $v(0) = 0$, *the inequality*

$$\int_{-\pi}^{\pi} |v(re^{it})|^p \, dt \leq C_p \int_{-\pi}^{\pi} |u(re^{it})|^p \, dt$$

holds for every $0 < r < 1$.

Proof: Write $F(z) = |F(z)| \, e^{i\psi(z)}$, where $\psi(z) = \arg F(z)$ is determined in such a way that $|\psi(z)| < \pi/2$. Then $u(z) = |F(z)| \cos\psi(z)$ and $v(z) = |F(z)| \sin\psi(z)$, so that: $|u(z)|^p = |F(z)|^p \, |\cos\psi(z)|^p$ and $|v(z)|^p = |F(z)|^p \, |\sin\psi(z)|^p$.

We are going to prove that we can choose positive constants C_p and D_p for which the inequality

$$(5.4) \qquad |\sin\theta|^p \leq C_p \, |\cos\theta|^p - D_p \cos(p\theta)$$

holds for $|\theta| < \pi/2$.

Let $\delta > 0$ be small enough to have $\pi/2 < p((\pi/2) - \delta)$. Then if $(\pi/2) - \delta < |\theta| < \pi/2$, we have: $\pi/2 < p((\pi/2) - \delta) < |p\theta| < \pi$, so that $\cos(p\theta) \leq \cos(p((\pi/2) - \delta)) < 0$. Thus $0 < -\cos(p((\pi/2) - \delta)) \leq -\cos(p\theta)$. If we take D_p big enough, we certainly have (5.4) for $(\pi/2) - \delta < |\theta| < \pi/2$. But if $|\theta| \leq (\pi/2) - \delta$, we have $0 < \sin\delta = \cos((\pi/2) - \delta) \leq \cos\theta$. We just need to choose C_p big enough to have $1 \leq C_p(\sin\delta)^p - D_p$. Once this is done, we are sure that (5.4) holds also for $|\theta| \leq (\pi/2) - \delta$.

Now we use (5.4) to prove the theorem:

$$\int_{-\pi}^{\pi} |v(re^{it})|^p\,dt = \int_{-\pi}^{\pi} |F(re^{it})|^p\,|\sin\psi(re^{it})|^p\,dt \leq$$

$$\leq C_p \int_{-\pi}^{\pi} |F(re^{it})|^p |\cos\psi(re^{it})|^p dt - D_p \int_{-\pi}^{\pi} |F(re^{it})|^p \cos(p\psi(re^{it}))dt =$$

$$= C_p \int_{-\pi}^{\pi} |u(re^{it})|^p\,dt - D_p \int_{-\pi}^{\pi} \operatorname{Re}(F(re^{it})^p) =$$

$$= C_p \int_{-\pi}^{\pi} |u(re^{it})|^p\,dt - 2\pi\, D_p \operatorname{Re}(F(0)^p) \leq C_p \int_{-\pi}^{\pi} |u(re^{it})|^p\,dt.$$

since $F(0) = u(0) > 0$. □

We are ready to present the Marcel Riesz inequality

COROLLARY 5.5. For every $1 < p < \infty$, *there exists a constant* B_p *such that, for each* $f \in L^p(T)$:

$$\int_{-\pi}^{\pi} |\tilde{f}(t)|^p\,dt \leq B_p \int_{-\pi}^{\pi} |f(t)|^p\,dt$$

Proof: First, let $1 < p \leq 2$. The linearity allows us to assume $f \geq 0$, not identically 0. Let u be the Poisson integral of f and v the harmonic conjugate of u determined by the condition $v(0) = 0$. Then, we can apply theorem 5.3 to the function $F(z) = u(z) + iv(z)$. Since $\tilde{f}(t) = \lim_{r\to 1} v(re^{it})$, Fatou's lemma yields:

$$\int_{-\pi}^{\pi} |\tilde{f}(t)|^p\,dt \leq \liminf_{r\to 1} \int_{-\pi}^{\pi} |v(re^{it})|^p\,dt \leq$$

$$\leq C_p \liminf_{r\to 1} \int_{-\pi}^{\pi} |u(re^{it})|^p\,dt \leq C_p \int_{-\pi}^{\pi} |f(t)|^p\,dt.$$

This proves M. Riesz's inequality for $1 < p \leq 2$.

Suppose now $2 < p < \infty$ and let p' be the conjugate exponent, so that $1 < p' < 2$. For $f \in L^p(T)$, we denote by u the Poisson integral of f and by v the harmonic conjugate to u such that $v(0) = 0$. We can write, for every $r < 1$:

$$\left(\int_{-\pi}^{\pi} |v(re^{it})|^p\,dt\right)^{1/p} = \sup\left\{\left|\int_{-\pi}^{\pi} v(re^{it})\,g(t)\,dt\right| : \ g \text{ with}\right.$$

$$\int_{-\pi}^{\pi} |g(t)|^{p'}\, dt \le 1\Big\} .$$

But

$$(5.6) \qquad \int_{-\pi}^{\pi} v(re^{it})\, g(t)\, dt = - \int_{-\pi}^{\pi} u(re^{it})\, \tilde{g}(t)\, dt$$

Indeed, let $h(z)$ be the Poisson integral of g and $w(z)$ the harmonic conjugate to h with $w(0) = 0$. Then, for $0 < s < 1$:

$$(5.7) \qquad \int_{-\pi}^{\pi} (v(rse^{it})h(se^{it})+u(rse^{it})w(se^{it}))\, dt = 0, \text{ since}$$

$v(rz)h(z)+u(rz)w(z) = \mathrm{Im}((u(rz)+iv(rz))\cdot(h(z)+iw(z))$ is a harmonic function vanishing ar $z = 0$. Now $1 < p' < 2$ and $h = P(g)$ with $g \in L^{p'}(T)$. We can apply theorem 5.3, with p' instead of p, to the holomorphic function $h + iw$ and conclude that it belongs to $H^{p'}$. It follows that $w = P(\tilde{g})$. Therefore $h(se^{it}) \to g(t)$ in $L^{p'}$ and $w(se^{it}) \to \tilde{g}(t)$ in $L^{p'}$ as $s \to 1$, so that the integrand in (5.7) converges to $v(re^{it})\, g(t) + u(re^{it})\, \tilde{g}(t)$ in L^1 as $s \to 1$ and (5.6) follows. Now, using (5.6) and observing that:

$$\left|\int_{-\pi}^{\pi} u(re^{it})\tilde{g}(t)\, dt\right| \le \left(\int_{-\pi}^{\pi} |f(t)|^{p}\, dt\right)^{1/p}\left(\int_{-\pi}^{\pi} |\tilde{g}(t)|^{p'}\, dt\right)^{1/p'} \le$$

$\le B_{p'}^{1/p'}\left(\int_{-\pi}^{\pi} |f(t)|^{p}\, dt\right)^{1/p}$, we conclude that:

$$\left(\int_{-\pi}^{\pi} |v(re^{it})|^{p}\, dt\right)^{1/p} \le B_{p'}^{1/p'} \left(\int_{-\pi}^{\pi} |f(t)|^{p}\, dt\right)^{1/p}$$

From this we get, by using Fatou's lemma:

$$\int_{-\pi}^{\pi} |\tilde{f}(t)|^{p}\, dt \le B_{p'}^{p-1} \int_{-\pi}^{\pi} |f(t)|^{p}\, dt. \square$$

<u>*COROLLARY 5.8. If*</u> $f \in H(D)$, <u>*then, for every*</u> $1 < p < \infty$ <u>*and every*</u> $0 \le r < 1$:

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\mathrm{Im}\, F(re^{it})|^{p}\, dt\right)^{1/p} \le B_{p}^{1/p}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathrm{Re} F(re^{it})|^{p}\, dt\right)^{1/p} +$$

$$+ |\mathrm{Im}\, F(0)| .$$

<u>Proof</u>: Let $u = \mathrm{Re}\, F$ and let v be the harmonic conjugate of u determined by the condition $v(0) = 0$. Then $\mathrm{Im}\, F(z) = v(z) + \mathrm{Im} F(0)$.

Now one just needs to realize that $v(re^{it})$ is the conjugate function of $u(re^{it})$ and apply corollary 5.5. $\square$

We can easily see that corollary 5.5. does nos extend to $p = 1$. For example, if $f(t) = P_r(t) = (1-r^2)/(1-r^2+2r\cos t)$, then

$$\tilde{f}(t) = \frac{2r \sin t}{1+r^2-2r\cos t}$$

the so called conjugate Poisson kernel, which we shall denote by $Q_r(t)$. Indeed,

$$P_r(t)+i\,Q_r(t) = \frac{1-r^2+i\,2r \operatorname{sen} t}{|1-re^{it}|^2} = \frac{1+z}{1-z} \qquad (z = re^{it})$$

is holomorphic and $\int_{-\pi}^{\pi} Q_r(t)\,dt = 0$. Now

$$\int_{-\pi}^{\pi} |Q_r(t)|\,dt = 4\log\frac{1+r}{1-r} \to \infty \qquad \text{as} \qquad r \to 1$$

whereas $\int_{-\pi}^{\pi} |P_r(t)|\,dt = 2\pi$ independently of r.

By duality, that is, by applying 5.6, we see that the conjugate function operator is not bounded in L^∞.

However, for $p = 1$, we have the following substitute result due to Kolmogorov:

THEOREM 5.9. The conjugate function operator is of weak type (1.1), *which means that there exists a constant* C, *such that for every* $f \in L^1([-\pi,\pi])$ *and every* $\lambda > 0$, *the set* $E_\lambda = \{t \in [-\pi,\pi]: |\tilde{f}(t)| > \lambda\}$ *has Lebesgue measure* $|E_\lambda|$ *satisfying*

$$|E_\lambda| \leq C\lambda^{-1}\int_{-\pi}^{\pi} |f(t)|\,dt.$$

Proof: Because of the linearity of the conjugate function operator, we may assume that $f \geq 0$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = 1$.

Fix $\lambda > 0$. The function $z \longmapsto \frac{z-i\lambda}{z+i\lambda}$ clearly maps the half-plane $\operatorname{Re} z > 0$ into the half-plane $\operatorname{Im} z > 0$. In this half-plane $\operatorname{Im} z > 0$ we choose the determination of the argument such that $-\pi < \arg z <$ < 0. With this choice, we define for $\operatorname{Re} z > 0$:

$$h_\lambda(z) = 1 + \frac{1}{\pi} \arg \frac{z - i\lambda}{z + i\lambda}$$

Then, h_λ is a harmonic function such that $0 < h_\lambda(z) < 1$.

The level lines $h_\lambda(z) = k$ are, obviously, arcs of circles passing through $i\lambda$ and $-i\lambda$. In particular, the level line $h_\lambda(z) = 1/2$ is the semi-circle $\{\lambda e^{i\theta}: -\pi/2 < \theta < \pi/2\}$ and we see that $|z| \geq \lambda$ implies $h_\lambda(z) \geq 1/2$. It will also be convenient to observe that:

$$h_\lambda(1) = 1 + \frac{1}{\pi} \arg \frac{1 - i\lambda}{1 + i\lambda} = 1 - \frac{1}{\pi} \operatorname{arc\,tg} \frac{2\lambda}{1 - \lambda^2} = 1 - \frac{2}{\pi} \operatorname{arc\,tg} \lambda =$$

$$= \frac{2}{\pi} \left(\frac{\pi}{2} - \operatorname{arc\,tg} \lambda\right) \leq \frac{2}{\pi} \cdot \frac{1}{\lambda}$$

Let $u = P(f)$, and let v be the harmonic conjugate of u with $v(0) = 0$. Then $u(z) > 0$ for every $z \in D$ and we may consider $h_\lambda(u(z) + iv(z))$, which is a positive harmonic function in D, and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} h_\lambda(u(re^{it}) + iv(re^{it}))\, dt = h_\lambda(u(0)) = h_\lambda(1) \leq \frac{2}{\pi} \cdot \frac{1}{\lambda}$$

since $u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt = 1$.

On the otrher hand, if $|v(re^{it})| > \lambda$, then also $|u(re^{it}) + iv(re^{it})| > \lambda$ and, consequently, $h_\lambda(u(re^{it}) + iv(re^{it})) > 1/2$. Thus

$$\frac{1}{2\pi} \cdot \frac{1}{2} \cdot |\{t \in [-\pi,\pi] : |v(re^{it})| > \lambda\}| \leq$$

$$\leq \frac{1}{2\pi} \int_{\{t:\ |v(re^{it})| > \lambda\}} h_\lambda(u(re^{it})+iv(re^{it}))\, dt \leq \frac{2}{\pi} \cdot \frac{1}{\lambda},$$

so that

$$|\{t \in [-\pi,\pi] : |v(re^{it})| > \lambda\}| \leq 8/\lambda \quad \text{for all} \quad 0 < r < 1.$$

Take $r_j \uparrow 1$. Since $\tilde{f}(t) = \lim_{r\to 1} v(re^{it})$:

$$|E_\lambda| = \left| \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} \{t \in [-\pi, \pi]: |v(r_j e^{it})| > \lambda\} \right| =$$

$$= \lim_{n\to\infty} \left| \bigcap_{j=n}^{\infty} \{t \in [-\pi,\pi] : |v(r_j e^{it})| > \lambda\} \right| \leq 8/\lambda .$$

In this way, we see that Kolmogorov's inequality is valid for $f \geq 0$ with $C = 4/\pi$. To pass to an arbitrary f is straightforward. Clearly $C = \frac{64}{\pi}$ will suffice. □

As a consequence, we can show that the conjugate function operator is bounded from L^1 to L^p for $0 < p < 1$.

COROLLARY 5.10. For $f \in L^1([-\pi,\pi])$ *and* $0 < p < 1$:

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\tilde{f}(t)|^p \, dt\right)^{1/p} \leq C(1-p)^{-1/p} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)| \, dt$$

where C *is the same constant appearing in theorem* 5.9.

Proof: We write $\int_{-\pi}^{\pi} |\tilde{f}(t)|^p \, dt$ in terms of the distribution function of $\tilde{f}$, which is the function

$$\lambda \longmapsto |E_\lambda| = |\{t \in [-\pi,\pi] : |\tilde{f}(t)| > \lambda\}|$$

defined for $\lambda > 0$. Fubini's theorem implies that

$$\int_{-\pi}^{\pi} |\tilde{f}(t)|^p \, dt = p\int_0^\infty \lambda^{p-1} |E_\lambda| \, d\lambda$$

We use the estimate obtained in theorem 5.9 when

$$\lambda > \lambda_0 = \frac{C}{2\pi}\int_{-\pi}^{\pi} |f(t)| \, dt$$

For $\lambda < \lambda_0$ we use the better estimate $|E_\lambda| \leq 2\pi$. We get:

$$\int_{-\pi}^{\pi} |\tilde{f}(t)|^p dt \leq 2\pi p \int_0^{\lambda_0} \lambda^{p-1} d\lambda + pC\left(\int_{-\pi}^{\pi} |f(t)| dt\right)\left(\int_{\lambda_0}^{\infty} \lambda^{p-2} d\lambda\right) =$$

$$= 2\pi\lambda_0^p + p\cdot 2\pi\lambda_0\cdot(1-p)^{-1}\lambda_0^{p-1} = 2\pi(1-p)^{-1}\lambda_0^p =$$

$$= 2\pi(1-p)^{-1}\left(\frac{C}{2\pi}\int_{-\pi}^{\pi} |f(t)| \, dt\right)^p$$

and this gives us the desired inequality. □

COROLLARY 5.11. Sppose $f \in L^1([-\pi,\pi])$ *is such that* $\tilde{f} \in L^1([-\pi,\pi])$. *Let* $u = P(f)$ *and let* v *be the conjugate harmonic function of* u *with* $v(0) = 0$. *Then* $v = P(\tilde{f})$.

Proof: Applying corollary 5.10 to the functions $u(re^{it})$ and its conjugate $v(re^{it})$, we see that $u + iv \in H^p$ for all $0 < p < 1$. Besides, its boundary function $f(t) + i\tilde{f}(t)$ belongs to L^1. It follows from corollary 3.7. that $u + iv \in H^1$ and, consequently $v = P(\tilde{f})$. $\square$

We can identify H^p with a subspace of $L^p(T)$ by means of the isometry sending the H^p function $F(z)$ to its boundary function $F(e^{it})$. There is no harm in denoting this space of boundary functions also by H^p. We shall also consider $\mathrm{Re}H^p$, the space formed by those functions in the boundary which are real parts of H^p functions.

We can give a description of H^p for $1 \leq p \leq \infty$ in terms of the conjugate function. We shall look at the H^p of the boundary, but note that, in this case, the Poisson integral sends us back to $H^p(D)$. Suppose $F(e^{it})$ is in H^p, $1 \leq p \leq \infty$. Let $u(z) = \mathrm{Re}\, F(z)$. Then $\mathrm{Im}\, F(z) = v(z) + c$, where v is the harmonic conjugate of u having $v(0) = 0$ and $c = \mathrm{Im}\, F(0)$. If we call $f(t) = u(e^{it})$, we have: $F(e^{it}) = f(t) + i\tilde{f}(t) + ic$. Thus, when $1 \leq p \leq \infty$, every function in H^p (of the boundary) is of the form $f + i\tilde{f} + ic$ for some $f \in \mathrm{Re}\, L^p$ such that $\tilde{f} \in L^p$, and some $c \in \mathbb{R}$. Conversely, let $f \in \mathrm{Re}\, L^p$, $1 \leq p \leq \infty$, be such that $\tilde{f} \in L^p$ (when $1 < p < \infty$, the M. Riesz inequality guarantees that $\tilde{f} \in L^p$ as soon as $f \in L^p$, but this is no longer true for $p = 1$ or $p = \infty$). Let $u(z) = P(f)(z)$ and let v be the conjugate harmonic function of u with $v(0) = 0$. We know from corollary 5.11. that $u + iv = P(f+i\tilde{f})$. Therefore $u + iv \in H^p(D)$ and consequently, its boundary function $f + i\tilde{f}$ is in H^p. Thus, for $1 < p < \infty$, we can write:

$$H^p = \{f + i\tilde{f} + ic: f \in \mathrm{Re}\, L^p,\ c \in \mathbb{R}\} \quad \text{and} \quad \mathrm{Re}\, H^p = \mathrm{Re}\, L^p.$$

However

$$H^1 = \{f + i\tilde{f} + ic: f \in \mathrm{Re}\, L^1,\ \tilde{f} \in L^1,\ c \in \mathbb{R}\}$$

and

$$\mathrm{Re}\, H^1 = \{f \in \mathrm{Re}\, L^1: \tilde{f} \in L^1\} \subsetneq \mathrm{Re}\, L^1.$$

That the inclusion is proper follows either from the remarks we made after Hardy's inequality or from the fact that the conjugate function operator is not bounded in L^1. The space $\operatorname{Re} H^1$ with the norm $\|f\|_{\operatorname{Re} H^1} = \|f\|_1 + \|\tilde{f}\|_1$, which is an equivalent copy of the subspace $\{F \in H^1 : F(0) \in \mathbb{R}\}$ of H^1, is a space different from $\operatorname{Re} L^1$. Similarly

$$H^\infty = \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^\infty,\ \tilde{f} \in L^\infty,\ c \in \mathbb{R}\} \quad \text{and}$$

$$\operatorname{Re} H^\infty \subsetneq \operatorname{Re} L^\infty .$$

To finish our discussion of the conjugate function operator $f \longrightarrow \tilde{f}$, let us see how we can define it without getting inside the disk. We have defined $\tilde{f}(\theta) = \lim\limits_{r\to 1} v(re^{i\theta})$ where:

$$v(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2r \sin t}{1+r^2-2r\cos t} f(\theta-t)\,dt =$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \frac{r \sin t}{1+r^2-2r\cos t} (f(\theta-t)-f(\theta+t))\,dt$$

But $\dfrac{r\sin t}{1+r^2-2r\cos t} \longrightarrow \dfrac{1}{2\,\operatorname{tg}(t/2)}$ as $r \longrightarrow 1$ and

$$\left|\frac{r\sin t}{1+r^2-2r\cos t}\right| \le \frac{1}{\operatorname{tg}|t/2|} \le \frac{2}{|t|}$$

Thus, if

$$(5.12) \qquad \int_0^{\pi} \frac{|f(\theta-t) - f(\theta+t)|}{t}\,dt < \infty$$

the Lebesgue dominated convergence theorem gives

$$(5.13) \qquad \tilde{f}(\theta) = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2\operatorname{tg}(t/2)} (f(\theta-t) - f(\theta+t))\,dt$$

Condition (5.12) holds, for instance, if $f'(\theta)$ exists. Besides, if f' is continuous in an open interval (a,b), the convergence is uniform on every closed subinterval of (a,b). This implies that $v(z)$ extends continuously to the arc $\{e^{it} : a < t < b\}$.

If (5.12) does not hold, the integral in (5.13) is no longer absolutely convergent, it becomes a singular integral. However, (5.13) is still valid for almost every θ, provided that the integral is interpreted in the sense of Cauchy's principal value.

THEOREM 5.14. If $f \in L^1(T)$, *then*

$$\tilde{f}(\theta) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{0<\varepsilon<|t|<\pi} \frac{1}{2\operatorname{tg}(t/2)} f(\theta-t)\, dt$$

for every θ *in the Lebesgue set of* f *and, consequently, for almost every* θ. *This limit is called the principal value of the singular integral and is customarily denoted by the initials* p.v. *preceding the integral.*

Proof: Let θ be a Lebesgue point for f. We shall show that the quantity

$$\sigma_r = v(re^{i\theta}) - \frac{1}{\pi} \int_{1-r<|t|<\pi} \frac{1}{2\operatorname{tg}(t/2)} f(\theta-t)\, dt$$

tends to 0 as r tends to 1. This will prove the theorem. But

$$\sigma_r = \frac{1}{2\pi} \int_{|t|<1-r} \frac{2r \sin t}{1+r^2-2r\cos t} f(\theta-t)\, dt +$$

$$+ \frac{1}{\pi} \int_{1-r<|t|<\pi} \left(\frac{r \sin t}{1+r^2-2r\cos t} - \frac{1}{2\operatorname{tg}(t/2)} \right) f(\theta-t)\, dt$$

and we can always substitute $f(\theta-t) - f(\theta)$ for $f(\theta-t)$ because our kernels are odd.

The first integral is bounded in absolute value by

$$\frac{1}{\pi} \frac{1}{1-r} \int_{|t|<1-r} |f(\theta-t) - f(\theta)|\, dt$$

which tends to 0 as r tends to 1, since θ is a Lebesgue point.

As for the second integral, it is bounded in absolute value by:

$$C(1-r)^2 \int_{1-r<|t|<\pi} \frac{|f(\theta-t) - f(\theta)|}{|t|^3}\, dt$$

for r close to 1, say $r > 1/2$. We just need to study, for example, the integral over the interval $1-r < t < \pi$. Integration by parts gives:

$$(1-r)^2 \int_{1-r}^{\pi} \frac{|f(\theta-t) - f(\theta)|}{t^3} \, dt =$$

$$= \frac{(1-r)^2}{\pi^3} \int_0^{\pi} |f(\theta-t) - f(\theta)| \, dt - \frac{1}{1-r} \int_0^{1-r} |f(\theta-t) - f(\theta)| \, dt +$$

$$+ 3(1-r)^2 \int_{1-r}^{\pi} \left(\int_0^s |f(\theta-t) - f(\theta)| \, dt\right) \frac{ds}{s^4}$$

The first two terms in this sum clearly tend to 0 as r tends to 1. The third term will be split as $I_1 + I_2$ where, for some δ to be chosen later, I_1 corresponds to integrating over the interval $1-r < s < \delta$ and I_2 comes from integration over the remaining portion $\delta < s < \pi$. Given $\varepsilon > 0$, δ can be chosen in such a way that, for $0 < s < \delta$

$$\frac{1}{s} \int_0^s |f(\theta-t) - f(\theta)| \, dt < \varepsilon$$

With this choice of δ,

$$I_1 \leq 3(1-r)^2 \int_{1-r}^{\delta} \varepsilon \frac{ds}{s^3} \leq \frac{3}{2} \varepsilon$$

and

$$I_2 \leq \frac{3\pi(1-r)^2}{\delta^4} \int_0^{\pi} |f(\theta-t) - f(\theta)| \, dt$$

which can also be made $< \varepsilon$ by choosing r close to 1. $\square$

Let us finish this section with a brief reference to the counterpart of the conjugate function in the setting of the upper falf plane. This is the so called Hilbert transform, which we shall denote by H. The definition is a follows. Given $f \in L^p(\mathbb{R})$ for some $1 \leq p < \infty$, we consider the Poisson integral of f, which is the function $u(x,t)$ defined in $\mathbb{R}^2_+$ as

$$u(x,t) = P_t * f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t}{t^2+(x-y)^2} f(y) \, dy$$

The study of Poisson integrals and harmonic functions in half-spaces will be carried out in detail in chapter II, section 4. There, the reader will find complete justification for the claims we are about to make. $u(x,t)$ is a harmonic function and, since $\mathbb{R}^2_+$ is simply connected, u will have a harmonic conjugate $v(x,t)$, which we shall choose in such a way that $v(x,t) \to 0$ as $t \to \infty$ uniformly in x. This is indeed possible, and uniquely determines v. We just have to set

$$v(x,t) = Q_t * f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x-y}{t^2+(x-y)^2} \, f(y)\, dy.$$

Since

$$P_t(x-y)+iQ_t(x-y) = \frac{i}{\pi} \frac{x-y-it}{(x-y)^2+t^2} = \frac{i}{\pi} \frac{1}{z-y}$$

where $z = x + it$, we see clearly that $F(z) = u(z) + iv(z)$ is the analytic function

$$F(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{z-y}\, dy$$

By using Fatou's theorem for $\mathbb{R}^2_+$ (corollary 4.19 in chapter II), we can establish the existence for a.e. $x \in \mathbb{R}$ of

$$Hf(x) = \lim_{t\to 0} v(x,t) = \lim_{t\to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x-y}{t^2+(x-y)^2} \, f(y)\, dy.$$

To do this, we assume $f \geq 0$ and either extend theorem 5.2. to $\mathbb{R}^2_+$ (which requires to make sure that the function $G(z) = 1/(1+F(z))$ cannot have boundary values vanishing on a set of positive measure) or else look at the bounded analytic function $e^{-F(z)}$.

As $t \to 0$, the kernel $\frac{1}{\pi} \frac{y}{t^2+y^2}$ tends to the singular kernel $\frac{1}{\pi y}$, and we can show that

$$Hf(x) = \text{p.v.}\ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x-y)}{y}\, dy = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\, dy.$$

for a.e.x

The key to this is to show that

$$v(x,t) - \frac{1}{\pi} \int_{|y|>t} \frac{1}{y} \, f(x-y)\, dy \to 0 \quad \text{as} \quad t \to 0$$

for a.e.x.

The proof is entirely similar to that of theorem 5.14. Actually, the computations are simpler in this case.

Estimates for a rather general class of singular integrals will be obtained in section 5 of chapter II.

6. H^p AS A LINEAR SPACE

Now, we are going to look at H^p as a topological vector space. We have seen that, for $p \geq 1$, $F \longmapsto \|F\|_{H^p}$ is a norm on H^p while if $0 < p < 1$, this is no longer true and we have, instead, a quasi-norm $F \longmapsto \|F\|^p_{H^p}$ and, consequently, $(F,G) \mapsto \|F-G\|^p_{H^p}$ is an invariant distance compatible with the vector structure. We shall write, for $p \geq 1$, $N_p(F) = \|F\|_{H^p}$, and for $0 < p < 1$, $N_p(F) = \|F\|^p_{H^p}$ and we shall see H^p, $0 < p \leq \infty$, as a metric linear space with the distance $(F,G) \mapsto N_p(F-G)$. We have also shown that the mapping

$$(6.1) \qquad \begin{aligned} H^p(D) &\longrightarrow L^p(T) \\ F(z) &\longmapsto F(e^{it}) \end{aligned}$$

sending each H^p function $F(z)$ into its boundary function $f(t) = F(e^{it})$, is an isometry

THEOREM 6.2. For $0 < p < 1$, H^p is not locally convex.

Proof: If it were locally convex, the ball $B_1 = \{F \in H^p: N_p(F) < 1\}$ would contain some convex neighbourhood of 0, say V, which would, in its turn, contain $B_\varepsilon = \{F \in H^p: N_p(F) < \varepsilon\}$ for some $\varepsilon > 0$. Let us divide $[-\pi,\pi]$ into n intervals of equal length $I_1, \ldots, I_n$ and construct continuous functions $f_1, \ldots, f_n$ such that each f_j is zero outside of I_j, takes a value $a > 0$ (to be specified later) at the center of I_j, and is piecewise linear

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_j(t)|^p\, dt = \frac{1}{p+1}\,\frac{a^p}{n}$$

Thus, if we take $a \equiv a_n$ (it will depend on n), in such a way that $a_n^p = (p+1)n\,\varepsilon/2$; then each of the functions $f_1, \ldots, f_n$ will satisfy $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f_j(t)|^p\, dt < \varepsilon$. Consider now

$g_n(t) = \sum_{j=1}^{n} \lambda_j f_j(t)$ with $\lambda_j \geq 0$ such that $\sum_{j=1}^{n} \lambda_j = 1$. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g_n(t)|^p \, dt = \sum_{j=1}^{n} \frac{1}{2\pi}\int_{I_j} |g_n(t)|^p \, dt = \sum_{j=1}^{n} \lambda_j^p \frac{1}{p+1} \frac{a^p}{n} =$$

$$= \sum_{j=1}^{n} \lambda_j^p \cdot \varepsilon/2.$$

If we make $\lambda_j = j^{-1/p} \Big/ \sum_{k=1}^{n} j^{-1/p}$, we shall have

$$\sum_{j=1}^{n} \lambda_j^p = \left(\sum_{j=1}^{n} j^{-1/p}\right)^{-p} \left(\sum_{j=1}^{n} j^{-1}\right) \longrightarrow \infty \quad \text{as } n \to \infty.$$

Thus, we can choose n in such a way that $\frac{1}{2\pi}\int_{-\pi}^{\pi} |g_n(t)|^p \, dt > 1$. This proves that $L^p(T)$ is not locally convex for $0 < p < 1$. To prove the same fact for H^p requires some more work. We keep n fixed. Each f_j is a uniform limit of trigonometric polynomials, i.e.: given $\alpha > 0$, we can find for each j

$$h_j(t) = \sum_{k=-m_j}^{m_j} a_{jk} e^{ikt}$$

so that $|h_j(t) - f_j(t)| < \alpha$ for every $-\pi \leq t \leq \pi$. Let $q_j(t) = e^{im_j t} h_j(t)$. Then $|q_j(t) - e^{im_j t} f_j(t)| < \alpha$ for $-\pi \leq t \leq \pi$.

Observe that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |e^{im_j t} f_j(t)|^p \, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_j(t)|^p \, dt < \varepsilon$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{j=1}^{n} \lambda_j e^{im_j t} f_j(t)\right|^p dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{j=1}^{n} \lambda_j f_j(t)\right|^p dt > 1$$

Let

$$Q_j(z) = \sum_{k=-m_j}^{m_j} a_{jk} z^{k+m_j}$$

so that $Q_j(e^{it}) = q_j(t)$ and

$$\|Q_j\|_{H^p}^p = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Q_j(e^{it})|^p dt \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} (\alpha + |e^{im_j t} f_j(t)|)^p dt \leq$$

$$\leq \alpha^p + \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_j(t)|^p \, dt < \varepsilon$$

if α is small.

$$\Big\|\sum_{j=1}^{n} \lambda_j Q_j \Big\|_{H^p}^{p} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|\sum_{j=1}^{n} \lambda_j Q_j(e^{it})\Big|^p \, dt \geq$$

$$\geq \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big(\Big|\sum_{j=1}^{n} \lambda_j e^{im_j t} f_j(t)\Big|^p - \alpha^p\Big)\, dt > 1$$

if α is small. We have arrived at the following contradiction: On the one hand $Q_1, \ldots, Q_n \in B_\varepsilon \subset V$ so that, since V is convex, $\sum_{j=1}^{n} \lambda_j Q_j \in V \subset B_{\frac{1}{n}}$ or, in other words $N_p(\sum_{j=1}^{n} \lambda_j Q_j) < 1$. On the other hand $N_p(\sum_{j=1} \lambda_j Q_j) > 1$. This proves the theorem. □

Of course, the first non locally convex spaces to be examined were the spaces L^p for $0 < p < 1$. M.M. Day ([1]) observed that, for $0 < p < 1$, the topological dual of $L^p([0,1])$ is zero. This is very easy to see. One just needs to realize that the only convex neighbourhood of 0 is the whole space. Indeed, let V be a convex neighbourhood of 0, $V \supset \{f \in L^p: \int_0^1 |f(x)|^p \, dx < \varepsilon\}$ for some $\varepsilon > 0$. If g is any function in $L^p([0,1])$, we can divide $[0,1]$ into n subintervals $I_1, \ldots, I_n$, in such a way that $\int_{I_j} |g(x)|^p dx = \frac{1}{n}\int_0^1 |g(x)|^p \, dx$ for $j = 1, 2, \ldots, n$. But then $\int_{I_j} |ng(x)|^p \, dx = n^{-(1-p)} \int_0^1 |g(x)|^p \, dx < \varepsilon$ if n is big. Thus, for n big, the functions $ng\chi_{I_j}$ are all in V and, consequently, the convex combination $g = \sum_{j=1}^{n} n^{-1} ng\chi_{I_j}$ will be in V too. We have sen that $V = L^p$.

However, the spaces H^p for $0 < p < 1$, in spite ob being non locally convex, have a great deal of continuous linear functionals.

In general, if $F \in H^p$ with $0 < p < \infty$ and F is not identically zero, we factor it as $F(z) = B(z)\,H(z)$ where B is the corresponding Blaschke product and H is nonvanishing with $\|H\|_{H^p} = \|F\|_{H^p}$. Then, let $A(z)$ be an analytic function such that $H(z) = \exp(A(z))$, and write $G(z) = \exp(pA(z))$, so that $|G(z)| = |H(z)|^p$ and, consequently, $\|G\|_{H^1} = \|H\|_{H^p}^p = \|F\|_{H^p}^p$. Since $G \in H^1$, it will have the Cauchy representation

$$G(z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{G(\zeta)}{\zeta - z}\, d\zeta$$

Therefore:

$$|F(z)|^p \leq |H(z)|^p = |G(z)| \leq \frac{1}{1-|z|}\frac{1}{2\pi}\int_{-\pi}^{\pi}|G(e^{it})|dt = (1-|z|)^{-1}\|F\|_{H^p}^p$$

We can state:

THEOREM 6.3. For $F \in H^p$, $0 < p \leq \infty$ *and* $z \in D$

$$|F(z)| \leq (1-|z|)^{-1/p}\ \|F\|_{H^p}$$

COROLLARY 6.4. For $0 < p \leq \infty$, H^p *is a complete space*

Proof: For $p \geq 1$, the image of H^p under the isometry (6.1) is, clearly, $\{f \in L^p\colon \hat{f}(j) = 0$ for each $j < 0\}$, which is, of course, closed. The completeness of H^p follows immediately. By using theorem 6.3, we can easily derive the completeness of H^p for any p. Indeed, theorem 6.3. implies that if $F \in H^p$ and K is a compact subset of D, then:

$$\sup_{z\in K}\ |F(z)| \leq (\sup_{z\in K}\ (1-|z|)^{-1/p}) \cdot \|F\|_{H^p}$$

Hence, if F_j is a Cauchy sequence in H^p, it will converge uniformly on compact subsets of D to a certain holomorphic function $F(z)$. Besides

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|F_j(re^{it})-F(re^{it})|^p\,dt = \lim_{k\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}|F_j(re^{it})-F_k(re^{it})|^p dt \leq$$

$$\leq \limsup_{k\to\infty}\ \|F_j-F_k\|_{H^p}^p < \varepsilon$$

if j is big. Thus, $F \in H^p$ and $F_j \to F$ in H^p as $j \to \infty$. $\square$

So, we can see H^p as a closed subspace of $L^p(T)$. If $0 < p < \infty$, H^p is the minimal closed subspace which contains $\{e^{ijt}\colon j = 0, 1, \ldots\}$ (If $F(z) = \sum_{j=0}^{\infty} a_j z^j$ is in H^p, $F(re^{it}) \to F(e^{it})$ in L^p as $r \to 1$ and, for r fixed, $\sum_0^n a_j r^j e^{ijt} \to F(re^{it})$ uniformly as $n \to \infty$).

For $1 \leq p < \infty$, this point of view can be used to describe the dual space $(H^p)^*$. We know that $L^p(T)^* = L^{p'}(T)$ with $(1/p) + (1/p') = 1$. To each $f \in L^{p'}(T)$, we can associate the functional

$\Lambda_f \in (H^p)^*$ obtained by restriction to H^p, that is, given by

$$\Lambda_f(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})\, f(t)\, dt$$

It follows from Hölder's inequality that

$$\|\Lambda_f\| \le \|f\|_{p'},$$

so that we have a continuous linear mapping

$$\begin{aligned} (6.5) \qquad & L^{p'}(T) \longrightarrow (H^p)^* \\ & f \longmapsto \Lambda_f \end{aligned}$$

The Hahn-Banach theorem tells us that every $\Lambda \in (H^p)^*$ is of the form $\Lambda = \Lambda_f$ for some f with $\|f\|_{p'} \le \|\Lambda\|$. The kernel of the mapping (6.5) is, clearly,

$$\{f \in L^{p'}(T) \colon \hat{f}(-j) = \int_{-\pi}^{\pi} e^{ijt} f(t)\, \frac{dt}{2\pi} = 0,\ j = 0, 1, \ldots\} =$$

$$= \{f \in H^{p'} \colon \int_{-\pi}^{\pi} f(e^{it})\, dt = 0\}.$$

We shall denote this space by $H^{p'}(0)$. Passing to the quotient in (6.5), we obtain an isometry

$$\begin{aligned} & L^{p'}/H^{p'}(0) \longrightarrow (H^p)^* \\ & f + H^{p'}(0) \longmapsto \Lambda_f \end{aligned}$$

Thus, we can write

$$(6.6) \qquad (H^p)^* = L^{p'}/H^{p'}(0), \qquad 1 \le p < \infty.$$

with the pairing

$$< f + H^{p'}(0),\ F > = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})\, f(t)\, dt$$

In the same way we can prove that:

$$(6.7) \qquad (H^p(0))^* = L^{p'}/H^{p'}, \qquad 1 \le p < \infty$$

If we also exclude $p' = \infty$ (i.e. $p = 1$), then the subspace $H^{p'}(0)$ has a topological complement in $L^{p'}$. This is a consequence of the boundedness of the conjugate function operator in $L^{p'}(T)$ for $1 < p' < \infty$, which was established in last section. Let us see this. If $f \in L^{p'}(T)$ has Fourier series $\sum_{-\infty}^{\infty} a_j e^{ijt}$, then its Poisson integral will be $u(re^{it}) = \sum_{-\infty}^{\infty} a_j r^{|j|} e^{ijt}$. The function $v(re^{it}) = -i \sum_{-\infty}^{\infty} (\operatorname{sgn} j)\, a_j r^{|j|} e^{ijt}$, where $\operatorname{sgn} j = 1$ for $j > 0$, $\operatorname{sgn} j = -1$ for $j < 0$ and $\operatorname{sgn} 0 = 0$, is the conjugate harmonic function of u with $v(0) = 0$, since $u(re^{it}) + iv(re^{it}) = a_0 + 2 \sum_{1}^{\infty} a_j (re^{it})^j$ is holomorphic. But corollary 5.11 tells us that $v = P(\tilde{f})$. Consequently $\tilde{f}$ has the Fourier series $\sum_{-\infty}^{\infty} (-i)(\operatorname{sgn} j)\, a_j\, e^{ijt}$. We see that the operator defined on $f \in L^{p'}(T)$ by setting

$$A(f) = (1/2)\,(f + i\tilde{f} - \hat{f}(0))$$

is a bounded operator in $L^{p'}(T)$ sending the function $f \in L^{p'}(T)$ with Fourier series $\sum_{-\infty}^{\infty} a_j e^{ijt}$ into the function $A(f)$ having Fourier series $\sum_{j>0} a_j e^{ijt}$. Thus A is the projection of $L^{p'}(T)$ onto $H^{p'}(0)$. The function $f - A(f)$ will have Fourier series $\sum_{j\le 0} a_j e^{ijt} = \sum_{j=0}^{\infty} a_{-j} e^{-ijt}$. If we write $F(z) = \sum_{j=0}^{\infty} a_{-j} z^j$, we have $F \in H^{p'}$ and

$$f(t) = Af(t) + F(e^{-it})$$

And if we write $G(z) = \sum_{j=0}^{\infty} \bar{a}_{-j} z^j$, we also have $G \in H^{p'}$ and $F(e^{-it}) = \overline{G(e^{it})}$, so that

$$f(t) = Af(t) + \overline{G(e^{it})}$$

By using the notations $\overline{H^{p'}} = \{\overline{h(t)} : h \in H^{p'}\}$ and $(H^{p'})^- = \{h(-t) : h \in H^{p'}\}$, we can conclude that $L^{p'}$ can be written as a topological direct sum in the two following ways:

$$L^{p'} = H^{p'}(0) \oplus \overline{H^{p'}}$$

$$L^{p'} = H^{p'}(0) \oplus (H^{p'})^-$$

Of course, we could have also writen topological direct sum decompositions

$$L^{p'} = H^{p'} \oplus \overline{H^{p'}(0)}$$

$$L^{p'} = H^{p'} \oplus (H^{p'}(0))^{-}$$

Naturally, the projection on $H^{p'}$ is the operator

$$B(f) = (1/2)(f+i\tilde{f}+\hat{f}(0))$$

sending each function $f \in L^{p'}(T)$ with Fourier series $\sum_{-\infty}^{\infty} a_j e^{ijt}$ into its "analytic part" $B(f)$ having Fourier series $\sum_{j=0}^{\infty} a_j e^{ijt}$. The topological direct sum decompositions give rise to topological isomorphisms

$$L^{p'}/H^{p'}(0) \approx H^{p'}$$

$$L^{p'}/H^{p'} \approx H^{p'}(0)$$

In each case two isomorphisms arise from the two different decompositions. In the first case we can either send the class $f + H^{p'}(0)$ into the function of t:

$$\overline{(f-Af)(t)} = (1/2)(f+i\tilde{f}+\hat{f}(0))(t) = Bf(t)$$

or else send it into this other function of t:

$$(f-Af)(-t) = (1/2)(f-i\tilde{f}+\hat{f}(0))(-t) = \overline{Bf(-t)}$$

In the second case we can either send the class $f+H^{p'}$ into the function of t:

$$\overline{(f-Bf)(t)} = (1/2)(f+i\tilde{f}-\hat{f}(0))(t) = Af(t)$$

or else into

$$(f-Bf)(-t) = (1/2)(f-i\tilde{f}-\hat{f}(0))(-t) = \overline{Af(-t)}$$

In view of these isomorphisms, (6.6) and (6.7) can be restated for

$1 < p < \infty$ by saying that

$$(6.8) \qquad (H^p)^* = H^{p'} \qquad\qquad 1 < p < \infty$$

with the pairing

$$< G,F > = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})\, G(e^{-it})\, dt, \qquad F \in H^p,\ G \in H^{p'}$$

or

$$< G,F > = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})\, \overline{G(e^{it})}\, dt, \qquad F \in H^p,\ G \in H^{p'}$$

$$(6.9) \qquad (H^p(0))^* = H^{p'}(0) \qquad\qquad 1 < p < \infty$$

We know that $H^\infty(0)$ is not a complemented subspace of L^∞, since this would clearly imply the boundedness of the conjugate function in L^∞ and, by duality, in L^1. So, we still have a problem to which we shall turn in section 9: To describe $(H^1)^*$ as a space of functions. For the time being, all we can say is that $(H^1)^* = L^\infty/H^\infty(0)$.

7. CANONICAL FACTORIZATION THEOREM

Let $F \in H^p$, $0 < p \le \infty$. We have used many times the factorization $F = B\cdot H$, where B is the Blaschke product formed with the zeroes of F, H is never zero and $\|H\|_{H^p} = \|F\|_{H^p}$. Starting with this factorization, we shall get a finer one.

We apply theorem 3.2 to the function H and conclude that, for any sequence $r_j \uparrow 1$, both $(\log^+ |H(r_je^{it})|)$ and $(\log^- |H(r_je^{it})|)$ are bounded sequences in the Banach space $L^1 \subset M$. They will have subsequences converging in the weak-* topology of measures. Thus, there exists $r_j \uparrow 1$ and positive measures μ_1, μ_2 such that

$$\log^+ |H(r_je^{it})| \longrightarrow d\mu_1(t) \quad \text{and} \quad \log^- |H(r_je^{it})| \longrightarrow d\mu_2(t)$$

both in the weak-* topology. But it turns out that $d\mu_1(t) = \log^+ |H(e^{it})|\, dt$. Indeed, we just need to observe that $\log^+ |H(r_je^{it})| \longrightarrow \log^+ |H(e^{it})|$ in the L^1 norm. To see this, write, if $p \geq 1$

$$\left|\log^+ |H(e^{it})| - \log^+ |H(r_j e^{it})|\right| \leq \left||H(e^{it})| - |H(r_j e^{it})|\right| \leq |H(e^{it}) - H(r_j e^{it})|$$

and if $0 < p < 1$:

$$\left|\log^+ |H(e^{it})| - \log^+ |H(r_j e^{it})|\right| \leq p^{-1}\left||H(e^{it})|^p - |H(r_j e^{it})|^p\right| \leq \leq p^{-1}|H(e^{it}) - H(r_j e^{it})|^p$$

Now, by corollary 3.9:

$$\log H(r_j z) = i \arg H(0) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log |H(r_j e^{it})|\, dt =$$

$$= ic + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log^+ |H(r_j e^{it})|\, dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log^- |H(r_j e^{it})|\, dt$$

Letting $j \to \infty$, we get:

$$\log H(z) = ic + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log^+ |H(e^{it})|\, dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu_2(t)$$

But $d\mu_2(t) = g(t)dt + d\sigma(t)$ where g is integrable ≥ 0 and σ is a non-negative singular measure. Thus

$$\log H(z) = ic + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} k(t)\, dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\sigma(t)$$

with $k(t) = \log^+ |H(e^{it})| - g(t)$, integrable. But, then, $\log |H(z)| = P(k(t)\, dt - d\sigma(t))$ and this implies that $\log|H(e^{it})| = k(t)$. We have proved the following result, known as canonical factorization theorem

THEOREM 7.1. Every $F \in H^p$, $0 < p \leq \infty$, *not identically zero, has a canonical factorization*

$$F(z) = e^{ic}B(z)\cdot\exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\, d\sigma(t)\right)\cdot\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log |F(e^{it})|\, dt\right)$$

where B *is the Blaschke product formed with the zeroes of* F, c *is a real constant, and* σ *is a non-negative singular measure.*

We shall use the notation

$$I_F(z) = e^{ic}B(z)\exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\,d\sigma(t)\right)$$

$$E_F(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log|F(e^{it})|\,dt\right)$$

and we shall say that I_F and E_F are, respectively, the inner and outer factors of F.

Since $|E_F(z)|^p = \exp P(\log(|F(e^{it})|^p)) \le P(|F(e^{it})|^p)$, it is clear that if $F \in H^p$, then also $E_F \in H^p$ and $\|E_F\|_{H^p} = \|F\|_{H^p}$ because $|E_F(e^{it})| = |F(e^{it})|$ for a.e.t.

We shall say that $F \in H^p$ is an *inner* function if and only if $E_F \equiv 1$; and we shall say that F is an *outer* function if and only if I_F is constant.

COROLLARY 7.2. The inner functions are precisely those functions $F \in H^\infty$ *for which* $|F(e^{it})| = 1$ *almost everywhere*

Proof: Since $|E_F(z)| = \exp P(\log|F(e^{it})|)$, we realize that $E_F \equiv 1 \Longleftrightarrow |E_F| \equiv 1 \Longleftrightarrow \log|F(e^{it})| = 0$ a.e. $\Longleftrightarrow |F(e^{it})| = 1$ a.e $\square$

COROLLARY 7.3. If $F \in H^p$, $0 < p \le \infty$, *and is not identically zero, then:*

$$(7.4)\qquad \log|F(0)| \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|F(e^{it})|\,dt$$

and equality holds if and only if F *is an outer function.*

Proof: It follows from theorem 7.1 that

$$|F(0)| = |B(0)|\exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi}d\sigma(t)\right)\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|F(e^{it})|\,dt\right) \le$$

$\le \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|F(e^{it})|\,dt\right)$ and equality holds if and only if $|B(0)| = 1$ (which implies $B \equiv 1$) and $\int_{-\pi}^{\pi}d\sigma = 0$ (which implies $\sigma = 0$). $\square$

We shall present some applications of the criterion just given for a function to be outer

THEOREM 7.5. Suppose that $F \in H^p$ *and* $F^{-1} \in H^p$ *for some* $0 < p \leqq \infty$. *Then* F *is an outer function.*

Proof: We know that

$$\log |F(0)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(e^{it})| \, dt$$

But also

$$-\log |F(0)| = \log(|F(0)|^{-1}) \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(|F(e^{it})|^{-1}) \, dt =$$

$$= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(e^{it})| \, dt$$

Therefore, equality holds in (7.4) and F is outer. □

THEOREM 7.6. Let $F_j \in H^p$ *be outer functions for* $j = 1, 2, \ldots$ *Suppose that* $|F_1(z)| \ge |F_2(z)| \ge \ldots$ *for every* $z \in D$ *and* $F_j(z) \to F(z)$ *uniformly over compact subsets of* D, *as* $j \to \infty$. *Then, if* F *is not identically zero,* F *in an outer function.*

Proof: For every j:

$$\log |F_j(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F_j(e^{it})| dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |F_j(e^{it})| \, dt -$$

$$- \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |F_j(e^{it})| \, dt$$

Now

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |F_j(e^{it})| dt \to \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |F(e^{it})| \, dt$$

$$\text{as } j \to \infty$$

by dominated convergence, and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |F_j(e^{it})| \, dt \to \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |F(e^{it})| \, dt \quad \text{as } j \to \infty$$

by monotone convergence. Letting $j \to \infty$, we get:

$$\log |F(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(e^{it})| \, dt$$

and, consequently, F is an outer function. □

THEOREM 7.7. Let $F \in H^p$, $0 < p \le \infty$, *not identically zero, and such that* $\operatorname{Re} F(z) \ge 0$ *for every* $z \in D$. *Then* F *is an outer function.*

Proof: If we had $\operatorname{Re} F(z) \ge \varepsilon > 0$ for every $z \in D$, then $|1/F(z)| \le 1/\varepsilon$, so that $F^{-1} \in H^\infty \subset H^p$. According to theorem 7.5., F would be outer. However, we only know that $\operatorname{Re} F(z) \geqq 0$. Let us define $F_j(z) = F(z) + (1/j)$. For $j = 1, 2, \ldots$, $F_j \in H^p$, and $\operatorname{Re} F_j(z) \ge 1/j > 0$, so that each F_j is outer. Besides, clearly $|F_1(z)| \ge |F_2(z)| \ge \ldots$ and $F_j(z) \to F(z)$ uniformly in D. Applying theorem 7.6., we conclude that F is outer. $\square$

Here is an interesting extension of corollary 3.7.

THEOREM 7.8 Let $K(z)$ *be the quotient of two outer functions, and suppose that* $K(e^{it})$ *belongs to* L^p *for some* $0 < p \le \infty$. *Then* $K \in H^p$.

Proof: Let $K(z) = E_F(z)/E_G(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log\frac{|F(e^{it})|}{|G(e^{it})|}\,dt\right)$

Then, if $p < \infty$, $|K(z)|^p = \exp P(\log(|K(e^{it})|^p)) \leqq P(|K(e^{it})|^p)$ so that, clearly, $K \in H^p$. If $p = \infty$, use 1 instead of p, and the result is obvious. $\square$

The inner factor plays an important rôle in some approximation problems. Here is the basic result

THEOREM 7.9. Let $F \in H^p$ *be an outer function* $0 < p < \infty$. *Let* $\mathcal{P} = \{P(z) = \sum_{j=0}^{n} a_j z^j\}$ *be the space of polynomials. Then* $F\cdot\mathcal{P} = \{F(z)\cdot P(z) : P \in \mathcal{P}\}$ *is dense in* H^p.

Proof: Suppose $F = E_F$.

a) Let $p \geqq 1$. Then we can see H^p as a closed subspace of $L^p(T)$ and use duality. To see that $E_F\cdot\mathcal{P}$ is dense in H^p, one just needs to show that if $k(e^{it})$ is in $L^{p'}(T)$ and

$$(7.10) \qquad \int_{-\pi}^{\pi} E_F(e^{it})e^{ijt}k(e^{it})dt = 0 \quad \text{for } j = 0, 1, 2, \ldots,$$

then $$\int_{-\pi}^{\pi} G(e^{it})k(e^{it})dt = 0 \quad \text{for each } G \in H^p.$$

But (7.10) implies that $E_F(e^{it})k(e^{it}) = e^{it}H(e^{it})$ for some $H \in H^1$. We can write:

$$E_F(e^{it})k(e^{it}) = e^{it}I_H(e^{it})E_H(e^{it})$$

Thus $\left|\frac{E_H(e^{it})}{E_F(e^{it})}\right| = |k(e^{it})|$, which belongs to $L^{p'}(T)$. Theorem 7.8 implies that $E_H/E_F \in H^{p'}$. Denote this function by S. Then

$$k(e^{it}) = e^{it}I_H(e^{it})S(e^{it})$$

Now if $G \in H^p$:

$$\int_{-\pi}^{\pi} G(e^{it})k(e^{it})dt = \int_{-\pi}^{\pi} G(e^{it})e^{it}I_H(e^{it})S(e^{it})dt = 0$$

since $GI_HS \in H^1$. This finishes the proof for $p \geq 1$.

b) Let now $0 < p < 1$. We shall show that if the result holds for $2p$, so it does for p. Once this is done, the theorem follows from a) by induction.

Given $H \in H^p$ and given $\varepsilon > 0$, we want to find $P \in \mathcal{P}$ such that $\|H - E_F\cdot P\|_{H^p} < \varepsilon$.

Observe that $E_F = K^2$ where $K \in H^{2p}$ and is outer, in fact $K = E_K$.

Now, given H, there is $R \in \mathcal{P}$ such that $\|H - R\|_{H^p}^p < \varepsilon/3$. Since $R \in H^{2p}$ and E_K is an outer function in H^{2p}, there will be $Q \in \mathcal{P}$ such that $\|R - E_K\cdot Q\|_{H^{2p}}^p < \varepsilon/3$. Of course, this implies that $\|R - E_K\cdot Q\|_{H^p}^p < \varepsilon/3$. Finally, there will exist $P \in \mathcal{P}$ such that $\|Q - E_K\cdot P\|_{H^{2p}} < \delta$ for δ to be specified later. But then

$$\|E_K\cdot Q - E_F\cdot P\|_{H^p}^p = \frac{1}{2\pi}\int_{-\pi}^{\pi} |E_K(Q - E_K\cdot P)|^p\, dt \leq$$

$$\leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|E_K|^{2p}\,dt\right)^{1/2}\cdot\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|Q-E_K\cdot P|^{2p}\,dt\right)^{1/2} =$$

$$= \|E_F\|_{H^p}^{p/2}\,\|Q-E_k\cdot P\|_{H^{2p}}^p$$

Thus $\|E_K\cdot Q - E_F\cdot P\|_{H^p}^p \leq \|E_F\|_{H^p}^{p/2}\cdot\delta^p < \varepsilon/3$ if we choose an appropriate δ. Finally

$$\|H-E_F\cdot P\|_{H^p}^p \leq \|H-R\|_{H^p}^p + \|R-E_K\cdot Q\|_{H^p}^p + \|E_K\cdot Q-E_F\cdot P\|_{H^p}^p < \varepsilon$$

This ends the proof. $\square$

COROLLARY 7.11. If $F \in H^p$, $0 < p < \infty$, *then the closure in* H^p *of the space* $F\cdot\mathcal{P}$ *is* $I_F\cdot H^p$.

Proof: We just need to realize that the mapping

$$\begin{aligned} H^p &\longrightarrow H^p \\ G &\longmapsto I_F\cdot G \end{aligned}$$

in an isometry. From this, it follows that $I_F\cdot H^p$ is closed. If $P \in \mathcal{P}$, $F\cdot P = I_F\cdot E_F\cdot P \in I_F\cdot H^p$. Thus $F\cdot\mathcal{P} \subset I_F\cdot H^p$ and, since $I_F\cdot H^p$ is closed, the closure $\overline{F\cdot\mathcal{P}}$ will also be contained in $I_F\cdot H^p$. On the other hand $\overline{E_F\cdot\mathcal{P}} = H^p$ implies $\overline{F\cdot\mathcal{P}} = \overline{I_F\cdot E_F\cdot\mathcal{P}} = I_F\cdot\overline{E_F\cdot\mathcal{P}} = I_F\cdot H^p$. $\square$

8. THE HELSON-SZEGÖ THEOREM

The Helson-Szegö theorem is a characterization of those positive (Borel) measures μ on $[-\pi,\pi]$ for which the conjugate function operator is bounded in $L^2(\mu)$. More specifically, denote by $\mathcal{T}$ the space formed by the trigonometric polynomials $f(t) = \sum_{-n}^{n} a_j e^{ijt}$ and for $f \in \mathcal{T}$ as above, we consider its conjugate function

$$\tilde{f}(t) = -i \sum_{-n}^{n} (\operatorname{sgn} j)\, a_j e^{ijt}$$

where $\operatorname{sgn} j = 1$ for $j > 0$, $\operatorname{sgn} j = -1$ for $j < 0$ and $\operatorname{sgn} 0 = 0$. The Helson-Szegö theorem provides an answer to the problem of finding those positive measures on $[-\pi,\pi]$ for which there is a constant C such that, for every $f \in \mathcal{T}$, the following inequality holds:

$$(8.1) \qquad \int_{-\pi}^{\pi} |\tilde{f}(t)|^2\, d\mu(t) \leq C \int_{-\pi}^{\pi} |f(t)|^2\, d\mu(t)$$

We shall call such measures Helson-Szegö measures. By now, we have met just one of them, namely Lebesgue measure $d\mu(t) = dt$, for which, an inequality like (8.1), is a particular instance of the M. Riesz theorem, and follows immediately from Plancherel's theorem.

Before presenting the Helson-Szegö theorem, we shall deal with another problem about measures, which has an interesting answer that will play a part in our treatment of the main question. The problem is this: given a finite positive measure μ on $[-\pi,\pi]$, find the distance in $L^p(\mu)$, $1 \le p < \infty$, from the constant function 1 to the space $\mathcal{P}(0)$ formed by (the restrictions to T of) those polynomials $P \in \mathcal{P}$ such that $P(0) = 0$. In other words, we want to find:

$$\operatorname{dist}(1,\mathcal{P}(0)) = \inf \{ (\frac{1}{2\pi} \int_{-\pi}^{\pi} |1-P(e^{it})|^p \, d\mu(t))^{1/p}, \quad P \in \mathcal{P}(0)\}$$

First of all we have:

THEOREM 8.2. If σ is a finite positive measure which is singular (with respect to Lebesgue measure), then:

$$\inf \{ \int_{-\pi}^{\pi} |1-P(e^{it})|^p \, d\sigma(t); \ P \in \mathcal{P}(0)\} = 0$$

i.e., the point 1 is adherent to $\mathcal{P}(0)$ in $L^p(\sigma)$

Proof: Let $f \in L^{p'}(\sigma)$ be such that $\int_{-\pi}^{\pi} f(t)\, e^{ijt}\, d\sigma(t) = 0$ for every $j = 1, 2, 3, \ldots$ If we can show that $\int_{-\pi}^{\pi} f(t)\, d\sigma(t) = 0$, we shall have the theorem proved. But what we are assuming is that $f(t)\, d\sigma(t)$ is a measure whose Fourier coefficients vanish for all negative frequencies. The F. and M. Riesz theorem (theorem 3.10) says that $f(t)\, d\sigma(t)$ is in this case absolutely continuous with respect to dt. On the other hand, it is clearly singular. Therefore, it is identically zero and the proof is finished. $\square$

We shall present now a result, due to Kolmogorov, which shows that we can limit our study to absolutely continuous measures. We need a previous lemma

LEMMA 8.3. Let E be a closed subset of T having Lebesgue measure 0. Then, there exists a function $F \in A$ (that is: F is holomorphic in D and continuous on $\overline{D}$) such that $F(z) = 1$ for every $z \in E$ and $|F(z)| < 1$ for every $z \in \overline{D} \setminus E$.

Proof: E is the complement in T of the union of a sequence of arcs $A_j = \{e^{it}: \alpha_j < t < \beta_j\}$. Since E has Lebesgue measure zero,

$\sum_j (\beta_j - \alpha_j) = 2\pi$. Let $m_j > 0$ be such that $m_j \to \infty$ while $\sum_j m_j(\beta_j - \alpha_j) < \infty$. Then, setting $a_j = (\beta_j - \alpha_j)/2$ and $\gamma_j = (\beta_j + \alpha_j)/2$, we define a 2π-periodic function $f(t)$ such that:

a) For $\alpha_j < t < \beta_j$: $f(t) = m_j a_j (a_j^2 - (t-\gamma_j)^2)^{-1/2}$

b) If $e^{it} \in E$: $f(t) = \infty$.

The function f is integrable in $[-\pi,\pi]$ since $\int_{\alpha_j}^{\beta_j} f(t)\,dt = \pi m_j a_j$ and E has measure zero. Actually, f is a continuous mapping from $\mathbb{R}$ to $[0,\infty]$. Indeed if a sequence of points not in E approaches a point in E it is easy to see that the corresponding values of f tend to ∞. We just need to consider two situations: when the points belong to just two of the arcs A_j and approach a common endpoint and when the points range over infinitely many of the A_j's. In the first case the formula for f gives $f(t) \to \infty$. In the second case we just need to observe that on A_j is $f(t) \geq m_j$ and $m_j \to \infty$.

Now, since f is integrable, we may consider its Poisson integral $u(re^{it}) = (P_r * f)(t)$. It is quite simple to see that if we set $u(e^{it}) = f(t)$, u becomes a continuous mapping from $\overline{D}$ into $[0,\infty]$. Since u is a harmonic function in D, we are able to consider v, the conjugate harmonic function of u such that $v(0) = 0$. Observe that f is smooth on the arcs A_j, so that, according to the observation we made after (5.13), the function v extends continuously up to each of those arcs, with finite limits.

It is now easy to see that the function

$$F(z) = \frac{u(z) + iv(z)}{u(z) + iv(z) + 1}$$

satisfies our requirements. Since $u \geq 0$, F is a well defined holomorphic function in D. It clearly extends continuously up to the arcs A_j since both u and v do. Now if $z \to e^{i\theta} \in E$, $|z| < 1$, we know that $u(z) \to \infty$ and this implies that $F(z) \to 1$ no matter how v behaves. As for the points e^{it} in the boundary such that $e^{it} \notin E$, we have

$$F(e^{it}) = \frac{f(t) + i\tilde{f}(t)}{f(t) + i\tilde{f}(t) + 1}$$

and again, if those points approach some point in E, the fact that $f(t) \to \infty$ guarantees by itself that $F(e^{it}) \to 1$. We have shown that F is a holomorphic function in D extending continuously up to the boundary of D in such a way that $F(e^{it}) = 1$ whenever $e^{it} \in E$. Also, the two formulas above for $F(z)$ $|z| < 1$ and $F(e^{it})$, $e^{it} \notin E$, clearly yield $|F(z)| < 1$ if $z \in \overline{D} \setminus E$. $\square$

Observe that lemma 8.3. provides a direct proof of the theorem of F. and M. Riesz (theorem 3.10) independent of the theory of the Hardy spaces. This was actually the original proof given by F. and M. Riesz. It is as follows: Let μ be a Borel measure with Fourier coefficients vanishing for all negative frequencies, that is:

$$\int_{-\pi}^{\pi} e^{ijt} d\mu(t) = 0 \quad \text{for} \quad j = 1, 2, 3, \ldots$$

We have to show that μ is absolutely continuous with respect to Lebesgue measure. For that purpose let $E \subset [-\pi,\pi]$ have $|E| = 0$. We have to show that $\mu(E) = 0$. We may assume that E is closed and consider the function F corresponding to E whose existence is granted by lemma 8.3. Then for each $n = 1, 2, 3, \ldots,$ since $(F(z))^n$ is in A

$$\int_{-\pi}^{\pi} (F(e^{it}))^n \, d\mu(t) = 0.$$

Now as $n \to \infty$, the dominated convergence theorem implies that

$$\int_{-\pi}^{\pi} (F(e^{it}))^n \, d\mu(t) \longrightarrow \int_E d\mu(t) = \mu(E)$$

Thus $\mu(E) = 0$ as was to be proved.

Now, we can state and prove Kolmogorov's result

<u>*THEOREM 8.4. Let*</u> $d\mu(t) = w(t)\,dt + d\sigma(t)$ <u>*with*</u> $w \geq 0$, $w \in L^1(T)$, <u>*and*</u> σ <u>*a finite positive singular measure. Then, for*</u> $1 \leq p < \infty$

$$\inf_{P \in \mathcal{P}(0)} \int_{-\pi}^{\pi} |1-P(e^{it})|^p \, d\mu(t) = \inf_{P \in \mathcal{P}(0)} \int_{-\pi}^{\pi} |1-P(e^{it})|^p w(t) \, dt$$

<u>Proof</u>: Let us denote respectively by A and B the infimum in the right and the infimum in the left of the identity to be proved. Clearly $A \leq B$, and what we have to see is that also $B \leq A$.

Let $\varepsilon > 0$. Then, there is $P \in \mathcal{P}(0)$ such that:

$$\int_{-\pi}^{\pi} |1-P(e^{it})|^p \; w(t) \; dt < A + \varepsilon$$

On the other hand, by theorem 8.2., there is $P_1 \in \mathcal{P}(0)$ such that

$$\int_{-\pi}^{\pi} |1-P_1(e^{it})|^p \; d\sigma(t) < \varepsilon$$

It will be convenient to write this as:

$$(8.5) \quad \int_{-\pi}^{\pi} |1-P(e^{it})-P_2(e^{it})|^p \; d\sigma(t) < \varepsilon \quad \text{with} \quad P_2 = P_1 - P \in \mathcal{P}(0)$$

Take a closed set E contained in the support of σ, such that:

$$(8.6) \qquad \int_{[-\pi,\pi]\setminus E} (1+|P(e^{it})|+|P_2(e^{it})|)^p \; d\sigma(t) < \varepsilon.$$

Since σ is singular, $|E| = 0$. Then, the preceding lemma gives us $F \in A$ such that $|F(z)| < 1$ for every $z \in \overline{D}\setminus E$ and $F(z) = 1$ for every $z \in E$ (we identify E with a subset of T in the natural way).

Since $F(z) = 1$ on E, (8.5) implies that

$$\int_E |1-P(e^{it})-F(e^{it})^n P_2(e^{it})|^p \; d\sigma(t) < \varepsilon$$

for every positive integer n. This, together with (8.6) yields:

$$\int_{-\pi}^{\pi} |1-P(e^{it})-F(e^{it})^n P_2(e^{it})|^p \; d\sigma(t) < 2\varepsilon.$$

On the other hand, if e^{it} is not in E, one has $F(e^{it})^n \to 0$ as $n \to \infty$ and, consequently $1-P(e^{it})-F(e^{it})^n P_2(e^{it}) \to 1-P(e^{it})$ as $n \to \infty$. Since E has Lebesgue measure 0, we can say that $1-P(e^{it})-F(e^{it})^n P_2(e^{it}) \to 1-P(e^{it})$ as $n \to \infty$ for almost every t. It follows that

$$\int_{-\pi}^{\pi} |1-P(e^{it})-F(e^{it})^n P_2(e^{it})|^p \; w(t)dt \to \int_{-\pi}^{\pi} |1-P(e^{it})|^p \; w(t)dt < A+\varepsilon$$

as $n \to \infty$. Thus, for n sufficiently large

$$\int_{-\pi}^{\pi} |1-P(e^{it})-F(e^{it})^n P_2(e^{it})|^p \; w(t) \; dt < A + \varepsilon$$

Adding up the corresponding estimate obtained for the singular part, we get:

$$\int_{-\pi}^{\pi} |1-P(e^{it})-F(e^{it})^n P_2(e^{it})|^p \, d\mu(t) < A + 3\varepsilon \quad \text{for large} \quad n.$$

Now fix n large enough so that the last inequality holds. Since $F^n \in A$, given $\delta > 0$, there will exist a polynomial $R(z) = \sum_0^N a_j z^j$ such that $|R(e^{it})-F(e^{it})^n| < \delta$ uniformly in t. Then $R(e^{it})P_2(e^{it})$ is close to $F(e^{it})^n P_2(e^{it})$. Observe that $RP_2 \in \mathcal{P}(0)$ and δ can be chosen in such a way that

$$\int_{-\pi}^{\pi} |1-P(e^{it})-R(e^{it})P_2(e^{it})|^p \, d\mu(t) < A + 3\varepsilon.$$

Let $P + RP_2 = Q \in \mathcal{P}(0)$. We have seen that

$$\int_{-\pi}^{\pi} |1-Q(e^{it})|^p \, d\mu(t) < A + 3\varepsilon. \text{ Thus } B < A + 3\varepsilon.$$

Since this is true for every $\varepsilon > 0$, we finally get $B \leq A$, which is what we wanted. $\square$

The next result, due to Szegö, gives a precise formula for the distance we are looking for.

THEOREM 8.7. For $w \geq 0$, $w \in L^1(T)$:

$$\inf_{P\in\mathcal{P}(0)} \frac{1}{2\pi} \int_{-\pi}^{\pi} |1-P(e^{it})|^p \, w(t)dt = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log w(t)dt\right), \quad 1 \leq p < \infty$$

Proof: a) Let us assume that $\int_{-\pi}^{\pi} \log^- w(t) \, dt < \infty$. Of course

$$\int_{-\pi}^{\pi} \log^+ w(t) \, dt \leq \int_{-\pi}^{\pi} w(t) \, dt < \infty.$$

Thus, the integrability of $\log w$ is equivalent to that of $\log^- w$. Since we have assumed $\int_{-\pi}^{\pi} \log^- w(t) \, dt < \infty$, we have

$$\exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log w(t) \, dt\right) = K, \text{ a positive number.}$$

Dividing w by K in the formula to be proved, we see that there is no loss of generality in assuming that $K = 1$, or, what is the same, that $\int_{-\pi}^{\pi} \log w(t) \, dt = 0$. Then we have to see that the left hand side in the formula of the statement is in this case equal to 1.

Let us consider the function

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log w(t)\, dt$$

Then, $F \in H(D)$, $F(0) = 0$, and the function $\exp(F(z)/p)$ is in H^p since

$$|\exp(F(z)/p)| = \exp(\operatorname{Re} F(z)/p) = \exp(P(\log(w^{1/p}))) \leq P(w^{1/p})$$

which is a harmonic function whose restrictions to circles of radius $r < 1$ have uniformly bounded L^p norms or, as we shall say for short, which is uniformly in L^p. Besides, $\exp(F(z)/p)$ is, by definition, an outer function, and $|\exp(F(e^{it})/p)| = w(t)^{1/p}$.

Let $P \in \mathcal{P}(0)$. We set $G(z) = (1-P(z)) \exp(F(z)/p)$. Then:

$$1 = G(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(re^{it})dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(re^{it})|dt \leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |G(re^{it})|^p dt\right)^{\frac{1}{p}}$$

for every $0 < r < 1$. Letting $r \to 1$, we get:

$$1 \leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{it})|^p dt\right)^{1/p}$$

since $G \in H^p$. Now

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{it})|^p\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} |1-P(e^{it})|^p\, w(t)\, dt,$$

so that we have obtained:

$$\inf_{P\in \mathcal{P}(0)} \frac{1}{2\pi} \int_{-\pi}^{\pi} |1-P(e^{it})|^p\, w(t)\, dt \geq 1$$

which is half of what we wanted to prove, the easy part. The converse inequality is deeper. We shall prove that equality holds by using the density theorem 7.9. Since $\exp(F(z)/p)$ is an outer function, there will be a sequence of polynomials $Q_j \in \mathcal{P}$ such that $Q_j(z) \exp(F(z)/p) \to 1$ in H^p as $j \to \infty$. In particular, $Q_j(0) \to 1$ as $j \to \infty$, so that $(Q_j(z)/Q_j(0)) \exp(F(z)/p) \to 1$ in H^p as $j \to \infty$. But $Q_j(z)/Q_j(0) = 1-P_j(z)$ for some $P_j \in \mathcal{P}(0)$. Therefore:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |(1-P_j(e^{it}))\exp(F(e^{it})/p)|^p dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} |1-P_j(e^{it})|^p w(t)\, dt \to 1$$

as $j \to \infty$, and we conclude that the infimum above is actually equal to 1. The theorem is proved with the restriction that $\log w$ is integrable.

b) In order to complete the proof of the theorem we need to consider the case when $\int_{-\pi}^{\pi} \log w(t)\, dt = -\infty$. Set, for $j = 1, 2, \ldots,$ $w_j(t) = \max(1/j, w(t))$. Then $w_1 \geq w_2 \geq \ldots \geq w$ and $w_j \to w$ as $j \to \infty$ at every point. $\int_{-\pi}^{\pi} \log w_j(t)\, dt > -\infty$ for every j and

$$\int_{-\pi}^{\pi} \log w_j(t)\, dt \longrightarrow \int_{-\pi}^{\pi} \log w(t)\, dt \quad \text{as} \quad j \to \infty$$

Now

$$\inf_{P\in\mathcal{P}(0)} \frac{1}{2\pi}\int_{-\pi}^{\pi} |1-P(e^{it})|^p w(t)dt \leq \inf_{P\in\mathcal{P}(0)} \frac{1}{2\pi}\int_{-\pi}^{\pi} |1-P(e^{it})|^p\, w_j(t)\, dt =$$

$$= \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log w_j(t)\, dt\right) \to 0 \quad \text{as} \quad j \to \infty.$$

Thus, even in this case the formula holds. $\square$

We are now ready to initiate our search for a characterization of the Helson-Szegö measures. The next two results will narrow the set of candidates.

THEOREM 8.8. Every Helson-Szegö measure is absolutely continuous (with respect to Lebesgue measure)

Proof: Let μ be a Helson-Szegö measure. Let E be a closed subset of $\mathbb{T}$ with Lebesgue measure $|E| = 0$. We have to show that $\mu(E) = 0$. Lemma 8.3. tells us that with E we can associate a function $F \in \mathcal{A}$ such that $F(z) = 1$ if $z \in E$ and $|F(z)| < 1$ if $z \in \overline{D} \setminus E$. Consider $F_j(z) = F(z)^j - F(0)^j$ for $j = 1, 2, \ldots$ Since $F_j \in \mathcal{A}$ and $F_j(0) = 0$, there is a sequence of polynomials $G_k \in \mathcal{P}(0)$ such that $G_k \to F_j$ uniformly as $k \to \infty$. Of course, the sequence depends on j. We have:

$$\operatorname{Re} G_k \to \operatorname{Re} F_j \quad \text{and} \quad \operatorname{Im} G_k \to \operatorname{Im} F_j \quad \text{uniformly as} \quad k \to \infty.$$

Looking at the boundary functions $\operatorname{Re} G_k$, $\operatorname{Im} G_k \in \mathcal{T}$, since $G_k \in \mathcal{P}(0)$, we have $\operatorname{Re} G_k = -(\operatorname{Im} G_k)^{\sim}$, so that

$$\int_{-\pi}^{\pi} |\mathrm{Re}\ G_k(e^{it})|^2\, d\mu(t) \le C \int_{-\pi}^{\pi} |\mathrm{Im}\ G_k(e^{it})|^2 d\mu(t)$$

and from this:

$$\int_{-\pi}^{\pi} |\mathrm{Re}\ F_j(e^{it})|^2\, d\mu(t) \le C \int_{-\pi}^{\pi} |\mathrm{Im}\ F_j(e^{it})|^2\, d\mu(t)$$

But if $e^{it} \in E$, $\mathrm{Re}\ F_j(e^{it}) = 1-\mathrm{Re}(F(0)^j) \to 1$ as $j \to \infty$, and if $e^{it} \notin E$, then $\mathrm{Re}\ F_j(e^{it}) \to 0$ as $j \to \infty$, whereas for every t, $\mathrm{Im}\ F_j(e^{it}) \to 0$ as $j \to \infty$. Thus, letting $j \to \infty$ in the last inequality, we get $\mu(E) = 0$. $\square$

THEOREM 8.9. Let μ *be a non-trivial Helson-Szegö measure*(*i.e.* $\mu \neq 0$). *Then* $d\mu(t) = w(t)\, dt$ *with* $w \ge 0$, $w \in L^1(T)$ *and* $\int_{-\pi}^{\pi} \log w(t)\, dt > -\infty$.

Proof: All we have to prove is the integrability of $\log w$. If it were $\int_{-\pi}^{\pi} \log w(t)\, dt = -\infty$, we should have, according to Szegö's theorem 8.7., a sequence $P_j \in \mathcal{P}(0)$ such that $\int_{-\pi}^{\pi} |1-P_j(e^{it})|^2\, w(t)\, dt \to 0$ as $j \to \infty$. Let $f_j(t) = 1-P_j(e^{it})$, $f_j \in \mathcal{T}$; then $\tilde{f}_j(t) = iP_j(e^{it})$ and also $\tilde{\tilde{f}}_j(t) = P_j(e^{it})$. Since $d\mu(t) = w(t)\, dt$ is a Helson-Szegö measure, the following inequalities hold:

$$\int_{-\pi}^{\pi} |P_j(e^{it})|^2 w(t)\, dt \le C \int_{-\pi}^{\pi} |1-P_j(e^{it})|^2\, w(t)\, dt \to 0 \quad \text{as} \quad j \to \infty.$$

So, we have, at the same time

$$\int_{-\pi}^{\pi} |P_j(e^{it})|^2\, w(t)\, dt \to 0 \quad \text{and} \quad \int_{-\pi}^{\pi} |1-P_j(e^{it})|^2\, w(t)\, dt \to 0$$

as $j \to \infty$. It follows that $\int_{-\pi}^{\pi} 1 \cdot w(t)\, dt = 0$, which is a contradiction. $\square$

From now on, we assume $w \ge 0$, $w \in L^1(T)$ and also $\log w \in L^1(T)$. Our problem is to find when $w(t)\, dt$ is a Helson-Szegö measure or, as we shall say equivalently, when is w a Helson-Szegö weight. For our final attack on the problem we shall be working in the Hilbert space $L^2(w)$. We shall denote the inner product in this space by

$$(f|g)_w = \int_{-\pi}^{\pi} f(t)\,\overline{g(t)}\,w(t)\,dt$$

and the norm by

$$\|f\|_w = \left(\int_{-\pi}^{\pi} |f(t)|^2\,w(t)\,dt\right)^{1/2}$$

It will be convenient to use the operator A appearing towards the end of section 6. A sends a trigonometric polynomial $f(t) = \sum_j a_j e^{ijt}$ to $Af(t) = \sum_{j>0} a_j e^{ijt}$.

We already know that the operator A is very closely related to the conjugate function operator.

Actually these two identities are very easy to check for $f \in \mathcal{T}$

$$\tilde{f} = -i(Af-\overline{A\bar{f}}) \quad \text{and} \quad Af = (-\tilde{\tilde{f}}+i\tilde{f})/2$$

They prove the following lemma

LEMMA 8.10. w *is a Helson-Szegö weight if and only if* A *is (or can be uniquely extended to) a bounded operator in* $L^2(w)$.

This can easily be translated into the following

THEOREM 8.11. w *is a Helson-Szegö weight if and only if there exists* $\alpha < 1$ *such that, for every* $P,Q \in \mathcal{P}(0)$

$$\left|\operatorname{Re}\int_{-\pi}^{\pi} P(e^{it})\,e^{-it}Q(e^{it})w(t)\,dt\right| \le \alpha\,\|P\|_w\|Q\|_w$$

Proof: Every $f \in \mathcal{T}$ is of the form $f(t) = P(e^{it}) + e^{it}\overline{Q(e^{it})}$ for some $P,Q \in \mathcal{P}(0)$ and, given this representation, $Af(t) = P(e^{it})$, and

$$\|f\|_w^2 = \left(P(e^{it}) + e^{it}\overline{Q(e^{it})}\,\middle|\,P(e^{it}) + e^{it}\overline{Q(e^{it})}\right)_w =$$

$$= \|P\|_w^2 + \|Q\|_w^2 + 2\operatorname{Re}\int_{-\pi}^{\pi} P(e^{it})e^{-it}Q(e^{it})\,w(t)\,dt$$

If the inequality in the statement holds with some $\alpha < 1$, then

$$||f||_w^2 \geq ||P||_w^2 + ||Q||_w^2 - 2\alpha ||P||_w ||Q||_w =$$

$$= (1-\alpha^2)\ ||P||_w^2 + (\alpha||P||_w - ||Q||_w)^2 \geq (1-\alpha^2)\ ||P||_w^2 =$$

$$= (1-\alpha^2)\ ||Af||_w^2. \text{ Thus}$$

$$||Af||_w^2 \leq (1-\alpha^2)^{-1}\ ||f||_w^2$$

and, consequently, w is a Helson-Szegö weight.

Conversely, let w be a Helson-Szegö weight. We want to get an inequality like the one in the statement of the theorem. Of course, we just need to look at $P,Q \in \mathcal{P}(0)$ with $||P||_w = ||Q||_w = 1$. If we call $f(t) = P(e^{it}) + e^{it}Q(e^{it})$, we know that $||P||_w^2 \leq C||f||_w^2$ where, of course, we may take $C > 1$. Thus $||f||_w^2 \geq 1/C$. Now, using again the formula for $||f||_w^2$ given at the beginning of the proof, we get

$$2(1+\mathrm{Re}\int_{-\pi}^{\pi} P(e^{it})\ e^{-it}Q(e^{it})w(t)\ dt) \geq 1/C$$

or, what is the same,

$$\mathrm{Re}\int_{-\pi}^{\pi} P(e^{it})e^{-it}Q(e^{it})w(t)dt \geq -(1-(1/2C))$$

Since this is also valid for $-P$, we conclude that

$$\left|\mathrm{Re}\int_{-\pi}^{\pi} P(e^{it})e^{-it}Q(e^{it})w(t)\ dt\right| \leq 1-(1/2C) = \alpha < 1. \quad \square$$

THEOREM 8.12. Define for $|z| < 1$

$$\Phi(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log(w(t)^{1/2})dt \right)$$

We know that $\Phi \in H^2$ *is an outer function and* $|\Phi(e^{it})|^2 = w(t)$ *for a.e.* t. *Let* $w(t)\Phi(e^{it})^{-2} = e^{i\phi(t)}$. *Then* w *is a Helson-Szegö weight if and only if the distance in* L^∞ *from* $e^{i\phi}$ *to* H^∞ *is strictly smaller than* 1.

Proof: According to theorem 8.11, w is a Helson-Szegö weight if and only if there is $\alpha < 1$ such that for every $P,Q \in \mathcal{P}(0)$:

$$\left| \mathrm{Re} \int_{-\pi}^{\pi} \Phi(e^{it})P(e^{it})\Phi(e^{it})e^{-it}Q(e^{it})e^{i\phi(t)}dt \right| \le \alpha\|P\|_w\|Q\|_w$$

This is equivalent to saying that, for every $P, Q \in \mathcal{P}(0)$:

$$\text{(8.13)} \quad \left| \int_{-\pi}^{\pi} \Phi(e^{it})P(e^{it})\Phi(e^{it})e^{-it}Q(e^{it})e^{i\phi(t)}dt \right| \le \alpha\|\Phi P\|_{H^2}\|\Phi Q\|_{H^2}$$

(just consider $e^{i\tau}P$ instead of P for an appropriate real number τ). It follows from theorem 7.9 that $\Phi\cdot\mathcal{P}(0)$ is dense in $H^2(0)$ and also $e^{-it}\Phi(e^{it})\cdot\mathcal{P}(0)$ is dense in H^2, so that (8.13) is equivalent to

$$\left| \int_{-\pi}^{\pi} e^{i\phi(t)}G(e^{it})H(e^{it})dt \right| \le \alpha\, \|G\|_{H^2}\, \|H\|_{H^2}$$

for every $G \in H^2(0)$ and every $H \in H^2$.

We know that every $F \in H^1(0)$ can be factored as $F = G\cdot H$ with $G \in H^2(0)$, $H \in H^2$ and $\|G\|_{H^2} = \|H\|_{H^2} = \|F\|_{H^1}^{1/2}$ so, we can say, finally, that w is a Helson-Szegö weight if and only if there is some $\alpha < 1$ such that

$$\left| \int_{-\pi}^{\pi} e^{i\phi(t)}F(e^{it})\, dt \right| \le \alpha\|F\|_{H^1} \quad \text{for every } F \in H^1(0)$$

This is the same as saying that the norm of $e^{i\phi(t)}$ in $H^1(0)^* = L^\infty/H^\infty$ is bounded by $\alpha < 1$ or, in other terms, that

$$\mathrm{dist}_{L^\infty}(e^{i\phi}, H^\infty) = \inf_{g\in H^\infty} \|e^{i\phi} - g\|_\infty \le \alpha < 1$$

as we wanted to show. $\square$

We are now ready to prove the Helson-Szegö theorem, which reads as follows.

<u>*THEOREM 8.14.*</u> w <u>*is a Helson-Szegö weight if and only if*</u> $w(t) = e^{u(t)+\tilde{v}(t)}$ <u>*with*</u> u <u>*and*</u> v <u>*real, bounded and*</u> $\|v\|_\infty < \pi/2$.

<u>Proof</u>: First of all, let us see that the condition is sufficient. Since u is bounded, e^u will be a positive function bounded away from 0 and ∞. Therefore, we just need to see that if $w(t) = e^{\tilde{v}(t)}$ with v real and $\|v\|_\infty < \pi/2$, then w is a Helson-Szegö

weight. In this case

$$\Phi(z) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \cdot \frac{1}{2} \tilde{v}(t)\, dt\right)$$

and, consequently, $\Phi(z)^2 = \exp(P(\tilde{v}-iv))\cdot e^{i\tau}$ for some real number τ. Then $w(t)\Phi(e^{it})^{-2} = e^{-i\tau}\cdot e^{iv(t)}$. The constant factor is irrelevant and we are reduced to seeing that $\mathrm{dist}_{L^\infty}(e^{iv},H^\infty) < 1$. But this is quite easy. We know that $\|v\|_\infty < \pi/2$, so that there is $\varepsilon > 0$ such that $|v(t)| \leq (\pi/2)-\varepsilon$ for a.e.t. Then $|e^{iv(t)} - \sin\varepsilon|^2 =$ $= 1 + \sin^2\varepsilon - 2\cos v(t)\sin\varepsilon \leq 1 + \sin^2\varepsilon - 2\sin^2\varepsilon = 1 - \sin^2\varepsilon$, since $\cos v(t) \geq \sin\varepsilon$. We have shown that $\mathrm{dist}_{L^\infty}(e^{iv},H^\infty) < 1$ and, consequently, w is a Helson-Szegö weight.

Next we shall prove the necessity. We start with a Helson-Szegö weight w and consider the functions Φ and ϕ associated to w in the statement of theorem 8.12. This theorem tells us that $\inf_{g\in H^\infty}\|e^{i\phi}-g\|_\infty = \alpha < 1$.

There will exist some $H \in H^\infty$ such that

$$|w(t)\Phi(e^{it})^{-2}-H(e^{it})| = |e^{i\phi(t)}-H(e^{it})| \leq \alpha < 1 \quad \text{for a.e.t.}$$

Since $|\Phi(e^{it})|^2 = w(t)$, it follows that

$$|\Phi(e^{it})^2H(e^{it})-w(t)| \leq \alpha w(t) \quad \text{for a.e.t.}$$

Note also that $w(t) > 0$ for a.e.t, since $\int_{-\pi}^{\pi} \log w(t)\, dt > -\infty$. We conclude that the boundary values $\Phi(e^{it})^2H(e^{it})$ of the H^1 function Φ^2H are, for almost every t, in the sector $|\arg z| \leq \arc\sin\alpha < \pi/2$. Since the values $\Phi(z)^2\cdot H(z)$ for $|z| < 1$ are obtained from the boundary values by means of the Poisson integral, they will also belong to the same sector. In particular $\mathrm{Re}(\Phi^2H) \geq 0$ and theorem 7.7. implies that Φ^2H is an outer function. Thus $\Phi^2H = e^{i\tau}e^{U+iV}$ where τ is a real number, U is a real harmonic function and V its conjugate. But

$$|V(e^{it})+\tau| \leq \arc\sin\alpha \text{ and } (V(e^{it})+\tau)^{\sim} = -U(e^{it}) + U(0)$$

Set $v(t) = -V(e^{it})-\tau$. Then $|v(t)| < \pi/2$ and $U(e^{it}) = \tilde{v}(t) + c$ with c a real constant. Thus

$$|H(e^{it})|\ |\Phi(e^{it})|^2 = e^{\tilde{v}(t)+c}$$

From $|e^{i\phi(t)} - H(e^{it})| \le \alpha < 1$, it follows that $0 < 1/(1+\alpha) \le$ $\le 1/|H(e^{it})| \le 1/(1-\alpha) < \infty$. We can set $e^c/|H(e^{it})| = e^{u(t)}$, with u real and bounded. Then $w(t) = |\Phi(e^{it})|^2 = e^{u(t)+\tilde{v}(t)}$ with $\|v\|_\infty \le \arcsin \alpha < \pi/2$. $\square$

9. THE DUAL OF H^1.

Every $f \in L^\infty(T)$ gives rise to a functional $\Lambda_f \in (H^1)^*$ obtained by assigning to each $F \in H^1$, the complex number

$$\Lambda_f(F) = \int_{-\pi}^{\pi} F(e^{it})f(t)dt$$

Conversely, every $\Lambda \in (H^1)^*$ is of the form $\Lambda = \Lambda_f$ for some $f \in L^\infty(T)$. The kernel of the mapping

$$L^\infty \longrightarrow (H^1)^*$$

$$f \longmapsto \Lambda_f$$

is $H^\infty(0)$, so that we can write

$$(H^1)^* = L^\infty/H^\infty(0)$$

Similarly

$$(H^1(0))^* = L^\infty/H^\infty.$$

The purpose of this section is to characterize these duals as spaces of functions. We shall look at $H^1(0)$.

Every continuous $\mathbb{C}$-linear functional over $H^1(0)$ is of the form

$$\Lambda(F) = \mathrm{Re}\Lambda(F) + i\mathrm{Re}\Lambda(-iF) = L(F) + iL(-iF)$$

where L is a continuous $\mathbb{R}$-linear functional.

Conversely, every continuous $\mathbb{R}$-linear functional L is of the form $L(F) = \mathrm{Re}\Lambda(F)$ for some continuous $\mathbb{C}$-linear functional Λ. Therefore, for some real functions $\phi, \psi \in L^\infty$

$$L(F) = \mathrm{Re}\Lambda(F) = \mathrm{Re} \int_{-\pi}^{\pi} F(e^{it})(\phi(t)+i\psi(t))\, dt =$$

$$= \lim_{r\to 1} \mathrm{Re} \int_{-\pi}^{\pi} F(re^{it})(\phi(t)+i\psi(t))\, dt =$$

$$= \lim_{r\to 1} \int_{-\pi}^{\pi} (\mathrm{Re}F(re^{it})\phi(t) - \mathrm{Im}F(re^{it})\psi(t))\, dt =$$

$$= \lim_{r\to 1} \int_{-\pi}^{\pi} \mathrm{Re}F(re^{it})(\phi(t)+\tilde{\psi}(t))\, dt$$

since, observing that $F(0) = 0$, we have

$$\int_{-\pi}^{\pi} (\mathrm{Re}F(re^{it})\tilde{\psi}(t) + \mathrm{Im}F(re^{it})\psi(t))\, dt =$$

$$= \int_{-\pi}^{\pi} \mathrm{Im}(F(re^{it})(\psi(t)+i\tilde{\psi}(t)))\, dt = 0$$

The correspondence $F \longmapsto \mathrm{Re}F$ is a one to one mapping from $H^1(0)$ onto $\mathrm{Re}H^1(0)$. We consider $\mathrm{Re}H^1(0)$ as a real Banach space with the norm

$$\|\mathrm{Re}F\| = \|F\|_{H^1}.$$

The continuous $\mathbb{R}$-linear functionals on $H^1(0)$ correspond to the continuous functionals on $\mathrm{Re}H^1(0)$. So, our problem is really to determine the real space $(\mathrm{Re}H^1(0))^*$.

We already know that the mapping

$$\mathrm{Re}L^\infty + (\mathrm{Re}L^\infty)^\sim \longrightarrow (\mathrm{Re}H^1(0))^*$$

sending the function $\phi + \tilde{\psi}$ into the functional L given by:

$$L(\mathrm{Re}F) = \lim_{r\to 1} \int_{-\pi}^{\pi} \mathrm{Re}F(re^{it})(\phi(t)+\tilde{\psi}(t))dt = \mathrm{Re} \int_{-\pi}^{\pi} F(e^{it})(\phi(t)+i\psi(t))dt$$

is onto. Let us see what is the kernel of this mapping. If

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \mathrm{Re}F(re^{it})(\phi(t)+\tilde{\psi}(t))\, dt = 0$$

for every $F \in H^1(0)$, then taking $F(z) = z^n$, $n = 1, 2, 3, \ldots$, we obtain

$$\int_{-\pi}^{\pi} (\cos n\, t)(\phi(t) + \tilde{\psi}(t))\, dt = 0$$

and, taking $F(z) = iz^n$, $n = 1, 2, 3, \ldots$, we obtain

$$\int_{-\pi}^{\pi} (\sin n\, t)(\phi(t) + \tilde{\psi}(t))\, dt = 0$$

It follows that $\phi(t) + \tilde{\psi}(t)$ is actually a constant. We can write

$$(\mathrm{Re}H^1(0))^* = (\mathrm{Re}L^\infty + (\mathrm{Re}L^\infty)^\sim)/\mathbb{R}.$$

the norm in $\mathrm{Re}L^\infty + (\mathrm{Re}L^\infty)^\sim$ being

$$\inf(\|\phi\|_\infty + \|\psi\|_\infty)$$

where the infimum is taken over all representations of the function as $\phi + \tilde{\psi}$ with $\phi, \psi \in L^\infty$.

What we have done is to transform the problem of characterizing the dual space, into the problem of characterizing those functions g which can be written in the form

$$g = \phi + \tilde{\psi}$$

with $\phi, \psi \in L^\infty$. These functions will turn out to be the functions of bounded mean oscillation (B.M.O.).

DEFINITION 9.1. Let g *be a locally integrable* 2π-*periodic function on* $\mathbb{R}$. *We shall say that* g *is of bounded mean oscillation if and only if there is a constant* C *such that for every bounded interval* $I \subset \mathbb{R}$:

$$\frac{1}{|I|}\int_I |g(t) - g_I|\, dt \le C \tag{9.2}$$

where g_I *stands for the average* $\frac{1}{|I|}\int_I g(t)\, dt$.

The space formed by all 2π-*periodic functions of bounded mean oscillation will be denoted by* B.M.O. (T) *or, simply, by* B.M.O.

Given $g \in$ B.M.O., the infimum of all the constants C for which (9.2) holds will be denoted by $\|g\|_*$. The mapping $g \longmapsto \|g\|_*$ is a seminorm on B.M.O., vanishing only for the constants. Thus, in order to get a normed space, we have to consider the quotient of B.M.O. modulo constants. We shall call this space also B.M.O., hoping that this ambiguity does not cause any problem.

LEMMA 9.3 *For a locally integrable 2π-periodic function g to be in B.M.O., it is sufficient that condition (9.2) holds for all intervals I having length $|I| \leq 2\pi$. In that case, actually, $\|g\|_* \leq 6C$.*

Proof: Suppose I is an interval with $|I| > 2\pi$. Let n be an integer such that $n.2\pi \leq |I| < (n+1)\,2\pi$. Take an interval $J \supset I$ with $|J| = (n+1).2\pi$. Then

$$\frac{1}{|I|}\int_I |g(t)-g_I|\,dt \leq \frac{n+1}{n}\,\frac{1}{|J|}\int_J |g(t)-g_I|\,dt \leq$$

$$\leq \frac{2}{|J|}\int_J |g(t)-g_I|\,dt \leq \frac{2}{|J|}\int_J |g(t)-g_J|\,dt + 2|g_J-g_I|.$$

But $|g_J-g_I| = \left|\frac{1}{|I|}\int_I (g(t)-g_J)dt\right| \leq \frac{2}{|J|}\int_J |g(t)-g_J|\,dt$

Therefore:

$$\frac{1}{|I|}\int_I \left|g(t)-g_I\right|\,dt \leq \frac{6}{|J|}\int_J |g(t)-g_J|dt = \frac{6}{2\pi}\int_{-\pi}^{\pi}\left|g(t)-\frac{1}{2\pi}\int_{-\pi}^{\pi} g(s)ds\right| dt \leq$$

$\leq 6C.$ $\square$

LEMMA 9.4. *For g as in lemma 9.3, to be in B.M.O. it is even sufficient that condition (9.2) holds for all intervals I having length $|I| \leq \delta$ for some $\delta > 0$.*

Proof: We can assume $\delta < 2\pi$. Let I be an interval with $\delta < |I| \leq 2\pi$. Then

$$\frac{1}{|I|}\int_I |g(t)-g_I|\,dt \leq \frac{2}{|I|}\int_I |g| \leq \frac{2}{\delta}\int_{-\pi}^{\pi}|g(t)|\,dt. \square$$

LEMMA 9.5. *For g as above to be in B.M.O., it is sufficient that there are: a constant C, and for every interval I, a constant C_I such that:*

$$\frac{1}{|I|}\int_I |g(t) - C_I|\, dt \le C.$$

Proof:

$$\frac{1}{|I|}\int_I |g(t)-g_I|\, dt \le \frac{1}{|I|}\int_I |g(t)-C_I|\, dt + |C_I-g_I| \le$$

$$\le \frac{2}{|I|}\int_I |g(t)-C_I|\, dt \le 2C. \square$$

The fact that $L^\infty+(L^\infty)^\sim \subset$ B.M.O. is fairly easy

THEOREM 9.6. Let $g = \phi + \tilde{\psi}$ *with* ϕ *and* ψ 2π-*periodic* L^∞-*functions. Then* $g \in$ B.M.O. *Actually*

$$||g||_* \le C(||\phi||_\infty+||\psi||_\infty)$$

with C *an absolute constant.*

Proof: It is obvious that $L^\infty \subset$ B.M.O. and $||\phi||_* \le 2||\phi||_\infty$. So, the only non-trivial part of the theorem is that the conjugate function operator sends L^∞ boundedly into B.M.O. We shall see in the 5^{th} section of chapter II that this is a standard feature of singular integrals. The proof given there can easily be adapted to our present situation. We shall not give the details here, since we shall find another proof of this result shortly. $\square$

LEMMA 9.7. Let ϕ *be a real* 2π-*periodic locally* L^2 *function. Then for* $|z| < 1$:

$$P(\phi^2)(z)-(P(\phi)(z))^2 = \frac{1}{8\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(\phi(t)-\phi(s))^2\,\frac{1-|z|^2}{|z-e^{it}|^2}\,\frac{1-|z|^2}{|z-e^{is}|^2}\,dt\,ds.$$

Proof: It is enough to look at the right hand side after expanding $(\phi(t)-\phi(s))^2 = \phi(t)^2+\phi(s)^2 - 2\phi(t)\phi(s)$. $\square$

DEFINITION 9.8. For ϕ *as in the lemma, we define*

$$\mathcal{N}(\phi) = \sup_{|z|<1}\,(P(\phi^2)(z) - (P(\phi)(z))^2)^{1/2}$$

The following properties are immediately verified

a) $N(\lambda\phi) = |\lambda|\ N(\phi)$ for every $\lambda \in \mathbb{R}$

b) $N(\phi+\psi) \leq N(\phi) + N(\psi)$ (use the lemma and Minkowski's inequality).

c) $N(\phi) = 0$ if and only if ϕ is a constant.

It follows that N is a seminorm on the space of functions on which it is finite.

We can see rather easily that $N(g) < \infty$ is a necessary condition for g to be of the form $g = \phi + \tilde{\psi}$ with ϕ and ψ bounded. Indeed:

THEOREM 9.9. Let ϕ and ψ be real 2π-periodic L^∞ functions. Then

$$N(\phi+\tilde{\psi}) \leq \sqrt{2}(\|\phi\|_\infty+\|\psi\|_\infty)$$

Proof: Lemma 9.7. implies that $N(\phi) \leq \sqrt{2}\ \|\phi\|_\infty$. On the other hand,

$$(P(\psi+i\tilde{\psi}))^2 = (P(\psi))^2 - (P(\tilde{\psi}))^2 + 2iP(\psi).P(\tilde{\psi})$$

is a holomorphic function belonging to H^p for every $p < \infty$.

Thus, it will be the Poisson integral of its boundary function, and the same will happen to its real part. Therefore

$$(P(\psi)(z))^2-(P(\tilde{\psi})(z))^2 = P(\psi^2-\tilde{\psi}^2)(z) = P(\psi^2)(z)-P(\tilde{\psi}^2)(z).$$

so that

$$P(\tilde{\psi}^2)(z) - (P(\tilde{\psi})(z))^2 = P(\psi^2)(z) - (P(\psi)(z))^2$$

and, consequently:

$$N(\tilde{\psi}) = N(\psi) \leq \sqrt{2}\ \|\psi\|_\infty. \quad \square$$

Next, we shall see that the finiteness of $N(g)$ implies that $g \in$ B.M.O.

THEOREM 9.10. For ϕ *real* 2π-*periodic*

$$\|\phi\|_* \le C\, N(\phi)$$

with C *an absolute constant*

Proof: For any interval I

$$\frac{1}{|I|}\int_I |\phi(t)-\phi_I|\,dt \le \left(\frac{1}{|I|}\int_I (\phi(t)-\phi_I)^2\,dt\right)^{1/2} =$$

$$= \left(\frac{1}{|I|}\int_I \phi(t)^2 dt - \phi_I^2\right)^{1/2} =$$

$$= \left(\frac{1}{2}\cdot\frac{1}{|I|^2}\int_I\int_I (\phi(t)-\phi(s))^2\,dt\,ds\right)^{1/2}$$

We may assume $I = [-a,a]$ with $0 < a \le \pi$. Then, we use the formula

$$P(\phi^2)(r)-(P(\phi)(r))^2 = \frac{1}{8\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(\phi(t)-\phi(s))^2\frac{1-r^2}{1+r^2-2r\cos s}\,\frac{1-r^2}{1+r^2-2r\cos t}\,ds\,dt$$

for $0 \le r < 1$. This follows from lemma 9.7.

If $|t| \le a$, we have:

$$\frac{1-r^2}{1+r^2-2r\cos t} = \frac{1-r^2}{(1-r)^2+2r(1-\cos t)} = \frac{1-r^2}{(1-r)^2+4r\sin^2(t/2)} \ge$$

$$\ge \frac{1-r^2}{(1-r)^2+4r\sin^2(a/2)} \ge \frac{1-r}{(1-r)^2+4\sin^2(a/2)}$$. In particular, choosing

$r = 1-\sin(a/2)$, we have: $\dfrac{1-r^2}{1+r^2-2r\cos t} \ge \dfrac{\sin(a/2)}{5\sin^2(a/2)} = \dfrac{1}{5\sin(a/2)} \ge$

$\ge \dfrac{2}{5a}$. Thus, for our choice of $r = 1-\sin(a/2)$:

$$P(\phi^2)(r)-(P(\phi)(r))^2 \ge \frac{1}{8\pi^2}\int_{-a}^{a}\int_{-a}^{a}(\phi(t)-\phi(s))^2\left(\frac{2}{5a}\right)^2 ds\,dt =$$

$$= \left(\frac{2}{5\pi}\right)^2\frac{1}{2|I|^2}\int_I\int_I(\phi(t)-\phi(s))^2\,ds\,dt \ge$$

$$\ge \left(\frac{2}{5\pi}\right)^2\left(\frac{1}{|I|}\int_I|\phi(t)-\phi_I|\,dt\right)^2$$

We conclude that, for $|I| \leq 2\pi$:

$$\frac{1}{|I|}\int_I |\phi(t) - \phi_I|\, dt \leq \frac{5\pi}{2} N(\phi)$$

As in lemma 9.3, if $|I| > 2\pi$:

$$\frac{1}{|I|}\int_I |\phi(t) - \phi_I|\, dt \leq 15\cdot\pi\, N(\phi)$$

So, $\phi \in$ B.M.O. with $\|\phi\|_* \leq 15\cdot\pi\, N(\phi)$. □

Combining the last two theorems, we obtain the promised alternative proof of theorem 9.6.:

$$\|\phi + \tilde{\psi}\|_* \leq 15\cdot\pi\, N(\phi+\tilde{\psi}) \leq 15\cdot\sqrt{2}\cdot\pi(\|\phi\|_\infty + \|\psi\|_\infty)$$

The difficult part of the duality theorem is to show that every f with $N(f) < \infty$ can indeed be written in the form $f = \phi + \tilde{\psi}$ with $\phi, \psi \in L^\infty$, so that f gives rise to a continuous linear functional over $\mathrm{Re}H^1(0)$. To reduce the proof of this fact to its essential core we shall need several simple preliminary lemmas.

LEMMA 9.11. Let $W \in C^2(\overline{D})$ *with* $W(0) = 0$. *Then:*

$$\int_{-\pi}^{\pi} W(e^{it})\, dt = \iint_D \left(\log \frac{1}{|z|}\right) \Delta W(z)\, dx\, dy$$

Proof: $W \in C^2(\overline{D})$ means that W is C^2 on some open set containing $\overline{D}$.

For small $\varepsilon > 0$, we apply Green's theorem on the domain $D_\varepsilon = \{z \in \mathbb{C}: \varepsilon < |z| < 1\}$, obtaining

$$\iint_{\varepsilon<|z|<1} \left(\log\frac{1}{|z|}\right) \Delta W(z)dxdy = \iint_{\varepsilon<|z|<1} \left(\left(\log\frac{1}{|z|}\right)\Delta W(z) -\right.$$

$$\left.- \left(\Delta \log\frac{1}{|z|}\right) W(z)\right) dxdy = \int_{\partial D_\varepsilon} \left(\log\frac{1}{|z|}\right)\frac{\partial}{\partial n} W - \left(\frac{\partial}{\partial n}\log\frac{1}{|z|}\right) W(z) d\sigma(z) =$$

$$= \int_{-\pi}^{\pi} W(e^{it})dt + \varepsilon(\log\varepsilon)\int_{-\pi}^{\pi} \frac{\partial}{\partial r}(W(re^{it}))\Big|_{r=\varepsilon} dt - \int_{-\pi}^{\pi} W(\varepsilon e^{it})\, dt$$

We have denoted by $\frac{\partial}{\partial n}$ the derivative in the direction of the outer normal.

Now, since $W(0) = 0$, we have $|W(\varepsilon e^{it})| = O(\varepsilon)$ and we only have to make $\varepsilon \to 0$ to get what we wanted. $\square$

REMARK 9.12. Observe that the lemma is valid, with the same proof, for $W(z) = |z|\, W_1(z)$ *with* $W_1 \in C^2(\bar{D})$

LEMMA 9.13. Let U *and* V *be harmonic functions on* $D(0,R)$, $R > 1$ *and let* $U(0) = 0$. *Then*

$$\int_{-\pi}^{\pi} U(e^{it})\, V(e^{it})\, dt = 2 \iint_D \left(\log \frac{1}{|z|} \right) < \nabla U | \nabla V > dxdy$$

where $< | >$ *is used to denote the scalar product and* ∇ *to denote the gradient*

Proof: We just need to apply lemma 9.11 with $W = UV$, and observe that

$$\Delta(UV) = (\Delta U)V + U(\Delta V) + 2 < \nabla U | \nabla V > = 2 < \nabla U | \nabla V >. \square$$

LEMMA 9.14. Let $F \in H(D(0,R))$, $R > 1$ *such that* $F(z) = 0$ *only for* $z = 0$. *Then* $|F(z)|$ *is* C^∞ *in* $0 < |z| < R$, *and*

$$\Delta(|F|) = |F|^{-1}\, |\nabla \mathrm{Re} F|^2$$

Proof: Writing $F(z) = U(z) + iV(z)$ with U and V real, we see that $|F(z)| = (U(z)^2 + V(z)^2)^{1/2}$ is clearly C^∞ for $z \neq 0$. To compute the Laplacian is a simple exercise. $\square$

LEMMA 9.15. Let $F \in H(D(0,R))$, $R > 1$, *and suppose that* F *does not vanish except at* $z = 0$, *where it has a simple zero. Then:*

$$\int_{-\pi}^{\pi} |F(e^{it})|\, dt = \iint_D \left(\log \frac{1}{|z|}\right) |F(z)|^{-1} |\nabla \mathrm{Re} F(z)|^2\, dxdy.$$

Proof: We just need to apply remark 9.12. with $W(z) = F(z)$ and use lemma 9.14. $\square$

Now we are ready to prove that the condition $N(f) < \infty$ is sufficient for f to be of the form $f = \phi + \tilde{\psi}$ with $\phi, \psi \in L^\infty$. In order to prove this, it is enough to see that, if $N(f) < \infty$, then, for every $F \in H^1(0)$ there exists

(9.16) $$\lim_{r\to 1}\int_{-\pi}^{\pi} \operatorname{Re}F(re^{it})\, f(t)\, dt$$

and provides a continuous linear functional on $H^1(0)$. Indeed, we know that this implies the existence of $\phi,\psi \in L^\infty$ such that, for every $F \in H^1(0)$

$$\lim_{r\to 1}\int_{-\pi}^{\pi} \operatorname{Re}F(re^{it})\, (f(t) - \phi(t) - \tilde{\psi}(t))\, dt = 0$$

In particular, taking $F(z) = z^n$, $n = 1, 2, \ldots$ and $F(z) = iz^n$ $n = 1, 2, \ldots$, we arrive at $f(t) - \phi(t) - \tilde{\psi}(t) = \text{constant}$.

On the other hand, in order to see that the limit (9.16) exists and provides a continuous linear functional on $H^1(0)$, it is enough to show that there is a constant C such that for every $F \in H^1(0)$ and every $0 < r < 1$:

(9.17) $$\left| \int_{-\pi}^{\pi} \operatorname{Re}F(re^{it})\, f(t)\, dt \right| \le C\, \|F\|_{H^1}.$$

Indeed, suppose we know that (9.17) holds independently of r. Then, given $F \in H^1(0)$ and given $\varepsilon > 0$, there will exist $r_0 < 1$ such that for every r, $r_0 < r < 1$

$$\int_{-\pi}^{\pi} |F(re^{it}) - F(e^{it})|\, dt < \varepsilon.$$

If r_1, r_2 are such that $r_0 < r_1 < r_2 < 1$, then $r = r_1/r_2$ satisfies $r_0 < r < 1$ and $G(z) = F(z) - F(rz)$ is a function belonging to $H^1(0)$ with $\|G\|_{H^1} < \varepsilon$. Therefore

$$\left|\int_{-\pi}^{\pi} \operatorname{Re}F(r_2e^{it})f(t)dt - \int_{-\pi}^{\pi} \operatorname{Re}F(r_1e^{it})f(t)dt\right| = \left|\int_{-\pi}^{\pi} \operatorname{Re}G(r_2e^{it})f(t)dt\right| \le$$

$\le C\,\varepsilon$

and we conclude that the limit in (9.16) exists and provides a continuous linear functional on $H^1(0)$.

Also note that, in order to see (9.17), it is enough to show that there exists a constant C such that for every $F \in H^1(0)$, every $0 < r < 1$ and every $0 < s < 1$:

(9.18) $$\left|\int_{-\pi}^{\pi} \operatorname{Re}F(re^{it})\, P(f)\, (se^{it})\, dt\right| \le C\, \|F\|_{H^1}$$

We can even assume that $F \in H^1(0)$ has a simple zero at $z = 0$ and is different from 0 elsewhere. Indeed, we can write $F(z) = zG(z)$ with $G \in H^1$ and then, if $G(z) = B(z)\,H(z)$ is the Riesz factorization of G, we have $F(z) = z((B(z)-1)/2)\,H(z)+z((B(z)+1)/2)H(z)$. The two terms in this sum are functions in $H^1(0)$ having just a simple zero at $z = 0$ and with H^1 norms bounded by $\|F\|_{H^1}$.

Fix $F \in H^1(0)$ with just a simple zero at $z = 0$, and fix also r and s. Set

$$U(z) = \operatorname{Re}F(rz) \text{ and}$$

$$V(z) = P(f)(sz)$$

Then, after writing the integral in (9.18) as in lemma 9.13, we see that it is enough to prove the inequality

$$\text{(9.19)} \qquad \left| \iint_D \left(\log \frac{1}{|z|} \right) < \nabla U | \nabla V > dxdy \right| \le C\, \|F\|_{H^1}$$

with C independent of r,s or F. We shall eventually prove 9.19 with $C = k\mathcal{N}(f)$ and k independent of r,s or F.

As a preliminary step we use the Cauchy-Schwarz inequality, obtaining as a bound for the left hand side of (9.19):

$$\iint_D \left(\log \frac{1}{|z|} \right)^{1/2} \frac{|\nabla U|}{|F(rz)|^{1/2}} \left(\log \frac{1}{|z|} \right)^{1/2} |\nabla V|\,|F(rz)|^{1/2} dxdy \le$$

$$\le \left[\iint_D \left(\log \frac{1}{|z|} \right) \frac{|\nabla U|^2}{|F(rz)|}\, dxdy \right]^{\frac{1}{2}} \cdot \left[\iint_D \left(\log \frac{1}{|z|}\right) |\nabla V|^2 |F(rz)|\, dxdy \right]^{\frac{1}{2}}$$

According to lemma 9.15, the first factor equals

$$\left[\int_{-\pi}^{\pi} |F(re^{it})|\, dt \right]^{1/2} \le \|F\|_{H^1}^{1/2}.$$

Thus, it will be enough to show that the second factor is bounded by $C\mathcal{N}(f)\,\|F\|_{H^1}^{1/2}$ with C an absolute constant. We shall indeed show that

$$\text{(9.20)} \quad \iint_D |z| \left(\log \frac{1}{|z|} \right) |\nabla V(z)|^2 |G(z)|\, dxdy \le C\mathcal{N}(f)^2\, \|G\|_{H^1}.$$

holds for every $G \in H^1$, C being an absolute constant.
(9.20) will be a consequence of the following two results:

THEOREM 9.21. The measure $d\mu(z) = |z| \left(\log \frac{1}{|z|} \right) |\nabla V(z)|^2 dxdy$ *in* D *satisfies the following condition:*

If for every interval I, *we denote by* $R(I)$ *the set*

$$R(I) = \{re^{it} : t \in I,\ 1 - \frac{|I|}{2\pi} < r < 1\}, \quad \textit{then}$$

$$\mu(R(I)) \leq C\mathcal{N}(f)^2 \, |I|$$

with C *an absolute constant*

In general, a positive Borel measure ν on D having $\sup_I (\nu(R(I))/|I|) < \infty$ is customarily called a Carleson measure and the sup. above is called the Carleson constant of the measure.

THEOREM 9.22. Let ν *be a Carleson measure with Carleson constant* $k(\nu)$. *Then*

a) *For every* $1 < p < \infty$ *and every* $g \in L^p(T)$

$$\iint_D |P(g)(z)|^p \, d\nu(z) \leq Ck(\nu) \int_{-\pi}^{\pi} |g(t)|^p \, dt$$

with C *independent of* ν *and* g.

b) *For every* $G \in H^1$:

$$\iint_D |G(z)| \, d\nu(z) \leq Ck(\nu) \int_{-\pi}^{\pi} |G(e^{it})| \, dt.$$

We give first the

Proof of theorem 9.22

We just need to prove a). Indeed, once we have a), if $G \in H^1$, we write $G = G_1 \cdot G_2$ with $G_1, G_2 \in H^2$ and $\|G_1\|_{H^2} = \|G_2\|_{H^2} = \|G\|_{H^1}^{1/2}$. Then

$$\iint_D |G(z)|\, d\nu(z) \le \left[\iint_D |G_1(z)|^2\, d\nu(z)\right]^{1/2}\left[\iint_D |G_2(z)|^2\, d\nu(z)\right]^{1/2} \le$$

$\le Ck(\nu)\, \|G_1\|_{H^2} \|G_2\|_{H^2} = Ck(\nu)\, \|G\|_{H^1}$, and we obtain b).

Before presenting the proof of a) let us say that the analog of a) for the euclidean space $\mathbb{R}^n$ is given as theorem 2.18 in chapter II. The proof that follows is based upon the same ideas.

For $0 < \alpha < \pi/2$ and for $e^{i\theta} \in T$, we shall denote by $\Gamma_\alpha(\theta)$ the Stolz region based on $e^{i\theta}$ with aperture α which is, by definition, the interior of the minimal convex set containing $e^{i\theta}$ and the disk $\overline{D(0,\sin\alpha)}$. For $f \in L^1(T)$ we define

$$N_\alpha f(\theta) = \sup\{|P(f)(z)|: z \in \Gamma_\alpha(\theta)\}.$$

$N_\alpha f(\theta)$ is the non-tangential Poisson maximal function of aperture α . Its euclidean counterpart will be denoted $P^*_{\nabla,\alpha}(f)$, but for the moment we do not want to introduce unnecessary complications.

The main fact is that

$$(9.23) \qquad N_\alpha f(\theta) \le C_\alpha Mf(\theta)$$

where

$$Mf(\theta) = \sup_{I \ni \theta} \frac{1}{|I|} \int_I |f(t)|\, dt$$

is the Hardy-Littlewood maximal function, to be systematically by studied in chapter II.

The proof of (9.23) is essentially the same as that of (4.15) in chapter II. We omit the details.

We shall use the operator $N = N_\alpha$ with $\alpha = \pi/6$, so that $\sin\alpha = 1/2$. Let $1 < p < \infty$ and $g \in L^p(T)$. We want to prove a). If $|z| \le \frac{1}{2}$, then

$$|P(g)(z)| \le \sup_{0<r<1/2} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} (P_r(t))^{p'} dt\right)^{1/p'} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(t)|^p\, dt\right)^{1/p}.$$

so that

$$\int_{|z|\leq 1/2} |P(g)(z)|^p \, d\nu(z) \leq C\nu(D) \int_{-\pi}^{\pi} |g(t)|^p dt \text{ and } \nu(D) \leq 2\pi k(\nu).$$

Thus, we can forget about the region $|z| \leq 1/2$ and concentrate on $|z| > 1/2$.

It is a simple geometrical observation that if $|P(g)(z)| > \lambda$ for a given z with $|z| > 1/2$, then $N(f)(\theta) > \lambda$ if $e^{i\theta}$ belongs to an arc centered at $z/|z|$ and having length $\geq c(1-|z|)$ where c is some geometric constant ($c = \sqrt{3}/2$ is enough)

Given $\lambda > 0$, let $E_\lambda = \{\theta \in [-\pi,\pi] : N(f)(\theta) > \lambda\}$, and let I_j be the connected components of E_λ. It follows from the last observation that

$$\{z: 1/2 \leq |z| < 1, |f(z)| > \lambda\} \subset \bigcup_j R(I_j^*)$$

where I_j^* is an interval concentric with I_j but having length $4\pi|I_j|$. Then

$$\nu(\{z: 1/2 \leq |z| < 1, |f(z)| > \lambda\}) \leq \sum_j \nu(R(I_j^*)) \leq$$

$$\leq \sum_j k(\nu) \, |I_j^*| \leq Ck(\nu) \sum_j |I_j| = Ck(\nu) \, |E_\lambda|$$

Next we shall use the fact that M is a bounded operator in $L^p(T)$. This fact can be established essentially as in section 1 of chapter II. After (9.23), N is also bounded in $L^p(T)$, and we can write

$$\int_{1/2\leq|z|<1} |P(g)(z)|^p d\nu(z) =$$

$$= p \int_0^\infty \lambda^{p-1} \nu(\{z: 1/2 \leq |z| < 1, |f(z)| > \lambda\}) \, d\lambda \leq$$

$$\leq Ck(\nu) \; p \int_0^\infty \lambda^{p-1} |E_\lambda| \, d\lambda = Ck(\nu) \int_{-\pi}^{\pi} (N(g)(t))^p dt \leq$$

$$\leq Ck(\nu) \int_{-\pi}^{\pi} |g(t)|^p \, dt. \ \square$$

We come finally to the

<u>Proof of theorem 9.21:</u> Observe, first of all, that

the inequality

(9.24) $\quad \mu(R(I)) \leq C\mathcal{N}(f)^2\ |I|$

needs to be established only for small intervals. The reason is that, given $0 < r_0 < 1$, we have:

(9.25) $$\iint_{|z|<r_0} |z|(\log \frac{1}{|z|})\ |\nabla V(z)|^2\ dxdy \leq C(r_0)\mathcal{N}(f)^2$$

so that, once we have proved (9.24) for small intervals, say for all I such that $|I| \leq \delta$ for some fixed $\delta > 0$, if we are given an interval J with $|J| > \delta$, we can write J as the union of intervals $I_1, \ldots, I_N$ with $|I_j| \leq \delta$ and $N \leq 2\pi/\delta$. Then, since $R(J) \subset R(I_1) \cup \ldots \cup R(I_N) \cup D(0,r_0)$ provided we take $1-(\delta/(2\pi)) < r_0 < 1$, we get

$$\mu(R(J)) \leq C\mathcal{N}(f)^2\ |J|$$

with a bigger C.

To prove (9.25) is very simple: if we write $f_0(t) = f(t) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f$ and $V_0(z) = P(f_0)(sz)$ we have $\nabla V(z) = \nabla V_0(z)$ since V and V_0 differ only by a constant. But, for $|z| \leq r_0 < 1$,

$$|\nabla V_0(z)| \leq \frac{C}{2\pi}\int_{-\pi}^{\pi} |f_0(t)|dt \leq \frac{C}{2\pi}\int_{-\pi}^{\pi} \left|f(t) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f\right|\ dt \leq$$
$$\leq C\ \|f\|_* \leq C\mathcal{N}(f)$$

by theorem 9.10.

Naturally the constant in the first inequality depends on r_0 and tends to ∞ as $r_0 \uparrow 1$. (9.25) follows after realizing that $|z|\ \log \frac{1}{|z|} \leq 1$.

To prove (9.24) we shall assume $|I| = h < 1/2$ and also that I is centered at 0. Then

$$R(I) \subset R_h = \{re^{it}: |t| < h/2 \text{ and } 1-h < r < 1\}$$

and the proof will be finished if we are able to show that

$$\mu(R_h) \leq C\mathcal{N}(f)^2 h$$

It will be more convenient to work in the upper half plane $\mathbb{R}^2_+$. To that effect we map the disk $|z| < 1$ onto $\mathbb{R}^2_+$ by means of the conformal mapping:

$$\Phi(z) = i\,\frac{1-z}{1+z}$$

On $\mathbb{R}^2_+$ we shall use the coordinate $\zeta = \xi + i\eta$, with $\xi,\eta \in \mathbb{R}$. The inverse of Φ will be the mapping

$$\Psi(\zeta) = \frac{i-\zeta}{i+\zeta}$$

For $1/2 \leq |z| < 1$ note that

$$|z| \log \frac{1}{|z|} \leq \frac{1}{2} \log \frac{1}{|z|^2} \leq \frac{1}{2}\left(\frac{1}{|z|^2} - 1\right) \leq 2(1-|z|^2)$$

and

$$1 - |z|^2 = 1 - \frac{\xi^2+(1-\eta)^2}{\xi^2+(1+\eta)^2} = \frac{4\eta}{\xi^2+(1+\eta)^2} \leq 4\eta$$

so that

$$|z| \log \frac{1}{|z|} \leq 8\eta \qquad \text{for } 1/2 \leq |z| < 1$$

A simple computation shows that

$$\Phi(R_h) \subset Q_h = \{\zeta = \xi + i\eta : |\xi| < h/2,\ 0 < \eta < h\}$$

Now, after writing $W(\zeta) = U(\Phi(\zeta))$, the change of variables formula implies

$$\iint_{R_h} |z|\left(\log \frac{1}{|z|}\right)\left(\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2\right) dx\,dy =$$

$$= \iint_{\Phi(R_h)} |\Psi(\zeta)| \left(\log \frac{1}{|\Psi(\zeta)|}\right)\left(\left(\frac{\partial W}{\partial \xi}\right)^2+\left(\frac{\partial W}{\partial \eta}\right)^2\right) d\xi\, d\eta$$

This is most easily seen by observing that the Jacobian of Ψ is $|\Psi'(\zeta)|^2$ and, if we write $V = \operatorname{Re} G$ with G holomorphic, then $|\nabla V(\Psi(\zeta))|^2|\Psi'(\zeta)|^2 = |G'(\Psi(\zeta))|^2|\Psi'(\zeta)|^2 = |(G\circ\Psi)'(\zeta)|^2 = |\nabla W(\zeta)|^2$

We arrive at:

$$\iint_{R_h} |z| \left(\log \frac{1}{|z|}\right) |\nabla V(z)|^2 \, dx\, dy \leq$$

$$\leq 8 \iint_{Q_h} \eta \left(\left(\frac{\partial W}{\partial \xi}\right)^2 + \left(\frac{\partial W}{\partial \eta}\right)^2\right) d\xi\, d\eta$$

Now $Q_h \subset \{\zeta \in \mathbb{R}^2_+ : |\zeta| < 2h\}$. Actually, for $\zeta \in Q_h$, we have $1 - \frac{|\zeta|}{2h} > \frac{1}{4}$

Then

$$\iint_{Q_h} \eta |\nabla W(\zeta)|^2 d\xi d\eta \leq 4 \iint_{\substack{|\zeta|<2h \\ \eta \geq 0}} \eta\left(1 - \frac{|\zeta|}{2h}\right) |\nabla W(\zeta)|^2 \, d\xi d\eta$$

Since W is harmonic: $\Delta(W^2) = 2\,|\nabla W|^2$. We just need to show that:

$$\iint_{\substack{|\zeta|<2h \\ \eta>0}} \eta\left(1 - \frac{|\zeta|}{2h}\right) \Delta(W^2)\, d\xi d\eta \leq C\mathcal{N}(f)^2 h$$

Remember that $\mathcal{N}(f) = \sup(P(f^2)(z)-(P(f)(z))^2)^{1/2}$ so that, if we set $A(z) = P(f^2)(z)-(P(f)(z))^2$, then

$$0 \leq A(z) \leq \mathcal{N}(f)^2$$

Now $W(\zeta)^2 = V(z)^2 = (P(f)(sz))^2 = P(f^2)(sz)-A(sz)$. Set $B(\zeta) = A(sz)$. It follows that

$$\Delta(W^2) = -\Delta B$$

and the integral to be estimated equals

$$-\iint_{\substack{|\zeta|<2h \\ \eta>0}} \eta\left(1 - \frac{|\zeta|}{2h}\right) \Delta B(\zeta)\, d\xi\, d\eta$$

which, using Green's theorem and writing Γ for the boundary of our domain, is seen to be equal to:

$$-\iint_{\substack{|\zeta|<2h \\ \eta>0}} B(\zeta)\Delta\left(\eta\left(1-\frac{|\zeta|}{2h}\right)\right) d\xi d\eta - \int_\Gamma \eta\left(1-\frac{|\zeta|}{2h}\right) \frac{\partial B}{\partial n}\, d\sigma(\zeta) +$$

$$+ \int_\Gamma B(\zeta) \frac{\partial\left(\eta\left(1-\frac{|\zeta|}{2h}\right)\right)}{\partial n}\, d\sigma(\zeta) \leq -\iint_{\substack{|\zeta|<2h \\ \eta>0}} B(\zeta)\Delta\left(\eta\left(1-\frac{|\zeta|}{2h}\right)\right) d\xi\, d\eta$$

The inequality is due to the fact that the first of the two integrals over Γ is zero, since $\eta(1-\frac{|\zeta|}{2h})$ vanishes on Γ, and the second one is ≤ 0, since $\frac{\partial}{\partial n}(\eta(1-\frac{|\zeta|}{2h}))\leq 0$ on Γ. Using the polar representation $\zeta=\rho e^{i\theta}$, the Laplacian becomes $\Delta=(\frac{1}{\rho}\frac{\partial}{\partial\rho})^2+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}$. Then it is very easy to compute

$$\Delta(\eta(1-\frac{|\zeta|}{2h})) = -\frac{3}{2}\frac{1}{h}\sin\theta$$

Finally

$$-\iint_{\substack{|\zeta|<2h\\ \eta>0}} B(\zeta)\Delta(\eta(1-\frac{|\zeta|}{2h}))\,d\xi d\eta = \frac{3}{2h}\iint_{\substack{|\zeta|<2h\\ \eta>0}} B(\zeta)\sin\theta\,d\xi\,d\eta \leq$$

$$\leq \frac{3}{2h}\mathcal{N}(f)^2\int_0^{2h}\int_0^{\pi}\sin\theta\,d\theta\,\rho\,d\rho = 6\mathcal{N}(f)^2h$$

as we wanted to prove $\square$

We have finally identified $(H^1(0))^*$ as the space of functions f on T for which $\mathcal{N}(f)<\infty$. In order to see that this space actually coincides with B.M.O., some more work is needed.

For the next result we shall use the fact that, for $f\in$ B.M.O.:

$$\sup_I\left(\frac{1}{|I|}\int_I|f(t)-f_I|^2\,dt\right)^{1/2}\leq C\,\|f\|_*$$

with C an absolute constant.

This is contained in corollary 3.10 of chapter II and is a consequence of the John-Nirenberg theorem, a basic result for B.M.O. functions, which shall be studied in chapter II within its most natural framework.

We shall also make use of the following observation: If $f\in$ B.M.O., and $I\subset J$ are concentric intervals, then:

$$|f_I-f_J|\leq 2\,\|f\|_*(1+\log_2(|J|/|I|))$$

This is clear if $|J|\leq 2\,|I|$ since

$$|f_I - f_J| \leq \frac{1}{|I|}\int_I |f(t)-f_J|\,dt \leq \frac{|J|}{|I|}\frac{1}{|J|}\int_J |f(t)-f_J|\,dt \leq 2\,\|f\|_*$$

The general case follows by writing

$$|f_I - f_J| \leq |f_I - f_{J_1}| + |f_{J_1} - f_{J_2}| + \dots + |f_{J_N} - f_J|$$

where $I \subset J_1 \subset J_2 \subset \dots \subset J_N \subset J$ and

$$|J| \leq 2\,|J_N| \leq \dots \leq 2^N\,|J_1| \leq 2^{N+1}\,|I|$$

We are ready to prove the following

THEOREM 9.26. For f *real and* 2π-*periodic*

$$\mathcal{N}(f) \leq C\,\|f\|_*$$

with C *an absolute constant.*

Proof: We have to obtain an estimate

$$P(f^2)(z) - (P(f)(z))^2 \leq C\,\|f\|_*^2$$

with C independent of f and of $z \in D$. It is clearly enough to consider $z = r$ with $0 \leq r < 1$.

According to lemma 9.7

$$P(f^2)(r) - (P(f)(r))^2 = \frac{1}{8\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)^2(f(s)-f(t))^2}{(1+r^2-2r\cos s)(1+r^2-2r\cos t)}\,ds\,dt$$

which, after integrating by parts and writing $1-r = u$, is seen to be bounded by:

$$C\int_0^\infty\int_0^\infty \frac{us^2}{(u^2+s^2)^2}\,\frac{ut^2}{(u^2+t^2)^2}\left(\frac{1}{4st}\int_{-s}^{s}\int_{-t}^{t}(f(\sigma)-f(\tau))^2\,d\sigma\,d\tau\right)ds\,dt$$

Call $I = [-s,s]$, $J = [-t,t]$. Then, we have to estimate

$$\left(\frac{1}{|J|}\frac{1}{|I|}\int_J\int_I (f(\sigma)-f(\tau))^2\,d\sigma\,d\tau\right)^{1/2} \leq$$

$$\leq \left(\frac{1}{|I|}\int_I |f(\sigma)-f_I|^2\,d\sigma\right)^{1/2} + \left(\frac{1}{|J|}\int_J |f(\tau)-f_J|^2\,d\tau\right)^{1/2} + |f_I - f_J| \leq$$

$$\leq C \|f\|_* (1 + |\log (|J|/|I|)|).$$

Finally

$$P(f^2)(r) - (P(f)(r))^2 \leq$$

$$\leq C \int_0^\infty \int_0^\infty \frac{u^2 s^2 t^2 \|f\|_*^2 (1+\log^2(s/t))}{(u^2+s^2)^2(u^2+t^2)^2} \, ds \, dt =$$

$$= C \int_0^\infty \int_0^\infty \frac{x^2 y^2 (1+2(\log x)^2 + 2(\log y)^2)}{(1+x^2)^2 (1+y^2)^2} \, dx \, dy \, \|f\|_*^2 =$$

$$= C \|f\|_*^2. \quad \square$$

All the work we have done in this section can be summarized in the following result:

THEOREM 9.27, Let f *be a real* 2π-*periodic function integrable on the torus. Then, the following conditions are equivalent:*

(a) *The mapping*

$$F \longmapsto \lim_{r\to 1} \int_{-\pi}^{\pi} \operatorname{Re} F(re^{it}) \, f(t) \, dt$$

is a continuous linear functional on $H^1(0)$

(b) $f = \phi + \tilde{\psi}$ *for certain* $\phi, \psi \in L^\infty$.

(c) $f \in$ B.M.O.

(d) $N(f) < \infty$. (*see definition* 9.8)

(e) $d\mu(z) = |z| \, (\log \frac{1}{|z|}) \, |\nabla P(f)(z)|^2 \, dx \, dy$

is a Carleson measure

Besides, the natural norms associated to f *in* (a), (b), (c), (d) *and* (e) *are all equivalent*. $\square$

10. THE CORONA THEOREM

In this section we are going to view H^∞ as a Banach algebra with pointwise multiplication. It is easy to see that all the algebra homomorphisms of H^∞ onto $\mathbb{C}$ are, actually, continuous. They form a weak-* closed subset S of the unit ball of the dual space $(H^\infty)^*$. S, with the weak-* topology is, consequently, a compact space which is customarily called the maximal ideal space of H^∞. The reason for this name is the one-to-one correspondence between S and the set of maximal ideals of H^∞ obtained by associating to an algebra homomorphism L from H^∞ onto $\mathbb{C}$, the maximal ideal $I = \{f \in H^\infty : L(f) = 0\}$.

The corona theorem is an important step in understanding the space S. For each $z \in D = \{z \in \mathbb{C}: |z| < 1\}$, the evaluation at z is an algebra homomorphism δ_z of H^∞ onto $\mathbb{C}$ given by $\delta_z(f) = f(z)$. The mapping

$$\begin{aligned} D &\longrightarrow S \\ z &\longmapsto \delta_z \end{aligned}$$

is a homeomorphism between D and a part of S, so that D can be considered as a subspace of S. The purpose of this section is to prove the following result, known as the corona theorem

THEOREM 10.1 D *is dense in* S.

We shall base our proof upon three lemmas. The first one is simply a reformulation of the condition that D is dense in S.

LEMMA 10.2. The following statements are equivalent:

a) D *is dense in* S.

b) *For every* $f_1, f_2, \ldots, f_n \in H^\infty$ *such that* $\inf_{z\in D} \max_j |f_j(z)| > 0$, *there exist* $g_1, g_2, \ldots, g_n \in H^\infty$, *such that*

$$g_1(z)f_1(z) + g_2(z)f_2(z) + \ldots + g_n(z)f_n(z) = 1$$

for every $z \in D$.

Proof: Assuming a), let $f_1, f_2, \ldots, f_n \in H^\infty$ be such that $\inf_{z\in D} \max_j |f_j(z)| = \delta > 0$. Consider the ideal

$I = \{g_1 f_1 + \ldots + g_n f_n : g_1, \ldots, g_n \in H^\infty\}$. Now, I is not contained in any maximal ideal J, since it follows from a) that $\max_j |f_j(J)| \geq \delta$. Therefore: $I = H^\infty$ and, consequently: $1 = g_1 f_1 + \ldots$ $\ldots + g_n f_n$.

Now assume, conversely, that b) holds. Given $J \in S$, consider the neighbourhood of J $V = J + \{L: |h_j(L)| < \delta\}$ for some $h_1, \ldots, h_n \in H^\infty$ and $\delta > 0$, that is $V = \{L: |h_j(L) - h_j(J)| < \delta\}$. Now, the functions $f_j(z) = h_j(z) - h_j(J)$ are in J and, since J does not contain 1, it follows from b) that $\inf_{z\in D} \max_j |f_j(z)| \leq 0$, from which it is clear that there exists $z \in D$ such that for every $j = 1, \ldots, n$: $|f_j(z)| < \delta$. This means that $z \in V$. Since this can be done for J and V arbitrary, we conclude that a) holds. $\square$

For the next lemma we shall use the differential operators $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. We shall be interested in solving the equation

$$(10.3) \qquad \bar{\partial} f(z) = g(z)$$

LEMMA 10.4. Let $g(z)$ *be a* C^∞ *function with compact support in* $\mathbb{C}$. *Then, the function*

$$f(z) = \frac{1}{\pi}\iint_{\mathbb{C}} \frac{g(\zeta)}{z-\zeta}\, d\xi\, d\eta \ ; \quad \zeta = \xi + i\eta$$

is a C^∞ *solution of the equation* (10.3) *for (say)* $|z| < 2$

Proof: Since g has compact support and we are only interested in $|z| < 2$, we can write for an appropriate $R > 0$:

$$f(z) = \frac{1}{\pi}\iint_{|w|<R} \frac{g(z-w)}{w}\, du\, dv; \quad w = u + iv$$

So

$$\bar{\partial} f(z) = \frac{1}{\pi}\iint_{|w|<R} \frac{\bar{\partial} g(z-w)}{w}\, du\, dv$$

Without loss of generality, we can take $z = 0$, and compute

$$\bar{\partial} f(0) = \lim_{r\to 0} \frac{1}{\pi} \iint_{r<|w|<R} \frac{\bar{\partial} g(-w)}{w}\, du\, dv =$$

$$= \frac{i}{2\pi} \lim_{r\to 0} \iint_{r<|w|<R} \left(\frac{\partial}{\partial u}\left(i \frac{g(-w)}{w}\right) - \frac{\partial}{\partial v}\left(\frac{g(-w)}{w} \right)\right) du\, dv =$$

$$= -\frac{i}{2\pi} \lim_{r\to 0} \int_{|w|=r} \frac{g(-w)}{w}\, dw = \lim_{r\to 0} \frac{1}{2\pi} \int_0^{2\pi} g(-re^{it})\,dt = g(0)$$

where we have used Green's theorem to pass from the two-dimensional integral to the integral over the circle. $\square$

The third and last lemma contains the basic idea for the proof of theorem 10.1

LEMMA 10.5. Let h *be a* C^∞ *function on* $D(0,R)$, $R > 1$. *Suppose that in* D, *both*

$$|z|\left(\log \frac{1}{|z|}\right) |h(z)|^2 \, dx\, dy \qquad \textit{and}$$

$$|z|\left(\log \frac{1}{|z|}\right) |\partial h(z)| \, dx\, dy$$

are Carleson measures with Carleson constants A *and* B *respectively. Then we can find a function* $v(z)$, *which is* C^∞ *on some disk* $D(0,R')$, $R' > 1$ *and satisfies*

1) $\bar{\partial} v(z) = h(z)$ *for* $|z| < 1$, *and*

2) $|v(e^{it})| < C(A^{1/2}+B)$ *for every real* t *where* C *is an absolute constant.*

Proof: We redefine $h(z)$ for $|z| > (1+R)/2$ so as to obtain a function, which we still call h, C^∞ in all of $\mathbb{C}$ and having compact support. Then, the formula in lemma 10.4 with h in place of g, gives us a C^∞ function v_0 such that $\bar{\partial} v_0(z) = h(z)$ for $|z| < 2$. The problem is that v_0 may be very large on the boundary of the unit disk, and we have to change it so as to satisfy condition 2). Now if f is holomorphic in $D(0,R')$, $R' > 1$ and we set $v = v_0 - f$, we still have $\bar{\partial} v(z) = h(z)$ for $|z| < 1$, since $\bar{\partial} f = 0$. The idea is to choose f in such a way that 2) holds. $v_0(e^{it})$

belongs to the class C of all continuous functions on the boundary of D. Consider $A = H^\infty \cap C$. Let d be the L^∞ distance of $v_0(e^{it})$ from A, that is

$$d = \inf \{\|v_0 - f\|_\infty = \sup_t |v_0(e^{it}) - f(e^{it})| : f \in A\}.$$

If $d' > d$, we can find $f \in A$ such that $|v_0(e^{it}) - f(e^{it})| \leq d'' < d'$ for every t, and, consecuently, there is $r < 1$ such that for every t: $|v_0(e^{it}) - f(re^{it})| < d'$. The function $v = v_0 - f_r$ where $f_r(z) = f(rz)$ satisfies 1) and also $|v(e^{it})| < d'$. So, our problem is really to compute d, or rather, to see that $d < C(A^{1/2}+B)$. Now, d is the norm of $v_0(e^{it})$ in the quotient space C/A. It follows immediately from the F. and M. Riesz theorem (3.10), that the dual of C/A can be identified with $H^1(0)$. Thus, according to the Hahn-Banach theorem,

$$d = \sup\{|\int v_0(e^{it})F(e^{it})\,dt| \; ; F \in H^1(0), \; \|F\|_{H^1} \leq 1\}.$$

We can further restrict the set of F's, taking only those which are analytic on some D(0,R) $R > 1$, where, of course, R varies with F. For such an F, we apply Green's theorem, obtaining (lemma 9.11)

$$\int_0^{2\pi} v_0(e^{it})F(e^{it})\,dt = \iint_D (\log \frac{1}{|z|})\, \Delta(v_0(z)F(z))\,dx\,dy.$$

But $\Delta(v_0F) = 4\partial\bar\partial(v_0F) = 4hF' + 4\partial hF$. Thus

$$\int_0^{2\pi} v_0(e^{it})\,F(e^{it})\,dt = 4\iint_D F(z)\,\partial h(z) \log \frac{1}{|z|}\,dx\,dy +$$

$$+ 4 \iint_D F'(z)\,h(z)\log\frac{1}{|z|}\,dx\,dy.$$

The first term in the sum is bounded in absolute value by

$$4\iint_D \left|\frac{F(z)}{z}\right| |\partial h(z)|\,|z| \log\frac{1}{|z|}\,dx\,dy \leq CB\,\|F\|_{H^1}$$

since we are dealing with a Carleson measure, so that theorem 9.22. b) applies.

For the second term in the sum we use first the Cauchy-Schwarz inequality to dominate it by

$$4\left(\iint_D \left(\log \frac{1}{|z|}\right)\frac{|F'(z)|^2}{|F(z)|}\,dx\,dy\right)^{1/2}\left(\iint_D \left|\frac{F(z)}{z}\right| |h(z)|^2|z| \log \frac{1}{|z|}dx\,dy\right)^{1/2}$$

Observe that we just need to estimate this for $F \in H^1(0)$ with no zeroes in D except for a simple zero at the origin (see the expla̲nation at the beginning of page 107). For this kind of F, we have, according to lemma 9.15:

$$\iint_D \left(\log \frac{1}{|z|}\right) \frac{|F'(z)|^2}{|F(z)|}\,dx\,dy = \|F\|_{H^1}$$

Also, applying theorem 9.22. b) again, we get

$$\iint_D \left|\frac{F(z)}{z}\right| |h(z)|^2|z| \log \frac{1}{|z|}dx\,dy \le CA\,\|F\|_{H^1}$$

Thus, for every $F \in H^1(0)$, analytic on $\overline{D}$:

$$\left|4 \iint_D F'(z)\,h(z) \log \frac{1}{|z|}\,dx\,dy\right| \le CA^{1/2}\,\|F\|_{H^1}.$$

Adding the two estimates we conclude that $d < C(A^{1/2} + B)$ as we wanted to prove. □

Now we shall prove a theorem, which, according to lemma 10.2, implies theorem 10.1.

THEOREM 10.6 Let $f_1, f_2, \ldots, f_n \in H^\infty$ *be such that*

a) $\|f_j\|_\infty \le 1$. *for every* j

b) *There is* $\delta > 0$ *so that* $\max_j |f_j(z)| > \delta$ *for each* $z \in D$.

Then, there exists a constant $C = C(\delta,n)$, *depending only on* n *and* δ, *such that*

$$g_1(z)f_1(z) + g_2(z)f_2(z) + \ldots + g_n(z)f_n(z) = 1$$

for every $z \in D$ *with some functions* $g_1, g_2, \ldots, g_n \in H^\infty$, *satisfying* $\|g_j\|_\infty \le C(\delta,n)$ *for every* j.

Proof: We start by analyzing the case $n = 2$.

It is enough to prove the theorem for functions f_j analytic on a disk slightly larger than D. Indeed, the uniform bound on the g_j's allows us to use the typical convergence theorem for normal families (see Rudin [1]).

Suppose we have f_1 and f_2 with $\|f_1\|_\infty \leq 1$, $\|f_2\|_\infty \leq 1$ and, for every $z \in D$:

$$\text{either} \quad |f_1(z)| > \delta \quad \text{or} \quad |f_2(z)| > \delta$$

Let $U(z)$ be a C^∞ function, depending only on $|z|$, such that $U(z) \equiv 0$ for $|z| \leq \delta/2$, $U(z) \equiv 1$ for $|z| \geq \delta$ and $0 \leq U(z) \leq 1$ everywhere. We set

$$\phi_j(z) = \frac{U(f_j(z))}{U(f_1(z))+U(f_2(z))} \quad \text{for} \quad j = 1, 2.$$

Each ϕ_j is a C^∞ function on some $D(0,R)$, $R > 1$, and

$$\phi_1(z) + \phi_2(z) \equiv 1$$

Consequently $(\phi_1/f_1)f_1 + (\phi_2/f_2)\, f_2 \equiv 1$

We would have finished the proof if the functions ϕ_j/f_j were analytic in D. But they are ony C^∞ (Note that if f_j has a zero at z_0, it is an isolated zero, and z_0 has a neighbourhood at which $|f_j(z)| < \delta/2$, so that $\phi_j(z) = 0$ in that neighbourhood.)

Now we shall try to find v such that

$$g_1 = (\phi_1/f_1) + vf_2 \quad \text{and} \quad g_2 = (\phi_2/f_2) - vf_1$$

become analytic in $|z| < 1$. Once we achieve this, we shall have

$$g_1f_1 + g_2f_2 \equiv 1$$

and the control on v will give us control over g_1 and g_2.

For the analyticity of g_1 and g_2 we need $\bar{\partial} g_1 = \bar{\partial} g_2 = 0$ in D. Since $\bar{\partial} f_1 = \bar{\partial} f_2 = 0$, we obtain the conditions:

$$(\bar\partial\phi_1/f_1) + f_2\bar\partial v = 0, \quad (\bar\partial\phi_2/f_2) - f_1\bar\partial v = 0.$$

But $\phi_1 + \phi_2 \equiv 1$ implies $\bar\partial\phi_1 + \bar\partial\phi_2 \equiv 0$.

Thus, the two conditions are equivalent to this single one:

$$\bar\partial v = \bar\partial\phi_2/(f_1 f_2)$$

We shall apply lemma 10.5 with

$$\text{(10.7)} \qquad h(z) = \bar\partial\phi_2(z)/(f_1(z)f_2(z))$$

This is clearly a C^∞ function on $D(0,R)$, $R > 1$ and it satisfies the estimate

$$|h(z)| \leq (4/\delta^2)\ |\bar\partial\phi_2(z)| \qquad \text{for} \quad |z| < 1$$

A simple computation shows that:

$$|\bar\partial\phi_2(z)| = \frac{|U(f_1(z))\bar\partial(U(f_2(z)))-U(f_2(z))\bar\partial(U(f_1(z)))|}{|U(f_1(z)) + U(f_2(z))|^2} \leq C(|f_1'(z)| +$$

$$+ |f_2'(z)|)$$

Then $|h(z)|^2\ |z|\ \log\frac{1}{|z|} \leq C_\delta(|f_1'(z)|^2 + |f_2'(z)|^2)\ |z|\ \log\frac{1}{|z|}$

where C_δ is a constant depending on δ (we shall use the same symbol for, possibly different, constants depending on δ only). Observe that theorem 9.21. implies that $(|f_1'(z)|^2 + |f_2'(z)|^2)\cdot$ $\cdot|z|\ \log\frac{1}{|z|}\ dx\ dy$ is a Carleson measure with Carleson constant bounded by $C(\|f_1\|_\infty + \|f_2\|_\infty) = C$. Therefore:

(10.8) $\quad |h(z)|^2\ |z|\ \log\frac{1}{|z|}\ dx\ dy$ *is a Carleson measure with Carleson constant* C_δ.

From (10.7) we compute immediately

$$\partial h(z) = \frac{\partial\bar\partial\phi_2}{f_1 f_2} - \frac{\bar\partial\phi_2}{f_1 f_2}\ \left(\frac{f_1'}{f_1} + \frac{f_2'}{f_2}\right)$$

The second term in the right hand side vanishes identically on the open set where either $|f_1|$ or $|f_2|$ is $<\delta/2$. In the complement of that open set, it is bounded in absolute value by $C_\delta(|f_1'(z)|^2 + |f_2'(z)|^2)$.

As for the first term, it also vanishes identically wherever $|f_1|$ or $|f_2|$ is $< \delta/2$. In the complementary, it is given by

$$\frac{\Delta\phi_2(z)}{4f_1(z)f_2(z)} = \frac{1}{4f_1(z)f_2(z)} \cdot \Delta\left(\frac{U(f_2(z))}{U(f_1(z)) + U(f_2(z))} \right)$$

After some computation (write $\Delta = 4\partial\,\bar\partial$), this is seen to be bounded also by $C_\delta(|f_1'(z)|^2 + |f_2'(z)|^2)$

Using theorem 9.21 again, we conclude that:

(10.9) $|\partial h(z)||z| \log \frac{1}{|z|}\, dx\, dy$ *is a Carleson measure with Carleson constant* C_δ.

(10.8) and (10.9) allow us to apply lemma 10.5 to our particular h, obtaining v, C^∞ in $D(0,R)$, $R > 1$, such that $\bar\partial v = h$ in D and $|v(e^{it})| \leq C_\delta$ for every t. Then

$$g_1 = (\phi_1/f_1) + vf_2 \quad \text{and} \quad g_2 = (\phi_2/f_2) - vf_1$$

are analytic in D and satisfy

$$g_1f_1 + g_2f_2 \equiv 1 \quad \text{in } D.$$

Also, since $|\phi_j/f_j| \leq 2/\delta$ and $|v(e^{it})| \leq C_\delta$, it follows that $|g_j(e^{it})| \leq C_\delta$, and therefore $g_j \in H^\infty$ with $\|g_j\|_\infty \leq C_\delta$.

This finishes the proof for the case $n = 2$.

The case $n > 2$ is only slightly more complicated. Suppose we have $f_1, f_2, \ldots, f_n \in H^\infty$ such that $\|f_j\|_\infty \leq 1$ for each j and $\inf_{z\in D} \max_j |f_j(z)| > \delta > 0$. We may further assume, as in the case $n = 2$ that the functions f_j are holomorphic in $\bar D$. We use the same function U that we used for the case $n = 2$, and write

$$\phi_j(z) = \frac{U(f_j(z))}{U(f_1(z))+U(f_2(z))+ \ldots + U(f_n(z))}$$

We have $\sum_{j=1}^{n} \phi_j(z) \equiv 1$ in D. Now, we try to find functions v_{jk} such that each of the functions

$$g_j = (\phi_j/f_j) + \sum_{k=1}^{n} v_{jk} f_k$$

becomes analytic. We also impose the conditions

$$(10.10) \qquad v_{jk} = -v_{kj}, \; v_{jj} \equiv 0$$

which guarantee that $\sum_{j=1}^{n} g_j f_j \equiv 1$

For g_j to be analytic, we must have:

$$(\bar\partial\phi_j/f_j) + \sum_{k=1}^{n} f_k \bar\partial v_{jk} = 0$$

This may be achieved if we have

$$(10.11) \qquad \bar\partial v_{jk} = \frac{\phi_j}{f_j f_k} \bar\partial\phi_k - \frac{\phi_k}{f_k f_j} \bar\partial\phi_j$$

since $\sum_{k=1}^{n} \bar\partial\phi_k = 0$ and $\sum_{k=1}^{n} \phi_k \equiv 1$

In order to get the v_{jk} satisfying (10.10) and (10.11), we first find, as in the case $n = 2$, functions $w_{jk}(z)$, C^∞ on $D(0,R)$, $R > 1$, such that $\bar\partial w_{jk} = \dfrac{\phi_j}{f_j f_k} \bar\partial\phi_k$ on D and $|w_{jk}(e^{it})| \leq C_\delta$ for every t.

Then set $v_{jk} = w_{jk} - w_{kj}$

This automatically gives us (10.10) and (10.11). The corresponding g_j are analytic on $\bar D$, satisfy $\sum_{j=1}^{n} g_j f_j \equiv 1$ on D and also $\|g_j\|_\infty \leq C_\delta$ for every j.

This completes the proof of theorem 10.6, and, consequently, of the corona theorem. □

11. NOTES AND FURTHER RESULTS

11.1.- The material in sections 1 to 8 is already classical. For this part, we can give four references in book form, namely:

P. Duren [1], K. Hoffman [1], P. Koosis [1] and A. Zygmund [1].

11.2.- Theorem 1.20 is due to Fatou [1]. However, most people call Fatou's theorem to the result asserting that every bounded holomorphic function in D has non-tangential boundary values a.e.

If we assume that $F'(\theta_1)$ is infinite in theorem 1.20., we can still conclude that $u(re^{i\theta_1}) \longrightarrow F'(\theta_1)$ as $r \to 1$; but it might not be true that $u(z) \longrightarrow F'(\theta_1)$ as $z \xrightarrow{N.T.} e^{i\theta_1}$. However, if in addition $u(z) \geq 0$ for every $z \in D$, the non-tangential convergence holds. See Koosis [1] for the details.

In chapter II, section 4, we shall discuss the boundary behaviour of harmonic functions in the upper-half-space and, in general, we shall give results for approximate identities in $\mathbb{R}^n$. The approach there will be based upon the Hardy-Littlewood maximal function, which will be studied in the first two sections of chapter II. It is instructive to compare the two different approaches.

11.3.- Theorem 2.16 is due to Hardy [1]. This paper contains also the so called Hardy's convexity theorem, which says that $\log m_p(F,r)$ is a convex function of $\log r$.

The proof we give of theorem 2.16 using subharmonic functions, is due to F. Riesz [1]. In this paper one can also find a proof of Hardy's convexity theorem based upon subharmonicity in an annulus. A.E. Taylor [1] proves Hardy's theorems with Banach space methods. A related result is Littlewood's subordination theorem, which says that if $f(z) = F(w(z))$ with F and w holomorphic in D and $|w(z)| \leq |z|$, then $m_p(f,r) \leq m_p(F,r)$. This is contained in Littlewood [1].

11.4.- The classical source for the theory of subharmonic functions of two variables is T. Radó [1].

11.5.- Fatou [1] showed that each $F \in H^\infty$ has a non-tangential limit $F(e^{it})$ at a.e.t, and that $F(e^{it})$ cannot vanish on an arc unless $F(z)$ vanishes identically. F. and M. Riesz [1] improved this result by extending it to H^1 functions and showing that $F(z)$ vanishes identically if $F(e^{it})$ vanishes on a set of positive measure. Szegö [1] showed that if $F \in H^2$ is not identically 0, then

$\log |F(e^{it})|$ is integrable. Finally F. Riesz was able to extend this result from H^2 to H^p for any $p > 0$. The key for this extension is his factorization theorem (theorem 3.3), which appears in F. Riesz [2]. This factorization allowed him to obtain theorem 3.6., which clarifies the boundary behaviour of an H^p function both in the pointwise sense and in the L^p "norm".

11.6.- Theorem 2.21 was proved by Blaschke [1]. The convergence to 0 of the integrals appearing in theorem 2.22, characterizes Blaschke products among all analytic functions F satisfying $|F(z)| \leq 1$. This theorem is due to M. Riesz, but was first published by Frostman (see Zygmund [1], vol. I p. 281)

11.7.- M. and G. Weiss [1] show how to obtain Riesz Factorization theorem $F = B \cdot H$ without having to worry about the behaviour of the zeroes of F to construct the Blaschke product B. They start by constructing H by a limiting process, and then, they obtain the properties of B, which they show has to be essentially the Blaschke product.

11.8.- F. and R. Nevanlinna [1] proved that an analytic function is is the class N if and only if it is the quotient of two bounded analytic functions. This allows them to show that for $F \in N$, the nontangential limit $F(e^{it})$ exists almost everywhere and $\log |F(e^{it})|$ is integrable unless $F(z)$ is identically 0.

Paley and Zygmund [1] gave examples of functions which are very close to being in the class N and for which the radial limit does not exist anywhere.

11.9.- Theorem 3.10. was proved originally by F. and M. Riesz [1]. The proof we give, which is possible after F. Riesz factorization theorem, is not the original one. However, the original proof of F. and M. Riesz, is also contained in this chapter (see section 8, right before theorem 8.4). There are many other proofs of this remarkable theorem. For a proof which uses functional analysis instead of function theory see Helson [1] and Helson and Lowdenslager [1]. This proof is also contained in the books of Hoffman and Koosis. A very short and elementary proof is given by Øksendal [1]. See also Doss [1].

11.10.- Theorem 3.12. and its corollary are due to Privalov (see his book [1]). The theorem of Caratheodory which appears in the proof of corollary 3.13. can be seen in Zygmund [1], Chapter VII, Sec. 10. Koosis [1] contains a very nice treatment of this theorem and of conformal mapping in general.

11.11.- Hardy's inequality (theorem 4.1) was proved for the first time in Hardy and Littlewood [2], and theorem 4.5. in Fejér and Riesz [1], although Hardy [2] anticipated it for the case $p = 2$. The classical book on Inequalities is Hardy, Littlewood and Polya [1]. The conformality of F at the boundary points where there is a tangent (corollary 4.7) was first proved by Lindelöf [1].

11.12.- Theorem 5.3., and consequently M. Riesz inequality (corollary 5.5) were first proved by M. Riesz [1]. The proof given in the text comes from A.P. Calderón [5]. A nice proof based upon Green's theorem was given by P. Stein [1]. By a careful modification of Calderón's argument, Pichorides [1] found the best constants C_p in the inequality $\|\tilde{f}\|_p \leq C_p\|f\|_p$. These are: $C_p = \operatorname{tg}\frac{\pi}{2p}$ if $1 < p \leq 2$ and $C_p = \operatorname{cotg}\frac{\pi}{2p}$ if $1 \leq p < \infty$. Theorem 5.9 is due to Kolmogorov [2] and corollary 5.10 is usually called Kolmogorov's inequality. Since $f \in L^1$ does not guarantee that $\tilde{f} \in L^1$, it is interesting to mention here a result of A. Zygmund, according to which

$$\int_{-\pi}^{\pi}|\tilde{f}(t)|\,dt \leq C\int_{-\pi}^{\pi}|f(t)|(1+\log^+|f(t)|)\,dt + 2\pi$$

Actually A. Zygmund proved also that, conversely, if f is in L^1 and $f \geq 0$, the integrability of $\tilde{f}$ implies that of $f\log^+ f$, or, in other words, that $f \in L\log^+ L$. These theorems can be found in chapter VII of A. Zygmund [1], together with many other results about the conjugate function.

11.13.- The canonical factorization (theorem 7.1) is due to Smirnov [1]. It was A. Beurling [1] who introduced the terms "inner" and "outer" functions. His paper contains the approximation theorem 7.9. for H^2. The main result of this paper is the identification of those closed subspaces E of H^2 which are invariant under multiplication by the function z. They are of the form $E = FH^2$ for F an inner function. The invariant subspace theory has been extended to H^p, $0 < p < 1$ by Gamelin [1] and to $H^p(\mu)$, the closure in

$L^p(\mu)$ of the polynomials in z, by J.J. Guadalupe [1]. Theorem 7.9. for H^p appears in Srinivasan and Wang [1].

11.14.- When we move from the unit disk D to a general simply connected domain G properly contained in $\mathbb{C}$, there are three natural candidates for $H^p(G)$

(1) The space consisting of those $F \in H(G)$ such that $|F|^p$ has a harmonic majorant in G.

(2) The one formed by the functions $F \in H(G)$ for which the averages of $|F|^p$ over a family of rectifiable Jordan curves tending to the boundary, are bounded.

(3) The closure of the polynomials in L^p of the boundary (assumed rectifiable).

These spaces are studied in chapter 10 of P. Duren [1]. If ϕ is a conformal equivalence between the unit disk and G, there are conditions on ϕ for the different types of spaces to coincide. For example, the spaces in (1) and (2) coincide if and only if $0 < a \leq |\phi'(z)| \leq b$ for every $z \in D$.

As for the spaces in (2) and (3), they coincide if and only if ϕ' is an outer function. Of course, this is a property only of the domain G, and it is independent of the particular mapping ϕ. The domains enjoying this property are called Smirnov domains.

The spaces $H^p(G)$ for G multiply connected were studied by W. Rudin [3], D. Sarason [1] and Coifman and Weiss [2].

11.15.- The Helson-Szegö theorem is, in a sense, the starting point of the subject of weighted norm inequalities. It was proved in Helson-Szegö [1]. In that paper, the problem is presented as one of "Prediction Theory". Let us give a brief justification for this. Consider a discrete, zero-mean, stationary Gaussian process, that is, a sequence (X_n), $n \in \mathbb{Z}$ of real random variables (i.e.: measurable functions) in $L^p(P)$, P a probability measure, satisfiyng the following conditions:

1) $\int X_n \, dP = 0$ for every n

2) $\int X_n X_k \, dP$ depends only on $n-k$

3) Every function in the linear span of $\{X_n\}$ is normally distributed.

It is readily seen that the numbers $c_n = \int X_0 X_n \, dP$ form a positive definite sequence. According to a theorem of Herglotz (see Karznelson [1] p. 38) the c_n's are the Fourier coefficients of a finite positive measure μ on the torus. Consequently, we get an isometry between $L^2(\mu)$ and the span in $L^2(P)$ of the functions X_n by sending X_n to the function $e_n(t) = e^{int}$. In this way, the "past" and the "future" of the process (we think of n as representing time) can be identified with the spans in $L^2(\mu)$ of the exponentials $\{e_n;\ n \leq 0\}$ and $\{e_n;\ n > 0\}$ respectively. The name Prediction theory is then applied to the study of the relation between the past and the future. For example, as theorem 8.11. shows, the boundedness of the conjugate function in $L^2(\omega)$ is equivalent to the fact that the "past" and the "future" are at a positive angle in $L^2(\omega)$. The reader interested in this prediction-theoretic aspect in advised to look at Dym-McKean [2]. See also Sarason [3], Ch. 7.

Lemma 8.3. was already in Fatou [1]. A related result is the so-called Rudin-Carleson theorem (see Carleson [4], Rudin [4] and Doss [1]). For an extension of theorem 8.4., see Guadalupe and Rubio de Francia [1].

11.16.- Theorem 9.27 is due to C. Fefferman [1]. Section 9 is devoted to proving it along the lines set up by C. Fefferman and E.M. Stein [2]. This basic paper studies H^p for a euclidean half space. All we have done is to adapt Fefferman and Stein's proof to the context of the disk. We shall take up the subject of Hardy spaces of several variables in chapter III, but there, our approach will differ from the original one. The proof of the duality will be entirely by real variable methods. There will be a third proof of the duality theorem in section 5 of chapter IV. It will be based upon a weak version of the Helson-Szegö theorem. Observe that condition (b) in theorem 9.27. provides a link between Helson-Szegö weights and B.M.O. functions, namely: the logarithm of a Helson-Szegö weight is a B.M.O.

function. This relation will be further exploited in chapters II and IV.

11.17.- The dual space of H^p for $0 < p < 1$ was known before the dual of H^1. This was due to Duren, Romberg and Shields [1]. See also P. Duren [1]. The dual turns out to be a Lipschitz space. The corresponding duality results in several variables will be obtained together with the dual of H^1 in section 5 of chapter III.

11.18.- The B.M.O. function f is said to have vanishing mean oscillation or to belong to V.M.O. if and only if

$$\frac{1}{|I|}\int_I |f(x)-f_I|\,dx \to 0 \quad \text{as} \quad |I| \to 0$$

It can be seen that V.M.O. is a proper closed subspace of B.M.O. It clearly contains the B.M.O. closure of the space C formed by the continuous functions. V.M.O. is to B.M.O. what C is to L^∞. This space has been studied by D. Sarason [2] (see also D. Sarason [3]).

He has proved this theorem:

The following conditions on the B.M.O. *function* f, *are equivalent*:

(a) $f \in$ V.M.O.

(b) $\|f-f(\cdot-t)\|_* \to 0$ *as* $t \to 0$.

(c) f *is in the* B.M.O. *closure of* C.

(d) $f = \phi + \tilde{\psi}$ *for certain* $\phi, \psi \in C$.

(e) *The Carleson measure* μ *associated to* f *as in* (e) *of theorem* 9.27 *satisfies*

$$\mu(R(I))/|I| \to 0 \quad \textit{as} \quad |I| \to 0$$

where R(I) *has the same meaning as in theorem* 9.21.

From the characterization (d) above, we can easily see that the dual of V.M.O. can be identified, in the natural way, with $H^1(0)$. Let us see this: Of course $H^1(0)$ acts on V.M.O. because it acts on B.M.O., which is a larger space. Conversely, if L is a linear functional on V.M.O., we can form the composition

$$C \times C \longrightarrow \text{V.M.O.} \longrightarrow \mathbb{R}$$

$$(\phi,\psi) \longmapsto \phi + \tilde{\psi} \longmapsto L(\phi+\tilde{\psi})$$

which is a bounded linear functional on $C \times C$. Consequently, there will exist measures μ and ν such that

$$L(\phi+\tilde{\psi}) = \int \phi \, d\mu + \int \psi \, d\nu.$$

It follows that, whenever ϕ and $\tilde{\phi}$ are continuous:

$$\int \tilde{\phi} \, d\mu = \int \phi \, d\nu.$$

In particular, for $n = 1, 2, 3, \dots$

$$\int e^{int} (d\nu + i d\mu) = 0$$

so that, according to the F. and M. Riesz theorem (3.10), $d\nu + i d\mu = (f + i\tilde{f})\, dx$ for some $f \in H^1$.

Now $L(\phi + \tilde{\psi}) = \int \phi \, \tilde{f} + \int \psi \, f = \int (\phi + \tilde{\psi})\tilde{f}$ and $\tilde{f} \in H^1(0)$.

We can write $(\text{V.M.O.})^* = H^1(0)$. The space B.M.O., strictly larger, cannot have the same dual. Thus, $H^1(0)$ is not a reflexive space.

11.19.- The corona theorem (10.1) was conjectured by Kakutani around 1941. It was finally proved by L. Carleson [1]. Hörmander [2] gave another proof, in which he introduced the relation between the corona theorem and the solution of the $\bar{\partial}$-equation (10.3). However the proof we have given, based upon the fundamental lemma 10.5, is due to T. Wolff (circa 1979).

CHAPTER II

CALDERON-ZYGMUND THEORY

In 1952, A.P. Calderón and A. Zygmund [1] invented a simple but powerful method which has become widely known as the Calderón-Zygmund decomposition. We aim to describe here this method together with some of its most immediate and interesting applications.

The first two sections give a description of the method in connection with the (very closely related) Hardy-Littlewood maximal operator. Apart from the usual estimates for this maximal function, we also obtain some weighted inequalities which anticipate the A_p theory to be developed in Chapter IV, and we study some variants of the Hardy-Littlewood operator when Lebesgue measure is replaced by a more general measure. This leads us in a natural way to the definition and study of Carleson measures.

This is not the only maximal operator to appear in this chapter. The so-called sharp maximal function shares enough properties with the Hardy-Littlewood operator, but behaves in a different way in L^∞, which is somehow replaced by our friend the space BMO. There is an interesting relation between weights and BMO which will be further exploited in Chapter IV. This relation comes to light after proving the John-Niremberg inequality for BMO functions, which is yet another application of the Calderón-Zygmund decomposition.

The last two sections contain a very condensed account on singular integral operators and their application to multiplier theorems, since these operators originally motivated the Calderón-Zygmund decomposition, and they are still the most striking and important application of the method.

1. THE HARDY-LITTLEWOOD MAXIMAL FUNCTION AND THE CALDERON-ZYGMUND DECOMPOSITION

Let f be a locally integrable function in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ we define

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|dy$$

where the supremum is taken over all cubes Q containing x (cube will always mean a compact cube with sides parallel to the axes and nonempty interior), and $|Q|$ stands for the Lebesgue measure of Q.

Mf will be called the (Hardy-Littlewood) maximal function of f, and the operator M sending f to Mf, the (Hardy-Littlewood) maximal operator.

Observe that we obtain the same value $Mf(x)$, which can be $+\infty$, if we allow in the definition only those cubes Q for which x is an interior point. It follows from this remark that the function Mf is lower semicontinuous, i.e., for every $t > 0$, the set

$$E_t = \{x \in \mathbb{R}^n : Mf(x) > t\}$$

is open.

In order to study the size of Mf, we shall look at its distribution function $\lambda(t) = |E_t|$. It will be instructive to start with the case $n = 1$, which is particularly simple. Let $f \in L^1(\mathbb{R})$. The open set E_t is a disjoint union of open intervals I_j: its connected components. Let us look at one of the I_j's, and let us call it I. Take any compact set $K \subset I$. For each $x \in K$, we have some compact interval Q_x containing x in its interior and satisfying

$$\frac{1}{|Q_x|} \int_{Q_x} |f(y)|dy > t.$$

Since $Q_x \subset E_t$, it follows that $Q_x \subset I$. Since K is compact, we can cover it with the interiors of just finitely many of the Q_x's, say $\{Q_j\}$. We can even assume that this finite covering is minimal in the sense that no Q_j is superfluous. Then, no point is in more than two of the interiors of the Q_j's. It follows that:

$$|K| \leq \sum_j |Q_j| < \frac{1}{t} \sum_j \int_{Q_j} |f(y)|dy \leq \frac{2}{t} \int_{\bigcup_j Q_j} |f(y)|dy \leq$$

$$\leq \frac{2}{t} \int_I |f(y)|dy.$$

Since this is true for every compact $K \subset I$, we obtain:

$$(1.1) \qquad |I| \leq \frac{2}{t} \int_I |f(y)|dy$$

This implies, in particular, that I is bounded. Let $I = (a,b)$. Then, since $b \in \bar{I}$ and $b \notin E_t$, we can write:

$$\frac{1}{|I|} \int_I |f(y)|dy \leq Mf(b) \leq t$$

Finally (1.1) implies:

$$|E_t| = \sum_j |I_j| \leq \frac{2}{t} \sum_j \int_{I_j} |f(y)|dy = \frac{2}{t} \int_{E_t} |f(y)|dy$$

We have obtained the following result:

THEOREM 1.2. Let $f \in L^1(\mathbb{R})$. *Then, for every* $t > 0$, *the set* $E_t = \{x \in \mathbb{R} : Mf(x) > t\}$ *can be written as a disjoint union of bounded open intervals* I_j, *such that, for every* $j = 1,2, \ldots$:

$$(1.3) \qquad \frac{t}{2} \leq \frac{1}{|I_j|} \int_{I_j} |f(y)|dy \leq t$$

and, as a consequence:

$$|E_t| \leq \frac{2}{t} \int_{E_t} |f(x)|dx. \qquad \square$$

Now we seek an analogue of theorem 1.2 in dimension $n > 1$. The extension is not straightforward.

Let $f \in L^1(\mathbb{R}^n)$, $n > 1$, and let $t > 0$. Instead of looking at the maximal function Mf, we shall try to obtain directly a family of cubes $\{Q_j\}$ such that the average of $|f|$ over each is comparable to t in the sense that a relation like (1.3) holds. This is quite easy and it will be done most effectively by considering only dyadic cubes. For $k \in \mathbb{Z}$, we consider the lattice $\Lambda_k = 2^{-k}\mathbb{Z}^n$ formed by those points of $\mathbb{R}^n$ whose coordinates are integral multiples of 2^{-k}. Let $\mathcal{D}_k$ be the collection of the cubes determined by Λ_k, that is, those cubes with side length 2^{-k} and vertices in Λ_k. The cubes belonging to $\mathcal{D} = \bigcup_{-\infty}^{\infty} \mathcal{D}_k$ are called dyadic cubes. Observe that if $Q,Q' \in \mathcal{D}$ and $|Q'| \leq |Q|$, then either $Q' \subset Q$ or else Q and Q' do not overlap (by which we mean that their interiors are disjoint). Each $Q \in \mathcal{D}_k$ is the union of 2^n nonoverlapping cubes belonging to $\mathcal{D}_{k+1}$. We shall call $\mathcal{C}_t$ the family formed by the cubes $Q \in \mathcal{D}$ which satisfy the condition:

$$t < \frac{1}{|Q|}\int_Q |f(x)|dx \tag{1.4}$$

and are maximal among those which satisfy it. Every $Q \in \mathcal{D}$ satisfying (1.4) is contained in some $Q' \in \mathcal{C}_t$ because condition (1.4) imposes an upper bound on the size of $Q(|Q| \leq t^{-1}\|f\|_1)$. The cubes in $\mathcal{C}_t$ are, by definition, nonoverlapping. Also, if $Q \in \mathcal{D}_k$ is in $\mathcal{C}_t$ and Q' is the only cube in $\mathcal{D}_{k-1}$ containing Q, we shall have:

$$\frac{1}{|Q'|}\int_{Q'} |f(x)|dx \leq t$$

but, since $|Q'| = 2^n|Q|$, we get:

$$\frac{1}{|Q|}\int_Q |f(x)|dx \leq \frac{2^n}{|Q'|}\int_{Q'} |f(x)|dx \leq 2^n t$$

We have achieved our purpose by obtaining a family $\mathcal{C}_t = \{Q_j\}$ of cubes such that, for every j:

$$t < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|dx \leq 2^n t \tag{1,5}$$

Next, we shall investigate the relation with the maximal function Mf. Suppose $x \in \mathbb{R}^n$ is such that $Mf(x) > t$. There will be some cube R containing x in its interior and satisfying

$$t < \frac{1}{|R|}\int_R |f(y)|dy .$$

We look for a dyadic cube of comparable size over which the average of $|f|$ is comparably big. Let k be the only integer such that

$$2^{-(k+1)n} < |R| \leq 2^{-kn}.$$

There is at most one point of Λ_k interior to R. Consequently, there is some cube in $\mathcal{D}_k$, and at most 2^n of them, meeting the interior of R. It follows that there is some $Q \in \mathcal{D}_k$ meeting the interior of R and satisfying:

$$\int_{R\cap Q} |f(y)|dy > \frac{t|R|}{2^n}$$

Since $|R| \leq |Q| < 2^n|R|$, we have:

$$\int_{R\cap Q} |f(y)|dy > \frac{t|R|}{2^n} > \frac{t|Q|}{4^n}$$

and therefore:

$$\frac{1}{|Q|}\int_Q |f(y)|dy > \frac{t}{4^n}$$

It follows that $Q \subset Q_j \in \mathcal{C}_{4^{-n}t}$ for some j.

In general for any cube Q and any $\alpha > 0$, we shall denote by Q^α the cube with the same center as Q but with side length α times that of Q. In our particular situation, since R and Q meet and $|R| \leq |Q|$, it follows that $R \subset Q^3 \subset Q_j^3$. We conclude that if $\mathcal{C}_{4^{-n}t} = \{Q_j\}$, then $E_t \subset \bigcup_j Q_j^3$ and this leads to the estimate

$$|E_t| \leq \sum_j |Q_j^3| = 3^n \sum_j |Q_j| < \frac{3^n \cdot 4^n}{t} \sum_j \int_{Q_j} |f(y)|dy \leq \frac{C}{t}\int_{\mathbb{R}^n} |f(y)|dy .$$

We have obtained the following result:

THEOREM 1.6. Let $f \in L^1(\mathbb{R}^n)$. *Then, for every* $t > 0$, *the set* $E_t = \{x \in \mathbb{R}^n : M f(x) > t\}$ *is contained in the union of a family of cubes* $\{Q_j^3\}$ *which result from expanding by a factor of* 3 *the nonoverlapping cubes* $\{Q_j\}$ *which satisfy:*

$$\frac{t}{4^n} < \frac{1}{|Q_j|} \int_{Q_j} |f(x)|dx \leq \frac{t}{2^n} \tag{1.7}$$

It follows that:

$$|E_t| \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|dx \tag{1.8}$$

where the constant C *depends only on the dimension* n. □

We shall derive some consequences of the basic inequality (1.8) which illustrate the role played by the maximal operator M. The importance of the operator M stems from the fact that it controls many operators arising naturally in Analysis. As an example, we are going to prove an extension of Lebesgue's differentiation theorem.

THEOREM 1.9. Let $f \in L^1_{loc}(\mathbb{R}^n)$. *For* $x \in \mathbb{R}^n$ *and* $r > 0$, *let* $Q(x;r) = \{y \in \mathbb{R}^n : |y-x|_\infty \equiv \max_j |y_j - x_j| \leq r\}$. *Then, for almost every* $x \in \mathbb{R}^n$:

$$\frac{1}{|Q(x;r)|} \int_{Q(x;r)} |f(y) - f(x)|dy \to 0 \quad \text{as} \quad r \to 0. \tag{1.10}$$

Proof: We may assume $f \in L^1(\mathbb{R}^n)$. It will be enough to show that, for every $t > 0$, the set

$$A_t = \{x \in \mathbb{R}^n : \limsup_{r\to 0} \frac{1}{|Q(x;r)|} \int_{Q(x;r)} |f(y) - f(x)|dy > t\}$$

has zero measure. Indeed, the set where (1.10) does not hold, is precisely $\bigcup_{j=1}^{\infty} A_{1/j}$.

Given $\varepsilon > 0$, we can write $f = g + h$, where g is continuous with compact support and $\int|h| < \varepsilon$. For g we clearly have:

$$\frac{1}{|Q(x;r)|} \int_{Q(x;r)} |g(y) - g(x)|dy \to 0 \quad \text{as} \quad r \to 0.$$

Therefore

$$\limsup_{r\to 0} \frac{1}{|Q(x;r)|} \int_{Q(x;r)} |g(y) - g(x)|\,dy \leq M\,h\,(x) + |h(x)|$$

and

$$A_t \subset \{x \in \mathbb{R}^n : M\,h\,(x) > t/2\} \cup \{x \in \mathbb{R}^n : |h(x)| > t/2\}.$$

But

$$|\{x \in \mathbb{R}^n : M\,h\,(x) > t/2\}| \leq C\|h\|_1/t < C\varepsilon/t$$

and

$$|\{x \in \mathbb{R}^n : |h(x)| > t/2\}| \leq \int_{\mathbb{R}^n} \frac{2|h(x)|}{t}\,dx \leq 2\varepsilon/t.$$

Thus A_t is contained in a set of measure $\leq C\varepsilon/t$. Since this can be done for any $\varepsilon > 0$, we get $|A_t| = 0$. □

The points x for which (1.10) holds are called Lebesgue points for f. We can rephrase theorem 1.9 by saying that almost every point $x \in \mathbb{R}^n$ is a Lebesgue point for f.

COROLLARY 1.11. Let $f \in L^1_{loc}(\mathbb{R}^n)$. *Then, for every Lebesgue point* x *for* f *and, therefore, for a.e. point* $x \in \mathbb{R}^n$:

a) $$f(x) = \lim_{r\to 0} \frac{1}{|Q(x;r)|} \int_{Q(x;r)} f(y)\,dy$$

b) $$|f(x)| \leq M\,f\,(x).$$

Proof: In order to prove a), just note that

$$\left|\frac{1}{|Q(x;r)|} \int_{Q(x;r)} f(y)\,dy - f(x)\right| \leq \frac{1}{|Q(x;r)|} \int_{Q(x;r)} |f(y) - f(x)|\,dy$$

whilst b) is an immediate consequence of a). □

If x is a Lebesgue point for f and we have a sequence of cubes $Q_1 \supset Q_2 \supset \dots$ with $\bigcap_j Q_j = \{x\}$, then:

$$f(x) = \lim_{j\to\infty} \frac{1}{|Q_j|} \int_{Q_j} f(y)\,dy$$

Indeed if Q_j has side length r_j, we have $Q_j \subset Q(x;r_j)$ and

$r_j^n = |Q_j| \downarrow 0$, so that $r_j \to 0$. Therefore

$$\left|\frac{1}{|Q_j|}\int_{Q_j} f(y)dy - f(x)\right| \leq \frac{1}{|Q_j|}\int_{Q_j} |f(y) - f(x)|dy \leq$$

$$\leq \frac{2^n}{|Q(x;r_j)|}\int_{Q(x;r_j)} |f(y) - f(x)|dy \to 0 \quad \text{as} \quad j \to \infty.$$

Let $f \in L^1(\mathbb{R}^n)$ and let $\mathcal{C}_t \equiv \mathcal{C}_t(f) = \{Q_j\}$ be the collection formed by those maximal dyadic cubes over which the average of $|f|$ is $> t$ (called Calderón-Zygmund cubes for f corresponding to t). Let $x \notin \bigcup_j Q_j$. Then the average of $|f|$ over any dyadic cube will be $\leq t$. Let $\{R_k\}$ be a sequence of dyadic cubes of decreasing size such that $\bigcap_k R_k = \{x\}$. For each of them

$$\frac{1}{|R_k|}\int_{R_k} |f(y)|dy \leq t$$

If, besides, x is a Lebesgue point for f (and hence for $|f|$) we get, by passing to the limit : $|f(x)| \leq t$. Thus $|f(x)| \leq t$ for a.e. $x \notin \bigcup_j Q_j$.

The splitting of the space $\mathbb{R}^n$ into a subset Ω made up of non-overlapping cubes Q_j over each of which the average of $|f|$ is between t and $2^n t$, and a complementary subset F where $|f(x)| \leq t$ a.e., is the first step of the so-called Calderón-Zygmund decomposition which will be a tool of constant use in the sequel. Let us record the following:

THEOREM 1.12. Given $f \in L^1(\mathbb{R}^n)$ *and* $t > 0$, *there is a family of non-overlapping cubes* $\mathcal{C}_t = \mathcal{C}_t(f)$ *consisting of those maximal dyadic cubes over which the average of* $|f|$ *is* $> t$. *This family satisfies:*

(i) *for every* $Q \in \mathcal{C}_t$: $t < \frac{1}{|Q|}\int_Q |f(x)|dx \leq 2^n t$

(ii) *for a.e.* $x \notin \bigcup Q$, *where* Q *ranges over* $\mathcal{C}_t$, *is* $|f(x)| \leq t$.

Besides, for every $t > 0$, $E_t = \{x \in \mathbb{R}^n : Mf(x) > t\} \subset \bigcup Q^3$ *where* Q *ranges over* $\mathcal{C}_{4^{-n}t}$. $\square$

Next we are going to study a useful generalization of the maximal function. Let μ be a positive Borel measure on $\mathbb{R}^n$, finite on compact sets and satisfying the following "doubling" condition:

$$(1.13) \qquad \mu(Q^2) \leq C\mu(Q)$$

for every cube Q, with $C > 0$ independent of Q. We shall often say simply that μ is a doubling measure. This implies, of course, that for every $\alpha > 0$, there is a constant $C = C_\alpha > 0$, depending only on α, such that $\mu(Q^\alpha) \leq C\mu(Q)$ for every cube Q. Since we are in $\mathbb{R}^n$, the finiteness of μ on compact subsets implies that μ is regular. Notice that for every cube Q, $\mu(Q) > 0$. Indeed, if we had $\mu(Q) = 0$ for some cube Q, we would have $\mu(Q^k) \leq C_k\mu(Q) = 0$, from which $\mu(\mathbb{R}^n) = 0$, which is excluded as trivial.

Now, for μ as above, $f \in L^1_{loc}(\mu)$ and $x \in \mathbb{R}^n$, define:

$$M_\mu f(x) = \sup_{x\in Q} \frac{1}{\mu(Q)} \int_Q |f(y)|\,d\mu(y)$$

where the sup. is taken over all cubes Q containing x. As before, we obtain the same value $M_\mu f(x)$ if we just take in the definition, those cubes Q containing x in their interior. This is a consequence of the regularity of μ.

Let $f \in L^1(\mu)$ and $t > 0$. We want to obtain a Calderón-Zygmund decomposition for f and t relative to the measure μ and, at the same time, we want to estimate the μ-measure of the set $E_t = \{x \in \mathbb{R}^n : M_\mu f(x) > t\}$, which is, of course, open. We are going to apply the same ideas that led to theorem 1.12. We need to make two observations.

First, we are going to see that there is a constant $K > 1$ such that, every time we have dyadic cubes $Q' \subsetneq Q$, it follows that $\mu(Q) \geq K\mu(Q')$. To see this, let Q'' be a dyadic cube contained in Q, contiguous to Q' and with the same diameter. Then $Q' \subset Q''^3$ and consequently, for $C = C_3$ we have: $\mu(Q') \leq C\mu(Q'') \leq C(\mu(Q) - \mu(Q'))$. This implies $(1 + C)\mu(Q') \leq C\mu(Q)$, which gives $\mu(Q) \geq K\mu(Q')$ with $K = (1 + C)/C > 1$, and our claim is justified. As a consequence, if we have a strictly increasing sequence of dyadic cubes $Q_0 \subsetneq Q_1 \subsetneq Q_2 \subsetneq \cdots$, we have the inequality $\mu(Q_k) \geq K^k \mu(Q_0) \to \infty$ as $k \to \infty$. The conclusion is that if a chain of dyadic cubes is such that the μ-measure of the cubes is bounded above, then their diameter is also bounded above or, what is the same, the chain terminates at a given cube containing all the others.

The other observation we need is the following: for every $A > 0$ there is $B > 0$ such that every time we have cubes Q and R which meet and satisfy $|Q| < A|R|$, then they also satisfy $\mu(Q) < B\mu(R)$. To see this, just note that $Q \subset R^\alpha$ with $\alpha = 2A^{1/n} + 1$, and choose B accordingly.

Now we go back to our problem. Denote by $\mathcal{C}_t = \mathcal{C}_t(f;\mu)$ the collection formed by the maximal dyadic cubes Q satisfying the condition

$$t < \frac{1}{\mu(Q)} \int_Q |f(y)|d\mu(y).$$

Since this condition forces $\mu(Q)$ to be bounded by $t^{-1} \int_{\mathbb{R}^n} |f(y)|d\mu(y) < \infty$, our first observation implies that every dyadic cube satisfying our condition is contained in some member of $\mathcal{C}_t$. Take $Q \in \mathcal{C}_t$. If $Q \in \mathcal{D}_k$ and Q' is the only cube in $\mathcal{D}_{k-1}$ containing Q, we have

$$\frac{1}{\mu(Q')} \int_{Q'} |f(y)|d\mu(y) \leq t.$$

But $Q' \subset Q^3$, so that $\mu(Q') \leq C\mu(Q)$. Therefore

$$\frac{1}{\mu(Q)} \int_Q |f(x)|d\mu(x) \leq \frac{C}{\mu(Q')} \int_{Q'} |f(x)|d\mu(x) \leq Ct$$

Thus, for every $Q \in \mathcal{C}_t$:

$$t < \frac{1}{\mu(Q)} \int_Q |f(x)|d\mu(x) \leq Ct.$$

Let $x \in E_t$, that is: $M_\mu f(x) > t$. Then there will be some cube R containing x in its interior such that

$$\frac{1}{\mu(R)} \int_R |f(y)|d\mu(y) > t.$$

As we did for the case when μ = Lebesgue measure, let Q be a dyadic cube which overlaps with R and satisfies $|R| \leq |Q| < |R|2^n$

and

$$\int_{R\cap Q} |f|\,d\mu > 2^{-n} t\mu(R).$$

Let B be the constant corresponding to $A = 2^n$ in our second observation. Then

$$\int_{R\cap Q} |f|\,d\mu > B^{-1}2^{-n} t\mu(Q)$$

and hence

$$\frac{1}{\mu(Q)} \int_Q |f|\,d\mu > \frac{t}{2^n B} .$$

It follows that $Q \subset Q_j$ for some $Q_j \in \mathcal{C}_{2^{-n}B^{-1}t}$ and $R \subset Q^3 \subset Q_j^3$. If $\mathcal{C}_{2^{-n}B^{-1}t} = \{Q_j\}$, then $E_t \subset \bigcup_j Q_j^3$.

We finally obtain the estimate:

$$\mu(E_t) \leq \sum_j \mu(Q_j^3) \leq C \sum_j \mu(Q_j) \leq \frac{C\cdot 2^n \cdot B}{t} \sum_j \int_{Q_j} |f|\,d\mu \leq \frac{C}{t} \int_{\mathbb{R}^n} |f|\,d\mu .$$

This basic estimate can be used to extend theorem 1.9 and its corollary, obtaining:

THEOREM 1.14. With μ *as above, let* $f \in L^1_{loc}(\mu)$. *Then, for almost every* $x \in \mathbb{R}^n$ *(with respect to* μ*) :*

a) $$\lim_{r\to 0} \frac{1}{\mu(Q(x;r))} \int_{Q(x;r)} |f(y)-f(x)|\,d\mu(y) = 0$$

b) $$f(x) = \lim_{r\to 0} \frac{1}{\mu(Q(x;r))} \int_{Q(x;r)} f(y)\,d\mu(y)$$

c) $$|f(x)| \leq M_\mu f(x) \qquad \square$$

In particular, if $\mathcal{C}_t(f,\mu) = \{Q_j\}$, we have $|f(x)| \leq t$ for a.e. $x \notin \bigcup_j Q_j$ (with respect to μ).

We can finally state the following:

THEOREM 1.15. For μ *as above, let* $f \in L^1(\mu)$ *and* $t > 0$. *Then, there is a family of non-overlapping cubes* $\mathcal{C}_t = \mathcal{C}_t(f,\mu)$, *consist-*

ing of those maximal dyadic cubes over which the average of $|f|$ *relative to* μ *is* $> t$, *which satisfies*

i) for every $Q \in \mathcal{C}_t : t < \frac{1}{\mu(Q)} \int_Q |f| d\mu \leq Ct$

ii) for a.e. $x \notin \cup Q$ *where* Q *ranges over* $\mathcal{C}_t$ *(and a.e. is with respect to* μ*) we have:* $|f(x)| \leq t$. *Besides, for every* $t > 0$, *the set* $E_t = \{x \in \mathbb{R}^n : M_\mu f(x) > t\}$ *is contained in* $\cup Q^3$ *where* Q *ranges over* $\mathcal{C}_{t/C}$, *and we have an estimate:*

$$\mu(E_t) \leq Ct^{-1} \int_{\mathbb{R}^n} |f| d\mu.$$

Here C represents an absolute constant, possibly different at each occurrence.

2. NORM ESTIMATES FOR THE MAXIMAL FUNCTION

THEOREM 2.1. Let f *be a measurable function on* $\mathbb{R}^n$ *and let* $t > 0$. *Then we have the following estimates for the Lebesgue measure of the set* $E_t = \{x \in \mathbb{R}^n : Mf(x) > t\}$:

$$(2.2) \qquad |E_t| \leq \frac{C}{t} \int_{\{x \in \mathbb{R}^n : |f(x)| > t/2\}} |f(x)| dx.$$

$$(2.3) \qquad |E_t| \geq \frac{C'}{t} \int_{\{x \in \mathbb{R}^n : |f(x)| > t\}} |f(x)| dx$$

with constants C *and* C' *which do not depend on* f *or* t.

Proof: Write $f = f_1 + f_2$, where $f_1(x) = f(x)$ if $|f(x)| > t/2$, and $f_1(x) = 0$ otherwise. Then $Mf(x) \leq Mf_1(x) + Mf_2(x) \leq Mf_1(x) +$

$+ t/2$, since $f_2 \leq t/2$ implies that $Mf_2 \leq t/2$ also. Thus

$$|E_t| \leq |\{x \in \mathbb{R}^n : Mf_1(x) > t/2\}| \leq \frac{3^n \cdot 4^n}{t/2} \int_{\mathbb{R}^n} |f_1(x)|dx =$$

$$= \frac{C}{t} \int_{\{x:|f(x)| > t/2\}} |f(x)|dx$$

which gives (2.2).

As for (2.3), we may assume that $f \in L^1(\mathbb{R}^n)$ (otherwise we truncate and apply a limiting process). Then we use the Calderón-Zygmund decomposition for f and t. We have non-overlapping cubes Q_j, such that

$$t < \frac{1}{|Q_j|} \int_{Q_j} |f(x)|dx \leq 2^n t$$

for every j, and $|f(x)| \leq t$ for a.e. $x \notin \bigcup_j Q_j$. Now, since $x \in Q_j$ implies that $Mf(x) > t$, we can write:

$$|E_t| \geq \sum_j |Q_j| \geq \frac{1}{2^n t} \sum_j \int_{Q_j} |f(x)|dx \geq \frac{1}{2^n t}\int_{\{x:|f(x)|>t\}} |f(x)|dx$$

and (2.3) is proved with $C' = 2^{-n}$. □

The next result is proved in exactly the same way.

THEOREM 2.4. Suppose μ *is a regular positive Borel measure in* $\mathbb{R}^n$ *satisfying a "doubling" condition like (1.13). Then, there are constants,* C, C', *such that, for any Borel function* f *and any* $t > 0$*:*

$$\frac{C'}{t} \int_{\{x\in\mathbb{R}^n : |f(x)|>t\}} |f(x)|d\mu(x) \leq \mu(\{x : M_\mu f(x) > t\}) \leq$$

$$\leq \frac{C}{t} \int_{\{x\in\mathbb{R}^n : |f(x)|>t/2\}} |f(x)|d\mu(x) \qquad \square$$

From theorem 2.1. we can easily derive several norm estimates for the maximal function.

THEOREM 2.5. For every p *with* $1 < p < \infty$, *there is a constant* $C_p > 0$ *such that, for every* $f \in L^p(\mathbb{R}^n)$:

$$\left(\int_{\mathbb{R}^n} (Mf(x)^p dx\right)^{1/p} \leq C_p\left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}$$

Proof:

$$\int_{\mathbb{R}^n} (Mf(x))^p dx = p\int_0^\infty t^{p-1}|\{x : Mf(x) > t\}|dt \leq$$

$$\leq Cp\int_0^\infty t^{p-2}\int_{\{x:|f(x)|>t/2\}} |f(x)|dxdt = Cp\int_{\mathbb{R}^n}\left(\int_0^{2|f(x)|} t^{p-2}dt\right)|f(x)|dx =$$

$$= \frac{C\cdot 2^{p-1}p}{p-1}\int_{\mathbb{R}^n} |f(x)|^p dx. \qquad \square$$

In exactly the same way we obtain the following

THEOREM 2.6. Let μ *be a regular positive Borel measure in* $\mathbb{R}^n$ *satisfying a "doubling" condition like (1.13). Then, for each* p *with* $1 < p < \infty$, *there is a constant* $C_p > 0$ *such that for every* $f \in L^p(\mu)$:

$$\left(\int_{\mathbb{R}^n} (M_\mu f(x))^p d\mu(x)\right)^{1/p} \leq C_p \left(\int_{\mathbb{R}^n} |f(x)|^p d\mu(x)\right)^{1/p} \qquad \square$$

We have seen that the operator M is bounded in $L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. However, it is not bounded in $L^1(\mathbb{R}^n)$. As a matter of fact, for $f \geq 0$, Mf fails to be in L^1 unless $f(x) = 0$ for a.e. x, since $Mf(x) \geq C|x|^{-n}$ for large x, with $C > 0$ if f is not trivial. The basic estimate (1.8) acts as a substitute for L^1-boundedness, as will be seen below. As regards local integrability of the maximal function, we have the following result:

THEOREM 2.7. Let f *be an integrable function supported in a ball* $B \subset \mathbb{R}^n$. *Then* Mf *is integrable over* B *if and only if:*

$$(2.8) \qquad \int_B |f(x)|\log^+|f(x)|dx < \infty.$$

Proof: If (2.8) holds, then

$$\int_B Mf(x)dx = \int_0^\infty |\{x \in B : Mf(x) > t\}|dt = 2\int_0^\infty |\{x \in B : Mf(x) > 2t\}|dt \leq$$

$$\leq 2\left(\int_0^1 |B|dt + \int_1^\infty |E_{2t}|dt\right) \leq 2|B| + C\int_1^\infty \frac{1}{t}\int_{\{x:|f(x)|>t\}} |f(x)|dxdt =$$

$$= 2|B| + C\int_{\mathbb{R}^n} |f(x)|\int_1^{|f(x)|}\frac{dt}{t}\,dx = 2|B| + C\int_{\mathbb{R}^n} |f(x)|\log^+|f(x)|dx.$$

Observe that for this part of the proof we do not need to use the fact that f is supported in B. Indeed, the same proof shows that if $\int_{\mathbb{R}^n} |f(x)|\log^+|f(x)|dx < \infty$, which we shall indicate by saying that $f \in L\log L(\mathbb{R}^n)$, then Mf is locally integrable.

Going back to the proof of the theorem, suppose that $\int_B Mf(x)dx < \infty$. If we denote by B' the ball concentric with B but with radius twice as big, we can easily see that $\int_{B'} Mf(x)dx < \infty$. It is sufficient to realize that for $x \in B'\setminus B$, $Mf(x) \leq CMf(x^*)$, where x^* is the point symmetric to x with respect to the boundary of B. In the complement of B', Mf is bounded, since we are at least a fixed distance far from the support of f. It is also obvious that $Mf(x) \to 0$ as $|x| \to \infty$. Thus, we conclude that, for any $t_0 > 0$

$$\int_{\{x:Mf(x)>t_0\}} Mf(x)dx < \infty.$$

For $t_0 = 1$, this integral is bigger than or equal to

$$\int_1^\infty |\{x : Mf(x) > t\}|dt \geq \int_1^\infty \frac{1}{2^n t}\int_{\{x\in\mathbb{R}^n : |f(x)|>t\}} |f(x)|dxdt =$$

$$= \frac{1}{2^n}\int_{\mathbb{R}^n} |f(x)|\int_1^{|f(x)|}\frac{dt}{t}dx = \frac{1}{2^n}\int_{\mathbb{R}^n} |f(x)|\log^+|f(x)|dx$$

Thus (2.8) holds and the proof is complete. □

The theorem extends clearly to M_μ for a measure μ satisfying a doubling condition. There is no need to write a new statement.

Suppose now that we have two measure spaces with respective measures μ and ν, and that T is an operator bounded from $L^p(\mu)$ to $L^q(\nu)$, that is:

$$\text{(2.9)} \qquad \left(\int |Tf|^q d\nu\right)^{1/q} \leq C\left(\int |f|^p d\mu\right)^{1/p}$$

Then,

$$\nu(\{x : |Tf(x)| > t\}) \leq \int_{\{x:|Tf(x)|>t\}} (|Tf(x)|/t)^q d\nu \leq$$

$$\leq \frac{1}{t^q}\int |Tf|^q d\nu \leq \frac{C^q}{t^q}\left(\int |f|^p d\mu\right)^{q/p}$$

and we obtain

$$\nu(\{x : |Tf(x)| > t\}) \leq \left(\frac{C\|f\|_{L^p(\mu)}}{t}\right)^q \tag{2.10}$$

When T satisfies (2.10) we say that the operator T is of weak type (p,q) with respect to the pair of measures (ν,μ). For example, (1.8) is read by saying that M is of weak type $(1,1)$ (with respect to Lebesgue measure). However, we know that M fails to be bounded in L^1. In general, (2.10) may hold whereas (2.9) does not hold for a given operator T. It is convenient to see (2.10) as a substitute or a weakening of (2.9). With this in mind, when (2.9) holds, we say that T is of strong type (p,q) with respect to the pair of measures (ν,μ). Sometimes it is convenient to indicate that (2.10) holds by saying that T sends $L^p(\mu)$ boundedly into $L^q_*(\nu)$. This means that we are considering the space $L^q_*(\nu)$ (called weak-$L^q(\nu)$) formed by those functions g for which

$$n_q(g) = \sup_{t>0}\,(t\cdot\nu(\{x : |g(x)| > t\})^{1/q}) < \infty.$$

This space with the "norm" n_q (fails to be subadditive even though we have a similar inequality with a constant in the right hand side) is bigger (topologically also) than $L^q(\nu)$.

Weak type inequalities such as (2.10) can be used to obtain strong type inequalities. This is what we have done to prove theorem 2.5. We are going to present a result, which is a particular case of the Marcinkiewicz interpolation theorem and is based upon the same idea as our proof of 2.5.

THEOREM 2.11. Suppose we have two measure spaces with respective measures μ *and* ν. *Let* T *be an operator sending functions in* $L^{p_0}(\mu) + L^{p_1}(\mu)$ *to* ν*-measurable functions*, $1 \leq p_0 < p_1 \leq \infty$. *Suppose that:*

i) T *is subadditive, that is, for* $f_1, f_2 \in L^{p_0}(\mu) + L^{p_1}(\mu)$,

$$|T(f_1 + f_2)(x)| \leq |Tf_1(x)| + |Tf_2(x)|,\ \nu\text{-a.e.}$$

ii) T *is of weak type* (p_o,p_o), *that is:*

$$\nu(\{x : |Tf(x)| > t\}) \leq \frac{C_o\int |f|^{p_o}d\mu}{t^{p_o}},$$

with C_o *independent of* $f \in L^{p_o}(\mu)$ *and* $t > 0$.

iii) T *is of weak type* (p_1,p_1) *which means the same as above if* $p_1 < \infty$, *while, if* $p_1 = \infty$, *weak type and strong type coincide by definition:* $\|Tf\|_{L^\infty(\nu)} \leq C_1\|f\|_{L^\infty(\mu)}$

Then, for every p *such that* $p_o < p < p_1$, T *is of strong type* (p,p), *that is:* $\int |Tf|^p d\nu \leq C_p\int |f|^p d\mu$.

Proof: Fix p with $p_o < p < p_1$ and let $f \in L^p(\mu) \subset L^{p_o}(\mu)+L^{p_1}(\mu)$. For every $t > 0$ write $f(x) = f^t(x) + f_t(x)$ where $f^t(x) = f(x)$ if $|f(x)| > t$ and $f^t(x) = 0$ otherwise. Clearly $f^t \in L^{p_o}(\mu)$, and since $|f_t(x)| \leq t$, $f_t \in L^{p_1}(\mu)$. Suppose $p_1 < \infty$. Then, since $|Tf(x)| \leq |T(f^t)(x)| + |T(f_t)(x)|$, we can write:

$$\nu(\{x : |Tf(x)| > t\}) \leq \nu(\{x : |T(f^t)(x)| > t/2\}) +$$
$$+ \nu(\{x : |T(f_t)(x)| > t/2\}) \leq \frac{C_o\int |f^t|^{p_o}d\mu}{(t/2)^{p_o}} + \frac{C_1\int |f_t|^{p_1}d\mu}{(t/2)^{p_1}}$$

Thus

$$\int |Tf(x)|^p d\nu = p\int_0^\infty t^{p-1}\nu(\{x: |Tf(x)| > t\})dt \leq$$
$$\leq 2^{p_o}C_o p\int_0^\infty t^{p-p_o-1}\int_{\{x:|f(x)|>t\}}|f(x)|^{p_o}d\mu(x)dt +$$
$$+ 2^{p_1}C_1 p\int_0^\infty t^{p-p_1-1}\int_{\{x:|f(x)|<t\}}|f(x)|^{p_1}d\mu(x)dt \leq$$
$$\leq 2^{p_o}C_o p\int\left(\int_0^{|f(x)|} t^{p-p_o-1}dt\right)|f(x)|^{p_o}d\mu(x) +$$
$$+ 2^{p_1}C_1 p\int\left(\int_{|f(x)|}^\infty t^{p-p_1-1}dt\right)|f(x)|^{p_1}d\mu(x) =$$
$$= \frac{2^{p_o}C_o p}{p-p_o}\int |f(x)|^p d\mu(x) + \frac{2^{p_1}C_1 p}{p_1-p}\int |f(x)|^p d\mu(x) = C_p\int |f(x)|^p d\mu(x).$$

For the case $p_1 = \infty$, we simply observe that

$$\nu(\{x : |Tf(x)| > t\}) \leq \nu(\{x : |T(f^{\alpha t})(x)| > t/2\})$$

where $\alpha = 2^{-1} C_1^{-1}$ and, consequently

$$\int |Tf(x)|^p \, d\nu \leq 2^{p_0} C_0 \, p \int_0^\infty t^{p-p_0-1} \int_{\{x : |f(x)|>\alpha t\}} |f(x)|^{p_0} d\mu(x)\, dt =$$

$$= 2^{p_0} C_0 \, p \int \left(\int_0^{|f(x)|/\alpha} t^{p-p_0-1} dt \right) |f(x)|^{p_0} \, d\mu(x) =$$

$$= C_p \int |f(x)|^p d\mu \qquad \square$$

Next we shall establish a general inequality for the maximal function. This inequality involves a weight function $\phi(x)$.

THEOREM 2.12. For every p *with* $1 < p < \infty$, *there is a constant* C_p *such that, for any measurable functions on* $\mathbb{R}^n$, $\phi \geq 0$ *and* f, *we have the inequality:*

$$\int_{\mathbb{R}^n} (Mf(x))^p \phi(x)\, dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^p (M\phi)(x)\, dx \tag{2.13}$$

Proof: Except when $M\phi(x) = \infty$ a.e., in which case (2.13) holds trivially, $M\phi$ is the density of a positive measure $\mu(d\mu(x) = M\phi(x)dx)$ and ϕ is the density of another positive measure $\nu(d\nu(x) = \phi(x)dx)$. (2.13) means that M is a bounded operator from $L^p(\mu)$ to $L^p(\nu)$. This is clearly true for $p = \infty$. Indeed, if $M\phi(x) = 0$ for some x, then $\phi(x) = 0$ a.e. and $L^\infty(\nu) = 0$, so that nothing needs to be proved. If $M\phi(x) > 0$ for every x and $\alpha > \|f\|_{L^\infty(\mu)}$, we have $\int_{\{|f|>\alpha\}} M\phi = 0$ and consequently $|\{x : |f(x)|>\alpha\}| = 0$ or, what is the same, $|f(x)| \leq \alpha$ a.e., from which $Mf(x) \leq \alpha$ a.e. Thus $\|Mf\|_{L^\infty(\nu)} \leq \alpha$ and, finally,

$$\|Mf\|_{L^\infty(\nu)} \leq \|f\|_{L^\infty(\mu)} .$$

Having the (∞,∞) result, if we are able to show that M is of weak type $(1,1)$ with respect to the pair (ν,μ), the interpolation theorem 2.11 will give (2.13). Thus, all we need is to show that:

$$\int_{\{x:Mf(x)>t\}} \phi(x)dx \le \frac{C}{t}\int_{\mathbb{R}^n} |f(x)|(M\phi)(x)dx \tag{2.14}$$

We can obviously assume that $f \ge 0$. We can also assume that $f \in L^1(\mathbb{R}^n)$. Indeed, we can find integrable functions f_j, such that $f_1 \le f_2 \le \cdots \uparrow f$ a.e. and observe that

$$\{x : Mf(x) > t\} = \bigcup_j \{x : Mf_j(x) > t\}.$$

So, let $f \in L^1(\mathbb{R}^n)$ and $f \ge 0$. Given $t > 0$, we know (Theorem 1.6) that there is a family of non-overlapping cubes $\{Q_j\}$ such that, for each j:

$$\frac{t}{4^n} < \frac{1}{|Q_j|}\int_{Q_j} f(x)dx \le \frac{t}{2^n}$$

and also

$$\{x : Mf(x) > t\} \subset \bigcup_j Q_j^3$$

Then

$$\int_{\{x:Mf(x)>t\}} \phi(x)dx \le \sum_j \int_{Q_j^3} \phi(x)dx \le$$

$$\le \sum_j \frac{1}{|Q_j^3|}\int_{Q_j^3} \phi(x)dx\, \frac{3^n\cdot 4^n}{t}\int_{Q_j} f(x)dx \le$$

$$\le \frac{3^n\cdot 4^n}{t}\sum_j \int_{Q_j} f(x)(M\phi)(x)dx \le \frac{C}{t}\int_{\mathbb{R}^n} f(x)(M\phi)(x)dx . \qquad \square$$

The theorem we have just proved identifies a whole class of weight functions ϕ for which the operator M is bounded in $L^p(\phi)$ for every $p \in (1,\infty]$ and of weak type $(1,1)$ with respect to ϕ, namely, the class, customarily denoted by A_1, of those $\phi \ge 0$ satisfying $M\phi(x) \le C\phi(x)$ a.e. for some constant C. For instance, it is easy to check that $\phi(x) = |x|^\alpha$ is an A_1 weight in $\mathbb{R}^n$ provided $-n < \alpha \le 0$. We shall go back to weights in chapter IV.

There is an interesting extension of theorem 2.12 whose proof is but a repetition of the arguments which led to 1.6, 2.5 and 2.12. In order to present this result we make several definitions:

Given a function f in $\mathbb{R}^n$, we define a function $\mathcal{M}f$ in $\overline{\mathbb{R}^{n+1}_+} = \{(x,t) : x \in \mathbb{R}^n,\ t \geq 0\}$ by setting

$$\mathcal{M}f(x,t) = \sup\{\frac{1}{|Q|}\int_Q |f(y)|dy \ :\ x \in Q \text{ and side lenght } (Q) \geq t\}$$

Given a positive Borel measure μ in $\overline{\mathbb{R}^{n+1}_+}$, we define a function $\mathcal{N}(\mu)$ in $\mathbb{R}^n$ by setting

$$\mathcal{N}(\mu)(x) = \sup_{x \in Q} \frac{\mu(\tilde{Q})}{|Q|} ,$$

where the sup is taken over all cubes Q containing x and for a cube Q,

$$\tilde{Q} = \{(x,t) \in \overline{\mathbb{R}^{n+1}_+} : x \in Q \text{ and } 0 \leq t \leq \text{side length } (Q)\} ,$$

that is, $\tilde{Q}$ is the cube in $\overline{\mathbb{R}^{n+1}_+}$ having Q as a face. With the above definitions, we can state the following:

<u>*THEOREM 2.15. For every*</u> p <u>*with*</u> $1 < p < \infty$, <u>*there is a constant*</u> C_p <u>*such that, for every*</u> f <u>*and every*</u> μ:

$$(2.16) \quad \left(\int_{\overline{\mathbb{R}^{n+1}_+}} \{\mathcal{M}f(x,t)\}^p d\mu(x,t)\right)^{1/p} \leq C_p\left(\int_{\mathbb{R}^n} |f(x)|^p \mathcal{N}\mu(x)dx\right)^{1/p}$$

<u>*Proof:*</u> Before we start the proof, let us note that this result includes the previous one. Just take $d\mu(x,t) = \phi(x)dx \otimes d\delta(t)$ where δ is the unit mass concentrated at the origin in the t axis. Then since $\mathcal{M}f(x,0) = Mf(x)$ and, for our particular μ, $\mathcal{N}\mu(x) = M\phi(x)$, (2.16) reduces to (2.13) in this case.

Now we prove the theorem. As in the proof of the preceding result, if we exclude the trivial case when $\mathcal{N}\mu(x) = \infty$ a.e., $\mathcal{N}\mu(x)dx$ is a measure and $\mathcal{M}$ is bounded from $L^\infty(\mathcal{N}\mu(x)dx)$ to $L^\infty(\overline{\mathbb{R}^{n+1}_+},\mu)$. All we need to do is to prove a weak type (1,1) inequality and then use interpolation. If we call $E_\alpha = \{(x,t) \in \overline{\mathbb{R}^{n+1}_+} : \mathcal{M}f(x,t) > \alpha\}$, we have to show that there is a constant C such that, for every $\alpha > 0$

$$\mu(E_\alpha) \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(x)| \, \mathcal{N}\mu(x)\,dx.$$

Let us see this. We may assume that f is integrable. Fix $\alpha > 0$ and suppose that $(x,t) \in E_\alpha$. Then, there is a cube R containing x, with side length $(R) \geq t$ and such that

$$\frac{1}{|R|} \int_R |f(y)|\,dy > \alpha.$$

Let k be the only integer such that $2^{-(k+1)n} < |R| \leq 2^{-kn}$. As in the proof that led to theorem 1.6, there is some $Q \in \mathcal{D}_k$ which meets the interior of R and satisfies

$$\int_{R\cap Q} |f(y)|\,dy > \frac{\alpha|R|}{2^n} > \frac{\alpha|Q|}{4^n},$$

so that

$$\frac{1}{|Q|} \int_Q |f(y)|\,dy > \frac{\alpha}{4^n}.$$

It follows that $Q \subset Q_j \in \mathcal{C}_{4^{-n}\alpha}$ for some j and $x \in R \subset Q^3 \subset Q_j^3$. On the other hand $t \leq$ side length $(R) \leq$ side length (Q_j^3), so that $(x,t) \in \widetilde{Q_j^3}$. Thus, we have seen that $E_\alpha \subset \bigcup_j \widetilde{Q_j^3}$ where $\mathcal{C}_{4^{-n}\alpha} = \{Q_j\}$. Then

$$\mu(E_\alpha) \leq \sum_j \mu(\widetilde{Q_j^3}) \leq \sum_j \frac{\mu(\widetilde{Q_j^3})}{|Q_j^3|} \, \frac{3^n \cdot 4^n}{\alpha} \int_{Q_j} |f(y)|\,dy \leq$$

$$\leq \frac{C}{\alpha} \sum_j \int_{Q_j} |f(x)| \, \mathcal{N}\mu(x)\,dx \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(x)| \, \mathcal{N}\mu(x)\,dx. \qquad \square$$

In particular, if the measure μ is such that

$$(2.17) \quad \mu(\tilde{Q}) \leq C|Q|$$

for every cube $Q \subset \mathbb{R}^n$ with C independent of Q, then $\mathcal{N}\mu(x) \leq C$ and (2.16) implies that $f \longmapsto \mathcal{M}f$ is an operator bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^{n+1}_+, \mu)$ for every p with $1 < p < \infty$. Actually, given any p with $1 < p < \infty$, (2.17) is not only sufficient but also necessary for $\mathcal{M}$ to be bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^{n+1}_+, \mu)$. Indeed, since $\mathcal{M}(\chi_Q)(x,t) \geq 1$ for every $(x,t) \in \tilde{Q}$, the boundedness of $\mathcal{M}$ implies that

$$\mu(\tilde{Q}) \leq \int_{\tilde{Q}} (M(\chi_Q)(x,t))^p d\mu(x,t) \leq C|Q|$$

The importance of the operator M stems from the fact that Mf controls the Poisson integral of f, $P(f)$, defined by:

$$P(f)(x,t) = c_n \int_{\mathbb{R}^n} \frac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} f(y)dy$$

for $x \in \mathbb{R}^n$ and $t > 0$, where

$$c_n = \left(\int_{\mathbb{R}^n} \frac{dx}{(|x|^2+1)^{\frac{n+1}{2}}}\right)^{-1} = \Gamma(\tfrac{n+1}{2})/\pi^{\frac{n+1}{2}}$$

Indeed:

$$|P(f)(x,t)| \leq c_n \left\{ \int_{|y-x|\leq t} \frac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} |f(y)|dy \; + \right.$$

$$\left. + \sum_{k=0}^{\infty} \int_{2^k t<|y-x|\leq 2^{k+1}t} \frac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} |f(y)|dy \right\} \leq$$

$$\leq c_n \left\{ \frac{1}{t^n} \int_{|y-x|\leq t} |f(y)|dy + \sum_{k=0}^{\infty} \frac{t}{(2^k t)^{n+1}} \int_{|y-x|\leq 2^{k+1}t} |f(y)|dy \right\} \leq C\, Mf(x,t)$$

Actually, for $f \geq 0$

$$P(f)(x,t) = c_n \int_{\mathbb{R}^n} \frac{t}{(|x-y|^2+t^2)^{\frac{n+1}{2}}} f(y)dy \geq \frac{C}{t^n} \int_{|x-y|\leq t} f(y)dy$$

In particular if $f = \chi_Q$ and $(x,t) \in \tilde{Q}$, we get $P(\chi_Q)(x,t) \geq a_n > 0$, where a_n depends only on the dimension n. Then, the same argument that we used for M shows that if the operator $f \longmapsto P(f)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\overline{\mathbb{R}^{n+1}_+},\mu)$, then μ satisfies (2.17). The measures μ satisfying (2.17) are called Carleson measures. We can state the following:

THEOREM 2.18. Let μ *be a positive Borel measure on* $\overline{\mathbb{R}^{n+1}_+}$ *and let* $1 < p < \infty$. *Then* $f \longmapsto P(f)$ *is a bounded operator from* $L^p(\mathbb{R}^n)$ *to* $L^p(\overline{\mathbb{R}^{n+1}_+},\mu)$ *if and only if* μ *is a Carleson measure, that is, if and only if* (2.17) *holds for some constant* C. □

Observe that the condition obtained does not depend on p and is also equivalent to the fact that $f \longmapsto P(f)$ sends $L^1(\mathbb{R}^n)$ boundedly into $L^1_*(\overline{\mathbb{R}^{n+1}_+},\mu)$.

3. THE SHARP MAXIMAL FUNCTION AND THE SPACE B.M.O.

For a real locally integrable function f in $\mathbb{R}^n$, the sharp maximal function $f^\#$ is defined at $x \in \mathbb{R}^n$ by setting

$$f^\#(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,dy$$

where the sup. is taken over all the cubes Q containing x, and f_Q stands for the average of f over Q, that is:

$$f_Q = \frac{1}{|Q|} \int_Q f(x)\,dx.$$

The sharp maximal operator $f \longmapsto f^\#$ is an analogue of the Hardy-Littlewood maximal operator M, but it has certain advantages over it which we shall presently see. Of course, $f^\#(x) \leq 2Mf(x)$. It is also clear that in the definition of $f^\#(x)$ one can take only those cubes Q containing x in its interior

Actually

$$(3.1) \qquad f^\#(x) \cong \sup_{x\in Q} \inf_{a\in\mathbb{R}} \frac{1}{|Q|} \int_Q |f(y) - a|\,dy$$

where $\cong$ is used to indicate that each side is bounded by the other times an absolute constant. It is clear that the right hand side of (3.1) is $\leq f^{\#}(x)$. For the opposite inequality we see that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq \frac{1}{|Q|}\int_Q |f(x) - a|dx + |f_Q - a| \leq$$

$$\leq \frac{2}{|Q|}\int_Q |f(x) - a|dx$$

for every $a \in \mathbb{R}$. It follows that $f^{\#}(x)$ is bounded by twice the right hand side of (3.1). We also note that:

$$(3.2) \qquad (|f|)^{\#}(x) \leq 2f^{\#}(x)$$

Indeed

$$\frac{1}{|Q|}\int_Q \big||f(x)| - |f_Q|\big|dx \leq \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq f^{\#}(x)$$

for each $x \in Q$. By using (3.1), we get (3.2).

If f is such that $f^{\#}$ is bounded, we say that f is a function of bounded mean osciallation, and we denote by the initials B.M.O. the space formed by these functions. Thus

$$\text{B.M.O.} = \{f \in L^1_{loc}(\mathbb{R}^n) : f^{\#} \in L^\infty\}$$

We write $\text{B.M.O.}(\mathbb{R}^n)$ when we need to specify the underlying space.

For $f \in$ B.M.O. we write

$$\|f\|_* = \|f^{\#}\|_\infty = \sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx.$$

Of course we get after (3.1):

$$\frac{1}{2}\|f\|_* \leq \sup_Q .\inf_{a\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f(x) - a|dx \leq \|f\|_*$$

Thus, in order to be able to say that $f \in$ B.M.O., it suffices to make sure that there exists $C < \infty$ and, for each Q, a constant a_Q such that

$$\frac{1}{|Q|}\int_Q |f(x) - a_Q|dx \leq C.$$

Then $\|f\|_* \leq 2C$. This is the usual way to see that a certain $f \in$ B.M.O.

Clearly, $f \longmapsto \|f\|_*$ is a seminorm and $\|f\|_* = 0$ if and only if f is constant. It is natural to consider the quotient space of B.M.O. modulo constants, which is a normed space and, actually a Banach space (the completeness is left as an exercise for the reader. It can be seen in Neri [1]). This space of equivalence classes modulo constants will also be called B.M.O. The ambiguity does not cause any problem.

Of course $L^\infty \subset$ B.M.O. and $\|f\|_* \leq 2\|f\|_\infty$. However, there are unbounded B.M.O. functions as we shall soon see. We shall give two results which provide many examples of B.M.O. functions.

THEOREM 3.3. If w *is an* A_1 *weight, that is, if* $Mw(x) \leq Cw(x)$ *a.e., then* $\log w$ *is in* B.M.O. *with a norm depending only on the* A_1 *constant for* w, *i.e. the smallest* C.

Proof: Call $\log w = \phi$, that is $w = e^\phi$. We have, for every cube Q

$$\frac{1}{|Q|}\int_Q e^{\phi(x)}dx \leq Ce^{\phi(x)} \qquad \text{for a.e. } x \in Q$$

or, equivalently

$$\left(\frac{1}{|Q|}\int_Q e^{\phi(x)}dx\right) \cdot \operatorname*{ess.\,sup.}_{x\in Q}(e^{-\phi(x)}) \leq C$$

But $\operatorname*{ess.\,sup.}_{x\in Q}(e^{-\phi(x)}) = \exp(-\operatorname*{ess.inf.}_{x\in Q}\phi(x))$, and Jensen's inequality implies

$$\frac{1}{|Q|}\int_Q e^{\phi(x)}dx \geq \exp(\phi_Q)$$

Thus, $\exp(\phi_Q - \text{ess. inf. } \phi(x)) \leq C$ and, consequently, ϕ satisfies, for some other constant C independent of Q, the property

$$\phi_Q - \operatorname*{ess.\,inf.}_{x\in Q}\phi(x) \leq C$$

We express this by saying that ϕ is of bounded lower oscillation, and denote by B.L.O. the class formed by all the functions of bounded lower oscillation. Now, we see that B.L.O. $\subset$ B.M.O. Indeed

$$|\phi(x) - \phi_Q| \leq (\phi(x) - \operatorname*{ess.\,inf.}_Q \phi) + (\phi_Q - \operatorname*{ess.\,inf.}_Q \phi) \text{ a.e.}$$

Averaging over Q we obtain

$$\frac{1}{|Q|}\int_Q |\phi(x) - \phi_Q|dx \leq 2(\phi_Q - \operatorname*{ess.\,inf.}_Q \phi)$$

and the inclusion B.L.O. $\subset$ B.M.O follows readily. $\square$

Observe that the class B.L.O. introduced above fails to be a vector space even though it is stable under the sum and the product by a non-negative number. Actually B.L.O. $\cap$ (-B.L.O.) = L^∞. Indeed, if both ϕ and $-\phi$ are in B.L.O., we have at the same time

$$\phi_Q - \operatorname*{ess.\,inf.}_Q \phi \leq C \quad \text{and} \quad -\phi_Q + \operatorname*{ess.\,sup.}_Q \phi \leq C$$

Adding up we get: $\operatorname*{ess.sup}_Q \phi - \operatorname*{ess.\,inf}_Q \phi \leq C$ independent of the cube Q. This is only possible if ϕ is essentially bounded.

Theorem 3.3 gives us B.M.O. functions from A_1 weights. We shall presently see a nice way to produce A_1 weights by using the Hardy-Littlewood maximal operator M.

Let μ be a positive Borel measure on $\mathbb{R}^n$, finite on compact sets, and hence regular. It makes sense to consider the maximal function

$$M\mu(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q d\mu$$

where the sup is taken over all cubes containing x. Exactly as in the case of integrable functions, one obtains for measures the fundamental estimate

$$|\{x \in \mathbb{R}^n : M\mu(x) > t\}| \leq \frac{C}{t}\int_{\mathbb{R}^n} d\mu$$

with C depending only on the dimension. We can state the following:

THEOREM 3.4. Let μ *be a positive Borel measure such that* $M\mu(x) < \infty$ *for a.e.* $x \in \mathbb{R}^n$, *and let* $0 < \gamma < 1$. *Then the function* $w(x) = (M\mu(x))^\gamma$ *is an* A_1 *weight with a constant depending only on*

γ *and the dimension* n.

Proof: Let Q be a fixed cube. We shall see that

$$\frac{1}{|Q|}\int_Q w(x)\,dx \leq Cw(x) \quad \text{for a.e. } x \in Q$$

with C independent of Q. Let $\tilde{Q} = Q^3$, the 3-dilated of Q. We write $\mu = \mu_1 + \mu_2$ with $\mu_1 = \chi_{\tilde{Q}}\mu$, the restriction of μ to $\tilde{Q}$. Then $M\mu(x) \leq M\mu_1(x) + M\mu_2(x)$ and, since $0 < \gamma < 1$, also $(M\mu(x))^\gamma \leq (M\mu_1(x))^\gamma + (M\mu_2(x))^\gamma$. Therefore, it will be enough to see that the averages of $(M\mu_1(x))^\gamma$ and $(M\mu_2(x))^\gamma$ over Q are both bounded by $Cw(x)$ for any $x \in Q$ with C depending only on γ and the dimension. We carry out these two estimates separately:

$$\frac{1}{|Q|}\int_Q (M\mu_1(x))^\gamma dx = \frac{1}{|Q|}\int_0^\infty \gamma t^{\gamma-1}|\{x \in Q : M\mu_1(x) > t\}|\,dt =$$

$$= \frac{1}{|Q|}\left(\int_0^R + \int_R^\infty\right)$$

(we split the integral by using an arbitrary $R > 0$). Near 0 we use the trivial estimate for the distribution function, which is, clearly, $\leq |Q|$. From R to ∞ we use the weak type estimate, the distribution function is $\leq Ct^{-1}\|\mu_1\|$, where $\|\mu_1\| = \mu_1(\mathbb{R}^n)$. Thus

$$\frac{1}{|Q|}\int_Q (M\mu_1(x))^\gamma dx \leq \frac{1}{|Q|}\left(|Q|R^\gamma + C\int_R^\infty \gamma t^{\gamma-2}dt\,\|\mu_1\|\right) =$$

$$= R^\gamma\left(1 + \frac{C\gamma}{1-\gamma}\,\frac{\|\mu_1\|}{R|Q|}\right)$$

Taking $R = \|\mu_1\|\ |Q|^{-1}$, we get:

$$\frac{1}{|Q|}\int_Q (M\mu_1(x))^\gamma dx \leq C(\|\mu_1\|\ |Q|^{-1})^\gamma \leq C(\mu(\tilde{Q})/|\tilde{Q}|)^\gamma \leq Cw(x)$$

for every $x \in Q$. What we have just done is to realize that every operator of weak type $(1,1)$ in a finite measure space, actually takes L^1 boundedly into L^p, if $p < 1$. This fact is known as Kolmogorov's inequality (see also Chapter V, Lemma 2.8.)

To deal with $M\mu_2$ is even simpler. It is enough to see that, because of the fact that μ_2 lives far from Q (outside $\tilde{Q}$), for any two points $x,y \in Q$, we have $M\mu_2(x) \leq CM\mu_2(y)$, with C an absolute constant. Indeed if Q' is a cube containing x and meeting $\mathbb{R}^n \setminus \tilde{Q}$, then $Q \subset Q'^3$, so that

$$\frac{1}{|Q'|}\int_{Q'} d\mu_2 \leq \frac{3^n}{|Q'^3|}\int_{Q'^3} d\mu_2 \leq 3^n M\mu_2(y).$$

Thus

$$\frac{1}{|Q|}\int_Q (M\mu_2(x))^\gamma dx \leq C(M\mu_2(x))^\gamma \leq Cw(x)$$

for every $x \in Q$. $\square$

For example, take $\mu = \delta$, the Dirac delta or unit mass at the origin in $\mathbb{R}^n$. Then $M\delta(x) = |x|_\infty^{-n}$ where $|x|_\infty = \max_{1\leq j\leq n} |x_j|$ for $x = (x_1, \dots, x_n)$. Thus $M\delta(x) \cong |x|^{-n}$.

It follows that for any γ with $0 \leq \gamma < 1$, $|x|^{-n\gamma}$ is an A_1 weight, or, in other words $|x|^\alpha$ is an A_1 weight for any α with $-n < \alpha \leq 0$ (only for these α's actually, since w has to be locally integrable and w^{-1} locally bounded). However, our main concern here is the fact that $\log |x|$ is an example of an unbounded B.M.O. function. Note that $\log \frac{1}{|x|}$ is actually in B.L.O. In general we have

COROLLARY 3.5. a) For any positive Borel measure μ *such that* $M\mu(x) < \infty$ *for a.e.* $x \in \mathbb{R}^n$, $\log M\mu(x)$ *is a* B.M.O. *function with norm depending only on the dimension.*

b) $\log |x|$ *is in* B.M.O. $\square$

Since $(|f|)^\#(x) \leq 2f^\#(x)$, we know that $f \in$ B.M.O. implies $|f| \in$ B.M.O. Consequently B.M.O. is a lattice (if $f,g \in$ B.M.O., then the functions $\max(f,g) = (|f-g|+f+g)/2$ and $\min(f,g) = (f+g-|f-g|)/2$ will also be in B.M.O.). However, we may have $|f| \in$ B.M.O. without having $f \in$ B.M.O. For example if:

$$f(x) = \begin{cases} 0 & \text{for } |x| > 1 \\ -\log|x| & \text{if } 0 < x < 1 \\ \log|x| & \text{if } -1 < x < 0 \end{cases}$$

it is clear that $|f(x)| = \max(\log \frac{1}{|x|}, 0)$ is in B.M.O. However, f is not in B.M.O. Since f is odd and, consequently, has average 0 on every interval $[-a,a]$, we just need to observe that

$$\frac{1}{2a}\int_{-a}^{a}|f(x)|dx = \frac{1}{a}\int_{0}^{a}\log\frac{1}{x}dx = 1 - \log a \to \infty \quad \text{for} \quad a \to 0.$$

There is an intimate relation between the operator $f \longmapsto f^{\#}$ and the Hardy-Littlewood maximal operator M. It is contained in the following statement:

<u>*THEOREM 3.6. If*</u> f <u>*is such that*</u> $Mf \in L^{p_0}$ <u>*for some*</u> p_0 <u>*with*</u> $0 < p_0 < \infty$, <u>*then for every*</u> p <u>*such that*</u> $p_0 \leq p < \infty$, <u>*we have:*</u>

$$\int_{\mathbb{R}^n}(Mf(x))^p dx \leq C\int_{\mathbb{R}^n}(f^{\#}(x))^p dx$$

<u>*with*</u> C <u>*independent of*</u> f.

<u>*Proof:*</u> We may assume that $f \geq 0$ since $Mf = M(|f|)$ and $(|f|)^{\#} \leq 2f^{\#}$.

The proof is based upon the Calderón-Zygmund decomposition. First we see that this decomposition can be carried out for our function f. Let $t > 0$ and suppose Q is a cube such that $f_Q > t$. Then, for every $x \in Q$

$$t < \frac{1}{|Q|}\int_Q f(y)dy \leq Mf(x)$$

and thus

$$t^{p_0} \leq \frac{1}{|Q|}\int_Q (Mf(x))^{p_0}dx \leq \frac{1}{|Q|}\int_{\mathbb{R}^n}(Mf(x))^{p_0}dx = \frac{C}{|Q|}$$

It follows that if $Q_1 \subsetneq Q_2 \subsetneq \dots$ is an increasing family of dyadic cubes for each of which is

$$\frac{1}{|Q_k|}\int_{Q_k} f(y)dy > t,$$

then the family is necessarily finite since $|Q_k|$ is bounded independently of k. Thus, each dyadic cube Q satisfying $f_Q > t$ will be contained in a maximal one. If $\{Q_j\}$ is the family consisting of these maximal dyadic cubes, for each of them will be

$$t < \frac{1}{|Q_j|}\int_{Q_j} f(y)\,dy \le 2^n t.$$

In order to indicate the dependence on t, we shall denote this family by $\{Q_{t,j}\}$. For a.e. $x \notin \bigcup_j Q_{t,j}$ is $f(x) \le t$.

Observe that if $t < s$, then each $Q_{s,j} \subset Q_{t,k}$ for some k. Given $t > 0$ we fix $Q_o = Q_{2^{-n-1}t,j_o}$ and take $A > 0$. There are two possibilities: either

$$Q_o \subset \{x : f^{\#}(x) > t/A\} \quad \text{or} \quad Q_o \not\subset \{x : f^{\#}(x) > t/A\}.$$

In the first case

$$\sum_{\{j:Q_{t,j} \subset Q_o\}} |Q_{t,j}| \le |\{x : f^{\#}(x) > t/A\}|$$

In the second case

$$\frac{1}{|Q_o|}\int_{Q_o} |f(y) - f_{Q_o}|\,dy \le t/A$$

and, taking into account that $f_{Q_o} \le 2^n 2^{-n-1} t = t/2$, we can write:

$$\sum_{\{j:Q_{t,j} \subset Q_o\}} (t-t/2)|Q_{t,j}| \le \sum_{\{j:Q_{t,j} \subset Q_o\}} \int_{Q_{t,j}} |f(y)-f_{Q_o}|\,dy \le$$

$$\le \int_{Q_o} |f(y) - f_{Q_o}|\,dy \le A^{-1}t|Q_o|.$$

Adding up in all the possible Q_o's, we get:

$$\sum_j |Q_{t,j}| \le |\{x : f^{\#}(x) > t/A\}| + 2A^{-1}\sum_k |Q_{2^{-n-1}t,k}|.$$

Call $\alpha(t) = \sum_j |Q_{t,j}|$ and $\beta(t) = |\{x : Mf(x) > t\}|$

We know that $\alpha(t) \le \beta(t)$ and

$$\beta(t) \le \sum_j |Q^3_{4^{-n}t,j}| \le C_1\alpha(t/C_2).$$

In terms of α we have got the following inequality:

$$\alpha(t) \leqq |\{x : f^{\#}(x) > t/A\}| + 2A^{-1}\alpha(2^{-n-1}t)$$

Now, for $N > 0$, we consider

$$I_N = \int_0^N pt^{p-1}\alpha(t)dt \leqq \int_0^N pt^{p-1}\beta(t)dt \leqq$$

$$\leqq pp_o^{-1}N^{p-p_o}\int_0^N p_o t^{p_o-1}\beta(t)dt \leqq pp_o^{-1}N^{p-p_o}\int_{\mathbb{R}^n}(Mf(x))^{p_o}dx < \infty$$

since we are assuming $Mf \in L^{p_o}$. Also

$$I_N \leqq \int_0^N pt^{p-1}|\{x : f^{\#}(x) > t/A\}|dt + 2A^{-1}\int_0^N pt^{p-1}\alpha(2^{-n-1}t)dt =$$

$$= \int_0^N pt^{p-1}|\{x : f^{\#}(x) > t/A\}|dt + CA^{-1}\int_0^{2^{-n-1}N} pt^{p-1}\alpha(t)dt$$

from which:

$$I_N \leqq \int_0^N pt^{p-1}|\{x : f^{\#}(x) > t/A\}|dt + CA^{-1}I_N$$

with a C depending only on n and p. Take $A = 2C$ and obtain:

$$I_N \leqq 2\int_0^N pt^{p-1}|\{x : f^{\#}(x) > t/A\}|dt.$$

Letting $N \to \infty$, we arrive at

$$\int_0^\infty pt^{p-1}\alpha(t)dt \leqq 2\int_0^\infty pt^{p-1}|\{x : f^{\#}(x) > t/A\}|dt$$

and then

$$\int_{\mathbb{R}^n}(Mf(x))^p dx = \int_0^\infty pt^{p-1}\beta(t)dt \leqq C_1\int_0^\infty pt^{p-1}\alpha(t/C_2)dt =$$

$$= C\int_0^\infty pt^{p-1}\alpha(t)dt \leqq C\int_0^\infty pt^{p-1}|\{x : f^{\#}(x) > t\}|\,dt =$$

$$= C\int_{\mathbb{R}^n}(f^{\#}(x))^p dx. \qquad \square$$

We have seen that the maximal functions Mf and $f^{\#}$ are closely related. We have the trivial pointwise estimate $f^{\#}(x) \leq 2Mf(x)$, but we also have an estimate going in the opposite direction, this time an L^p estimate. For f "good enough", and $0 < p < \infty$, it suffices to see that $f^{\#} \in L^p$ to conclude that Mf, and therefore f, are also in L^p. For f "good" and $1 < p < \infty$, saying that $Mf \in L^p$ is equivalent to saying that $f^{\#} \in L^p$ and this is also equivalent to saying that $f \in L^p$. The situation is different for $p = \infty$. Saying that $Mf \in L^\infty$ is the same as saying that $f \in L^\infty$. However, to say that $f^{\#} \in L^\infty$ is equivalent to saying that $f \in$ B.M.O. The space B.M.O. is a natural substitute for L^∞, and the operator $f \longmapsto f^{\#}$ allows us to treat the spaces L^p and B.M.O. in a uniform way. We shall see that many classical operators carry L^∞ to B.M.O. boundedly. This is what makes the following interpolation theorem interesting.

THEOREM 3.7. Let T *be a linear operator bounded in* L^{p_0} *for some* p_0 *with* $1 < p_0 < \infty$. *Assume also that* T *carries* L^∞ *to B.M.O. boundedly. Then, for every* p *with* $p_0 < p < \infty$, T *is bounded in* L^p.

Proof: We consider the operator $f \longmapsto (Tf)^{\#}$. This is a sublinear operator bounded in L^{p_0} and also in L^∞. By Marcinkiewicz's interpolation theorem, it will also be bounded in L^p. Let $f \in L^p \cap L^{p_0}$. Then $Tf \in L^{p_0}$ and $M(Tf) \in L^{p_0}$. On the other hand $(Tf)^{\#} \in L^p$ and $\int (Tf)^{\# p} \leq C\int |f|^p$. The preceding theorem gives:

$$\int (M(Tf))^p \leq C\int (Tf)^{\# p} \leq C\int |f|^p$$

Thus $\int |Tf|^p \leq C\int |f|^p$ and this inequality extends to every $f \in L^p$. □

The most important result regarding B.M.O. is the following theorem of F. John and L. Nirenberg.

THEOREM 3.8. There exist constants C_1, C_2 *depending only on the dimension* n, *such that for every* $f \in \text{B.M.O.} = \text{B.M.O.}(\mathbb{R}^n)$, *every cube* Q *and every* $t > 0$:

$$(3.9) \qquad |\{x \in Q : |f(x) - f_Q| > t\}| \leq C_1 e^{-(C_2/\|f\|_*)t} |Q|$$

Proof: It is again an application of the Calderón-Zygmund decomposition. Observe, first of all, that we can assume $\|f\|_* = 1$, because the inequality (3.9) does not change if we replace f by a constant times it. We fix Q and take $\alpha > 1$. We know that

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \leq 1 < \alpha.$$

We make the Calderón-Zygmund decomposition of Q for the function $f - f_Q$ relative to α, obtaining cubes $Q_{1,j}$ (dyadic subcubes of Q) for each of which:

$$\alpha < \frac{1}{|Q_{1,j}|}\int_{Q_{1,j}} |f(x) - f_Q|dx \leq 2^n\alpha$$

Besides, for a.e. $x \notin \bigcup_j Q_{1,j}$ is $|f(x) - f_Q| \leq \alpha$. It follows that for each $Q_{1,j}$ is
$|f_{Q_{1,j}} - f_Q| \leq 2^n\alpha$. Also:

$$\sum_j |Q_{1,j}| \leq \frac{1}{\alpha}\sum_j \int_{Q_{1,j}} |f(x) - f_Q|dx \leq \frac{1}{\alpha}\int_Q |f(x) - f_Q|dx \leq \frac{1}{\alpha}|Q|$$

On each $Q_{1,j}$ we make the Calderón-Zygmund decomposition for the function $f - f_{Q_{1,j}}$ relative to α. Thus we obtain a family $Q_{2,k}$ of dyadic subcubes of $Q_{1,j}$, for each of which is $|f_{Q_{2,k}} - f_{Q_{1,j}}| \leq 2^n\alpha$, and also for a.e. $x \in Q_{1,j} \setminus (\bigcup_k Q_{2,k})$ is $|f(x) - f_{Q_{1,j}}| \leq \alpha$. Besides, $\sum_k |Q_{2,k}| \leq \frac{1}{\alpha}|Q_{1,j}|$. Now we put together all the families $\{Q_{2,k}\}$ corresponding to different $Q_{1,j}$'s and call the resulting family also $\{Q_{2,k}\}$. Then, outside the union of the $Q_{2,k}$'s we have:

$$|f(x) - f_Q| \leq |f(x) - f_{Q_{1,j}}| + |f_{Q_{1,j}} - f_Q| \leq \alpha + 2^n\alpha \leq 2.2^n\alpha$$

and also

$$\sum_k |Q_{2,k}| \leq (\tfrac{1}{\alpha})^2|Q|.$$

Subsequently, we obtain for each natural number N, a family of non-overlapping cubes $\{Q_{N,j}\}$ in such a way that outside of their union is $|f(x) - f_Q| \leq N.2^n\alpha$ and such that $\sum_j |Q_{N,j}| \leq \alpha^{-N}|Q|$.

If $N\cdot 2^n\alpha \leq t < (N+1)\cdot 2^n\alpha$ with $N = 1,2,\ldots,$ then

$$|\{x \in Q : |f(x) - f_Q| > t\}| \leq \sum_j |Q_{N,j}| \leq \alpha^{-N}|Q| = e^{-N\log\alpha}|Q| \leq$$
$$\leq e^{-C_2 t}|Q|$$

(with $C_2 = 2^{-n-1}(\log\alpha)/\alpha$) since $t < (N+1)2^n\alpha \leq N2^{n+1}\alpha$. On the other hand, if $t < 2^n\alpha$, then $C_2 t < (\log\alpha)/2$, and we use the trivial majorization

$$|\{x \in Q : |f(x) - f_Q| > t\}| \leq |Q| < e^{(\log\alpha)/2 - C_2 t}|Q|$$

Thus, we get (3.9) for every t by choosing C_2 as above and $C_1 = \sqrt{\alpha}$. Finally, α can be chosen in order to get an optimal value of the constant C_2 $(\alpha = e)$. $\square$

COROLLARY 3.10. If $f \in$ B.M.O. *then:*

i) For every p *with* $0 < p < \infty$:

$$\|f\|_{*,p} \equiv \sup_Q\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx\right)^{1/p} \leq C_p\|f\|_*$$

with C_p *independent of* f, *in such a way that, for* $1 < p < \infty$, $f \mapsto \|f\|_{*,p}$ *is a norm equivalent to* $f \mapsto \|f\|_*$ *on* B.M.O.

ii) For every λ *such that* $0 < \lambda < C_2/\|f\|_*$, *where* C_2 *is the same constant appearing in (3.9), we have:*

$$\sup_Q \frac{1}{|Q|}\int_Q e^{\lambda|f(x)-f_Q|}\,dx < \infty$$

Proof: i) $\displaystyle\int_Q |f(x) - f_Q|^p dx = \int_0^\infty pt^{p-1}|\{x \in Q : |f(x) - f_Q| > t\}|dt \leq$

$$\leq C_1\int_0^\infty pt^{p-1} e^{-(C_2/\|f\|_*)t}\,dt\cdot|Q|.$$

After a change of variables

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx \leq C_1\cdot p\cdot(\|f\|_*/C_2)^p\int_0^\infty s^{p-1}e^{-s}\,ds =$$

$$= C_1 p\Gamma(p) C_2^{-p} \|f\|_*^p = C_p^p \|f\|_*^p$$

which gives i) with $C_p = (C_1 p\Gamma(p) C_2^{-p})^{1/p}$.

If $1 < p$, we have $\|f\|_* \leq \|f\|_{*,p} \leq C_p \|f\|_*$, so that the norms $\|\ \|_*$ and $\|\ \|_{*,p}$ are equivalent over B.M.O. Also, if $p > 1$, Stirling's formula $\Gamma(p) \cong \sqrt{2\pi}\, e^{-p}\, p^{p-1/2}$ can be used to conclude that $C_p \leq C\cdot p$ with an absolute constant C.

$$\text{ii)} \quad \int_Q e^{\lambda|f(x)-f_Q|}\, dx = \int_0^\infty \lambda e^{\lambda t} |\{x \in Q : |f(x)-f_Q| > t\}|\, dt \leq$$

$$\leq \int_0^\infty \lambda e^{\lambda t}\, C_1\, e^{-(C_2/\|f\|_*)t}\, dt\, |Q| = C_1 \lambda \int_0^\infty e^{(\lambda - C_2/\|f\|_*)t}\, dt\, |Q| =$$

$$= C_1 \lambda (C_2/\|f\|_* - \lambda)^{-1} |Q| \quad \text{if } 0 < \lambda < C/\|f\|_*. \quad \square$$

4. HARMONIC AND SUBHARMONIC FUNCTIONS IN A HALF-SPACE

In the first two sections of chapter I, we studied harmonic and subharmonic functions in a disk or in a euclidean ball, and solved the Dirichlet problem for such domains. Our purpose in this section will be to extend this study to the half-space $\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1} : t > 0\}$. We already posed the Dirichlet problem for this domain in the first section of Chapter I, and started looking for a solution. We shall complete this task in the present section. The main difference between the situation analized in Chapter I and our present setting is the unboundedness of the domain, which forces us to look carefully at the behaviour of our functions at infinity.

We start by showing that Poisson's kernel does indeed solve the Dirichlet problem. At the end of section 1 in Chapter I we introduced the Poisson kernel for $\mathbb{R}^{n+1}_+$

$$P_t(x) = C_n \frac{t}{(t^2 + |x|^2)^{(n+1)/2}}, \quad t > 0, \quad x \in \mathbb{R}^n$$

with

$$C_n = \Gamma((n+1)/2)/(\pi^{(n+1)/2}).$$

Of course $P_t(x) = t^{-n}P_1(t^{-1}x) \geq 0$

so that

$$\int_{\mathbb{R}^n} P_t(x)dx = \int_{\mathbb{R}^n} P_1(x)dx = C_n\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{(n+1)/2}}\,dx = 1$$

as can be seen by integrating in polar coordinates.

We also have, for any $\delta > 0$

$$\int_{|x|>\delta} P_t(x)dx = \int_{|x|>\delta/t} P_1(x)dx \to 0 \quad \text{as} \quad t \to 0$$

because P_1 is integrable. Thus, we have:

i) $P_t(x) \geq 0$, $x \in \mathbb{R}^n$, $t > 0$

ii) $\int_{\mathbb{R}^n} P_t(x)dx = 1$ for all $t > 0$

iii) $\int_{|x|>\delta} P_t(x)dx \to 0$ as $t \to 0$ for every $\delta > 0$.

In other words, $\{P_t\}_{t>0}$ is an approximate identity in $\mathbb{R}^n$.

Also, the function $(x,t) \longmapsto P_t(x)$ is harmonic in $\mathbb{R}^{n+1}_+$. Indeed

$$P_t(x) = -\frac{C_n}{n-1}\frac{\partial}{\partial t}\left\{\frac{1}{(t^2+|x|^2)^{(n-1)/2}}\right\}$$

and

$$(t^2+|x|^2)^{-(n-1)/2} = |(x,t)|^{2-(n+1)}$$

is "the" radial harmonic function in $\mathbb{R}^{n+1}\setminus\{0\}$.

THEOREM 4.1. For $f \in \bigcup_{p=1}^{\infty} L^p(\mathbb{R}^n)$, *we consider the Poisson integral of* f:

$$P(f)(x,t) = P_t * f(x) = \int_{\mathbb{R}^n} P_t(x-y)f(y)dy = u(x,t).$$

Then $u(x,t)$ *is harmonic in* $\mathbb{R}^{n+1}_+$ *and:*

a) *If* $f \in L^p(\mathbb{R}^n)$ *with* $1 \leq p < \infty$, *it follows that* $u(\cdot,t) \to f$ *in* L^p *as* $t \to 0$.
b) *If* f *is continuous and bounded, it follows that* $u(\cdot,t) \to f$ *uniformly on compact subsets as* $t \to 0$.
c) *If* f *is uniformly continuous and bounded, the convergence* $u(\cdot,t) \to f$ *as* $t \to 0$ *is uniform in* $\mathbb{R}^n$.
d) *If* $f \in L^\infty(\mathbb{R}^n)$, *all we can say is that* $u(\cdot,t) \to f$ *in the weak-* topology of* L^∞.
Also if μ *is a (finite) Borel measure in* $\mathbb{R}^n$ *and* $u(x,t) = P(\mu)(x,t) = P_t * \mu(x)$, *then* u *is harmonic and* $u(\cdot,t) \to \mu$ *in the weak-* topology of* $M(\mathbb{R}^n)$.

Proof: The harmonicity of $u(x,t)$ is a consequence of the harmonicity of $P_t(x)$. Besides

$$u(x,t) - f(x) = \int_{\mathbb{R}^n} (f(x - y) - f(x))P_t(y)dy.$$

Thus, by using Minkowski's inequality for integrals, we get:

$$\| u(\cdot,t) - f\|_p \leq \int_{\mathbb{R}^n} \| f(\cdot - y) - f\|_p P_t(y)dy = \int_{|y|<\delta} + \int_{|y|>\delta}$$

In cases a) and c), given $\varepsilon > 0$, $\delta > 0$ can be chosen in such a way that $\|f(\cdot - y) - f\|_p < \varepsilon/2$ whenever $|y| < \delta$. With this choice of δ, the first term in our sum of integrals is $< \varepsilon/2$ independently of t. The second term in that sum is $\leq 2\|f\|_p \int_{|y|>\delta} P_t(y)dy \to 0$ as $t \to 0$ as we know from property iii) of the Poisson kernel. Therefore, the second term in the sum can also be made $< \varepsilon/2$ by taking t close to 0. This proves a) and c). The proof of b) is entirely analogous. Part d) and the final statement concerning a Borel measure μ are proved by duality exactly as in theorem 1.18 in Chapter I. Now $M(\mathbb{R}^n)$ is the dual of $C_0(\mathbb{R}^n)$, the space of continuous functions vanishing at ∞ with the supremum norm, and these functions are uniformly continuous, so that part c) applies. □

COROLLARY 4.2. *For* f *continuous and bounded in* $\mathbb{R}^n$, *the function* $u(x,t)$ *defined by* $u(x,t) = P(f)(x,t)$ *if* $t > 0$ *and* $u(x,0) = f(x)$, *is continuous in* $\overline{\mathbb{R}^{n+1}_+}$ *and harmonic in* $\mathbb{R}^{n+1}_+$. *It is therefore a solution of the Dirichlet problem in* $\mathbb{R}^{n+1}_+$ *with* *boundary function* f.

Proof: It follows immediately from part b) of the previous theorem. □

PROPOSITION 4.3. With the same hypothesis of theorem 4.1, we have $\|u(\cdot,t)\|_p \leq \|f\|_p$ *for every* $t > 0$.

Proof: Since the function $x \longmapsto u(x,t)$ is the convolution of P_t and $f (u(\cdot,t) = P_t * f)$, this is a particular instance of Young's inequality and it is easily obtained by writing $u(x,t) = \int_{\mathbb{R}^n} P_t(y) f(x-y)\,dy$ and applying Minkowski's inequality for integrals. □

Now we shall obtain Poisson's representation for a class of nice harmonic functions.

THEOREM 4.4. Let $u(x,t)$ *be a continuous function in* $\overline{\mathbb{R}^{n+1}_+}$, *harmonic in* $\mathbb{R}^{n+1}_+$ *and bounded. Then* u *coincides necessarily with the Poisson integral of the boundary function* $x \longmapsto u(x,0)$, *that is, we have the Poisson representation:*

$$u(x,t) = \int_{\mathbb{R}^n} P_t(x-y)\,u(y,0)\,dy .$$

Proof: Consider the function $v(x,t)$ given by:

$$v(x,t) = \begin{cases} u(x,t) - \int_{\mathbb{R}^n} P_t(x-y)\,u(y,0)\,dy & \text{if } t > 0 \\ \\ 0 \quad \text{if} \quad t = 0 \end{cases}$$

This function is continuous in $\overline{\mathbb{R}^{n+1}_+}$, bounded, harmonic in $\mathbb{R}^{n+1}_+$ and vanishes at every point of the boundary. Theorem 1.33 in chapter I tells us that v is identically 0. □

Theorem 4.4. can be read by saying that the Dirichlet problem in $\mathbb{R}^{n+1}_+$ for a bounded and continuous boundary function has a unique bounded solution, but, of course, we know that uniqueness disappears as soon as we allow unbounded solutions.

In order to extend to our situation Theorems 1.3 and 1.8 from Chapter I, and also for other applications, we shall make some basic estimates for harmonic and subharmonic functions in $\mathbb{R}^{n+1}_+$.

THEOREM 4.5. Let $v(x,t)$ *be a non-negative subharmonic function in* $\mathbb{R}^{n+1}_+$ *which is uniformly in* $L^p(\mathbb{R}^n)$ *for some* $1 \leq p < \infty$, *by which we mean that:*

$$\sup_{t>0} \int_{\mathbb{R}^n} v(x,t)^p dx = M^p < \infty .$$

Then there is a constant C *depending only on* p *and* n, *such that* $v(x,t) \leq CMt^{-n/p}$ *for every* $(x,t) \in \mathbb{R}^{n+1}_+$. *In particular* $v(x,t)$ *is bounded in each proper sub-half-space* $\{(x,t) \in \mathbb{R}^{n+1}_+ : t \geq t_0 > 0\}$. *Actually, the following stronger property holds:* $v(x,t) \to 0$ *as* $(x,t) \to \infty$ *in each proper sub-half-space.*

Proof: Given $(x_0, t_0) \in \mathbb{R}^{n+1}_+$, let $B_0 = B((x_0, t_0), t_0/2)$ denote the ball centered at (x_0, t_0) with radius $t_0/2$. Then

$$v(x_0,t_0) \leq \frac{1}{|B_0|} \int_{B_0} v(x,t)\,dx\,dt \leq$$

$$\leq \left(\frac{1}{|B_0|} \int_{B_0} v(x,t)^p dx\,dt\right)^{1/p} \leq$$

$$\leq \left(C\, t_0^{-n-1} \int_{t_0/2}^{3t_0/2} \int_{|x-x_0|<t_0/2} v(x,t)^p dx\,dt\right)^{1/p} \leq$$

$$\leq C\, M\, t_0^{-n/p} .$$

This proves the first part of the theorem. Let us prove now that $v(x,t) \to 0$ as $(x,t) \to \infty$, $t \geq t_0 > 0$. Fix $t_0 > 0$. Since $v(x,t) \leq CMt^{-n/p}$, we know that given $\varepsilon > 0$, we can find $t_1 > t_0$ such that $v(x,t) \leq \varepsilon$ provided $t \geq t_1$. We just need to show that for $t_0 \leq t \leq t_1$, $v(x,t)$ can be made smaller than ε by taking x big. Proceeding as in the first part of the proof, we see that for $|x| > t_1$, say, and for $t_0 \leq t \leq t_1$

$$v(x,t)^p \leq \frac{C}{t_0^{n+1}} \int_{t_0/2}^{3t_1/2} \int_{|y|>|x|-(t_1/2)} v(y,t)^p dy\,dt \to 0 \qquad \text{as} \quad |x| \to \infty$$

and the proof is completed. $\square$

We shall see below that theorem 4.5 still holds for $v(x,t) = |u(x,t)|$ with u harmonic in $\mathbb{R}^{n+1}_+$ and any $p > 0$. When $0 < p < 1$, the proof of theorem 4.5 cannot be used, since the second inequality in that proof is Jensen's inequality which is no longer available for $p < 1$. We shall use, instead, the following

LEMMA 4.6. Let u *be harmonic in a ball* $B \subset \mathbb{R}^{n+1}$ *and continuous in* $\bar{B}$. *Then, if* (x_o,t_o) *is the center of* B, *we have, for any* $p > 0$:

$$|u(x_o,t_o)|^p \leq C_p \frac{1}{|B|} \int_B |u(x,t)|^p dxdt$$

where C_p *is a constant depending only on* p *and* n.

Proof: Of course, we already know that the lemma is true when $p \geq 1$ with $C_p = 1$. Let us look now at the case $0 < p < 1$. We may assume that B is the unit ball centered at the origin and also that $\int_B |u(x,t)|^p dxdt = |B|$. Write

$$m_p(r) = \left(\frac{1}{|\Sigma_n|} \int_{\Sigma_n} |u(r\sigma)|^p d\sigma \right)^{1/p}$$

$d\sigma$ being Lebesgue measure on the unit sphere, and

$$m_\infty(r) = \sup \{ |u(x,t)| : |x|^2 + t^2 = r^2 \} .$$

We may also assume that $m_\infty(r) \geq 1$ for every r with $0 < r < 1$, since otherwise the maximum principle would immediately give us the required inequality

$$m_1(r) = \frac{1}{|\Sigma_n|} \int_{\Sigma_n} |u(r\sigma)| d\sigma = \frac{1}{|\Sigma_n|} \int_{\Sigma_n} |u(r\sigma)|^p |u(r\sigma)|^{1-p} d\sigma \leq$$

$$\leq (m_\infty(r))^{1-p} (m_p(r))^p .$$

From the representation of u as a Poisson integral over the sphere of radius r (see the formula right before theorem 1.30 in Chapter I), we obtain, for $0 < s < r$: $m_\infty(s) \leq 2(1 - sr^{-1})^{-n} m_1(r)$. Now take $s = r^a$ with $a > 1$. We have

$$m_\infty(r^a) \leq 2(1 - r^{a-1})^{-n}\, m_1(r) \leq 2(1 - r^{a-1})^{-n}(m_\infty(r))^{1-p}(m_p(r))^p .$$

By taking logarithms and integrating we get:

$$\int_{1/2}^1 \log m_\infty(r^a)\frac{dr}{r} \leq C_a + (1 - p)\int_{1/2}^1 \log m_\infty(r)\frac{dr}{r} +$$

$$+ p\int_{1/2}^1 \log m_p(r)\frac{dr}{r}$$

The last integral is bounded by a constant depending only on p and n, since our normalizing assumption is equivalent to

$$(n + 1) \int_0^1 m_p(r)^p\, r^n\, dr = 1.$$

Therefore, after a change of variables in the integral on the left hand side, we can write:

$$\frac{1}{a}\int_{(1/2)^a}^1 \log m_\infty(r) \frac{dr}{r} \leq C_a + (1 - p) \int_{1/2}^1 \log m_\infty(r) \frac{dr}{r}$$

But $a > 1$ implies $(1/2)^a < 1/2$ and, since we have assumed $m_\infty(r) \geq 1$, it follows that

$$\int_{1/2}^1 \log m_\infty(r) \frac{dr}{r} \leq \int_{(1/2)^a}^1 \log m_\infty(r) \frac{dr}{r} .$$

Thus

$$\frac{1}{a} \int_{(1/2)^a}^1 \log m_\infty(r) \frac{dr}{r} \leq C_a + (1 - p) \int_{(1/2)^a}^1 \log m_\infty(r) \frac{dr}{r}$$

Now we choose a so close to 1 that $1/a > 1-p$, that is, we take $1 < a < (1-p)^{-1}$. Then we obtain:

$$\int_{(1/2)^a}^1 \log m_\infty(r) \frac{dr}{r} \leq C_p$$

This implies that for some $r_o > 0$ $((1/2)^a \leq r_o \leq 1)$, $m_\infty(r_o) \leq C_p$, a constant different from the previous one, but still depending only on p and n. Then the maximum principle gives us $|u(0,0)| \leq C_p$ as we wanted to prove. $\square$

Now we can give the promised extension of theorem 4.5.

THEOREM 4.7. Let $u(x,t)$ *be a harmonic function in* $\mathbb{R}^{n+1}_+$ *which is uniformly in* $L^p(\mathbb{R}^n)$ *for some* $0 < p < \infty$, *that is:*

$$\sup_{t>0} \int_{\mathbb{R}^n} |u(x,t)|^p dx = M^p < \infty.$$

Then there is a constant C *depending only on* p *and* n, *such that* $|u(x,t)| \leq CMt^{-n/p}$ *for every* $(x,t) \in \mathbb{R}^{n+1}_+$. *In particular* $u(x,t)$ *is bounded in each proper sub-half-space* $\{(x,t) \in \mathbb{R}^{n+1}_+ : t \geq t_0 > 0\}$. *Actually, the following stronger property holds* $u(x,t) \to 0$ *as* $(x,t) \to \infty$ *in each proper sub-half-space.*

Proof: It is exactly the same as the proof of theorem 4.5 starting with the inequality:

$$|u(x_0,t_0)|^p \leq C\frac{1}{|B_0|} \int_{B_0} |u(x,t)|^p dxdt$$

with $B_0 = B((x_0,t_0),t_0/2))$, which was established in lemma 4.6. □

We can now present the analogues of theorems 1.3 and 1.8 in Chapter I, providing characterizations of the Poisson integrals of L^p functions, $p > 1$, or measures in $\mathbb{R}^n$.

THEOREM 4.8. Let $1 < p \leq \infty$. *Then the following two conditions on the function* $u(x,t)$ *defined in* $\mathbb{R}^{n+1}_+$ *are equivalent:*

a) u *is harmonic in* $\mathbb{R}^{n+1}_+$ *and is uniformly in* $L^p(\mathbb{R}^n)$

b) u *is the Poisson integral of some function* $f \in L^p(\mathbb{R}^n)$.

Proof: We already know, from 4.1 and 4.3, that (b) implies (a). Let us prove that, conversely, (a) implies (b). Assuming that (a) holds, consider the functions $f_j(x) = u(x,t_j)$ for a sequence $t_j \downarrow 0$. From (a), (f_j) is a bounded sequence in $L^p(\mathbb{R}^n) = L^{p'}(\mathbb{R}^n)^*$. As in the proof of theorem 1.3 in Chapter I, (f_j) has a subsequence converging in the weak-$*$ topology of L^p. We may assume that (f_j) itself converges in the weak-$*$ topology to a certain $f \in L^p(\mathbb{R}^n)$, that is, for every $g \in L^{p'}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} g(x)f_j(x)dx \to \int_{\mathbb{R}^n} g(x)f(x)dx \qquad \text{as } j \to \infty.$$

Now, for each j, the function $(x,t) \longmapsto u(x,t+t_j)$ is continuous in $\overline{\mathbb{R}^{n+1}_+}$, harmonic in $\mathbb{R}^{n+1}_+$ and, according to theorem 4.7, it is also bounded. It follows from theorem 4.4 that this function is the Poisson integral of its boundary function, that is, for every j

$$u(x,t+t_j) = \int_{\mathbb{R}^n} P_t(x-y)u(y,t_j)\,dy = \int_{\mathbb{R}^n} P_t(x-y)f_j(y)\,dy.$$

Letting j go to ∞, we get

$$u(x,t) = \int_{\mathbb{R}^n} P_t(x-y)f(y)\,dy = P(f)\,(x,t)$$

since the function $y \longmapsto P_t(x-y)$ belongs to $L^{p'}(\mathbb{R}^n)$. $\square$

As in the torus, the case $p = 1$ is different, and we get, with a similar proof, the following

THEOREM 4.9. The following two conditions on the function $u(x,t)$ *defined in* $\mathbb{R}^{n+1}_+$ *are equivalent:*

a) u *is harmonic in* $\mathbb{R}^{n+1}_+$ *and is uniformly in* $L^1(\mathbb{R}^n)$

b) u *is the Poisson integral of some (finite) Borel measure* μ *in* $\mathbb{R}^n$, *that is* $u(x,t) = P(\mu)(x,t) = \int_{\mathbb{R}^n} P_t(x-y)\,d\mu(y)$. $\square$

Based upon the same ideas is the following result on harmonic majorization, which will be a very useful tool in the sequel

THEOREM 4.10. Let v *be a nonnegative subharmonic function in* $\mathbb{R}^{n+1}_+$ *which is uniformly in* $L^p(\mathbb{R}^n)$ *for some* $1 \leq p < \infty$. *Then* v *has a least harmonic majorant in* $\mathbb{R}^{n+1}_+$. *Moreover, if* $p > 1$, *this harmonic majorant is the Poisson integral of some function in* $L^p(\mathbb{R}^n)$ *and if* $p = 1$, *it is the Poisson integral of some (finite) Borel measure on* $\mathbb{R}^n$.

Proof: As before, consider the functions $f_j(x) = v(x,t_j)$ for a sequence $t_j \downarrow 0$. We may assume that f_j converges as $j \to \infty$ in the weak-* topology of $L^p(\mathbb{R}^n)$ if $p > 1$ or in the weak-* topology of $M(\mathbb{R}^n)$ if $p = 1$. Just to fix the notation, let $p > 1$. Then $f_j \to f \in L^p(\mathbb{R}^n)$ in the weak-* topology. Now we claim that for every j there is

$$(4.11)\quad v(x,t+t_j) \leq \int_{\mathbb{R}^n} P_t(x-y)v(y,t_j)\,dy = \int_{\mathbb{R}^n} P_t(x-y)f_j(y)\,dy.$$

The way to see this is to observe that both functions

$$(x,t) \longmapsto v(x,t+t_j) \quad \text{and} \quad P(f_j)(x,t)$$

tend to 0 as $(x,t) \to \infty$ in $\mathbb{R}^{n+1}_+$. Thus, given $\varepsilon > 0$, we just have to make $R > 0$ big enough so that

$$v(x,t+t_j) - P(f_j)(x,t) \leq \varepsilon$$

in the boundary of the region $K_R = \{(x,t) \in \mathbb{R}^{n+1}_+ : |x|^2 + t^2 \leq R^2\}$.

Then we can apply corollary 2.6 from Chapter I to conclude that $v(x,t+t_j) - P(f_j)(x,t) \leq \varepsilon$ for every $(x,t) \in K_R$. Since this is true for all large enough R's, we have actually $v(x,t+t_j) -$ $- P(f_j)(x,t) \leq \varepsilon$ for every $(x,t) \in \mathbb{R}^{n+1}_+$. Now, since ε was arbitrary, we get (4.11). That the function $(x,t) \longmapsto v(x,t+t_j)$ tends to 0 as $(x,t) \to \infty$ in $\mathbb{R}^{n+1}_+$ follows from theorem 4.5. As for the function $P(f_j)$, we just need to observe that $f_j \in C_0(\mathbb{R}^n)$, the space of continuous functions vanishing at ∞, and this already guarantees that $P(f_j)(x,t) \to 0$ as $(x,t) \to \infty$ in $\mathbb{R}^{n+1}_+$. Indeed, let $h \in C_0(\mathbb{R}^n)$. Take $\varepsilon > 0$. Then

$$P(h)(x,t) = \int_{\mathbb{R}^n} P_t(x-y)h(y)\,dy = \int_{|y|<R} + \int_{|y|>R} .$$

By taking R big enough, the second integral can be made $<\varepsilon/2$ in absolute value. For the first integral we make the observation that $P_t(x) \to 0$ as $(x,t) \to \infty$ in $\mathbb{R}^{n+1}_+$. Thus, we have established (4.11) for every j and every $(x,t) \in \mathbb{R}^{n+1}_+$. Letting $j \to \infty$, we have:

$$v(x,t) = \limsup_{j\to\infty} v(x,t+t_j) \leq \lim_{j\to\infty} \int_{\mathbb{R}^n} P_t(x-y)f_j(y)\,dy =$$
$$= \int_{\mathbb{R}^n} P_t(x-y)f(y)\,dy.$$

Thus $u(x,t) = P(f)(x,t)$ is a harmonic majorant of v. For the case $p = 1$ we get a harmonic majorant $u = P(\mu)$ with $\mu \in M(\mathbb{R}^n)$. That u is indeed the least harmonic majorant of v follows from the fact that $P(f_j)(x,t)$ is the least harmonic majorant of $(x,t) \mapsto v(x,t+t_j)$ because, if h is a harmonic majorant of v, we would have $h(x,t+t_j) \geq v(x,t+t_j)$ and, consequently $h(x,t+t_j) \geq$ $\geq P(f_j)(x,t)$. Letting $j \to \infty$, we get: $h(x,t) \geq u(x,t)$. $\square$

We shall next study the pointwise convergence of Poisson integrals.

The approach will be the same as that we have already used to prove the Lebesgue theorem (corollary 1.11). Given $f \in \bigcup_{1\le p\le\infty} L^p(\mathbb{R}^n)$, we are interested in studying the convergence of the averages $P_t * f(x)$, for x fixed, as $t \to 0$. It is natural to look at the supremum of the absolute values of those averages, that is, to consider the Poisson maximal function $P^*f(x) = \sup_{t>0} |P_t*f(x)|$. At the end of section 2 we proved that $|P_t*f(x)| \le C\mathcal{M}f(x,t)$ with

$$\mathcal{M}f(x,t) = \sup \{\frac{1}{|Q|}\int_Q |f| : x \in Q, \text{ side length } (Q) \ge t\}$$

Of course, it follows that

$$P^*f(x) \le C \sup_{t>0} \mathcal{M}f(x,t) = CMf(x)$$

Thus, P^* is a bounded operator in L^p for any $p > 1$ and it is also of weak type (1,1). This is all we need to prove the following

<u>*THEOREM 4.12. If*</u> $f \in \bigcup_{1\le p\le\infty} L^p(\mathbb{R}^n)$, <u>*then*</u> $P_t*f(x) \to f(x)$ <u>*as*</u> $t \to 0$ <u>*for almost every*</u> $x \in \mathbb{R}^n$.

<u>*Proof:*</u> Assume first that $f \in L^p(\mathbb{R}^n)$ for some p such that $1 \le p < \infty$. It will be enough to show that for each $s > 0$, the set

$$A_s = \{x \in \mathbb{R}^n : \limsup_{t\to 0} |P_t*f(x) - f(x)| > s\}$$

has measure 0. Indeed, the set of points $x \in \mathbb{R}^n$ where it is not true that $P_t*f(x) \to f(x)$ as $t \to 0$, is precisely $\bigcup_{j=1}^{\infty} A_{1/j}$.

Given $\varepsilon > 0$, we can write $f = g + h$, where g is continuous with compact support and $\int_{\mathbb{R}^n} |h|^p < \varepsilon$. We have, for every x and t:

$$|P_t*f(x) - f(x)| \le |P_t*h(x)| + |P_t*g(x) - g(x)| + |h(x)|.$$

According to theorem 4.1, part c), $P_t*g(x) \to g(x)$ as $t \to 0$. Consequently,

$$\limsup_{t\to 0} |P_t*f(x) - f(x)| \le \limsup_{t\to 0} |P_t*h(x)| + |h(x)| \le$$

$\leq P^*(h)(x) + |h(x)|.$

It follows that

$$A_s \subset \{x \in \mathbb{R}^n : P^*(h)(x) > s/2\} \cup \{x \in \mathbb{R}^n : |h(x)| > s/2\},$$

and we get:

$$|A_s| \leq Cs^{-p}\int_{\mathbb{R}^n} |h(x)|^p dx < Cs^{-p}\varepsilon .$$

Since this is true for every $\varepsilon > 0$, we actually have $|A_s| = 0$. Thus, the theorem is proved in this case.

The case $f \in L^\infty(\mathbb{R}^n)$ can be reduced to the previous one. Let us see how. Take $B = B(0,R)$, $R > 0$. We shall prove that $P_t*f(x)\to f(x)$ as $t \to 0$ for almost every $x \in B$. This will be enough, since R is arbitrary. Let $B' = B(0,R+1)$. Write $f = f_1 + f_2$ with $f_1(x) = f(x)$ if $x \in B'$ and $f_1(x) = 0$ elsewhere. Then $f_1 \in L^1(\mathbb{R}^n)$ and, consequently $P_t*f_1(x) \to f_1(x)$ as $t \to 0$ for a.e. $x \in \mathbb{R}^n$. Thus, we just need to prove that $P_t * f_2(x) \to f_2(x)=0$ for a.e. $x \in B$. But for $x \in B$:

$$P_t*f_2(x) = \int_{\mathbb{R}^n} P_t(y)f_2(x-y)dy = \int_{|y|>1} P_t(y)f_2(x-y)dy$$

and

$$|P_t*f_2(x)| \leq \int_{|y|>1} P_t(y)|f(x-y)|dy \leq \|f\|_\infty \int_{|y|>1} P_t(y)dy =$$
$$= \|f\|_\infty \int_{|y|>1/t} P_1(y)dy \to 0 \quad \text{as} \quad t \to 0. \qquad \square$$

In general, if $\phi \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}\phi(x)dx = 1$, we may consider the approximate identity ϕ_t, $t > 0$, obtained by making $\phi_t(x) = t^{-n}\phi(t^{-1}x)$. Then, if we now set

$$u(x,t) = \phi_t*f(x) = \int_{\mathbb{R}^n}\phi_t(x-y)f(y)dy$$

all parts of theorem 4.1, with the exception of the harmonicity of u, still hold, since they depend only on these three facts:

i) $\displaystyle\int_{\mathbb{R}^n}|\phi_t(x)|dx = \int_{\mathbb{R}^n}|\phi(x)|dx < \infty$ for every $t > 0$.

ii) $\int_{\mathbb{R}^n} \phi_t(x)dx = \int_{\mathbb{R}^n} \phi(x)dx = 1$ for every $t > 0$, and

iii) For every $\delta > 0$: $\int_{|x|>\delta} |\phi_t(x)|dx = \int_{|x|>\delta/t} |\phi(x)|dx \to 0$ as $t \to 0$.

Theorem 4.12, however, depends upon the inequality $P^*f(x) \leq \leq CMf(x)$. If we want to extend it to other approximate identities ϕ_t, we shall have to study the maximal function $\phi^*(f)(x) = = \sup_{t>0} |\phi_t * f(x)|$. If we get an inequality $\phi^*(f)(x) \leq CMf(x)$, theorem 4.12 extends immediately to the approximate identity $\{\phi_t\}$. If ϕ is radial and decreasing, the inequality is very easy to obtain.

THEOREM 4.13. Let $\psi(x) \geq 0$ *be a radial, decreasing, integrable function in* $\mathbb{R}^n$. *Then* $\psi^*(f)(x) \leq C\|\psi\|_1 Mf(x)$ *for every* $f \in \bigcup_{1\leq p\leq\infty} L^p(\mathbb{R}^n)$ *and every* $x \in \mathbb{R}^n$, *with* C *depending only on the dimension* n.

Proof: Let

$$\phi(x) = \sum_{j=1}^{m} a_j \chi_{B(0,r_j)}(x) \quad \text{with } a_j > 0.$$

Call $B_j = B(0,r_j)$. Then

$$\int_{\mathbb{R}^n} \phi(x)dx = \sum_{j=1}^{m} a_j|B_j| < \infty$$

and for every $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$,

$$\left|\int f(x)\phi(x)dx\right| \leq \sum a_j \int_{B_j} |f| = \sum a_j|B_j|\left(\frac{1}{|B_j|}\int_{B_j}|f|\right) \leq$$

$$\leq C\left(\int_{\mathbb{R}^n}\phi\right)Mf(0).$$

By applying this to the function $f(x - \cdot)$ we obtain

$$\left|\int_{\mathbb{R}^n} f(x-y)\phi(y)dy\right| \leq C\left(\int_{\mathbb{R}^n}\phi\right)Mf(x).$$

Now observe that, for ϕ as above and $t > 0$, ϕ_t is a function of the same type and $\|\phi_t\|_1 = \|\phi\|_1$. Therefore

$$\left|\int_{\mathbb{R}^n} f(x-y)\phi_t(y)\,dy\right| \leq C\left(\int_{\mathbb{R}^n}\phi\right) Mf(x).$$

Now if ψ is radial, decreasing and integrable, we can find a sequence of functions $\phi^{(j)}$, each of which is a finite linear combination of characteristic functions of balls centered at 0 with coefficients ≥ 0, such that, for every x, $\phi^{(j)}(x)$ increases monotonically to $\psi(x)$. Then

$$\left|\int_{\mathbb{R}^n} f(x-y)\psi_t(y)\,dy\right| \leq \int_{\mathbb{R}^n} |f(x-y)|\psi_t(y)\,dy =$$

$$= \lim_{j\to\infty}\int_{\mathbb{R}^n}|f(x-y)|\phi_t^{(j)}(y)\,dy$$

Since, for each j

$$\int_{\mathbb{R}^n}|f(x-y)|\phi_t^{(j)}(y)\,dy \leq C\left(\int_{\mathbb{R}^n}\psi\right)Mf(x)$$

we finally get:

$$\psi^*(f)(x) \leq C\|\psi\|_1 Mf(x). \qquad \square$$

COROLLARY 4.14. Let $\phi \in L^1(\mathbb{R}^n)$ *with* $\int_{\mathbb{R}^n}\phi(x)\,dx = 1$ *and such that the function* $\psi(x) = \sup\{|\phi(y)| : |y| \geq |x|\}$ *is integrable* (ψ *is called the least radial decreasing majorant of* ϕ). *Then, for every* $f \in \bigcup_{1\leq p\leq\infty} L^p(\mathbb{R}^n)$, $\phi_t * f(x) \to f(x)$, *as* $t \to 0$, *for almost every* $x \in \mathbb{R}^n$.

Proof: It is identical to that of theorem 4.12. The fact that ϕ is integrable and $\int_{\mathbb{R}^n}\phi = 1$ guarantees the uniform convergence for f continuous with compact support. Then, we just have to observe that $\phi^*(f)(x) \leq \psi^*(|f|)(x) \leq C\|\psi\|_1 Mf(x)$. $\square$

Actually, the pointwise convergence of a Poisson integral at a boundary point holds in a stronger form than the one we have discussed. We can allow any non-tangential approach. Let us give precise definitions. For a point $x \in \mathbb{R}^n$ and any real number $N > 0$, we shall call $\Gamma_N(x)$ to the open vertical cone with vertex x and aperture N, that is

$$\Gamma_N(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < Nt\}.$$

Given a function u defined on $\mathbb{R}^{n+1}_+$ and a point $x \in \mathbb{R}^n$, we shall say that $u(y,t)$ converges to ℓ as (y,t) tends to x (actually to $(x,0)$) nontangentially, and we shall write $u(y,t) \to \ell$ as $(y,t) \overset{N.T.}{\to} x$, if and only if, for every $N > 0$, $u(y,t)$ converges to ℓ as (y,t) tends to x remaining always in $\Gamma_N(x)$. This is the natural extension to our context of the notion of non-tangential convergence introduced for the disk in Chapter I (in theorem 1.20). To prove the non-tangential convergence of Poisson integrals we shall look at the nontangential Poisson maximal function of aperture $N > 0$, defined for any $f \in \bigcup_{1\leq p\leq\infty} L^p(\mathbb{R}^n)$ as:

$$P^*_{\nabla,N}f(x) = \sup_{(y,t)\in\Gamma_N(x)} |P_t*f(y)|$$

We know that $|(P_t*f)(y)| \leq C\mathcal{M}f(y,t)$ with C depending only on n. However, if $|y - x| < Nt$, we can easily see that $\mathcal{M}f(y,t) \leq$ $\leq C_N Mf(x)$, where C_N is a constant depending on N. To prove this inequality we just need to observe that if Q is a cube containing y and having side length $\geq t$, then $x \in Q^{2N+1}$, so that we have

$$\frac{1}{|Q|}\int_Q |f| \leq \frac{(2N+1)^n}{|Q^{2N+1}|}\int_{Q^{2N+1}} |f| \leq (2N+1)^n Mf(x),$$

and, consequently,

$$\mathcal{M}f(y,t) \leq (2N+1)^n Mf(x).$$

Thus we can write:

$$(4.15)\qquad P^*_{\nabla,N}f(x) \leq C_N Mf(x).$$

This inequality is all we need to prove the following:

THEOREM 4.16. If $f \in \bigcup_{1\leq p\leq\infty} L^p(\mathbb{R}^n)$, *then* $P_t*f(y) \to f(x)$ *as* $(y,t) \overset{N.T.}{\to} x$ *for almost every* $x \in \mathbb{R}^n$.

Proof: The proof is entirely similar to that of theorem 4.12. Suppose first that $f \in L^p(\mathbb{R}^n)$ and $1 \leq p < \infty$. Then, after fixing $N > 0$, we proceed as in 4.12, with the operator $P^*_{\nabla,N}$ instead of P^*. In this way, we show that the set

$$E_N = \{x \in \mathbb{R}^n : \limsup_{(y,t)\to x, (y,t)\in\Gamma_N(x)} |P_t*f(y) - f(x)| > 0\}$$

has measure 0. If we call $E = \bigcup_{N=1}^{\infty} E_N$, we shall have $|E| = 0$ and $P_t * f(y) \to f(x)$ as $(y,t) \xrightarrow{N.T.} x$ for every $x \notin E$.

As for the case $f \in L^\infty(\mathbb{R}^n)$, we realize that the argument given in 4.12 extends to our case, since, as can be easily seen, $P_t * f_2(y) \to 0$ as $(y,t) \xrightarrow{N.T.} x$ if $x \in B$. $\square$

We shall be able to give a more precise result by using a more direct approach based upon the inequality:

$$P_t(x - y) \leqq C_N P_t(x) \tag{4.17}$$

valid for every $x \in \mathbb{R}^n$ and every $(y,t) \in \mathbb{R}^{n+1}_+$ satisfying $|y| < Nt$, with a constant C_N that depends only on N and the dimension n. To prove (4.17), we may assume $t = 1$. Then, we need to show that, for every $x \in \mathbb{R}^n$,

$$\frac{|x|^2 + 1}{|x - y|^2 + 1} \leqq C_N$$

provided $|y| < N$. This is very simple, since for $|x| > 2N$, we can make

$$\frac{|x|^2 + 1}{|x - y|^2 + 1} \leqq \frac{|x|^2 + 1}{(|x| - N)^2 + 1} \leqq \frac{|x|^2 + 1}{(|x|/2)^2 + 1} \leqq 4$$

and for $|x| \leqq 2N$ we simply use

$$\frac{|x|^2 + 1}{|x - y|^2 + 1} \leqq |x|^2 + 1 \leqq 4N^2 + 1.$$

The more precise version of the nontangential convergence is as follows.

THEOREM 4.18. If $f \in \bigcup_{1 \leqq p \leqq \infty} L^p(\mathbb{R}^n)$, *then* $P_t * f(y) \to f(x)$ *as* $(y,t) \xrightarrow{N.T.} x$, *for every Lebesgue point* x *of* f.

Proof: Let x be a Lebesgue point for f. Given $\varepsilon > 0$, by definition of Lebesgue point, there is a $\delta > 0$ such that for every $0 < r < \delta$

$$\frac{1}{r^n} \int_{|y-x|<r} |f(y) - f(x)| dy < \varepsilon$$

Consider the function g defined by $g(y) = f(y) - f(x)$ if $|y - x| < \delta$ and $g(y) = 0$ otherwise. The inequality above guarantees that $Mg(x) \leq C\varepsilon$. Now, if $|y - x| < Nt$,

$$|P_t * f(y) - f(x)| = \left|\int_{\mathbb{R}^n} (f(z) - f(x))P_t(y - z)dz\right| \leq$$

$$\leq \int_{\mathbb{R}^n} |f(z) - f(x)|P_t(y - z)dz.$$

But, since $y - z = x - z - (x - y)$ and $|x - y| < Nt$, inequality (4.17) applies to yield $P_t(y - z) \leq C_N P_t(x - z)$. Hence

$$|P_t * f(y) - f(x)| \leq C_N\int_{\mathbb{R}^n} |f(z) - f(x)|P_t(x - z)dz =$$

$$= C_N\left(\int_{|z-x|<\delta} + \int_{|z-x|>\delta}\right) = C_N\int_{\mathbb{R}^n} |g(z)|P_t(x - z)dz +$$

$$+ C_N\int_{|z-x|>\delta} |f(z) - f(x)|P_t(x - z)dz .$$

The first integral in this sum is $P_t * |g|(x) \leq CMg(x) \leq C\varepsilon$. Now we claim that the second integral in the sum can also be made $< \varepsilon$ by making t close enough to 0. Indeed, this second integral can be bounded by

$$\|f\|_p\left(\int_{|z|>\delta} P_t(z)^{p'}dz\right)^{1/p'} + |f(x)|\int_{|z|>\delta} P_t(z)dz$$

and we just need to observe that $\left(\int_{|z|>\delta} P_t(z)^q dz\right)^{1/q} \to 0$ as $t \to 0$ for every $q \geq 1$. This is a very simple computation

$$\left(\int_{|z|>\delta} P_t(z)^q dz\right)^{1/q} = t^{-n/q'}\left(\int_{|x|>\delta/t} P(x)^q dx\right)^{1/q} \leq$$

$$\leq t^{-n/q'}\left(\int_{|x|>\delta/t} |x|^{-(n+1)q}dx\right)^{1/q} = t^{-n/q'}C(\delta)\,t^{n/q'+1} = Ct \qquad \square$$

As a consequence, we obtain an extension to our context of Fatou's classical theorem (stated after corollary 1.21 in Chapter I.)

COROLLARY 4.19. *Let the function* $u(x,t)$ *be harmonic and bounded in* $\mathbb{R}^{n+1}_+$. *Then* u *has non-tangential limits at almost every point* $x \in \mathbb{R}^n$, *the boundary of* $\mathbb{R}^{n+1}_+$.

Proof: According to theorem 4.8, $u(x,t) = P_t * f(x)$ for some

$f \in L^\infty(\mathbb{R}^n)$. Then the conclusion of the corollary follows by applying theorems 4.16 or 4.18. $\square$

We shall present now a local version of Fatou's theorem. This will imply, in particular, that, for a harmonic function $u(x,t)$ in $\mathbb{R}^{n+1}_+$, the existence of non-tangential limits at almost every boundary point is equivalent to the non-tangential boundedness almost everywhere. First of all, we make precise the notion of non-tangential boundedness. For a point $x \in \mathbb{R}^n$ and real numbers $N > 0$, $h > 0$, we shall call $\Gamma_N^h(x)$ to the open vertical cone with vertex x and aperture N, truncated at height h, that is

$$\Gamma_N^h(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y - x| < Nt \text{ and } 0 < t < h\}.$$

Then we make the following:

DEFINITION 4.20. A function u *defined in* $\mathbb{R}^{n+1}_+$ *is said to be non-tangentially bounded at the point* $x \in \mathbb{R}^n$ *if there are numbers* $N > 0$ *and* $h > 0$ *such that* u *is bounded in* $\Gamma_N^h(x)$, *that is, such that*

$$\sup\{|u(y,t)| : (y,t) \in \Gamma_N^h(x)\} < \infty$$

Observe that, while the non-tangential boundedness of u at x requires only the existence of a truncated cone $\Gamma_N^h(x)$ on which u is bounded, for non-tangential convergence one has to consider all possible apertures. Thus, even though the existence of a non-tangential limit of u at x clearly implies the non-tangential boundedness of u at x, the converse is by no means obvious. Actually the converse is false if one looks at an individual point x, but it is true if one looks at a whole set of points and disregards a set of measure 0. This is the content of the following result, which can be viewed as a local version of Fatou's theorem.

THEOREM 4.21. Let u *be a harmonic function in* $\mathbb{R}^{n+1}_+$. *Let* $E \subset \mathbb{R}^n$ *and suppose that* u *is non-tangentially bounded at every* $x \in E$. *Then* u *has non-tangential limits at almost every point* $x \in E$.

Proof: All we need to prove is that if u is non-tangentially bounded at every point of a set of positive measure, then there is some subset of positive measure such that u has non-tangential limits at every point of this subset. Indeed, once this is proved, if u is non-tangentially bounded at every point $x \in E$, then the set $\{x \in E : u$ does not have a non-tangential limit at $x\}$ necessarily must have measure 0.

Assume that u is non-tangentially bounded at every $x \in E$ with $|E| > 0$ and let us try to find $F \subset E$ with $|F| > 0$ such that u has non-tangential limits at every $x \in F$. We start by making several reductions for which the harmonicity of u does not play any role. For these preparatory steps we could assume that u is merely continuous.

First of all, E can be taken to be compact. Also, since $E = \bigcup E(N,h,M)$ where $E(N,h,M) = \{x \in E : |u(y,t)| \leq M$ for every $(y,t) \in \Gamma_N^h(x)\}$ and the union is taken over all N, h, M positive and rational; we may perfectly well assume that E coincides with a given $E(N,h,M)$ or, in other words, that there are $N > 0$, $h > 0$, $M > 0$, such that for every $x \in E : \sup\{|u(y,t)| : (y,t) \in \Gamma_N^h(x)\} \leq$ $\leq M$.

The next step will be to show that, for every positive integer $k > N$, there is a set $E_k \subset E$ with $|E_k| > 0$ and such that for every $x \in E_k$, u is bounded over $\Gamma_k^k(x)$ with a bound M_k independent of x. This proof will be based upon Lebesgue's differentiation theorem. We know that if $x \in E$ is a Lebesgue point for χ_E, the characteristic function of E, then

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}\chi_E(y)\,dy = \frac{|B(x,r) \cap E|}{|B(x,r)|} \to \chi_E(x) = 1, \text{ as } r \to 0.$$

Consequently, this convergence holds for almost every $x \in E$. The points $x \in E$ for which this holds are called density points of E. Take $0 < \eta < 1$. A further restriction on η will be imposed below. If $x \in E$ is a density point, we shall have

$$\frac{|B(x,r) \cap E|}{|B(x,r)|} \geq \eta \quad \text{for every} \quad r < \delta(x), \quad \delta(x) > 0$$

Now, if we take $\delta > 0$ small enough, we shall have a subset $F \subset E$ with $|F| > 0$ such that, for every $x \in F$ and every $0 < r < \delta$ is

$\frac{|B(x,r) \cap E|}{|B(x,r)|} \geq \eta$. Suppose we have a fixed integer $k > N$. We claim that if η is taken close enough to 1, depending on k, and if $s > 0$ is taken small enough, also depending on k, we have:

$$(4.22) \qquad \Gamma_k^s(x) \subset \bigcup_{z \in E} \Gamma_N^h(z) \quad \text{for every } x \in F.$$

Let us show this. Take $x \in F$ and $(y,t) \in \Gamma_k^s(x)$, that is: $|y - x| < kt$, $t < s$. We must prove that there is some $z \in E$ such that $|y - z| < Nt$ (we may assume, to start with, that $s < h$). If there existed no such z, we would have $B(y,Nt) = \{z: |z-y| < Nt\} \subset \mathbb{R}^n \setminus E$ and, consequently, $B(y,Nt) \cap B(x,kt) \subset B(x,kt) \setminus E$, in such a way that the set $B(x,kt) \setminus E$ contains at least a ball of diameter Nt. Thus:

$$\frac{|B(x,kt) \cap E|}{|B(x,kt)|} = 1 - \frac{|B(x,kt) \setminus E|}{|B(x,kt)|} \leq 1 - \left(\frac{N}{2k}\right)^n < \eta$$

if η has been so chosen. Now, if s is taken so small that $ks < \delta$, we would have a contradiction with the selection of δ and F depending on our η. It follows that, with our choice of F and s, (4.22) holds and hence, for every $x \in F$ and every $(y,t) \in \Gamma_k^s(x)$, $|u(y,t)| \leq M$. Suppose that $F \subset B(0,R)$ and take $M_k \geq M$ and also $M_k \geq \sup\{|u(y,t)| : |y| \leq R + k^2, s \leq t \leq k\}$ (since u is continuous, it will be bounded on the compact set $\bar{B}(0,R + k^2) \times [s,k] \subset \mathbb{R}_+^{n+1}$). Then for every $x \in F$ and every $(y,t) \in \Gamma_k^k(x)$, $|u(y,t)| \leq M_k$, so that, by setting $E_k = F$, we have the set we were looking for.

Observe that, given $k > N$ and $\varepsilon > 0$, the set E_k can be taken such that $|E \setminus E_k| < \varepsilon$. Indeed, once η has been selected, we realize that the set E_0 of density points of E can be written as $\bigcup_{j=1}^{\infty} A_j$ where

$$A_j = \{x \in E_0 : \frac{|B(x,r) \cap E|}{|B(x,r)|} \geq \eta \text{ for all } r < 1/j\} .$$

Since (A_j) is an increasing family, $|A_j| \to |E_0| = |E|$ as $j \to \infty$. Thus by taking $\delta > 0$ small enough, say $\delta = 1/j$, $F = A_j$ can be taken with measure as close to $|E|$ as we wish.

Our final preparatory step will be to realize that there is a set

$G \subset E$ with $|G| > 0$ such that for every $x \in G$ and every truncated cone $\Gamma_k^k(x)$, u is bounded on $\Gamma_k^k(x)$ with a bound M_k depending on k, but not on $x \in G$ for fixed k. Actually, given $\varepsilon > 0$, G can be taken in such a way that $|E \setminus G| < \varepsilon$. All we have to do is to consider for each integer $k > N$, a set $E_k \subset E$ satisfying, as in the previous paragraph that u is bounded on $\cup \{\Gamma_k^k(x) : x \in E_k\}$ and also $|E \setminus E_k| < 2^{-k}\varepsilon$. Then, we take $G = \bigcap_{k>N} E_k$ so that

$$|E \setminus G| = \left|\bigcup_{k>N} (E \setminus E_k)\right| \leq \sum_{k>N} 2^{-k}\varepsilon < \varepsilon$$

and, for every k, u is bounded on $\cup\{\Gamma_k^k(x) : x \in G\}$.

Once these preparatory steps have been taken, our task has been reduced to the following: We have a compact set $E \subset \mathbb{R}^n$, and the region $\Omega = \bigcup_{x \in E} \Gamma_N^h(x)$ for some $N > 0$ and $h > 0$ fixed. We know that u is a harmonic function in $\mathbb{R}_+^{n+1}$ such that $|u(x,t)| \leq 1$ for every $(x,t) \in \Omega$. Then we must show that for almost every $x \in E$, there exists $\lim u(y,t)$ as (y,t) tends to $(x,0)$ remaining always inside of Ω. Let us show this. For every positive integer j, define $\phi_j(x) = u(x,1/j)$ if $(x,1/j) \in \Omega$ and $\phi_j(x) = 0$ otherwise. For $(x,t) \in \mathbb{R}_+^{n+1}$ we define $\phi_j(x,t) = P_t * \phi_j(x)$, and we write $u(x,t + 1/j) = \phi_j(x,t) + \psi_j(x,t)$. The sequence of functions $\phi_j(x)$ is bounded in $L^\infty(\mathbb{R}^n)$ and, consequently, it has a subsequence $(\phi_{j'})$ converging in the weak-* topology of L^∞ to a function $\phi \in L^\infty(\mathbb{R}^n)$. For $(x,t) \in \mathbb{R}_+^{n+1}$ we define $\phi(x,t) = P_t * \phi(x)$. The weak-* convergence implies that $\phi_{j'}(x,t) \to \phi(x,t)$ for every $(x,t) \in \mathbb{R}_+^{n+1}$. Since $u(x,t + 1/j) \to u(x,t)$ and $\phi_{j'}(x,t) \to \phi(x,t)$, it follows that, also $\psi_{j'}(x,t) \to \psi(x,t)$ for every $(x,t) \in \mathbb{R}_+^{n+1}$, and we have: $u(x,t) = \phi(x,t) + \psi(x,t)$. Now, since $\phi(x,t) = P_t * \phi(x)$ and $\phi \in L^\infty(\mathbb{R}^n)$, the ordinary Fatou theorem (corollary 4.19) implies that $\phi(y,t)$ has non-tangential limits at almost every point $x \in \mathbb{R}^n$. We shall presently see that for almost every $x \in E$, $\psi(y,t)$ converges to 0 as $(y,t) \to (x,0)$ remaining inside of Ω. This will finish the proof of the theorem. Write the boundary of Ω as $\partial\Omega = E \cup B$ where $B = \{(x,t) \in \partial\Omega : t > 0\}$. We shall construct a function $H(x,t)$ with the following properties:

a) H is harmonic in $\mathbb{R}_+^{n+1}$

b) $H \geq 0$ in $\mathbb{R}_+^{n+1}$

c) $H \geq 2$ in B, and

d) For almost every $x \in E$ is $H(y,t) \to 0$ as $(y,t) \to (x,0)$ non-tangentially.

Assuming for a moment the existence of H, we shall see that $|\psi(x,t)| \leq H(x,t)$ for every $(x,t) \in \Omega$, from which the convergence $\psi(y,t) \to 0$ as $(y,t) \to (x,0)$ remaining inside of Ω, follows immediately for almost every $x \in E$. Now, to prove that $|\psi(x,t)| \leq H(x,t)$ for every $(x,t) \in \Omega$, we just need to see that, for a given $(x,t) \in \Omega$, and for an arbitrary $\varepsilon > 0$, we have $|\psi_j(x,t)| \leq H(x,t) + \varepsilon$ for j big enough. If $1/j < h$, call

$$\Omega_j = \bigcup_{x \in E} \Gamma_N^{h-1/j}(x) \qquad \text{and} \qquad \partial\Omega_j = E \cup B_j$$

where

$$B_j = \{(x,t) \in \partial\Omega_j : t > 0\}.$$

Given $(x,t) \in \Omega$, we have, for j big enough, $(x,t) \in \Omega_j$ and $(x,t+1/j) \in \Omega$, so that $|u(x,t+1/j)| \leq 1$. Also, $|\phi_j(x,t)| \leq 1$, and hence $|\psi_j(x,t)| \leq 2$, and this holds also for $(x,t) \in B_j$. We know that $2 \leq H(x,t)$ for $(x,t) \in B$. Thus, by continuity, given $\varepsilon > 0$, we shall have $2 \leq H(x,t) + \varepsilon$ for every $(x,t) \in B_j$ provided j is big enough. Thus, for j big enough $|\psi_j(x,t)| \leq H(x,t) + \varepsilon$ for every $(x,t) \in B_j$. Also, since

$$\left.\begin{aligned} u(x, t + \tfrac{1}{j}) &\to u(x, \tfrac{1}{j}) = \phi_j(x) \\ \phi_j(x,t) &\to \phi_j(x) \end{aligned}\right\} \quad \text{as} \quad t \to 0$$

with uniform convergence on compact sets, it turns out that $|\psi_j(x,t)| \leq \varepsilon$ for $(x,t) \in \Omega$ and t small enough. Then, the fact that $H \geq 0$ implies $|\psi_j(x,t)| \leq H(x,t) + \varepsilon$. Thus, given $\varepsilon > 0$, we know that for j big enough and for $t_o > 0$ small enough, the inequality $|\psi_j(x,t)| \leq H(x,t) + \varepsilon$ holds in the boundary of the set $\Omega_j^{t_o} = \{(x,t) \in \Omega_j : t > t_o\}$. But $|\psi_j|$ is a subharmonic function and $H + \varepsilon$ is harmonic. Consequently we have $|\psi_j(x,t)| \leq$ $\leq H(x,t) + \varepsilon$ all through $\Omega_j^{t_o}$ and, since this is valid for all sufficiently small $t_o > 0$, we actually have $|\psi_j(x,t)| \leq H(x,t) + \varepsilon$ for every $(x,t) \in \Omega_j$. We have succeeded in proving what we wanted: given $(x,t) \in \Omega$ and given $\varepsilon > 0$, we have $|\psi_j(x,t)| \leq H(x,t) + \varepsilon$ for big enough j. From this we immediately get $|\psi(x,t)| \leq H(x,t)$ for every $(x,t) \in \Omega$.

It only remains to construct the function H. Let X be the characteristic function of the complement of E, that is $\chi = \chi_{\mathbb{R}^n \setminus E}$. Then, we claim that, for an appropriate constant C, the function $H(x,t) = C\{(P_t * \chi)(x) + t\}$ satisfies the four properties required. In fact, a), b) and d) are clear and only c) needs to be verified. For those $(x,t) \in B$ with $t = h$, we have $H(x,t) \geq 2$ if we make $Ch \geq 2$. The points $(x,t) \in B$ with $0 < t < h$ are included in the set $\{(x,t) \in \mathbb{R}^{n+1}_+ : \text{dist}(x,E) = Nt\}$, so that, if $(x,t) \in B$ and $0 < t < h$, we have $B(x,Nt) \subset \mathbb{R}^n \setminus E$ and, consequently

$$P_t * \chi(x) = c_n \int_{\mathbb{R}^n} \chi(y)\, P_t(x-y)\, dy \geq$$

$$\geq c_n \int_{B(x,Nt)} P_t(x-y)\, dy =$$

$$= c_n t \int_{|y|<Nt} \frac{dy}{(t^2+|y|^2)^{(n+1)/2}}$$

$$= c_n \int_{|z|<N} \frac{dz}{(1+|z|^2)^{(n+1)/2}} = \text{constant} > 0.$$

Thus, we just need to make C appropriately big to guarantee that $H(x,t) \geq 2$ also for those $(x,t) \in B$ with $0 < t < h$. $\square$

By combining theorem 4.8 and theorem 4.1(a) we see that, if a function $u(x,t)$ is harmonic in $\mathbb{R}^{n+1}_+$ and it is uniformly in $L^p(\mathbb{R}^n)$ for some $1 < p < \infty$, then the functions $u_t(x) = u(x,t)$ converge to a given function $f \in L^p(\mathbb{R}^n)$ in the L^p norm as $t \to 0$. Besides, theorem 4.12 tells us that there is pointwise convergence almost everywhere, in such a way that, for almost every $x \in \mathbb{R}^n$ is $f(x) = \lim_{t\to 0} u(x,t)$.

In case $u(x,t)$ is harmonic in $\mathbb{R}^{n+1}_+$ and is uniformly in $L^p(\mathbb{R}^n)$ with $p = \infty$ or $p = 1$, we can still see by using theorem 4.8 or 4.9 and then part d) in theorem 4.1 or the closing sentence in this same theorem, that the functions u_t converge to a certain $f \in L^\infty(\mathbb{R}^n)$ or to a certain Borel measure $\mu \in M(\mathbb{R}^n)$ in the corresponding weak-* topology as $t \to 0$.

Thus, for u harmonic, the fact that u is uniformly in $L^p(\mathbb{R}^n)$

with $1 \leq p \leq \infty$ implies the convergence of the family u_t as $t \to 0$ in the sense of tempered distributions (all the knowledge about tempered distributions that we shall need is contained in Stein-Weiss [2], Chapter I, section 3). Also, the boundary distribution (f or μ as the case may be) uniquely determines u. Now suppose that u is still harmonic in $\mathbb{R}^{n+1}_+$ and u is uniformly in $L^p(\mathbb{R}^n)$ but $0 < p < 1$. Then theorem 4.7 tells us that each u_t is a bounded function, hence a tempered distribution. The question arises naturally: Does $\lim_{t\to 0} u_t$ exist in the sense of tempered distributions? The answer is contained in the following:

THEOREM 4.23. Let $u(x,t)$ *be a harmonic function in* $\mathbb{R}^{n+1}_+$ *and suppose that* u *is uniformly in* $L^p(\mathbb{R}^n)$ *for some* $p > 0$. *Then* $\lim_{t\to 0} u_t = f$ *exists in the sense of tempered distributions, and this limit* f *uniquely determines* u.

Proof: Since we know the theorem to hold for $1 \leq p \leq \infty$, we assume $0 < p < 1$. For $(x,t) \in \mathbb{R}^{n+1}_+$ and $\delta > 0$, we define $u_\delta(x,t) = u(x,t+\delta)$. Each u_δ is harmonic in $\mathbb{R}^{n+1}_+$. From theorem 4.7 we know that $|u(x,t)| \leq CMt^{-n/p}$. We can use this estimate to obtain

$$\int_{\mathbb{R}^n} |u(x,t)|dx \leq \int_{\mathbb{R}^n} |u(x,t)|^p (CMt^{-n/p})^{1-p} dx \leq$$
$$\leq CMt^{-n((1/p)-1)}$$

This implies, in particular, that for each $\delta > 0$ is

$$\sup_{t>0} \int_{\mathbb{R}^n} |u_\delta(x,t)|dx \leq CM\delta^{-n((1/p)-1)} < \infty$$

It follows that, for each $\delta > 0$, $u_\delta(x,t)$ will be the Poisson integral of some finite Borel measure. Actually, since this measure has to be the weak-* limit of the functions $x \mapsto u_\delta(x,t)$ as $t \to 0$, we see that it coincides with the integrable function $u_\delta(x) = u(x,\delta)$. If $\eta > \delta > 0$, we have $u_\eta(x) = u(x,\eta) = u_\delta(x,\eta-\delta) = P_{\eta-\delta} * u_\delta(x)$. Taking Fourier transforms we get:

$$\hat{u}_\eta(\xi)e^{2\pi|\xi|\eta} = \hat{u}_\delta(\xi)e^{2\pi|\xi|\delta}.$$

This allows us to define a function $\psi(\xi) = \hat{u}_\delta(\xi)e^{2\pi|\xi|\delta}$, $\xi \in \mathbb{R}^n$, $\delta > 0$. ψ is a well defined continuous function because the right hand side does not depend on $\delta > 0$. Now, since for any $t > 0$,

$$\psi(\xi)e^{-2\pi|\xi|t} = \widehat{u_t}(\xi) = \mathcal{F}[u(\cdot,t)](\xi),$$

we obtain the estimate

$$|\psi(\xi)e^{-2\pi|\xi|t}| \le \int_{\mathbb{R}^n} |u(x,t)|dx \le CMt^{-n((1/p)-1)} = Ct^{-N}$$

where we have made $N = n((1/p) - 1) > 0$. Thus

$$|\psi(\xi)| \le C \inf_{t>0} t^{-N}e^{2\pi|\xi|t} = C_N|\xi|^N .$$

This estimate implies that ψ is a slowly increasing function and, consequently, a tempered distribution. We know (see Stein-Weiss [2]) that the Fourier transform establishes an isomorphism from the topological vector space $\mathcal{S}'$, formed by the tempered distributions, onto itself. Therefore, there is a unique $f \in \mathcal{S}'$ such that $\psi = \hat{f}$. Now it is easy to see that $u_t \to f$ in $\mathcal{S}'$ as $t \to 0$. We have to see that, for every function ϕ in the Schwartz class $\mathcal{S}$

$$\int_{\mathbb{R}^n} u(x,t)\phi(x)dx \to \langle f,\phi\rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx$$

as $t \to 0$. But, by using the Fourier transform

$$\int_{\mathbb{R}^n} u(x,t)\phi(x)dx = \int_{\mathbb{R}^n} \widehat{u_t}(\xi)\mathcal{F}^{-1}\phi(\xi)d\xi =$$

$$= \int_{\mathbb{R}^n} \psi(\xi)e^{-2\pi|\xi|t}\, \mathcal{F}^{-1}\phi(\xi)d\xi \to \int_{\mathbb{R}^n} \psi(\xi)\, \mathcal{F}^{-1}\phi(\xi)d\xi =$$

$$= \langle f,\phi\rangle \quad \text{as} \quad t \to 0.$$

Observe that $f = 0$ forces $u = 0$, so that f uniquely determines u. $\square$

It is important to note that the estimate obtained for ψ in the above proof involves M, and it implies $|\langle f,\phi\rangle| \le C(\phi)M$. We shall have to appeal to this estimate in the next chapter, when we shall deal with the real variable theory of Hardy spaces.

5. SINGULAR INTEGRAL OPERATORS

All through this chapter we have seen the action of the Calderón-Zygmund decomposition in several situations. We shall now apply this powerful device to the problem for which it was originally created (in A. Calderón and A. Zygmund [1]), namely, the estimation of operators of the form

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \Omega(y)\,|y|^{-n} f(x - y)\,dy$$

where Ω is homogeneous of degree zero and its integral over the unit sphere is null. It turns out that the same method works for a wider class of operators which we shall now describe:

DEFINITION 5.1. Given a tempered distribution K, *the convolution operator*

$$Tf(x) = K * f(x) \qquad (f \in \mathcal{S}(\mathbb{R}^n))$$

is called a singular integral operator if the following two conditions are satisfied:

(a) $\hat{K} \in L^\infty(\mathbb{R}^n)$

(b) K *coincides in* $\mathbb{R}^n \setminus \{0\}$ *with a locally integrable function* $K(x)$ *satisfying Hörmander's condition:*

$$\int_{|x|>2|y|} |K(x - y) - K(x)|\,dx \le B_K \qquad (y \in \mathbb{R}^n)$$

The first condition gives, by Plancherel's theorem, the inequality

$$\|Tf\|_2 \le \|\hat{K}\|_\infty \|f\|_2 \qquad (f \in \mathcal{S}(\mathbb{R}^n))$$

and therefore, T extends to a bounded operator in $L^2(\mathbb{R}^n)$ with norm $\|\hat{K}\|_\infty$. Under the assumption (a) only, nothing more can be said about T, but if we add (b), then other estimates can be proved, as we shall see below.

Let us now look at the effect which Hörmander's condition is designed to produce: If $K \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$ and $a(x)$ is an integrable function supported in a cube Q centered at the origin and with mean value zero, then the convolution

$$K * a(x) = \int_Q K(x-y)a(y)dy = $$
$$= \int_Q \left[K(x-y) - K(x)\right] a(y)dy$$

makes sense for a.e. $x \in Q$ and is locally integrable away from Q. Writing $\tilde{Q} = Q^{2\sqrt{n}}$, it is obvious that $|x| > 2|y|$ whenever $y \in Q$ and $x \notin \tilde{Q}$, and condition 5.1(b) then implies

$$\int_{\mathbb{R}^n \setminus \tilde{Q}} |K * a(x)|dx \leq$$
$$\leq \int_Q \int_{|x|>2|y|} |K(x-y) - K(x)||a(y)|dx\, dy \leq B_K \int_Q |a(y)|dy$$

By translation invariance, the same is true if $a(x)$ is supported in an arbitrary cube. Thus, we have proved

LEMMA 5.2. If K *satisfies* 5.1(b) *and* $a \in L^1(\mathbb{R}^n)$ *is supported in a cube* Q *with* $\int_Q a(y)dy = 0$, *then*

$$\int_{\mathbb{R}^n \setminus \tilde{Q}} |K * a(x)|dx \leq B_K \|a\|_1 \qquad (\text{with } \tilde{Q} = Q^{2\sqrt{n}})$$

Before proving the estimates for singular integral operators, let us look for interesting examples to which the theory can be applied. More applications will be given in the next section.

EXAMPLES: (a) We are going to describe the singular integral operators defined by convolution with principal value distributions. These distributions $K = \text{p.v.}k(x)$ act in the following manner:

$$K(\phi) = \text{p.v.} \int k(x)\phi(x)dx = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} k(x)\phi(x)dx$$

where $k \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$ is a function to which we impose the growth condition

$$(5.3) \qquad \int_{r<|x|<2r} |k(x)|dx \leq C_1 \qquad (r > 0)$$

(which is natural in view of the basic example mentioned at the beginning of this section). Assuming (5.3), it is easy to check that $K(\phi)$ is well defined for all Schwartz functions ϕ and K is a tempered distribution if

$$\text{(5.4)}\quad \begin{cases} \left|\int_{r<|x|<R} k(x)dx\right| \leqq C_2 & (0 < r < R) \\ \text{and } \lim\limits_{r\downarrow 0} \int_{r<|x|<1} k(x)dx \quad \text{exists.} \end{cases}$$

Indeed, if (5.3) and (5.4) hold, with $\ell = \text{p.v.}\int_{|x|<1} k(x)dx$, then

$$K(\phi) = \lim_{\varepsilon\to 0}\{\phi(0)\int_{\varepsilon<|x|<1} k(x)dx + \int_{\varepsilon<|x|<1} k(x)\left[\phi(x) - \phi(0)\right]dx +$$

$$+ \int_{|x|\geqq 1} k(x)\phi(x)dx\} = \ell\phi(0) + \int_{|x|<1} k(x)\left[\phi(x) - \phi(0)\right]dx +$$

$$+ \int_{|x|\geqq 1} k(x)\phi(x)dx = \ell\phi(0) + I_1 + I_2$$

where the integrals I_1 and I_2 are absolutely convergent, and moreover

$$|I_1| \leqq \|\nabla\phi\|_\infty \sum_{k=0}^{\infty} \int_{2^{-k-1}\leqq|x|<2^{-k}} |k(x)||x|dx \leqq 2C_1\|\nabla\phi\|_\infty$$

and

$$|I_2| \leqq \sup_{x\in\mathbb{R}^n} (|x||\phi(x)|) \sum_{k=0}^{\infty} \int_{2^k\leqq|x|<2^{k+1}} |k(x)|2^{-k}dx \leqq$$

$$\leqq 2C_1 \sup_{x\in\mathbb{R}^n} (|x||\phi(x)|)$$

All these principal value distributions give rise to singular integral operators provided that Hörmander's condition is satisfied:

PROPOSITION 5.5. If $k \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$ *satisfies (5.1(b)),(5.3) and (5.4), and if* $K = \text{p.v. } k(x)$, *then*

$$Tf(x) = K * f(x) = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon} k(x-y)f(y)dy \qquad (f \in \mathcal{S}(\mathbb{R}^n))$$

is a singular integral operator.

In order not to delay too much the study of the properties of singular integral operators, the proofs of this and the next proposition are postponed to the end of the section.

(b) Let us consider now the particular case in which $k(x)$ is homogeneous of degree $-n$, i.e.

$$k(x) = \Omega(x)|x|^{-n} = \Omega(x')|x|^{-n}$$

with Ω homogeneous of degree 0 (we shall systematically use the notation: $x' = x/|x|$). The operator T obtained by convolution with k will then be, wherever it may be defined, dilation invariant:

$$T(f^\delta) = (Tf)^\delta \qquad (\text{with } f^\delta(x) = f(\delta x),\ \delta > 0)$$

Now, conditions (5.3) and (5.4) are respectively equivalent to

$$\|\Omega\|_1 = \int_{|x'|=1} |\Omega(x')|\,d\sigma(x') < \infty$$

and

$$\int_{|x'|=1} \Omega(x')\,d\sigma(x') = 0$$

where $d\sigma$ denotes the $(n-1)$-dimensional Lebesgue measure on the unit sphere.

Finally, some continuity condition must be imposed on Ω in order to ensure that 5.1(b) holds. To this end, we introduce the following L^1-modulus of continuity:

$$\omega_1(\Omega;t) = \sup_{h\in\mathbb{R}^n,\ |h|\le t} \int_{|x'|=1} |\Omega(x'+h) - \Omega(x')|\,d\sigma(x')$$

PROPOSITION 5.6. Suppose that Ω *is homogeneous of degree* 0 *and integrable over the unit sphere with integral equal to zero. If the* L^1-Dini *condition*

$$\int_0^1 \omega_1(\Omega;t)\,\frac{dt}{t} < \infty$$

holds, then

$$Tf(x) = \text{p.v.} \int \Omega(y)|y|^{-n} f(x-y)\,dy \qquad (f \in \mathcal{S}(\mathbb{R}^n))$$

is a singular integral operator. Moreover, if Ω *is odd, then the extension of* T *to* $L^2(\mathbb{R}^n)$ *is given, in terms of the Fourier transform, by* $(Tf)\hat{}(\xi) = \hat{f}(\xi)m(\xi)$, *where*

$$m(\xi) = -\frac{\pi i}{2}\int_{|x'|=1}\Omega(x')\operatorname{sign}(x'\cdot\xi)d\sigma(x')$$

Observe that $m(\xi) = (\text{p.v. } \Omega(x)|x|^{-n})\hat{}(\xi)$ is homogeneous of degree zero, something which could have been predicted due to the homogeneity of $\Omega(x)|x|^{-n}$ and the behaviour of the Fourier transform with respect to dilations. An explicit formula for $m(\xi)$ when Ω is not necessarily odd can be found in Stein [1] or Stein-Weiss [2]. We also point out that the L^1-Dini condition is rather weak; in particular, it is verified if $|\Omega(x) - \Omega(y)| \leq C|x' - y'|^\alpha$ for some $\alpha > 0$.

c) In $\mathbb{R}^1$, there is basically one homogeneous singular integral operator, corresponding to $\Omega(t) = \frac{1}{\pi}\operatorname{sign}(t)$ (or $k(t) = \frac{1}{\pi t}$), namely

$$Hf(x) = \text{p.v. } \frac{1}{\pi}\int_{-\infty}^{\infty} f(x - t)\frac{dt}{t}$$

This is the Hilbert transform, which is also defined in $L^2(\mathbb{R})$ by

$$(Hf)\hat{}(\xi) = -i \operatorname{sign}(\xi)\hat{f}(\xi)$$

In fact, the formula of Proposition 5.6 gives in this case

$$m(\xi) = -\frac{\pi i}{2}\left[\Omega(1)\operatorname{sign}(\xi) + \Omega(-1)\operatorname{sign}(-\xi)\right] = -i \operatorname{sign}(\xi) .$$

In $\mathbb{R}^n$, we have a family of analogues of the Hilbert transform corresponding to $\Omega_h(x') = c_n(x'\cdot h)$, where h is any fixed vector in $\mathbb{R}^n$, and the normalizing constant c_n is chosen (for a reason which will become clear in a moment) so that

$$\int_{|x'|=1}|x'_1|d\sigma(x') = \int_{|x'|=1}|x'\cdot u|d\sigma(x') = \frac{2}{\pi c_n}$$

for every unit vector u (an Advanced Calculus computation shows that $c_n = \pi^{-(n+1)/2}\Gamma(\frac{n+1}{2})$). Since Ω_h depends linearly on h, it suffices to consider the operators corresponding to the vectors of the standard basis: $h = e_1, e_2, \ldots, e_n$, namely

$$R_jf(x) = \text{p.v. } c_n\int\frac{y_j}{|y|^{n+1}} f(x-y)dy \qquad (j = 1,2,\ldots,n)$$

which are called the Riesz transforms. The action of R_j is specially simple in terms of the Fourier transform:

$$(R_j f)^{\widehat{}}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi) \qquad (f \in L^2 \; ; \; j = 1, 2, \dots, n)$$

This is a consequence of (5.6) and the following identity

$$\frac{\pi}{2} c_n \int_{|x'|=1} (x' \cdot h) \operatorname{sign}(x' \cdot \xi) \, d\sigma(x') = \frac{\xi \cdot h}{|\xi|} \qquad (\xi, h \in \mathbb{R}^n)$$

whose proof is now, with our choice of c_n, almost trivial: We can fix ξ with $|\xi| = 1$, and then, the left hand side is a linear function of h, say $\ell(h)$, such that $|\ell(h)| \leq |h|$ and $\ell(\xi) = 1$; this necessarily implies $\ell(h) = \xi \cdot h$.

We can now state the main result concerning singular integral operators:

THEOREM 5.7. Every singular integral operator satisfies the inequalities

$$(5.8) \quad \|Tf\|_p \leq C_p \|f\|_p \qquad (f \in L^2 \cap L^p; \; 1 < p < \infty)$$

$$(5.9) \quad |\{x : |Tf(x)| > t\}| \leq C_1 t^{-1} \|f\|_1 \qquad (f \in L^2 \cap L^1)$$

and

$$(5.10) \quad \|Tf\|_{BMO} \leq C_\infty \|f\|_\infty \qquad (f \in L^2 \cap L^\infty)$$

where C_p, $1 \leq p \leq \infty$, *depends only on* p, n *and on the constants* $\|\hat{K}\|_\infty$ *and* B_K *of the kernel.*

Thus, we can consider every singular integral operator extended as a bounded operator from L^p to itself, from L^1 to weak-L^1 and from $L^\infty_c = \{L^\infty$ functions with compact support$\}$ to BMO.

The plan of the proof is as follows: We obtain first the basic estimate (5.9). Interpolating by Marcinkiewicz' theorem 2.11 between this and the L^2-boundedness of T, the inequalities (5.8) follow for $1 < p \leq 2$. We then prove a technical lemma (5.11) which will also be used later, and from such a result, the L^∞-BMO estimate will easily be derived. Finally, we use (5.10) and the interpolation theorem 3.7 to obtain the remaining L^p-inequalities: $2 < p < \infty$.

Proof of (5.9): Take $f \in L^1 \cap L^2$ and, given $t > 0$, let $\{Q_j\}$ be the cubes of the Calderón-Zygmund decomposition corresponding to $|f|$ at height t. (see 1.12). This decomposition of the underlying space into cubes allows to define what is called the Calderón-Zygmund decomposition, $f(x) = g(x) + b(x)$, of the function f into its "good part" g and its "bad part" b, which are given as follows:

$$g(x) = \sum_j \{\frac{1}{|Q_j|} \int_{Q_j} f\}\chi_{Q_j}(x) + f(x)\chi_{\mathbb{R}^n\setminus\Omega}(x)$$

with $\Omega = \bigcup_j Q_j$ (observe that $|\Omega| \leq Ct^{-1}\|f\|_1$)

$$b(x) = f(x) - g(x) = \sum_j b_j(x)$$

where b_j is supported in Q_j and has mean value zero; precisely:

$$b_j(x) = \{f(x) - \frac{1}{|Q_j|} \int_{Q_j} f\}\chi_{Q_j}(x)$$

To estimate Tg we use the fact that T is bounded in L^2 together with the observation that $|g(x)| = |f(x)| \leq t$ for a.e. $x \notin \Omega$, and $|g(x)| \leq 2^n t$ for all $x \in \Omega$; thus

$$|\{x : |Tg(x)| > \tfrac{t}{2}\}| \leq 4t^{-2}\int |Tg(x)|^2 dx$$

$$\leq 4\|\hat{K}\|_\infty^2 t^{-2} \int |g(x)|^2 dx \leq$$

$$\leq 4\|\hat{K}\|_\infty^2 t^{-2} \{\int_{\mathbb{R}^n\setminus\Omega} t|f(x)|dx + (2^n t)^2|\Omega|\} \leq$$

$$\leq Ct^{-1}\|f\|_1$$

Next, we observe that Lemma 5.2 can be applied to each piece $b_j(x)$ of the bad part, because $Tb_j(x) = K * b_j(x)$ for a.e. $x \notin Q_j$ (this can be seen by approximating b_j by test functions supported in Q_j). Thus, if $\tilde{Q}_j = Q_j^{2\sqrt{n}}$

$$\int_{\mathbb{R}^n\setminus\tilde{Q}_j} |Tb_j(x)|dx \leq B_K\|b_j\|_1 \leq 2B_K \int_{Q_j} |f(x)|dx$$

On the other hand, since $b \in L^2(\mathbb{R}^n)$, the series $\sum_j b_j$ and $\sum_j Tb_j$ converge in the L^2-norm to b and Tb, respectively. In particular, $|Tb(x)| \leq \sum_j |Tb_j(x)|$ a.e., and

$$\int_{\mathbb{R}^n \setminus \tilde{\Omega}} |Tb(x)|dx \leq \sum_j 2B_K \int_{Q_j} |f(x)|dx \leq 2B_K\|f\|_1$$

where $\tilde{\Omega} = \bigcup_j \tilde{Q}_j$ (so that $|\tilde{\Omega}| \leq (2\sqrt{n})^n|\Omega|$). Finally

$$|\{x : |Tb(x)| > \tfrac{t}{2}\}| \leq |\tilde{\Omega}| + \frac{2}{t}\int_{\mathbb{R}^n \setminus \tilde{\Omega}} |Tb(x)|dx \leq$$

$$\leq Ct^{-1}\|f\|_1. \quad \square$$

Before stating the auxiliary lemma, we introduce a variant of the Hardy-Littlewood maximal operator which will also appear in the sequel:

NOTATION: For $1 \leq p < \infty$, we denote $M_pf = M(|f|^p)^{1/p}$, i.e.

$$M_pf(x) = \sup_{x\in Q} \left(\frac{1}{|Q|}\int_Q |f|^p\right)^{1/p}$$

LEMMA 5.11. Let $f \in L_c^\infty(\mathbb{R}^n)$, *so that the integral*

$$I_\varepsilon = \int_{|y|>\varepsilon} K(-y)f(y)dy$$

exists for every $\varepsilon > 0$. *Then, for all* $1 < p < \infty$, *we have*

$$\varepsilon^{-n}\int_{|x|<\varepsilon/2} |Tf(x) - I_\varepsilon|dx \leq C_pM_pf(0) +$$

$$+ \varepsilon^{-n}\iint_{2|x|<\varepsilon<|y|} |K(x-y) - K(-y)||f(y)|dx\,dy$$

with C_p *independent of* ε *and* f.

Proof: Since M_pf increases as p increases, we can assume that $1 < p \leq 2$, and in this case, we have already proved that T is bounded in L^p. Set

$$f_1 = f\chi_{\{x:|x|<\varepsilon\}}, \qquad f_2 = f - f_1$$

Then

$$|Tf(x) - I_\varepsilon| \leq |Tf_1(x)| + \left|\int_{|y|>\varepsilon} [K(x-y) - K(-y)]f(y)dy\right|$$

and we only have to observe that, if $p' = \frac{p}{p-1}$ is the dual exponent of p,

$$\int_{|x|<\varepsilon/2} |Tf_1(x)|\,dx \leq C_p \varepsilon^{n/p'} \|f_1\|_p \leq C_p\, \varepsilon^n\, M_p f(0). \qquad \square$$

The final step to establish Theorem 5.7 is:

Proof of (5.10): We only need to use the previous lemma with (say) $p = 2$, because $M_2 f(0) \leq \|f\|_\infty$ and

$$(Tf)^{\#}(0) \leq A_n \sup_{\varepsilon>0} \varepsilon^{-n} \int_{|x|<\varepsilon/2} |Tf(x) - I_\varepsilon|\,dx$$

with A_n depending only on the dimension of the space. Thus, for all $f \in L_c^\infty(\mathbb{R}^n)$, we obtain

$$(Tf)^{\#}(0) \leq A_n (C_2 + B_K) \|f\|_\infty$$

and since everything is translation invariant, we also have

$$(Tf)^{\#}(x) \leq C\|f\|_\infty \qquad (x \in \mathbb{R}^n;\quad f \in L_c^\infty)$$

which is (5.10). If one wishes to get the same inequality for all $f \in L^2 \cap L^\infty$ (even though this is a rather irrelevant question), just define $f_j = f\chi_{\{x:|x|<j\}}$ so that $Tf = \lim_{j\to\infty} Tf_j$ (in L^2) and

$$\|Tf\|_{BMO} \leq \limsup_{j\to\infty} \|Tf_j\|_{BMO} \leq C\|f\|_\infty . \qquad \square$$

The estimates of Theorem 5.7 seem specially satisfactory in L^p with $1 < p < \infty$. It is not true, however, that singular integral operators map L^1 (or L^∞) to itself, and this is what makes the substitute results (5.9) and (5.10) interesting. We can easily see this in the case of the Hilbert transform applied to the function $f = \chi_{[a,b]} \in L^1 \cap L^\infty$:

$$Hf(x) = \text{p.v. } \frac{1}{\pi}\int_{x-b}^{x-a} \frac{dt}{t} = \frac{1}{\pi} \log \left|\frac{x-a}{x-b}\right|$$

It is clear that $Hf \notin L^\infty$, and since $Hf(x) \sim \frac{b-a}{\pi(x-b)}$, $|x| \to \infty$, it also follows that $Hf \notin L^1$ (but $Hf \in (\text{weak-}L^1) \cap BMO$).

In order to get a better understanding of the behaviour of our operators in L^1 and L^∞, we pose ourselves two questions:

(Q1) <u>Are there functions</u> $f \in L^1$ <u>for which we can assert that</u> $Tf \in L^1$ <u>for all singular integral operators</u> T?

(Q.2) <u>If</u> T <u>is a singular integral operator, is it possible to define "in a natural way"</u> Tf <u>for all</u> $f \in L^\infty$, <u>so that</u> $T : L^\infty \to BMO$?.

Looking for an answer to the first question, we immediately find a necessary condition: If $f \in L^1$ is such that $R_j f \in L^1$ for some Riesz transform R_j, we must have

$$(R_j f)\hat{}(\xi) = (-i\xi_j/|\xi|)\hat{f}(\xi)$$

and since the left hand side is continuous, so must be the right hand side, which forces: $\hat{f}(0) = \int f = 0$.

This suggests the possibility of using Lemma 5.2, provided that we also assume that f is supported in a cube Q. In fact

$$\|Tf\|_1 \leq \int_{\tilde{Q}} |Tf(x)|dx + B_K\|f\|_1 \leq$$
$$\leq \|\hat{K}\|_\infty \|f\|_2 |\tilde{Q}|^{\frac{1}{2}} + B_K\|f\|_1$$

and thus, a uniform estimate will be obtained, for instance, if $\|f\|_\infty \leq \frac{1}{|Q|}$. Let us collect the conditions imposed so far and give a name to the functions satisfying them:

DEFINITION 5.12. An atom(*) *is a bounded function supported in a cube* Q *and such that* $\|a\|_\infty \leq \frac{1}{|Q|}$ *and* $\int a(x)dx = 0$.

Now, what the preceding remarks prove is that

$$(5.13) \qquad \|Ta\|_1 \leq C$$

for every atom $a(x)$ and every singular integral operator T, with C depending only on n and on the constants B_K and $\|\hat{K}\|$ of the kernel of T.

One may argue that (5.13) is not a good answer to (Q1) due to the very special conditions imposed to $a(x)$. A trivial (but nevertheless interesting) extension consists in defining the Banach

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(*) This will be called a $(1,\infty)$-atom in Chapter III.

space $H^1_{at}(\mathbb{R}^n)$ (atomic H^1) as follows:

$$f \in H^1_{at} \quad \text{iff} \quad f(x) = \sum_j \lambda_j\, a_j(x)$$

for some $(\lambda_j) \in \ell^1$ and atoms $a_j(x)$, and

$$\|f\|_{H^1_{at}} = \inf \{\sum_j |\lambda_j| \;:\; f(x) = \sum_j \lambda_j\, a_j(x)\}$$

The series $\sum_j \lambda_j a_j$ converges in L^1, so that $H^1_{at} \subset L^1$, and clearly $\int f = 0$ for every $f \in H^1_{at}$. Now, we have

<u>*COROLLARY 5.14:*</u> <u>*Every singular integral operator*</u> T <u>*maps*</u> $H^1_{at}(\mathbb{R}^n)$ <u>*boundedly into*</u> $L^1(\mathbb{R}^n)$, <u>*i.e.*</u>

$$\|Tf\|_1 \leq C\|f\|_{H^1_{at}} \qquad (f \in H^1_{at})$$

More insight into H^1_{at} will be gained in Chapter III. For the time being, let us merely observe that a Schwartz function f belongs to H^1_{at} if and only if $\int f = 0$.

Turning to (Q2), let us first observe that it is not a trivial question. In fact, the proof of Theorem 5.7 shows that we can actually define $T : L^p \cap L^\infty \to BMO$ for arbitrary $p < \infty$, but since $L^p \cap L^\infty$ is not dense in L^∞, this does not allow to extend T by continuity to all bounded functions. Following the argument in the proof of (5.10), one would be tempted to define $Tf = \lim T(f\chi_{B_j})$, where $B_j = \{x : |x| \leq j\}$. Such a limit, however, may not exist in any reasonable sense (see the example below Proposition 5.15). But, since we want to consider Tf and $T(f\chi_{B_j})$ as elements of BMO, we may try to add a constant to each term $T(f\chi_{B_j})$ so as to make the corrected sequence converge. This, we can achieve:

<u>*PROPOSITION 5.15.*</u> <u>*Given a singular integral operator*</u> T <u>*with kernel*</u> K, <u>*for each*</u> $f \in L^\infty$ <u>*we define*</u>

$$(5.16) \qquad Tf(x) = \lim_{j\to\infty} \left(T(f\chi_{B_j})(x) - \int_{1<|y|<j} K(-y)f(y)\,dy\right)$$

<u>*The sequence to the right converges locally in*</u> L^1 <u>*and also pointwise a.e., and the extended operator*</u> T <u>*satisfies*</u>

$$\|Tf\|_{BMO} \leq C_\infty \|f\|_\infty \qquad (f \in L^\infty)$$

where C_∞ *is the same constant of* (5.10).

Observe that, if $f \in L_c^\infty$, the old and new definitions of $Tf(x)$ differ only in a constant: $\int_{|y|>1} K(-y)f(y)dy$; thus they agree as elements of BMO. In general, if $f \in L^\infty$, we denote by Tf the class in BMO represented by the function defined by (5.16). In this way, we extend the operator $T : L_c^\infty \to \mathrm{BMO}$ to an operator $T : L^\infty \to \mathrm{BMO}$ with the same norm.

Proof of (5.15): Let us denote by $g_j(x)$ the j-th term of the sequence in the right hand side of (5.16). Given a compact set F, let ℓ be the first natural number such that $\ell \geq 2 \sup_{x\in F}|x|$. Then, for all $x \in F$ and $j > \ell$:

$$g_j(x) = g_\ell(x) + \int_{\ell<|y|\leq j} \left[K(x-y) - K(-y)\right] f(y)dy$$

The integral to the right is uniformly bounded (when $x \in F$) by $B_K\|f\|_\infty$, and converges when $j \to \infty$ to the same integral extended over all $|y| > \ell$. Thus, $g_j(x)$ converges pointwise and in $L^1(F)$. As a consequence, we have

$$\|Tf\|_{\mathrm{BMO}} \leq \limsup_{j\to\infty} \|g_j\|_{\mathrm{BMO}} =$$

$$= \limsup_{j\to\infty} \|T(f\chi_{B_j})\|_{\mathrm{BMO}} \leq C_\infty \|f\|_\infty$$

where we have used (5.10) and the fact that g_j and $T(f\chi_{B_j})$ differ in a constant. $\square$

The necessity of substracting the constant terms in (5.16) can be seen in the following instructive example: Compute the Hilbert transform of $f(x) = \mathrm{sign}(x)$. Since $f\chi_{B_j} = \chi_{[0,j]} - \chi_{[-j,0]}$, we have

$$g_j(x) = \frac{1}{\pi}\{\log\left|\frac{x}{x-j}\right| - \log\left|\frac{x+j}{x}\right|\} + \frac{2}{\pi}\int_1^j \frac{dt}{t} =$$

$$= \frac{2}{\pi}\log|x| - \frac{1}{\pi}\log|x^2-j^2| + \frac{2}{\pi}\log j$$

and letting $j \to \infty$, we find: $Hf(x) = \frac{2}{\pi}\log|x|$.

Our next objective is to give more precise estimates for some singular integral operators for which more regularity of the kernel is assumed.

DEFINITION 5.17. A singular integral operator $Tf = K * f$ *is called regular if its kernel satisfies the following two conditions:*

(5.18) $\quad |K(x)| \leq B|x|^{-n} \qquad (x \in \mathbb{R}^n \setminus \{0\})$

(5.19) $\quad |K(x-y) - K(x)| \leq B|y||x|^{-n-1} \qquad (|x| > 2|y| > 0)$

Observe that (5.19) trivially implies Hörmander's condition 5.1(b), and it is automatically verified if $|\nabla K(x)| \leq \text{Const.}|x|^{-n-1}$ away from the origin. For homogeneous operators as those described in Proposition 5.6, a sufficient condition to be a regular singular integral operator is: $\Omega \in C^1(\mathbb{R}^n \setminus \{0\})$.

The growth limitation (5.18) enables us to define the truncated operators

$$T_\varepsilon f(x) = \int_{|y|>\varepsilon} K(y)f(x-y)\,dy \qquad (\varepsilon > 0)$$

directly for L^p functions $f(x)$, $1 \leq p < \infty$, and one naturally wishes to have a more explicit definition of Tf as a limit of $T_\varepsilon f$. Thus, we are interested in uniform estimates for $(T_\varepsilon)_{\varepsilon>0}$, and we are going to see that we can actually estimate the maximal operator: $T^*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$.

THEOREM 5.20. If T *is a regular singular integral operator and* $f \in L^p$, $1 \leq p < \infty$, *then the following inequalities are verified:*

(i) $\quad (Tf)^\#(x) \leq C_q M_q f(x) \qquad (1 < q)$

(ii) $\quad T^*f(x) \leq C_q M_q f(x) + CM(Tf)(x) \qquad (1 < q)$

(iii) $\quad \|T^*f\|_p \leq C_p\|f\|_p \qquad (1 < p < \infty)$

(iv) $\quad |\{x : T^*f(x) > t\}| \leq Ct^{-1}\|f\|_1, \qquad t > 0$

Proof: By translation invariance, we only need to prove the pointwise estimates (i) and (ii) at $x = 0$. The basic point is that we can use Lemma 5.11 (which holds now for every $f \in L^p$, since the assumption $f \in L_c^\infty$ was only imposed to ensure the existence of I_ε) with an improved estimate for the last term, namely

$$(5.21) \qquad \int_{|y|>2|x|} |K(x-y) - K(-y)||f(y)|dy \leq CMf(0)$$

Since $Mf \leq M_q f$ for every $q > 1$, we obtain

$$\varepsilon^{-n} \int_{|x|<\varepsilon/2} |Tf(x) - I_\varepsilon|dx \leq C_q M_q f(0)$$

which implies (i) (at $x = 0$). On the other hand, we observe that $I_\varepsilon = T_\varepsilon f(0)$, and it also follows that

$$C_q M_q f(0) \geq \varepsilon^{-n} \int_{|x|<\varepsilon/2} (|T_\varepsilon f(0)| - |Tf(x)|)dx \geq$$

$$\geq A_n |T_\varepsilon f(0)| - M(Tf)(0)$$

and taking the supremum over all $\varepsilon > 0$, we get (ii).

Let us now prove (5.21), which is a consequence of the stronger regularity provided by (5.19). In fact,

$$\int_{|y|>2|x|} (\ldots) \leq \sum_{j=1}^{\infty} \int_{2^j|x|<|y|\leq 2^{j+1}|x|} B|x||y|^{-n-1}|f(y)|dy \leq$$

$$\leq B \sum_{j=1}^{\infty} 2^{-j}(2^j|x|)^{-n} \int_{|y|\leq 2^{j+1}|x|} |f(y)|dy \leq$$

$$\leq C \sum_{j=1}^{\infty} 2^{-j} Mf(0) = CMf(0)$$

Finally, (iii) is a trivial consequence of (ii) and the L^p inequalities for the operators M and T, and the proof of (iv) is postponed to Chapter V, where it will appear in a natural context. $\square$

COROLLARY 5.22. Let $k \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$ *be a function satisfying* (5.4), (5.18) *and* (5.19). *Then, the singular integral operator* $Tf = (\text{p.v.}k(x)) * f$ *(whose existence is guaranteed by Proposition 5.5) satisfies*

$$Tf(x) = \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} k(y)f(x-y)\,dy \qquad \text{a.e.}$$

for every $f \in L^p$, $1 \leq p < \infty$.

Proof: For all $f \in S(\mathbb{R}^n)$, we know that $\lim_{\varepsilon\to 0} T_\varepsilon f(x) = Tf(x)$ at every $x \in \mathbb{R}^n$. Since $S(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, the result follows from 5.20 (iii) and (iv) by a standard technique already used to derive the differentiation theorem 1.9 from the estimates for the Hardy-Littlewood maximal operator. $\square$

To conclude this section, we remember that the results needed to establish some examples of singular integral operators were stated without proofs. Here we provide the proofs:

Proof of (5.5): We must only show that $\hat{K} \in L^\infty(\mathbb{R}^n)$, where $K = \text{p.v.}k(x)$. Let us define the truncated kernels

$$k_\varepsilon^R(x) = \begin{cases} k(x) & \text{if} \quad \varepsilon < |x| < R \\ 0 & \text{if} \quad |x| \leq \varepsilon \quad \text{or} \quad |x| \geq R \end{cases}$$

Since $\lim_{\varepsilon\to 0, R\to\infty} k_\varepsilon^R = K$ (in the sense of tempered distributions), it suffices to prove that the Fourier transforms of $\{k_\varepsilon^R\}_{0<\varepsilon<R}$ are uniformly bounded. We begin by observing that Hörmander's condition transfers to the truncated kernels in the following sense:

$$\int_{\mathbb{R}^n} |k_\varepsilon^R(x-y) - k_\varepsilon^R(x)|\,dx \leq 2\int_{\varepsilon<|x|<\varepsilon+|y|} |k(x)|\,dx \,+$$

$$+\, 2\int_{R-|y|<|x|<R} |k(x)|\,dx + \int_{\varepsilon<|x|<R} |k(x-y)-k(x)|\,dx$$

and therefore,

$$|y| \leq \frac{\varepsilon}{2} \quad \text{implies} \quad \int_{\mathbb{R}^n} |k_\varepsilon^R(x-y) - k_\varepsilon^R(x)|\,dx \leq 4C_1 + B_K$$

Now, we fix $\xi \in \mathbb{R}^n$ and decompose

$$(k_\varepsilon^R)^\wedge(\xi) = (k_\varepsilon^r)^\wedge(\xi) + (k_r^R)^\wedge(\xi)$$

where $r = 1/|\xi|$; we also define $y = -\xi/2|\xi|^2$, so that $|y| = r/2$ and $e^{-2\pi i y\cdot\xi} = -1$. Then

$$|(k_r^R)\hat{\ }(\xi)| = \frac{1}{2}|(k_r^R)\hat{\ }(\xi)(1 - e^{-2\pi i y\cdot\xi})| =$$

$$= \frac{1}{2}|\left[k_r^R(\cdot) - k_r^R(\cdot-y)\right]\hat{\ }(\xi)| \leq$$

$$\leq \frac{1}{2}\int |k_r^R(x) - k_r^R(x-y)|dx \leq 2C_1 + \frac{1}{2}B_K$$

by the preceding observation. On the other hand, (5.3) implies

$$\int_{|x|<r} |k(x)|\ |x|\ dx \leq 4C_1 r \qquad (r > 0)$$

(this type of inequality is actually equivalent to (5.3)) and thus

$$|(k_\varepsilon^r)\hat{\ }(\xi)| \leq |\int_{\varepsilon<|x|<r} k(x)(e^{-2\pi i x\cdot\xi}-1)dx| +$$

$$+ |\int_{\varepsilon<|x|<r} k(x)dx| \leq$$

$$\leq \int_{|x|<r} |k(x)|2\pi|x|\ |\xi|\ dx + C_2 \leq 8\pi C_1 + C_2 \quad . \qquad \square$$

Proof of (5.6): The first assertion will follow from (5.5) if we show that $k(x) = \Omega(x)|x|^{-n}$ satisfies Hörmander's condition. But it does, because

$$\int_{|x|>2|y|} |k(x-y) - k(x)|dx \leq \int_{|x|>2|y|} \frac{|\Omega(x-y) - \Omega(x)|}{|x-y|^n}dx +$$

$$+ \int_{|x|>2|y|} |\Omega(x)(|x-y|^{-n} - |x|^{-n})|dx \leq$$

$$\leq \int_{|y|}^{\infty} \{\int_{|x'|=1} |\Omega(tx'+y) - \Omega(tx')|d\sigma(x')\}\ \frac{dt}{t} +$$

$$+ C_n|y| \int_{|x|>2|y|} |\Omega(x)|\ |x|^{-n-1}dx \leq$$

$$\leq \int_{|y|}^{\infty} \omega_1\left(\frac{|y|}{t}\right)\frac{dt}{t} + \frac{1}{2}C_n\ \|\Omega\|_1 = \int_0^1 \omega_1(t)\frac{dt}{t} + \frac{1}{2}C_n\|\Omega\|_1$$

For the second assertion, we have to prove that $m(\xi)$ is the Fourier transform of $K = \text{p.v.}(\Omega(x)|x|^{-n})$. Since k is odd,

$$(k_\varepsilon^R)^\wedge(\xi) = \frac{1}{2}\int k_\varepsilon^R(x)(e^{-2\pi i x\cdot\xi} - e^{2\pi i x\cdot\xi})dx =$$

$$= -i\int_{|x'|=1}\Omega(x')\{\int_\varepsilon^R \sin(2\pi t x'\cdot\xi)\frac{dt}{t}\}\, d\sigma(x')$$

The inner integral is always bounded in absolute value by π, and its limit when $\varepsilon \to 0$, $R \to \infty$, is $\frac{\pi}{2}\text{sign}(x'\cdot\xi)$. This gives the desired expression for $m(\xi)$. $\square$

6. MULTIPLIERS

In the preceding section we obtained L^p inequalities for convolution operators whose kernels are not integrable. Of course, if $k \in L^1(\mathbb{R}^n)$, then $Tf = k*f$ is also a singular integral operator, since

$$\|\hat{k}\|_\infty \leq \|k\|_1 \quad , \quad B_k \leq 2\|k\|_1$$

(B_k denoting the constant in Hörmander's condition 5.1(b)). In this case, Young's inequality gives L^p estimates for all $1 \leq p \leq \infty$, namely

$$\|k*f\|_p \leq \|k\|_1 \|f\|_p \qquad (1 \leq p \leq \infty)$$

and theorem 5.7 seems to be of no interest here. There is, however, one point in that theorem which makes it useful even for this simple situation, and it is the fact that, for a fixed p with $1 < p < \infty$, the estimate of $\|k*f\|_p$ depends only on the constants $\|\hat{k}\|_\infty$ and B_k, which may be considerably smaller than $\|k\|_1$.

To put it in a more precise form: If we are given a family of kernels $k_N \in L^1$, and we wish to obtain uniform estimates for $\|k_N*f\|_p$ by means of Young's inequality, we are forced to impose the strong condition $\sup_N \|k_N\|_1 < \infty$. However, theorem 5.7 has the following consequence which will be used for our main result below:

LEMMA 6.1. Let (k_N) *be a sequence of kernels in* $L^1(\mathbb{R}^n)$, *and suppose that there exists* $C > 0$ *such that, for all* $N = 1,2, \ldots$

$$\text{a)} \quad |\hat{k}_N(\xi)| \leq C \qquad\qquad (\xi \in \mathbb{R}^n)$$

$$\text{b)} \quad \int_{|x|>2|y|} |k_N(x-y) - k_N(x)|\,dx \leq C \qquad\qquad (y \in \mathbb{R}^n)$$

Then, $T_N f = k_N * f$, $N = 1,2, \ldots$, *are uniformly bounded operators from* L^p *to itself* $(1 < p < \infty)$, *from* L^1 *to weak*-L^1 *and from* L^∞ *to* BMO.

This suggests a different (though equivalent) approach to singular integral operators defined by principal value distributions. Indeed, under the hypothesis of Proposition 5.5, the kernels $k_N = k\chi_{\{x:1/N\leq x\leq N\}}$ satisfy (a) and (b) of the preceding lemma. Therefore

$$\|k_N * f\|_p \leq C_p \|f\|_p \qquad\qquad (f \in L^p;\ 1 < p < \infty)$$

with C_p independent of $N = 1,2,\ldots$, and for all $f \in L^p$, the limit

$$Tf = \lim_{N\to\infty} k_N * f \qquad\qquad (\text{in } L^p\text{-norm})$$

exists (because it obviously exists for Schwartz functions) defining an operator T which is bounded in L^p, $1 < p < \infty$, and which certainly agrees with the unique bounded extension to L^p of the convolution operator: $f \longmapsto (\text{p.v. } k(x)) * f$, initially defined in $\mathcal{S}(\mathbb{R}^n)$.

This is not, however, our reason for stating lemma 6.1. here. We aim at a different application of the Calderón-Zygmund method which gives sufficient conditions for Fourier multipliers in L^p.

DEFINITION 6.2. Let $1 \leq p < \infty$. *Given* $m \in L^\infty(\mathbb{R}^n)$, *we say that* m *is a (Fourier) multiplier for* L^p *if the operator* T_m, *initially defined in* $L^2(\mathbb{R}^n)$ *by the relation*

$$(T_m f)\hat{}(\xi) = m(\xi)\hat{f}(\xi)$$

satisfies the inequality

$$\|T_m f\|_p \leq C\|f\|_p \qquad (f \in L^2 \cap L^p)$$

In this case, T_m *has a unique bounded extension to* L^p *which we still denote by* T_m.

Some general facts about multipliers are contained in 7.15 and at the end of this section. By Plancherel's theorem, every $m \in L^\infty$ is a multiplier for L^2, and the norm of T_m as an operator in L^2 is equal to $\|m\|_\infty$. The fact that the Riesz transforms are bounded operators in L^p, can be restated by saying that $m_j(\xi) = \xi_j/|\xi|$ $(j = 1,2,\ldots,n)$ are multipliers for L^p if $1 < p < \infty$. The Hörmander-Mihlin multiplier theorem provides a generalization of this:

THEOREM 6.3. Let $a = \left[\frac{n}{2}\right] + 1$ *be the first integer greater than* $\frac{n}{2}$. *If* $m \in L^\infty(\mathbb{R}^n)$ *is of class* C^a *outside the origin and satisfies*

$$\text{(6.4)} \qquad \{R^{-n}\int_{R<|x|<2R} |D^\alpha m(\xi)|^2 d\xi\}^{\frac{1}{2}} \leq CR^{-|\alpha|} \qquad (0 < R < \infty)$$

for every multi-index α *such that* $|\alpha| \leq a$, *then* m *is a multiplier for* L^p, $1 < p < \infty$.

For many applications, one merely verifies the condition

$$\text{(6.4')} \qquad |D^\alpha m(\xi)| \leq C|\xi|^{-\alpha} \qquad (|\alpha| \leq a)$$

which is stronger than (6.4). This is satisfied by every function $m(\xi)$ of class C^a outside the origin and homogeneous of degree ib (with $b \in \mathbb{R}$), i.e.

$$m(t\xi) = t^{ib}m(\xi) \qquad (t > 0)$$

When $b = 0$ (which is the most interesting case), the operator T_m is dilation invariant, i.e., $T_m(f_\delta) = (T_m f)_\delta$, where $f_\delta(x) = \delta^{-n}f(\frac{x}{\delta})$.

We wish to prove the theorem by reducing it to an application of Lemma 6.1. Two standard techniques are needed for this reduction.

The first one is the smooth cutting of the multiplier into dyadic pieces, which is based on the following

LEMMA 6.5. There is a non negative function $\psi \in C^\infty(\mathbb{R}^n)$ *supported in the spherical shell* $\{\xi : \frac{1}{2} < |\xi| < 2\}$ *such that*

$$\sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1 \qquad\qquad (\xi \neq 0)$$

Proof: Take $\Phi \in C^\infty$ supported in $\frac{1}{2} < |\xi| < 2$, non negative and such that $\Phi(\xi) > 0$ when $\frac{1}{\sqrt{2}} \leq |\xi| \leq \sqrt{2}$. Then,

$$\Phi_0(\xi) = \sum_{j\in\mathbb{Z}} \Phi(2^{-j}\xi)$$

is a C^∞ function strictly positive at every $\xi \neq 0$ and such that $\Phi_0(2\xi) = \Phi_0(\xi)$. Thus, $\psi(\xi) = \Phi(\xi)/\Phi_0(\xi)$ satisfies the required conditions. □

The second previous result is an explicit formulation of the well known fact that the regularity of the multiplier is translated into control of the size of the kernel:

LEMMA 6.6. Let $a = \left[\frac{n}{2}\right] + 1$, *and let* s *be such that* $a = \frac{n}{s} + \frac{1}{2}$ *(so that* $s \leq 2$*). If* $k \in L^2(\mathbb{R}^n)$ *is such that* $\hat{k}$ *is of class* C^a, *then,*

$$\int_{|x|>t} |k(x)|dx \leq C_n t^{-\frac{1}{2}} \max_{|\alpha|=a} \|D^\alpha \hat{k}\|_s \qquad\qquad (0 < t < \infty)$$

Proof: Since $|x|^a \leq C(|x_1|^a + \dots + |x_n|^a)$, and since the Fourier transform of $x_j^a k(x)$ is $(-2\pi i)^{-a} D_j^a \hat{k}(\xi)$ (with $D_j = \partial/\partial\xi_j$), we use Hausdorff-Young's inequality to get

$$\int_{|x|>t} |k(x)|dx \leq C_n \sum_{j=1}^n \left(\int |x_j^a k(x)|^{s'} dx\right)^{1/s'} .$$

$$\cdot \left(\int_{|x|>t} |x|^{-as} dx\right)^{1/s} \leq C'_n \sum_{j=1}^n \|D_j^a \hat{k}\|_s\, t^{-a+n/s} . \qquad \square$$

We can now proceed to prove the multiplier theorem.

Proof of 6.3: We use the partition of the unity defined by the function ψ of 6.5 in order to decompose the multiplier as

$m = \sum_j m_j$, where

$$m_j(\xi) = m(\xi)\psi(2^{-j}\xi) \qquad (j \in \mathbb{Z})$$

Observe that $\operatorname{supp}(m_j) \subset \{\xi : 2^{j-1} < |\xi| < 2^{j+1}\}$, and that we can control the size of the derivatives of m_j as follows:

$$\|D^\alpha m_j\|_s \le C\, 2^{j(-|\alpha|+n/s)} \qquad (|\alpha| \le a;\ 1 \le s \le 2) \tag{6.7}$$

Indeed, by the Leibnitz rule of differentiation, if $|\alpha| \le a$

$$|D^\alpha m_j(\xi)| = \Big|\sum_{\alpha=\beta+\gamma} D^\beta m(\xi) 2^{-j|\gamma|} D^\gamma \psi(2^{-j}\xi)\Big| \le$$

$$\le C\, 2^{-j|\alpha|} \sum_{|\beta|\le|\alpha|} 2^{j|\beta|} |D^\beta m(\xi)|$$

and it is now that the hypothesis (6.4) is used, together with Hölder's inequality, to obtain

$$\Big(\int_{2^{j-1}\le|\xi|\le 2^{j+1}} |D^\beta m(\xi)|^s d\xi\Big)^{1/s} \le C(2^{jn})^{1/s-1/2} \cdot (2^j)^{-|\beta|+n/2} =$$

$$= C\, 2^{j(-|\beta|+n/s)}$$

from which (6.7) follows.

Now, our plan is to apply lemma 6.1 to the kernels $k^N(x) = \sum_{-N}^{N} k_j(x)$, with

$$k_j(x) = \hat{m}_j(-x) = \int e^{2\pi i x\cdot\xi} m_j(\xi)\, d\xi$$

Since $\sum_j |\hat{k}_j(\xi)| = |m(\xi)| \in L^\infty(\mathbb{R}^n)$, we only have to prove

$$\sum_{j=-\infty}^{\infty} \int_{|x|>2|y|} |k_j(x-y) - k_j(x)|\, dx \le C \qquad (y \in \mathbb{R}^n) \tag{6.8}$$

If we are able to do so, the operators

$$T^N f(x) = \sum_{j=-N}^{N} k_j * f(x) \qquad (N = 1, 2, \dots)$$

will be uniformly bounded in L^p, $1 < p < \infty$, and since $\hat{k}_j = m_j$,

T^N is the operator defined by the multiplier

$$m^N(\xi) = \sum_{j=-N}^{N} m_j(\xi) = m(\xi) \sum_{j=-N}^{N} \psi(2^{-j}\xi)$$

which converges to $m(\xi)$ when $N \to \infty$. By Plancherel's theorem, $T^N f \to T_m f$ (in L^2) for every $f \in L^2$, and in particular

$$\liminf_{N \to \infty} |T^N f(x) - T_m f(x)| = 0 \qquad \text{a.e.}$$

Then, Fatou's lemma gives, for every $f \in L^2 \cap L^p$

$$\|T_m f\|_p \leq \liminf_{N \to \infty} \|T^N f\|_p \leq C_p \|f\|_p \qquad (1 < p < \infty)$$

Thus, matters are reduced to proving (6.8). For a fixed $y \in \mathbb{R}^n$, we shall separate the sum into two parts:

$$\sum_{j=-\infty}^{\infty} \int_{|x|>2|y|} |k_j(x-y) - k_j(x)|dx = \sum_{j\in I} + \sum_{j \in II}$$

where $I = \{j : 2^j|y| \geq 1\}$ and $II = \{j : 2^j|y| < 1\}$. For the terms in the first sum, we use the trivial majorization

$$\int_{|x|>2|y|} |k_j(x-y) - k_j(x)|dx \leq$$

$$\leq 2\int_{|x|>|y|} |k_j(x)|dx \leq C|y|^{-\frac{1}{2}} 2^{-j/2}$$

which is based on (6.6) and (6.7) (with s such that $a - \frac{n}{s} = \frac{1}{2}$). Therefore

$$\sum_{j\in I} (\ldots) \leq C|y|^{-\frac{1}{2}} \sum_{2^j \geq |y|^{-1}} 2^{-j/2} \leq C\sqrt{2}/(\sqrt{2} - 1)$$

For the terms corresponding to $j \in II$, the cancellation of $k_j(x - y) - k_j(x)$ must play a role. Define

$$\tilde{k}_j(x) = k_j(x-y) - k_j(x)$$

$$\tilde{m}_j(\xi) = (\tilde{k}_j)\hat{\ }(\xi) = m_j(\xi)(e^{-2\pi i y\cdot\xi} - 1)$$

Our task consists now in obtaining an estimate for the derivatives of $\tilde{m}_j$ which is good enough, namely

$$\sup_{|\alpha|=a} \|D^\alpha \tilde{m}_j\|_s \le C|y|2^{j(1-a+n/s)} \qquad (1\le s\le 2;\ 2^j|y|<1) \tag{6.9}$$

To prove this, we first observe that, for all $\xi \in \operatorname{supp}(\tilde{m}_j)$

$$|D_\xi^\gamma(e^{-2\pi i y\cdot\xi}-1)| \le C|y|2^{j(1-|\gamma|)} \qquad (j \in II)$$

(this is obvious when $\gamma = 0$ because $|y\cdot\xi| \le 2^{j+1}|y|$, and when $|\gamma| > 0$ we have $|D^\gamma(e^{-2\pi i y\cdot\xi})| = |(2\pi y)^\gamma| \le C|y|^\gamma$). By the Leibnitz rule

$$\sup_{|\alpha|=a} |D^\alpha \tilde{m}_j(\xi)| \le C|y|2^{j(1-a)} \sum_{|\beta|\le a} 2^{j|\beta|}|D^\beta m_j(\xi)|$$

and inserting the estimates obtained in (6.7) for $D^\beta m_j$, we have proved (6.9). Now, we are in position to invoke again (6.6), which gives

$$\int_{|x|>2|y|} |k_j(x-y) - k_j(x)|dx \le C(2|y|)^{-\frac{1}{2}}2^{j/2}|y|$$

and therefore,

$$\sum_{j\in II}(\dots) \le C|y|^{\frac{1}{2}} \sum_{2^j<|y|^{-1}} 2^{(j-1)/2} \le C/(\sqrt{2}-1). \qquad \square$$

Under the hypothesis of 6.3, the operator T_m is also of weak type (1,1), and satisfies

$$\|T_m f\|_{BMO} \le C\|f\|_\infty \qquad (f \in L_c^\infty)$$

It is natural to ask whether the last inequality can be improved to obtain the same kind of estimate that regular singular integral operators satisfy (see 5.20(i)). We shall now prove, with almost no additional effort, a more precise form of 6.3 which shows how higher smoothness of the multiplier m implies higher regularity of the operator T_m. In particular, the operators described in the last theorem always satisfy: $(T_m f)^\# \le CM_2 f$, and they behave exactly as regular singular integral operators if we assume the decay (6.4) for all the derivatives of order $\le n$ of $m(\xi)$.

THEOREM 6.10. Let a *be an integer such that* $\frac{n}{2} < a \le n$, *and* *suppose that the multiplier* $m(\xi)$ *satisfies* (6.4) *for all* $|\alpha| \le a$. *Then, for every* $q > \frac{n}{a}$, *the operator* T_m *satisfies*

$$(T_m f)^{\#}(x) \le C_q M_q f(x) \qquad (f \in \bigcup_{1<p<\infty} L^p(\mathbb{R}^n))$$

Proof: We use the same notation as in the proof of 6.3. It suffices to obtain the pointwise estimate for every operator

$$T^N f(x) = k^N * f(x) = \sum_{j=-N}^{N} k_j * f(x)$$

(with C_q independent of N) and this will be an easy consequence of the crucial inequality for the kernels k^N:

$$(6.11) \quad \{\int_{|x|>2t} |k^N(x-y) - k^N(x)|^{q'}dx\}^{1/q'} \le Ct^{-\varepsilon-n/q}|y|^{\varepsilon} \qquad (t \ge |y|)$$

where $\varepsilon > 0$ will be fixed below. In fact, given $y \in \mathbb{R}^n$, we denote $t_j = 2^j|y|$, and for an arbitrary f, it follows from (6.11) that

$$\int_{|x|>2|y|} |k^N(x-y) - k^N(x)||f(x)|dx \le$$

$$\le \sum_{j=1}^{\infty} \{\int_{|x|>t_j} |k^N(x-y) - k^N(x)|^{q'}dx\}^{1/q'}\{\int_{t_j<|x|\le 2t_j} |f(x)|^q dx\}^{1/q}$$

$$\le C|y|^{\varepsilon} \sum_{j=1}^{\infty} t_j^{-\varepsilon-n/q} t_j^{n/q} M_q f(0) \le C' M_q f(0)$$

This is a substitute, for our present problem of the inequality (5.21), and the rest of the proof is as in theorem 5.20(i).

Now, in order to establish (6.11), given q, we take $s \le 2$ such that $\frac{n}{a} < s < q$. We can also assume that $\varepsilon = a - \frac{n}{s} < 1$, and this is our choice of ε. The variant of lemma 6.6 which we need here is as follows: If $k \in L^2(\mathbb{R}^n)$ and $\hat{k}(\xi)$ is of class C^a, then

$$\{\int_{|x|>t} |k(x)|^{q'}dx\}^{1/q'} \le Ct^{-\varepsilon-n/q} \max_{|\alpha|=a} \|D^{\alpha}\hat{k}\|_s$$

The proof is entirely similar, but Hölder's inequality is applied in the form: $\|\cdot\|_{q'} \leq \|\cdot\|_r \, \|\cdot\|_{s'}$, with $\frac{1}{r} = \frac{1}{s} - \frac{1}{q}$. Since (6.7) and (6.9) remain to hold for the derivatives of m_j and $\tilde{m}_j$ up to order a, if $t \geq |y|$, we have

$$\{\int_{|x|>2t} |k_j(x-y) - k_j(x)|^{q'} dx\}^{1/q'} \leq \begin{cases} C2^{-j\varepsilon} t^{-\varepsilon-n/q} & (\text{if } 2^j|y| \geq 1) \\ C|y|2^{j(1-\varepsilon)} t^{-\varepsilon-n/q} & (\text{if } 2^j|y| < 1) \end{cases}$$

Therefore, the left hand side of (6.11) is majorized by

$$Ct^{-\varepsilon-n/q}\{\sum_{2^j \geq |y|^{-1}} 2^{-j\varepsilon} + |y| \sum_{2^j < |y|^{-1}} 2^{j(1-\varepsilon)}\} \leq$$

$$\leq C_\varepsilon t^{-\varepsilon-n/q}|y|^\varepsilon$$

and the proof is ended. □

To conclude this section ,we list some elementary properties of multipliers:

(6.12). Let $1 < p < \infty$. *Then* m *is a multiplier for* L^p *if and only if it is a multiplier for* $L^{p'}$, *and the norms of* T_m *as an operator on* L^p *and on* $L^{p'}$ *are identical.*

This follows from the identity

$$\int T_m f(x) \overline{g(x)} \, dx = \int f(x) \overline{T_m g(x)} \, dx$$

($f \in L^2 \cap L^p$, $g \in L^2 \cap L^{p'}$), which is obtained by using the definition of T_m and Plancherel's theorem.

(6.13). If m *is a multiplier for* L^p, *and if* $|\frac{1}{q} - \frac{1}{2}| \leq |\frac{1}{p} - \frac{1}{2}|$ *then* m *is a multiplier for* L^q.

The condition on q means that, either $p \leq q \leq p'$ or $p' \leq q \leq p$. The result is a consequence of the preceding statement and

Marcinkiewicz' interpolation theorem, 2,11.. If one uses instead the Riesz-Thorin interpolation theorem (see 7.5 below), it follows that the norm of T_m in L^q is not greater than the norm in L^p. In particular, taking $q = 2$, we always have $\|T_m\|_{p,p} \geq \|m\|_\infty$.

(6.14). The multipliers for L^p *form a subalgebra of* $L^\infty(\mathbb{R}^n)$ *which is invariant under translations, rotations and dilations.*

The first assertion is obvious, because $T_{m_1 m_2} f = T_{m_1}(T_{m_2} f)$. The invariance properties follow from the behaviour of the Fourier transform with respect to affine transformations: $A(x) = h + L(x)$ (L linear), which is very simple

$$(f \circ A)\hat{\ }(\xi) = e^{2\pi i h \cdot \tilde{L}(\xi)} |\det L|^{-1} \hat{f}(\tilde{L}(\xi))$$

(with $\tilde{L} = (L^*)^{-1}$).

(6.15). The following functions are multipliers for $L^p(\mathbb{R}^n)$ *for all* $1 < p < \infty$:

$$m_j(\xi) = \operatorname{sign}(\xi_j) \quad , \qquad j = 1,2,\dots,n$$

$$m(\xi) = \chi_P(\xi)$$

where P *is a (bounded or unbounded) polyhedron in* $\mathbb{R}^n$, *i.e.*, P *is the intersection of a finite number of half-spaces*

$$S_u^r = \{\xi \in \mathbb{R}^n : \xi \cdot u \geq r\} \qquad (u \in \mathbb{R}^n, \ r \in \mathbb{R})$$

In fact, the multiplier m_1 corresponds to the Hilbert transform acting on the first variable:

$$T_{m_1} f(x_1, x_2, \dots, x_n) = iH(f(\cdot, x_2, \dots, x_n))(x_1)$$

which is bounded in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, by the boundedness of H and Fubini's theorem. Similarly for m_j, $2 \leq j \leq n$.

Concerning χ_P, it suffices to prove the result for the characteristic funciton of a half-space (because the product of multipliers is again a multiplier), and by translation and dilation invariance, we only have to prove it for

$$S = S^{o}_{e_1} = \{\xi \in \mathbb{R}^n : \xi_1 = \xi \cdot e_1 \geq 0\}$$

But $\chi_S = \frac{1}{2}(1 + m_1)$.

7. NOTES AND FURTHER RESULTS

7.1.- The definition of the maximal function in $\mathbb{R}^n$ is in Wiener [1]. It is the n-dimensional analogue of the maximal function defined by Hardy and Littlewood [1] for functions in the torus T. The class $L \log L$ was introduced by Zygmund to give a sufficient condition for the local integrability of Mf. That this condition is actually necessary (theorem 2.7) was observed by Stein [4].

7.2.- There are many variants of the maximal function in $\mathbb{R}^n$. It is equivalent, for instance, to consider balls instead of cubes. More generally, given a family of sets $\mathcal{F}$ which covers $\mathbb{R}^n$ in the "narrow sense" (i.e., every $x \in \mathbb{R}^n$ belongs to sets $F \in \mathcal{F}$ of arbitrarily small diameter), we can define

$$M_{\mathcal{F}}f(x) = \sup_{x\in F\in\mathcal{F}} \frac{1}{|F|}\int_F |f(y)|dy$$

If every set $F \in \mathcal{F}$ can be packed within two cubes: $Q_1 \subset F \subset Q_2$ in such a way that diam $(Q_2) \leq C$ diam (Q_2), C = fixed constant, then $M_{\mathcal{F}}$ and M are equivalent, in the sense that

$$C^{-n}Mf(x) \leq M_{\mathcal{F}}f(x) \leq C^nMf(x)$$

This is not the case for the family $\mathcal{R}$ of all bounded n-dimensional intervals: $R = [a_1,b_1] \times [a_2,b_2] \times \dots \times [a_n,b_n]$. $M_{\mathcal{R}}f$ is called the *strong maximal function* and controls the differentiation with respect to arbitrary intervals in $\mathbb{R}^n$, as well as the unrestricted summability of multiple Fourier series (see Zygmund [1], Chap. XVII). It was first studied by Jessen, Marcinkiewicz and Zygmund [1], who proved that $M_{\mathcal{R}}$ is a bounded operator from $L^p(\mathbb{R}^n)$ to itself, $1 < p \leq \infty$, from $L(\log L)^n$ to L^1 and from $L(\log L)^{n-1}$ to weak L^1. As a corollary of the last result:

$$\lim_{\substack{x \in R \\ \operatorname{diam}(R) \to 0}} \frac{1}{|R|} \int_R f(y)\,dy = f(x) \qquad \text{a.e.}$$

for every f which is locally in $L(\log L)^{n-1}(\mathbb{R}^n)$. The estimates for M_R are based on its majorization by the n-fold application of one-dimensional maximal functions (see section IV.6).

7.3.- The bounds obtained in this chapter for the maximal function in $L^p(\mathbb{R}^n)$, for a fixed $p > 1$, are of the order of a^n for large n (with $a > 1$). However, if $M^{(n)}$ denotes the maximal operator over *centered balls* in $\mathbb{R}^n$, namely

$$M^{(n)}f(x) = \sup_{0<r<\infty} \frac{1}{|B|r^n} \int_{rB} |f(x+y)|\,dy$$

(with $B = \{x \in \mathbb{R}^n : |x| < 1\}$), then, the following inequalities hold

$$(*) \qquad \|M^{(n)}f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \qquad (1 < p < \infty;\quad n \in \mathbb{N})$$

with C_p independent of n. This striking result is due to Stein [7], [8]. As an application, one can obtain pointwise summability of Fourier series of infinitely many variables.

We mention two open questions in this direction

(Q1) Is there a weak type analogue of $(*)$ for $p = 1$?

(Q2) Is $(*)$ true if we replace B by a convex, centrally symmetric, open set in $\mathbb{R}^n$?

Concerning (Q1), a weak type estimate with a factor n has been obtained by Stein and Stromberg [1]. As for (Q2), Bourgain [3] has recently given an affirmative answer for $p \geq 2$ when $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, $\|\cdot\|$ being an unconditional norm: $\|(\pm x_1, \pm x_2, \ldots, \pm x_n)\| = \|(x_1, x_2, \ldots, x_n)\|$. This includes the maximal function over centered cubes.

7.4.- Calderón and Zygmund [1] introduced the decomposition which bears their name in order to study the existence and boundedness of singular integrals. The approach followed here to derive the estimates for the maximal function from such a decomposition is also suggested in the same paper. The more usual approach to such estimates is via *covering lemmas*, a particular example of which is

Besicovitch Covering Lemma: Given a set $A \subset \mathbb{R}^n$, *suppose that for each* $x \in A$ *we have a cube* $Q(x)$ *centered at* x. *Then, from the family* $\{Q(x)\}_{x\in A}$ *it is possible to select a subfamily of cubes* $\{Q_j\}$ *which still cover* A *and such that* $\sum_j \chi_{Q_j}(x) \leq 4^n$ *(i.e., no more than* 4^n *cubes* Q_j *can overlap).*

For this, similar results and their application to estimates of maximal functions, we refer to the monographs by de Guzmán [1], Stein [1], Stein and Weiss [2]. The Besicovitch lemma can be used to give an improved version of theorems 2.4 and 2.6:

"*Let* μ *be a regular positive Borel measure in* $\mathbb{R}^n$, *and let* $\tilde{M}_\mu$ *be the maximal operator defined as in section 1, but with centered cubes only. Then* $\tilde{M}_\mu$ *is bounded in* $L^p(\mu)$ *and of weak type* (1,1)".

Observe that no doubling condition is imposed. For the (non-centered) operator M_μ, the result is still true in $\mathbb{R}^1$, but false in $\mathbb{R}^n$ for $n > 1$ (Sjögren [1]).

Covering lemmas for families F such as those considered in 7.2 are sometimes much harder to obtain. One such lemma for the family R has been obtained by A. Córdoba and R. Fefferman [3], yielding a different (geometric) proof of the Jessen-Marcinkiewicz-Zygmund theorem.

7.5.- The Marcinkiewicz interpolation theorem (2.11) can be stated more generally for operators of weak type (p_0,q_0) and (p_1,q_1) with $p_0 \leq q_0$, $p_1 \leq q_1$. A companion of this interpolation theorem whose proof (based on the maximum modulus principle for analytic functions) is completely different in spirit, is:

Riesz-Thorin Theorem: If T *is a linear operator of strong types* (p_0,q_0) *and* (p_1,q_1), *with* $1 \leq p_i,q_i \leq \infty$, *then* T *is of strong type* (p_θ,q_θ), *where*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\ , \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\ ,$$

and we have

$$\|T\|_{p_\theta,q_\theta} \leq \|T\|^{1-\theta}_{p_0,q_0} \; \|T\|^{\theta}_{p_1,q_1}.$$

The Marcinkiewicz and Riesz-Thorin theorems are the prototypes of the real and complex methods of interpolation, respectively. These are described in Bergh and Löfstrom [1]. An extension of the Riesz-Thorin theorem due to Stein [5] allows the operator T to change with the exponents p,q: If T^z are linear operators depending analytically on z, $0 \leq \mathrm{Re}(z) \leq 1$, the hypothesis

$$\|T^{it}f\|_{q_0} \leq M_0(t)\,\|f\|_{p_0} \quad , \quad \|T^{1+it}f\|_{q_1} \leq M_1(t)\,\|f\|_{p_1}$$

imply $\|T^{\theta}f\|_{q_\theta} \leq M\,\|f\|_{p_\theta}$, provided that $M_0(t)$ and $M_1(t)$ have a moderate growth when $|t| \to \infty$. An application of this theorem will be indicated in Chapter V, 5.19 and 7.7. Further generalizations have been obtained by Coifman, Cwickel, Rochberg, Sagher and Weiss [1].

7.6.- The inequality (2.13) is due to C. Fefferman and Stein [1]. The idea of relating inequalities of this type with Carleson measures is taken from the same paper.

The space BMO was introduced in John and Niremberg [1], where they proved the theorem bearing their name, (3.8). C. Fefferman and Stein [2] defined the sharp maximal function $f^{\#}$ and obtained the equivalence $\|Mf\|_p \sim \|f^{\#}\|_p$ and the interpolation between L^p and BMO (theorems 3.6 and 3.7).

7.7.- The condition 5.1(b) was already implicit in Calderón and Zygmund [1], but it was Hörmander [1] who first formulated it explicitly in this form and pointed out that other operators apart from the homogeneous singular integrals could be dealt with by the same method. He used it, in particular, to prove the multiplier theorem 6.3, improving a previous result of Mihlin [1]. One of the early reasons to develop the theory of singular integrals was its applicability to problems in linear partial differential equations. Just to show this connection, we write a simple consequence of theorem 6.3:

Let $P(D)$ be an elliptic constant coefficient operator homogeneous of degree k. Then, the a priori inequalities

$$\|D^{\alpha}f\|_p \leq C_p\|P(D)f\|_p \qquad (|\alpha| = k,\ 1 < p < \infty)$$

hold for every $f \in C^k$ with compact support (observe that $D^\alpha f = T_m(P(D)f)$ with the multiplier $m(\xi) = \xi^\alpha/P(\xi)$ satisfying (6.4')).

The theory of singular integral operators has been generalized along many different lines. Just a few of them are described in the next five notes. Others (singular integrals for vector valued functions or in product domains) will be considered in Chapters V and IV.

7.8.- One can consider "non-isotropic" dilations in $\mathbb{R}^n$, defined by

$$x \mapsto \delta(x) = \delta^A x \qquad , \qquad \delta > 0$$

where A is a positive-definite $n \times n$ matrix (When A = identity, we get the usual "isotropic" dilations). If $d = \text{trace}(A)$ is the homogeneous dimension of $\mathbb{R}^n$ with respect to these dilations, then the analogue of Proposition 5.6 is true for kernels $k(x)$ with the right homogeneity: $k(\delta(x)) = \delta^{-d}k(x)$. See de Guzmán [2], Chapter 11, and the references cited there.

This notion of dilation can be carried over to the more general setting of locally compact groups (Rivière [1]). These ideas have been specially fruitful when applied to some nilpotent Lie groups (like the Heisenberg group) in connection with solvability and regularity properties of non-elliptic partial differential operators (see Folland and Stein [1]).

7.9.- The Hilbert transform along a fixed direction $u \in \mathbb{R}^n$, $|u| = 1$, is defined as follows

$$H_u f(x) = \text{p.v.}\ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x-tu)}{t}dt \qquad , \qquad f \in L^2(\mathbb{R}^n) \ .$$

This is the simplest example of a singular integral whose kernel is supported in a submanifold of $\mathbb{R}^n$: the line $\{tu : t \in \mathbb{R}\}$. In this case, the rotation invariance and Fubini's theorem immediately prove that H_u is bounded in $L^p(\mathbb{R}^n)$ and of weak type (1,1). A non-trivial extension consists in replacing the line by a curve like

$$\gamma(t) = (t^{a_1}, t^{a_2}, \ldots, t^{a_n}) \qquad , \qquad t \in \mathbb{R}$$

The Hilbert transform along γ is

$$H_\gamma f(x) = \text{p.v.} \int_{-\infty}^{\infty} f(x-\gamma(t))\frac{dt}{t}$$

It is known that $\|H_\gamma f\|_p \leq C_p\|f\|_p$, $1 < p < \infty$, but the weak type inequality for $p = 1$ is still an open question. We refer to Stein and Wainger [1] for this and related results.

7.10.- The fact that $\|H_u f\|_p \leq C_p\|f\|_p$ suggests that the same inequality will be true for any "convex combination" of directional Hilbert transforms:

$$T_\Omega f(x) = \int_{|u|=1} H_u f(x)\Omega(u)\,d\sigma(u)$$

(with $\Omega \in L^1(\Sigma_{n-1})$). When Ω is odd, integration in polar coordinates shows that

$$T_\Omega f(x) = \text{p.v. } \frac{2}{\pi}\int_{\mathbb{R}^n} \Omega(y)|y|^{-n}f(x-y)\,dy$$

which are the operators considered in Proposition 5.6, no regularity being now assumed on Ω. Thus, for Ω odd

$$(*) \qquad \|T_\Omega f\|_p \leq C_p\|\Omega\|_1\,\|f\|_p \quad , \qquad 1 < p < \infty$$

with C_p = Norm in $L^p(\mathbb{R})$ of the Hilbert transform. This simple and elegant method, called the *method of rotations*, is also due to Calderón and Zygmund [2], its major drawback being that no weak type result is obtained for $p = 1$. When Ω is not odd, the method still works provided that we assume $\Omega \in L \log L(\Sigma_{n-1})$.

If we apply (*) to the Riesz transforms R_j, $j = 1,2,\dots,n$, it turns out that $\|\Omega_j\|_1 = 1$, and therefore, $\|R_j f\|_p \leq C_p\|f\|_p$ with C_p independent of n. This has been improved by Stein [8] to the following:

$$C_p^{-1}\|f\|_{L^p(\mathbb{R}^n)} \leq \Big\|\Big(\sum_{j=1}^n |R_j f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \leq C_p\|f\|_{L^p(\mathbb{R}^n)}$$

for all $n \in \mathbb{N}$ and $1 < p < \infty$. For a proof of this result based on the method of rotations, see Duoandikoetxea and Rubio de Francia [1].

7.11.- The Calderón-Zygmund method can also be applied to operators which are not translation invariant. A general setting provided by Coifman and Weiss [1] is the following:

We consider a *space of homogeneous type* X, which means that X is endowed with a pseudo-metric ρ and a "doubling" measure $dm(x)$, and we try to study operators of the form

$$Tf(x) = \int_X K(x,y)f(y)\,dm(y)$$

Now, Hörmander's condition splits into two parts:

$$\text{(H)} \qquad \int_{\rho(x,y_0)>k\rho(y,y_0)} |K(x,y) - K(x,y_0)|\,dm(x) \leq C$$

for some fixed k, $C > 0$ and for all $y, y_0 \in X$, and (H') which is the condition (H) for the kernel: $K'(x,y) = K(y,x)$. Then

i) $\|Tf\|_2 \leq C\|f\|_2$ plus (H) imply that T is bounded in L^p, $1 < p < 2$, and of weak type $(1,1)$
ii) $\|Tf\|_2 \leq C\|f\|_2$ plus (H') imply that T is bounded in L^p, $2 < p < \infty$, and from L^∞ to BMO.

Examples of operators included in this general framework are those considered in 7.8 and singular integrals in some compact groups or in homogeneous spaces under the action of such groups, like the unit sphere Σ_{n-1} of $\mathbb{R}^n$, the unit ball B_n of $\mathbb{C}^n$, etc.

When $X = \mathbb{R}^n$, the conditions analogous to those of regular singular integral operators (see 5.17) are

$$\text{(CZ.1)} \qquad |K(x,y)| \leq C|x-y|^{-n}$$

$$\text{(CZ.2)} \qquad |\nabla_x K(x,y)| + |\nabla_y K(x,y)| \leq C|x-y|^{-n-1}$$

(it is obvious that (CZ.2) implies (H) and (H')). Operators bounded in $L^2(\mathbb{R}^n)$ and defined by a kernel satisfying (CZ.1) and (CZ.2) are sometimes called *Calderón-Zygmund operators*. See Coifman and Meyer [1], Journé [1].

7.12.- When T is not a convolution operator, the Fourier transform is no longer an effective tool for proving L^2-boundedness, and this is usually the difficult part when analysing whether T is a Calderón-Zygmund operator. A nice necessary and sufficient condition has been found by David and Journé [1]. For operators with antisymmetric kernels, this criterion takes the following specially simple form:

"T1 Theorem": If T *is given by a kernel satisfying* (CZ.1), (CZ.2) *and* $K(x,y) = -K(y,x)$, *then* T *is bounded in* $L^2(\mathbb{R}^n)$ *if and only if* $T(1) \in BMO$.

As an application, we consider the commutator of order k:

$$T_k f(x) = \text{p.v.} \int_{-\infty}^{\infty} \left(\frac{A(x)-A(y)}{x-y}\right)^k \frac{f(y)}{x-y}dy \quad , \quad f \in L^2(\mathbb{R})$$

where A is a Lipschitz function, i.e. $A' \in L^\infty$ (such commutators appear naturally when trying to construct an algebra of pseudo-differential operators, see A.P. Calderón [4]). Now, since T_0 is the Hilbert transform and $T_k(1) = T_{k-1}(A')$, the "T1 theorem" plus induction gives

$$(*) \qquad \|T_k f\|_2 \leq (C\|A'\|_\infty)^k \|f\|_2$$

This is the theorem of A.P. Calderón [3], which can also be formulated in terms of another important operator: The Cauchy integral along a Lipschitz curve Γ in the complex plane, defined by

$$Gf(x) = \frac{1}{\pi i} \text{ p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{z(x)-z(y)}dy \quad , \quad f \in L^2(\mathbb{R})$$

where $z(t) = t + iA(t)$ is the parametrization of Γ. Now, if $\|A'\|_\infty < 1$, we can expand the integrand into a geometric series

$$Gf(x) = \frac{1}{\pi} \sum_{k=o}^{\infty} i^{-k-1} \, T_k f(x)$$

and, writing $\varepsilon = 1/C$, (*) is equivalent to

$$(**) \qquad \|Gf\|_2 \leq \text{Const. } \|f\|_2 \quad \text{provided} \quad \|A'\|_\infty < \varepsilon$$

The estimates (*) have been improved by Coifman, McIntosh and Meyer [1], allowing to remove the restriction $\|A'\|_\infty < \varepsilon$ in (**). This has also been obtained by David [1]. We refer to Y.Meyer [1], [2] for an excellent exposition of these and related results.

7.13.- Lemma 4.6 is due to Hardy and Littlewood, but we have presented here the proof of Fefferman and Stein [2]. The argument used to prove 4.14 comes from Calderón and Zygmund [1]. By using Stein's spherical means, which we shall discuss in Chapter V, 7.7, this result can be partially improved: For ψ radial and integrable in $\mathbb{R}^n$, the operator $f(x) \mapsto \sup_{t>0} |\psi_t * f(x)|$ is bounded in $L^p(\mathbb{R}^n)$, $p > \frac{n}{n-1}$. A different result for approximations of the identity is Zo's theorem (see section V.4).

7.14.- Theorem 4.21 is due to A.P. Calderón [1] who succeeded in extending to harmonic functions in $\mathbb{R}^{n+1}_+$ a previous theorem of Privalov [1] valid for harmonic functions in the unit disk D. Privalov's theorem was proved originally via conformal mapping. We shall give an outline of this proof below. The method is no longer available in higher dimensions. This is what makes Calderón's contribution a basic one. In order to given an account of Privalov's proof, let as assume, without loss of generality, that u is a harmonic function in D such that for every t for which $e^{it} \in E$, a subset of T = boundary of D with $|E| > 0$, $|u(z)|$ is bounded by a fixed number n on the Stolz domain $\Gamma_\alpha(t)$, where α is also fixed. (The Stolz domains are defined in page 109). We may also assume that E is compact. We consider the set $U = \cup\Gamma_\alpha(t)$ where e^{it} ranges over E. Then U is an open set bounded by a rectifiable Jordan curve. We know that $|u(z)| \leq n$ on U. Now the idea is to use a conformal equivalence $\psi : D \to U$ and look at the function $u(\psi(z))$ which is holomorphic and bounded in D. We can apply Fatou's theorem to conclude that $u(\psi(z))$ converges non-tangentially at almost every point in the boundary of D. We know (corollary 4.7 in chapter I) that, except for the points where the boundary of U has no tangent, which form a set of arc length 0, the conformal mapping remains conformal at the boundary points. We also know (see after corollary 3.13 in chapter I) that the homeomorphism induced between the boundaries of U and D carries sets of length 0 to sets of Lebesgue measure 0. Thus, we have that for almost every t and for every β, $\psi(\Gamma_\beta(t))$ is a region non-tangential at $\psi(e^{it})$ with respect to U, and we finally get the existence of non-tangential limits for u at almost every point of E, which is part of the boundary of U. Note that in E arc length coincides with Lebesgue measure.

The method works even for a function meromorphic in D. Indeed, after uniformizing the situation, so that we can assume α and n

fixed, we realize that the fact that $|u(z)| \leq n$ for each $z \in \Gamma_\alpha(t)$ with $e^{it} \in E$, implies that the singularities of u in U are all inside a disk $D(o,r)$ with $r < 1$ and, consequently, there is only a finite number of them. Multiplying by an appropriate polynomial, we get a holomorphic function on U, for which non-tangential boundary values exist at a.e. point of E. Obviously, the same thing happens for the original function u.

Note that the same technique of changing the domain conformally, can be used to prove the following theorem, also due to Privalov [1]:

If a function $u(z)$ *meromorphic in* D *has non-tangential limit* 0 *at each point of a subset of* T *of positive measure, then* u *is identically* 0 *on* D.

By using the two mentioned theorems, one can easily obtain the following result, due to Plessmer [1] (see also Zygmund [1], Chap. XIV).

Let $u(z)$ *be an analytic function in* D. *Then, except for a set of points of Lebesgue measure* 0, *at every point* e^{it} *of the boundary, either the function has a finite limit as* z *tends non-tangentially to* e^{it} *or else, the image* $u(\Gamma_\alpha(t))$ *is dense in the plane for every* α.

Actually A.P. Calderón [1] extends Privalov's theorem to pluriharmonic functions in a Cartesian product of euclidean half-spaces and obtains non-tangential convergence at a.e. point of the distinguished boundary. By using this result, he is able to extend Plessner's theorem to analytic functions of several complex variables.

There is a convenient way to reflect analytically the fact that a harmonic function u has non-tangential limits at a.e. point of a subset E of the boundary with $|E| > 0$. This is achieved through the *area function* $A(u)$ introduced by Lusin.

For u harmonic in D, we define

$$A_\alpha(u)(t) = \iint_{\Gamma_\alpha(t)} |\nabla u(x,y)|^2\, dxdy = \iint_{\Gamma_\alpha(t)} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dxdy \; .$$

If we just fix α, we can simply write $\Gamma(t)$ and $A(u)(t)$.

Observe that if $F = u + iv$ is analytic in D, then

$$|\nabla u(x,y)|^2 = |F'|^2 = \frac{\partial(u,v)}{\partial(x,y)} ,$$

the Jacobian of the mapping F. Thus, we see that $A_\alpha(u)(t)$ is the area (points counted according to their multiplicity) of $F(\Gamma_\alpha(t))$. This explains the name "area function".

The following result holds:

Let u *be harmonic in* D *and let* E *be a subset of the boundary of* D *with* $|E| > 0$. *Then, for* u *to have a non-tangential limit at a.e.* $x \in E$, *it is necessary and sufficient that* $A(u)(x)$ *is finite for a.e.* $x \in E$.

The necessity was proved by Marcinkiewicz and Zygmund [2] and the sufficiency by Spencer [1].

Note that if $F = u + iV$ is analytic in D then:

$$|\nabla u|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = |\nabla v|^2$$

by the Cauchy-Riemann equations. Thus $A(u) = A(v)$ and we get:

If u *has a non-tangential limit for a.e. point of* E, *then the same is true for* v *and conversely.*

For a function $u(x,t)$ harmonic in the upper half-space $\mathbb{R}^{n+1}_+$, the (generalized) area function $A(u)$ is defined as follows:

$$Au(x) = \iint_{\Gamma(x)} |\nabla u(x,t)|^2 \, t^{1-n} \, dxdt =$$

$$= \iint_{\Gamma(x)} \left(\left|\frac{\partial u}{\partial t}\right|^2 + \sum_{j=1}^{n} \left|\frac{\partial u}{\partial x_j}\right|^2\right) t^{1-n} \, dxdt$$

where $\Gamma(x) = \Gamma_\alpha^h(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \alpha t, \ 0 < t < h\}$.

A.P. Calderón [7] extended the theorem of Marcinkiewica and Zygmund to this higher dimensional setting and the extension of Spencer's result was carried out by E.M. Stein [9]. The generalized area function was previously considered in connection with other related operators in E.M. Stein [10].

In section 8 of chapter III we shall indicate some results on the area function which are connected with the theory of Hardy spaces.

7.15.- Apart from the trivial case of L^2, the Fourier multipliers for L^p are completely characterized only when $p = 1$: the class of L^1 multipliers coincides with the Fourier-Stieltjes transforms of finite Borel measures in $\mathbb{R}^n$. Sufficient conditions for $m(\xi)$ to be an L^p multiplier for all $1 < p < \infty$ are given by the Hörmander-Mihlin theorem, 6.3, and by another classical result: the Marcinkiewicz multiplier theorem, whose 1-dimensional version will be proved in chapter V, 5.13. There is an extensive literature concerning more specific multipliers for different ranges of p's. We shall briefly discuss here those ones related to summability of Fourier series or integrals.

Given $\alpha \geq 0$, is it true that

$$\int \hat{f}(\xi)\left(1-\frac{|\xi|}{R}\right)^{\alpha} e^{2\pi i x.\xi}\, d\xi \xrightarrow[\text{in } L^p]{} f(x) \qquad (R \to \infty)$$

for every $f \in L^p(\mathbb{R}^n)$? The question is equivalent to the study of the range of p's corresponding to the *Bochner-Riesz multipliers:*

$$m_\alpha(\xi) = (1 - |\xi|^2)_+^\alpha = \left[\max\,(1 - |\xi|^2\ ;\ 0)\right]^\alpha$$

Our present knowledge about this challenging problem is the following:

i) $\alpha > \frac{n-1}{2}$ (critical index). Then $m_\alpha(\xi) = \hat{K}_\alpha(\xi)$ with a kernel K_α which can be explicitly computed and belongs to $L^1(\mathbb{R}^n)$. Thus, m_α is an L^p multiplier for all $1 \leq p < \infty$. See Stein and Weiss [2].

ii) $\alpha = 0$. For $n = 1$, $m_0 = \chi_{[-1,1]}$ is an L^p multiplier for $1 < p < \infty$ (this is equivalent to M. Riesz theorem), while for $n \geq 2$, $m_0 = \chi_{\{\xi: |\xi| \leq 1\}}$ is not an L^p multiplier for any $p \neq 2$ (C. Fefferman [2])

iii) $0 < \alpha \leq \frac{n-1}{2}$. The conjecture is that m_α is an L^p multiplier if and only if

$$p_\alpha = \frac{2n}{n+1+2\alpha} < p < \frac{2n}{n-1-2\alpha} = p'_\alpha$$

The necessity of this condition is easily seen, because $K_\alpha \notin L^{p_\alpha}$. For $n = 2$, the conjecture is true, and it was first proved by Carleson and Sjölin [1]; different proofs have been given by C. Fefferman [4] and Córdoba [1]. For $n > 2$, the conjecture has been proved when $\alpha > \frac{n-1}{2n+2}$. This follows from results of Stein and Tomas on restriction of Fourier transforms (see P. Tomas [1]); it can also be proved by using the approach in Córdoba [1].

CHAPTER III

REAL VARIABLE THEORY OF HARDY SPACES

In this chapter we shall develop an analogue of the H^p theory studied in chapter I, where the torus $\mathbb{T}$ will be replaced by the euclidean space $\mathbb{R}^n$.

In the new context, we can no longer rely upon the powerful methods of Complex Analysis. Our basic tool will be harmonic majorization (theorem 4.10 in chapter II), together with the real-variable techniques centering around the "Calderón-Zygmund theory", which were also developed in chapter II.

Instead of dealing directly with the case of $\mathbb{R}^n$ for arbitrary n, we have chosen to start with the one-dimensional situation, to which the first three sections are devoted. Beginning with a definition of $H^p(\mathbb{R}^2_+)$ as a space of analytic functions, which parallels that of $H^p(D)$ for the disk D, we soon realize that $H^p(\mathbb{R}^2_+)$ can be identified with a real space of tempered distributions on $\mathbb{R}$ denoted by $\mathrm{Re}H^p(\mathbb{R})$ or simply $H^p(\mathbb{R})$. The main achievement is the identification of the basic building blocks, called "atoms", into which any H^p distribution can be decomposed. This atomic decomposition is carried out in section 3. We give two proofs of the atomic decomposition. The first is based on a smooth version of the Calderón-Zygmund decomposition (theorem 3.6). The second, given in 3.11, is obtained working directly with the harmonic extension, and it carries over to the n-dimensional case with only minor modifications.

Two proofs are also given for the Burkholder-Gundy-Silverstein theorem asserting that a distribution is in $H^p(\mathbb{R})$ if and only if it is the boundary value of a harmonic function in $\mathbb{R}^2_+$ whose non-tangential maximal function belongs to $L^p(\mathbb{R})$. The first proof follows from the work done in all three sections, but we also give a direct proof due to P. Koosis [2] based upon Cauchy's integral theorem of Complex Function Theory.

The characterization of $H^p(\mathbb{R})$ given by the Burkholder-Gundy-Silverstein theorem provides a natural way to define $H^p(\mathbb{R}^n)$ for n>1. Section 4 is devoted to the study of these spaces for higher dimensions.

The rest of the chapter can be seen as a test for the power of the atomic description of the Hardy spaces.

Section 5 contains a thorough investigation of the duals of the spaces $H^p(\mathbb{R}^n)$

$0<p\leq 1$. After the atomic description, the identification of the duals can be achieved by simple real-variable techniques. In particular, section 5 contains the n-dimensional version of the duality H^1-B.M.O. which was established for the torus in section 10 of chapter I (C.Fefferman's theorem).

Section 6 contains some interpolation theorems for operators sending H^p to L^p (or weak-L^p).

Finally, section 7 is the culmination of our work. It shows how to use atoms to obtain estimates for different kinds of operators acting on H^p. It contains results for singular integrals and multipliers, extensions to $\mathbb{R}^n$ of the inequalities of Hardy and Fejer-Riesz already encountered in section 4 of chapter I; and also the molecular description of H^p which allows us to obtain some H^p-H^p results.

1. H^p SPACES FOR THE UPPER HALF PLANE

Let $F(z)$ be an analytic function of $z = x + it$ in the upper half plane $\mathbb{R}^2_+ = \{z = x + it;\ t > 0\}$. Define, for any $p > 0$ $\|F\|_{H^p} = \sup_{t>0}\{\int_{-\infty}^{\infty}|F(x + it)|^p dx\}^{1/p}$, and call $m_F(x) = \sup_{|x-y|<t} |F(y+it)|$.

Then, we have

THEOREM 1.1. For F *analytic in* $\mathbb{R}^2_+$ *and* $p>0$, *the following are equivalent:*

a) $\|F\|_{H^p} < \infty$

b) $m_F \in L^p(\mathbb{R}) \equiv L^p$

Moreover, we have the inequalities: $\|F\|_{H^p} \leq \|m_F\|_p \leq C\|F\|_{H^p}$.

Proof: Clearly b) implies a) because $|F(x + it)| \leq m_F(x)$ for every $t > 0$ and, consequently $\|F\|_{H^p} \leq \|m_F\|_p$. That a) implies b) can be seen in the following way: consider $s(z) = |F(z)|^\varepsilon$ with some ε satisfying $0 < \varepsilon < p$. Then $s(x + it)$ is a non-negative subharmonic function on $\mathbb{R}^2_+$, which is uniformly in L^q with $q = p/\varepsilon > 1$. It follows from theorem 4.10 in Chapter II that $s(x + it) \leq (s_0 * P_t)(x)$ for some $s_0 \in L^q$. Thus (see (4.15) in Chapter II)

$$m_F(x) \leq (P^*_\nabla(s_0)(x))^{1/\varepsilon} \leq C.(M(s_0)(x))^{1/\varepsilon}$$

and

$$\int_{-\infty}^{\infty} m_F(x)^p dx \leq C.\int_{-\infty}^{\infty}|s_0(x)|^q dx < \infty,$$

so b) holds.

To prove the inequality $\|m_F\|_p \leq C\|F\|_{H^p}$, observe that $m_F \in L^p$ implies that $m_F(x) < \infty$ a.e. and, by theorem 4.21 in Chapter II this implies that F has non-tangential limits a.e. In particular, there exists $F(x) = \lim_{t\to 0} F(x+it)$ for a.e.x. Recall that s_o is obtained as the weak-* limit in L^q of the functions $x \mapsto s(x+it_n)$ for some sequence $t_n \downarrow 0$. But $s(x+it_n) \to |F(x)|^\varepsilon$ for a.e.x and, since all the functions $s(x+it_n)$ are bounded by the same L^q function, namely m_F^ε, it follows that $s(x+it_n)$ tends to $|F(x)|^\varepsilon$ in the L^q norm. Therefore $s_o(x) = |F(x)|^\varepsilon$ for a.e.x and

$$\int_{-\infty}^{\infty} |s_o(x)|^q dx = \int_{-\infty}^{\infty} |F(x)|^p dx \le \liminf_{t\to 0} \int_{-\infty}^{\infty} |F(x+it)|^p dx \le \|F\|_{H^p}^p. \quad \square$$

We shall denote by $H^p(\mathbb{R}^2_+)$, or simply H^p when no confusion is likely to arise, the class formed by those functions $F(z)$ analytic in $\mathbb{R}^2_+$, for which $\|F\|_{H^p} < \infty$. Clearly, for any F, G analytic in $\mathbb{R}^2_+$ and any $\lambda \in \mathbb{C}$:

$$\|\lambda F\|_{H^p} = |\lambda| \, \|F\|_{H^p}$$

$$\|F+G\|_{H^p} \le \|F\|_{H^p} + \|G\|_{H^p} \quad \text{if} \quad 1 \le p < \infty, \quad \text{and}$$

$$\|F+G\|_{H^p}^p \le \|F\|_{H^p}^p + \|G\|_{H^p}^p \quad \text{if} \quad 0 < p \le 1 .$$

It follows that H^p is a linear space, $F \longmapsto \|F\|_{H^p}$ is a norm in H^p if $1 \le p < \infty$ and $F \longmapsto \|F\|_{H^p}^p$ is a p-norm in H^p if $0 < p < 1$ (a p-norm is a non-negative function which is 0 only at 0, satisfies the triangle inequality and is p-homogeneous. It gives rise to a metric compatible with the linear structure). We shall view H^p as a metric linear space, the distance function being $(F,G) \longmapsto \|F-G\|_{H^p}$ if $1 \le p < \infty$ and $(F,G) \longmapsto \|F-G\|_{H^p}^p$ if $0 < p < 1$.

Let us collect the main facts regarding the boundary behaviour of an H^p function. They follow very simply from theorem 1.1 and its proof.

COROLLARY 1.2. Let $F \in H^p$. *Then:*

a) $\lim_{t\to 0} F(x+it)$ *exists for a.e.* $x \in \mathbb{R}$ *and the function* $F(x) = \lim_{t\to 0} F(x+it)$ *is in* L^p. *Actually* F *converges non-tangentially at a.e.* $x \in \mathbb{R}$.

b) $\lim_{t\to 0} \int_{-\infty}^{\infty} |F(x+it) - F(x)|^p dx = 0$ *and*

$$\lim_{t\to 0}\int_{-\infty}^{\infty}|F(x+it)|^p dx = \int_{-\infty}^{\infty}|F(x)|^p dx$$

c) $\|F\|^p_{H^p} \sim \int_{-\infty}^{\infty}(m_F(x))^p dx \sim \int_{-\infty}^{\infty}|F(x)|^p dx$

Proof: Everything is a consequence of the fact that $m_F \in L^p$. a) follows from theorem 4.21 in Chapter II. For b) we just need to apply the Lebesgue dominated convergence theorem, and c) has been proved in theorem 1.1. $\square$

LEMMA 1.3. For every $F \in H^p$: $|F(x,t)| \leq C\|F\|_{H^p}\cdot t^{-1/p}$ *with a constant* C *which does not depend on* F.

Proof: We just need to apply theorem 4.5 in Chapter II to $s(x,t) = |F(x,t)|^\varepsilon$, $0 < \varepsilon < p$. $\square$

An important consequence of lemma 1.3 is that the imbedding of H^p into the space $H(\mathbb{R}^2_+)$ of all the functions holomorphic in the upper half plane, endowed with the topology of uniform convergence on compact subsets, is continuous. This can be used to prove the following.

LEMMA 1.4. For every $p > 0$, H^p *is a complete space.*

Proof: Let (F_n) be a Cauchy sequence in H^p. It follows from 1.3 that F_n converges uniformly on each compact subset of $\mathbb{R}^2_+$ to a certain analytic function F. Now, given $\varepsilon > 0$

$$\int_{-\infty}^{\infty}|F_n(x+it) - F(x+it)|^p dx \leq \liminf_{m\to\infty}\int_{-\infty}^{\infty}|F_n(x+it)-F_m(x+it)|^p dx \leq$$

$$\leq \liminf_{m\to\infty}\|F_n - F_m\|^p_{H^p} < \varepsilon$$

if n is big enough. $\square$

LEMMA 1.5. Suppose that $F \in H^{p_1}$ *and its boundary function* F(x) *belongs to* L^{p_2} *for some* $p_2 > p_1$. *Then* $F \in H^{p_2}$.

Proof: Consider the function $|F(x+it)|^\varepsilon$ for some ε such that $0 < \varepsilon < p_1$. This is a subharmonic function which is uniformly in $L^{p_1/\varepsilon}$, where $p_1/\varepsilon > 1$. It follows that

$$|F(x+it)|^{\epsilon} \leq P_t * (|F|^{\epsilon})(x)$$

so that $m_F(x) \leq C.(M(|F|^{\epsilon})(x))^{1/\epsilon}$. But $|F(x)|^{\epsilon}$ is in $L^{p_2/\epsilon}$, $p_2/\epsilon > 1$, and, consequently, $m_F \in L^{p_2}$. We conclude that $F \in H^{p_2}$. $\square$

Suppose $F \in H^p$ and $1 \leq p < \infty$. It follows from part b) of corollary 1.2 that F is the Poisson integral of its boundary function, which is an L^p function $F(x) = f(x) + ig(x)$, with f and g real. Indeed, for each $t > 0$, the function $F_t(z) = F(z + it)$ is the Poisson integral of its boundary function $F_t(x) = F(x + it)$ and the functions $F_t(x)$ converge to $F(x)$ in the L^p norm as $t \downarrow 0$. Thus $F(x + it) = P_t * (f + ig)(x)$. On the other hand, if $Q_t(x) = H(P_t)(x) = \frac{1}{\pi}\frac{x}{t^2+x^2}$, the function $G(x + it) = P_t * f(x) + iQ_t * f(x)$ is analytic and has the same real part as F. Then $F - G$ has to be a constant and, since F and G tend to 0 for $t \to \infty$, $F - G$ is actually 0. We get $F(x + it) = P_t * f(x) + iQ_t * f(x)$ and, letting $t \downarrow 0$, we find $g = Hf$. We have seen that every $F \in H^p$, $1 \leq p < \infty$, is the Poisson integral of an L^p function of the form $f + iHf$ for some $f \in ReL^p$ and $\|F\|_{H^p} \sim \|f\|_{L^p} + \|Hf\|_{L^p}$. Conversely, for any L^p function of the form $f + iHf$ with f real, the function $F(x + it) = P_t * (f + iHf)(x) = P_t * f(x) + iQ_t * f(x)$ is in H^p. There is an important difference between the cases $p > 1$ and $p = 1$. For $p > 1$ we have an inequality $\|Hf\|_{L^p} \leq C_p\|f\|_{L^p}$, whereas the corresponding inequality for $p = 1$ does not hold, as we saw in the previous chapter. This implies that, when $p > 1$, any $f \in ReL^p$ has $Hf \in L^p$ and consequently it gives rise to $F(x + it) = P_t * (f + iHf)(x)$ belonging to H^p and the mapping $F \longmapsto f$ is an isomorphism between the Banach spaces H^p and ReL^p. When $p = 1$, the range of the mapping $F \longmapsto f$ from H^1 to ReL^1 is strictly smaller than ReL^1. We shall denote this range by ReH^1. It is the space formed by those functions $f \in ReL^1$ for which $Hf \in L^1$. In ReH^1 we shall consider the norm $\|f\|_{L^1} + \|Hf\|_{L^1}$. With this norm ReH^1 becomes a Banach space. The completeness follows from the fact that the Hilbert transform H is closed in L^1 or, more simply, from the isomorphism $F \longmapsto f$ between H^1 and ReH^1, which is the reason for the name ReH^1. What we have done is to see H^1 as a space of real functions in $\mathbb{R}$. As such, it is quite different from ReL^1, whereas for $p > 1$ H^p is simply ReL^p when viewed as a space of real functions. The aim of this chapter is precisely to study H^p from this real variable point of view.

Now let $0 < p < 1$. Will it be possible in this case to view H^p as a space of real functions on $\mathbb{R}$?. A first difficulty we encounter is that the mapping sending the function $F(z)$ to the function $\mathrm{Re}F(x)$ is not one to one. For example, let $F(z) = \frac{-i}{z(z-1)}$. Since $|F(z)| \leq \frac{1}{|x||x-1|}$ and the right hand side of this inequality is in L^p if and only if $1/2 < p < 1$, it follows that $F \in H^p$ for $1/2 < p < 1$ and, actually, for no other p since $m_F(x) \sim (|x||x-1|)^{-1}$. But $\mathrm{Re}F(x) = 0$. The way to avoid this difficulty is to consider $\mathrm{Re}F(x)$ as a tempered distribution rather than as a function. Indeed, $\mathrm{Re}F(x + it)$ is a harmonic function which is uniformly in L^p. We know that the functions $x \mapsto \mathrm{Re}F(x+it)$ have a limit in the sense of tempered distributions as $t \downarrow 0$ and that this tempered distribution $\mathrm{Re}F(x)$ uniquely determines $\mathrm{Re}F(z)$ and, consequently $F(z)$ (see theorem 4.23 in Chapter II). In our concrete example

$$\mathrm{Re}F(z) = \mathrm{Re}\left(i\left(\frac{1}{z} - \frac{1}{z-1}\right)\right) = -\mathrm{Im}\left(\frac{1}{z} - \frac{1}{z-1}\right) =$$

$$= \frac{t}{x^2 + t^2} - \frac{t}{(x-1)^2 + t^2}$$

which converges in the sense of tempered distributions as $t \downarrow 0$ to the measure $\delta_0 - \delta_1$, where δ_a denotes the Dirac delta at a, that is, the unit mass concentrated at a. For $0 < p < 1$, we shall denote by $\mathrm{Re}H^p$ the space formed by the boundary distributions $\mathrm{Re}F(x)$ corresponding to the functions $F \in H^p$, with the p-norm $\mathrm{Re}F(x) \longmapsto \|F\|^p_{H^p}$.

Observe that theorem 4.23 in Chapter II implies that the space $\mathrm{Re}H^p$ is continuously imbedded in the space $\mathcal{S}'$ of tempered distributions. Indeed, if $f \in \mathrm{Re}H^p$ is the boundary distribution corresponding to $F \in H^p$, the theorem mentioned above shows that for every $\psi \in \mathcal{S}$

$$|\langle f, \psi\rangle| = \left|\lim_{t\to 0} \int \mathrm{Re}F(x,t)\psi(x)dx\right| \leq C\|F\|_{H^p}$$

where, of course, C depends on ψ. See the note following the proof of theorem 4.23 in Chapter II.

A convenient way to work with $\mathrm{Re}H^p$ for $0 < p < 1$ is to consider a dense subspace whose members are functions. Even for $p = 1$ it will be useful to look for a dense class in $\mathrm{Re}H^1$ formed by nice functions (smooth, vanishing rapidly at ∞ etc.). Let us give some typical density results.

THEOREM 1.6. For every $N > 0$, *let* H^p_N *be the space formed by those* $F \in H^p$ *such that:*

i) F *is* C^∞ *in* $\overline{\mathbb{R}^2_+} = \{(x,t) \mid t \geq 0\}$

ii) $F(x + it) = O(|x|^{-N})$ *as* $|x| \to \infty$, *uniformly in* $t > 0$.

Then H^p_N *is dense in* H^p.

Proof: Since every $F(z)$ in H^p is the limit in H^p of the functions $z \longmapsto F(z + it)$ as $t \to 0$, we may start by assuming that F is C^∞ and bounded in $\overline{\mathbb{R}^2_+}$. For such an F in H^p, we shall exhibit a sequence of functions in H^p_N converging to F in H^p. Let

$$G_n(z) = \left(\frac{ni}{z + ni}\right)^N$$

Since each G_n is C^∞ in $\overline{\mathbb{R}^2_+}$ and also $O(|x|^{-N})$ as $|x| \to \infty$ uniformly in t, it follows that $G_nF \in H^p_N$ for each n. Also $G_nF \to F$ in H^p as $n \to \infty$. Indeed

$$\|G_nF - F\|^p_{H^p} = \int_{-\infty}^{\infty}\left|\left(\frac{ni}{x + ni}\right)^N - 1\right|^p |F(x)|^p dx \to 0 \qquad (n \to \infty)$$

because $\left|\left(\frac{ni}{x + ni}\right)^N - 1\right| \to 0$ boundedly as $n \to \infty$. $\square$

COROLLARY 1.7. a) For any $p,q > 0$ $H^p \cap H^q$ *is dense in* H^p *and, consequently, for* $q > 1$ $(\mathrm{Re}H^p) \cap L^q$ *is dense in* $\mathrm{Re}H^p$.
b) $(\mathrm{Re}H^p) \cap C^\infty_N$ *is dense in* $\mathrm{Re}H^p$, *where* C^∞_N *stands for the space formed by those* C^∞ *functions in* $\mathbb{R}$ *which vanish at* ∞ *as fast as* $|x|^{-N}$.

Proof: a) Let $N > 1/q$. Then $H^p_N \subset H^p \cap H^q$ since for $F \in H^p_N$, the boundary function belongs to $C^\infty_N \subset L^q$ and so, according to 1.5, $F \in H^q$. It follows that $H^p \cap H^q$ is dense in H^p. As for the second assertion in a), just note that the boundary distributions corresponding to the functions F in $H^p \cap H^q$, for $q > 1$, are precisely the functions in $(\mathrm{Re}H^p) \cap L^q$. b) follows if we observe that the boundary distributions corresponding to the functions in H^p_N are the functions in $(\mathrm{Re}H^p) \cap C^\infty_N$. $\square$

For $p = 1$ we can easily obtain a much nicer dense class.

THEOREM 1.8. The space consisting of those real functions f *whose Fourier transform* $\hat{f}$ *is* C^∞ *and has compact support not containing the origin, is dense in* ReH^1.

Proof: First of all, we observe that every $f \in \mathrm{ReH}^1$ has $\hat{f}(0) = \int f(x)dx = 0$. Indeed, we know that $Hf \in L^1$ and, consequently, $(Hf)^\wedge(x) = -i(\operatorname{sgn} x)\,\hat{f}(x)$ must be a continuous function. But, since $\operatorname{sgn} x$ has a jump at 0, Hf cannot be continuous at 0 unless $\hat{f}(0) = 0$.

As a second step, we shall show that if $f \in L^1$ has $\hat{f}(0) = 0$ and if for $\psi \in L^1$ and $t > 0$ we consider $\psi_t(x) = t^{-1}\psi(t^{-1}x)$, then $f * \psi_t \to 0$ in L^1 as $t \to \infty$. Indeed

$$f * \psi_t(x) = \int_{-\infty}^{\infty} f(y)\psi_t(x - y)dy = \int_{-\infty}^{\infty} f(y)(\psi_t(x - y) - \psi_t(x))dy.$$

Therefore:

$$\|f * \psi_t\|_1 \le \int_{-\infty}^{\infty} |f(y)|\, \|\psi_t(\cdot - y) - \psi_t\|_1 dy =$$

$$= \int_{-\infty}^{\infty} |f(y)|\, \|\psi(\cdot - t^{-1}y) - \psi\|_1 dy.$$

Since for $t \to \infty$, $\|\psi(\cdot - t^{-1}y) - \psi\|_1 \to 0$ for every y and $\|\psi(\cdot - t^{-1}y) - \psi\| \le 2\|\psi\|_1$, an appeal to the Lebesgue dominated convergence theorem yields $\|f * \psi_t\|_1 \to 0$ as $t \to \infty$. If we choose ψ such that $\hat{\psi}(x) = 1$ for every x in a neighbourhood of 0, say for $|x| < \eta$, $\eta > 0$, and consider the functions $g_t = f - f * \psi_t$, it follows from what we just proved that $g_t \to f$ in L^1 as $t \to \infty$. Besides, we have now $\hat{g}_t(x) = \hat{f}(x)(1 - \hat{\psi}(tx)) = 0$ provided $|x| < \eta/t$. Thus, we have been able to approximate in L^1 every function f having $\hat{f}(0) = 0$ by means of functions each of which has a Fourier transform vanishing in a whole neighbourhood of 0.

Suppose next that $\hat{\psi}$ is C^∞ with compact support, and still $\hat{\psi}(x) = 1$ for every x such that $|x| < \eta$. For $f \in L^1$ with $\hat{f}(0) = 0$, $t > 0$, $\varepsilon > 0$, consider now the functions $g_{t,\varepsilon} = (f - f * \psi_t) * \psi_\varepsilon$. We can see that $g_{t,\varepsilon} \to f$ in L^1 as $t \to \infty$ and $\varepsilon \to 0$. Indeed,

$$\|f - g_{t,\varepsilon}\|_1 = \|f - f * \psi_\varepsilon + f * \psi_t * \psi_\varepsilon\|_1 \leq$$

$$\leq \|f - f * \psi_\varepsilon\|_1 + \|f * \psi_t * \psi_\varepsilon\|_1 \leq \|f - f * \psi_\varepsilon\|_1 + \|f * \psi_t\|_1 \to 0$$

$$(t \to \infty, \quad \varepsilon \to 0)$$

Now $(g_{t,\varepsilon})\hat{}(x) = \hat{f}(x)(1 - \hat{\psi}(tx))\hat{\psi}(\varepsilon x) = 0$ if $|x| < \eta/t$ or if εx does not belong to the support of $\hat{\psi}$, a compact set. We have shown that every $f \in L^1$ and having $\hat{f}(0) = 0$ can be approximated in L^1 by the functions $g_{t,\varepsilon}$, each of which has a Fourier transform with compact support not containing 0. Actually, if $f \in \mathrm{ReH}^1$, we have $g_{t,\varepsilon} \to f$ in ReH^1 as $t \to \infty$ and $\varepsilon \to 0$; that is: not only $g_{t,\varepsilon} \to f$ in L^1, but also $H(g_{t,\varepsilon}) \to H(f)$ in L^1. This is so because $H(g_{t,\varepsilon}) = (Hf - Hf * \psi_t) * \psi_\varepsilon$ and Hf is an L^1 function with $(Hf)\hat{}(0) = 0$. So far we have proved that the space formed by the functions in ReH^1 whose Fourier transform has compact support not containing 0, is dense in ReH^1. All that remains to be done is to take $f \in \mathrm{ReH}^1$ such that $\hat{f}$ has compact support not containing 0, and show that f can be approximated in ReH^1 by functions whose Fourier transform is C^∞ and has compact support not containing 0. This can be done very easily. Let ψ be as above (all we shall need is that $\hat{\psi}$ is C^∞ with compact support and $\hat{\psi}(0) = 1$). For $\varepsilon > 0$ define $f_\varepsilon(x) = f(x)\hat{\psi}(-\varepsilon x)$. Then $(f_\varepsilon)\hat{}(x) = \hat{f} * \psi_\varepsilon(x)$. Since ψ is C^∞ (because $\hat{\psi}$ is in the Schwartz class $\mathcal{S}$), it is clear that $(f_\varepsilon)\hat{}$ is C^∞. Also, since the support of ψ_ε shrinks to 0 as $\varepsilon \to 0$, by taking $\varepsilon > 0$ sufficiently small, we shall guarantee that the compact support of f_ε does not contain 0.

Let us see, finally, that $f_\varepsilon \to f$ in ReH^1 as $\varepsilon \to 0$.

$$\int_{-\infty}^{\infty} |f(x) - f_\varepsilon(x)|\,dx = \int_{-\infty}^{\infty} |f(x)||1 - \hat{\psi}(-\varepsilon x)|\,dx \to 0 \quad \text{as} \quad \varepsilon \to 0$$

because the integrand tends to 0 as $\varepsilon \to 0$ for a.e.x, and is bounded by $2\|\hat{\psi}\|_\infty |f(x)|$, an integrable function. Thus $f_\varepsilon \to f$ in L^1 as $\varepsilon \to 0$. Now $(H(f_\varepsilon))\hat{}(x) = -i(\operatorname{sgn} x)(f_\varepsilon)\hat{}(x)$. For $\varepsilon > 0$ sufficiently small, say for $0 < \varepsilon \leq \varepsilon_0$, the supports of all the functions $(f_\varepsilon)\hat{}$ are contained in the single set $\{x : 0 < \alpha < |x| < \beta\}$ for appropriate α and β. Let $m(x)$ be a C^∞ function with compact support such that $m(x) = -i(\operatorname{sgn} x)$ for every x satisfying $\alpha < |x| < \beta$. Then $(H(f_\varepsilon))\hat{}(x) = m(x)(f_\varepsilon)\hat{}(x)$ for all x provided $0 < \varepsilon \leq \varepsilon_0$. But $m = \hat{K}$ for some $K \in L^1$. Therefore $H(f_\varepsilon) = K * f_\varepsilon$

and consequently the convergence $f_\varepsilon \to f$ in L^1 and the fact that $K \in L^1$, imply that $H(f_\varepsilon) \to H(f)$ in L^1 as $\varepsilon \to 0$. This ends the proof. $\square$

2. MAXIMAL FUNCTION CHARACTERIZATIONS OF H^p

For $p > 1$, the space $\mathrm{Re}H^p$ has been completely identified with $\mathrm{Re}L^p$. From now on, we shall concentrate our attention on $\mathrm{Re}H^p$ for $p \leq 1$. Our goal will be to understand $\mathrm{Re}H^p$, which is a space of distributions in $\mathbb{R}$, without going out of $\mathbb{R}$. Of course this has already been done for $p = 1$, since we know that $\mathrm{Re}H^1 = \{f \in \mathrm{Re}L^1 \ / \ Hf \in L^1\}$ with the norm $f \longmapsto \|f\|_1 + \|Hf\|_1$. However, even for $p = 1$, we shall get a clearer real variable characterization.

On $\mathrm{Re}L^2$ we consider the gauge:

$$(2.1) \qquad f \longmapsto \|P_\nabla^*(f + iHf)\|_p^p$$

The space of functions for which it is finite becomes, once completed, the space $\mathrm{Re}H^p$. In this section, we introduce several gauges which will turn out to be equivalent to (2.1) (in the obvious sense of dominating each other up to multiplicative constants), thus providing new ways to look at $\mathrm{Re}H^p$. The starting point will be the gauge:

$$(2.2) \qquad f \longmapsto \|P_\nabla^*(f)\|_p^p$$

which is, of course, dominated by (2.1). The gauge (2.2) is given by the non-tangential maximal operator P_∇^* associated to the Poisson kernel P_t. Now we shall consider other approximations to the identity.

In general if ψ is such that $|\psi(x)| \leq (\mathrm{const})\ (1 + |x|)^{-\alpha}$ with $\alpha > 1$, we can associate with each $f \in \mathrm{Re}L^2$, a function defined on $\mathbb{R}^2_+$ by $f(x,t) = (f * \psi_t)(x)$ where, as usual, $\psi_t(x) = t^{-1}\psi(t^{-1}x)$. Then we can use the function $f(x,t)$ to define the following maximal functions:

(i) The non-tangential maximal function

$$\psi_\nabla^*(f)(x) = \sup_{|x-y|<t} |f(y,t)|$$

ii) In general, for $N \geq 1$, the non-tangential maximal function of amplitude N

$$\psi_{\nabla,N}^*(f)(x) = \sup_{|x-y|<Nt} |f(y,t)|$$

(iii) For $M \geq 1$, the tangential maximal function with exponent M

$$\psi_M^{**}(f)(x) = \sup_{(y,t)\in\mathbb{R}_+^2} |f(y,t)|\left(\frac{t}{|x-y|+t}\right)^M$$

Let us study the relations among these different maximal operators.

LEMMA 2.3. $\|\psi_{\nabla,N}^*(f)\|_p^p \leq N\|\psi_\nabla^*(f)\|_p^p$

Proof: For any $\lambda > 0$, consider the open set $E_\lambda = \{x \in \mathbb{R} : \psi_\nabla^*(f)(x) > \lambda\}$. Then $E_\lambda = \bigcup_j I_j$ where the I_j's are disjoint open intervals. It is geometrically obvious that $\{x \in \mathbb{R} : \psi_{\nabla,N}^*(f)(x) > \lambda\} \subset \bigcup_j I_j^N$, and, consequently

$$|\{x \in \mathbb{R} : \psi_{\nabla,N}^*(f)(x) > \lambda\}| \leq N \sum_j |I_j| = N|E_\lambda|.$$

Thus:

$$\|\psi_{\nabla,N}^*(f)\|_p^p = p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{R} : \psi_{\nabla,N}^*(f)(x) > \lambda\}|d\lambda \leq$$

$$\leq pN\int_0^\infty \lambda^{p-1}|E_\lambda|d\lambda = N\|\psi_\nabla^*(f)\|_p^p \qquad \square$$

LEMMA 2.4. If $Mp > 1$, *then:*

$$\|\psi_M^{**}(f)\|_p^p \leq C\,\|\psi_\nabla^*(f)\|_p^p$$

where C *depends only on* M *and* p.

Proof: Fix $x \in \mathbb{R}$. Then breaking up $\mathbb{R}_+^2$ as the union of the sets $\{(y,t) : |x-y| < t\}$, $\{(y,t) : 2^k t \leq |x-y| < 2^{k+1}t\}$, $k = 0,1,2,..;$

we obtain:

$$(\psi_M^{**}(f)(x))^p \leq (\psi_\nabla^*(f)(x))^p + \sum_{k=0}^{\infty} 2^{-kMp}(\psi^*_{\nabla,2^{k+1}}(f)(x))^p$$

Thus, by the previous lemma

$$\|\psi_M^{**}(f)\|_p^p \leq (1 + 2\sum_{k=0}^{\infty}(2^{1-Mp})^k)\|\psi_\nabla^*(f)\|_p^p = C \cdot \|\psi_\nabla^*(f)\|_p^p$$

Since $1 - Mp < 0$ □

As a consequence of 2.4, we see that the gauge $f \mapsto \|P_M^{**}(f)\|_p^p$ is dominated by the gauge $f \mapsto \|P_\nabla^*(f)\|_p^p$ multiplied by some constant, provided $Mp > 1$. We shall indicate this by writing

$$\|P_M^{**}(f)\|_p^p << \|P_\nabla^*(f)\|_p^p$$

Now we shall use the Poisson kernel to construct a function σ in the class $\mathcal{S}$ of Schwartz, such that the maximal function $\sigma_\nabla^*(f)$ is dominated by a constant times $P_M^{**}(f)$. We shall eventually prove that σ can be used in place of the Poisson kernel to characterize H^p. Indeed, the same can be said about any $\eta \in \mathcal{S}$ with $\int \eta \neq 0$. We shall use σ as a bridge to pass from the Poisson kernel P to an arbitrary smooth kernel η. The construction of σ is as follows: Let ψ be a real continuous function defined on $[1,\infty)$ which is rapidly decreasing at ∞ and such that $\int_1^\infty \psi(s)ds = 1$ and $\int_1^\infty s^k\psi(s)ds = 0$ for $k = 1,2,...$ (a concrete ψ with the required properties is given in Stein [1] p. 182). Then define $\sigma(x) = \int_1^\infty \psi(s)P_s(x)dx$. It can be seen that $\sigma \in \mathcal{S}$. Indeed, σ is clearly integrable and it follows easily from the expression

$\hat{\sigma}(x) = \int_1^\infty \psi(s)e^{-2\pi s|x|}ds$, that $\hat{\sigma} \in \mathcal{S}$ and, consequently: $\sigma \in \mathcal{S}$.

Other properties of σ that we shall be using are:

i) σ is even

ii) σ has vanishing moments of every order ≥ 1.
Indeed, for $k = 1,2,...,$ we have:

$$\int_{-\infty}^{\infty} x^k\sigma(x)dx = (2\pi)^k \hat{\sigma}^{(k)}(0) = 0$$

iii) $\int_{-\infty}^{\infty}\sigma(x)dx = \hat{\sigma}(0) = \int_1^{\infty}\psi(s)ds = 1$

σ gives rise to the approximate identity σ_t, $t > 0$, where

$$\sigma_t(x) = t^{-1}\sigma(t^{-1}x) = \int_1^{\infty}\psi(s)P_{st}(x)ds$$

The corresponding maximal function satisfies

LEMMA 2.5. $\sigma_\nabla^*(f)(x) \le C\cdot P_M^{**}(f)(x)$

with C *depending only upon* M.

Proof: $\sigma_\nabla^*(f)(x) = \sup_{|x-y|<t} |(\sigma_t * f)(y)| =$

$$= \sup_{|x-y|<t}\left|\int_1^{\infty}\psi(s)(P_{st} * f)(y)ds\right| \le \sup_{|x-y|<t}\int_1^{\infty}|\psi(s)|\left|P_{st} * f(y)\right|ds \le$$

$$\le P_M^{**}(f)(x)\int_1^{\infty}|\psi(s)|(1+s)^M dx = C\cdot P_M^{**}(f)(x) \qquad \square$$

Thus, if $Mp > 1$, as we shall assume from now on, we have the following chain of gauges:

$$\|\sigma_\nabla^*(f)\|_p^p << \|P_M^{**}(f)\|_p^p << \|P_\nabla^*(f)\|_p^p$$

Now we pass from σ_t to a very general kind of approximate identity. We do it in several steps.

LEMMA 2.6. Let $\psi \in S$ *and suppose that* $\eta = \xi * \psi_s$ *with* $\xi \in S$ *and* $0 < s \le 1$. *Then:*

$$\eta_\nabla^*(f)(x) \le 2^M s^{-M}\left(\int_{-\infty}^{\infty}|\xi(\lambda)|(1+|\lambda|)^M d\lambda\right)\psi_M^{**}(f)(x)$$

Proof: Let $(y,t) \in \mathbb{R}_+^2$ with $|x-y| < t$. Then:

$$|\eta_t * f(y)| = |\xi_t * \psi_{st} * f(y)| = \left|\int_{-\infty}^{\infty}\xi_t(y-u)\psi_{st} * f(u)du\right| \le$$

$$\le \int_{-\infty}^{\infty}|\xi_t(y-u)||\psi_{st} * f(u)|du \le \int_{-\infty}^{\infty}|\xi_t(y-u)|\psi_M^{**}(f)(x)\left(\frac{|x-u|+st}{st}\right)^M du$$

$$\leq \psi_M^{**}(f)(x)s^{-M}\int_{-\infty}^{\infty}|\xi_t(y-u)|\left(s+\left|\frac{x-u}{t}\right|\right)^M du \leq$$

$$\leq \psi_M^{**}(f)(x)s^{-M}\int_{-\infty}^{\infty}\frac{1}{t}\left|\xi\left(\frac{y-u}{t}\right)\right|\left(1+\left|\frac{x-y}{t}\right|+\left|\frac{y-u}{t}\right|\right)^M du \leq$$

$$\leq 2^M\psi_M^{**}(f)(x)s^{-M}\int_{-\infty}^{\infty}|\xi(\lambda)|(1+|\lambda|)^M d\lambda$$

as we wanted to prove. □

LEMMA 2.7. Let σ *be the function appearing in lemma 2.5. Let* η *be* C^∞ *with support contained in* $[-1,1]$. *Then:*

$$\eta_\nabla^*(f)(x) \leq C\left(\int_{-\infty}^{\infty}|\eta(u)|du + \int_{-\infty}^{\infty}|\eta^{(M+1)}(u)|du\right)\sigma_M^{**}(f)(x)$$

where C *depends on* M *(a positive integer) but does not depend on either* η *or* f.

Proof: $\eta = \eta*\sigma*\sigma + \sum_{k=0}^{\infty}(\eta*\sigma_{2^{-k-1}}*\sigma_{2^{-k-1}} - \eta*\sigma_{2^{-k}}*\sigma_{2^{-k}})$

Indeed, the partial sum of the right hand side is

$$\eta*\sigma_{2^{-k-1}}*\sigma_{2^{-k-1}} = \eta*(\sigma*\sigma)_{2^{-k-1}}$$

which converges uniformly to η as $k\to\infty$, since

$$\int_{-\infty}^{\infty}(\sigma*\sigma)(x)dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(x-y)\sigma(y)dydx = 1.$$

The above expansion for η can also be written as

$$\eta = \eta*\sigma*\sigma + \sum_{k=0}^{\infty}\eta*(\sigma_{2^{-k-1}}-\sigma_{2^{-k}})*(\sigma_{2^{-k-1}}+\sigma_{2^{-k}}) =$$

$$= \eta*\sigma*\sigma + \sum_{k=0}^{\infty}\eta*(\sigma_-)_{2^{-k}}*(\sigma_+)_{2^{-k}}$$

where $\sigma_- = \sigma_{2^{-1}} - \sigma$ and $\sigma_+ = \sigma_{2^{-1}} + \sigma$. Now we apply lemma 2.6 with σ and σ_+ in place of ψ and, observing that

$$(\sigma_+)^{**}_M(f)(x) \leq (2^M + 1)\sigma^{**}_M(f)(x),$$

we get:

$$(2.8) \qquad \eta^*_\nabla(f)(x) \leq C\cdot\sigma^{**}_M(f)(x)\ \{\int_{-\infty}^{\infty}|\eta*\sigma(\lambda)|(1+|\lambda|)^M d\lambda + \\ + \sum_{k=0}^{\infty} 2^{kM}\int_{-\infty}^{\infty}|\eta*(\sigma_-)_{2^{-k}}(\lambda)|(1+|\lambda|)^M d\lambda\ \}$$

where C depends only on M.

$$\int_{-\infty}^{\infty}|\eta*\sigma(\lambda)|(1+|\lambda|)^M d\lambda = \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\eta(u)\sigma(\lambda-u)du\right|(1+|\lambda|)^M d\lambda \leq$$
$$\leq \int_{-\infty}^{\infty}|\eta(u)|\int_{-\infty}^{\infty}|\sigma(\lambda-u)|(1+|\lambda|)^M d\lambda du =$$
$$= \int_{-1}^{1}|\eta(u)|\int_{-\infty}^{\infty}|\sigma(\lambda)|(1+|\lambda+u|)^M d\lambda du \leq C\int_{-\infty}^{\infty}|\eta(u)|du$$

with C depending only on M.

Now, using the fact that σ_- is even and has all moments equal to zero, we can write:

$$\eta*(\sigma_-)_s(\lambda) = \int_{-\infty}^{\infty}\{\eta(\lambda+u)-\eta(\lambda)-u\eta'(\lambda)-\dots-\frac{u^M}{M!}\eta^{(M)}(\lambda)\}(\sigma_-)_s(u)du=$$
$$= \int_{-\infty}^{\infty}\{\frac{u^{M+1}}{M!}\int_0^1(1-r)^M\eta^{(M+1)}(\lambda+ru)dr\}(\sigma_-)_s(u)du.$$

Therefore:

$$\int_{-\infty}^{\infty}|\eta*(\sigma_-)_s(\lambda)|(1+|\lambda|)^M d\lambda \leq$$
$$\leq \frac{1}{M!}\int_0^1(1-r)^M\int_{-\infty}^{\infty}|u|^{M+1}|(\sigma_-)_s(u)|\int_{-\infty}^{\infty}(1+|\lambda|)^M|\eta^{(M+1)}(\lambda+ru)|d\lambda du dr$$

But, taking into account that η has support contained in $[-1,1]$ and $0 \leq r \leq 1$, the following estimate holds:

$$\int_{-\infty}^{\infty}(1+|\lambda|)^M|\eta^{(M+1)}(\lambda+ru)|d\lambda \leq C\cdot(1+|u|)^M\int_{-\infty}^{\infty}|\eta^{(M+1)}(\lambda)|d\lambda$$

with C depending only on M, so that:

$$\int_{-\infty}^{\infty}|\eta*(\sigma_-)_s(\lambda)|(1+|\lambda|)^M d\lambda \leq$$

$$\leq C\cdot\left(\int_{-\infty}^{\infty}|u|^{M+1}(1+|u|)^M|(\sigma_-)_s(u)|du\right)\left(\int_{-\infty}^{\infty}|\eta^{(M+1)}(\lambda)|d\lambda\right) \leq$$

$$\leq C\cdot s^{M+1}\int_{-\infty}^{\infty}|\eta^{(M+1)}(\lambda)|d\lambda$$

for every $s \in [0,1]$, with C depending only on M. Now we use in 2.8 the estimate we have just obtained with $s = 2^{-k}$, $k = 0,1,2,\ldots$. We get

$$\eta_\nabla^*(f)(x) \leq C\cdot\sigma_M^{**}(f)\left(\int_{-\infty}^{\infty}|\eta(u)|du + \sum_{k=0}^{\infty}2^{kM}2^{-k(M+1)}\int_{-\infty}^{\infty}|\eta^{(M+1)}(u)|du\right) \leq$$

$$\leq C\cdot\left(\int_{-\infty}^{\infty}|\eta(u)|du + \int_{-\infty}^{\infty}|\eta^{(M+1)}(u)|du\right)\sigma_M^{**}(f)(x) ,$$

as we wanted to show. $\square$

LEMMA 2.9. With σ *as above, let* η *be* C^∞ *with support contained in* $[-A,A]$ *for some* $A \geq 1$. *Then*

$$\eta_{\nabla,A}^*(f)(x) \leq C\cdot\left(\int_{-\infty}^{\infty}|\eta(u)|du + A^{M+1}\int_{-\infty}^{\infty}|\eta^{(M+1)}(u)|du\right)\sigma_M^{**}(f)(x)$$

where C *depends on* M *but does not depend on either* η, f *or* A.

Proof: Just observe that $\eta_{\nabla,A}^*(f)(x) \leq (\eta_{1/A})_\nabla^*(f)(x)$ where $\eta_{1/A}(x) = A\eta(Ax)$ is a C^∞ function with support contained in $[-1,1]$. Now we just need to apply 2.7 with $\eta_{1/A}$ in place of η, and realize that

$$\int_{-\infty}^{\infty}|\eta_{1/A}| = \int_{-\infty}^{\infty}|\eta| \quad\text{and}\quad \int_{-\infty}^{\infty}|(\eta_{1/A})^{(M+1)}| = A^{M+1}\int_{-\infty}^{\infty}|(\eta^{(M+1)})_{1/A}| =$$

$$= A^{M+1}\int_{-\infty}^{\infty}|\eta^{(M+1)}|. \qquad \square$$

Now, let ψ be any C^∞ function with compact support. Let I_ψ be the smallest closed interval containing the support of ψ, and denote by x_ψ the centre of I_ψ. Suppose that the point x is such that $\mathrm{dist}(x,I_\psi) < |I_\psi|$. Then

$$\left|\int_{-\infty}^{\infty}f(t)\psi(t)dt\right| = |(f * \psi(x_\psi - \cdot))(x_\psi)|$$

Observe that $|x - x_\psi| < (3/2)|I_\psi|$. Let $A = (3/2)|I_\psi|$. Consider the function $\eta(y) = \psi(x_\psi - y)$. The support of η is clearly contained in $[-A,A]$. Since

$$\left|\int_{-\infty}^{\infty} f(t)\psi(t)dt\right| \leq \eta_{\nabla}^{*}(f)(x)$$

we can use lemma 2.9 to conclude that

$$\left|\int_{-\infty}^{\infty} f(t)\psi(t)dt\right| \leq C\cdot\left(\int_{-\infty}^{\infty}|\psi(u)|du + |I_{\psi}|^{M+1}\int_{-\infty}^{\infty}|\psi^{(M+1)}(u)|du\right)\sigma_M^{**}(f)(x)$$

Let us now define the maximal operator:

$$S_M^{*}(f)(x) = \sup \frac{\left|\int_{-\infty}^{\infty} f(t)\psi(t)dt\right|}{\int_{-\infty}^{\infty}|\psi(u)|du + |I_{\psi}|^{M+1}\int_{-\infty}^{\infty}|\psi^{(M+1)}(u)|du}$$

where the supremum is taken over all the C^{∞} functions ψ with compact support and such that $\operatorname{dist}(x,I_{\psi}) < |I_{\psi}|$. Then we have proved the following

THEOREM 2.10. $S_M^{*}(f)(x) \leq C\cdot\sigma_M^{**}(f)(x)$ *with* C *depending only on* M. $\square$

Taking $M > 1/p$, this result adds a new link to our chain of gauges which now looks like this:

$$\|S_M^{*}(f)\|_p^p << \|\sigma_M^{**}(f)\|_p^p << \|\sigma_{\nabla}^{*}(f)\|_p^p << \|P_M^{**}(f)\|_p^p << \|P_{\nabla}^{*}(f)\|_p^p.$$

Actually, all these gauges are equivalent; but this fact will be the consequence of the atomic decomposition to be proved in the next section.

3. ATOMIC DECOMPOSITIONS

We are going to introduce certain functions called atoms which will turn out to be the basic "building blocks" into which any H^p function or distribution can be decomposed.

For $0 < p \leq 1$ and r such that $p < r$ and $1 \leq r$, a (p,r)-atom is going to be a real-valued function a, with support contained in an interval I and satisfying:

i) a "size condition": $\left(\frac{1}{|I|}\int_I |a(x)|^r dx\right)^{1/r} \leq |I|^{-1/p}$ if $r < \infty$ or $\|a\|_\infty \leq |I|^{-1/p}$ if $r = \infty$.

ii) a "cancellation condition": $\int_{-\infty}^{\infty} a(x)x^k dx = 0$ for every integer $k = 0,1,\dots,[1/p] - 1$, where $[1/p]$ stands for the biggest integer $\leq 1/p$.

Note that if condtiion i) holds for some interval I containing the support of a, then it also holds for any other interval $J \subset I$ such that J contains the support of a. The reason is that $(1/r) - (1/p) < 0$. In particular, condition i) holds for the smallest closed interval containing the support of a.

For the remainder of the section we shall write $N_0 = [1/p]$. A (p,∞)-atom is always a (p,r)-atom for any other r. Thus, when proving a general result about a (p,r)-atom, we do not need to write a separate proof for the case $r = \infty$. In the next five results we study the action of our basic operators on a single atom. The notation will be as follows: a will be a (p,r)-atom, I the smallest closed interval containing the suppport of a, x_I the centre of I and, as usual, $I^2 = x_I + 2(I - x_I)$. Observe that

$$\left(\frac{1}{|I|}\int_I |a(x)|^p dx\right)^{1/p} \leq \left(\frac{1}{|I|}\int_I |a(x)|^r dx\right)^{1/r} \leq |I|^{-1/p}$$

and consequently $\|a\|_p \leq 1$. We shall see that, for many operators, the image of a satisfies a similar inequality with a possibly bigger constant which does not depend on the particular atom. We start by looking at the action of the Hilbert transform H.

<u>LEMMA 3.1</u>. $\int_{I^2} |Ha(x)|^p dx \leq C$, <u>*with*</u> C <u>*independent of*</u> a.

Proof: We just use the fact that H is of weak type (r,r) (since $r \geq 1$), and also that $p < r$. We have:

$$\int_{I^2} |Ha(x)|^p dx = p\int_0^\infty \lambda^{p-1} |\{x \in I^2 : |Ha(x)| > \lambda\}| d\lambda \leq$$

$\leq$ (for any $R > 0$) $\leq$

$$\leq p\int_0^R \lambda^{p-1}d\lambda\ |I^2| + p\int_R^\infty \lambda^{p-1}\ C\lambda^{-r}\int_{-\infty}^\infty |a(x)|^r\ dx\ d\lambda \leq 2\ |I|\ R^p +$$

$$+\frac{Cp}{r-p}R^{p-r}\ |I|^{\ 1-r/p} = (\text{for the choice}\quad R = |I|^{-1/p}) =$$

$$= 2 + Cp/(r-p) = C \quad \square$$

LEMMA 3.2. For $x \notin I^2$:

$$|Ha(x)| \leq C\cdot |I|^{-1/p}(|I|/|x-x_I|)^{N_0+1}$$

with C *independent of* a.

Proof:

$$|Ha(x)| = \frac{1}{\pi}\left|\int_I \frac{a(y)}{x-y}\,dy\right| = \frac{1}{\pi}\left|\int_I a(y)\ \left(\frac{1}{x-y} - \sum_{k=1}^{N_0}\frac{(y-x_I)^{k-1}}{(x-x_I)^k}\right)dy\right| \leq$$

$$\leq C\ \left(\int_I |a(y)|^r\ dy\right)^{1/r}\left\|\frac{(y-x_I)^{N_0}}{(x-x_I-\theta_y(y-x_I))^{N_0+1}}\quad \chi_I(y)\right\|_{r'}$$

where $0 < \theta_y < 1$. But $|y-x_I| < |x-x_I|/2$, and consequently $|x-x_I-\theta_y(y-x_I)| \geq |x-x_I|/2$, so that:

$$|Ha(x)| \leq C\ |I|^{1/r-1/p}\ \frac{|I|^{N_0}}{|x-x_I|^{N_0+1}}\ |I|^{1/r'} = C\ |I|^{-1/p}(|I|/|x-x_I|)^{N_0+1}.$$

$\square$

Combining the estimates obtained in the last two lemmas we get:

THEOREM 3.3. $\int_{-\infty}^{\infty} |Ha(x)|^p\ dx \leq C$ *with* C *depending only on* p *and* r *but not on the particular* a.

Proof:

$$\int_{x\notin I^2} |Ha(x)|^p dx \leq C\cdot |I|^{p(N_0+1)-1}\int_{|x-x_I|>|I|} |x-x_I|^{-p(N_0+1)}dx =$$

$= C < \infty$, since $p(N_0+1) > 1$. This estimate, together with the one obtained in 3.1 yields what we wanted to prove. $\square$

THEOREM 3.4. $a \in ReH^p$ *and* $\|a\|_{ReH^p} \leq C$, *constant that depends on* p *and* r, *but not on the particular atom.*

Proof: Consider the function $A(z)$ of $z = x + it,\ t > 0$ given by

$$A(z) = -\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{1}{z-y}\ a(y)\ dy = (P_t+iQ_t)*a(x)$$

We shall show that $\int_{-\infty}^{\infty} |A(x + it)|^p dx \leq C$ with C independent of $t > 0$ and a. Since $|A(x + it)|^p \leq |P_t * a(x)|^p + |Q_t * a(x)|^p$, it will be enough to prove the estimates:

1) $\int_{-\infty}^{\infty} |P_t * a(x)|^p dx \leq C$ and

2) $\int_{-\infty}^{\infty} |Q_t * a(x)|^p dx \leq C$

But these estimates can be obtained very much in the same way as the estimate for Ha. Here are the details:

By observing that $|P_t * a(x)| \leq C \cdot M(a)(x)$ as we showed in Chapter II, and using the same argument of the proof of 3.1 with M instead of H, we get:

$$\int_{I^2} |P_t * a(x)|^p dx \leq C$$

Now let $x \notin I^2$. Then

$$|P_t * a(x)| = \left|\int_{-\infty}^{\infty} P_t(x - y)a(y)dy\right| =$$

$$= \left| \int_{-\infty}^{\infty} \left(P_t(x - y) - \sum_{k=0}^{N_o-1} \frac{(-1)^k}{k!} (P_t)^{(k)}(x-x_I)(y-x_I)^k\right) a(y)dy \right| \leq$$

$$\leq C \cdot \int_{-\infty}^{\infty} |(P_t)^{(N_o)}(x-x_I-\theta_y(y-x_I))|\, |y-x_I|^{N_o} |a(y)| dy \leq$$

$$\leq C \cdot |I|^{1/r-1/p} t^{-N_o-1} \left\| P^{(N_o)}\left(\frac{x-x_I-\theta_y(y-x_I)}{t}\right) |y-x_I|^{N_o} \chi_I(y) \right\|_{r'}$$

with $0 \leq \theta_y \leq 1$. Using the estimate $|P^{(N)}(s)| \leq C_N(1+|s|)^{-N-1}$, we get

$$\left|P^{(N_o)}\left(\frac{x-x_I-\theta_y(y-x_I)}{t}\right)\right| \leq C\, t^{N_o+1} (t+|x-x_I-\theta_y(y-x_I)|)^{-N_o-1}$$

and since $|x-x_I-\theta_y(y-x_I)| \geq |x-x_I|/2$ we finally have:

$$|P_t * a(x)| \leq C \cdot |I|^{1/r-1/p} \frac{|I|^{N_o}}{|x-x_I|^{N_o+1}} |I|^{1/r'} =$$

$$= C\, |I|^{-1/p} (|I|/|x-x_I|)^{N_o+1}$$

This is the same estimate as in 3.2. We proceed as in 3.3 to obtain

$$\int_{x \notin I^2} |P_t * a(x)|^p dx \leq C \quad .$$

This estimate combines with that previously obtained for I^2 to yield 1).

Next we shall try to get 2). First of all,

$$\int_{I^2} |Q_t * a(x)|^p dx = \int_{I^2} |H(P_t * a)(x)|^p dx.$$

To estimate this integral we proceed as in the proof of 3.1. and observe that, since $r \geq 1$, we have:

$$\int_{-\infty}^{\infty} |P_t * a(x)|^r dx \leq \int_{-\infty}^{\infty} |a(x)|^r dx.$$

This is enough to conclude that

$$\int_{I^2} |Q_t * a(x)|^p dx \leq C \quad .$$

The estimate of the integral for $x \notin I^2$ is carried out exactly as the one for $|P_t * a(x)|^p$. We just need to realize that the kernel $Q(s) = s/(1 + s^2)$ satisfies the same estimates that we used for $P(s) = 1/(1 + s^2)$, namely, $|Q^{(N)}(s)| \leq C_N(1 + |s|)^{-N-1}$. This ends the proof. □

COROLLARY 3.5. $\|P_\nabla^*(a + iHa)\|_p^p \leq C$, *independent of* a.

Proof: With the notation of the last proof, $P_\nabla^*(a + iHa) = m_A$. Thus, our corollary follows from theorem 1.1 and its corollary 1.2. □

We have seen that (p,r)-atoms provide very simple examples of functions in $\mathrm{Re}H^p$. The fundamental result will be that every function or distribution in $\mathrm{Re}H^p$ can be decomposed into (p,∞)-atoms. In order to achieve such a decomposition, we shall use the following result, which contains a refinement of the classical Calderón-Zygmund decomposition.

THEOREM 3.6. Let f *be a function in* $\mathrm{Re}L^2$ *(2 could be replaced by* $q > 1$*). Let* λ *be a real number* > 0 *and* N *and* M *integers*

≥ 0. Then

$$f(x) = g_\lambda(x) + \sum_{i=1}^{\infty} b_\lambda^i(x)$$

a.e. and in L^2*-norm, with* $|g_\lambda(x)| \leq C\lambda$, b_λ^i *supported in an interval* $I_i = I_i(\lambda)$ *and* $\int b_\lambda^i(x)x^k dx = 0$ *for every integer* k *such that* $0 \leq k \leq N$. *Besides, the intervals* $I_i = I_i(\lambda)$ *are going to be precisely the connected components of the open set*: $\{x \in \mathbb{R}: S_M^*(f)(x) > \lambda\}$.

Proof: For every I_i, a Whitney type decomposition $\{I_{i,j}\}$, j ranging over all the integers, is obtained in the following way: We first divide I_i into three intervals of equal length, of which, the one in the middle will be called $I_{i,0}$ and the remaining ones will be split in half. Of the two intervals next to $I_{i,0}$, the one to the right will be $I_{i,1}$ and the one to the left will be $I_{i,-1}$. The remaining intervals are split again and the process continues indefinitely. With this decomposition we associate a partition of unity as follows: Let η be an even C^∞ function with support in $[-1-\delta, 1+\delta]$ for some $\delta < 1/2$, identically equal to 1 on $[-1,1]$ and decreasing to 0 from 1 to $1+\delta$. Then if $x_{i,j}$ is the center of $I_{i,j}$, let

$$\eta_j^i(t) = \eta(2|I_{i,j}|^{-1}(t-x_{i,j})) = \eta(2^{|j|+1}|I_{i,0}|^{-1}(t-x_{i,j})),$$

that is, η_j^i is the result of "adapting" the function η to the interval $I_{i,j}$. Our choice of δ guarantees that every point in I_i has a neighbourhood that meets at most two of the supports of the η_j^i's. Thus $\sum_j \eta_j^i(t)$ is C^∞ in I_i and it satisfies the inequalities $1 \leq \sum_j \eta_j^i(t) \leq 2$ for every $t \in I_i$. Let

$$\phi_j^i(t) = \eta_j^i(t) \Big/ \Big\{ \sum_k \eta_k^i(t) \Big\}.$$

The function ϕ_j^i is C^∞ and has the same support as η_j^i. Clearly $\sum_j \phi_j^i = \chi_{I_i}$.

Next we shall use the family $(\phi_j^i)_{i,j}$ to decompose our function f. Let $P_j^i(x)$ be the unique polynomial of degree $\leq N$ such that

$$\int_{-\infty}^{\infty} (f(x) - P_j^i(x))x^k \phi_j^i(x)dx = 0 \qquad (k = 0,1, \ldots, N)$$

Then, if E denotes the complement of $\bigcup_i I_i$, we can write

$$f(x) = f(x)\chi_E(x) + \sum_i \sum_j P_j^i(x)\phi_j^i(x) + \sum_i \sum_j (f(x) - P_j^i(x))\phi_j^i(x)$$

Define

$$g_\lambda(x) = f(x)\chi_E(x) + \sum_i \sum_j P_j^i(x)\phi_j^i(x)$$

and for every i:

$$b_\lambda^i(x) = \sum_j (f(x) - P_j^i(x))\phi_j^i(x)$$

Then $f(x) = g_\lambda(x) + \sum_i b_\lambda^i(x)$ a.e.. The important fact is that $|P_j^i(x)\phi_j^i(x)| \leqq C\lambda$, where C is an absolute constant, that is, it does not depend on i, j, λ or f. Let us prove this. If $N = 0$, $P_j^i(x)$ is just the constant

$$m_j^i = \int_{-\infty}^{\infty} f(x)\phi_j^i(x)dx \Big/ \int_{-\infty}^{\infty} \phi_j^i(x)dx.$$

It follows from the definition of $S_M^*(f)(x)$ that, taking x to be the extremity of I_i closest to $I_{i,j}$, we have:

$$\left|\int_{-\infty}^{\infty} f(x)\phi_j^i(x)dx\right| \leqq S_M^*(f)(x)\left(\int_{-\infty}^{\infty}\phi_j^i(u)du + C|I_{i,j}|^{M+1}\int_{-\infty}^{\infty}|(\phi_j^i)^{(M+1)}(u)|du\right)$$

But, by the definition of the I_i's, $S_M^*(f)(x) \leqq \lambda$. Also, for the kind of functions that we are using:

$$\int_{-\infty}^{\infty}\phi_j^i(u)du + C|I_{i,j}|^{M+1}\int_{-\infty}^{\infty}|(\phi_j^i)^{(M+1)}(u)|du \leqq C\int_{-\infty}^{\infty}\phi_j^i(u)du$$

(indeed, just from the fact that each ϕ_j^i is adapted to $I_{i,j}$, it follows that, for every k, $\int_{-\infty}^{\infty}|(\phi_j^i)^{(k)}(u)|du \leqq C|I_{i,j}|^{-k+1}$. Besides, since ϕ_j^i is essentially constant on $I_{i,j}$, it follows that $\int\phi_j^i \sim |I_{i,j}|$). Consequently $|m_j^i| \leqq C\lambda$. For the case $N > 0$, we see that $|P_j^i(x)\,\phi_j^i(x)| \leqq C\cdot|m_j^i|$. Indeed, this inequality is invariant under translations and dilations so that we just need to prove it for a fixed ϕ, for example, ϕ can be the η considered above. Then

$$P(x) = \sum_{k=0}^{N}\left(\int_{-\infty}^{\infty} f(s)\xi_k(s)\phi(s)ds\right)\xi_k(x)$$

where (ξ_k) is an orthonormal basis for the span of $\{1, x, \ldots, x^N\}$ in $L^2(\phi(x)dx)$, say, the one obtained by the Gramm-Schmidt process. Clearly

$$|P(x)\phi(x)| \leqq C\frac{|\int_{-\infty}^{\infty} f(s)\phi(s)ds|}{\int_{-\infty}^{\infty}\phi(s)ds} = C|m|$$

From $|P_j^i(x)\phi_j^i(x)| \leqq C\lambda$, it follows that $|g_\lambda(x)| \leqq C\lambda$ since, of course, $|f(x)| \leq CS_M^*(f)(x)$ a.e. (just write $f(x) = \lim_{\varepsilon\to 0} f * \psi_\varepsilon(x)$ with ψ a C^∞ function with compact support, integral 1 and identically equal to 1 in some neighbourhood of 0). It also follows that $|b_\lambda^i(x)| \leqq C\cdot S_M^*(f)(x)$. Then, since $S_M^*(f) \in L^2$ (it is dominated by the Hardy-Littlewood maximal function), we can appeal to the Lebesgue dominated convergence theorem to conclude that the series $f(x) = g_\lambda(x) + \sum_i b_\lambda^i(x)$ is also convergent in L^2. Finally

$$\int_{-\infty}^{\infty} b_\lambda^i(x)x^k dx = \sum_j \int_{-\infty}^{\infty}(f(x) - P_j^i(x))x^k\phi_j^i(x)dx = 0$$

for $k = 0,1,\ldots,N$ by the definition of the P_j^j's. □

<u>*THEOREM 3.7*</u>. <u>*Let*</u> $f \in ReL^2$ <u>*with*</u> $S_M^*(f) \in L^p$, <u>*where*</u> M <u>*is an integer*</u> $\geqq 0$ <u>*and*</u> $0 < p \leqq 1$. <u>*Then there exist:*</u>

a) <u>*A sequence*</u> (a_i) <u>*of*</u> (p,∞) -<u>*atoms, and*</u>
b) <u>*A sequence*</u> (λ_i) <u>*of real numbers satisfying*</u>

$$\sum_i |\lambda_i|^p \leqq C\int_{-\infty}^{\infty}|S_M^*(f)(x)|^p dx$$

<u>*such that*</u> $f(x) = \sum_i \lambda_i a_i(x)$ <u>*for a.e.*</u> x, <u>*and the series also converges to*</u> f <u>*in the space*</u> ReH^p.

<u>*Proof:*</u> For each integer k, let

$$f(x) = g_k(x) + \sum_{i=1}^{\infty} b_k^i(x)$$

be the decomposition obtained in theorem 3.6 with $\lambda = 2^k$ and some fixed $N \geqq [1/p] - 1$. Now we put all these decompositions together. As $k \to -\infty$, $g_k(x) \to 0$ uniformly, since $|g_k(x)| \leqq C\cdot 2^k$. We even have convergence in L^2 because $|g_k(x)| \leqq C\cdot S_M^*(f)(x)$, which is in L^2. On the other hand $g_k(x) \to f(x)$ a.e. as $k \to \infty$ because $f(x) - g_k(x)$ lives in the set $\{x : S_M^*(f)(x) > 2^k\}$ which decreases to a set of measure 0 as $k \to \infty$. As before, there is convergence in L^2 as well, since $|f(x) - g_k(x)| \leqq CS_M^*(f)(x)$. All these facts imply that f is the sum of a telescopic series converging a.e. and in L^2, namely

$$f(x) = \sum_{-\infty}^{\infty}(g_{k+1}(x) - g_k(x)) = \sum_k \Big(\sum_i b_k^i(x) - \sum_j b_{k+1}^j(x) \Big) =$$

$$= \sum_k \sum_i \beta_k^i(x) ,$$

where

$$\beta_k^i(x) = b_k^i(x) - \sum_{\{j : I_j(k+1) \subset I_i(k)\}} b_{k+1}^j(x) .$$

Of course this can be done because each $I_j(k+1)$ is contained in a unique $I_i(k)$. If $x \in I_i(k)$, $\beta_k^i(x) = g_{k+1}(x) - g_k(x)$, and for all other x, $\beta_k^i(x) = 0$. Therefore $|\beta_k^i(x)| \leqq C\cdot 2^k$, with some absolute constant. With the same choice of C, let us define

$$a_k^i(x) = (C\, 2^k |I_i(k)|^{1/p})^{-1} \beta_k^i(x).$$

Then, clearly, each a_k^i is a (p,∞)-atom and $f(x) = \sum_k \sum_i \lambda_k^i a_k^i(x)$, where $\lambda_k^i = C\, 2^k |I_i(k)|^{1/p}$. The coefficients λ_k^i satisfy:

$$\sum_k \sum_i |\lambda_k^i|^p = \sum_k \sum_i C^p 2^{kp} |I_i(k)| =$$

$$= C \sum_{-\infty}^{\infty} 2^{kp} |\{x \in \mathbb{R} : S_M^*(f)(x) > 2^k\}| \leqq$$

$$\leqq C \int_0^\infty \lambda^{p-1} |\{x \in \mathbb{R} : S_M^*(f)(x) > \lambda\}| d\lambda = C \int_{-\infty}^\infty (S_M^*(f)(x))^p dx.$$

Thus, we have found a double series $\sum_k \sum_i \lambda_k^i a_k^i(x)$ which converges a.e. and in L^2 to our function f and is such that

$$\sum_k \sum_i |\lambda_k^i|^p \leqq C \cdot \int_{-\infty}^\infty |S_M^*(f)(x)|^p dx < \infty.$$

Observe that for fixed x and k, the corresponding sum in i contains at most one non zero element and we can find an enumeration of the double indices which allows us to view our double series as an ordinary series $\sum_j \lambda_j a_j(x)$ converging to $f(x)$ a.e. However the summation in the L^2 norm has to be performed first in the index i and then in the k. Still this L^2 convergence is useful for us. Let us consider the following analytic functions of

$$z = x + it,\ t > 0 : F(z) = P_t * (f + iHf)(x),$$

and for each i,k,

$$A_k^i(z) = P_t * (a_k^i + iHa_k^i)(x).$$

Then the L^2 convergence implies that $F(z) = \sum_k \sum_i \lambda_k^i A_k^i(z)$ where the double series converges in H^2 and, therefore, uniformly on compact subsets. Now theorem 3.4 tells us that (p,∞)-atoms are uniformly in H^p. Thus in our series $\|A_k^i\|_{H^p} \leqq C$, some absolute constant. This fact together with the finiteness of the sum $\sum_k \sum_i |\lambda_k^i|^p$ allows us to conclude that the series converges normally in H^p. it follows that the simple series $\Sigma\lambda_j a_j$ converges in $\mathrm{Re}H^p$ and the limit is, naturally, our function f. $\square$

For $f \in \mathrm{Re}L^2$, let

$$N_{p,r}(f) = \inf \{ (\sum_i |\lambda_i|^p)^{1/p} : f(x) = \sum_i \lambda_i a_i(x) \text{ a.e. and in } \mathrm{Re}H^p \}$$

the a_i's being arbitrary (p,r)-atoms. Since for $r_1 < r_2$, a (p,r_2)-atom is always a (p,r_1)-atom, we clearly have: $N_{p,r_1}(f) \leq N_{p,r_2}(f)$. We have just proved in theorem 3.7 that $(N_{p,\infty}(f))^p \leqq C\cdot \|S_M^*(f)\|_p^p$. This allows us to extend our chain of gauges over $\mathrm{Re}L^2$, obtaining, if $M > 1/p$:

$$(N_{p,r}(f))^p << (N_{p,\infty}(f))^p << \|S_M^*(f)\|_p^p << \|\sigma_M^{**}(f)\|_p^p << \|\sigma_\nabla^*(f)\|_p^p <<$$
$$<< \|P_M^{**}(f)\|_p^p << \|P_\nabla^*(f)\|_p^p << \|P_\nabla^*(f + iHf)\|_p^p,\quad 1 \leqq r < \infty, p < r.$$

Actually all these gauges are equivalent as we shall see now. Let $f \in \mathrm{Re}L^2$ be the sum of the series $\sum_i \lambda_i a_i$ both in the a.e. sense and in $\mathrm{Re}H^p$, with $\sum_i |\lambda_i|^p < \infty$, and the a_i's being (p,r)-atoms. Then, considering the analytic functions of $z = x + it,\ t > 0$, $F(z) = P_t * (f + iHf)(x)$, and for each index i, $A_i(z) = (P_t + iQ_t) * a_i(x)$ we have $F(z) = \sum_i \lambda_i A_i(z)$, the series converging in H^p and, consequently, uniformly on compact subsets of $\mathbb{R}_+^2$. It follows that

$$\|P_\nabla^*(f + iHf)\|_p^p \leqq C\ \|F\|_{H^p}^p \leqq C \sum_i |\lambda_i|^p,$$

the last inequality being a consequence of theorem 3.4. Since this is true for any such representation of f, we finally arrive at:

$$\|P_\nabla^*(f + iHf)\|_p^p \leq C(N_{p,r}(f))^p ,$$

which closes the chain of gauges and shows that all of them are equivalent. The completion of the space determined on $\mathrm{Re}L^2$ by any of the equivalent gauges above, is an equivalent copy of $\mathrm{Re}H^p$. Thus, we have several real-variable characterizations of $\mathrm{Re}H^p$. Let us state separately the corresponding theorems.

THEOREM 3.8. a) Let $f \in \mathrm{Re}H^1$. *Then there is a sequence of $(1,\infty)$-atoms* (a_j) *and a sequence of real numbers* (λ_j) *with* $\sum_j|\lambda_j| \leq C\,\|f\|_{\mathrm{Re}H^1}$, *such that* $f(x) = \sum_j \lambda_j a_j(x)$ *a.e., the series converging to* f *also in* $\mathrm{Re}H^1$.

b) Conversely if f *is a real measurable function such that* $f(x) = \sum_j \lambda_j a_j(x)$ *a.e., with* $\sum_j|\lambda_j| < \infty$, *and the* a_j*'s being $(1,r)$-atoms for some fixed* r, $1 < r \leq \infty$; *then* $f \in \mathrm{Re}H^1$ *and* $\|f\|_{\mathrm{Re}H^1} \leq C\cdot\sum_j|\lambda_j|$.

Proof: a) We can find a sequence $\{f_n\}$ of functions in $\mathrm{Re}H^1 \cap L^2$ such that $\|f_n\|_{\mathrm{Re}H^1} \leq \|f\|_{\mathrm{Re}H^1}$ and $f_n \to f$ in $\mathrm{Re}H^1$ as $n \to \infty$. We can always assume, by taking a subsequence if necessary, that $f_n(x) \to f(x)$ a.e. and

$$\|f_n - f_{n+1}\|_{\mathrm{Re}H^1} < 2^{-n}\|f\|_{\mathrm{Re}H^1} .$$

Then,

$$f(x) = \sum_{n=0}^{\infty}(f_{n+1}(x) - f_n(x))$$

with $f_0(x) = 0$. According to theorem 3.7, we can write, for each $n \geq 0$

$$f_{n+1}(x) - f_n(x) = \sum_i \lambda_{ni}a_{ni}(x)$$

with $\sum_i|\lambda_{ni}| \leq C2^{-n}\|f\|_{\mathrm{Re}H^1}$, and the a_{ni}'s being $(1,\infty)$-atoms. All the representations considered are convergent in $\mathrm{Re}H^1$. Then

$$f(x) = \sum_i \lambda_{1i}a_{1i}(x) + \sum_{n=1}^{\infty}\sum_i\lambda_{ni}a_{ni}(x)$$

and since

$$\sum_i |\lambda_{1i}| + \sum_n \sum_i |\lambda_{ni}| \leqq C \; \|f\|_{ReH^1}$$

this representation can be arranged as a series converging to f in ReH^1.

b) If $f(x) = \sum_j \lambda_j a_j(x)$ a.e. with $\sum_j |\lambda_j| < \infty$ and the a_j's being $(1,r)$-atoms, then the series converges necessarily to f in ReH^1 and $\|f\|_{ReH^1} \leqq C\cdot\Sigma|\lambda_j|$. Indeed, the estimate in 3.4 implies that there is convergence in ReH^1 and the rest is straightforward. □

There is a corresponding result for $0 < p < 1$.

THEOREM 3.9. a) *Let* $f \in ReH^p$, $0 < p < 1$. *Then, there is a sequence of (p,∞)-atoms* (a_j) *and a sequence of real numbers* (λ_j) *with* $\sum_j|\lambda_j|^p \leqq C \; \|f\|^p_{ReH^p}$ *such that* $f = \sum_j \lambda_j a_j$, *the series converging to* f *in* ReH^p *and, consequently, also in the sense of tempered distributions.*
b) *Conversely if* f *is a tempered distribution such that* $f = \sum_j \lambda_j a_j$ *in the sense of tempered distributions with* $\sum_j |\lambda_j|^p < \infty$, *and the* a_j's *being (p,r)-atoms for some fixed* r, $1 \leqq r \leqq \infty$, *then* $f \in ReH^p$ *and* $\|f\|^p_{ReH^p} \leqq C \sum_j |\lambda_j|^p$.

Proof: It is very much like the proof of the previous theorem with convergence in the sense of tempered distributions instead of convergence a.e. □

The equivalence between the gauges $f \mapsto \|P^*_\nabla(f)\|^p_p$ and $f \mapsto \|P^*_\nabla(f + iHf\|^p_p$ gives us the following characterization of ReH^p, which is known as the Burkholder-Gundy-Silverstein theorem.

THEOREM 3.10. *A tempered distribution* f *is in* ReH^p, $0 < p \leqq 1$, *if andonly if it is the boundary distribution corresponding to a real harmonic function* $u(x,t)$ *in* $\mathbb{R}^2_+$ *for which the maximal function*

$$m_u(x) = \sup_{|y-x|<t} |u(y,t)|$$

belongs to L^p. *Besides* $\|m_u\|_p \sim \|f\|_{ReH^p}$.

Proof: Of course we know that every $f \in ReH^p$, $0 < p \leqq 1$, is of the form $f = \lim_{t\to 0} u(\cdot,t)$ in the sense of tempered distributions, with u harmonic in $\mathbb{R}^2_+$ such that $\|m_u\|_p \leqq C\|f\|_{ReH^p}$.

Conversely let u be real, harmonic in $\mathbb{R}^2_+$, with $\|m_u\|_p < \infty$. Then we know from theorem 4.23 in chapter II that there exists $f = \lim_{t\to 0} u(\cdot,t)$ in the sense of tempered distributions. Write $u_t(x) = u(x,t)$. Clearly each $u_t \in L^2(\mathbb{R}^n)$. Let $v_t = H(u_t)$ and define $v(x,t) = v_t(x)$. Then $F(x+it) = u(x,t) + iv(x,t)$ is holomorphic in $\mathbb{R}^2_+$. Since $u(x,t+s) = P_s * u_t(x)$, we have $P^*_\nabla(u_t)(x) \leqq$ $\leqq m_u(x)$. It follows that

$$\|P^*_\nabla(u_t + iv_t)\|_p^p \leqq C\|P^*_\nabla(u_t)\|_p^p \leqq C\|m_u\|_p^p < \infty$$

But we also have $v(x,t+s) = P_s * v_t(x)$, so that

$$m_F(x) = \sup_{t>0} P^*_\nabla(u_t + iv_t)(x) = \lim_{t\downarrow 0} P^*_\nabla(u_t + iv_t)(x)$$

Then, Fatou's lemma gives $\|m_F\|_p^p \leqq C\|m_u\|_p^p$, in such a way that $F \in H^p$ and $\|f\|_{ReH^p} = \|F\|_{H^p} \leqq \|m_F\|_p \leqq C\|m_u\|_p$. $\square$

Now we shall make several observations about the real variable characterizations obtained for ReH^p.

First of all we notice that, even though a p-atom needs to have by definition vanishing moments only up to order $[1/p]-1$, the (p,∞)-atoms appearing in theorem 3.7 and, consequently in part a) of theorems 3.8 and 3.9, can be chosen to have as many vanishing moments as we wish. All we have to do is to use theorem 3.6 with N as big as we wish. This observation will be important for the interpolation theorems between Hardy spaces.

Looking back at the process by which we have obtained an atomic decomposition for $f \in L^2 \cap ReH^p$, we have to admit that it is rather long and somehow artificial. The question naturally arises of whether or not it is possible to obtain an atomic decomposition directly from the Poisson integral of f. We shall see that the answer is yes if one contents himself with $(p,2)$-atoms. This poses the additional problem of being able to decompose directly a $(p,2)$-atom into (p,∞)-atoms, a problem which has an independent interest, and that we shall also deal with.

3.11. Different Approach to the Atomic Decomposition:

First, let us try to decompose $f \in L^2 \cap \mathrm{Re}H^p$, $0 < p \leqq 1$, by using $u(x,t) = P_t * f(x)$. Our basic maximal function will be $P^*_{\nabla,2}(f)(x)$ (the 2 will be important!). Of course $\|P^*_{\nabla,2}(f)\|_p \leqq C\|f\|_{\mathrm{Re}H^p}$ with C an absolute constant. For each integer k we shall consider the open set

$$E^k = \{x \in \mathbb{R} : P^*_{\nabla,2}(f)(x) > 2^k\} = \bigcup_j I_j(k)$$

where the $I_j(k)$'s are the connected components of E^k. To each interval I, we associate the "tent"

$$\hat{I} = \{(y,t) \in \mathbb{R}^2_+ : [y-t,\ y+t] \subset I\}$$

and let

$$\widehat{E^k} = \bigcup_j \widehat{I_j(k)} \quad ; \quad T^k_j = \widehat{I_j(k)} \setminus \widehat{E^{k+1}}$$

Also, for each $\varepsilon > 0$, call $T^{k(\varepsilon)}_j = \{(y,t) \in T^k_j : t \geq \varepsilon\}$ (see Figure III. 3.1).

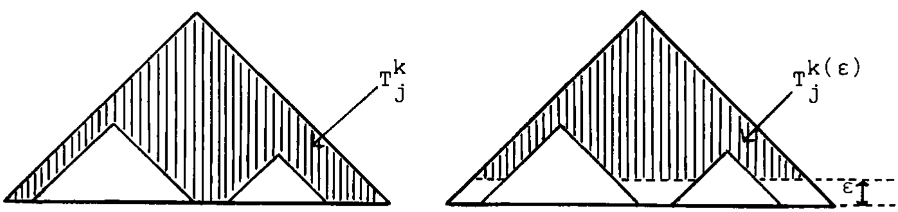

Figure III.3.1

We shall use a function $\psi \in C^\infty(\mathbb{R})$, real, even, supported in $[-1,1]$, with moments vanishing up to order $[1/p]-1$ and satisfying:

$$\int_0^\infty e^{-2\pi s}\,\hat{\psi}(s)\,ds = -1/(2\pi).$$

The existence of such a ψ is a very easy matter. First we can obtain a function in L^2, even, supported in $[-1/2,1/2]$ say, with the appropriate number of vanishing moments. Then we can make it smooth by convolving it with a smooth function, also even and supported in $[-1/2,1/2]$. Then we just have to normalize to get the condition on $\hat{\psi}$. With the aid of ψ we shall construct our atoms.

For each k, j, ε, we define the function:

$$g_j^{k(\varepsilon)}(x) = \int_{T_j^{k(\varepsilon)}} \frac{\partial u}{\partial t}(y,t)\psi_t(x-y)dydt \quad .$$

Since the support of ψ is contained in $[-1,1]$, the function $(y,t) \mapsto \psi_t(x-y)$, for x fixed, can be different from 0 only for points of the cone $\Gamma(x) = \{(y,t): |y-x| < t\}$. Thus $g_j^{k(\varepsilon)}(x)$ can be different from 0 only for points $x \in I_j(k)$. Now we estimate $\|g_j^{k(\varepsilon)}\|_2$. For any $\psi \in S$, we have:

$$\left\langle g_j^{k(\varepsilon)}, \phi \right\rangle = \int_{T_j^{k(\varepsilon)}} \frac{\partial u}{\partial t}(y,t) \int_{-\infty}^{\infty} \psi_t(x-y)\phi(x)dxdydt =$$

$$= \int_{T_j^{k(\varepsilon)}} \frac{\partial u}{\partial t}(y,t)(\psi_t * \phi)(y)dydt \ .$$

Then we use Cauchy-Schwarz's inequality to get:

$$|\langle g_j^{k(\varepsilon)}, \phi \rangle| \leq \left(\int_{T_j^{k(\varepsilon)}} t|\nabla u|^2 dydt\right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}_+^2} |\psi_t * \phi(y)|^2 \frac{dydt}{t}\right)^{\frac{1}{2}} \ .$$

But

$$\int_{\mathbb{R}_+^2} |\psi_t * \phi(y)|^2 \frac{dydt}{t} = \int_0^{\infty} \int_{-\infty}^{\infty} |\psi_t * \phi(y)|^2 dy \frac{dt}{t} =$$

$$= \int_0^{\infty} \int_{-\infty}^{\infty} |\hat{\psi}(t\xi)|^2 |\hat{\phi}(\xi)|^2 d\xi \frac{dt}{t} = \int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 \int_0^{\infty} |\hat{\psi}(t|\xi|)|^2 \frac{dt}{t} d\xi =$$

$$= \left(\int_0^{\infty} |\hat{\psi}(s)|^2 \frac{ds}{s}\right)\left(\int_{-\infty}^{\infty} |\hat{\phi}(\xi)|^2 d\xi\right) = C\|\phi\|_2^2$$

On the other hand, since u is harmonic, by Green's theorem, and denoting by $\frac{\partial}{\partial \nu}$ the derivative in the direction of the outer normal for points of the boundary $\partial T_j^{k(\varepsilon)}$ of $T_j^{k(\varepsilon)}$, we have

$$\int_{T_j^{k(\varepsilon)}} t|\nabla u|^2 dydt = \frac{1}{2} \int_{T_j^{k(\varepsilon)}} t\Delta(u^2)dydt =$$

$$= \frac{1}{2} \int_{\partial T_j^{k(\varepsilon)}} \left(t\frac{\partial(u^2)}{\partial \nu} - u^2 \frac{\partial t}{\partial \nu} \right) \leq$$

$$\leq \int_{\partial T_j^{k(\varepsilon)}} \left(|u| t \left|\frac{\partial u}{\partial \nu}\right| + \frac{1}{2} u^2 \left|\frac{\partial t}{\partial \nu}\right| \right) \leq C \ 2^k |I_j(k)|$$

with C an absolute constant. The reasons for this last inequality are the following:

a) $|u|$ and $t|\frac{\partial u}{\partial \nu}|$ are bounded by $C2^k$ on $\partial T_j^{k(\varepsilon)}$.

b) $|\frac{\partial t}{\partial \nu}| \leqq 1$ on $\partial T_j^{k(\varepsilon)}$, and

c) $|\partial T_j^{k(\varepsilon)}| \leqq C|I_j(k)|$.

Among these three properties, only the part of a) that refers to $t|\frac{\partial u}{\partial \nu}|$ needs further justification. We shall see that $t|\nabla u| \leqq C\cdot 2^k$ on $\partial T_j^{k(\varepsilon)}$ with C an absolute constant. Let us write the estimate for $t\frac{\partial u}{\partial t}$, the one for $t\frac{\partial u}{\partial x}$ being essentially the same. We use the mean value property of harmonic functions to write $\frac{\partial u}{\partial t}(x,t)$ for $(x,t) \in \partial T_j^{k(\varepsilon)}$ as the average of $\frac{\partial u}{\partial t}$ over a ball $B_{x,t}$ centered at (x,t) with radius t/c where c is a geometric constant (say $c = 3$), big enough to guarantee that $B_{x,t}$ is contained in $\Gamma_2(z) = \{(y,s): |y-z| < 2s\}$ for some $z \notin E^{k+1}$ and, consequently, that $|u| \leqq P^*_{\nabla,2}(f)(z) \leqq 2^{k+1}$ on $B_{x,t}$. Then we use Green's theorem to write:

$$\int_{B_{x,t}} \frac{\partial u}{\partial t} = -\int_{\partial B_{x,t}} u dx .$$

We get:

$$|t\frac{\partial u}{\partial t}| = |\frac{c}{t}\int_{B_{x,t}} \frac{\partial u}{\partial t}| = |\frac{c}{t}\int_{\partial B_{x,t}} u dx| \leqq C2^k .$$

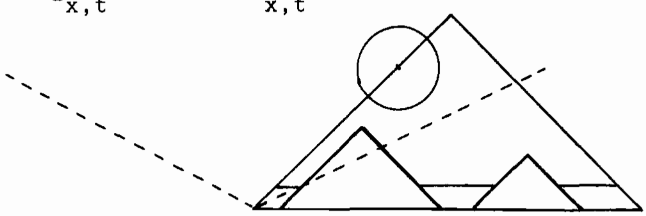

Figure III.3.2

It was essential that aperture 2 was used to define E^k and only aperture 1 to define the "tents" (see Figure III. 3.2).

Thus $g_j^{k(\varepsilon)} \in L^2$, is supported in $\overline{I_j(k)}$ and

$$\|g_j^{k(\varepsilon)}\|_2 \leqq C\, 2^k |I_j(k)|^{½} .$$

Actually, all that we did, can be done also for $\varepsilon = 0$, the only difference being that this time we cannot define

$$g_j^k(x) = \int_{T_j^k} \frac{\partial u}{\partial t}(y,t)\psi_t(x-y)dydt$$

because the integral may not converge absolutely. Instead, we define g_j^k as a distribution, its action on a function $\phi \in S$ being given by the formula:

$$\left\langle g_j^k, \phi\right\rangle = \int_{T_j^k} \frac{\partial u}{\partial t}(y,t)\int_{-\infty}^{\infty}\psi_t(x-y)\phi(x)dxdydt =$$

$$= \int_{T_j^k}\frac{\partial u}{\partial t}(y,t)(\psi_t * \phi)(y)dydt .$$

Then we proceed very much like before, obtaining that g_j^k is an L^2 function supported in $\overline{I_j(k)}$ with $\|g_j^k\|_2 \leqq C2^k|I_j(k)|^{1/2}$. A similar computation shows that $g_j^{k(\varepsilon)} \to g_j^k$ in L^2 as $\varepsilon \to 0$. Observe also that it follows from the cancellation properties of ψ, that the $g_j^{k(\varepsilon)}$ and g_j^k have vanishing moments up to order $[1/p]-1$ and, consequently, they belong to ReH^p. Now for a function $h \in L^2$, supported in an interval I and having vanishing moments up to order $[1/p]-1$, the function $a(x) = |I|^{(1/2)-(1/p)}\|h\|_2^{-1}h(x)$ is a $(p,2)$-atom. Thus, it follows from theorem 3.4. that

$$\|h\|_{\mathrm{ReH}^p} \leqq C|I|^{(1/p)-(1/2)}\|h\|_2$$

with C an absolute constant. By using this inequality we obtain $g_j^{k(\varepsilon)} \to g_j^k$ in ReH^p as $\varepsilon \to 0$.

We shall show below that $f = \sum_{k,j} g_j^k$ in S'. Once this is done, if we set

$$a_j^k(x) = C^{-1}2^{-k}|I_j(k)|^{-1/p}g_j^k(x)$$

then, each a_j^k is a $(p,2)$-atom, and we would then write

$$f = \sum_{k,j}\lambda_j^k a_j^k \quad \text{with} \quad \lambda_j^k = C2^k|I_j(k)|^{1/p}$$

This would be the desired decomposition, since

$$\sum_{k,j} |\lambda_j^k|^p = \sum_{k,j} C^p 2^{kp} |I_j(k)| =$$

$$= \sum_k C^p 2^{kp} \sum_j |I_j(k)| = C^p \sum_k 2^{kp} |E^k| =$$

$$= C^p 2 \sum_k 2^{k-1} 2^{k(p-1)} |\{x \in \mathbb{R} : P^*_{\nabla,2}(f)(x) > 2^k\}| \leq$$

$$\leq C \int_0^\infty \lambda^{p-1} |\{x \in \mathbb{R} : P^*_{\nabla,2}(f)(x) > \lambda\}| d\lambda = C \|P^*_{\nabla,2}(f)\|_p^p \leq$$

$$\leq C \|f\|^p_{\mathrm{Re}H^p} .$$

All that remains is to show that $f = \sum_{k,j} g_j^k$ in S'. We shall show first that $f = \lim_{\varepsilon \to 0} \sum_{k,j} g_j^{k(\varepsilon)}$ in S'. Indeed, for $\phi \in S$

$$\left\langle \sum_{k,j} g_j^{k(\varepsilon)}, \phi \right\rangle = \sum_{k,j} \left\langle g_j^{k(\varepsilon)}, \phi \right\rangle = \int_\varepsilon^\infty \int_{-\infty}^\infty \frac{\partial u}{\partial t}(y,t)(\psi_t * \phi)(y)\,dy\,dt =$$

$$=(\text{by Parseval's formula}) =$$

$$= \int_\varepsilon^\infty \int_{-\infty}^\infty (-2\pi|\xi|)e^{-2\pi|\xi|t}\hat{f}(\xi)\hat{\psi}(t|\xi|)\hat{\phi}(-\xi)\,d\xi\,dt =$$

$$= \int_{-\infty}^\infty \int_\varepsilon^\infty (-2\pi|\xi|t)e^{-2\pi|\xi|t}\hat{\psi}(t|\xi|)\frac{dt}{t}\hat{f}(\xi)\hat{\phi}(-\xi)\,d\xi =$$

$$= \int_{-\infty}^\infty \int_{\varepsilon|\xi|}^\infty (-2\pi s)e^{-2\pi s}\hat{\psi}(s)\frac{ds}{s}\hat{f}(\xi)\hat{\phi}(-\xi)\,d\xi \to \int_{-\infty}^\infty \hat{f}(\xi)\hat{\phi}(-\xi)\,d\xi \text{ as } \varepsilon \to 0$$

since $\int_{\varepsilon|\xi|}^\infty (-2\pi s)e^{-2\pi s}\hat{\psi}(s)\frac{ds}{s} \to 1$ boundedly as $\varepsilon \to 0$.

The interchanging of the order of integration is legitimate. Indeed, the first integral containing the Fourier transform signs is absolutely convergent as one verifies by taking absolute values and then interchanging the order of integration exactly as we did above. After all $\int_0^\infty e^{-2\pi s}|\hat{\psi}(s)|\,ds < \infty$. Thus, we have shown that

$$\left\langle \sum_{k,j} g_j^{k(\varepsilon)}, \phi \right\rangle \to \int_{-\infty}^{\infty} \hat{f}(\xi)\hat{\phi}(-\xi)d\xi = \int_{-\infty}^{\infty} f(x)\phi(x)dx \quad \text{as } \varepsilon \to 0.$$

In other words: $f = \lim_{\varepsilon\to 0} \sum_{k,j} g_j^{k(\varepsilon)}$ in S'. Unfortunately, we cannot just make $\varepsilon = 0$ in the above computation because we cannot guarantee the convergence of the series $\sum_{k,j} \langle g_j^k, \phi\rangle$ since the integral to its right is not necessarily absolutely convergent. But this is only a minor complication. It is clear that the series $\sum_{k,j} g_j^k$ converges normally in ReH^p. Indeed

$$\sum_{k,j} \|g_j^k\|_{\mathrm{ReH}^p}^p \leqq C\cdot \sum_{k,j} |I_j(k)|^{1-(p/2)} \|g_j^k\|_2^p \leqq$$

$$\leqq C \sum_{k,j} 2^{kp}|I_j(k)| \leqq C\|f\|_{\mathrm{ReH}^p}^p \quad .$$

Let us write $\sum_{k,j} g_j^k = g$ in ReH^p. By the same token $\sum_{k,j} g_j^{k(\varepsilon)}$ converges normally in ReH^p. Say $\sum_{k,j} g_j^{k(\varepsilon)} = g^{(\varepsilon)}$ in ReH^p. Now we just need to observe that $g^{(\varepsilon)} \to g$ in ReH^p as $\varepsilon \to 0$. Indeed

$$\|g - g^{(\varepsilon)}\|_{\mathrm{ReH}^p}^p \leqq \sum_{k,j} \|g_j^k - g_j^{k(\varepsilon)}\|_{\mathrm{ReH}^p}^p \to 0$$

as $\varepsilon \to 0$ because $g_j^{k(\varepsilon)} \to g_j^k$ in ReH^p (as $\varepsilon \to 0$), and

$$\|g_j^k - g_j^{k(\varepsilon)}\|_{\mathrm{ReH}^p}^p \lesssim 2C2^{kp}|I_j(k)|$$

which has a finite sum in k,j. Thus we have: $g^{(\varepsilon)} \to f$ in S' and $g^{(\varepsilon)} \to g$ in ReH^p. Since convergence in ReH^p implies convergence in S', it follows that $f = g$ as tempered distributions and, consequently $f = \sum_{k,j} g_j^k$ in S' as we wanted to show. $\square$

The proof we have just given is not what one would call a "real-variable" proof, since it relies upon the harmonicity of $u(x,t)$. Still, it is a nice direct proof of the inequality $N_{p,2}(f)^p \ll \|P_\nabla^*(f)\|_p^p$, without going through the "grand" maximal function $S_M^*(f)$. This inequality is enough for many purposes. For example, it leads immediately to the equivalence between the gauges $\|P_\nabla^*(f)\|_p^p$ and $\|P_\nabla^*(f + iHf)\|_p^p$ and, consequently, to the characterization of ReH^p contained in theorem 3.10. However, in order to give a complete alternative approach to atomic decompositions, we shall also give a direct proof of the inequality $N_{p,\infty}(f)^p \ll N_{p,2}(f)^p$. All we have to do is to show that every $(p,2)$-atom a can be written as $a = \sum \lambda_j a_j$ in $\mathrm{Re}H^p$ with the a_j's being (p,∞)-atoms and $\Sigma|\lambda_j|^p \leq C$, an absolute constant. This can be done quite simply by using the ordinary Calderón-Zygmund decomposition (not the smooth version of theorem 3.6). Here are the details:

3.12 <u>Decomposition of $(p,2)$-atoms into (p,∞)-atoms</u>
To make things as simple as possible, we start by assuming that $1/2 < p \leq 1$, so that $(p,2)$-atoms and (p,∞)-atoms need to have just one vanishing moment: the average.

Let a be a $(p,2)$-atom, and denote by I the smallest closed interval containing the support of a, so that $\|a\|_2 \leq |I|^{(1/2)-(1/p)}$. We shall consider the function $b(x) = |I|^{1/p}a(x)$, which has $\|b\|_2^2 \leq |I|$. Our basic operator will be M_2 defined as $M_2 f = M(|f|^2)^{1/2}$. For $\alpha > 0$, which will have to be taken appropriately big at the end of the proof, we look at the open set $U_\alpha = \{x \in \mathbb{R} : M_2 b(x) > \alpha\} = \{x \in \mathbb{R} : M(b^2)(x) > \alpha^2\}$. It has to be noted that for α big enough (say, $\alpha > \sqrt{2}$) is $U_\alpha \subset I^2$. Indeed, for $x \notin I^2$ is $M(b^2)(x) \leq \|b\|_2^2 / \mathrm{dist}(x,I) \leq 2$. Now, let $U_\alpha = \bigcup_j I_j$ be the decomposition of U_α into its connected components. We know that, for each j, is:

$$\alpha / \sqrt{2} \leq \left(\frac{1}{|I_j|}\int_{I_j} |b(x)|^2 dx\right)^{1/2} \leq \alpha$$

We write $b(x) = g_o(x) + \sum_j h_j(x)$, where:

$$g_o(x) = \begin{cases} b(x) & \text{if } x \notin U_\alpha \\ \dfrac{1}{|I_j|}\displaystyle\int_{I_j} b(y)dy & \text{if } x \in I_j \end{cases}$$

and $h_j(x) = b(x) - \dfrac{1}{|I_j|}\displaystyle\int_{I_j} b(y)dy$ if $x \in I_j$ and $h_j(x) = 0$ elsewhere. Clearly $|g_o(x)| \leq \alpha$ and

$$\frac{1}{|I_j|}\int_{I_j} |h_j(x)|dx \leq \left(\frac{1}{|I_j|}\int_{I_j} |h_j(x)|^2 dx\right)^{1/2} \leq 2\alpha$$

since $h_j(x) = (b(x) - g_o(x))\,\chi_{I_j}(x)$.

If we call $(2\alpha)^{-1}h_j(x) = b_j(x)$, we have a function b_j living in

I_j with $\|b_j\|_2^2 \leqq |I_j|$. The idea will be now to do for each b_j the same kind of decomposition that we have performed for b (with the same α!) and to build an induction process which will eventually lead to the decomposition of a into (p,∞)-atoms. We shall use multi-indices for the successive decompositions, in the following way:

$$b(x) = g_o(x) + \sum_{j_o} h_{j_o}(x) = g_o(x) + 2\alpha \sum_{j_o} b_{j_o}(x) =$$

$$= g_o(x) + 2\alpha \sum_{j_o} (g_{j_o}(x) + \sum_{j_1} h_{j_o,j_1}(x)) =$$

$$= g_o(x) + 2\alpha \sum_{j_o} g_{j_o}(x) + (2\alpha)^2 \sum_{j_o,j_1} b_{j_o,j_1}(x) = \quad \cdots\cdots \quad =$$

$$= g_o(x) + 2\alpha \sum_{j_o} g_{j_o}(x) + (2\alpha)^2 \sum_{j_o,j_1} g_{j_o,j_1}(x) + \ldots$$

$$+ (2\alpha)^n \sum_{j_o,j_1,\ldots,j_{n-1}} g_{j_o,j_1,\ldots,j_{n-1}}(x) +$$

$$+ (2\alpha)^n \sum_{j_o,j_1,\ldots,j_n} h_{j_o,j_1,\ldots,j_n}(x)$$

The function $h_{j_o,\ldots,j_n}$ lives in an interval $I_{j_o,\ldots,j_n}$. For $j_o,\ldots,j_{n-1}$ fixed, the intervals $I_{j_o,\ldots,j_{n-1},j_n}$ with varying j_n, are the connected components of the set

$$\{x \in \mathbb{R} : M(b^2_{j_o,\ldots,j_{n-1}})(x) > \alpha^2\}.$$

We have

$$\int_{-\infty}^{\infty} |h_{j_o,\ldots,j_n}(x)|dx \leq \Big(\frac{1}{|I_{j_o,\ldots,j_n}|}\int_{I_{j_o,\ldots,j_n}} |h_{j_o,\ldots,j_n}(x)|^2 dx\Big)^{1/2} \cdot |I_{j_o,\ldots,j_n}| \leq$$

$$\leqq (2\alpha)|I_{j_o,\ldots,j_n}|, \quad \text{and hence}$$

$$\int_{-\infty}^{\infty}(2\alpha)^n\Big|\sum_{j_o,\ldots,j_n} h_{j_o,\ldots,j_n}(x)\Big|dx =$$

$$= (2\alpha)^n \sum_{j_o,\ldots,j_n} \int_{-\infty}^{\infty}|h_{j_o,\ldots,j_n}(x)|dx \leqq$$

$$\leqq (2\alpha)^{n+1} \sum_{j_o,\ldots,j_n} |I_{j_o,\ldots,j_n}| =$$

$$= (2\alpha)^{n+1} \sum_{j_o,\ldots,j_{n-1}} |\{x \in \mathbb{R}: M(b^2_{j_o,\ldots,j_{n-1}})(x) > \alpha^2\}| \leqq$$

$$\leqq (2\alpha)^{n+1} C\alpha^{-2} \sum_{j_o,\ldots,j_{n-1}} \int_{-\infty}^{\infty}|b_{j_o,\ldots,j_{n-1}}(x)|^2 dx \leqq$$

$$\leqq (2\alpha)^{n+1} C\alpha^{-2} \sum_{j_o,\ldots,j_{n-1}} |I_{j_o,\ldots,j_{n-1}}| \leqq$$

$$\leqq (2\alpha)^{n+1}(C\alpha^{-2})^2 \sum_{j_o,\ldots,j_{n-2}} |I_{j_o,\ldots,j_{n-2}}| \leqq \; - - - \; \leqq$$

$$\leqq (2\alpha)^{n+1}(C\alpha^{-2})^n \sum_{j_o} |I_{j_o}| \leqq (2C\alpha^{-1})^{n+1}|I| \to 0 \quad \text{as} \quad n \to \infty$$

provided $\alpha > 2C$. Thus, when $\alpha > 2C$ we can write:

$$b(x) = \lim_{n\to\infty}\Big(g_o(x)+2\alpha\sum_{j_o} g_{j_o}(x)+\ldots+(2\alpha)^n\sum_{j_o,\ldots,j_{n-1}} g_{j_o,\ldots,j_{n-1}}(x)\Big)=$$

$$= g_o(x) + 2\alpha\sum_{j_o} g_{j_o}(x)+\ldots+(2\alpha)^n\sum_{j_o,\ldots,j_{n-1}} g_{j_o,\ldots,j_{n-1}}(x)+\ldots$$

where "lim" indicates convergence in L^1. We shall presently see that this series provides the desired atomic decomposition for $a(x)$. Observe that $|g_o(x)| \leqq \alpha$ and g_o lives in I^2 and has average 0, in such a way that $\alpha^{-1}(2|I|)^{-1/p}g_o(x) = a_o(x)$ is a (p,∞)-atom. Likewise, for every j_o, since $|g_{j_o}(x)| \leqq \alpha$ and g_{j_o} lives in $I^2_{j_o}$ and has average 0, the function $a_{j_o}(x) = \alpha^{-1}(2|I_{j_o}|)^{-1/p}\cdot g_{j_o}(x)$ will be a (p,∞)-atom. In general, for every n and every

$j_o,\dots,j_{n-1}$, the function

$$a_{j_o,\dots,j_{n-1}}(x) = \alpha^{-1}(2|I_{j_o,\dots,j_{n-1}}|)^{-1/p} g_{j_o,\dots,j_{n-1}}(x)$$

is a (p,∞)-atom. We can write:

$$a(x) = |I|^{-1/p} b(x) = |I|^{-1/p} 2^{1/p} \alpha \{ |I|^{1/p} a_o(x) +$$

$$+ 2\alpha \sum_{j_o} |I_{j_o}|^{1/p} a_{j_o}(x) +$$

$$+ (2\alpha)^2 \sum_{j_o,j_1} |I_{j_o,j_1}|^{1/p} a_{j_o,j_1}(x) + \dots +$$

$$+ (2\alpha)^n \sum_{j_o,\dots,j_{n-1}} |I_{j_o,\dots,j_{n-1}}|^{1/p} a_{j_o,\dots,j_{n-1}}(x) + \dots \}.$$

We have already seen that $\sum_{j_o,\dots,j_{n-1}} |I_{j_o,\dots,j_{n-1}}| \leq (C\alpha^{-2})^n |I|$.
Therefore, the sum of the p-th powers of the coefficients in the above decomposition will be:

$$2\alpha^p |I|^{-1} \{ |I| + (2\alpha)^p \sum_{j_o} |I_{j_o}| + (2\alpha)^{2p} \sum_{j_o,j_1} |I_{j_o,j_1}| + \dots +$$

$$+ (2\alpha)^{np} \sum_{j_o,\dots,j_{n-1}} |I_{j_o,\dots,j_{n-1}}| + \dots \} \leq$$

$$\leq 2\alpha^p \sum_{n=0}^{\infty} (2\alpha)^{np} (C\alpha^{-2})^n = 2\alpha^p \sum_{n=0}^{\infty} (2^p C \alpha^{p-2})^n = C < \infty$$

provided $\alpha > (2^p C)^{1/(2-p)}$. Actually the previous requirement on α, namely $\alpha > 2C$, is stronger than this one. This finishes the proof of the inequality $N_{p,\infty}(f)^p << N_{p,2}(f)^p$ for the case $1/2 < p \leq 1$.

In case $0 < p \leq 1/2$, we need to change the definition of the functions $g_o, g_{j_o}, \dots, g_{j_o,j_1,\dots,j_n}, \dots$. With the same notation used at the beginning of the proof, we have to define, for $x \in I_j$ $g_o(x) = P_{I_j}(b)(x)$, where $P_{I_j}(b)(x)$ is the unique polynomial of degree $\leq [1/p]-1$, such that for every $k = 0,1,\dots,[1/p]-1$ is:

$$\int_{I_j} b(x) x^k dx = \int_{I_j} P_{I_j}(b)(x) x^k dx.$$

Since $|P_{I_j}(b)(x)| \leq C\alpha$ for every $x \in I_j$ with C an absolute constant (the proof of this fact is contained in the proof of theorem 3.6), we still have with our new definition $|g_o(x)| \leq C\alpha$. Besides, now the h_j's, and consequently also g_o, have vanishing moments up to order $[1/p]-1$. We proceed in the same fashion with the definition of each $g_{j_o,\dots,j_{n-1}}$. That way we can guarantee that the corresponding $a_{j_o,\dots,j_{n-1}}$'s are indeed (p,∞)-atoms, because they have the right cancellation. The rest of the proof remains with only minor changes in the constants. $\square$

Naturally this proof can be suitably modified to obtain a direct proof of the inequality $N_{p,\infty}(f)^p << N_{p,r}(f)^p$ for any r such that $1 \leq r$ and $p < r$.

3.13. <u>Direct Proof of Burkholder -Gundy -Silverstein Theorem</u>:

We have seen how theorem 3.10 follows immediately from the inequality $\|P_\nabla^*(f + iHf)\|_p^p << \|P_\nabla^*(f)\|_p^p$ for, say, $f \in L^2$. There is a beautiful direct proof of this inequality due to P. Koosis. Even though it is a complex-variables argument, it is worth presenting an account of it here. By setting $u(x + it) = P_t * f(x)$ and $v(x + it) = P_t * (Hf)(x)$, we see that what we need to prove is that $\|m_v\|_p^p \leq C\|m_u\|_p^p$ for C an absolute constant. We shall prove this inequality for $0 < p \leq 1$. Call $F(z) = u(z) + iv(z)$, analytic in the upper half plane $\operatorname{Im} z > 0$. If we are able to prove the inequality

$$(3.14) \qquad \int_{-\infty}^{\infty} |F(x + it)|^p dx \leq C\|m_u\|_p^p$$

with C independent of $t > 0$, we can appeal to theorem 1.1 to conclude that $\|m_F\|_p^p \leq C\|m_u\|_p^p$ and, consequently, $\|m_v\|_p^p \leq C\|m_u\|_p^p$. Now to prove (3.14) it is enough to see that

$$(3.15) \qquad \int_{-\infty}^{\infty} |v(x + it)|^p dx \leq C\|m_u\|_p^p.$$

Calling $v_t(z) = v(z + it)$, $u_t(z) = u(z + it)$ and using the same names for the corresponding boundary functions, we shall see that

$$(3.16) \qquad \int_{-\infty}^{\infty} |v_t(x)|^p dx \leq C\int_{-\infty}^{\infty} |m_{u_t}(x)|^p dx$$

provided $\|m_u\|_p^p < \infty$.

This is clearly enough, since $m_{u_t}(x) \leqq m_u(x)$. To prove (3.16) for a fixed $t > 0$, we shall try to compare the distribution functions of v_t and m_{u_t}. For each $\lambda > 0$, we consider the open set $E_\lambda = \{x \in \mathbb{R} : m_{u_t}(x) > \lambda\}$. Observe that E_λ is bounded. Indeed, theorem 4.7 from chapter II implies that $|u_t(x+is)| \leqq \lambda$ in the complement of the triangle $T_N = \{x + is \in \mathbb{R}^2_+ : |x| < N - s\}$ if N is big enough, in such a way that E_λ is contained in the interval $[-N,N]$. Let $E_\lambda = \bigcup_j I_j$ be the decomposition of E_λ into its connected components I_j. Over each I_j consider the path $T_j = \{x + i\,\mathrm{dist}(x, \mathbb{R} \setminus I_j) : x \in I_j\}$ formed by the two legs of the isosceles right triangle whose other side is I_j. Denote by Γ the oriented path consisting of the points of $\mathbb{R} \setminus E_\lambda$ plus the points of $T = \bigcup_j T_j$ with the orientation corresponding to increasing x as shown in Figure III.3.3.

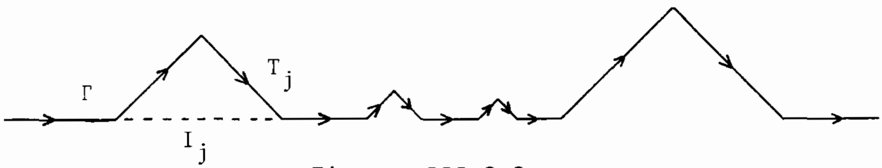

Figure III.3.3

We write

$$|\{x \in \mathbb{R} : |v_t(x)| > \lambda\}| \leqq |E_\lambda| + |\{x \in \mathbb{R} \setminus E_\lambda : |v_t(x)| > \lambda\}|$$

and try to estimate the second term in this sum. The function $(u_t(z) + iv_t(z))^2$ is in H^1. Therefore, as was pointed out in section 1:

$$\int_{-\infty}^{\infty} (u_t(x) + iv_t(x))^2 dx = 0$$

This, together with Cauchy's theorem, implies that

$$\int_\Gamma (u_t(z) + iv_t(z))^2 dz = 0$$

which, by taking real parts, leads to:

$$0 = \int_{\mathbb{R}\setminus E_\lambda} (u_t^2 - v_t^2)dx + \int_T (u_t^2 - v_t^2)dx - 2\int_T u_t v_t dy.$$

On T it is $dy = dx$ or $dy = -dx$, consequently

$$|2\int_T u_t v_t dy| \leqq \int_T 2|u_t||v_t|dx \leqq \int_T (u_t^2 + v_t^2)dx$$

Carrying this inequality to the previous identity we get

$$0 \leqq \int_{\mathbb{R}\setminus E_\lambda} (u_t^2 - v_t^2)dx + 2\int_T u_t^2 dx,$$

from which:

$$\int_{\mathbb{R}\setminus E_\lambda} v_t^2 dx \leqq \int_{\mathbb{R}\setminus E_\lambda} u_t^2 dx + 2\int_T u_t^2 dx \leqq \int_{\mathbb{R}\setminus E_\lambda} (m_{u_t})^2 dx + 2\lambda^2 |E_\lambda|$$

since $u_t(z) \leqq \lambda$ for every $z \in T$ and $\int_T dx = |E_\lambda|$. But

$$\int_{\mathbb{R}\setminus E_\lambda} (m_{u_t})^2 dx = \int_0^\lambda 2s|\{x \in \mathbb{R} : s < m_{u_t}(x) \leqq \lambda\}|ds =$$

$$= \int_0^\lambda 2s(|E_s| - |E_\lambda|)ds = \int_0^\lambda 2s|E_s|ds - \lambda^2|E_\lambda|$$

and therefore

$$\int_{\mathbb{R}\setminus E_\lambda} v_t^2 dx \leqq 2\int_0^\lambda s|E_s|ds + \lambda^2|E_\lambda|$$

which implies

$$|\{x \in \mathbb{R} : |v_t(x)| > \lambda\}| \leqq |E_\lambda| + |\{x \in \mathbb{R}\setminus E_\lambda : |v_t(x)| > \lambda\}| \leqq$$

$$\leqq |E_\lambda| + \lambda^{-2}\int_{\mathbb{R}\setminus E_\lambda} v_t^2 dx \leqq 2|E_\lambda| + 2\lambda^{-2}\int_0^\lambda s|E_s|ds.$$

Then

$$\int_{-\infty}^\infty |v_t(x)|^p dx = p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{R} : |v_t(x)| > \lambda\}|d\lambda \leqq$$

$$\leqq 2p\int_0^\infty \lambda^{p-1}|E_\lambda|d\lambda + 2p\int_0^\infty \lambda^{p-3}\int_0^\lambda s|E_s|ds\,d\lambda = C\|m_{u_t}\|_p^p$$

since

$$\int_0^\infty \lambda^{p-3}\int_0^\lambda s|E_s|ds\,d\lambda = \int_0^\infty \int_s^\infty \lambda^{p-3}d\lambda\, s|E_s|ds =$$

$$= (2-p)^{-1}\int_0^\infty s^{p-1}|E_s|ds = C\|m_{u_t}\|_p^p.$$

This finishes the proof. $\square$

4. H^p SPACES IN HIGHER DIMENSIONS

For $n > 1$, we turn theorem 3.10 into a definition of $H^p(\mathbb{R}^n)$.

DEFINITION 4.1. Let f *be a tempered distribution in* $\mathbb{R}^n$ *and let* $0 < p \leq 1$. *Then we say that* f *belongs to* $H^p(\mathbb{R}^n)$ *if and only if* f *is the boundary distribution corresponding to a real harmonic function* $u(x,t)$ *in* $\mathbb{R}^{n+1}_+$ *for which the maximal function* $m_u(x) = \sup_{|y-x|<t} |u(y,t)|$ *belongs to* $L^p(\mathbb{R}^n)$. *In that case, we define* $\|f\|_{H^p(\mathbb{R}^n)} = \|m_u\|_p$.

This makes sense since we know from theorem 4.23 in chapter II that f uniquely determines u. Clearly $H^p(\mathbb{R}^n)$ is a p-normed space with p-norm given by $f \mapsto \|f\|^p_{H^p(\mathbb{R}^n)}$.

For $n = 1$ the corresponding space was denoted by $\mathrm{Re}H^p$ in the preceding sections. From now on, we shall abandon this notation, using $H^p(\mathbb{R})$ instead, unless there is a risk of confusion.

If $f \in H^p(\mathbb{R}^n)$ and $u(x,t)$ is the unique harmonic function in $\mathbb{R}^{n+1}_+$ whose boundary distribution is f, theorem 4.7 of chapter II implies that:

$$|u(x,t)| \leq C\|f\|_{H^p(\mathbb{R}^n)} \cdot t^{-n/p}$$

with C an absolute constant. It follows from this inequality that $f \mapsto u$ is a continuous mapping from $H^p(\mathbb{R}^n)$ into the space $\mathrm{Harm}(\mathbb{R}^{n+1}_+)$ formed by the functions harmonic in $\mathbb{R}^{n+1}_+$ with the topology of uniform convergence over each compact subset of $\mathbb{R}^{n+1}_+$.

That $H^p(\mathbb{R}^n)$ is a complete space can be seen in the following way: Let (f_j) be a sequence of distributions in $H^p(\mathbb{R}^n)$ such that $\|f_j - f_k\|_{H^p(\mathbb{R}^n)} \to 0$ as $j,k \to \infty$. Denote by u_j the harmonic function corresponding to f_j. Then, the inequality above guarantees that the functions u_j converge uniformly over compact subsets of $\mathbb{R}^{n+1}_+$ to a certain u, which is, of course, necessarily harmonic. Actually, the inequality tells us that $u_j \to u$ uniformly over each proper sub-half-space $\{(x,t) \in \mathbb{R}^{n+1}_+ : t \geq t_0\}$, $t_0 > 0$. For $s > 0$ call $u_s(x,t) = u(x,t+s)$ and $u_{j,s}(x,t) = u_j(x,t+s)$. Then, for fixed $s > 0$ and $\varepsilon > 0$, we have $|u(y,t+s) - u_j(y,t+s)| \leq \varepsilon$ for every $(y,t) \in \mathbb{R}_+^{n+1}$ provided $j \geq$ certain j_o depending on ε and s. It follows, in particular, that $|m_{u_s}(x) - m_{u_{j,s}}(x)| \leq \varepsilon$ for every x, provided $j \geq j_o$. In other words: $m_{u_{j,s}}(x) \to m_{u_s}(x)$ as $j \to \infty$. By using Fatou's lemma we obtain:

$$\int_{\mathbb{R}^n}(m_{u_s}(x)^p dx \leq \sup_j \int_{\mathbb{R}^n}(m_{u_{j,s}}(x))^p dx \leq \sup_j \int_{\mathbb{R}^n}(m_{u_j}(x))^p dx =$$

$$= \sup_j \|f_j\|^p_{H^p(\mathbb{R}^n)} < \infty,$$

and, since $m_u(x) = \lim_{s\to 0} m_{u_s}(x)$, another application of Fatou's lemma yields

$$\int_{\mathbb{R}^n}(m_u(x))^p dx \leq \sup_j \|f_j\|^p_{H^p(\mathbb{R}^n)} < \infty.$$

Thus, u gives rise to a boundary distribution $f \in H^p(\mathbb{R}^n)$. To end the proof we must show that $f_j \to f$ in $H^p(\mathbb{R}^n)$ as $j \to \infty$. But

$$\|f_j - f\|^p_{H^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n}(m_{u_j - u}(x))^p dx$$

We proceed very much as before. For each $s > 0$, $m_{u_{j,s}-u_{k,s}}(x) \to$ $\to m_{u_{j,s}-u_s}(x)$ as $k \to \infty$ for j fixed. Thus

$$\int_{\mathbb{R}^n}(m_{u_{j,s}-u_s}(x))^p dx \leq \sup_{k>j_o} \|f_j - f_k\|^p_{H^p(\mathbb{R}^n)} \leq \varepsilon$$

provided $j \geq j_o$ big enough. Since this is true for every $s > 0$, we finally get:

$$\|f_j - f\|^p_{H^p(\mathbb{R}^n)} < \varepsilon \qquad (j \geq j_o)$$

The observation right after the proof of theorem 4.23 in chapter II implies that, for $f \in H^p(\mathbb{R}^n)$ and $\phi \in \mathcal{S}$, is

$$|\langle f, \phi \rangle| \leq C(\phi)\|f\|_{H^p(\mathbb{R}^n)}$$

where $C(\phi)$ is one of the semi-norms defining the topology of $\mathcal{S}$. This has as a consequence that the inclusion $H^p(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is continuous or, in other words, that convergence in $H^p(\mathbb{R}^n)$ implies convergence in the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions.

We shall give a characterization of $H^p(\mathbb{R}^n)$ in terms of atoms. This characterization will imply, in particular, that $H^1(\mathbb{R}^n)$ coincides with the space $H^1_{at}(\mathbb{R}^n)$ already introduced in chapter II, with equivalence of norms. First, let us extend the definition of atom to the case $n > 1$.

DEFINITION 4.2. For $0 < p \leq 1$ *and* r *such that* $p < r$ *and* $1 \leq r$, *a* (p,r)-*atom in* $\mathbb{R}^n$ *is going to be a real valued function* a, *with support contained in a cube* $Q \subset \mathbb{R}^n$ *and satisfying:*

i) a "size condition":

$$\left(\frac{1}{|Q|}\int_Q |a(x)|^r dx\right)^{1/r} \leq |Q|^{-1/p}$$

if $r < \infty$ *or* $\|a\|_\infty \leq |Q|^{-1/p}$ *if* $r = \infty$.

ii) a "cancellation condition":

$$\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$$

for every multi-index $\alpha = (\alpha_1,\ldots,\alpha_n)$ *formed by natural numbers* α_j *such that* $|\alpha| = \alpha_1+\ldots+\alpha_n \leq n(p^{-1}-1)$, *where* x^α *means* $x_1^{\alpha_1}\cdot x_2^{\alpha_2}\cdot\ldots\cdot x_n^{\alpha_n}$.

As we noted after the definition of atom for $n=1$, here also i) holds for every Q being a minimal cube containing the support of a.

The first thing will be to check that, for r fixed, (p,r)-atoms are uniformly in $H^p(\mathbb{R}^n)$. Since every (p,r)-atom a in $\mathbb{R}^n$ is the boundary distribution corresponding to the harmonic function $a(x,t) = P_t * a(x)$ for which $m_a(x) = P^*_\nabla(a)(x)$, what we need to prove is the following:

THEOREM 4.3. Let a *be a* (p,r)-*atom in* $\mathbb{R}^n$. *Then* $\|P^*_\nabla(a)\|_p^p \leq C$, *a constant independent of* a.

Proof: Let Q be a minimal cube containing the support of a. Denote by x_o the center of Q and set $N = [n(p^{-1}-1)]$, the biggest integer $\leq n(p^{-1}-1)$. The result will be a consequence of two facts:

1) $\displaystyle\int_{\tilde{Q}} |P^*_\nabla(a)(x)|^p dx \leq C$ (where $\tilde{Q} = Q^{2\sqrt{n}}$)

2) For every x such that $|x-x_o| > \operatorname{diam}(Q)$ is:

$$P_\nabla^*(a)(x) \leqq C|Q|^{(N+1)n^{-1}+1-p^{-1}} \cdot |x-x_o|^{-n-N-1}$$

Indeed, 2) implies that

$$\int_{\mathbb{R}^n \setminus \tilde{Q}} (P_\nabla^*(a)(x))^p dx \leqq \int_{|x-x_o|>\text{diam }(Q)} (P_\nabla^*(a)(x))^p dx \leqq$$

$$\leqq C(\text{diam }(Q))^{(N+1+n)p-n} \cdot \int_{\text{diam }(Q)}^\infty r^{-(N+1+n)p+n-1} dr = C < \infty$$

since $(N+1+n)p-n+1 > 1$, or, equivalently $N+1 > n(p^{-1}-1)$. Thus, everything reduces to proving 1) and 2).

1) is proved exactly as lemma 3.1 by using the fact that $P_\nabla^*(a)(x) \leqq CM(a)(x)$.

In order to prove 2) we shall see that for every $R > \text{diam }(Q)$ and for every (x,t) such that $|x-x_o| \geqq R-t$, we have:

$$(4.4) \qquad |P_t * a(x)| \leqq C|Q|^{(N+1)n^{-1}+1-p^{-1}} \cdot R^{-n-N-1}$$

with C an absolute constant.
This will be enough since $|y-x| < t$ implies

$$|y-x_o| = |x-x_o+y-x| \geqq |x-x_o|-|y-x| > |x-x_o|-t,$$

so that, once (4.4) is proved, we get, for $|x-x_o| > \text{diam }(Q)$ and any (y,t) such that $|y-x| < t$:

$$|P_t * a(y)| \leqq C|Q|^{(N+1)n^{-1}+1-p^{-1}} \cdot |x-x_o|^{-n-N-1}$$

Taking the sup. in (y,t) yields 2).

To prove (4.4) we take advantage of the cancellation condition ii) for the atom a, which allows us to subtract from $P_t(x-y)$ its Taylor polynomial of degree N around $x-x_o$ in the expression of $P_t * a(x)$. The computations are very similar to those carried out in the proof of theorem 3.4. Here are the details:

$$P_t * a(x) = \int_Q P_t(x-y)a(y)dy =$$

$$= \int_Q \{ P_t(x-y) - \sum_{|\alpha| \leq N} \frac{(-1)^{|\alpha|}}{\alpha!} D^\alpha(P_t)(x-x_o)\cdot(y-x_o)^\alpha \} a(y)dy =$$

$$= (-1)^{N+1} \int_Q \{ \sum_{|\alpha|=N+1} \frac{1}{\alpha!} D^\alpha(P_t)(x-x_o-\theta_y(y-x_o))\cdot(y-x_o)^\alpha \} a(y)dy$$

where, for a multi-index $\alpha = (\alpha_1,\ldots,\alpha_n)$, we have used the standard notation $\alpha! = (\alpha_1!)\cdot \ldots .(\alpha_n!)$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \quad \text{and where} \quad 0 \leq \theta_y \leq 1.$$

Now, since $P_t(x) = t^{-n}P(x/t)$, we have for $|\alpha| = N+1$:

$$D^\alpha(P_t)(x-x_o-\theta_y\cdot(y-x_o)) = t^{-n-N-1} D^\alpha P\left(\frac{x-x_o-\theta_y\cdot(y-x_o)}{t} \right)$$

From the formula

$$P(x) = c_n(1+|x|^2)^{-(n+1)/2} ,$$

we easily obtain:

$$|D^\alpha P(x)| \leq C_\alpha(1+|x|^2)^{-\frac{n+|\alpha|}{2}} \leq C'_\alpha(1+|x|)^{-n-|\alpha|}$$

Therefore, for $|\alpha| = N+1$, we get

$$|D^\alpha(P_t)(x-x_o-\theta_y\cdot(y-x_o))| \leq t^{-n-N-1}\cdot C'_\alpha \left(1+\left|\frac{x-x_o-\theta_y\cdot(y-x_o)}{t}\right|\right)^{-n-N-1}$$

$$= C(t+|x-x_o-\theta_y\cdot(y-x_o)|)^{-n-N-1} \leq CR^{-n-N-1}$$

since

$$t+|x-x_o-\theta_y\cdot(y-x_o)| \geq t+|x-x_o|-|y-x_o| \geq R-|y-x_o| \geq R/2$$

because $y \in Q$ implies that $|y-x_o| \leq \text{diam}(Q)/2 < R/2$.

Going back to the expression for $P_t * a(x)$, we conclude that:

$$|P_t * a(x)| \leq CR^{-n-N-1} \int_Q |y-x_o|^{N+1}|a(y)|dy \leq$$

$$\leq CR^{-n-N-1}(\text{diam }(Q)/2)^{N+1}\int_Q |a(y)|\,dy \leq$$

$$\leq CR^{-n-N-1}|Q|^{(N+1)n^{-1}+1}\left(\frac{1}{|Q|}\int_Q |a(y)|^r dy\right)^{1/r} \leq$$

$$\leq CR^{-n-N-1}|Q|^{(N+1)n^{-1}+1-p^{-1}}$$

as we wanted to show. $\square$

COROLLARY 4.5. Every (p,r)-*atom* a *in* $\mathbb{R}^n$ *is in* $H^p(\mathbb{R}^n)$ *with* $\|a\|^p_{H^p(\mathbb{R}^n)} \leq C$, *a constant depending only on* p,r *and* n, *but not on the particular atom.* $\square$

Next we obtain atomic decompositions (with $r = 2$) for nice functions in $H^p(\mathbb{R}^n)$.

THEOREM 4.6. Let $f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ *for some* p, $0 < p \leq 1$. *Then, there are:*

a) A sequence (a_j) *of* $(p,2)$-*atoms in* $\mathbb{R}^n$, *and*

b) A sequence (λ_j) *of real numbers with*

$$\sum_j |\lambda_j|^p \leq C\|f\|^p_{H^p(\mathbb{R}^n)}$$

C *being an absolute constant; such that*

$$f = \sum_j \lambda_j a_j \ ,$$

the convergence of the series holding in the space $S'(\mathbb{R}^n)$ *of tempered distributions.*

Proof: For $n = 1$ we have seen two ways to obtain atomic decompositions. First we gave a decomposition based upon the grand maximal function and then we presented a more direct approach based upon a non-tangential Poisson maximal function. Here we shall see how this second approach extends with non essential changes to the case $n > 1$. We shall consider the function $u(x,t) = P_t * f(x)$. This time our basic maximal function will be $P^*_{\nabla,N}(f)(x)$, where the number $N > 1$ will have to be chosen appropriately depending on the dimension n. In any case

$$(4.7)\quad \|P^*_{\nabla,N}(f)\|^p_p \leq C\|P^*_\nabla(f)\|^p_p = C\,\|f\|^p_{H^p(\mathbb{R}^n)}$$

where $C = C(n,N)$ is a constant that depends on n and N, but not on f. The proof of this inequality is essentially the same as that of lemma 2.3. The only difference is that, in dimension $n > 1$, we have to give an explicit way to decompose the open set $E_\lambda = \{x \in \mathbb{R}^n : P_\nabla^*(f)(x) > \lambda\}$ into non-overlapping cubes. We shall use Whitney's decomposition lemma (see 7. in chapter II) which tells us that any open set $\Omega \subsetneq \mathbb{R}^n$ can be written as $\Omega = \bigcup_j Q_j$ where (Q_j) is a sequence of non-overlapping cubes satisfying $\operatorname{diam}(Q_j) \leqq \operatorname{dist}(Q_j, \mathbb{R}^n \setminus \Omega) \leqq 4 \operatorname{diam}(Q_j)$. Doing this decomposition for $\Omega = E_\lambda$, we observe that, if we call $\tilde{Q}_j = \{(x,t) \in \mathbb{R}_+^{n+1} : x \in Q_j$ and $0 < t \leqq 6 \operatorname{diam}(Q_j)\}$ then $\{(x,t) \in \mathbb{R}_+^{n+1} : u(x,t) > \lambda\} \subset \bigcup_j \tilde{Q}_j$. Indeed, if $(x,t) \notin \bigcup_j \tilde{Q}_j$, we have either $x \notin \Omega$ or else $x \in Q_j$ and $t > 6 \operatorname{diam}(Q_j)$ for some j. In the first case, it is obvious that $u(x,t) \leqq P_\nabla^*(f)(x) \leqq \lambda$. In the second case, since $\operatorname{dist}(Q_j, \mathbb{R}^n \setminus E_\lambda) \leqq 4 \operatorname{diam}(Q_j)$, there will be some point $y \in \mathbb{R}^n \setminus E_\lambda$ such that $\operatorname{dist}(Q_j, y) < 5 \operatorname{diam}(Q_j)$. It follows that $|x-y| \leqq 6 \operatorname{diam}(Q_j) < t$. Thus $(x,t) \in \Gamma(y)$, and, consequently $u(x,t) \leqq P_\nabla^*(f)(y) \leqq \lambda$. Next we claim that, for an appropriate constant $M = M(n,N)$ we have

$$(4.8) \qquad \{x \in \mathbb{R}^n : P_{\nabla,N}^*(f)(x) > \lambda\} \subset \bigcup_j Q_j^M .$$

We just need to show that, by taking M big enough, we can guarantee that whenever $x \notin Q_j^M$, the cone $\Gamma_N(x)$ does not meet $\tilde{Q}_j$. This is quite easy: we just need to take $M > 12 N\sqrt{n} + 1$. Once we have proved (4.8), the proof of (4.7) goes on exactly as the proof of lemma 2.3.

With N still to be determined, let $E^k = \{x \in \mathbb{R}^n : P_{\nabla,N}^*(f)(x) > 2^k\}$ for each integer k. Use Whitney's lemma again to write $E^k = \bigcup_j Q_j(k)$ where the $Q_j(k)$'s are non-overlapping cubes such that:

$$\operatorname{diam}(Q_j(k)) \leqq \operatorname{dist}(Q_j(k), \mathbb{R}^n \setminus E^k) \leqq 4 \operatorname{diam}(Q_j(k)).$$

To each n-dimensional cube Q, associate the $(n+1)$-dimensional cube $\hat{Q} = \{(x,t) : x \in Q,\ 0 \leqq t \leqq |Q|^{1/n}\}$ and let

$$\widehat{E^k} = \bigcup_j \widehat{Q_j(k)} \qquad \text{and} \qquad T_j^k = \widehat{Q_j(k)} \setminus \widehat{E^{k+1}}$$

Also, for each $\varepsilon > 0$ call $T_j^{k(\varepsilon)} = \{(y,t) \in T_j^k : t \geqq \varepsilon\}$.

We shall use a function $\psi \in C^{\infty}(\mathbb{R}^n)$, real, radial, supported in $\{x \in \mathbb{R}^n : |x| \leq 1\}$, with moments vanishing up to order $[n((1/p)-1)]$ and satisfying, if $\hat{\psi}(\xi) = \phi(|\xi|)$:

$$\int_0^{\infty} e^{-2\pi s}\phi(s)ds = -1/(2\pi) \quad .$$

For each k,j and ε, we define the function:

$$g_j^{k(\varepsilon)}(x) = \int_{T_j^{k(\varepsilon)}} \frac{\partial u}{\partial t}(y,t)\psi_t(x-y)dydt \quad .$$

Since $\psi(x) = 0$ whenever $|x| \geq 1$, we see that the support of $g_j^{k(\varepsilon)}$ is contained in $Q_j(k)^3$. Now to obtain the estimate

$$\|g_j^{k(\varepsilon)}\|_2 \leq C\, 2^k|Q_j(k)|^{\frac{1}{2}}$$

we proceed exactly as in the one-dimensional case. We just need to get the estimate

$$\int_{T_j^{k(\varepsilon)}} t|\nabla u(y,t)|^2 dydt \leq C\, 2^{2k}|Q_j(k)|$$

and this will follow once we are able to show that, by choosing N appropriately, we can guarantee that every point (x,t) belonging to the boundary $\partial T_j^{k(\varepsilon)}$ of $T_j^{k(\varepsilon)}$ is the center of a ball of radius t/C which is totally contained in the cone $\Gamma_N(z)$ for some $z \in \mathbb{R}^n \setminus E^{k+1}$. The points $(x,t) \in \partial T_j^{k(\varepsilon)}$ are of two types: either $x \in E^k \setminus E^{k+1}$, in which case the required property obviously holds with $z = x$, or else $x \in Q_i(k+1)$ for some i. In this latter case, we claim that t is bigger or equal than one fifth the side length of $Q_i(k+1)$. The reason is that $Q_i(k+1)$ will be surrounded by other cubes belonging to the Whitney decomposition of E^{k+1}, all of them having diameter at least as big as one fifth the diameter of $Q_i(k+1)$. Indeed, if Q_1 and Q_2 are two cubes touching each other, both of them members of a Whitney decomposition of an open set Ω, we must have:

$$\operatorname{diam}(Q_1) \leq \operatorname{dist}(Q_1,\mathbb{R}^n \setminus \Omega) \leq \operatorname{diam}(Q_2) + \operatorname{dist}(Q_2,\mathbb{R}^n\setminus\Omega) \leq$$

$$\leq 5 \operatorname{diam}(Q_2).$$

Let us see how N can be chosen to guarantee that whenever

$x \in Q_i(k+1)$ and $t \geqq \frac{1}{5}$ side $(Q_i(k+1))$, then $(x,t) \in \Gamma_N(z)$ for some $z \in \mathbb{R}^n \setminus E^{k+1}$. Since dist $(Q_i(k+1), \mathbb{R}^n \setminus E^{k+1}) \leqq$ $\leqq 4$ diam $(Q_i(k+1))$, we can surely find $z \in \mathbb{R}^n \setminus E^{k+1}$ such that dist$(z, Q_i(k+1)) \leqq 5$ diam $(Q_i(k+1))$. Then, for any $x \in Q_i(k+1)$, we shall have $|z-x| \leqq 6$ diam $(Q_i(k+1))$. Thus, for (x,t) as above we shall always have $(x,t) \in \Gamma_N(z)$ provided $6\ \mathrm{diam}(Q_i(k+1)) <$ $< (N/5)$ side $(Q_i(k+1))$. This is achieved by making $N > 30\sqrt{n}$. By taking N even bigger, say $N = 30n$, we make sure that there exists a ball centered in (x,t) with radius t/C, which is totally contained in $\Gamma_N(z)$.

Thus, with the choice $N = 30n$, we have shown that $g_j^{k(\varepsilon)} \in L^2$, is supported in $Q_j(k)^3$ and satisfies

$$\|g_j^{k(\varepsilon)}\|_2 \leqq C\, 2^k |Q_j(k)|^{1/2}$$

Now, exactly as in the one-dimensional case, we define g_j^k as a distribution, its action on a function $\phi \in S(\mathbb{R}^n)$ being given by the formula:

$$\left\langle g_j^k, \phi \right\rangle = \int_{T_j^k} \frac{\partial u}{\partial t}(y,t)(\psi_t * \phi)(y)\,dy\,dt \quad .$$

Then we see, just like before, that g_j^k is an L^2 function supported in $Q_j(k)^3$ and satisfying the same estimate as each $g_j^{k(\varepsilon)}$, namely:

$$\|g_j^k\|_2 \leqq C\, 2^k |Q_j(k)|^{1/2}$$

A similar computation shows that $g_j^{k(\varepsilon)} \to g_j^k$ in L^2 as $\varepsilon \to 0$. The functions $g_j^{k(\varepsilon)}$ and g_j^k inherit the cancellation properties of ψ. Therefore, they belong to $H^p(\mathbb{R}^n)$ by corollary 4.5. Now, for a function $h \in L^2(\mathbb{R}^n)$, supported in a cube Q and having vanishing moments up to order $[n((1/p)-1)]$, the function $a = |Q|^{(1/2)-(1/p)}\|h\|_2^{-1}h$ is a $(p,2)$-atom. Thus, it follows from corollary 4.5 that

$$\|h\|_{H^p(\mathbb{R}^n)} \leqq C|Q|^{(1/p)-(1/2)}\|h\|_2 \quad .$$

That way we see that $g_j^{k(\varepsilon)} \to g_j^k$ in $H^p(\mathbb{R}^n)$ as $\varepsilon \to 0$.

If we are able to show that $f = \sum_{k,j} g_j^k$ in $S'(\mathbb{R}^n)$, then, setting $a_j^k = C^{-1}2^{-k}3^{n/2}|Q_j(k)^3|^{-1/p} g_j^k$ we shall have $\|a_j^k\|_2 \leq |Q_j(k)^3|^{(1/2)-(1/p)}$, so that a_j^k is a $(p,2)$-atom. We would then write

$$f = \sum_{k,j} \lambda_j^k a_j^k \quad \text{with} \quad \lambda_j^k = C\, 2^k |Q_j(k)|^{1/p} \quad ,$$

and this would be the desired decomposition, since

$$\sum_{k,j} |\lambda_j^k|^p = \sum_{k,j} C^p 2^{kp} |Q_j(k)| =$$

$$= \sum_k C^p 2^{kp} |\{x \in \mathbb{R}^n : P_{\nabla,N}^*(f)(x) > 2^k\}| \leq$$

$$\leq C \|P_{\nabla,N}^*(f)\|_p^p \leq C \|f\|_{H^p(\mathbb{R}^n)}^p$$

All that remains is to show that $f = \sum_{k,j} g_j^k$ in $S'(\mathbb{R}^n)$. First one shows that $f = \lim_{\varepsilon \to 0} \sum_{k,j} g_j^{k(\varepsilon)}$ in $S'(\mathbb{R}^n)$. The proof is formally the same as in dimension 1. Then we have $\sum_{k,j} g_j^{k(\varepsilon)} \to \sum_{k,j} g_j^k$ as $\varepsilon \to 0$ in $H^p(\mathbb{R}^n)$. But, as was pointed out at the beginning of the section, convergence in $H^p(\mathbb{R}^n)$ implies convergence in $S'(\mathbb{R}^n)$. Thus $f = \sum_{k,j} g_j^k$ in $S'(\mathbb{R}^n)$ and the proof is finished. $\square$

COROLLARY 4.9. Let $f \in H^p(\mathbb{R}^n)$, $0 < p \leq 1$. *Then, there is a sequence of* $(p,2)$-*atoms* (a_j) *in* $\mathbb{R}^n$ *and a sequence of real numbers* (λ_j) *with*

$$\sum_j |\lambda_j|^p \leq C \|f\|_{H^p(\mathbb{R}^n)}^p$$

for C *an absolute constant, such that* $f = \sum_j \lambda_j a_j$ *in the sense of tempered distributions.*

Proof: If f is the boundary distribution corresponding to the harmonic function $u(x,t)$ in $\mathbb{R}_+^{n+1}$ with $m_u \in L^p(\mathbb{R}^n)$, we know that for each $t > 0$, the function $x \longmapsto u(x,t) = u_t(x)$ belongs to $L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, and also to $H^p(\mathbb{R}^n)$. Thus, we can produce a sequence f_j of functions in $H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that

$$\|f_j\|_{H^p(\mathbb{R}^n)} \leq \|f\|_{H^p(\mathbb{R}^n)} \quad ,$$

$$\|f_j - f_{j+1}\|_{H^p(\mathbb{R}^n)}^p \leq 2^{-j} \|f\|_{H^p(\mathbb{R}^n)}^p \quad \text{and} \quad f_j \to f \quad \text{in} \quad S'(\mathbb{R}^n).$$

We write $f = f_1 + \sum_{j=1}^{\infty}(f_{j+1}-f_j)$ and use theorem 4.6 for each of the functions f_1, $f_{j+1}-f_j$. The proof is finished exactly as that of theorem 3.8 part a). $\square$

We actually have the following general result:

THEOREM 4.10. Let $0 < p \leq 1$, $p < r$, $1 \leq r \leq \infty$. *Then*
a) For every $f \in H^p(\mathbb{R}^n)$ *there is a sequence of* (p,r)-*atoms* a_j *in* $\mathbb{R}^n$ *and a sequence of real numbers* λ_j *with*

$$\sum_j |\lambda_j|^p \leq C\|f\|^p_{H^p(\mathbb{R}^n)}$$

for C *an absolute constant, such that* $f = \sum_j \lambda_j a_j$ *in the sense of tempered distributions.*
b) Conversely if $f \in S'(\mathbb{R}^n)$ *is such that* $f = \sum_j \lambda_j a_j$ *in the sense of tempered distributions, with* $\sum_j |\lambda_j|^p < \infty$ *and the* a_j*'s being* (p,r)-*atoms, then* $f \in H^p(\mathbb{R}^n)$ *and*

$$\|f\|^p_{H^p(\mathbb{R}^n)} \leq C \sum_j |\lambda_j|^p$$

for C *an absolute constant.*

Proof: Let us prove b) first. We have (p,r)-atoms a_j and real numbers λ_j such that $\sum_j |\lambda_j|^p < \infty$. By corollary 4.5, these conditions guarantee that the series $\sum_j \lambda_j a_j$ converges normally in $H^p(\mathbb{R}^n)$, say to $g \in H^p(\mathbb{R}^n)$. On the other hand, since $f = \sum_j \lambda_j a_j$ in $S'(\mathbb{R}^n)$ and we know that convergence in $H^p(\mathbb{R}^n)$ implies convergence in $S'(\mathbb{R}^n)$; it follows that $f = g \in H^p(\mathbb{R}^n)$ and $\|f\|^p_{H^p(\mathbb{R}^n)} \leq C \sum_j |\lambda_j|^p$.

a) will follow for every r once we prove it for $r = \infty$. Up to now we only have it for $r = 2$. We just need to see that every (p,2)-atom a in $\mathbb{R}^n$ can be written as $a = \sum_j \lambda_j a_j$ in $S'(\mathbb{R}^n)$ with the a_j's being (p,∞)-atoms and $\sum_j |\lambda_j|^p \leq C$, some absolute constant. This has been done already for $n = 1$ in the previous section. But the proof given there can easily be adapted to higher dimensions. We decompose the function $b(x) = |Q|^{1/p}a(x)$, where Q is a minimal cube containing the support of a, by using the Calderón-Zygmund cubes Q_j for b^2 at height α^2. For each of them we shall have:

$$\alpha^2 < \frac{1}{|Q_j|}\int_{Q_j}|b(x)|^2dx \leq 2^n\alpha^2.$$

We write $b(x) = g_o(x) + \sum_j h_j(x)$, where:

$$g_o(x) = \begin{cases} b(x) & \text{if } x \notin \bigcup_j Q_j \\ P_{Q_j}(b)(x) & \text{if } x \in Q_j \end{cases}$$

$P_{Q_j}(b)$ being the unique polynomial of degree $\leq [n((1/p)-1)]$ such that

$$\int_{Q_j} b(x)x^\beta dx = \int_{Q_j} P_{Q_j}(b)(x)x^\beta dx$$

for every multi-index $\beta = (\beta_1, \ldots, \beta_n)$ with $|\beta| \leq n((1/p)-1)$; and $h_j(x) = b(x) - P_{Q_j}(b)(x)$ if $x \in Q_j$ and $h_j(x) = 0$ elsewhere. Then $|g_o(x)| \leq C\alpha$ and

$$\left(\frac{1}{|Q_j|} \int_{Q_j} |h_j(x)|^2 dx\right)^{1/2} \leq C\alpha .$$

We call $(C\alpha)^{-1}h_j(x) = b_j(x)$, so that $\|b_j\|_2^2 \leq |Q_j|$ and go on by decomposing b_j with the same α^2.

The two basic facts are:

i) Each Q_j is a subcube of Q, since α is at least ≥ 1

ii) On each Q_j is $M(b^2)(x) > \alpha^2$, so that

$$\sum_j |Q_j| \leq |\{x \in \mathbb{R}^n : M(b^2)(x) > \alpha^2\}| \leq$$

$$\leq \frac{C}{\alpha^2} \int_{\mathbb{R}^n} |b(x)|^2 dx \leq \frac{C}{\alpha^2} |Q|$$

After this, the proof is practically identical to the one given for dimension 1. $\square$

We know that $H^1(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : Hf \in L^1(\mathbb{R})\}$. It is time to extend this characterization to $\mathbb{R}^n$ with $n > 1$. Of course the role of the Hilbert transform H will be played now by the system formed by the n Riesz transforms $R_1, \ldots, R_n$, defined in chapter II, section 5. We shall eventually prove that:

(4.11) $$H^1(\mathbb{R}^n) = \{ f \in L^1(\mathbb{R}^n) : R_j f \in L^1(\mathbb{R}^n) \text{ for every } j=1,2,..,n\}$$

That $f \in H^1(\mathbb{R}^n) \Rightarrow f \in L^1(\mathbb{R}^n)$ and $R_jf \in L^1(\mathbb{R}^n)$ for every $j = 1,2,\dots,n$ follows from the atomic decomposition and corollary 5.14 in chapter II. To establish the converse will require some work.

Suppose $f \in L^1(\mathbb{R}^n)$ is such that $R_jf \in L^1(\mathbb{R}^n)$ for every $j = 1,2,\dots,n$. With f we associate the following $n+1$ harmonic functions in $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n,\ t > 0\}$: $u_1(x,t) = P_t * R_1f(x),\dots,u_n(x,t) = P_t * R_nf(x)$, $u_{n+1}(x,t) = u(x,t) = P_t * f(x)$. These $n+1$ harmonic functions satisfy the conditions:

a) $\dfrac{\partial u_j}{\partial x_k} = \dfrac{\partial u_k}{\partial x_j}$, $\quad j,k \in \{1,\dots,n\}$, $\quad j \neq k$

b) $\dfrac{\partial u}{\partial x_j} = \dfrac{\partial u_j}{\partial t}$, $\quad j = 1,\dots,n$

c) $\displaystyle\sum_{j=1}^{n} \frac{\partial u_j}{\partial x_j} + \frac{\partial u}{\partial t} = 0$

as one easily sees by using Fourier transforms, taking into account that $(R_jf)\hat{}(\xi) = -i(\xi_j/|\xi|)\hat{f}(\xi)$. If one writes x_{n+1} instead of t, the equations a) b) and c) can be written as:

$$\frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j} \quad ; \quad j,k \in \{1,\dots,n+1\}, \quad j \neq k \tag{4.12}$$

$$\sum_{j=1}^{n+1} \frac{\partial u_j}{\partial x_j} = 0 \tag{4.13}$$

or, in terms of the vector field $F = (u_1,\dots,u_n,u_{n+1})$:

curl $F = 0$; div $\mathbf{F} = 0$.

These are the so called generalized Cauchy-Riemann equations for the vector field F. A C^2 vector field $F = (u_1,\dots,u_{n+1})$ satisfying (4.12) and (4.13) is called a conjugate system of harmonic functions. Observe that the harmonicity of each u_k follows from (4.12) and (4.13) since

$$\Delta u_k = \sum_{j=1}^{n+1} \frac{\partial^2 u_k}{\partial x_j^2} = \sum_{j=1}^{n+1} \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_k}\right) = \frac{\partial}{\partial x_k}(\text{div } F) = 0$$

The terminology comes from the case $n = 1$. In that case equations (4.12) and (4.13) for $F(x,y) = (v(x,y),u(x,y))$ reduce to $v_y = u_x$ and $v_x = -u_y$ which are the Cauchy-Riemann equations, necessary and sufficient for $u + iv$ to be an analytic function of $x + iy$.

In general, for a C^1 vector field $F = (u_1,\dots,u_{n+1})$ to satisfy (4.12) and (4.13) it is necessary and sufficient that F is the gradient of some harmonic function h. Indeed (4.12) implies that F is the gradient of the function

$$h(x_1,x_2,\dots,x_{n+1}) = \int_1^{x_1} u_1(s,x_2,\dots,x_{n+1})ds + \int_1^{x_2} u_2(1,s,x_3,\dots,x_{n+1})ds +$$
$$+ \dots + \int_1^{x_{n+1}} u_{n+1}(1,1,\dots,1,s)ds.$$

Then (4.13) implies that h is harmonic.

The crucial fact is the following:

LEMMA 4.14. Let $F = (u_1,u_2,\dots,u_{n+1})$ *be a system of conjugate harmonic functions in* $\mathbb{R}^{n+1}_+$ *and call*

$$|F| = \Big(\sum_{j=1}^{n+1} |u_j|^2\Big)^{1/2}$$

Then for every $\varepsilon \geq (n-1)/n$, $|F|^\varepsilon$ *is a subharmonic function in* $\mathbb{R}^{n+1}_+$.

Proof: We just need to examine the case $(n-1)/n \leq \varepsilon \leq 1$. It will be enough to prove that $\Delta(|F|^\varepsilon) \geq 0$ in the set where $F(x,t) \neq 0$ (see the observation immediately before theorem 2.11 in chapter I). We shall use the notation $\langle\ |\ \rangle$ for the scalar product in $\mathbb{R}^{n+1}$. Then, writing $|F|^\varepsilon = \langle F|F\rangle^{\varepsilon/2}$ we get

$$\frac{\partial}{\partial x_j}(|F|^\varepsilon) = \varepsilon|F|^{\varepsilon-2} \left\langle F\Big|\frac{\partial F}{\partial x_j}\right\rangle$$

and

$$\frac{\partial^2}{\partial x_j^2}(|F|^\varepsilon) = \varepsilon|F|^{\varepsilon-2}\left(\left\langle F\Big|\frac{\partial^2 F}{\partial x_j^2}\right\rangle + \left\langle\frac{\partial F}{\partial x_j}\Big|\frac{\partial F}{\partial x_j}\right\rangle\right) +$$
$$+ \varepsilon(\varepsilon-2)|F|^{\varepsilon-4} \left\langle F\Big|\frac{\partial F}{\partial x_j}\right\rangle^2 .$$

Thus, adding in $j = 1,\dots,n+1$:

$$\Delta(|F|^\varepsilon) = \varepsilon|F|^{\varepsilon-4}\left(|F|^2 \sum_{j=1}^{n+1} \left|\frac{\partial F}{\partial x_j}\right|^2 + (\varepsilon-2)\sum_{j=1}^{n+1} \left\langle F\Big|\frac{\partial F}{\partial x_j}\right\rangle^2\right)$$

We need to see that

$$(4.15) \qquad \sum_{j=1}^{n+1} \langle F | \frac{\partial F}{\partial x_j} \rangle^2 \leqq \frac{1}{2-\varepsilon} |F|^2 \sum_{j=1}^{n+1} |\frac{\partial F}{\partial x_j}|^2$$

But $\varepsilon \geqq (n-1)/n$ is equivalent to $1/(2-\varepsilon) \geqq n/(n+1)$, so (4.15) will be established as soon as we see that

$$(4.16) \qquad \sum_{j=1}^{n+1} \langle F | \frac{\partial F}{\partial x_j} \rangle^2 \leqq \frac{n}{n+1} |F|^2 \sum_{j=1}^{n+1} |\frac{\partial F}{\partial x_j}|^2$$

This will be seen by observing that for any $(n+1)$ - dimensional vector v with euclidean norm $|v| \leqq 1$

$$(4.17) \qquad \sum_{j=1}^{n+1} \langle v | \frac{\partial F}{\partial x_j} \rangle^2 \leqq \frac{n}{n+1} \sum_{j=1}^{n+1} |\frac{\partial F}{\partial x_j}|^2$$

To prove (4.17) denote by A the $(n+1) \times (n+1)$ matrix with entries $a_{j,k} = \partial u_k/\partial x_j$. Then the left hand side in (4.17) is simply $|Av|^2$, the square of the euclidean norm of the vector Av; and the sum in the right hand side equals $\Sigma_{j,k} |a_{j,k}|^2 = |||A|||^2$, the square of the Hilbert-Schmidt norm $|||A|||$ of the matrix A. Thus, if we write $||A|| = \sup \{ |Av| ; \ |v| \leqq 1 \}$, the norm of A as a linear operator in the euclidean space $\mathbb{R}^{n+1}$, what we need to prove is:

$$(4.18) \qquad ||A||^2 \leqq \frac{n}{n+1} |||A|||^2 \quad .$$

Note that (4.12) and (4.13) imply that our matrix A is symmetric and has trace equal to 0. We shall conclude by showing that any matrix A with these two properties satisfies (4.18). Since the two norms appearing in (4.18) are invariant under multiplication by an orthogonal matrix, we may assume that A is a diagonal matrix. Denote by $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$ the eigenvalues of A. By the invariance of the trace, we must have $\sum_{j=1}^{n+1} \lambda_j = 0$. Now, clearly $||A|| = \max_j |\lambda_j|$, whereas

$$|||A|||^2 = \sum_{j=1}^{n+1} |\lambda_j|^2 .$$

Let $\max_j |\lambda_j| = |\lambda_{j_o}|$. Then

$$|\lambda_{j_o}|^2 = |-\sum_{j \neq j_o} \lambda_j|^2 \leqq \Big(\sum_{j \neq j_o} |\lambda_j| \Big)^2 \leqq n \sum_{n \neq j_o} |\lambda_j|^2 .$$

By adding $n|\lambda_{j_o}|^2$ to both sides we get

$$(n+1)|\lambda_{j_o}|^2 \leqq n \sum_{j=1}^{n+1} |\lambda_j|^2 \quad ,$$

which readily gives (4.18). $\square$

Let us go back to our problem. We have $f \in L^1(\mathbb{R}^n)$ such that $R_j f \in L^1(\mathbb{R}^n)$ for every $j = 1,2,\dots,n$ and we want to prove that $f \in H^1(\mathbb{R}^n)$. We have associated with f the harmonic functions in $\mathbb{R}^{n+1}_+$: $u_1(x,t) = P_t * R_1 f(x),\dots,u_n(x,t) = P_t * R_n f(x)$, $u_{n+1}(x,t) = u(x,t) = P_t * f(x)$. These functions form a conjugate system $F = (u_1,\dots,u_{n+1})$. Since the functions $x \longmapsto u(x,t)$ converge to f in $L^1(\mathbb{R}^n)$ as $t \to 0$, to see that $f \in H^1(\mathbb{R}^n)$ it will be enough to show that the function $m_u(x) = \sup_{|y-x|<t} |u(y,t)|$ belongs to $L^1(\mathbb{R}^n)$. By using the fact, established in lemma 4.14, that $|F|^\varepsilon$ is subharmonic for some $0 < \varepsilon < 1$, we shall be able to show that, actually, the function $m_F(x) = \sup_{|y-x|<t} |F(y,t)|$ belongs to $L^1(\mathbb{R}^n)$. Since $m_u(x) \leqq m_F(x)$, this will be the end of the proof that $f \in H^1(\mathbb{R}^n)$. Now to see that m_F is integrable, fix ε such that $(n-1)/n \leqq \varepsilon < 1$. The key point is that $\varepsilon < 1$ but still $|F|^\varepsilon$ is subharmonic. Since each u_j is the Poisson integral of an integrable function, we have, for every $t > 0$

$$\int_{\mathbb{R}^n} |F(x,t)|\,dx \leqq C$$

with C a constant independent of t.

This implies that $v(x,t) = |F(x,t)|^\varepsilon$ is a non-negative subharmonic function in $\mathbb{R}^{n+1}_+$ which is uniformly in $L^p(\mathbb{R}^n)$ with $p = 1/\varepsilon > 1$. Thus, according to theorem 4.10 in chapter II, there is $h \in L^p(\mathbb{R}^n)$ such that $v(x,t) \leqq P_t * h(x)$ or, in other words $|F(x,t)| \leqq (P_t * h(x))^p$. Then $m_F(x) \leqq (P^*_\nabla(h)(x))^p \leqq C(M(h)(x))^p$ and this is an integrable function since $M(h) \in L^p(\mathbb{R}^n)$.

We can extend this treatment of Hardy spaces to $p < 1$. Suppose F is a system of conjugate harmonic functions in $\mathbb{R}^{n+1}_+$ such that

$$(4.19) \qquad \sup_{t>0} \int_{\mathbb{R}^n} |F(x,t)|^p\,dx < \infty$$

for some p with $(n-1)/n < p < 1$. Then taking $(n-1)/n < \varepsilon < p$ we have a non-negative subharmonic function $v(x,t) = |F(x,t)|^\varepsilon$ which is uniformly in $L^q(\mathbb{R}^n)$ where $q = p/\varepsilon > 1$. Proceeding as before we conclude that $m_F \in L^p(\mathbb{R}^n)$. Thus $u = u_{n+1}$, or any other component of F, gives rise to a distribution $f \in H^p(\mathbb{R}^n)$. Conversely, if $f \in H^p(\mathbb{R}^n)$. we can associate to f a conjugate system F of harmonic functions in $\mathbb{R}^{n+1}$ such that (4.19) holds and f is the boundary distribution corresponding to u_{n+1}, the last component of f. This takes us back to function theory. Actually this was the way in which the space $H^p(\mathbb{R}^n)$ was first introduced by Stein and Weiss [1] as a natural extension of the case $n = 1$. This approach works only for $(n-1)/n < p \leq 1$. When p gets smaller one needs to consider more and more complicated notions of conjugate systems (higher order gradients).

5. DUAL SPACES

In general if E is a p-normed space $(0 < p \leq 1)$ with p-norm N and L is a linear functional on E, it is clear that the continuity of L is equivalent to the existence of a constant C such that for every $x \in E$ $|L(x)| \leq C(N(x))^{1/p}$. The linear space formed by all the continuous linear functionals on E is called the dual space of E and shall be denoted by E^*. For $L \in E^*$, we shall call $\|L\|$ to the infimum of all the constants C in the above inequality or, equivalently

$$\|L\| = \sup\{|L(x)| : x \in E \text{ and } N(x) \leq 1\}.$$

The mapping $L \longmapsto \|L\|$ is a norm in E^*. Besides, it is easy to see that E^* is also complete, i.e., a Banach space.

In this section our aim is to characterize the dual spaces $(H^p(\mathbb{R}^n))^*$, $0 < p \leq 1$, $n \geq 1$. The atomic decomposition for $H^p(\mathbb{R}^n)$ will allow us to settle this question in a rather simple way.

Suppose $L \in (H^p(\mathbb{R}^n))^*$. Then, for every $f \in H^p(\mathbb{R}^n)$, $|L(f)| \leq \leq \|L\| \, \|f\|_{H^p(\mathbb{R}^n)}$. In particular, if $1 \leq r \leq \infty$ and $p < r$, we shall have, for every (p,r)-atom a: $|L(a)| \leq C\|L\|$ with C depending on p,r,n but not on the particular atom. For $N = = 0,1,2,\dots$ and Q a cube, $L^r_N(Q)$ will be the space consisting of those L^r functions supported in Q and having vanishing moments up to order N. We shall write $N_p(n) = [n((1/p) - 1)]$, the biggest integer $\leq n((1/p) - 1)$. When the dimension is kept fixed, we shall drop the n and write simply N_p. Given $f \in L^r_{N_p}(Q)$, it is clear that the function

$$a(x) = |Q|^{(1/r)-(1/p)}\|f\|_r^{-1}f(x)$$

is a (p,r)-atom. Therefore

$$|L(f)| \le C\|L\|\ |Q|^{(1/p)-(1/r)}\|f\|_r \ .$$

We have shown that our $L \in (H^p(\mathbb{R}^n))^*$ provides a continuous linear functional on $L^r_{N_p}(Q)$ with norm bounded by $C\|L\|\ |Q|^{(1/p)-(1/r)}$. Now the Hahn-Banach theorem allows us to extend this functional to the whole space $L^r(Q)$ without increasing the norm. Suppose, momentarily, that $r < \infty$. Then we know that every continuous linear functional on $L^r(Q)$ is given by a function belonging to $L^{r'}(Q)$ with norm dominated by the norm of the functional. In our case, we conclude that for every cube $Q \subset \mathbb{R}^n$, there is a function $g \in L^{r'}(Q)$ with:

$$\|g\|_{r'} \le C\|L\|\ |Q|^{(1/p)-(1/r)} \tag{5.1}$$

such that our $L \in (H^p(\mathbb{R}^n))^*$ is represented on functions $f \in L^r_{N_p}(Q)$ by the integral:

$$L(f) = \int_{\mathbb{R}^n} f(x)g(x)dx \ . \tag{5.2}$$

Naturally, the function g is not uniquely determined. If we add to g a polynomial of degree $\le N_p$, (5.2) continues to hold for every $f \in L^r_{N_p}(Q)$, since such functions f have vanishing moments up to order N_p. Conversely, if g_1 and g_2 are such that (5.2) holds for each of them in place of g, and every $f \in L^r_{N_p}(Q)$, then necessarily $g_1 - g_2$ is a polynomial of degree $\le N_p$. Indeed, the function $h = g_1 - g_2$ satisfies $\int_{\mathbb{R}^n} f(x)h(x)dx = 0$ for every $f \in L^r_{N_p}(Q)$. In general, for $f \in L^1(Q)$ and N a positive integer, we shall denote by $P^N_Q(f)$ the unique polynomial of degree $\le N$ having over Q the same moments as f up to order N. In our case we need to take $N = N_p$ fixed and we shall simply write $P_Q(f)$. Then, if $f \in L^r(Q)$, we shall have $f - P_Q(f)\chi_Q \in L^r_{N_p}(Q)$. Therefore

$$\int_Q (f(x) - P_Q(f)(x))h(x)dx = 0.$$

But

$$0 = \int_Q (f(x) - P_Q(f)(x))h(x)dx =$$

$$= \int_Q (f(x)-P_Q(f)(x))\cdot(h(x)-P_Q(h)(x))dx = \int_Q f(x)(h(x)-P_Q(h)(x))dx.$$

Since this is true for every $f \in L^r(Q)$, it follows that $h(x) = P_Q(h)(x)$ for every $x \in Q$.

By writing $\mathbb{R}^n = \bigcup_j Q_j$ where (Q_j) is an increasing sequence of cubes, choosing $g_j \in L^{r'}(Q_j)$ which represents L over Q_j and such that each g_{j+1} is an extension of g_j; we can find $g \in L^{r'}_{loc}(\mathbb{R}^n)$ such that (5.2) holds for every f belonging to the class $\Theta = \Theta^r_{N_p} = \{f \in L^r(\mathbb{R}^n) : f$ is compactly supported and has vanishing moments up to order $N_p\}$. Notice that, even though we have excluded the case $r = \infty$ to be able to use the duality $(L^r(Q))^* = L^{r'}(Q)$, the conclusion we have obtained holds trivially also for the case $r = \infty$, since $L^\infty(Q) \subset L^q(Q) \subset L^1(Q)$ for $1 < q < \infty$, and since (5.1) continues to hold when r increases as one sees most easily by writing (5.1) in the form

$$\left(\frac{1}{|Q|} \int_Q |g|^{r'} dx \right)^{1/r'} \leq C \;||L||\; |Q|^{(1/p)-1}$$

In what follows we allow also $r = \infty$, i.e. $r' = 1$. Let Q be a cube in $\mathbb{R}^n$. Then:

$$||(g - P_Q(g))\chi_Q||_{r'} = \sup \{ |\int_Q (g(x) - P_Q(g)(x))f(x)dx| :$$

$$: \text{supp } f \subset Q \text{ and } ||f||_r \leq 1\}$$

But

$$|\int_Q (g(x) - P_Q(g)(x))f(x)dx| = |\int_Q g(x)(f(x) - P_Q(f)(x))dx| \leq$$

$$\leq ||g\chi_Q||_{r'} \; ||(f - P_Q(f))\chi_Q||_r \leq C||L|| \; |Q|^{(1/p)-(1/r)} \quad \text{since}$$

$$||P_Q(f)\chi_Q||_r \leq C|f_Q| \; |Q|^{1/r} \leq C .$$

We have obtained the following estimate :

$$(5.3) \qquad \left(\frac{1}{|Q|} \int_Q |g(x) - P_Q(g)(x)|^{r'} dx \right)^{1/r'} \leq C||L|| \; |Q|^{(1/p)-1}$$

where, for $r' = \infty$, the left hand side has to be understood simply as $\|(g - P_Q(g))\chi_Q\|_\infty$.

DEFINITION 5.4. In general, for $1 \leq q \leq \infty$ *and* $0 \leq \alpha < \infty$, *we shall denote by* $L^q_\alpha(\mathbb{R}^n)$ *the space consisting of those functions* $g \in L^q_{loc}(\mathbb{R}^n)$ *for which there is a constant* C *such that for every cube* $Q \subset \mathbb{R}^n$,

$$(5.5) \qquad \left(\frac{1}{|Q|}\int_Q |g(x) - P_Q^{[\alpha]}(g)(x)|^q dx\right)^{1/q} \leq C|Q|^{\alpha/n}$$

where the left hand side is understood to be $\|(g - P_Q^{[\alpha]}(g))\chi_Q\|_\infty$ *in case* $q = \infty$. *For* $g \in L^q_\alpha(\mathbb{R}^n)$, *we shall denote by* $\|g\|_{L^q_\alpha(\mathbb{R}^n)}$ *the infimum of all the constants* C *which make (5.5) valid for every* Q. *The mapping* $g \longmapsto \|g\|_{L^q_\alpha(\mathbb{R}^n)}$ *is a semi-norm in* $L^q_\alpha(\mathbb{R}^n)$ *and* $\|g\|_{L^q_\alpha(\mathbb{R}^n)} = 0$ *if and only if* g *coincides almost everywhere with a polynomial of degree* $\leq [\alpha]$. *Therefore, if we form the quotient space of* $L^q_\alpha(\mathbb{R}^n)$ *modulo the space of functions agreeing a.e. with polynomials of degree* $\leq [\alpha]$, *we shall have a normed space* $\widetilde{L^q_\alpha}(\mathbb{R}^n)$.

Observe that $\widetilde{L^q_0}(\mathbb{R}^n) = \text{B.M.O.}(\mathbb{R}^n)$ provided $q < \infty$. It is important to note that on $L^q_\alpha(\mathbb{R}^n)$ the semi-norm $\|g\|_{L^q_\alpha(\mathbb{R}^n)}$ is equivalent to this other semi-norm:

$$\sup_{Q \text{ cube}} |Q|^{-\alpha/n} \cdot \inf\{\left(\frac{1}{|Q|}\int_Q |g(x) - P(x)|^q dx\right)^{1/q} :$$
$$: P \text{ polynomial of degree } \leq [\alpha]\}$$

Indeed, for every polynomial P of degree $\leq [\alpha]$:

$$\left(\frac{1}{|Q|}\int_Q |g(x) - P_Q^{[\alpha]}(g)(x)|^q dx\right)^{1/q} \leq$$
$$\leq \left(\frac{1}{|Q|}\int_Q |g(x) - P(x)|^q dx\right)^{1/q} + \sup_{x \in Q}|P(x) - P_Q^{[\alpha]}(g)(x)| \leq$$
$$\leq C\left(\frac{1}{|Q|}\int_Q |g(x) - P(x)|^q dx\right)^{1/q} \quad \text{since}$$

$$P(x) - P_Q^{[\alpha]}(g)(x) = P_Q^{[\alpha]}(P-g)(x)$$

and, consequently, for every $x \in Q$ is:

$$|P(x) - P_Q^{[\alpha]}(g)(x)| \le \frac{C}{|Q|}\int_Q |g(x)-P(x)|\, dx \le C\left(\frac{1}{|Q|}\int_Q |g(x)-P(x)|^q dx\right)^{\frac{1}{q}}$$

for some absolute constant C.

Therefore, in order to see that a certain g belongs to $\mathcal{L}_\alpha^q(\mathbb{R}^n)$ it is enough to find a constant C such that for every cube Q, there is a polynomial of degree $\le [\alpha]$, say $P_Q(x)$, satisfying the inequality:

$$\left(\frac{1}{|Q|}\int_Q |g(x)-P_Q(x)|^q\, dx\right)^{1/q} \le C\, |Q|^{\alpha/n}$$

This is where we stand on our way to identifying $(H^p(\mathbb{R}^n))^*$:

THEOREM 5.6 Let $0 < p \le 1$. *With* p *associate* $\alpha = n((1/p)-1)$. *Fix also* $1 \le q \le \infty$ *if* $p < 1$ *and* $1 \le q < \infty$ *if* $p = 1$. *Then for every* $L \in (H^p(\mathbb{R}^n))^*$ *there is a unique class* $\langle g\rangle \in \tilde{\mathcal{L}}_\alpha^q(\mathbb{R}^n)$ *such that for every* $f \in \Theta_{[\alpha]}^{q'}$ *and for every* $g \in \langle g\rangle$, *(5.2) holds. Besides* $\|g\|_{\mathcal{L}_\alpha^q(\mathbb{R}^n)} \le C\, \|L\|$ *with an absolute constant* C. □

In other words, we have found a continuous linear mapping

$$(5.7) \qquad (H^p(\mathbb{R}^n))^* \longrightarrow \tilde{\mathcal{L}}^q_{n((1/p)-1)}(\mathbb{R}^n)$$

$$L \longmapsto \langle g\rangle$$

Notice that $\Theta_{[\alpha]}^{q'} = \Theta_{N_p}^{q'}$ is dense in $H^p(\mathbb{R}^n)$ since it contains all the finite linear combinations of (p,q')-atoms. This implies that the mapping (5.7) is one to one. We shall eventually see that this mapping is also onto, so that it provides an equivalence of Banach spaces. Once this is done, we shall be able to write: $(H^p(\mathbb{R}^n))^* = \tilde{\mathcal{L}}^q_{n((1/p)-1)}(\mathbb{R}^n)$. Actually this identification implies that the spaces $\tilde{\mathcal{L}}_\alpha^q(\mathbb{R}^n)$ for a fixed α and varying q's, are all the same, and the corresponding norms are equivalent. So we could simply write $\tilde{\mathcal{L}}_\alpha(\mathbb{R}^n)$. Remember, however, that for $\alpha = 0$ (that is: $p = 1$), the index $q = \infty$ is not allowed. This is a natural restriction, since $\tilde{\mathcal{L}}_0^\infty = L^\infty$, whereas, $\tilde{\mathcal{L}}_0^q = $ B.M.O. for any $q < \infty$. The equivalence of the spaces $\tilde{\mathcal{L}}_0^q$ for different finite q's was already established in chapter II as a consequence of the John and Nirenberg theorem. Now it will be obtained for free together with the corresponding results for $\alpha > 0$, once we prove that the mapping (5.7) is an equivalence. This is our next goal. Let $0 < p \leq 1$. Call $\alpha = n((1/p)-1)$ and fix q admissible for our p, that is: $1 \leq q \leq \leq \infty$ if $p < 1$ and $1 \leq q < \infty$ if $p = 1$. Let $g \in \mathcal{L}^q_\alpha(\mathbb{R}^n)$. We want to construct $L \in (H^p(\mathbb{R}^n))^*$ such that for every $f \in \Theta^{q'}_{[\alpha]}$, (5.2) holds. Of course, for such functions f we can always define $L(f)$ by means of formula (5.2). Now, since the space $\Theta^{q'}_{[\alpha]}$ is dense in $H^p(\mathbb{R}^n)$, we shall be able to extend our L to a continuous linear functional on $H^p(\mathbb{R}^n)$ if we are able to prove that, for every $f \in \Theta^{q'}_{[\alpha]}$:

$$(5.8) \qquad |L(f)| \leq C \, \|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} \|f\|_{H^p(\mathbb{R}^n)}$$

with C an absolute constant.

This will also prove that the extended L satisfies $\|L\| \leq C \, \|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)}$, that way completing the proof that (5.7) is an equivalence.

The first step towards (5.8) is the following:

LEMMA 5.9. If a *is a* (p,q')-*atom, then*

$$\left| \int_{\mathbb{R}^n} a(x)\, g(x)\, dx \right| \leq \|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)}$$

Proof: Let Q be a minimal cube supporting a. Then

$$\left| \int_Q a(x)g(x)dx \right| = \left| \int_Q a(x)(g(x)-P_Q^{[\alpha]}(g)(x))dx \right| \leq$$

$$\leq \|a\|_{q'} \, \|(g-P_Q^{[\alpha]}(g))\chi_Q\|_q \leq |Q|^{(1/q')-(1/p)} \|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} |Q|^{(1/q)+(1/p)-1} =$$

$$= \|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)}. \qquad \square$$

Now, given $f \in \Theta^{q'}_{[\alpha]}$, we can write $f = \sum_j \lambda_j a_j$ where the a_j's are (p,q')-atoms, the λ_j's are real numbers satisfying $\sum_j |\lambda_j|^p \leq C \, \|f\|^p_{H^p(\mathbb{R}^n)}$ with C an absolute constant; and the convergence is in the sense of tempered distributions. If we are able to show that $L(f) = \sum_j \lambda_j L(a_j)$, we shall have:

$$|L(f)| \le \sum_j |\lambda_j||L(a_j)| \le (\sum_j |\lambda_j|)\ \|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} \le C\ \|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} \|f\|_{H^p(\mathbb{R}^n)}$$

since $\sum_j |\lambda_j| \le (\sum_j |\lambda_j|^p)^{1/p} \le C\ \|f\|_{H^p(\mathbb{R}^n)}$. That way we would get (5.8).

Therefore, the identification $(H^p(\mathbb{R}^n))^* = \widetilde{\mathcal{L}^q_{n((1/p)-1)}}(\mathbb{R}^n)$ will have been totally justified once we prove that $L(f) = \sum_j \lambda_j L(a_j)$ or, in other words: $\int_{\mathbb{R}^n} f(x)g(x)dx = \sum_j \lambda_j \int_{\mathbb{R}^n} a_j(x)g(x)dx.$

Of course this identity holds with any Schwartz function in place of g since $f = \sum_j \lambda_j a_j$ in the sense of tempered distributions. If we were able to find a sequence g_k of Schwartz functions such that for every $f \in \Theta^{q'}_{[\alpha]}$ is $\int_{\mathbb{R}^n} f(x)g_k(x)dx \longrightarrow \int_{\mathbb{R}^n} f(x)g(x)dx$ as $k \to \infty$ and also $\|g_k\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} \le C$, we would have finished since it would be $\sum_j \lambda_j \int_{\mathbb{R}^n} a_j(x)g_k(x)dx \longrightarrow \sum_j \lambda_j \int_{\mathbb{R}^n} a_j(x)g(x)dx$ by dominated convergence. So, we have reduced everything to a problem of approximation.

To approximate $g \in \mathcal{L}^q_\alpha(\mathbb{R}^n)$ by C^∞ functions with uniformly bounded $\mathcal{L}^q_\alpha$-norms is not difficult. We simply need to take the functions $g*\phi_\varepsilon$ where ϕ is C^∞ with compact support and has $\int\phi = 1$. It is clear that $g*\phi_\varepsilon \to g$ as $\varepsilon \to 0$ in $L^q_{loc}(\mathbb{R}^n)$, so that $\int_{\mathbb{R}^n} f(x)g*\phi_\varepsilon(x)dx \longrightarrow \int_{\mathbb{R}^n} f(x)g(x)dx$ as $\varepsilon \to 0$ for every $f \in \Theta^{q'}_{[\alpha]}$. We just have to see that $\|g*\phi_\varepsilon\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} \le C\ \|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)}$

We know that for each cube Q, there is a polynomial $P_Q^{[\alpha]}(g) = P_Q(g)$ such that

$$\left(\frac{1}{|Q|}\int_Q |g(x)-P_Q(g)(x)|^q dx\right)^{1/q} \le \|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)}\ |Q|^{\alpha/n}$$

Given a cube Q and $\varepsilon > 0$, consider the function

$$P_{Q,\varepsilon}(x) = \int_{\mathbb{R}^n} P_{Q-y}(g)(x-y)\,\phi_\varepsilon(y)\ dy$$

By writing $P_{Q-y}(g)(x-y) = \sum_{|\beta|\le[\alpha]} a_\beta(y)(x-y)^\beta$ and observing that the coefficients a_β are continuous, we realize that $P_{Q,\varepsilon}$ is a polinomial of degree $\le[\alpha]$. By using Minkowski's inequality for integrals, we get:

$$\left(\frac{1}{|Q|}\int_Q |g*\phi_\varepsilon(x)-P_{Q,\varepsilon}(x)|^q\,dx\right)^{1/q} =$$

$$\left(\frac{1}{|Q|}\int_Q \left|\int_{\mathbb{R}^n}(g(x-y)-P_{Q-y}(g)(x-y))\phi_\varepsilon(y)dy\right|^q dx\right)^{1/q} \le$$

$$\le \int_{\mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q |g(x-y)-P_{Q-y}(g)(x-y)|^q dx\right)^{1/q}|\phi_\varepsilon(y)|dy \le C\|g\|_{L^q_\alpha(\mathbb{R}^n)}|Q|^{\alpha/n}$$

since $\left(\frac{1}{|Q|}\int_Q |g(x-y)-P_{Q-y}(g)(x-y)|^q dx\right)^{1/q} =$

$$= \left(\frac{1}{|Q-y|}\int_{Q-y}|g(z)-P_{Q-y}(g)(z)|^q dz\right)^{1/q} \le \|g\|_{L^q_\alpha(\mathbb{R}^n)}|Q|^{\alpha/n}$$

We have obtained the desired inequality $\|g*\phi_\varepsilon\|_{L^q_\alpha(\mathbb{R}^n)} \le C\,\|g\|_{L^q_\alpha(\mathbb{R}^n)}$. We still have the problem of approximating a function $f \in C^\infty\cap L^q_\alpha(\mathbb{R}^n)$ by Schwartz functions with uniformly bounded L^q_α-norms. The idea should be to use a cut-off function ψ radial, C^∞, supported in $B(0,1)$ such that $\psi(x) = 1$ for every $x \in B(0,1/2)$ and $0 \le \psi(x) \le$ ≤ 1 for all x. The function $g(x)\psi(\varepsilon x)$ coincides with $g(x)$ for $|x| < 1/(2\varepsilon)$, is C^∞, and is supported in $B(0,1/\varepsilon)$. The functions $g\psi(\varepsilon\cdot)$ approximate g as $\varepsilon \to 0$ uniformly on compact subsets. It would be very nice if we could get a uniform estimate for the L^q_α-norm of these functions. We shall be able to do this very simply for the case $0 \le \alpha < 1$, which corresponds to $n/(n+1) < p \le 1$. To deal with the general case, we shall need to get a better understanding of the spaces $L^q_\alpha(\mathbb{R}^n)$.

Suppose that $0 \le \alpha < 1$. Then $[\alpha] = 0$, and this implies that the space $L^q_\alpha(\mathbb{R}^n)$ is a lattice. Indeed if $g \in L^q_\alpha(\mathbb{R}^n)$, we have, for every cube Q:

$$\left(\frac{1}{|Q|}\int_Q \big||g(x)|-|g_Q|\big|^q dx\right)^{1/q} \le \left(\frac{1}{|Q|}\int_Q |g(x)-g_Q|^q dx\right)^{1/q} \le$$

$$\le \|g\|_{L^q_\alpha(\mathbb{R}^n)}|Q|^{\alpha/n}.$$

Thus $|g| \in L^q_\alpha(\mathbb{R}^n)$. It follows easily that for $f_1,f_2 \in L^q_\alpha(\mathbb{R}^n)$, the functions $\max(f_1,f_2)$ and $\min(f_1,f_2)$ are both in $L^q_\alpha(\mathbb{R}^n)$, with norms bounded by $C\max(\|f_1\|_{L^q_\alpha}, \|f_2\|_{L^q_\alpha})$. We already pointed out that this holds for B.M.O. in chapter II. The advantage is that now we can approximate a function of $\mathcal{L}^q_\alpha$ by means of bounded functions with uniformly bounded $\mathcal{L}^q_\alpha$-noms. For $g \in \mathcal{L}^q_\alpha(\mathbb{R}^n)$ and $k > 0$

we simply take $g_k(x) = \begin{cases} k & \text{if } g(x) \geq k \\ g(x) & \text{if } -k \leq g(x) \leq k \\ -k & \text{if } g(x) \leq -k \end{cases}$ and

Then if we regularize by convolution each of the g_k's, we end up with bounded C^∞ functions. Therefore, we are reduced to applying our cut-off process to a function $g \in C^\infty \cap L^\infty \cap \mathcal{L}^q_\alpha(\mathbb{R}^n)$. For such a function we get a simple estimate

$$\|g\psi\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} \leq C(\|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} + \|g\|_\infty) \tag{5.10}$$

where C depends only on ψ, the cut-off function. To get this estimate we write for a cube Q with center x_0:

$$\left(\frac{1}{|Q|}\int_Q |g(x)\psi(x)-P_Q(g)(x)\psi(x_0)|^q dx\right)^{1/q} \leq$$
$$\leq \left(\frac{1}{|Q|}\int_Q |g(x)\psi(x)-P_Q(g)(x)\psi(x)|^q dx\right)^{1/q} +$$
$$+ \left(\frac{1}{|Q|}\int_Q |P_Q(g)(x)(\psi(x)-\psi(x_0))|^q dx\right)^{1/q} \leq C(\|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} + \|g\|_\infty)\ |Q|^{\alpha/n}$$

Indeed, the first term in the sum is bounded by $\|\psi\|_\infty\ \|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} |Q|^{\alpha/n}$. The second term in the sum is bounded by $2\|\psi\|_\infty\|g\|_\infty$. This estimate is good for big cubes, say for $|Q| > 1$. For small cubes, $|Q| \leq 1$, we use this other estimate for the second term: $C\|\nabla\psi\|_\infty\|g\|_\infty\ |Q|^{1/n} \leq C\|g\|_\infty|Q|^{\alpha/n}$. This estimate is obtained by using the mean value theorem for ψ.

We shall also need the following estimate for the dilated $x \longmapsto h(\varepsilon x)$ of an $\mathcal{L}^q_\alpha$ function h.

$$\|h(\varepsilon\cdot)\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} \leq C\varepsilon^\alpha\|h\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} \tag{5.11}$$

Indeed for a cube Q

$$\left(\frac{1}{|Q|}\int_Q |h(\varepsilon x)-P_{\varepsilon Q}(\varepsilon x)|^q dx\right)^{1/q} = \left(\frac{1}{|\varepsilon Q|}\int_{\varepsilon Q} |h(y)-P_{\varepsilon Q}(y)|^q dy\right)^{1/q} \leq$$
$$\leq \|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} |\varepsilon Q|^{\alpha/n} = \varepsilon^\alpha\ \|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} |Q|^{\alpha/n}$$ and (5.11) follows.

Now, since $g(x)\psi(\varepsilon x) = g(\varepsilon\varepsilon^{-1}x)\psi(\varepsilon x)$, we have:

$$\|g\psi(\varepsilon\cdot)\|_{L^q_\alpha(\mathbb{R}^n)} \leq C\varepsilon^\alpha\|g(\varepsilon^{-1}\cdot)\psi\|_{L^q_\alpha(\mathbb{R}^n)} \leq C\varepsilon^\alpha(\|g(\varepsilon^{-1}\cdot)\|_{L^q_\alpha(\mathbb{R}^n)} + \|g\|_\infty) \leq$$

$\leq C\,\|g\|_{L^q_\alpha(\mathbb{R}^n)} + C\,\varepsilon^\alpha\|g\|_\infty$. It is clear that, for $\varepsilon \to 0$ this norms are uniformly bounded. This ends the proof of the duality $(H^p(\mathbb{R}^n))^* = \tilde{L}^q_{n((1/p)-1)}(\mathbb{R}^n)$ for $n/(n+1) < p \leq 1$.

The case $p = 1$ deserves special mention. In this case $\alpha = n((1/p)-1) = 0$ and the dual space obtained is $\tilde{L}^q_0 = $ B.M.O. This is in agreement with the duality theorem proved in chapter I for the space H^1 of the disk. Let us record the result obtained for this special case.

THEOREM 5.12. For every $\langle g\rangle \in$ B.M.O. *the linear functional* L *defined by* (5.2) *over bounded functions with compact support, extends to a continuous linear functional* L *over* $H^1(\mathbb{R}^n)$ *with*

$$\|L\|_{(H^1(\mathbb{R}^n))^*} \leq C\,\|g\|_*$$

C *being an absolute constant.* □

This, in conjunction with the case $p = 1$ of theorem 5.6., can be stated as follows.

THEOREM 5.13. $(H^1(\mathbb{R}^n))^* = $ B.M.O. □

In general, with α and q related to p as in theorem 5.6., what we have already proved is

THEOREM 5.14. For $n/(n+1) < p \leq 1$:

$$(H^p(\mathbb{R}^n))^* = \tilde{L}^q_\alpha(\mathbb{R}^n). \;\square$$

In order to extend this result to $0 < p \leq 1$ we shall try to understand better the nature of the spaces $L^q_\alpha(\mathbb{R}^n)$ for $\alpha > 0$. They turn out to be Lipschitz spaces in disguise. Let us define the Lipschitz spaces $\Lambda_\alpha(\mathbb{R}^n)$.

DEFINITION 5.15. a) Let $\alpha > 0$, α *not an integer. Call* $[\alpha] = N$ *and* *let* $\alpha = N + \alpha'$, *where, of course,* $0 < \alpha' < 1$. $\Lambda_\alpha(\mathbb{R}^n)$ *will be the space consisting of those functions* $g \in C^N(\mathbb{R}^n)$ *for which there is a constant* C *such that for every multi-index* β *with* $|\beta| = N$, *the derivative* $D^\beta g$ *satisfies the Lipschitz condition*

$$(5.16) \qquad |D^\beta g(x+h) - D^\beta g(x)| \le C|h|^{\alpha'}$$

for every $x, h \in \mathbb{R}^n$. *For* $g \in \Lambda_\alpha(\mathbb{R}^n)$ *we denote by* $\|g\|_{\Lambda_\alpha(\mathbb{R}^n)}$ *the smallest constant* C *which makes (5.16) valid for all* β, x, h. *Equivalently, using the notation* $\Delta_h f(x) = f(x+h) - f(x)$ *we write:*

$$\|g\|_{\Lambda_\alpha(\mathbb{R}^n)} = \sup_{|\beta| = [\alpha]} \sup_{h \in \mathbb{R}^n \setminus \{0\}} |h|^{-\alpha'} \|\Delta_h D^\beta g\|_\infty$$

$\| \ \|_{\Lambda_\alpha(\mathbb{R}^n)}$ *is a semi-norm on* $\Lambda_\alpha(\mathbb{R}^n)$ *which vanishes precisely over the space formed by all the polynomials of degree* $\le N$. ((5.16) *with* $C = 0$ *implies that* $D^\beta g$ *is a constant). Denote by* $\tilde{\Lambda}_\alpha(\mathbb{R}^n)$ *the corresponding quotient space. It will be a normed space.*

b) Let $\alpha > 0$ *be an integer, say* $\alpha = N$. *Then* $\Lambda_\alpha(\mathbb{R}^n)$ *is going to be the space consisting of those functions* $g \in C^{N-1}(\mathbb{R}^n)$ *for which there is a constant* C *such that for every multi-index* β *with* $|\beta| = N-1$, *the derivative* $D^\beta g$ *satisfies the Lipschitz condition*

$$(5.17) \qquad |D^\beta g(x+2h) - 2D^\beta g(x+h) + D^\beta g(x)| \le C\,|h|$$

for every $x, h \in \mathbb{R}^n$. *Observe that the expression whose absolute value appears in the left hand side of (5.17) is* $\Delta_h^2 D^\beta g(x) = \Delta_h \Delta_h D^\beta g(x)$. *For* $g \in \Lambda_\alpha(\mathbb{R}^n)$, *we denote by* $\|g\|_{\Lambda_\alpha(\mathbb{R}^n)}$ *the smallest constant* C *which makes (5.17) valid for all* β, x, h. *Equivalently*

$$\|g\|_{\Lambda_\alpha(\mathbb{R}^n)} = \sup_{|\beta| = [\alpha]-1} \sup_{h \in \mathbb{R}^n \setminus \{0\}} |h|^{-1} \|\Delta_h^2 D^\beta g\|_\infty.$$

$\| \ \|_{\Lambda_\alpha(\mathbb{R}^n)}$ *is a semi-norm on* $\Lambda_\alpha(\mathbb{R}^n)$ *wich vanishes precisely over the space formed by all the polynomials of degree* $\le N$ ((5.17) *with* $C = 0$ *implies that* $D^\beta g$ *is an affine-linear function). In this case we shall also denote by* $\tilde{\Lambda}_\alpha(\mathbb{R}^n)$ *the normed space obtained by taking the quotient.*

We shall see that the spaces Λ_α and L^q_α coincide as sets and the corresponding norms are equivalent. The first step towards this equivalence is the following.

LEMMA 5.18. Let $g \in \Lambda_\alpha(\mathbb{R}^n)$. *Then for every ball* $B(x_0,r)$, *there exists a polynomial* P *of degree* $\le [\alpha]$ *such that for every multi-index* β *with* $|\beta| < \alpha$:

$$|D^\beta g(x) - D^\beta P(x)| \le C r^{\alpha-|\beta|} \|g\|_{\Lambda_\alpha(\mathbb{R}^n)} \quad \textit{for every} \quad x \in B(x_0,r)$$

with C *depending only upon* α *and* n.

Proof: a) Suppose that α is not an integer. We shall see that the Taylor polynomial of degree $[\alpha]$ at x_0 of the function g is good enough.

Indeed, let $P(x) = \sum_{0 \le |\gamma| \le [\alpha]} \frac{1}{\gamma!} D^\gamma g(x_0)(x-x_0)^\gamma$

Then $g(x)-P(x) = \sum_{|\gamma|=[\alpha]} \frac{1}{\gamma!} (D^\gamma g(\xi_x) - D^\gamma g(x_0))(x-x_0)^\gamma$

where ξ_x is a point in the segment joining x_0 and x. If $x \in B(x_0,r)$, we have $|D^\gamma g(\xi_x) - D^\gamma g(x_0)| \le \|g\|_{\Lambda_\alpha} r^{\alpha'}$ and $|(x-x_0)^\gamma| \le$ $\le |x-x_0|^{|\gamma|} \le r^{[\alpha]}$ for $|\gamma| = [\alpha]$. Therefore $|g(x)-P(x)| \le Cr^\alpha \|g\|_{\Lambda_\alpha}$ for $x \in B$.

The estimate for $|D^\beta g(x) - D^\beta P(x)|$ with $|\beta| \le [\alpha]$ is obtained in exactly the same way by observing that $D^\beta P(x)$ is the Taylor polynomial at x_0 of degree $[\alpha]-|\beta|$ for the function $D^\beta g$.

b) Suppose now that α is an integer, $\alpha = 1,2,\ldots$. Let ϕ be an even C^∞ function supported in $B(0,1)$ with $\int_{\mathbb{R}^n} \phi = 1$. Consider the function $h = g * \phi_r$ and let P_1 be the Taylor polynomial of h at x_0 of degree α. For $|\beta| = \alpha-1$, $D^\beta P_1$ will be the Taylor polynomial of degree 1 at x_0 for the function $D^\beta h = D^\beta g * \phi_r$. For $|\gamma| = 2$, $D^\gamma(\phi_r) = r^{-2}(D^\gamma\phi)_r$ is an even function with integral 0. So

$$\left|2D^{\beta+\gamma}h(x)\right| = \left|\int_{\mathbb{R}^n} D^\gamma\phi_r(y)(D^\beta g(x-y) + D^\beta g(x+y) - 2D^\beta g(x))\,dy\right| \le$$

$\le \|g\|_{\Lambda_\alpha} \cdot r \cdot r^{-2} \int_{\mathbb{R}^n} |D^\gamma\phi|\,dy \le C\|g\|_{\Lambda_\alpha} r^{-1}$. This allows us to obtain the estimate

$$|D^\beta h(x)-D^\beta P_1(x)| \le C \|g\|_{\Lambda_\alpha} \cdot r \quad \text{for every} \quad x \in B(x_0,r)$$

We also have

$$|2D^\beta(h-g)(x)| \le \left| \int_{\mathbb{R}^n} \phi_r(y)(D^\beta g(x-y)+D^\beta g(x+y)-2D^\beta g(x))\, dy \right| \le$$

$\le C \|g\|_{\Lambda_\alpha} \cdot r$. Thus $|D^\beta g(x)-D^\beta P_1(x)| \le C \|g\|_{\Lambda_\alpha} \cdot r$ for $x \in B(x_0,r)$ and $|\beta| = \alpha-1$. Now let P be a polynomial having the same terms of order α and $\alpha-1$ as those of P_1 but such that $D^\beta P(x_0) = D^\beta g(x_0)$ for every β with $|\beta| \le \alpha-2$. Then we just need to integrate the inequalities obtained already for $|\beta| = \alpha - 1$ in order to obtain the required inequalities for derivatives of lower order.

□

COROLLARY 5.19. $\Lambda_\alpha(\mathbb{R}^n) \subset L^\infty_\alpha(\mathbb{R}^n) \subset L^q_\alpha(\mathbb{R}^n)$ *for every* $\alpha > 0$ *and every* $1 \le q < \infty$. *Besides the inclusions are continuous.*

Proof: Let $g \in \Lambda_\alpha(\mathbb{R}^n)$. If Q is a cube with center x_0 and diameter $2r$, consider $B(x_0,r)$ and apply lemma 5.18. with $\beta = 0$. For $x \in Q$ is

$$|g(x)-P(x)| \le Cr^\alpha \|g\|_{\Lambda_\alpha(\mathbb{R}^n)} = C\|g\|_{\Lambda_\alpha(\mathbb{R}^n)} |Q|^{\alpha/n}$$

Thus $g \in L^\infty_\alpha(\mathbb{R}^n)$ with $\|g\|_{L^\infty_\alpha(\mathbb{R}^n)} \le C\|g\|_{\Lambda_\alpha(\mathbb{R}^n)}$

The other inclusion is obvious. □

In order to prove the converse we need the following

LEMMA 5.20. Let ϕ *be a* C^∞ *function with compact support in* $\mathbb{R}^n$. *Let* $\alpha > 0$ *and* $0 < p < 1$ *such that* $\alpha = n((1/p)-1)$. *Suppose* k *in an integer* ≥ 0 *and* β *is a multi-index in* n *variables, such that* $k + |\beta| > \alpha$. *Then*

$$\frac{\partial^k}{\partial t^k} D^\beta_x (\phi_t(x)) = Ct^{\alpha-k-|\beta|} a_t(x), \quad x \in \mathbb{R}^n,\ t > 0$$

where a_t *is a* (p,∞)-*atom for every* t, *and* C *is a constant depending on* ϕ, α, n *and the order of differentiation, but not on* t

Proof: $\frac{\partial^k}{\partial t^k} D_x^\beta(\phi_t(x)) = t^{-n-|\beta|-k}\ \Phi(x/t)$ for some function Φ, C^∞ with compact support. Then the function

$$t^{|\beta|+k-\alpha}\frac{\partial^k}{\partial t^k}\ D_x^\beta(\phi_t(x)) = t^{-n-\alpha}\Phi(x/t) = t^{-n/p}\ \Phi(x/t)$$

is bounded in absolute value by $Ct^{-n/p}$ where $C = \|\Phi\|_\infty$ does not depend on t. But, if Φ is supported in a cube Q centered at 0, then $t^{-n/p}\Phi(x/t)$ is supported in tQ. Since $|tQ| = Ct^n$ for some constant independent of t, we conclude that

$$\frac{\partial^k}{\partial t^k}\ D_x^\beta(\phi_t(x)) = Ct^{\alpha-k-|\beta|}\ a_t(x) \quad \text{where} \quad a_t(x) \text{ lives in } tQ \text{ and}$$

satisfies $|a_t(x)| \leq |tQ|^{-1/p}$ and C does not depend on t. Now we shall see that $a_t(x)$ is actually $a(p,\infty)$-atom by proving that $a_t(x)$ has vanishing moments up to order $[\alpha]$. For a multi-index γ with $|\gamma| \leq [\alpha]$, consider the moment

$$(5.21) \qquad \int_{\mathbb{R}^n} x^\gamma \frac{\partial^k}{\partial t^k}\ D_x^\beta(\phi_t(x))\ dx = \frac{\partial^k}{\partial t^k}\ \int_{\mathbb{R}^n} x^\gamma D_x^\beta(\phi_t(x))dx$$

In case $\beta \leq \gamma$, which means that $\beta_j \leq \alpha_j$ for each j, integration by parts shows that (5.21) equals a constant times
$\frac{\partial^k}{\partial t^k}\int_{\mathbb{R}^n} x^{\gamma-\beta}\phi_t(x)dx = C\,\frac{\partial^k}{\partial t^k}\,(t^{|\gamma|-|\beta|}) = 0$ since $k > |\gamma| - |\beta|$.
Indeed $|\beta| + k > \alpha \geq |\gamma|$.

In case it is not $\beta \leq \gamma$, there will be $\gamma_j < \beta_j$ for some j. Integration by parts in x_j gives immediately that (5.21) is 0. □

Now we are ready to prove the

THEOREM 5.22. For every $\alpha > 0$ *and every* $1 \leq q \leq \infty$ $\tilde{L}_\alpha^q(\mathbb{R}^n) \subset \tilde{\Lambda}_\alpha(\mathbb{R}^n)$ *and the inclusion is continuous.*

Proof: Let $g \in L_\alpha^q(\mathbb{R}^n)$. We shall show that g coincides almost everywhere with a function in $\Lambda_\alpha(\mathbb{R}^n)$ whose Λ_α-norm we shall estimate.

We shall use a function ϕ, radial, C^∞, supported in $B(0,1)$ such that $0 \leq \phi(x) \leq 1$ for all x and having $\int_{\mathbb{R}^n}\phi(x)dx = 1$. If for

$t > 0$ is $\phi_t(x) = t^{-n}\phi(x/t)$ the corresponding approximate identity, we shall consider the function $u(x,t) = (g*\phi_t)(x)$ in $\mathbb{R}^{n+1}_+$. Lemma 5.20. together with lemma 5.9. guarantee that for any integer $k \geqq 0$ and any multi-index β in n variables such that $k + |\beta| > \alpha$, we have:

$$(5.23)\quad \left|\frac{\partial^k}{\partial t^k} D_x^\beta u(x,t)\right| = \left|\int_{\mathbb{R}^n} g(x-y)\frac{\partial^k}{\partial t^k} D_y^\beta(\phi_t(y))\,dy\right| \leqq$$

$\leqq C\,\|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)}\, t^{\alpha-k-|\beta|}$, with C independent of t and g.

a) We study first the case when α is not an integer. Let $\alpha = [\alpha] + \alpha'$ with $0 < \alpha' < 1$. Let β be any multi-index with $|\beta| = [\alpha]$. Estimate (5.23) gives

$\left|\frac{\partial}{\partial t} D_x^\beta\, u(x,t)\right| \leqq C\,\|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)}\, t^{\alpha'-1}$. Then for $0 < t_0 < t_1$, we have

$$(5.24)\quad |D^\beta u(x,t_1) - D^\beta u(x,t_0)| \leqq C\|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} \int_{t_0}^{t_1} t^{\alpha'-1}\,dt \leqq$$

$$\leqq C\,\|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)}\, t_1^{\alpha'}.$$

This implies that the functions $x \longmapsto D^\beta u(x,t)$ converge uniformly as $t \to 0$ to a certain (continuous) function $u_\beta(x)$. On the other hand $D^\beta u = (D^\beta g) * \phi_t \longrightarrow D^\beta g$ as $t \to 0$ in the sense of distributions. Therefore $D^\beta g = u_\beta$ as distributions. Now, for $x, h \in \mathbb{R}^n$

$$|u_\beta(x+h) - u_\beta(x)| \leqq |u_\beta(x+h) - D^\beta u(x+h,|h|)| +$$

$$+ |D^\beta u(x+h,|h|) - D^\beta u(x,|h|)| + |D^\beta u(x,|h|) - u_\beta(x)| \leqq$$

$\leqq C\,\|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} |h|^{\alpha'}$. Indeed, estimate (5.24) for $t_0 \to 0$ implies that the first and third terms in the sum are bounded by $C\,\|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)} |h|^{\alpha'}$. To estimate the second term we use the mean value theorem together with the estimate, obtained from (5.23)

$$\left|\nabla_x D_x^\beta u(x,t)\right| \leqq C\,\|g\|_{\mathcal{L}^q_\alpha(\mathbb{R}^n)}\, t^{\alpha'-1}$$

Thus, after modifying g if necessary on a set of measure 0, we have $g \in \Lambda_\alpha(\mathbb{R}^n)$ and

$$\|g\|_{\Lambda_\alpha(\mathbb{R}^n)} \leq C \|g\|_{L^q_\alpha(\mathbb{R}^n)}.$$

b) Suppose now that α is an integer. Let β be a multi-index such that $|\beta| = \alpha - 1$. Since $|\beta| + 2 > \alpha$, estimate (5.23) holds with $k = 2$. Thus

$$\left| \frac{\partial^2}{\partial t^2} D^\beta_x u(x,t) \right| \leq C \|g\|_{L^q_\alpha(\mathbb{R}^n)} t^{-1}$$

By integrating in t (say between t and 1), we get a local estimate

$$\left| \frac{\partial}{\partial t} D^\beta_x u(x,t) \right| \leq C \log (t^{-1})$$

Here C depends on x but it can be taken to be the same for each compact subset of $\mathbb{R}^n$. This estimate is still good to obtain this other local estimate for $0 < t_0 < t_1$.

$$|D^\beta_x u(x,t_1) - D^\beta_x u(x,t_0)| \leq C \int_{t_0}^{t_1} \log(t^{-1})dt \leq C\, t_1(1-\log t_1) \to 0$$

as $t_1 \to 0$. This is enough to guarantee that the functions $x \longmapsto D^\beta_x u(x,t)$ converge uniformly over each compact, as $t \to 0$, to a certain (continuous) function $u_\beta(x)$. As before $D^\beta g = u_\beta$ as distributions and we just need to look at the second differences of u_β. We write, using integration by parts

$$\begin{aligned} u_\beta(x) &= D^\beta_x u(x,t) - \int_0^t \frac{\partial}{\partial s} D^\beta_x u(x,s)\, ds = \\ &= D^\beta_x u(x,t) - t \frac{\partial}{\partial t} D^\beta_x u(x,t) + \int_0^t s \frac{\partial^2}{\partial s^2} D^\beta_x u(x,s)\, ds. \end{aligned}$$

Now we estimate the second order difference $\Delta^2_h u_\beta(x)$ by estimating the corresponding second order difference of the right hand side with $t = |h|$. The third term is bounded by $C \|g\|_{L^q_\alpha(\mathbb{R}^n)} |h|$ and so will be its second order difference. Next we observe that for a smooth function, the second order differences Δ^2_h are dominated by a bound for the second derivatives times $|h|^2$. We get, by using (5.23):

$$|\Delta^2_h D^\beta_x u(x,|h|)| \leq C \|g\|_{L^q_\alpha(\mathbb{R}^n)} |h|^{-1}|h|^2 = C \|g\|_{L^q_\alpha(\mathbb{R}^n)} |h| \quad \text{and}$$

$|\Delta_h^2\,(|h|\,\frac{\partial}{\partial t}\,D_x^\beta u(x,|h|))| \le |h|\,C\,\|g\|_{\mathcal{L}_\alpha^q(\mathbb{R}^n)}|h|^{-2}|h|^2 =$

$= C\,\|g\|_{\mathcal{L}_\alpha^q(\mathbb{R}^n)}|h|$. We conclude that, after modifying g if necessary on a null set, $g \in \Lambda_\alpha(\mathbb{R}^n)$ and

$$\|g\|_{\Lambda_\alpha(\mathbb{R}^n)} \le C\,\|g\|_{\mathcal{L}_\alpha^q(\mathbb{R}^n)}. \quad \square$$

We have finally proved, that for every $\alpha > 0$ and every $1 \le q \le \infty$:

(5.25) $\quad \tilde{\mathcal{L}}_\alpha^q(\mathbb{R}^n) = \tilde{\Lambda}_\alpha(\mathbb{R}^n)$

in the sense that the two spaces coincide as sets and the norms are equivalent. Sometimes the Λ_α-norm is easier to handle. This is particularly the case in the approximation problem which we still have to solve in order to complete the proof of the duality theorem

The solution is contained in the following

LEMMA 5.26. Let $g \in \Lambda_\alpha(\mathbb{R}^n)$. *Lemma 5.18 guarantees that for each* $r > 0$ *there exists* $g_r \in \Lambda_\alpha(\mathbb{R}^n)$ *such that* $g_r - g$ *is a polynomial of degree* $\le [\alpha]$ *and* $|D^\beta g_r(x)| \le Cr^{\alpha-|\beta|}\,\|g\|_{\Lambda_\alpha(\mathbb{R}^n)}$ *for every* x *with* $|x| < r$ *and every multi-index* β *with* $|\beta| < \alpha$. *Then if* ψ *is our cut-off function (* ψ *is radial,* C^∞, *supported in* $B(0,1)$, *such that* $\psi(x) = 1$ *for every* $x \in B(0,1/2)$ *and* $0 \le \psi(x) \le 1$ *for all* x*), the functions* $x \mapsto g_r(x)\,\psi(x/r)$ *have uniformly bounded* Λ_α*-norms. Actually their* Λ_α*-norms are bounded by* $C\|g\|_{\Lambda_\alpha(\mathbb{R}^n)}$ *with* C *an absolute constant.*

Proof: a) Suppose that α is not an integer. Let $\alpha = [\alpha] + \alpha'$ with $0 < \alpha' < 1$. Let β be a multi-index with $|\beta| = [\alpha]$. We have to see that the first difference Δ_h of the function $D^\beta(g_r(x)\,\psi(x/r)) = \sum_{\gamma+\delta=\beta} C_{\gamma,\delta}\,D^\gamma g_r(x) D^\delta(\psi(x/r))$ satisfies the estimate $|\Delta_h| \le C\,|h|^{\alpha'}$. We just need to prove these estimates for the first differences of each of the terms $D^\gamma g_r(x)\cdot D^\delta(\psi(x/r))$. But the first difference of a product is clearly

(5.27) $\quad \Delta_h(F\cdot G)(x) = \Delta_h F(x)\cdot G(x+h) + F(x)\Delta_h G(x)$. Let us apply this

formula with $F(x) = D^\gamma g_r(x)$ and $G(x) = D^\delta(\psi(x/r))$. First of all note that the function $F\cdot G$ lives in $|x| < r$ and, consequently it satisfies

$$|D^{\gamma}g_r(x)D^{\delta}(\psi(x/r))| \leq Cr^{\alpha-|\gamma|}\ r^{-|\delta|} = Cr^{\alpha-|\beta|} = Cr^{\alpha'}$$

It follows that we only need the estimate for $|h| \leq r$. In this case we use (5.27). The first term in this sum will be bounded by $C|h|^{\alpha'}$ if $\gamma = \beta$ and otherwise by $Cr^{\alpha-|\gamma|-1}|h|r^{-|\delta|} = Cr^{\alpha'-1}|h| = C(|h|/r)^{1-\alpha'}|h|^{\alpha'} \leq C\ |h|^{\alpha'}$. As for the second term, it is bounded by $Cr^{\alpha-|\gamma|}\ r^{-|\delta|-1}|h| \leq C\ |h|^{\alpha'}$. This concludes the proof for α not an integer.

b) Let α be an integer >0. Let β be a multi-index with $|\beta| = \alpha - 1$. We shall have to see that for every γ and δ multi-indices such that $\gamma + \delta = \beta$, the second order difference Δ_h^2 of the function $D^{\gamma}g_r(x)D^{\delta}(\psi(x/r))$ satisfies the estimate $|\Delta_h^2| \leq C|h|$. We only need to consider $|h| \leq r$ since the function itself is bounded by $Cr^{\alpha-|\gamma|}\ r^{-|\delta|} = Cr^{\alpha-|\beta|} = Cr$. By iterating (5.27) we get the following formula for the second order difference of a product:

$$(5.28)\quad \Delta_h^2(F\cdot G)(x) = \Delta_h^2F(x)\cdot G(x+2h)+2\Delta_hF(x)\Delta_hG(x+h)+F(x)\Delta_h^2G(x).$$

We apply it for $F(x) = D^{\gamma}g_r(x)$ and $G(x) = D^{\delta}(\psi(x/r))$. The first term in the sum is bounded by $C|h|$ if $\gamma = \beta$ and otherwise by $Cr^{\alpha-|\gamma|-1}\ |h|\ r^{-|\delta|} = C\ |h|$. The second term is bounded by $Cr\ r^{-1}\ |h| = C|h|$ if $\gamma = \beta$ and otherwise by $Cr^{\alpha-|\gamma|-1}|h|\ r^{-|\delta|} = C|h|$. Finally the third term is bounded by $Cr^{\alpha-|\gamma|}\ r^{-|\delta|-1}|h| = C|h|$. This ends the proof. $\square$

Observe that for $f \in \Theta^{q'}_{[\alpha]}$ and r big enough

$$\int_{\mathbb{R}^n} f(x)g_r(x)\psi(x/r)\ dx = \int_{\mathbb{R}^n} f(x)g_r(x)\ dx = \int_{\mathbb{R}^n} f(x)g(x)\ dx$$

Thus, we have solved our approximation problem. Given $g \in C^{\infty}\cap\mathcal{L}^q_{\alpha}(\mathbb{R}^n) = C^{\infty}\cap\Lambda_{\alpha}(\mathbb{R}^n)$, we have been able to find a sequence of functions $g_k(x) = g_{r_k}(x)\psi(x/r_k)$, $r_k \uparrow \infty$, which are C^{∞} with compact support, have $\|g_k\|_{\mathcal{L}^q_{\alpha}(\mathbb{R}^n)} \leq C\ \|g_k\|_{\Lambda_{\alpha}(\mathbb{R}^n)} \leq C$ and are such that, for every $f \in \Theta^{q'}_{[\alpha]}$: $\int_{\mathbb{R}^n} f(x)g_k(x)dx \to \int_{\mathbb{R}^n} f(x)g(x)dx$.

This completes the proof of the converse of theorem 5.6, which can now be stated in the following way, the notation being the same as in theorem 5.6.

THEOREM 5.29. For every $\langle g\rangle \in \widetilde{L}^q_\alpha(\mathbb{R}^n)$, *the linear functional defined by* (5.2) *on functions* $f \in \Theta^{q'}_{[\alpha]}$, *extends to a continuous linear functional* L *on* $H^p(\mathbb{R}^n)$ *with*

$$\|L\|_{(H^p(\mathbb{R}^n))^*} \leq C\,\|g\|_{L^q_\alpha(\mathbb{R}^n)}$$

C *being an absolute constant.* □

Altogether we can write in a single statement the fruit of all our efforts in this section, which is the following duality theorem

THEOREM 5.30. For every $0 < p \leq 1$, *with* $\alpha = n((1/p)-1)$, $1 \leq q \leq \infty$ *if* $p < 1$ *and* $1 \leq q < \infty$ *if* $p = 1$:

$$(H^p(\mathbb{R}^n))^* = \widetilde{L}^q_\alpha(\mathbb{R}^n) = \begin{cases} \widetilde{\Lambda}_\alpha(\mathbb{R}^n) & \text{if } \alpha > 0 \\ \\ \text{B.M.O. } (\mathbb{R}^n) & \text{if } \alpha = 0. \end{cases}$$ □

6. INTERPOLATION OF OPERATORS BETWEEN H^p SPACES

The atomic characterization of $H^p(\mathbb{R}^n)$ allows us to obtain estimates for the norm of an operator from H^p to L^p by simply checking its action on an atom. It is natural to study how these estimates can be interpolated.

Here is a simple variant of the Marcinkiewicz interpolation theorem

THEOREM 6.1. Let T *be an operator sending functions in* $H^1(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$ *to measurable functions in* $\mathbb{R}^n$ *where* $1 < p_1 \leq \infty$. *Suppose that:*

i) T *is subadditive, that is, for* $f_1, f_2 \in H^1(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$ $|T(f_1+f_2)(x)| \leq |Tf_1(x)|+|Tf_2(x)|$ *for almost every* x.

ii) T *is of weak type* $(H^1,1)$, *by which we mean that there is a constant* C_0 *such that for every* $f \in H^1(\mathbb{R}^n)$ *and for every* $t > 0$:

$$\left|\{x \in \mathbb{R}^n : |Tf(x)| > t\}\right| \le C_0 t^{-1} \|f\|_{H^1(\mathbb{R}^n)}$$

iii) T *is of weak type* (p_1,p_1) *with constant* C_1.

Then for every p *such that* $1 < p < p_1$, T *is bounded in* $L^p(\mathbb{R}^n)$ *with norm depending on* C_0, C_1, p_1 *and* p.

Proof: Take a fixed p with $1 < p < p_1$. Let $f \in L^p(\mathbb{R}^n)$. For each $t > 0$ we shall split the function f into a good part g_t and a bad part b_t very much like in the proof of (5.9) in chapter II. The difference will be that we shall use now the Calderón-Zygmund cubes $\{Q_j\}$ of the function $|f|^q$ corresponding to the value t^q where q is chosen so that $1 < q < p$. That is: the Q_j's will be the maximal cubes in the family of all dyadic cubes Q satisfying

$$t^q < \frac{1}{|Q|}\int_Q |f(x)|^q\, dx \tag{6.2}$$

These maximal cubes exist since, by Jensen's inequality, every cube Q satisfying (6.2) also satisfies

$$t^p < \frac{1}{|Q|}\int_Q |f(x)|^p\, dx$$

and consequently its volume is bounded by $t^{-p}\|f\|_p^p$. The cubes Q_j will satisfy:

$$t^q < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|^q\, dx \le 2^n t^q \tag{6.3}$$

We shall write $E_t = \bigcup_j Q_j$. Note that $E_t \subset \{x \in \mathbb{R}^n : M_q(f)(x) > t\}$ where $M_q(f)(x) = (M(|f|^q)(x))^{1/q}$. Observe also that $|f(x)| \le t$ for almost every $x \in \mathbb{R}^n \setminus E_t$.

We decompose $f = g_t + b_t$, where the good part g_t is given by

$$g_t(x) = f(x)\,\chi_{\mathbb{R}^n\setminus E_t}(x) + \sum_j f_{Q_j}\chi_{Q_j}(x)$$

and the bad part b_t is

$$b_t(x) = \sum_j b_t^j(x) \text{ with } b_t^j(x) = (f(x)-f_{Q_j})\,\chi_{Q_j}(x)$$

Let us assume first that $p_1 < \infty$. The case $p_1 = \infty$ is simpler and shall be treated at the end.

Let us look separately at the two pieces of the decomposition

a) $g_t \in L^{p_1}(\mathbb{R}^n)$. This is clear since $|g_t(x)| \le 2^{n/q}t$ for a.e.x. Actually

$$\int_{\mathbb{R}^n} |g_t(x)|^{p_1}\, dx = \int_{\mathbb{R}^n\setminus E_t} |f(x)|^{p_1}dx + \sum_j |f_{Q_j}|^{p_1}|Q_j| \le$$

$$\le \int_{\{x\in\mathbb{R}^n:|f(x)|\le t\}} |f(x)|^{p_1}dx + 2^{np_1/q}\, t^{p_1}|E_t| \le Ct^{p_1-p}\|f\|_p^p.$$

b) $b_t \in H^1(\mathbb{R}^n)$. Let us see why this is so and let us estimate the norm. Observe that each b_t^j lives in Q_j, has average 0 and satisfies:

$$\left(\frac{1}{|Q_j|}\int_{Q_j} |b_t^j|^q dx\right)^{1/q} \le 2\left(\frac{1}{|Q_j|}\int_{Q_j} |f(x)|^q dx\right)^{1/q} \le 2\cdot 2^{n/q}t = Ct.$$

It follows that $C^{-1}t^{-1}|Q_j|^{-1}b_t^j(x) = a_j(x)$ is a $(1,q)$-atom. Then, since $b_t(x) = \sum_j Ct\,|Q_j|\,a_j(x)$, we obtain that $b_t \in H^1(\mathbb{R}^n)$ with $\|b_t\|_{H^1(\mathbb{R}^n)} \le Ct\sum_j |Q_j| = Ct\,|E_t| \le Ct^{1-p}\|f\|_p^p$.

a) and b) imply, for any particular $t > 0$, that $T(f)$ is well defined. Besides:

$$\|Tf\|_p^p = p\int_0^\infty t^{p-1}|\{x\in\mathbb{R}^n:|Tf(x)| > t\}|\, dt \le$$

$$\le p\int_0^\infty t^{p-1}\,|\{x\in\mathbb{R}^n: |T(g_t)(x)| > t/2\}|\, dt +$$

$$+ p\int_0^\infty t^{p-1}|\{x\in\mathbb{R}^n: |T(b_t)(x)| > t/2\}|dt \le$$

$$\le Cp\int_0^\infty t^{p-1-p_1}\int_{\mathbb{R}^n} |g_t(x)|^{p_1}\, dxdt + Cp\int_0^\infty t^{p-2}\,\|b_t\|_{H^1(\mathbb{R}^n)}\, dt \le$$

$$\le Cp\int_0^\infty t^{p-1-p_1}\int_{\{x\in\mathbb{R}^n:|f(x)|\le t\}} |f(x)|^{p_1}dxdt + Cp\int_0^\infty t^{p-1}|E_t|dt$$

But:

$$\int_0^\infty t^{p-1-p_1}\int_{\{x\in\mathbb{R}^n:\,|f(x)|\le t\}}|f(x)|^{p_1}dxdt =$$

$$= \int_{\mathbb{R}^n}\left(\int_{|f(x)|}^\infty t^{p-1-p_1}dt\right)|f(x)|^{p_1}\,dx = \frac{1}{p_1-p}\int_{\mathbb{R}^n}|f(x)|^p\,dx \quad \text{and}$$

$$p\int_0^\infty t^{p-1}\,|E_t|dt \le p\int_0^\infty t^{p-1}\,|\{x\in\mathbb{R}^n:M_q(f)(x)>t\}|\,dt =$$

$$= \int_{\mathbb{R}^n}(M_q(f)(x))^p\,dx = \int_{\mathbb{R}^n}(M(|f|^q)(x))^{p/q}dx \le C\int_{\mathbb{R}^n}|f(x)|^p\,dx.$$

Therefore $\|Tf\|_p \le C\|f\|_p$ with C depending only on C_0, C_1, p_1 and p. This settles the case $p_1 < \infty$. When $p_1 = \infty$, we make a change of variables in the formula for $\|Tf\|_p^p$, writing with $A > 0$:

$$\|Tf\|_p^p = C\int_0^\infty t^{p-1}|\{x\in\mathbb{R}^n:\ |Tf(x)| > At\}|dt \le$$

$$\le C\int_0^\infty t^{p-1}|\{x\in\mathbb{R}^n:\ |T(g_t)(x)| > At/2\}|dt +$$

$$+ C\int_0^\infty t^{p-1}|\{x\in\mathbb{R}^n:\ |T(b_t)(x)| > At/2\}|\,dt.$$

Now since $|T(g_t)(x)| \le C_1 2^{n/q}t$, if we take $A > 2C_1\cdot 2^{n/q}$, the first integral in the sum will vanish. The second integral is estimated as before and the proof is ended. $\square$

A similar result holds for $H^{p_0}(\mathbb{R}^n)$ with $p_0 < 1$ in place of $H^1(\mathbb{R}^n)$. In general we can extend theorem 6.1. in the following way:

THEOREM 6.4. Let T *be an operator sending* $H^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$ *into measurable functions in* $\mathbb{R}^n$, *where* $0 < p_0 \le 1 < p_1 \le \infty$. *Suppose that*

i) T *is subadditive*

ii) T *is of weak type* (H^{p_0}, p_0), *by which we mean that there is a constant* C_0 *such that for every* $f\in H^{p_0}(\mathbb{R}^n)$ *and for every* $t > 0$:

$$|\{x\in\mathbb{R}^n:\ |Tf(x)| > t\}| \le C_0 t^{-p_0}\,\|f\|_{H^{p_0}(\mathbb{R}^n)}^{p_0}$$

iii) T *is of weak type* (p_1, p_1) *with constant* C_1.

Then, for every p *such that* $1 < p < p_1$, T *is bounded in* $L^p(\mathbb{R}^n)$ *with norm depending on* C_0, C_1, p_0, p_1 *and* p.

Proof: For $p_0 = 1$, theorem 6.4. coincides with theorem 6.1. which has been proved already. Let $p_0 < 1$. We proceed as in the proof of theorem 6.1. but making now

$$g_t(x) = f(x)\chi_{\mathbb{R}^n\setminus E_t}(x) + \sum_j P_{Q_j}(f)(x)\chi_{Q_j}(x)$$

and $b_t(x) = \sum_j b_t^j(x)$ with $b_t^j(x) = (f(x)-P_{Q_j}(f)(x))\chi_{Q_j}(x)$, where $P_{Q_j}(f)(x)$ is the unique polynomial of degree $\leq [n(\frac{1}{p_0} - 1)]$ having the same moments as f on Q_j up to order $[n(\frac{1}{p_0} - 1)]$.

Suppose $p_1 < \infty$.

a) $\int_{\mathbb{R}^n} |g_t(x)|^{p_1}dx \leq \int_{\{x\in\mathbb{R}^n: |f(x)|\leq t\}} |f(x)|^{p_1}dx + Ct^{p_1}|E_t| \leq$
$\leq Ct^{p_1-p}\|f\|_p^p$ because, after all: $|P_{Q_j}(f)(x)| \leq C|f_{Q_j}|$ for all $x \in Q_j$ as was shown in the proof of theorem 3.6.

b) $C^{-1}t^{-1}|Q_j|^{-1/p_0}\, b_t^j(x) = a_j(x)$ is a (p_0,q)-atom. Then, since $b_t(x) = \sum_j Ct\, |Q_j|^{1/p_0}\, a_j(x)$, we obtain that $b_t \in H^{p_0}(\mathbb{R}^n)$ with

$$\|b_t\|^{p_0}_{H^{p_0}(\mathbb{R}^n)} \leq Ct^{p_0} \sum_j |Q_j| = Ct^{p_0}\, |E_t| \leq Ct^{p_0-p}\, \|f\|_p^p$$

It follows that $L^p(\mathbb{R}^n) \subset H^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$ and, in particular, Tf is well defined. Besides:

$$\|Tf\|_p^p = p\int_0^\infty t^{p-1}|\{x \in \mathbb{R}^n: |Tf(x)| > t\}|dt \leq$$

$$\leq p\int_0^\infty t^{p-1}\, |\{x \in \mathbb{R}^n: |T(g_t)(x)| > t/2\}|\, dt +$$

$$+ p\int_0^\infty t^{p-1}|\{x \in \mathbb{R}^n: |T(b_t)(x)| > t/2\}|dt \leq$$

$$\leq Cp\int_0^\infty t^{p-1-p_1}\int_{\mathbb{R}^n} |g_t(x)|^{p_1}\, dxdt + Cp\int_0^\infty t^{p-1-p_0}\|b_t\|^{p_0}_{H^{p_0}(\mathbb{R}^n)}\, dt \leq$$

$$\leq Cp\int_0^\infty t^{p-1-p_1}\int_{\{x\in\mathbb{R}^n: |f(x)|\leq t\}} |f(x)|^{p_1}\, dx\, dt +$$

$$+ Cp\int_0^\infty t^{p-1}\, |E_t|dt \leq C\int_{\mathbb{R}^n} |f(x)|^p\, dx$$ exactly as before.

The case $p_1 = \infty$ is treated as in the previous proof. $\square$

A natural question arises at this point. Suppose that we are under the hypotheses of theorem 6.4. with $0 < p_0 < 1$. Can we allow $p = 1$ in the conclusion?. The Riesz transforms show that the answer to this question is negative (see next section). T does not need to be bounded in L^1.

However we can easily see that T has to be of weak type (1,1). To show this we proceed as in the proof of theorem 6.4. but taking $q = 1$. We still have, assuming $p_1 < \infty$

a) $\int_{\mathbb{R}^n} |g_t(x)|^{p_1} dx \leq \int_{\{x\in\mathbb{R}^n: |f(x)|\leq t\}} |f(x)|^{p_1} dx + Ct^{p_1}|E_t| \leq$
$\leq Ct^{p_1-1} \|f\|_1$.

b) $C^{-1}t^{-1}|Q_j|^{-1/p_0} b_t^j(x) = a_j(x)$ is a $(p_0,1)$-atom. This leads to

$$\|b_t\|^{p_0}_{H^{p_0}(\mathbb{R}^n)} \leq Ct^{p_0}|E_t| \leq Ct^{p_0-1} \|f\|_1.$$

It follows that $L^1(\mathbb{R}^n) \subset H^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$ and, in particular Tf is well defined. Besides, $|\{x \in \mathbb{R}^n: |Tf(x)| > t\}| \leq$
$\leq |\{x \in \mathbb{R}^n: |T(g_t)(x)| > t/2\}| + |\{x \in \mathbb{R}^n: |T(b_t)(x)| > t/2\}| \leq$
$\leq Ct^{-p_1} \int_{\mathbb{R}^n} |g_t(x)|^{p_1} dx + Ct^{-p_0}\|b_t\|^{p_0}_{H^{p_0}(\mathbb{R}^n)} \leq Ct^{-1} \|f\|_1$

In any case, for $p_0 < p \leq 1$, the natural domain for the operator T is $H^p(\mathbb{R}^n)$. We have the following result

THEOREM 6.5. Let T *be an operator sending* $H^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$ *into measurable functions in* $\mathbb{R}^n$, *where* $0 < p_0 < 1 \leq p_1 \leq \infty$. *Suppose that*

i) T *is sublinear, which means that* T *is subadditive and also* $|T(\lambda f)(x)| = |\lambda||Tf(x)|$ *for every* $\lambda \in \mathbb{R}$ *and every* f.

ii) T *is of weak type* (H^{p_0},p_0) *and*

iii) T *is of weak-type* (p_1,p_1)

Then, for every p *such that* $p_0 < p \leq 1$ *and* $p < p_1$, T *is bounded* *from* $H^p(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$.

Proof: Remember that every $f \in H^p(\mathbb{R}^n)$ can be written as $f = \sum_j \lambda_j a_j$ where the a_j's are (p,p_1)-atoms having vanishing moments up to order $[n(\frac{1}{p_0}-1)]$ and $\sum_j |\lambda_j|^p \leq C \|f\|^p_{H^p(\mathbb{R}^n)}$, the series converging to f in the sense of tempered distributions. The main part of the proof will be to show that for every (p,p_1)-atom a with moments vanishing up to order $[n(\frac{1}{p_0} - 1)]$ is $\|Ta\|_p \leq C$, an absolute constant. Let us show this. Let Q be a minimal cube containing the support of a. Suppose, for definiteness, that $p_1 < \infty$

a) $\|a\|^{p_1}_{p_1} \leq |Q|^{1-(p_1/p)}$. Thus, by iii):

$$|\{x \in \mathbb{R}^n: |Ta(x)| > t\}| \leq C_1 t^{-p_1}|Q|^{1-(p_1/p)} \quad \text{for every } t > 0$$

b) $|Q|^{(1/p)-(1/p_0)}a$ is a (p_0,p_1)-atom. Thus, by ii) and i):

$$|\{x \in \mathbb{R}^n: |Ta(x)| > t\}| =$$

$$=|\{x \in \mathbb{R}^n: |T(|Q|^{(1/p)-(1/p_0)}a)(x)| > |Q|^{(1/p)-(1/p_0)}t\}| \leq Ct^{-p_0}|Q|^{1-(p_0/p)}$$

Now we estimate $\|Ta\|^p_p$ by using b) for t near 0 and a) for t big. We get, with $R > 0$:

$$\|Ta\|^p_p = \int_0^\infty pt^{p-1} |\{x \in \mathbb{R}^n: |Ta(x)| > t\}| \, dt = \int_0^R + \int_R^\infty \leq$$

$$\leq C \int_0^R t^{p-1-p_0}|Q|^{1-(p_0/p)} dt + C \int_R^\infty t^{p-1-p_1} |Q|^{1-(p_1/p)} dt \leq$$

$$\leq C(R^{p-p_0}|Q|^{1-(p_0/p)} + R^{p-p_1} |Q|^{1-(p_1/p)}) = C \quad \text{is we choose}$$

$$R = |Q|^{-1/p}.$$

The case $p_1 = \infty$ is even simpler. We just use $\|a\|_\infty \leq |Q|^{-1/p}$ in place of a). It follows that $|Ta(x)| \leq C|Q|^{-1/p}$. Thus

$$\|Ta\|^p_p = \int_0^\infty pt^{p-1}|\{x \in \mathbb{R}^n: |Ta(x)| > t\}|dt \leq$$

$$\leq \int_0^{C|Q|^{-1/p}} t^{p-1-p_0} |Q|^{1-(p_0/p)} dt \leq C.$$

Next we have to put together the estimates for the different atoms. First, we have to see that $H^p \subset H^{p_0} + L^{p_1}$. It will be enough to consider the case $p_1 = 1$, since we already know that, for $p_1 > 1$, $L^1 \subset H^{p_0} + L^{p_1}$ Write $f \in H^p$ as $f = \sum \lambda_j a_j$ like at the beginning of the proof. Let Q_j be a minimal cube supporting a_j. Let $J_0 = \{j: |Q_j| < |\lambda_j|^p\}$ and $J_1 = \{j: j \notin J_0\}$. We claim that: $\sum_{j\in J_0} \lambda_j a_j \in H^{p_0}$ and $\sum_{j\in J_1} \lambda_j a_j \in L^1$. Indeed: for every j, $|Q_j|^{(1/p)-(1/p_0)} a_j = b_j$ is a (p_0,p_1)-atom. So $\sum_{j\in J_0} \lambda_j a_j = \sum_{j\in J_0} \lambda_j |Q_j|^{(1/p_0)-(1/p)} b_j$ and

$$\sum_{j\in J_0} |\lambda_j|^{p_0} |Q_j|^{1-(p_0/p)} \leq \sum_{j\in J_0} |\lambda_j|^{p_0} |\lambda_j|^{p-p_0} \leq \sum_j |\lambda_j|^p \leq \|f\|_{H^p}^p .$$

It follows that $\sum_{j\in J_0} \lambda_j a_j \in H^{p_0}$. Also for $j \in J_1$, $\|a_j\|_1 \leq |Q_j|^{1-(1/p)} \leq |\lambda_j|^{p-1}$, from which $\sum_{j\in J_1} |\lambda_j| \|a_j\|_1 \leq$ $\leq \sum_j |\lambda_j|^p \leq \|f\|_{H^p}^p$. This implies that $\sum_{j\in J_1} \lambda_j a_j \in L^1$.

Finally to end the proof of the theorem, let $f \in H^p$ and write $f = \sum_j \lambda_j a_j$ as above. By splitting f as before into a sum belonging to H^{p_0} and another in L^1 and realizing that T is at least of weak type (1,1), we see that $|T(f)(x)| \leq \sum_j |\lambda_j| |Ta_j(x)|$. Thus $\|Tf\|_p^p \leq C \sum_j |\lambda_j|^p \leq C\|f\|_{H^p}^p$ as we wanted to show. $\square$

Observe that for $p_1 = 1$, the condition iii) in theorem 6.5 can be replaced by

iii)' T is of weak type $(H^1,1)$

and still the conclusion of the theorem holds.

Indeed, let a be a (p,∞)-atom with moments vanishing up to order $[n(\frac{1}{p_0} - 1)]$. As before, if Q is a minimal cube supporting a, then $|Q|^{(1/p)-(1/p_0)}a$ is a (p_0,∞)-atom and it follows that $|\{x \in \mathbb{R}^n: |Ta(x)| > t\}| \leq Ct^{-p_0} |Q|^{1-(p_0/p)}$. But also now $|Q|^{(1/p)-1}a$ is a $(1,\infty)$-atom, and this implies that $|\{x \in \mathbb{R}^n: |Ta(x)| > t\}| \leq Ct^{-1} |Q|^{1-(1/p)}$. From these two facts, we prove, exactly as before, that $\|Ta\|_p^p \leq C$. To end the proof we also need to make sure that $H^p \subset H^{p_0} + H^1$. The proof is very similar to what we did to show that $H^p \subset H^{p_0} + L^1$. We write $f \in H^p$ as

$f = \sum_j \lambda_j a_j$ with the a_j's being (p,∞)-atoms having moments vanishing up to order $[n(\frac{1}{p_0} - 1)]$ and $\sum_j |\lambda_j|^p \le C \|f\|^p_{H^p}$. We split the sum as before, having $\sum_{J_0} \lambda_j a_j \in H^{p_0}$ and also $\sum_{J_1} \lambda_j a_j \in H^1$. Indeed $|Q_j|^{(1/p)-1} a_j = b_j$ is a $(1,\infty)$-atom. So $\sum_{J_1} \lambda_j a_j = \sum_{J_1} \lambda_j |Q_j|^{1-(1/p)} b_j$ with $\sum_{J_1} |\lambda_j| \, |Q_j|^{1-(1/p)} \le \sum_{J_1} |\lambda_j| |\lambda_j|^{p-1} = \sum_{J_1} |\lambda_j|^p < \infty$.

It is clear that, in general, we can prove the following result.

THEOREM 6.6. Let T *be an operator sending* $H^{p_0}(\mathbb{R}^n) + H^{p_1}(\mathbb{R}^n)$ *into measurable functions in* $\mathbb{R}^n$, *where* $0 < p_0 < p_1 \le 1$. *Suppose that*

i) T *is sublinear*

ii) T *is of weak type* (H^{p_0}, p_0), *and*

iii) T *is of weak type* (H^{p_1}, p_1)

Then, for every p *such that* $p_0 < p < p_1$, T *is bounded from* $H^p(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$. $\square$

7. ESTIMATES FOR OPERATORS ACTING ON H^p SPACES

We shall start by considering singular integrals. We already know (chapter II, corollary 5.14) that any singular integral operator maps $H^1(\mathbb{R}^n)$ boundedly into $L^1(\mathbb{R}^n)$. We shall presently see that any regular singular integral operator of principal value type is bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided $n/(n+1) < p \le 1$, and that p can be pushed further down to 0 by requiring more and more regularity of the kernel.

LEMMA 7.1. Let T *be a regular singular integral operator in* $\mathbb{R}^n$ *(that means that* T *has a kernel* K *satisfying* (5.18) *and* (5.19) *from chapter* II*). Let* a *be a* (p,∞)-*atom with* $n/(n+1) < p \le 1$. *Then* $\|Ta\|_p \le C$, *an absolute constant.*

Proof: Let Q be a minimal cube supporting a. Denote by x_0 the center of Q and call $\tilde{Q} = Q^{2\sqrt{n}}$. Then, for $x \in \mathbb{R}^n \setminus \tilde{Q}$,

$$|Ta(x)| \leq \int_{\mathbb{R}^n} |K(x-y)-K(x-x_0)|\,|a(y)|dy \leq C|x-x_0|^{-n-1}\int_{\mathbb{R}^n}|y-x_0||a(y)|dy \leq$$

$$\leq C|x-x_0|^{-n-1}\,|Q|^{1+(1/n)-(1/p)}$$

This implies that Ta is in L^p on $\mathbb{R}^n \setminus \tilde{Q}$, since $p(n+1) > n$. Actually

$$\int_{\mathbb{R}^n\setminus\tilde{Q}}|Ta(x)|^p dx \leq C\int_{C|Q|^{1/n}}^{\infty} r^{-(n+1)p+n-1}dr\,|Q|^{p+(p/n)-1} = C.$$

Since, also

$$\int_{\tilde{Q}} |Ta(x)|^p dx \leq \left(\int_{\mathbb{R}^n}|Ta(x)|^2 dx\right)^{p/2}.|\tilde{Q}|^{1-(p/2)} \leq C\,\|a\|_2^p|Q|^{1-(p/2)} \leq C$$

we get $\|Ta\|_p \leq C$. $\square$

Note that the result also holds for a (p,r)-atom even if $r < \infty$. Indeed, we still have the estimate

$$(7.2)\quad |Ta(x)| \leq C|x-x_0|^{-n-1}|Q|^{1+(1/n)-(1/p)} \quad \text{for } x \in \mathbb{R}^n \setminus \tilde{Q}$$

which is now obtained by using the appropriate version of Hölder's inequality, and on $\tilde{Q}$, we can use the fact that T is bounded in L^r if $r > 1$ or the weak type $(1,1)$ in case $r = 1$. This is entirely similar to what we did at the beginning of section 3 for the Hilbert transform H in $\mathbb{R}$.

From the lemma and the atomic description of $H^p(\mathbb{R}^n)$, we get:

THEOREM 7.3. Let T *be a regular singular integral operator of principal value type in* $\mathbb{R}^n$ *with kernel* K. *Then, for every* p *such that* $n/(n+1) < p \leq 1$, T *can be extended to a bounded operator between* $H^p(\mathbb{R}^n)$ *and* $L^p(\mathbb{R}^n)$. *Indeed, if* $f = \sum_j \lambda_j a_j$, *with* a_j (p,∞)-*atoms and* $\sum_j |\lambda_j|^p \leq C\,\|f\|^p_{H^p(\mathbb{R}^n)}$ *is an atomic decomposition for* $f \in H^p(\mathbb{R}^n)$; *then* $Tf = \sum_j \lambda_j\, Ta_j$ *in* $L^p(\mathbb{R}^n)$.

Proof: We may assume also $p < 1$. Start with f in a nice dense class of functions, say $f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Let $f = \sum_j \lambda_j a_j$ be an atomic decomposition for f with $\sum_j |\lambda_j|^p \leq C\|f\|^p_{H^p(\mathbb{R}^n)}$. We know (chapter II, corollary 5.22.) that, for almost every $x \in \mathbb{R}^n$ $Tf(x) = \lim_{\varepsilon\to 0} T_\varepsilon f(x)$ where

$$T_\varepsilon f(x) = \int_{|y|>\varepsilon} K(y)f(x-y)\,dy$$

We shall need a smooth truncation of the kernel. For that purpose, we shall use a radial C^∞ function ϕ in $\mathbb{R}^n$ such that: $\phi(x) = 0$ for $|x| \leq 1/2$, $\phi(x) = 1$ for $|x| \geq 1$ and $0 \leq \phi(x) \leq 1$ everywhere. Then:

$$T_\varepsilon f(x) = \int_{|y|>\varepsilon} K(y)f(x-y)dy = \int_{|y|>\varepsilon} K(y)\phi(y/\varepsilon)f(x-y)\,dy =$$
$$= \int_{\mathbb{R}^n} K(y)\phi(y/\varepsilon)f(x-y)dy - \int_{\varepsilon/2<|y|<\varepsilon} K(y)\phi(y/\varepsilon)f(x-y)\,dy.$$

Call $\tilde{K}_\varepsilon(y) = K(y)\phi(y/\varepsilon)$ and $\tilde{T}_\varepsilon f(x) = \int_{\mathbb{R}^n} \tilde{K}_\varepsilon(y)f(x-y)\,dy$

Then, we shall prove that $Tf(x) = \lim_{\varepsilon\to 0} \tilde{T}_\varepsilon f(x)$ for a.e. $x \in \mathbb{R}^n$. To show this, we just need to see that:

$$(7.4)\quad \int_{\varepsilon/2<|y|<\varepsilon} K(y)\phi(y/\varepsilon)f(x-y)dy \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Now (7.4) follows once these two facts are established:

a) (7.4) holds with $\psi \in \mathcal{S}(\mathbb{R}^n)$ in place of f.

b) The integral in (7.4) is bounded in absolute value by

$$\frac{C}{\varepsilon^n} \int_{|y|<\varepsilon} |f(x-y)|\,dy \leq CM(f)(x).$$

The way to derive (7.4) from a) and b) is already familiar to us (see, for instance, the proof of theorem 1.9. in chapter II) To obtain b) we just use the estimate $|K(x)| \leq C|x|^{-n}$. As for a), it will follow once we realize that

$$(7.5)\qquad \int_{\varepsilon/2<|y|<\varepsilon} K(y)\phi(y/\varepsilon)dy \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Indeed if (7.5) holds, we shall just need to see that

$$\int_{\varepsilon/2<|y|<\varepsilon} K(y)\phi(y/\varepsilon)(\psi(x-y)-\psi(x))\,dy \to 0 \quad \text{as} \quad \varepsilon \to 0$$

But this is obvious since the smoothness of ψ guarantees that the integrand is absolutely integrable around 0.

Now, (7.5) follows easily from the fact that K satisfies (5.4) in chapter II, after integrating by parts. Writing $\phi(x) = \phi(|x|)$ and $F(r) = \int_{\Sigma_{n-1}} K(ry')\,dy'\, r^{n-1}$, we get

$$\int_{\varepsilon/2<|y|<\varepsilon} K(y)\phi(y/\varepsilon)dy = \int_{\varepsilon/2}^{\varepsilon} F(r)\Phi(r/\varepsilon)dr = \int_0^{\varepsilon} F(r)dr -$$

$$- \int_{\varepsilon/2}^{\varepsilon} \int_0^{s} F(r)\,dr\, \varepsilon^{-1}\phi'(s/\varepsilon)\,ds.$$

Now from the fact that $\int_0^{\varepsilon} F(r)\,dr = \lim_{\delta\to 0} \int_{\delta<|y|<\varepsilon} K(y)dy \to 0$ as $\varepsilon \to 0$, we clearly obtain (7.5).

Let us look at the kernels $\tilde{K}_\varepsilon$. They satisfy uniform estimates

$$|\tilde{K}_\varepsilon(x)| \leq C|x|^{-n} \quad \text{for every} \quad x \neq 0, \text{ and}$$

$$|\tilde{K}_\varepsilon(x-y)-\tilde{K}_\varepsilon(x)| \leq C|y||x|^{-n-1} \quad \text{for} \quad 0 < 2|y| < |x|$$

(uniform means, of course, that C does not depend on $\varepsilon > 0$). The first estimate is obvious and the second can be derived in the following way: Let $0 < 2|y| < |x|$

$$|\tilde{K}_\varepsilon(x-y)-\tilde{K}_\varepsilon(x)| = |K(x-y)\phi(\tfrac{x-y}{\varepsilon}) - K(x)\phi(\tfrac{x}{\varepsilon})| \leq$$

$$\leq |K(x-y)-K(x)||\phi(\tfrac{x-y}{\varepsilon})| + |K(x)||\phi(\tfrac{x-y}{\varepsilon}) - \phi(\tfrac{x}{\varepsilon})|$$

The first term in this sum is bounded by $C|y||x|^{-n-1}$ where C is the constant for K . The second term is bounded by $C|x|^{-n}|y/\varepsilon|$. Note, however, that this second term vanishes unless $\varepsilon/3 < |x| < 2\varepsilon$, so that the estimate can be rewritten as $C|y||x|^{-n-1}$.

The fact that the kernels $\tilde{K}_\varepsilon$ satisfy these uniform estimates implies, of course, that for every (p,∞)-atom, with our restriction on p, $\|\tilde{T}_\varepsilon a\|_p \leq C$ with C independent of ε.

Observe that each $\tilde{K}_\varepsilon$ is a bounded function with $|\tilde{K}_\varepsilon(x)| \le C\varepsilon^{-n}$. Also we have an inequality

$$(7.6) \qquad |\tilde{K}_\varepsilon(x-y) - \tilde{K}_\varepsilon(x)| \le C_\varepsilon\, |y|$$

with C_ε depending on ε, valid for every $x,y \in \mathbb{R}^n$. Indeed, since $\tilde{K}_\varepsilon$ is bounded, (7.6) needs to be proved only for $|y|$ small, say for $|y| < \varepsilon/8$. We can also assume $|x| > \varepsilon/4$, since otherwise, the left hand side of (7.6) vanishes. Then we have $2|y| < |x|$ and we can use the inequality proved above, to obtain:

$$|\tilde{K}_\varepsilon(x-y) - \tilde{K}_\varepsilon(x)| \le C|x|^{-n-1}|y| \le C\varepsilon^{-n-1}|y|.$$

Now, since $\tilde{K}_\varepsilon$ is bounded, (7.6) implies that

$$|\tilde{K}_\varepsilon(x-y) - \tilde{K}_\varepsilon(x)| \le C_{\varepsilon,\alpha}\, |y|^\alpha$$

for every $0 < \alpha \le 1$ (We only need to consider $|y| < 1$). In other words: $\tilde{K}_\varepsilon \in \Lambda_\alpha(\mathbb{R}^n)$ for all $0 < \alpha \le 1$. For $n/(n+1) < p < 1$, we have $0 < \alpha = n((1/p)-1) < 1$. Thus $\tilde{K}_\varepsilon \in \Lambda_{n((1/p)-1)}(\mathbb{R}^n)$. As we saw in section 5, this guarantees that $\tilde{K}_\varepsilon$ can be integrated against H^p functions, and provides a linear functional on $H^p(\mathbb{R}^n)$. This implies that

$$\tilde{T}_\varepsilon f(x) = \sum_j \lambda_j \tilde{T}_\varepsilon a_j(x)$$

and, consequently:

$$\|\tilde{T}_\varepsilon f\|_p^p \le \sum_j |\lambda_j|^p \|\tilde{T}_\varepsilon a_j\|_p^p \le C \sum_j |\lambda_j|^p \le C \|f\|^p_{H^p(\mathbb{R}^n)}.$$

Now, from $Tf(x) = \lim_{\varepsilon\to 0} \tilde{T}_\varepsilon f(x)$ a.e., we can get, by using Fatou's lemma: $\|Tf\|_p^p \le C\|f\|^p_{H^p(\mathbb{R}^n)}$. After this, the operator T can be extended by continuity to the whole space $H^p(\mathbb{R}^n)$.

Denote its extension also by T. Then, if $f \in H^p(\mathbb{R}^n)$ has an atomic decomposition $f = \sum_j \lambda_j a_j$ with $\sum_j |\lambda_j|^p \le C \|f\|^p_{H^p(\mathbb{R}^n)}$; since $f = \lim_{N\to\infty} \sum_{j=1}^N \lambda_j a_j$ in $H^p(\mathbb{R}^n)$, we have $Tf = \lim_{N\to\infty} \sum_{j=1}^N \lambda_j Ta_j = \sum_j \lambda_j Ta_j$ in $L^p(\mathbb{R}^n)$, as we wanted to prove. □

Observe that, even though the statement was made for (p,∞)-atoms, the conclusion holds also for a decomposition into (p,r)-atoms with any admissible r.

Now we shall impose more regularity to the kernel obtaining H^p, L^p boundedness for p's closer to 0. We get the following extension of theorem 7.3.

THEOREM 7.7. Let T *be a singular integral operator of principal value type in* $\mathbb{R}^n$ *with a kernel* K *of class* C^{k+1} *outside the origin,* $k = 0, 1, 2, \ldots,$ *and satisfying, for every multi-index* β *such that* $|\beta| \leq k+1$, *and every* $x \neq 0$:

$$|D^\beta K(x)| \leq C\, |x|^{-n-|\beta|}$$

Then, for every p *such that* $n/(n+k+1) < p \leq 1$, T *extends to a bounded operator between* $H^p(\mathbb{R}^n)$ *and* $L^p(\mathbb{R}^n)$. *This extended operator sends the distribution* $f \in H^p(\mathbb{R}^n)$ *having an atomic decomposition* $f = \sum \lambda_j a_j$ *in* $H^p(\mathbb{R}^n)$, *into the function* $Tf = \sum \lambda_j Ta_j$.

Proof: We may assume $p < 1$. The first thing we do is to extend lemma 7.1. to our general situation. We have a fixed p with $n/(n+k+1) < p < 1$. We take a (p,∞)-atom a and we have to make sure that $\|Ta\|_p \leq C$, an absolute constant. The main difference with the proof of lemma 7.1. is that now we estimate $|Ta(x)|$ for $x \in \mathbb{R}^n \setminus \tilde{Q}$ by substracting from $K(x-y)$ the Taylor polynomial of K at $x-x_0$ of degree $N = [n(\frac{1}{p}-1)]$. By doing this we get

$$|Ta(x)| \leq C|x-x_0|^{-n-1-N}|Q|^{1-(1/p)+((N+1)/n)}$$

All this is possible since $n/(n+k+1) < p$ is equivalent to $n(\frac{1}{p}-1) < k+1$, and consequently: $N \leq k$. The estimate implies that Ta is in L^p on $\mathbb{R}^n \setminus \tilde{Q}$ because $p(n+1+N) > n$. Actually

$$\int_{\mathbb{R}^n \setminus \tilde{Q}} |Ta(x)|^p\, dx \leq C \int_{C|Q|^{1/n}}^{\infty} r^{-(n+1+N)p+n-1} dr\, |Q|^{p-1+p((N+1)/n)} = C.$$

Combining this with the inequality $\int_{\tilde{Q}} |Ta(x)|^p\, dx \leq C$, which is obtained exactly as before, we arrive at $\|Ta\|_p \leq C$.

To end the proof we proceed as in the proof of theorem 7.3. Take $f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with an atomic decomposition $f = \sum_j \lambda_j a_j$, $\sum |\lambda_j|^p \leq C\|f\|^p_{H^p(\mathbb{R}^n)}$. We consider the smoothly truncated kernels $\tilde{K}_\varepsilon$ defined exactly as in the proof of theorem 7.3. and the corresponding operators $\tilde{T}_\varepsilon g = \tilde{K}_\varepsilon * g$.

Since T is a regular singular integral operator of principal value type, we still have $Tf(x) = \lim_{\varepsilon\to 0} \tilde{T}_\varepsilon f(x)$ for a.e. $x \in \mathbb{R}^n$. Now the kernels $\tilde{K}_\varepsilon$ satisfy:

1) $|D^\beta \tilde{K}_\varepsilon(x)| \leq C|x|^{-n-|\beta|}$, for every multi-index β such that $|\beta| \leq k+1$, and every $x \neq 0$, with C independent of $\varepsilon > 0$.

2) $\tilde{K}_\varepsilon \in \Lambda_\alpha(\mathbb{R}^n)$ with $\alpha = n(\frac{1}{p} - 1)$,

Indeed, the derivative appearing in 1) is a linear combination of terms ofthe form

$$\varepsilon^{-|\sigma|} D^\gamma K(x) D^\sigma \phi(x/\varepsilon), \text{ with } |\gamma| + |\sigma| = |\beta|$$

But each of these terms is bounded by $C\,|x|^{-n-|\gamma|}\,\varepsilon^{-|\sigma|}$. If $\sigma = 0$, this is the required estimate. If $\sigma \neq 0$, the corresponding term may be $\neq 0$ only for $1/2 < |x/\varepsilon| < 1$. Thus, we finally obtain the estimate $C|x|^{-n-|\beta|}$.

2) follows from the fact that each $\tilde{K}_\varepsilon$ is bounded and has bounded derivatives up to order $k+1$ and $[\alpha] = [n(\frac{1}{p} - 1)] \leq k$.

With the aid of 1) and 2), the proof is completed as follows: 2) implies that $\tilde{K}_\varepsilon$, and consequently $\tilde{K}_\varepsilon(x-\cdot)$, can be integrated against H^p functions and provides a linear functional on $H^p(\mathbb{R}^n)$. Thus, we get

$$\tilde{T}_\varepsilon f(x) = \sum_j \lambda_j \tilde{T}_\varepsilon a_j(x)$$

Then, 1) implies that we have uniform estimates $\|\tilde{T}_\varepsilon a_j\|^p_p \leq C$. Therefore, $\|\tilde{T}_\varepsilon f\|^p_p \leq \sum_j |\lambda_j|^p \|\tilde{T}_\varepsilon a_j\|^p_p \leq C \sum_j |\lambda_j|^p \leq C\|f\|^p_{H^p(\mathbb{R}^n)}$. By using Fatou's lemma we conclude that $\|Tf\|^p_p \leq C\,\|f\|^p_{H^p(\mathbb{R}^n)}$. The proof finishes as that of theorem 7.3. $\square$

Observe that any singular integral operator T actually maps $H^1(\mathbb{R}^n)$ boundedly into itself. In order to prove this we just need to see that $R_j(Tf) \in L^1(\mathbb{R}^n)$ for every $j = 1, \ldots, n$, as soon as $f \in H^1(\mathbb{R}^n)$. But $R_j(Tf) = T(R_jf)$ and it follows from the characterization of $H^1(\mathbb{R}^n)$ by means of systems of conjugate harmonic functions, that each $R_jf \in H^1(\mathbb{R}^n)$. Therefore $T(R_jf) \in L^1(\mathbb{R}^n)$.

It is natural to seek an extension of this result to $p < 1$ for nice singular integrals. As usual we start by analizing the action of the operator on an atom.

THEOREM 7.8. Let T *and* p *be as in theorem 7.7. and let* a *be a* (p,∞)-*atom. Then* $Ta \in H^p(\mathbb{R}^n)$ *and*

$$\|Ta\|_{H^p(\mathbb{R}^n)} \leq C, \text{ \textit{independent of} } a.$$

Proof: Since T commutes with translations, we may assume that a is supported in a ball $B = B(0,r)$ and $|a(x)| \leq C|B|^{-1/p}$ for every x, with C a geometric constant. We shall produce an atomic decomposition for Ta by decomposing the kernel K in a smooth way. Take functions $\phi_j(t)$, $j = 0, 1, 2, \ldots$ C^∞ in $(0,\infty)$, satisfying $\phi_j \geq 0$, $\sum_{j=0}^{\infty} \phi_j(t) = 1$ for every $t \in (0,\infty)$, ϕ_0 is supported in $[0,2r]$, ϕ_j is supported in $[2^{j-1}r, 2^{j+1}r]$ for $j \geq 1$ and $|\phi_j^{(k)}(t)| \leq C_k t^{-k}$ for every $t > 0$, every $k = 0, 1, 2, \ldots$ and every j, with C_k depending only on k and being independent of r (This partition of unity can be obtained as in chapter II, lemma 6.5.).

Now we define for each $j = 0, 1, 2, \ldots$: $K_j(x) = K(x)\phi_j(|x|)$, and observe that all the K_j's satisfy the same estimates as K with a uniform constant. We have a decomposition:

$$Ta(x) = \sum_{j=0}^{\infty} K_j * a(x)$$

valid a.e. and also in the sense of distributions, since it is actually a locally finite sum. Each $K_j * a$ has vanishing moments up to order $N = [n(\frac{1}{p} - 1)]$, since convolution preserves this property of a. Now we just need to study the relation between the size and the support of each K_j*a.

Convolution with K_0 is an operator bounded in L^2. Thus:

$$\|K_0*a\|_2 \le C\|a\|_2 \le C|B|^{1/2-1/p} = C(r^n)^{(1/2)-(1/p)}$$

Since K_0*a has support contained in $B(0,3r)$, we can write $K_0*a = = \lambda_0 a_0$ where a_0 is a $(p,2)$-atom.

To estimate K_j*a for $j \ge 1$ we use the fact that $|D^\beta K_j(x)| \le \le C(2^j r)^{-n-|\beta|^j}$ for every $|\beta| \le k+1$, in particular for $|\beta| = N+1 \le k+1$. We write

$$K_j*a(x) = \int_{\mathbb{R}^n} (K_j(x-y)-P(-y))a(y)\, dy$$

where P is the Taylor polynomial of degree N at x for the function K_j. We get

$$|K_j*a(x)| \le C(2^j r)^{-n-N-1}\int_{|y|<r} |y|^{N+1}|a(y)|dy \le$$
$$\le C(2^j r)^{-n-N-1} r^{N+1} r^{n-(n/p)} = C2^{j(n((1/p)-1)-N-1)}(2^j r)^{-n/p}$$

Since K_j*a is supported in $B(0,2^{j+2}r)$, we can write $K_j*a(x) = = \lambda_j a_j(x)$ where a_j is a (p,∞)-atom and $\lambda_j = C2^{j(n((1/p)-1)-N-1)}$.

Note that $n(\frac{1}{p} - 1) - N - 1 < 0$, and, consequently $\sum |\lambda_j|^p \le C$, a fixed finite number.

Thus, we have shown that $Ta = \sum_{j=0}^{\infty} \lambda_j a_j$ in the distribution sense, where the a_j's are $(p,2)$-atoms and $\sum_{j=0}^{\infty} |\lambda_j|^p \le C$. This proves that $Ta \in H^p(\mathbb{R}^n)$ with $\|Ta\|_{H^p(\mathbb{R}^n)} \le C$. □

We can finally give

THEOREM 7.9 Let T *and* p *be as in theorem 7.7. Then* T *extends to a bounded operator in* $H^p(\mathbb{R}^n)$.

Proof: Let $f \in L^2 \cap H^p(\mathbb{R}^n)$ and suppose

$$f = \sum_j \lambda_j a_j$$

is a decomposition of f into (p,∞)-atoms a_j with $\sum_j |\lambda_j|^p \leq$ $\leq C \|f\|^p_{H^p(\mathbb{R}^n)}$.

If we are able to show that

$$(7.10) \qquad Tf = \sum_j \lambda_j Ta_j$$

as tempered distributions; then, since theorem 7.8. implies that the series converges in $H^p(\mathbb{R}^n)$, we shall have: $Tf \in H^p(\mathbb{R}^n)$ and

$$\|Tf\|^p_{H^p(\mathbb{R}^n)} \leq \sum_j |\lambda_j|^p \|Ta_j\|^p_{H^p(\mathbb{R}^n)} \leq C \sum_j |\lambda_j|^p \leq$$
$$\leq C \|f\|^p_{H^p(\mathbb{R}^n)}.$$

(7.10) can be proved by using the smoothly truncated kernels $\tilde{K}_\varepsilon$ and the corresponding convolution operators $\tilde{T}_\varepsilon$ appearing in theorems 7.3. and 7.7. Let us see how. First of all, a simple modification of the argument given in theorem 7.3. implies that

$$\tilde{T}_\varepsilon f \longrightarrow Tf \quad \text{in} \quad L^2(\mathbb{R}^n) \quad \text{as} \quad \varepsilon \to 0.$$

This also follows from the fact (established in chapter V just before corollary 4.11.)that the maximal operator of the smooth truncations is bounded in L^2. Now

$$\tilde{T}_\varepsilon f = \sum_j \lambda_j \tilde{T}_\varepsilon a_j.$$

This is a pointwise identity and also an identity between tempered distributions. If we are able to show that

$$\tilde{T}_\varepsilon a_j \longrightarrow Ta_j \quad \text{in} \quad H^p(\mathbb{R}^n) \quad \text{as} \quad \varepsilon \to 0$$

for every j with $\|\tilde{T}_\varepsilon a_j\|^p_{H^p(\mathbb{R}^n)} \leq C$

we shall have

$$\sum_j \lambda_j \tilde{T}_\varepsilon a_j \longrightarrow \sum_j \lambda_j Ta_j \quad \text{in} \quad H^p(\mathbb{R}^n) \quad \text{as} \quad \varepsilon \to 0$$

from which 7.10 follows immediately, that way completing the proof of the theorem.

To see that, for an atom a

$$\tilde{T}_\varepsilon a \longrightarrow Ta \quad \text{in} \quad H^p(\mathbb{R}^n) \quad \text{as} \quad \varepsilon \to 0$$

is quite simple.

As in the proof of theorem 7.8., we decompose

$$Ta = \sum_j K_j * a$$

$$\tilde{T}_\varepsilon a = \sum_j \tilde{K}_{\varepsilon,j} * a$$

Now $\tilde{K}_{\varepsilon,j}(x) = \tilde{K}_\varepsilon(x)\phi_j(|x|) = K(x)\phi(x/\varepsilon)\phi_j(|x|) = (K_j)^\sim_\varepsilon$.

Thus, clearly, for each j

$$\tilde{K}_{\varepsilon,j} * a \longrightarrow K_j * a \quad \text{in} \quad L^2 \quad \text{as} \quad \varepsilon \to 0$$

and, since both functions have the same support:

$$\tilde{K}_{\varepsilon,j} * a \longrightarrow K_j * a \quad \text{in} \quad H^p(\mathbb{R}^n) \quad \text{as} \quad \varepsilon \to 0$$

Actually, as the proof of theorem 7.8. shows,

$$\tilde{K}_{\varepsilon,j} * a = \mu_j\, b_{j,\varepsilon} \quad \text{and} \quad K_j * a = \mu_j\, b_j$$

with $b_{j,\varepsilon} \longrightarrow b_j$ in $H^p(\mathbb{R}^n)$ as $\varepsilon \to 0$

and uniformly bounded norms, and $\sum_j |\mu_j|^p < \infty$. This clearly gives $\tilde{T}_\varepsilon a \longrightarrow Ta$ in $H^p(\mathbb{R}^n)$ as $\varepsilon \to 0$, with uniformly bounded norms. $\square$

We have seen that atoms are very convenient for studying the behaviour of certain operators T, like singular integrals, on H^p. To prove H^p, L^p estimates we just need to study the size of Ta for an atom a. We have also seen that these operators map H^p boundedly into itself. There is an unsatisfactory aspect in this second kind of results, namely, that for a general atom a, Ta is not an atom itself but has to be decomposed into atoms. Indeed, in general Ta will not have compact support. We are going to find a class of functions more general than atoms, which still generate H^p and include the functions Ta for every atom a. These functions will naturally decompose into atoms, and will be called molecules.

To motivate the definition of a molecule, let us look back at the estimates obtained for Ta in lemma 7.1. We assume for simplicity $n = 1$ and $p = 1$. Let a be an atom supported in an interval I centered at 0. Then we have the estimates:

$$\|Ta\|_2 \le C\,\|a\|_2 \le C|I|^{-1/2}$$

and

$$|Ta(x)| \le C\,|x|^{-2}|I|, \quad \text{for} \quad |x| > |I|$$

This second estimate can be rewritten as

$$|x|^2|Ta(x)|^2 \le C|x|^{-2}\,|I|^2, \quad \text{for} \quad |x| > |I|$$

which implies

$$\int_{|x|>|I|} |x|^2|Ta(x)|^2dx \le C|I|$$

Since $\displaystyle\int_{|x|<|I|} |x|^2|Ta(x)|^2dx \le |I|^2\cdot C|I|^{-1} = C|I|$

we get

$$\int_{-\infty}^{\infty} |x|^2|Ta(x)|^2\,dx \le C|I|.$$

This, combined with the first estimate, leads to

$$\int_{-\infty}^{\infty} |Ta(x)|^2\,dx \cdot \int_{-\infty}^{\infty} |x|^2|Ta(x)|^2dx \le C.$$

This is a very nice condition as the following lemmas show

LEMMA 7.11. Let f *be a function on the real line* $\mathbb{R}$ *such that both* $f(x)$ *and* $x\cdot f(x)$ *belong to* $L^2(\mathbb{R})$. *Then* $f \in L^1(\mathbb{R})$. *Actually*

$$\left(\int_{-\infty}^{\infty}|f(x)|dx\right)^2 \le 8\cdot\left(\int_{-\infty}^{\infty}|f(x)|^2dx\right)^{1/2}\left(\int_{-\infty}^{\infty}|x|^2|f(x)|^2dx\right)^{1/2}$$

Proof:

$$\int_{-\infty}^{\infty} |f(x)|dx \le \int_{|x|<r} |f(x)|dx + \int_{|x|>r} |f(x)|dx \le$$

$$\le \left(\int_{-\infty}^{\infty} |f(x)|^2dx\right)^{1/2}(2r)^{1/2} + \left(\int_{-\infty}^{\infty} |f(x)|^2|x|^2dx\right)^{1/2}\left(\int_{|x|>r} \frac{dx}{|x|^2}\right)^{1/2} =$$

$$= 2^{1/2}\ (r^{1/2}\|f\|_2 + r^{-1/2}\|xf(x)\|_2)$$

By taking $r = \|f\|_2^{-1}\ \|xf(x)\|_2$ we get

$$\int_{-\infty}^{\infty} |f(x)|dx \le 2^{1/2}.2\cdot\|f\|_2^{1/2}\ \|xf(x)\|_2^{1/2}.$$

whose square is the inequality in the statement. $\square$

LEMMA 7.12. Suppose f *is as in the previous lemma and* $\int_{-\infty}^{\infty} f(x)dx = 0$. *Then* $f \in H^1(\mathbb{R})$ *and*

$$\|f\|_{H^1(\mathbb{R})}^2 \le C\ \|f\|_2\ \|xf(x)\|_2$$

Proof: We already know that $f \in L^1(\mathbb{R})$. We shall prove that, also, $Hf \in L^1(\mathbb{R})$. This is quite easy. First of all, since $f \in L^2(\mathbb{R})$ and the Hilbert transform H is bounded in $L^2(\mathbb{R})$, we have $Hf \in L^2(\mathbb{R})$. Secondly

$$xHf(x) = \text{p.v.}\ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x-y}\ f(y)\ dy =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)dy + \text{p.v.}\ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{x-y}\ f(y)\ dy = H(yf(y))(x)$$

since f has vanishing integral. Again, by the L^2 boundedness of H, we get that $x\cdot Hf(x)$ belong to $L^2(\mathbb{R})$. Then, lemma 7.11. applied to Hf implies that $Hf \in L^1(\mathbb{R})$ as we wanted to show. $\square$

Now, it seems natural to call molecule (centered at $x_0 \in \mathbb{R}$) to any real valued function M on the line satisfying the two conditions:

i) $$\left(\int_{-\infty}^{\infty} |M(x)|^2dx\right)^{1/4}\left(\int_{-\infty}^{\infty} |x-x_0|^2|M(x)|^2dx\right)^{1/4} < \infty$$

ii) $$\int_{-\infty}^{\infty} M(x)\ dx = 0$$

The left hand side of i) will be called "molecular norm" of M and will be denoted by $N(M)$. It is clear that every (1,2)-atom a is a molecule with molecular norm bounded by a constant independent of the atom. Indeed, if I is the minimal interval containing the support of a, and x_0 is the center of I:

$$\|a\|_2 \leq |I|^{-1/2} \quad \text{and} \quad \| \, |\cdot - x_0| \, a\|_2 \leq |I|^{1/2}$$

so that $N(a) \leq 1$.

We have obtained the following characterization of H^1:

A function f is in H^1 if and only if $f(x) = \sum_{j=1}^{\infty} M_j(x)$ for a.e.x, with the M_j's being molecules and $\sum_{j=1}^{\infty} N(M_j) < \infty$.

Next, we face the task of extending the molecular characterization to $H^p(\mathbb{R}^n)$ for arbitrary p and n. It will also be convenient to consider L^q norms for arbitrary $q > 1$ rather than just L^2 norms.

DEFINITION 7.13. Let $0 < p \leq 1 < q \leq \infty$ and $b > 1/p - 1/q$. Then, a (p,q,b)-molecule centered at $x_0 \in \mathbb{R}^n$ is a real-valued function M defined on $\mathbb{R}^n$ and satisfying:

i) $$\|M\|_q^{1-\theta} \| \, |\cdot - x_0|^{nb} M\|_q^{\theta} < \infty$$

where $\theta = (1/p-1/q)/b$, *so that* $0 < \theta < 1$

ii) $$\int_{\mathbb{R}^n} x^{\beta} M(x)\,dx = 0$$ *for every multi-index* β *such that* $|\beta| \leq [n((1/p)-1)] = N$

The left hand side of i) will be called the "molecular norm" of M and it will be denoted by $N(M)$.

Observe that condition i) implies that all the integrals in ii) are absolutely convergent. Indeed, we just need to see that

$$\int_{|x|>1} |x|^N |M(x)| \, dx < \infty$$

and this follows from Hölder's inequality in this way:

$$\int_{|x|>1} |x|^N |M(x)|\,dx \le \| |x|^{nb} M\|_q \; \| |x|^{N-nb} \chi_{\{|x|>1\}} \|_{q'} < \infty$$

since $nb-N > n\left(\frac{1}{p} - \frac{1}{q} - \frac{1}{p} + 1\right) = \frac{n}{q'}$.

Note that the molecules we used to motivate the formal definition are, in our present notation (1,2,1)-molecules in $\mathbb{R}$.

LEMMA 7.14. A (p,q,b)*-molecule is also a* $(p,q,\bar{b})$*-molecule for any admissible* $\bar{b} < b$

Proof: Suppose $1/p - 1/q < \bar{b} < b$, and let M be a (p,q,b)-molecule centered at $x_0 \in \mathbb{R}^n$. Then

$$\| |\cdot - x_0|^{n\bar{b}} M\|_q \le R^{n\bar{b}} \|M\|_q + R^{n(\bar{b}-b)} \| |\cdot - x_0|^{nb} M\|_q$$

for every $R > 0$, as one sees by looking separately at the cases $|x-x_0| < R$ and $|x-x_0| > R$.

Taking $R = \left(\| |\cdot - x_0|^{nb} M\|_q \,/\, \|M\|_q\right)^{1/nb}$ we get:

$$\| |\cdot - x_0|^{n\bar{b}} M\|_q \le 2\|M\|_q^{1-(\bar{b}/b)} \| |\cdot - x_0|^{nb} M\|_q^{\bar{b}/b}$$

and, consequently, writing $\bar{\theta} = (1/p - 1/q)/\bar{b} = \theta\, b/\bar{b}$:

$$\|M\|_q^{1-\bar{\theta}} \| |\cdot - x_0|^{n\bar{b}} M\|_q^{\bar{\theta}} \le 2^{\bar{\theta}} \|M\|_q^{1-\theta} \| |\cdot - x_0|^{nb} M\|_q^{\theta}. \quad \square$$

LEMMA 7.15. Let $0 < p \le 1 < q \le \infty$. *Then, a* (p,q)*-atom is always a* (p,q,b)*-molecule for any admissible* b, *with molecular norm bounded by a constant independent of the atom*.

Proof: Let a be a (p,q)-atom. Let Q be a minimal cube supporting a, and let x_0 be the center of Q. Suppose $b > 1/p - 1/q$. Then:

$$\|a\|_q \le |Q|^{1/q-1/p} \quad \text{and}$$

$$\| |\cdot - x_0|^{nb} a\|_q \le C|Q|^{b+1/q-1/p}.$$

Therefore, with $\theta = (1/p - 1/q)/b$, we have

$$\|a\|_q^{1-\theta} \| |\cdot - x_0|^{nb} a\|_q^{\theta} \le C < \infty$$

since $(1-\theta)(1/q-1/p) + \theta(b+1/q-1/p) = 1/q - 1/p + b\theta = 0$.

It is clear that C is a geometric constant independent of a. $\square$

The main result is the following:

THEOREM 7.16. Let M *be a* (p,q,b)-*molecule in* $\mathbb{R}^n$. *Then* $M \in H^p(\mathbb{R}^n)$ *and*

$$\|M\|_{H^p(\mathbb{R}^n)} \leq C\, N(M)$$

with C *independent of the molecule.*

Proof: We may assume, without loss of generality, that M is centered at the origin, and also that $N(M) = 1$. We define σ by setting

$$\sigma^{n(1/p-1/q)} = \|M\|_q^{-1}$$

and consider the sets:

$$E_0 = \{x \in \mathbb{R}^n : |x| \leq \sigma\} \quad \text{and}$$

$$E_k = \{x \in \mathbb{R}^n : 2^{k-1}\sigma < |x| \leq 2^k\sigma\}, \text{ for } k = 1, 2, 3, \ldots$$

We shall denote by M_k the restriction of M to E_k, that is: $M_k(x) = M(x)\chi_{E_k}(x)$.

We shall also consider, for each k, the unique polynomial of degree $N = [n((1/p)-1)]$ whose restriction to E_k, denoted by P_k, satisfies

$$\int_{\mathbb{R}^n} (M_k(x)-P_k(x))x^\alpha dx = 0$$

for every multi-index α with $|\alpha| \leq N$.

The proof will consist of two parts

1) To show that each M_k-P_k is a multiple of a (p,q)-atom with a sequence of coefficients in l^p.

2) To show, using summation by parts, that $\sum_k P_k$ can be written as an infinite linear combination of (p,∞)-atoms, with a sequence of coefficients in l^p.

Since $M = \sum_k M_k = \sum_k (M_k - P_k) + \sum_k P_k$

and the series in 1) and 2) will be seen to converge locally in L^q, the theorem will be proved.

Let us start with part 1). It is clear that $M_k - P_k$ has the right cancellation properties, so we just need to worry about the relation between the size and the support.

$M_k - P_k$ is supported in the ball $B(0, 2^k\sigma)$. Thus, a minimal cube Q_k containing the support of $M_k - P_k$ has measure equal to $C(2^k\sigma)^n$, with C a geometric constant.

As for the size, we have

$$\|M_k - P_k\|_{L^q(dx/|Q_k|)} = C\|M_k - P_k\|_{L^q(dx/|E_k|)} \leq$$

$$\leq C\|M_k\|_{L^q(dx/|E_k|)} \text{ , since } P_k \text{ is bounded by a constant}$$

times the average of $|M_k|$ over E_k (see below).

The definition of σ together with the fact that M is a (p,q,b)-molecule centered at the origin with $N(M) = 1$, imply that

$$\sigma^{-nb(1-\theta)} \| |x|^{nb} M\|_q \leq 1$$

where $\theta = (1/p - 1/q)/b$ as in definition 7.13. It will be convenient to write this estimate as

$$\| |x|^{nb} M\|_q \leq \sigma^{na}$$

with $a = b(1-\theta) = b - (1/p - 1/q) > 0$.

Let us go back to the estimates for $M_k - P_k$

$$\|M_0\|_{L^q(dx/|E_0|)} \leq \|M\|_q |E_0|^{-1/q} = C\sigma^{-n/p}$$

$$\|M_k\|_{L^q(dx/|E_k|)} \leq C(2^k\sigma)^{-n/q}\|M\cdot(|x|/(2^k\sigma))^{nb}\|_q \leq$$

$$\leq C(2^k\sigma)^{-n/q-nb}\sigma^{na} = C(2^k\sigma)^{-n/p}(2^{na})^{-k}$$

It follows that, for $k = 0, 1, 2, \ldots$

$$C(2^{na})^k(M_k - P_k) = A_k$$

is a (p,q)-atom, C being an absolute constant. In other words:

$$M_k - P_k = \lambda_k A_k$$

where A_k is a (p,q)-atom and $\lambda_k = C(2^{na})^{-k}$. Observe that $\sum_{k=0}^{\infty} \lambda_k^p = \sum_{k=0}^{\infty} C^p(2^{nap})^{-k} < \infty$ as we wanted to show.

Next, we come to part 2), where we have to deal with $\sum_{k=0}^{\infty} P_k$

For $k = 0, 1, 2, \ldots$ and α a multi-index such that $|\alpha| \leq N$, let $\phi_\alpha^k(x)$ be the function on E_k (actually the restriction to E_k of a polynomial of degree $\leq N$) uniquely determined by the conditions

$$\frac{1}{|E_k|}\int_{E_k} \phi_\alpha^k(x)\, x^\beta\, dx = \delta_{\alpha,\beta}$$

for every multi-index β with $|\beta| \leq N$, where $\delta_{\alpha,\beta} = 1$ if $\beta = \alpha$ and 0 otherwise.

Then

$$P_k(x) = \sum_{|\alpha|\leq N} m_\alpha^k \phi_\alpha^k(x) \quad \text{with}$$

$$m_\alpha^k = \frac{1}{|E_k|}\int M_k(x)x^\alpha\, dx.$$

It is clear, by homogeneity, that:

$$|\phi_\alpha^k(x)| \leq C(2^k\sigma)^{-|\alpha|}$$

with C an absolute constant.

Note in passing that the above expansion for P_k and the estimates for the ϕ_α^k, immediately yield:

$$|P_k(x)| \leq \frac{C}{|E_k|} \int_{E_k} |M(x)|dx$$

estimate which we have already used in part 1.

Now

$$\sum_{k=0}^{\infty} P_k(x) = \sum_{k=0}^{\infty} \sum_{|\alpha|\leq N} m_\alpha^k \phi_\alpha^k(x) = \sum_{|\alpha|\leq N} \sum_{k=0}^{\infty} m_\alpha^k \phi_\alpha^k(x) =$$

$$= \sum_{|\alpha|\leq N} \sum_{k=0}^{\infty} (m_\alpha^k |E_k|)(|E_k|^{-1}\phi_\alpha^k(x))$$

After observing that $\sum_{k=0}^{\infty} m_\alpha^k |E_k| = \int_{\mathbb{R}^n} M(x)x^\alpha dx = 0$ for every α with $|\alpha| \leq N$, we can write:

$$\sum_{k=0}^{\infty} (m_\alpha^k |E_k|)(|E_k|^{-1}\phi_\alpha^k(x)) =$$

$$= \sum_{k=0}^{\infty} \Big(- \sum_{j=k+1}^{\infty} m_\alpha^j |E_j| + \sum_{j=k}^{\infty} m_\alpha^j |E_j|\Big)(|E_k|^{-1}\phi_\alpha^k(x)) =$$

$$= \sum_{k=0}^{\infty} (|E_{k+1}|^{-1}\phi_\alpha^{k+1}(x) - |E_k|^{-1}\phi_\alpha^k(x)) \sum_{j=k+1}^{\infty} m_\alpha^j |E_j| =$$

$$= \sum_{k=0}^{\infty} N_\alpha^k \psi_\alpha^k(x)$$

where

$$N_\alpha^k = \sum_{j=k+1}^{\infty} m_\alpha^j |E_j| \quad \text{and} \quad \psi_\alpha^k(x) = |E_{k+1}|^{-1}\phi_\alpha^{k+1}(x) - |E_k|^{-1}\phi_\alpha^k(x).$$

We have obtained

$$\sum_{k=0}^{\infty} P_k(x) = \sum_{|\alpha|\leq N} \sum_{k=0}^{\infty} N_\alpha^k \psi_\alpha^k(x)$$

It is clear that each ψ_α^k has the right cancellation properties to be a multiple of a p-atom. It is supported in a cube of measure equal to $C(2^k\sigma)^n$. Now we just have to estimate the size of $N_\alpha^k \psi_\alpha^k$

$$|N_\alpha^k| \leq \sum_{j=k+1}^{\infty} \int |M_j(x)||x|^{|\alpha|}dx \leq C \sum_{j=k+1}^{\infty} \|M_j\|_{L^q(dx/|E_j|)} (2^j\sigma)^{n+|\alpha|} \leq$$

$$\leq C \sum_{j=k+1}^{\infty} (2^j\sigma)^{n+|\alpha|-n/p}(2^{na})^{-j} = C\sigma^{n+|\alpha|-n/p} \sum_{j=k+1}^{\infty} (2^j)^{n(1-a-1/p)+|\alpha|} =$$

$$= C\sigma^{n+|\alpha|-n/p}(2^k)^{n(1-a-1/p)+|\alpha|}$$

since $n(1-a-1/p)+|\alpha|\leq n(1-a-1/p)+N\leq n(1-a-1/p+1/p-1) = -na < 0$.
Now $|N_\alpha^k\psi_\alpha^k(x)| \leq C\sigma^{n+|\alpha|-n/p}(2^k)^{n(1-a-1/p)+|\alpha|}(2^k\sigma)^{-n-|\alpha|} =$
$= C\sigma^{-n/p}(2^k)^{n(-a-1/p)} = C(2^k\sigma)^{-n/p}(2^k)^{-na}$

It follows that $N_\alpha^k\psi_\alpha^k(x) = \lambda_{k,\alpha} A_{k,\alpha}(x)$ where $A_{k,\alpha}$ is a (p,∞)-atom and $0 \leq \lambda_{k,\alpha} \leq C(2^{na})^{-k}$.

Then:

$$\sum_{k=0}^{\infty} P_k(x) = \sum_{|\alpha|\leq N} \sum_{k=0}^{\infty} \lambda_{k,\alpha} A_{k,\alpha}(x)$$

and

$$\sum_{|\alpha|\leq N} \sum_{k=0}^{\infty} (\lambda_{k,\alpha})^p \leq C \sum_{|\alpha|\leq N} \sum_{k=0}^{\infty} (2^{nap})^{-k} < \infty$$

as we wanted to show. $\square$

After this, we can give the following molecular characterization of H^p.

THEOREM 7.17. Let $0 < p \leq 1 < q \leq \infty$ *and* $b > 1/p - 1/q$. *Then, a tempered distribution* f *in* $\mathbb{R}^n$, *belongs to* $H^p(\mathbb{R}^n)$ *if and only if, as tempered distributions*

$$f = \sum_j M_j$$

where each M_j *is a* (p,q,b)*-molecule, and*

$$\sum_j N(M_j)^p < \infty$$

Actually, if the above decomposition holds, then

$$\|f\|^p_{H^p(\mathbb{R}^n)} \leq C \sum_j N(M_j)^p$$

with an absolute constant C. *Also, if* $f \in H^p(\mathbb{R}^n)$, *the above decomposition can be achieved with*

$$\sum_j N(M_j)^p \leq C\|f\|^p_{H^p(\mathbb{R}^n)}$$

C *being again an absolute constant.* $\square$

We may reformulate theorem 7.8. in the following way:

THEOREM 7.18. Let T *be a singular integral operator satisfying the hypotheses of theorem* 7.7. *Let* $n/(n+k+1) < p \leq 1 < q < \infty$ *and* $b > 1/p - 1/q$. *Then, for every* (p,q)-*atom* a, Ta *is a* (p,q,b)-*molecule with* $N(Ta) \leq C$, *independent of* a.

Proof: Let Q be a minimal cube supporting a and let x_0 be the center of Q. Then

$$\|Ta\|_q \leq C\|a\|_q \leq C|Q|^{1/q-1/p}$$

and

$$\||\cdot-x_0|^{nb}Ta\|_q \leq \left(\int_{\tilde{Q}}||x-x_0|^{nb}Ta(x)|^q dx\right)^{1/q} + \left(\int_{\mathbb{R}^n\setminus\tilde{Q}}||x-x_0|^{nb}Ta(x)|^q dx\right)^{1/q}$$

The first term in this sum is bounded by $C|Q|^{b+1/q-1/p}$. Then, as in the proof of theorem 7.7., we have, for $x \in \mathbb{R}^n\setminus\tilde{Q}$:

$$|x-x_0|^{nb}|Ta(x)| \leq C|x-x_0|^{-n-1-N+nb}|Q|^{1-(1/p)+((N+1)/n)}$$

This estimate implies that the second term in the sum above is also bounded by $C|Q|^{b+1/q-1/p}$. Consequently, with $\theta = (1/p-1/q)/b$, we have

$$\|Ta\|_q^{1-\theta}\||\cdot-x_0|^{nb}Ta\|_q^{\theta} \leq C|Q|^{(1-\theta)(1/q-1/p)+\theta(b+1/q-1/p)} = C$$

as we wanted to prove. $\square$

REMARKS 7.19. a) Observe that, if M is a function in the Schwartz class $\mathcal{S}$, then M is a (p,q,b)-molecule for any admissible (p,q,b) if and only if M satisfies ii) in definition 7.13., that is, if the right amount of moments of M vanish. This is clear since, for $M \in \mathcal{S}$, i) in 7.13 holds automatically.

b) The usefulness of molecules is enhanced in case $q = 2$ and

$b = k/n$, with k a positive integer. With these values of the parameters, if M is a molecule centered at 0, we shall have

$$\|M\|_2^{1-\theta} \; \||x|^k M\|_2^{\theta} < \infty$$

This is equivalent to saying that there is a constant C such that, for every multi-index α with $|\alpha| = k$:

$$\|M\|_2^{1-\theta} \|x^\alpha M\|_2^{\theta} \leq C.$$

since $|x|^k \sim \sum_{|\alpha|=k} |x^\alpha|$

The fact that $x^\alpha M(x)$ is in L^2 is equivalent to saying that the Fourier transform $\hat{M}$ of M has a derivative $D^\alpha \hat{M}$ in the L^2 sense (see Stein-Weiss [2], I.1.9.). Indeed

$$D^\alpha \hat{M}(\xi) = ((-2\pi i x)^\alpha M(x))\hat{\ }(\xi).$$

By using Plancherel's theorem, we arrive at the following condition for $\hat{M}$:

$$\|\hat{M}\|_2^{1-\theta} \; \|D^\alpha \hat{M}\|_2^{\theta} \leq C$$

Now we shall see that condition ii) can also be interpreted as a requirement on $\hat{M}$. Observe that i) implies that, for every multi-index β such that $|\beta| \leq k$, $D^\beta \hat{M}$ has L^2-derivatives up to order $k - |\beta|$ or, as it is said equivalently, $D^\beta \hat{M}$ belongs to the Sobolev space $L^2_{k-|\beta|}(\mathbb{R}^n)$. We know, after Sobolev's lemma (see Stein [1], p.124) that the Sobolev space $L^2_s(\mathbb{R}^n)$ is made up of continuous functions provided $s > n/2$. Now for $|\beta| \leq N \leq n(\frac{1}{p} - 1)$, since $\frac{k}{n} > \frac{1}{p} - \frac{1}{2}$, we have: $k-|\beta| > n(\frac{1}{p} - \frac{1}{2}) - n(\frac{1}{p} - 1) = \frac{n}{2}$. Thus $D^\beta \hat{M}$ is (equal a.e. to) a continuous function. Note, finally, that condition ii) in definition 7.13. is equivalent to saying that, for $|\beta| \leq N = [n((1/p) - 1)]$:

$$D^\beta \hat{M}(0) = 0,$$

where the evaluation of $D^\beta \hat{M}$ at 0 is meaningful since it is a continuous function.

We have reached the following conclusion:

The function F is the Fourier transform of a $(p,2,k/n)$-molecule centered at zero if and only if :

i)' $\qquad \|F\|_2^{1-\theta}\|D^\alpha F\|_2^\theta \leq C$

with C independent of α, $|\alpha| = k$, where $\theta = n(1/p-1/2)/k$ and

ii)' $\qquad D^\beta F(0) = 0$ for every $0 \leq |\beta| \leq N$

These two conditions will be the key to our study of the multiplier operator

$$f \longmapsto Tf = (m\hat{f})^\vee$$

where, as usual $^\vee$ denotes the inverse Fourier transform. We shall present below a version of Hörmander's multiplier theorem for H^p spaces. The result will consist in showing that, if we impose the usual conditions on the derivatives of m up to a sufficiently higher order (see theorem 6.3. in chapter II), then T carries a $(p,2)$-atom a (centered at 0), having a certain (large) number of vanishing moments, into a molecule. To see this we have to show that $(Ta)^\wedge = m\hat{a}$ satisfies the same conditions i)' and ii)' as F above.

Since the natural hypotheses of Hörmander's theorem are estimates for the derivatives of m, it is reasonable to expect that success will depend on our ability to estimate the size of the derivatives of $\hat{a}$ for a an atom. We shall give below several estimates of this type, which have also independent interest and extend results obtained in chapter I.

c) Observe that it follows from the previous remark that, for $F \in \mathcal{S}$ to be the Fourier transform of a $(p,2,k/n)$-molecule (centered at 0) it is necessary and sufficient that $D^\beta F(0) = 0$ for every $0 \leq |\beta| \leq N$.

In the next theorem we gather information about the Fourier transform $\hat{a}$ of an atom a.

<u>*THEOREM 7.20. Let*</u> a <u>*be a*</u> $(p,2)$<u>*-atom centered at the origin*</u> (<u>*this means, of course, that there is a minimal cube*</u> Q <u>*supporting*</u> a <u>*and*</u>

centered at zero). Suppose that a *has vanishing moments up to order* $k \geq N = [n(1/p - 1)]$. *(We shall abbreviate this restriction on the atom by saying that* a *is a* $(p,2,k)$*-atom). Then, if we set* $d = 1/(1/p-1/2) = 2p/(2-p)$, *we have the following estimates:*

a) *For every multi-index* α *such that* $0 \leq |\alpha| \leq k$:

$$|D^\alpha \hat{a}(\xi)| \leq C|\xi|^{k+1-|\alpha|}\|a\|_2^{-A(k,n)}$$

with $\quad A(k,n) = d(\frac{k+1}{n} + \frac{1}{2}) - 1 > 0$

b) *For every multi-index* α *and every* $1 \leq r \leq \infty$:

$$\||D^\alpha \hat{a}|^2\|_{r'} \leq C\|a\|_2^{-B(\alpha,r,n)}$$

with $\quad B(\alpha,r,n) = d(\frac{2|\alpha|}{n} + \frac{1}{r}) - 2$

Here, as usual, r' *stands for the exponent conjugate to* r. *The constants* C *is* a) *and* b) *do not depend on* a.

Proof: a) $D^\alpha \hat{a}(\xi) = \int_Q a(x)(-2\pi i x)^\alpha e^{-2\pi i x\cdot\xi} dx =$

$$= \int_Q a(x)(-2\pi i x)^\alpha (e^{-2\pi i x\cdot\xi} - P(x))\, dx$$

where $P(x)$ is the Taylor polynomial of degree $k - |\alpha|$ at 0 for the function $x \longmapsto e^{-2\pi i x\cdot\xi}$. The typical estimate for the remainder in Taylor's formula for this function yields:

$$|D^\alpha \hat{a}(\xi)| \leq C|\xi|^{k+1-|\alpha|}\int_Q |a(x)||x|^{k+1} dx \leq$$
$$\leq C|\xi|^{k+1-|\alpha|}|Q|^{\frac{k+1}{n} + \frac{1}{2}}\|a\|_2 \leq$$
$$\leq C|\xi|^{k+1-|\alpha|}\|a\|_2^{1-d(\frac{k+1}{n} + \frac{1}{2})}$$

since $\|a\|_2 \leq |Q|^{1/2-1/p}$ is equivalent to saying that $|Q| \leq \|a\|_2^{-d}$.

Observe that, since $k \geq N = [n(\frac{1}{p} - 1)]$, we shall have $k+1 > n(\frac{1}{p}-1)$ or, equivalently,

$$\frac{k+1}{n} + \frac{1}{2} > \frac{1}{p} - \frac{1}{2} = \frac{1}{d} \quad \text{so that:} \quad A(k,n) > 0.$$

b) Here we just use the fact that $D^\alpha \hat{a}$ is the Fourier transform of the function $a(x)\ (-2\pi i x)^\alpha$. First we deal with the cases $r = 1$ and $r = \infty$, and then we use interpolation.

If $r = 1, r' = \infty$, we have, by using the L^1, L^∞ boundedness of the Fourier transform:

$$|D^\alpha \hat{a}(\xi)| \leq C \int_Q |a(x)||x|^{|\alpha|}dx \leq C|Q|^{\frac{|\alpha|}{n}+\frac{1}{2}} \|a\|_2 \leq$$
$$\leq C\|a\|_2^{1-d(\frac{|\alpha|}{n}+\frac{1}{2})}$$

After raising to the second power, we obtain b) for $r = 1$.

If $r = \infty$, $r' = 1$, we use Plancherel's theorem to obtain:

$$\int_{\mathbb{R}^n} |D^\alpha \hat{a}(\xi)|^2 d\xi = C\int_Q |a(x)|^2|x|^{2|\alpha|}dx \leq C|Q|^{\frac{2|\alpha|}{n}} \|a\|_2^2 \leq$$
$$\leq C\|a\|_2^{2(1-d|\alpha|/n)}$$

which is what we wanted. Now, for $1 < r < \infty$, we write

$$\int_{\mathbb{R}^n} |D^\alpha \hat{a}(\xi)|^{2r'}d\xi = \int_{\mathbb{R}^n} |D^\alpha \hat{a}(\xi)|^2 |D^\alpha \hat{a}(\xi)|^{2r'-2}d\xi \leq$$
$$\leq C\|a\|_2^{-B(\alpha,\infty,n)-(r'-1)B(\alpha,1,n)}$$

and observe that

$$\frac{1}{r'}\left(B(\infty)+(r'-1)B(1)\right) = \frac{1}{r'}B(\infty)+\frac{1}{r}B(1) = B(r). \qquad \square$$

COROLLARY 7.21. Let $f \in H^p(\mathbb{R}^n)$, $0 < p \leq 1$. *Then the Fourier transform* $\hat{f}$ *of* f, *which always makes sense as a tempered distribution, is actually a continuous function satisfying the estimate*:

$$|\hat{f}(\xi)| \leq C\|f\|_{H^p(\mathbb{R}^n)} |\xi|^{n((1/p)-1)}$$

with C *independent of* f.

Proof: We use the preceding theorem with $|\alpha| = 0$ and $k = N = [n((1/p)-1)]$. Given a $(p,2)$-atom a centered at 0, part a) of 7.20. implies $|\hat{a}(\xi)| \leq C|\xi|^{N+1}\|a\|_2^{-A}$, where

$$A = A(N,n) = d\left(\frac{N+1}{n} + \frac{1}{2}\right) - 1$$

Also, part b) with $r = 1$, yields: $|\hat{a}(\xi)| \leq C\|a\|_2^{1-d/2}$.

Observe that, in the first estimate, the exponent of $\|a\|_2$ is negative, whereas in the second estimate, the exponent of $\|a\|_2$ is positive, because $d \leq 2$.

Note also that the estimates are valid for every $(p,2)$-atom a, even if it is not centered at zero. Indeed, when a function is translated, the modulus of its Fourier transform does not change.

Now, combining both estimates we obtain a better one.

We use the first one where $|\xi|^{N+1}\|a\|_2^{-A} \leq \|a\|_2^{1-d/2}$ or, equivalently, where $|\xi|^n \leq \|a\|_2^d$. In the rest of $\mathbb{R}^n$, we use the second estimate.

We obtain, for $|\xi|^n \leq \|a\|_2^d$:

$$|\hat{a}(\xi)| \leq C|\xi|^{N+1}\|a\|_2^{-A} \leq C|\xi|^{N+1-An/d} = C|\xi|^{n((1/p)-1)}$$

since $N+1-An/d = n/d-n/2 = n((1/p)-1)$.

For $|\xi|^n > \|a\|_2^d$, we get

$$|\hat{a}(\xi)| \leq C\|a\|_2^{1-d/2} \leq C|\xi|^{(n/d)(1-d/2)} = C|\xi|^{n((1/p)-1)}.$$

Thus, $(p,2)$-atoms satisfy uniformly the estimate:

$$|\hat{a}(\xi)| \leq C|\xi|^{n((1/p)-1)}.$$

Now if $f \in H^p(\mathbb{R}^n)$, we have a decomposition

$$f = \sum_j \lambda_j a_j$$

where the a_j's are $(p,2)$-atoms and

$$\sum_j |\lambda_j|^p \leq C\|f\|^p_{H^p(\mathbb{R}^n)}$$

Since the series converges in $\mathcal{S}'$ and the Fourier transform is continuous in S', we shall have

$$\hat{f} = \sum_j \lambda_j \hat{a}_j$$

The uniform estimate for the $\hat{a}_j$'s, which are, of course, continuous functions, together with the fact that

$$\sum_j |\lambda_j| \leq (\sum_j |\lambda_j|^p)^{1/p} \leq C\|f\|_{H^p(\mathbb{R}^n)}$$

imply that the series $\sum_j \lambda_j \hat{a}_j$ converges uniformly over compact subsets to a given continuous function. The distribution $\hat{f}$ must coincide with it. In this sense $\hat{f}$ is a continuous function and

$$|\hat{f}(\xi)| \leq \sum_j |\lambda_j||\hat{a}_j(\xi)| \leq C \sum_j |\lambda_j||\xi|^{n((1/p)-1)} \leq$$

$$\leq C\|f\|_{H^p(\mathbb{R}^n)} |\xi|^{n((1/p)-1)}. \quad \square$$

The estimate we have just obtained, can be written as:

$$|\hat{f}(x)|^p |x|^{n(p-2)} \leq C\|f\|^p_{H^p(\mathbb{R}^n)} |x|^{-n}$$

so that, for $f \in H^p(\mathbb{R}^n)$, the function $|\hat{f}(x)|^p|x|^{n(p-2)}$ is in weak L^1. We shall presently see that this function is actually integrable. We first prove it for an atom.

THEOREM 7.22. Let a *be a* $(p,2)$*-atom in* $\mathbb{R}^n$, $0 < p \leq 1$. *Then:*

$$\int_{\mathbb{R}^n} |\hat{a}(x)|^p |x|^{n(p-2)} dx \leq C$$

a constant independent of a.

Proof: We split the integral to be estimated as the sum of the integral over $B(0,R)$ and the integral over the complement, for any $R > 0$. For the first integral we use the first estimate of $\hat{a}$ as it appears in the proof of corollary 7.21.

$$\int_{|x|<R} |\hat{a}(x)|^p |x|^{n(p-2)} dx \leq C \int_{|x|<R} |x|^{(N+1)p+n(p-2)} dx \|a\|_2^{-pA}$$

where $A = d\,(\frac{N+1}{n} + \frac{1}{2})-1 = d\,(\frac{N+1}{n} - \frac{1}{p} + 1)$

A simple computation yields:

$$\int_{|x|<R} |\hat{a}(x)|^p |x|^{n(p-2)} dx \leq C(R\cdot \|a\|_2^{-d/n})^{(N+1+n)p-n}$$

Note that $(N+1+n)p-n > 0$, since $N+1 > n(1/p - 1)$.

For the second integral we start by using Hölder's inequality with exponents $2/p$ and its conjugate $2/(2-p)$. Then we use Plancherel's theorem, obtaining:

$$\int_{|x|>R} |\hat{a}(x)|^p |x|^{n(p-2)} dx \leq \left(\int_{\mathbb{R}^n} |\hat{a}(x)|^2 dx\right)^{p/2} \left(\int_{|x|>R} |x|^{-2n} dx\right)^{(2-p)/2} =$$

$$= C\|a\|_2^p \, R^{-np/d}$$

If we take $R = \|a\|_2^{d/n}$, the proof is finished. $\square$

<u>*COROLLARY 7.23. For*</u> $0 < p \leq 1$

$$\int_{\mathbb{R}^n} |\hat{f}(x)|^p |x|^{n(p-2)} dx \leq C\|f\|^p_{H^p(\mathbb{R}^n)}$$

<u>*where*</u> C <u>*does not depend on*</u> $f \in H^p(\mathbb{R}^n)$.

<u>*Proof*</u> Let $f \in H^p(\mathbb{R}^n)$. Write

$$f = \sum_j \lambda_j a_j$$

with a_j being $(p,2)$-atoms and $\sum_j |\lambda_j|^p \leq C\|f\|^p_{H^p(\mathbb{R}^n)}$

As we have seen in the proof of corollary 7.21., this implies that

$$\hat{f}(x) = \sum_j \lambda_j \hat{a}_j(x)$$

with the series converging uniformly on compact subsets. Then

$$|\hat{f}(x)|^p \leq \sum_j |\lambda_j|^p |\hat{a}_j(x)|^p$$

so that, applying theorem 7.22. to each a_j, we get:

$$\int_{\mathbb{R}^n} |\hat{f}(x)|^p |x|^{n(p-2)} dx \leq \sum_j |\lambda_j|^p \int_{\mathbb{R}^n} |\hat{a}_j(x)|^p |x|^{n(p-2)} dx \leq C\|f\|^p_{H^p(\mathbb{R}^n)}$$

as we wanted to prove. $\square$

For $p = 1$, the inequality just proved is:

$$\int_{\mathbb{R}^n} \frac{|\hat{f}(x)|}{|x|^n}\, dx \le C\, \|f\|_{H^1(\mathbb{R}^n)} \tag{7.24}$$

This is an extension to our context of Hardy's inequality, established in chapter I (theorem 4.1).

As a preparation for the multiplier theorem, we need the following:

LEMMA 7.25. Let k *be an integer* $> n/2$. *Suppose that* m *is a function on* $\mathbb{R}^n$ *satisfying*:

$$\left(\frac{1}{R^n} \int_{R<|x|\le 2R} |D^\beta m(x)|^2 dx \right)^{1/2} \le AR^{-|\beta|} \tag{7.26}$$

for every multi-index β *such that* $0 \le |\beta| \le k$ *and every* $R > 0$, *with* A *independent of* R *and* β. *Then there is a constant* C *independent of* m, *such that if* $r = 1$ *or* $n \ge n/r > 2(|\beta|-k) + n$, *the following inequality holds:*

$$\left(\frac{1}{R^n} \int_{R<|x|\le 2R} |D^\beta m(x)|^{2r} dx \right)^{\frac{1}{2r}} \le CAR^{-|\beta|} \tag{7.27}$$

In case $2(|\beta|-k) + n < 0$, *then* $|x|^{|\beta|}|D^\beta m(x)| \le CA$ *and* $D^\beta m$ *is continuous on* $\mathbb{R}^n \setminus \{0\}$.

Proof: Take a non-negative, radial, C^∞ function $\eta \le 1$, supported on $\{x \in \mathbb{R}^n : 1/2 < |x| < 4\}$ and equal to 1 for $1 \le |x| \le 2$.

Given $R > 0$ and β such that $0 \le |\beta| \le k$, let $f(x) = R^{|\beta|}\eta(x/R)D^\beta m(x)$, and $g(x) = f(Rx) = R^{|\beta|}\eta(x)D^\beta m(Rx)$. Call $s = k - |\beta|$. Then, if ν is a multi-index such that $0 \le |\nu| \le s$, condition 7.26. implies that the function $D^\nu g(x) = R^{|\nu|}D^\nu f(Rx)$ is in $L^2(\mathbb{R}^n)$ and satisfies $\|D^\nu g\|_2 \le CA$, with C independent of m, ν, s and R. In other words: g has derivatives in L^2 up to order s. This is usually expressed by saying that g belongs to the Sobolev space $L^2_s(\mathbb{R}^n)$. This is a normed space with norm given by $\sum_{0\le|\nu|\le s} \|D^\nu g\|_2$. We appeal to Sobolev's imbedding theorem (see Stein [1] p. 124), which says:

i) For $s > n/2$, the functions of $L^2_s(\mathbb{R}^n)$, belong to $C_0(\mathbb{R}^n)$, the space of functions continuous on $\mathbb{R}^n$ and vanishing at ∞, and the inclusion $L^2_s(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$ is continuous.

ii) There is a continuous inclusion $L^2_s(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ provided $s > n/2 - n/q$.

Since $s = k - |\beta|$, if we write $2r = q$, the condition $s > n/2 - n/q$ is equivalent to $n/r > 2(|\beta| - k) + n$. If this holds, then:

$$\left(\frac{1}{R^n} \int_{R<|x|\leq 2R} |D^\beta m(x)|^{2r} dx \right)^{\frac{1}{2r}} \leq R^{-|\beta|-n/(2r)} \|f\|_{2r} =$$

$$= R^{-|\beta|} \|g\|_{2r} \leq C \cdot A \; R^{-|\beta|}$$

Observe that this is valid with the usual interpretation for $r = \infty$. $\square$

The key to the multiplier theorem will be the following result.

THEOREM 7.28. Suppose m *satisfies the hypothesis of lemma* 7.25. *and* a *is a* $(p,2,k-1)$-*atom centered at the origin with* $0 < p \leqq 1$, $1/p - 1/2 < k/n$. *Then* $(m \cdot \hat{a})^\vee$ *is a* $(p,2,k/n)$-*molecule centered at the origin with*

$$N((m\hat{a})^\vee) \leq C \cdot A$$

where C *depends only on* p,k *and* n.

Proof: Observe that $k-1 \geq N = [n(\frac{1}{p} - 1)]$, since $k > n(1/p-1/2) >$ $> n(1/p-1)$.

We shall prove that $m\hat{a}$ is the Fourier transform of a $(p,2,k/n)$-molecule centered at the origin and having molecular norm bounded by $C \cdot A$. In order to do this, it will be enough (see remark 7.19. b)) to show that:

i) $$\left(\|m\hat{a}\|_2^{1/2-1/p+k/n} \, \|D^\nu (m\hat{a})\|_2^{1/p-1/2} \right)^{n/k} \leq C \cdot A$$

for $|\nu| = k$; and

ii) $D^\nu (m\hat{a})(0) = 0 \quad$ for $\quad 0 \leq |\nu| \leq N$

Let us start by proving i). It will be convenient to write $d = 1/(1/p-1/2)$ as we did already in theorem 7.20. We can write i) also in this way:

$$(7.29) \qquad \|D^\nu(m\hat{a})\|_2 \leq CA^{kd/n}\|m\hat{a}\|_2^{1-(kd/n)}; \qquad |\nu| = k$$

Observe that, since we have assumed $k/n > 1/d$, the exponent $1-(kd/n)$ is negative. Note also that we saw (lemma 7.25) that $|m(x)| \leq C\cdot A$. Therefore (7.29) will follow if we are able to see that:

$$\|D^\nu(m\hat{a})\|_2 \leq C\cdot A\|a\|_2^{1-(kd/n)}; \qquad |\nu| = k.$$

To establish this inequality will require to prove that whenever α and β are multi-indices satisfying $|\alpha| + |\beta| = k$, we have:

$$\|(D^\alpha\hat{a})(D^\beta m)\|_2 \leq C\cdot A \ \|a\|_2^{1-(kd/n)}$$

The case $\beta = 0$, $|\alpha| = k$, is quite easy. We use the estimate b) in theorem 7.20., with $r = \infty$, together with the fact that $\|m\|_\infty \leq C\cdot A$. We get:

$$\|(D^\alpha\hat{a})m\|_2 \leq \|m\|_\infty \|D^\alpha\hat{a}\|_2 \leq CA\|a\|_2^{1-(kd/n)}$$

Next we consider the case $0 < |\beta| \leq k$, $0 \leq |\alpha| < k$. We write:

$$\|(D^\alpha\hat{a})(D^\beta m)\|_2^2 = \sum_{j\in\mathbb{Z}} \int_{2^j<|x|\leq 2^{j+1}} |D^\alpha\hat{a}(x)|^2|D^\beta m(x)|^2\,dx =$$

$= S_1 + S_2$, where S_1 is the sum over $j \leq j_0$, and j_0 is an integer to be determined later.

Using theorem 7.20. a) to estimate $D^\alpha\hat{a}$ and condition (7.26) on m, we get:

$$\int_{2^j<|x|\leq 2^{j+1}} |D^\alpha\hat{a}(x)|^2|D^\beta m(x)|^2dx \leq$$

$$\leq C\|a\|_2^{-2A(k-1,n)}2^{2j(k-|\alpha|)}2^{nj}\left(2^{-nj}\int_{2^j<|x|\leq 2^{j+1}} |D^\beta m(x)|^2dx\right) \leq$$

$$\leq CA^2\|a\|_2^{-2A(k-1,n)}2^{nj}, \text{ where } A(k-1,n) = d(k/n+1/2)-1$$

These estimates can be added for $j \leq j_0$, obtaining:

$$S_1 \leq CA^2 \|a\|_2^{-2A(k-1,n)} \sum_{-\infty}^{j_0} 2^{nj} = CA^2 \|a\|_2^{-2A(k-1,n)} 2^{nj_0}$$

Since $-2A(k-1,n) = 2(1-(kd/n)-(d/2))$, if we choose j_0 such that $2^{nj_0} \sim \|a\|_2^d$, we shall have:

$$S_1 \leq CA^2 \|a\|_2^{2(1-(kd/n))}$$

which is the estimate we are looking for.

Complete success will be achieved if we are able to get the same estimate for S_2 with j_0 fixed as above.

We start by using Hölder's inequality to get:

$$\int_{2^j<|x|\leq 2^{j+1}} |D^\alpha \hat{a}(x)|^2 |D^\beta m(x)|^2 dx \leq$$

$$\leq \left(\int_{2^j<|x|\leq 2^{j+1}} |D^\alpha \hat{a}(x)|^{2r'} dx\right)^{1/r'} . 2^{jn/r} \left(2^{-jn} \int_{2^j<|x|\leq 2^{j+1}} |D^\beta m(x)|^{2r} dx\right)^{1/r}$$

Then we use theorem 7.20. b) and (7.27), after choosing r in such a way that (7.27) can be applied. Let us see how this can be done.

If $|\beta| > n/2$, we take $r = 1$. If $0 < |\beta| \leq n/2$ and $0 < |\beta| < k - (n/2)$, we take $r = \infty$. It only remains the case $k - (n/2) \leq |\beta| \leq n/2$. Of course this case only arises when $k \leq n$. In this situation we choose $1 < r < \infty$ such that $0 < 2|\beta| - (n/r) < 2k - n$.

Observe that, in all cases, r has been selected in such a way that $2|\beta| - (n/r) > 0$ and also β and r satisfy the hypothesis of lemma 7.25., so that (7.27) holds. We get:

$$S_2 \leq C\|a\|_2^{-B(\alpha,r,n)} A^2 \sum_{j_0}^{\infty} (2^{(n/r)-2|\beta|})^j \leq CA^2 \|a\|_2^{-B(\alpha,r,n)} (2^{(n/r)-2|\beta|})^{j_0}$$

where $B(\alpha,r,n) = d(2|\alpha|/n+1/r)-2$. Then, taking into account that $2^{nj_0} \sim \|a\|_2^d$, we arrive at:

$$S_2 \leq CA^2 \|a\|_2^{2(1-(kd/n))}$$

and i) is completely proved.

Now we have to prove ii). We already pointed out (remark 7.19. b) that i) implies the existence of $D^{\nu}(m\hat{a})(0)$ for every ν satisfying $0 \leq |\nu| \leq N$. We just need to see that these derivatives actually vanish. For $\nu = 0$, it suffices to realize that m is bounded and $|\hat{a}(x)| \leq C|x|^k$ (theorem 7.20. a), so that $(m\hat{a})(0) = \lim_{x\to 0} m(x)\hat{a}(x) = 0$. For higher order ν's, we proceed in a similar fashion, writing

$$D^{\nu}(m\hat{a})(0) = \lim_{t\to 0} t^{-|\nu|}\Delta_t^{\nu}(m\hat{a})(0)$$

where, for t real, $\Delta_t^{\nu} = \Delta_{te_1}^{\nu_1} \dots \Delta_{te_n}^{\nu_n}$, the difference operator obtained by applying Δ_{te_1}, ν_1 times, ..., Δ_{te_n}, ν_n times; the e_j's being the standard unit vectors of $\mathbb{R}^n$. Since, clearly $|\Delta_t^{\nu}(m\hat{a})(0)| \leq C|t|^k$ and $|\nu| \leq N < k$, we obtain $D^{\nu}(m\hat{a})(0) = 0$, as we wanted to prove. $\square$

We shall say that the measurable function m on $\mathbb{R}^n$ is a multiplier on $H^p(\mathbb{R}^n)$, $0 < p \leq 1$, if and only if for every $f \in H^p(\mathbb{R}^n)$, the function $m\hat{f}$ is the Fourier transform of some distribution belonging to $H^p(\mathbb{R}^n)$ and there is a constant C independent of f such that

$$\|(m\hat{f})^{\vee}\|_{H^p(\mathbb{R}^n)} \leq C\|f\|_{H^p(\mathbb{R}^n)}$$

Theorem 7.28. immediately yields the H^p multiplier theorem we were looking for. It can be stated in the following way:

THEOREM 7.30. Let k *be a positive integer. Suppose* m *is a function on* $\mathbb{R}^n$ *satisfying:*

$$\left(\frac{1}{R^n}\int_{R<|x|\leq 2R} |D^{\beta}m(x)|^2 dx\right)^{1/2} \leq AR^{-|\beta|}$$

for every multi-index β *such that* $0 \leq |\beta| \leq k$ *and every* $R > 0$, *with* A *independent of* R *and* β. *Then, for every* p *such that* $1/(k/n+1/2) < p \leq 1$, m *is a multiplier on* $H^p(\mathbb{R}^n)$, *and there is a constant* C *independent of* m *and* f *such that*:

$$\|(m\hat{f})^{\vee}\|_{H^p(\mathbb{R}^n)} \leq CA\|f\|_{H^p(\mathbb{R}^n)}$$

Proof: Fix p such that $1/(k/n+1/2) < p \leq 1$. This is equivalent to saying that $k/n > 1/p - 1/2$, and this implies that $k-1 \geq N = [n(\frac{1}{p}-1)]$ as we pointed out in the proof of theorem 7.28.

Let $f \in H^p(\mathbb{R}^n)$. Write $f = \sum_j \lambda_j a_j$ where each a_j is a $(p,2,k-1)$-atom and the λ_j's are real numbers such that $\sum_j |\lambda_j|^p \leq C\|f\|^p_{H^p}$. Then

$$(m\hat{f})^\vee = \sum_j \lambda_j (m\hat{a}_j)^\vee$$

in the sense of tempered distributions. It follows from theorem 7.28. and the translation-invariance of the operator $g \longmapsto (m\hat{g})^\vee$, that each $(m\hat{a}_j)^\vee$ is a $(p,2,k/n)$-molecule with $N((m\hat{a}_j)^\vee) \leq CA$. Then $\sum_j N(\lambda_j(m\hat{a}_j)^\vee)^p \leq CA^p \sum_j |\lambda_j|^p \leq CA^p \|f\|^p_{H^p(\mathbb{R}^n)}$ so that

$$\|(m\hat{f})^\vee\|_{H^p(\mathbb{R}^n)} \leq CA\|f\|^p_{H^p(\mathbb{R}^n)}$$

as we wanted to prove. $\square$

We shall see next that a multiplier on $H^p(\mathbb{R}^n)$, for some $0 < p \leq 1$, has to be necessarily a bounded function. Actually, the following is true:

THEOREM 7.31. Suppose m *is a multiplier on* $H^p(\mathbb{R}^n)$, *where* $0 < p \leq 1$ *with norm* A *(by norm we mean, of course, the infimum of all the constants* C *which make the inequality* $\|(m\hat{f})^\vee\|_{H^p(\mathbb{R}^n)} \leq C\|f\|_{H^p(\mathbb{R}^n)}$ *valid for all* $f \in H^p(\mathbb{R}^n)$*). Then* m *is a continuous function on* $\mathbb{R}^n \setminus \{0\}$ *and there is a constant* C *independent of* m, *such that* $|m(x)| \leq C\cdot A$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$.

Proof: Given p, associate with each function f its dilation by $t > 0$, defined as follows

$$f_t(x) = t^{-n/p} f(x/t)$$

Extend this definition to distributions in the usual way, that is:

$$< f_t, \phi > = < f, t^{n(1-(1/p))} \phi(t\cdot) >$$

Then, it is clear that, for every $t > 0$, and every $f \in H^p(\mathbb{R}^n)$:

$$\|f_t\|_{H^p(\mathbb{R}^n)} = \|f\|_{H^p(\mathbb{R}^n)}$$

To see this we just need to observe that if a is a (p,∞)-atom, then a_t is also a (p,∞)-atom. Indeed if Q is a minimal cube supporting a, then a_t is supported in tQ and

$$\|a_t\|_\infty = t^{-n/p}\|a\|_\infty \leq t^{-n/p}|Q|^{-1/p} = |tQ|^{-1/p}.$$

Observe that

$$(f_t)\hat{}(x) = t^{n(1-(1/p))}\hat{f}(tx)$$

Then, applying corollary 7.21. to $(m(f_t)\hat{})\check{}$ for $f \in H^p(\mathbb{R}^n)$, we get:

$$|m(x)(f_t)\hat{}(x)| \leq CA\|f\|_{H^p(\mathbb{R}^n)}|x|^{n((1/p)-1)}$$

and, after choosing $t = |x|^{-1}$:

$$|m(x)\hat{f}(x/|x|)| \leq CA\|f\|_{H^p(\mathbb{R}^n)} .$$

To conclude, we just need to realize that we can find $f \in H^p(\mathbb{R}^n)$ with $\|f\|_{H^p(\mathbb{R}^n)} \leq C$ (a geometric constant) and such that $\hat{f}(x) = 1$ for every x having $|x| = 1$. Indeed, if we take $F \in C^\infty(\mathbb{R}^n)$ such that F is supported on the set $\{x \in \mathbb{R}^n: 1/4 < |x| < 4\}$ and is identically 1 on $\{x \in \mathbb{R}^n: 1/2 < |x| < 2\}$, it is clear (see remarks 7.19. b and c) that $F = \hat{f}$ where f is a molecule with $N(f) \leq C$. The continuity of m follows from the continuity of the Fourier transform of an H^p distribution, which was established in corollary 7.21. $\square$.

The fact that m is a bounded function implies, of course, that m is a multiplier on L^2. There are interpolation theorems which allow us to conclude that m is a multiplier on H^q for every $p < q \leq 1$ (see, for example, Calderón and Torchinsky [2]) and also a multiplier on L^q for $1 < q \leq 2$. Then, by duality, m is a multiplier on L^q for $2 \leq q < \infty$ and on B.M.O.

We have not given interpolation results for operators bounded on H^p. We shall content ourselves with a simple application of molecules.

THEOREM 7.32. Let $0 < p_0 < 1 < q$ *Suppose* T *is a linear operator bounded in* $L^q(\mathbb{R}^n)$ *which sends each* (p_0,q,k)-*atom in* $\mathbb{R}^n$ *into a* (p_0,q,b)-*molecule in* $\mathbb{R}^n$ *with molecular norm bounded by a constant independent of the atom. Then* T *is bounded on* $H^p(\mathbb{R}^n)$ *for every* $p_0 < p \leq 1$.

Proof: We shall see that T sends each (p,q,k)-atom into a (p,q,b)-molecule with uniformly bounded molecular norm. Note that $k \geq [n((1/p_0)-1]] \geq [n((1/p)-1]]$ and $b > 1/p_0 - 1/q > 1/p - 1/q$, so that the parameters are still admissible.

Let a be a (p,q,k)-atom. Let Q be a minimal cube supporting a. Then $b = |Q|^{1/p-1/p_0}\, a$ is a (p_0,q,k)-atom, so that $Tb = |Q|^{1/p-1/p_0}\, Ta$ will be a (p_0,q,b)-molecule centered at a certain $x_0 \in \mathbb{R}^n$. We shall have:

$$|Q|^{1/p-1/p_0}\|Ta\|_q^{1-\theta_0}\||\cdot-x_0|^{nb}Ta\|_q^{\theta_0} = \|Tb\|_q^{1-\theta_0}\||\cdot-x_0|^{nb}Tb\|_q^{\theta_0} \leq C$$

where $\theta_0 = (1/p_0-1/q)/b$. We shall also write $\theta = (1/p-1/q)/b$ and $d = 1/(1/p-1/q)$.

We know that $\|a\|_q \leq |Q|^{1/q-1/p} = |Q|^{-1/d}$. Using that fact that T is bounded in $L^q(\mathbb{R}^n)$, we can write:

$$\|Ta\|_q^{d(1/p_0-1/p)} \leq C\|a\|_q^{d(1/p_0-1/p)} \leq C|Q|^{1/p-1/p_0}$$

since $1/p_0 - 1/p > 0$. Now, carrying this inequality into our previous inequality for Ta, we get:

$$\|Ta\|_q^{1-\theta_0+d(1/p_0-1/p)}\||\cdot-x_0|^{nb}Ta\|_q^{\theta_0} \leq C.$$

Observe that $d(1/p_0-1/p) = ((1/p_0-1/q)-(1/p-1/q))/(1/p-1/q) =$ $= (\theta_0/\theta) - 1$, so that $1-\theta_0+d(1/p_0-1/p) = (1-\theta)\,\theta_0/\theta$. Then our last inequality can be written as:

$$(\|Ta\|_q^{1-\theta}\||\cdot-x_0|^{nb}Ta\|_q^{\theta})^{\theta_0/\theta} \leq C.$$

and this implies that Ta is a (p,q,b)-molecule as we wanted to show. $\square$

We shall shift our attention from translation-invariant operators to dilation invariant operators. We shall start working on the real line. As a point of departure, we shall consider two basic inequalities due to Hardy

$$(7.33)\qquad \left(\int_0^\infty \left|\frac{F(x)}{x}\right|^p dx\right)^{1/p} \le \frac{p}{p-1}\left(\int_0^\infty |f(x)|^p dx\right)^{1/p}$$

where $F(x) = \int_0^x f(y)dy$ and $1 < p < \infty$

$$(7.34)\qquad \int_0^\infty \frac{|\hat{f}(\xi)|}{\xi}\, d\xi \le C\|f\|_{H^1}$$

This second inequality was contained in corollary 7.23.

We want to point out that these two well-known inequalities involve operators of the same type.

$$\frac{F(x)}{x} = \frac{1}{x}\int_0^x f(y)dy = \int_0^\infty K(x,y)f(y)\, dy$$

with $K(x,y) = \frac{1}{x}\chi_+(x-y)$, denoting by χ_+ the characteristic function of the half-line $\mathbb{R}_+ = [0,\infty)$

The left hand side of (7.34) is

$$\int_0^\infty \frac{|\hat{f}(\xi)|}{\xi}\, d\xi = \int_0^\infty |\hat{f}(\tfrac{1}{x})|\, \frac{dx}{x}$$

and $\frac{1}{x}\hat{f}(\frac{1}{x}) = \frac{1}{x}\int_{-\infty}^\infty f(y)\, e^{-2\pi iy/x}dy = \int_{-\infty}^\infty K(x,y)f(y)dy$

with $K(x,y) = \frac{1}{x}e^{-2\pi iy/x}$

Both kernels satisfy the condition

$$(7.35)\qquad K(\lambda x,\lambda y) = \lambda^{-1}K(x,y). \qquad (\lambda > 0)$$

This condition guarantees that the operator T given by

$$(7.36)\qquad Tf(x) = \int_{-\infty}^\infty K(x,y)f(y)\, dy$$

commutes with dilations (it sends the function $f(\lambda\cdot)$ into $Tf(\lambda\cdot)$).

It is well known that if the kernel K satisfies (7.35) and

(7.37) $$\int_{-\infty}^{\infty} |K(1,y)|\ |y|^{-1/p} dy = C < \infty$$

where $1 \leq p < \infty$, then the operator T given by (7.36) is bounded from $L^p(\mathbb{R})$ to $L^p(\mathbb{R}_+)$. Indeed:

$$Tf(x) = \int_{-\infty}^{\infty} \frac{1}{x} K(1,\frac{y}{x}) f(y) dy = \int_{-\infty}^{\infty} K(1,u) f(xu) du$$

so that, Minkowski's integral inequality yields:

$$\|Tf\|_p \leq \int_{-\infty}^{\infty} |K(1,u)|\ \|f(u\cdot)\|_p du = \int_{-\infty}^{\infty} |K(1,u)|\,|u|^{-1/p} du \|f\|_p = C\|f\|_p$$

(7.33) can be obtained in this way since

$$\int_0^{\infty} |K(1,y)| y^{-1/p} dy = \int_0^1 y^{-1/p} dy = \frac{p}{p-1}$$

Obviously (7.33) breaks down for $p = 1$. However, it can be extended to $f \in H^1$ as we are about to see

LEMMA 7.38. Let f *be a* $(1,\infty)$*-atom having interval-support* $[a,b] \subset \mathbb{R}_+$. *Then*:

$$\int_0^{\infty} \left| \frac{F(x)}{x} \right| dx \leq (\log 2) \frac{b-a}{b+a} \leq \log 2$$

Proof: Since f has integral zero, F lives also on $[a,b]$. For $a \leq x \leq b$, we have:

$$F(x) = \int_0^x f(y)dy = \int_a^x f(y)dy = -\int_x^b f(y)dy.$$

Thus:

$$|F(x)| \leq \frac{1}{b-a} \min (x-a, b-x)$$

and, consequently, setting $c = (a+b)/2$ and $\theta = \frac{b-a}{b+a}$, we get:

$$\int_0^{\infty} \left| \frac{F(x)}{x} \right| dx \leq \frac{1}{b-a} \left(\int_a^c \frac{x-a}{x} dx + \int_c^b \frac{b-x}{x} dx \right) =$$

$$= \frac{b}{b-a} \log \frac{2b}{b+a} + \frac{a}{b-a} \log \frac{2a}{b+a} =$$

$$= \frac{1}{2} \left((1+\tfrac{1}{\theta})\log(1+\theta)+(\tfrac{1}{\theta}-1)\log(1-\theta)\right) =$$

$$= \theta \sum_{j=0}^{\infty} \frac{\theta^{2j}}{(2j+1)(2j+2)} \leq \theta \sum_{j=0}^{\infty} \frac{1}{(2j+1)(2j+2)} = \theta \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} = \theta \log 2$$

as we wanted to prove. $\square$

Let us point out that, for the two inequalities contained in the lemma, the constant log 2 is sharp. Indeed, for

$$f(x) = \begin{cases} 1/2 & \text{for } 0 \leq x \leq 1 \\ -1/2 & \text{for } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

it will be

$$F(x) = \begin{cases} x/2 & \text{for } 0 \leq x \leq 1 \\ 1-(x/2) & \text{for } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\int_0^\infty \left| \frac{F(x)}{x} \right| dx = \int_1^2 \frac{dx}{x} = \log 2$$

If we want to pay attention to the constants appearing in the inequalities, we should specify which, of the many different equivalent norms, are we using on H^1. Sometimes we shall work with the atomic norm

$$\|f\|_{H^1_{at}} = \inf\{\sum_j |\lambda_j| : f = \sum_j \lambda_j a_j, \text{ with } a_j \ (1,\infty)\text{-atoms and } \sum_j |\lambda_j| < \infty\}.$$

while other times we shall take the norm

$$\|f\|_1 + \|Hf\|_1$$

LEMMA 7.39. Suppose $f \in H^1$ *and let* f_e *be its even part, that is:* $f_e(x) = (f(x)+f(-x))/2$. *Then both* f_e *and* $\chi_+ f_e$ *belong to* H^1.

Actually

$$\|\chi_+ f_e\|_{H^1_{at}} \leq \|f\|_{H^1_{at}}$$

Proof: If $f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x)$ with each a_j being a $(1,\infty)$-atom and $\sum_{j=1}^{\infty} |\lambda_j| < \infty$, we have:

$$f_e(x) = \frac{1}{2}\left(\sum_{j=1}^{\infty} \lambda_j a_j(x) + \sum_{j=1}^{\infty} \lambda_j a_j(-x)\right)$$

If a_j lives in $[0,\infty)$, then both a_j and $a_j(-\cdot)$ are $(1,\infty)$-atoms. The same happens if a_j lives in $(-\infty,0]$. If we are not in either of these cases, that is, if 0 is interior to the interval-support of a_j; then $\frac{1}{4}(a_j(x)+a_j(-x))$ is a $(1,\infty)$-atom. We reach the conclusion that $f_e \in H^1$ and $\|f_e\|_{H^1_{at}} \leq 2\|f\|_{H^1_{at}}$

Now, let us look at

$$f_e(x)\chi_+(x) = \frac{1}{2}\left(\sum_{j=1}^{\infty} \lambda_j a_j(x)\chi_+(x) + \sum_{j=1}^{\infty} \lambda_j a_j(-x)\chi_+(x)\right)$$

If a_j lives in $[0,\infty)$, $a_j = a_j\chi_+$ is a $(1,\infty)$-atom and $a_j(-x)\chi_+(x) = 0$. If a_j lives in $(-\infty,0]$, then $a_j\chi_+ = 0$ and $a_j(-x)\chi_+(x)$ is a $(1,\infty)$-atom. Finally, when 0 is interior to the interval-support of a_j, then $\frac{1}{2}(a_j(x)+a_j(-x))\chi_+(x)$ is a $(1,\infty)$-atom. Thus $f_e\chi_+ \in H^1$ and $\|f_e\chi_+\|_{H^1_{at}} \leq \|f\|_{H^1_{at}}$ $\square$

LEMMA 7.40. *For a function* f *supported on* $\mathbb{R}_+$, *the following properties are equivalent*:

i) $f \in H^1$

ii) $f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x)$, *where each* a_j *is a* $(1,\infty)$-*atom living in* $[0,\infty)$ *and* $\sum_{j=1}^{\infty} |\lambda_j| < \infty$.

iii) *The even extension* g *of* f *belongs to* H^1.

Besides, $\|f\|_{H^1}$, $\|g\|_{H^1}$ *and* $\inf\{\sum_{j=1}^{\infty} |\lambda_j|\}$, *where the inf is taken over all decompositions allowed in* ii); *are equivalent norms*.

Proof: Assume i). Let $f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x)$ be an atomic decomposition for f. Then, also $f(x) = f(x)\chi_+(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x)\chi_+(x)$.

Now each $a_j(x)\chi_+(x)$ is an atom living in $[0,\infty)$ and we have obtained ii) with an infimum bounded by $\|f\|_{H^1_{at}}$.

Assuming ii), we can write

$$g(x) = \sum_{j=1}^{\infty} \lambda_j \, a_j(x) + \sum_{j=1}^{\infty} \lambda_j \, a_j(-x).$$

Both $a_j(x)$ and $a_j(-x)$ are atoms, so that $g \in H^1$ with $\|g\|_{H^1_{at}} \leq 2\|f\|_{H^1_{at}}$ and iii) holds.

Finally if we assume iii), since $g = g_e$ and $f = g\chi_+$, lemma 7.39. yields $\|f\|_{H^1_{at}} \leq \|g\|_{H^1_{at}}$. □

Here is our extension of 7.33. to H^1:

THEOREM 7.41. Let $f \in H^1$ *be supported in* $[0,\infty)$. *Then*:

$$\int_0^{\infty} \left| \frac{F(x)}{x} \right| dx \leq (\log 2) \, \|f\|_{H^1_{at}}$$

Proof: Just write f as in lemma 7.40. ii) and apply lema 7.38. to each atom. □

Given $f \in H^1$, consider for $x > 0$

$$\frac{1}{x} \int_{-x}^{x} f(y)dy = \frac{1}{x} \left(\int_0^x f(y)dy + \int_0^x f(-y)dy \right) =$$

$$= \frac{2}{x} \int_0^x f_e(y)dy.$$

Applying theorem 7.41. to the function $\chi_+ f_e$, we obtain:

$$(7.42) \quad \int_0^{\infty} \left| \frac{1}{x} \int_{-x}^{x} f(y)dy \right| dx \leq 2(\log 2) \|\chi_+ f_e\|_{H^1_{at}} \leq 2(\log 2) \|f\|_{H^1_{at}}$$

$$\frac{1}{x} \int_{-x}^{x} f(y)dy = \int_{-\infty}^{\infty} K_0(x,y) f(y)dy \qquad \text{with}$$

$$K_0(x,y) = \frac{1}{x} \chi_+(x-|y|); \text{ that is:}$$

$$K_0(x,y) = \frac{1}{x} K_0(1,\tfrac{y}{x}) = \frac{1}{x} k_0(\tfrac{y}{x}) \quad \text{with} \quad k_0(u) = \chi_+(1-|u|) = \chi_{[-1,1]}(u).$$

DEFINITION 7.43. $K(x,y)$, $(x > 0,\ y \in \mathbb{R})$, *will be said to be a Hardy kernel for* H^1 *if and only if* K *satisfies condition* 7.35 *and also the operator* T *given by* 7.36. *is bounded from* $H^1(\mathbb{R})$ *to* $L^1(\mathbb{R}_+)$.

One of our basic examples of a Hardy kernel will be $K_0(x,y) = \frac{1}{x}\chi_+(x-|y|)$. It is indeed a Hardy kernel, as the estimate 7.42. shows. We shall denote by T_0 the corresponding operator bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R}_+)$.

Using T_0 and integration by parts we shall find a sufficient condition for $K(x,y)$ to be a Hardy kernel for H^1.

THEOREM 7.44. Suppose $K(x,y)$, $(x > 0,\ y \in \mathbb{R})$, *satisfies* 7.35 *and also assume that* $k(y) = K(1,y)$ *is an even function of bounded variation with total variation* $\leq B$. *Then* K *is a Hardy kernel for* H^1 *and the operator* T *defined by* 7.36. *satisfies*:

$$\int_0^\infty |Tf(x)|dx \leq (\log 2)\, B\|f\|_{H^1_{at}}$$

Proof:

$$k(y) = k(|y|) = k(\infty) - \int_{|y|}^\infty dk(t)$$

except for a denumerable set of points.

Let f be a $(1,\infty)$-atom. Then:

$$Tf(x) = \int_{-\infty}^\infty k(u)f(xu)du = -\int_{-\infty}^\infty\int_{|u|}^\infty dk(t)f(xu)du =$$

$$= -\int_0^\infty \int_{-t}^t f(xu)dudk(t) = -\int_0^\infty t\,\frac{1}{xt}\int_{-xt}^{xt} f(v)dvdk(t) =$$

$$= -\int_0^\infty t\, T_0(f)(xt)dk(t) = -\int_0^\infty T_0(tf(t\cdot))(x)dk(t),$$

the last identity being a consequence of the dilation-invariance of T_0.

Observe that $tf(t\cdot)$ is also a $(1,\infty)$-atom. Therefore:

$$\int_0^\infty |Tf(x)|dx \leq \int_0^\infty\int_0^\infty |T_0(tf(t\cdot))(x)|dx|dk(t)| \leq$$

$$\leq 2\ (\log 2) \int_0^\infty |dk(t)| \leq (\log 2)\ B.$$

The result follows, as usual, after decomposing any $f \in H^1$ into atoms. □

Let k be the even extension of a function monotone and bounded on $\mathbb{R}_+$. If we take the atom

$$f(x) = \begin{cases} 1/2 & \text{if } 0 \leq x \leq 1 \\ -1/2 & \text{if } 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

then all the inequalities in the proof of theorem 7.44. are, actually, equalities. This shows that the constant in the theorem is the best possible. It also allows us to give the following necessary condition:

THEOREM 7.45. Let $k(y)$ *be an even function on* $\mathbb{R}$, *monotone on each side of* 0. *Then, the corresponding kernel* $K(x,y) = (1/x)k(y/x)$ *is a Hardy kernel for* H^1 *if and only if* k *is bounded.*

To simplify statements as the one just given, we shall say sometimes that $k(y)$ is a Hardy kernel for H^1 to indicate, of course, that $K(x,y) = (1/x)k(y/x)$ is a Hardy kernel for H^1.

To give a simple application of theorem 7.44., consider the operator:

$$\mathcal{L}f(x) = \int_{-\infty}^{\infty} e^{-x|y|} f(y)dy.$$

We claim that:

$$\int_0^\infty \left| \frac{\mathcal{L}f(x)}{x} \right| dx \leq 2\ (\log 2)\ \|f\|_{H^1_{at}} \tag{7.46}$$

Indeed $$\int_0^\infty \left| \frac{\mathcal{L}f(x)}{x} \right| dx = \int_0^\infty \left| \frac{\mathcal{L}f(1/x)}{x} \right| dx$$

and $$(1/x)\mathcal{L}f(1/x) = \int_{-\infty}^{\infty} (1/x)e^{-|y|/x} f(y)\ dy$$

has a kernel $K(x,y) = (1/x)e^{-|y|/x}$, so that $k(u) = K(1,u) = e^{-|u|}$, which has total variation $B = 2$.

Instead of using the atomic characterization of H^1, we can use the Hilbert transform H. We obtain the following result:

THEOREM 7.47. Let $p \geq 1$ *and* $f \in L^p(\mathbb{R}_+)$. *Then, with* F *defined as in* 7.33.:

$$\left(\int_0^\infty \left| \frac{F(x)}{x} \right|^p dx \right)^{1/p} \leq C(\|f\|_p + \|Hf\|_p)$$

where C *is independent of* p *and* f.

Proof: Let f be a linear combination of atoms living in $[0,\infty)$. Then:

$$\frac{F(x)}{x} = \frac{1}{x} \int_0^x f(y)dy = \frac{1}{x} \int_0^x (f(y)+f(-y))dy = \frac{2}{x} \int_0^x f_e(y)dy =$$
$$= \frac{1}{x} \int_{-x}^x f_e(y)dy = \int_{-1}^1 f_e(xy)dy = \int_{-\infty}^\infty \chi_{[-1,1]}(y) f_e(xy)dy =$$
$$= \int_{-\infty}^\infty H\chi_{[-1,1]}(y)\, Hf_e(xy)dy.$$

A simple computation yields:

$$H\chi_{[-1,1]}(y) = \frac{1}{\pi} \log \left| \frac{y+1}{y-1} \right|$$

Then, since both Hilbert transforms are odd, we can write:

$$\frac{F(x)}{x} = \frac{2}{\pi} \int_0^\infty \log \left| \frac{y+1}{y-1} \right| Hf_e(xy)dy = \frac{2}{\pi} \int_0^\infty k(y) Hf_e(xy)dy$$

with $k(y) = \log \left| \dfrac{y+1}{y-1} \right| \cong \begin{cases} y & \text{near } 0 \\ \log|y-1| & \text{near } 1 \\ 1/y & \text{at } \infty \end{cases}$

Since

$$\int_0^\infty \log \left| \frac{y+1}{y-1} \right| y^{-1/p} dy \cong \int_1^\infty y^{-1-(1/p)} dy = p,$$

we immediately obtain:

$$\left\| \frac{F(x)}{x} \right\|_p \leq Cp\, \|Hf_e\|_p \leq Cp\|Hf\|_p$$

For $p > 1$, this inequality can be combined with 7.33. so that we have:

$$\left\|\frac{F(x)}{x}\right\|_p \leq \min\,(p'\|f\|_p,\ Cp\|Hf\|_p)$$

Then, invoking the simple inequality

$$\min\,(\tfrac{a}{b},\tfrac{c}{d}) \leq \frac{a+c}{b+d}, \quad a,\ b,\ c,\ d > 0$$

we get:

$$\left\|\frac{F(x)}{x}\right\|_p \leq \min\,\left(\frac{\|f\|_p}{1/p'}\ ,\ \frac{C\|Hf\|_p}{1/p} \right) \leq \|f\|_p + C\|Hf\|_p \leq$$

$$\leq C(\|f\|_p + \|Hf\|_p)$$

Since the linear combinations of atoms are dense in $L^p(\mathbb{R}^+)$, we obtain the result. $\square$

By examining more closely the case $p = 1$, we can give another version of 7.42. We write, as before:

$$T_0f(x) = \frac{1}{x}\int_{-x}^{x} f(y)dy = \frac{1}{x}\int_{-x}^{x} f_e(y)dy = \int_{-1}^{1} f_e(xy)dy =$$

$$= \frac{2}{\pi}\int_0^{\infty} \log\left|\frac{y+1}{y-1}\right| Hf_e(xy)dy. \text{ Then we observe that:}$$

$$\int_0^{\infty} \log\left|\frac{y+1}{y-1}\right| y^{-1}\,dy = \pi^2/2$$

(see Selby [1] p. 467), and, consequently:

$$\int_0^{\infty} \left|\frac{1}{x}\int_{-x}^{x} f(y)dy\right| dx \leq \pi\,\|Hf\|_1 \tag{7.48}$$

In the proof above we can take f to be an atom or an element of any other dense class for which it is legitimate to write:

$$T_0f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \log\left|\frac{y+1}{y-1}\right| Hf_e(xy)\,dy$$

If we use, for example, the dense class appearing in theorem 1.8., we obtain a proof of (7.48.) independent of the atomic decomposition.

Theorem 7.44. was inspired by inequality (7.33.). Now we shall present another basic sufficient condition, which is inspired by inequality (7.34).

THEOREM 7.49. Let T *be an operator bounded from* $L^2(\mathbb{R})$ *to* $L^2(\mathbb{R}_+)$ *with operator norm bounded by* A. *Suppose that* T *commutes with dilations and, at least for atoms, is given by* (7.36) *with* $k(y) = K(1,y)$ *satisfying a Lipschitz condition*

$$|k(x) - k(y)| \le B|x - y|$$

Then T *can be extended to an operator bounded from* $H^1_{at}(\mathbb{R})$ *to* $L^1(\mathbb{R}_+)$ *with operator norm bounded by* $CA^{2/3}B^{1/3}$, C *being an absolute constant*.

Proof: Let f be an atom with interval-support $I = [a,b]$. Then $\|f\|_2 \le |I|^{-1/2}$, so that, using the L^2-boundedness of T, we can obtain, for any $R > 0$:

$$\int_0^R |Tf(x)|dx \le R^{1/2}\|Tf\|_2 \le AR^{1/2}\|f\|_2 \le A(R/|I|)^{1/2}$$

To get an estimate for $x > R$, we use the function $F(t) = \int_{-\infty}^t f(s)ds$, and integrating by parts, write:

$$Tf(x) = \int_{-\infty}^{\infty} k(y)f(xy)dy = -\int_{-\infty}^{\infty} k'(y)\,\frac{1}{x}\,F(xy)dy$$

This leads to

$$|Tf(x)| \le (\|k'\|_\infty/x)\int_{-\infty}^{\infty} |F(xy)|dy \le (B/x^2)\int_{-\infty}^{\infty} |F(y)|dy$$

But

$$|F(t)| = \left|\int_{-\infty}^t f(y)dy\right| = \left|\int_t^{\infty} f(y)dy\right| \le \min(t-a,b-t)/|I|.$$

Setting $c = (a+b)/2$, we get:

$$\int_{-\infty}^{\infty} |F(y)|dy \le \left(\int_a^c (y-a)dy + \int_c^b (b-y)dy\right)/|I| = |I|/4$$

so that:

$$|Tf(x)| \le B|I|/(4x^2)$$

and, consequently:

$$\int_R^\infty |Tf(x)|\ dx \le B|I|/(4R)$$

Putting together both estimates, we arrive at:

$$\|Tf\|_1 \le \min_{R>0} \left(A(R/|I|)^{1/2} + (B/4)(R/|I|)^{-1}\right)$$

By elementary calculus, the function $r \longmapsto \alpha r^{1/2} + \beta r^{-1}$ attains its minimum over $r > 0$ for $r^{3/2} = 2\beta/\alpha$. In our case, the best estimate will correspond to $R/|I| = (B/(2A))^{2/3}$. With this choice for R, we get:

$$\|Tf\|_1 \le CA^{2/3}B^{1/3}$$

Splitting a general $f \in H^1$ into atoms, we obtain

$$\|Tf\|_1 \le CA^{2/3}B^{1/3}\|f\|_{H^1_{at}}$$

as we wanted to show. $\square$

The best example of application of this second criterion is to the operator

$$T_1 f(x) = \frac{1}{x}\hat{f}\left(\frac{1}{x}\right), \qquad x > 0$$

It is clearly bounded from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}_+)$ since:

$$\int_0^\infty \left|\frac{1}{x}\hat{f}\left(\frac{1}{x}\right)\right|^2 dx = \int_0^\infty |\hat{f}(x)|^2 dx \le \int_{-\infty}^\infty |\hat{f}(x)|^2 dx = \int_{-\infty}^\infty |f(x)|^2 dx$$

We also have:

$$T_1 f(x) = \int_{-\infty}^\infty K_1(x,y)\ f(y)dy$$

with $\quad K_1(x,y) = \frac{1}{x}\, e^{-2\pi i(y/x)}$

This kernel K_1 obviously satisfies (7.35), so that T_1 commutes with dilations. Also $k_1(y) = K_1(1,y) = e^{-2\pi iy}$ is clearly Lipschitz with $\|k_1'\|_\infty = 2\pi$.

Then, theorem 7.49. implies that K_1 is a Hardy kernel for H^1.

We can give another necessary and sufficient condition, this time for odd functions.

THEOREM 7.50. Let $k(y)$ *be an odd function (or measure) with constant sign to each side of* 0. *Then* k *is a Hardy kernel for* H^1 *if and only if*:

$$\int_0^\infty |k(y)|\ \frac{dy}{y} < \infty.$$

Proof: Of course, the condition is sufficient, because it is just (7.37) with $p = 1$ and we know that this implies that the operator is bounded from $L^1(\mathbb{R})$ to $L^1(\mathbb{R}_+)$.

To see that it is also necessary, consider the atom:

$$f(x) = \begin{cases} -1/2 & \text{if} \quad -1 \leq x < 0 \\ 1/2 & \text{if} \quad 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $Tf(x) = \int_{-\infty}^{\infty} k(y)f(xy)dy = \frac{1}{2}\int_0^{1/x} k(y)dy - \frac{1}{2}\int_{-1/x}^{0} k(y)dy$

so that $|Tf(x)| = \pm \int_0^{1/x} |k(y)|dy$, which belongs to L^1 if and only if $\int_0^\infty \int_0^{1/x} |k(y)|dy\ dx = \int_0^\infty |k(y)|\ \frac{dy}{y} < \infty$. $\square$

Apart from the three basic sufficient conditions for k to be a Hardy kernel for H^1 (7.37 with $p = 1$, 7.44. and 7.49), we can derive others by using mappings known to preserve H^1. We shall give two examples based upon de Hilbert transform and the Fourier transform.

The identity

$$\int_{-\infty}^{\infty} Hk(y)\ f(xy)dy = -\int_{-\infty}^{\infty} k(y)\ Hf(xy)dy$$

allows us to write that:

THEOREM 7.51. The Hilbert transform Hk *of a Hardy kernel for* H^1, k, *is also a Hardy kernel for* H^1. $\square$

This statement is not very precise because it does not say what do we mean by Hk for a general Hardy kernel k. This difficulty can be overcome by using the theory of distributions. We shall limit ourselves to using theorem 7.51. just for nice k's.

For example, let $k(y) = y\, \chi_{[0,1]}(y)$. Of course this $k(y)$ is a Hardy kernel for H^1, since

$$\int_0^\infty |k(y)| \frac{dy}{y} = \int_0^1 dy = 1 < \infty.$$

Now $Hk(y) = yH(\chi_{[0,1]})(y) - \frac{1}{\pi}\int_0^1 dy = \frac{1}{\pi}\left(y \log\left|\frac{y}{y-1}\right| - 1\right)$

will also be a Hardy kernel for H^1. However it does not satisfy any of the three basic conditions. First of all $\int_{-\infty}^{\infty} |Hk(y)| \frac{dy}{|y|} = \infty$. Actually the integral is ∞ even if we add a constant to Hk since $Hk(y) \to 0$ as $y \to \infty$ whereas $Hk(y) \to -1/\pi$ as $y \to 0$. The two other conditions fail since $Hk(y) \to \infty$ as $y \to 1$.

Next we shall use the Fourier transform, for which we have the formula:

$$\int_{-\infty}^{\infty} \hat{g}(y) f(xy)\,dy = \int_{-\infty}^{\infty} g(y)\,\frac{1}{x}\,\hat{f}\,(\tfrac{y}{x})\,dy =$$
$$= \int_{-\infty}^{\infty} g(y)\frac{1}{x}\left(\frac{1}{|y|} f(\tfrac{1}{y}\cdot)\right)^{\wedge}(\tfrac{1}{x})\,dy = \int_{-\infty}^{\infty} g(y)\,T_1\left(\frac{1}{|y|} f(\tfrac{1}{y}\cdot)\right)(x)\,dy$$

This formula cannot ve applied to every Hardy kernel g because T_1 does not preserve H^1. Still, for g integrable or, more generally, for a finite measure μ instead of g, we can take advantage of the fact that T_1 sends H^1 into L^1. We get the following result:

THEOREM 7.52. The Fourier transform $\hat{\mu}$ *of a finite measure* μ, *is a Hardy kernel for* H^1.

Proof:

$$Tf(x) = \int_{-\infty}^{\infty} \hat{\mu}(y) f(xy)\,dy = \int_{-\infty}^{\infty} T_1\left(\frac{1}{|y|} f(\tfrac{1}{y}\cdot)\right)(x)\,d\mu(y).$$

and consequently:

$$\|Tf\|_1 \le \int_{-\infty}^{\infty} \left\|T_1\left(\frac{1}{|y|} f(\tfrac{1}{y}\cdot)\right)\right\|_1 |d\mu(y)| \le C\|\mu\|\,\|f\|_{H^1}. \quad \square$$

For example, if k is an even function of bounded variation, then k' is a finite measure and, consequently $(k')^{\wedge}(y) = 2\pi y\hat{k}(y)$ is a Hardy kernel for H^1. Thus, for such k, both $k(y)$ and $y\hat{k}(y)$ are Hardy kernels for H^1.

We shall end the section by presenting some extensions of the Fejer-Riesz inequality which was studied already, for the torus, in chapter I (theorem 4.5.). First we deal with the one-dimensional case and later we shall move on to higher dimensions.

THEOREM 7.53. For $f \in H^1(\mathbb{R})$, *let* $u(x,t) = P_t * f(x)$ *be the Poisson integral of* f. *Then*:

$$\int_0^\infty |u(x,t)|\,dt \le (2(\log 2)/\pi)\ \|f\|_{H^1_{at}}$$

Proof:

$$u(x,t) = \int_{-\infty}^{\infty} f(x-y)P_t(y)\,dy = \int_{-\infty}^{\infty} K(t,y)f(x-y)\,dy$$

where $\quad K(t,y) = P_t(y) = \frac{1}{\pi}\frac{t}{y^2+t^2}$

Now K obviously satisfies (7.35) and $k(y) = K(1,y) = \frac{1}{\pi}\frac{1}{1+y^2}$ is an even function of bounded variation with total variation $2/\pi$. It follows from theorem 7.44. that

$$\int_0^\infty |u(x,t)|\,dt \le \left(\frac{2\log 2}{\pi}\right)\|f(x-\cdot)\|_{H^1_{at}} = \frac{2\log 2}{\pi}\|f\|_{H^1_{at}} \quad \Box$$

The more accurate estimate given by the first inequality in lemma 7.38. can be used to show that:

THEOREM 7.54. With the same hypothesis and notation of theorem 7.53., *we have*

$$\int_0^\infty |u(x,t)|\,dt \to 0 \qquad \text{as} \qquad |x| \to \infty$$

Proof: Let $f = \sum_j \lambda_j a_j$, where $\sum_j |\lambda_j| < \infty$ and each a_j is an atom with interval-support I_j, centered at c_j. Then, proceding as in the proof of theorem 7.44., we get, for a fixed x:

$$P_t * a_j(x) = \int_{-\infty}^{\infty} a_j(x-y)K(t,y)\,dy = -\int_0^\infty s\ T_0(f_j)(st)\,dk(s),$$

where we have used the notation:

$$K(t,y) = P_t(y),\ k(y) = K(1,y),\ f_j(y) = a_j(x-y)$$

Suppose $x \notin I_j$. Then

$$T_0(f_j)(s) = \frac{1}{s}\int_{-s}^{s} a_j(x-y)dy = \frac{1}{s}\int_0^s (a_j(x-y)+a_j(x+y))dy$$

The atom $y \longmapsto a_j(x-y)$ is supported in $x - I_j$ and the atom $y \longmapsto a_j(x+y)$ is supported in I_j-x. These two intervals $x-I_j$ and I_j-x are symmetric and, because $x \notin I_j$, they are disjoint. For $s > 0$, only one of the atoms $a_j(x-y)$ or $a_j(x+y)$ actually occurs in the expression for $T_0(f_j)(s)$. By using the first inequality in lemma 7.38., we get:

$$\int_0^\infty |T_0(f_j)(s)|ds \leq 2(\log 2)\,\frac{|I_j|}{|x-c_j|} \leq \frac{C|I_j|}{|x-c_j|+|I_j|}$$

and, consequently:

$$\int_0^\infty |P_t*a_j(x)|dt \leq \int_0^\infty \int_0^\infty |sT_0(f_j)(st)|dt|dk(s)| \leq \frac{C|I_J|}{|x-c_j|+|I_j|}$$

If $x \in I_j$, we use theorem 7.53 to write:

$$\int_0^\infty |P_t*a_j(x)|dt \leq C \leq \frac{C|I_j|}{|x-c_j|+|I_j|}$$

Finally, since $u(x,t) = \sum_j \lambda_j P_t*a_j(x)$, we get:

$$\int_0^\infty |u(x,t)|dt \leq C \sum_j |\lambda_j| \frac{|I_j|}{|x-c_j|+|I_j|} \to 0 \quad \text{as} \quad |x| \to \infty . \square$$

In order to give an n-dimensional version of the Fejer-Riesz inequality, we shall use the following

THEOREM 7.55. Let a *be a* $(1,\infty)$*-atom in* $\mathbb{R}^n$. *Then, the function* $\bar{a}$, *defined on* $\mathbb{R}$ *by*:

$$\bar{a}(r) = \begin{cases} r^{n-1}\displaystyle\int_{\Sigma_{n-1}} a(r\sigma)d\sigma & \text{if} \quad r > 0 \\ 0 \quad \text{if} \quad r \leq 0 \end{cases}$$

is a $(1,\infty)$*-atom in* $\mathbb{R}$ *times a geometric constant.*

Proof: The cancellation is immediate since

$$\int_{-\infty}^{\infty} \bar{a}(r)dr = \int_0^{\infty} r^{n-1} \int_{\Sigma_{n-1}} a(r\sigma)d\sigma dr = \int_{\mathbb{R}^n} a(x)dx = 0.$$

Suppose a is supported in a ball of diameter δ and $|a(x)| \leq C\delta^{-n}$ for every x. Let $\bar{a}$ be supported in $[r_0, r_0+\delta]$ with $r_0 \geq 0$. To estimate the size of $\bar{a}$ we shall distinguish two cases.

First, if $r_0 \leq \delta$, we immediately get

$$|\bar{a}(r)| \leq C\delta^{-1}$$

If, on the contrary, we have $\delta < r_0$, we have to be more clever. There are N rotations $\theta_1, \theta_2, \ldots, \theta_N$, such that the supports of the atoms $x \longmapsto a(\theta_j x)$ for $j = 1, \ldots, N$, do not overlap, and $N \geq C(r_0/\delta)^{n-1}$ with C a geometric constant. Then:

$$\bar{a}(r) = \frac{r^{n-1}}{N} \int_{\Sigma_{n-1}} \sum_{j=1}^{N} a(\theta_j r\sigma)d\sigma$$

from which we obtain the estimate:

$$|\bar{a}(r)| \leq C\left(\frac{r_0+\delta}{r_0}\right)^{n-1} \cdot \frac{1}{\delta} \leq C\delta^{-1}. \qquad \square$$

COROLLARY 7.56. For $f \in H^1(\mathbb{R}^n)$, *the function* $\bar{f}$ *defined on* $\mathbb{R}$ *by*:

$$\bar{f}(r) = \begin{cases} r^{n-1}\displaystyle\int_{\Sigma_{n-1}} a(r\sigma)d\sigma & \text{if } r > 0 \\ 0 & \text{if } r \leq 0 \end{cases}$$

belongs to $H^1(\mathbb{R})$ *and*: $\|\bar{f}\|_{H^1(\mathbb{R})} \leq C\, \|f\|_{H^1(\mathbb{R}^n)}$

with C *independent of* f.

Proof: Decompose f into atoms and use theorem 7.55. $\square$

Now we can give the promised extension of the Fejer-Riesz inequality to $\mathbb{R}^n$.

THEOREM 7.57. For $f \in H^1(\mathbb{R}^n)$, *let* $u(x,t) = P_t * f(x)$ *be the Poisson* *integral of* f. *Then*:

$$\int_0^\infty |u(x,t)|t^{n-1}dt \leq C\|f\|_{H^1(\mathbb{R}^n)}$$

where C *does not depend on* f.

Proof:

$$\begin{aligned} u(x,t)t^{n-1} &= c_n \int_{\mathbb{R}^n} \frac{t^n}{(t^2+|y|^2)^{\frac{n+1}{2}}} f(x-y)\,dy = \\ &= c_n \int_0^\infty \frac{t^n}{(t^2+r^2)^{\frac{n+1}{2}}} \int_{\Sigma_{n-1}} f(x-r\sigma)d\sigma\, r^{n-1}dr = \\ &= \int_0^\infty K(t,r)\bar{g}(r)dr \end{aligned}$$

where $K(t,r) = c_n \dfrac{t^n}{(t^2+r^2)^{\frac{n+1}{2}}}$ satisfies (7.35) with $k(r) = K(1,r) =$ $= c_n \dfrac{1}{(1+r^2)^{\frac{n+1}{2}}}$ obviously even and of bounded variation, so that theorem 7.44. can be applied; and $g(y) = f(x-y)$ has $\|g\|_{H^1(\mathbb{R}^n)} = \|f\|_{H^1(\mathbb{R}^n)}$, so that, the function $\bar{g}$ associated to g as in corollary 7.56, satisfies $\|\bar{g}\|_{H^1(\mathbb{R})} \leq C\,\|f\|_{H^1(\mathbb{R}^n)}$ Therefore, the theorem follows. $\square$

Exactly as in the one-dimensional case we can prove:

THEOREM 7.58. With the same hypothesis and notation of theorem 7.57:

$$\int_0^\infty |u(x,t)|\, t^{n-1}\, dt \to 0 \ \underline{\textit{as}}\ \ |x| \longrightarrow \infty. \ \square$$

8. NOTES AND FURTHER RESULTS

8.1.- In chapter I we viewed the spaces H^p as classes of functions analytic in the unit disk D. We saw that these spaces are useful tools in studying many questions of Fourier Analysis on T, the boundary of D. However, our methods were based on Complex Analysis, so that we always had to be changing from T to D and then back to T. This was the usual way to work in Fourier Analysis during the first half of our century. Now in chapter III we build upon the basic real variable techniques discussed in chapter II, and completely liberate the H^p theory from its dependence on Complex methods. The history of this liberation movement of the H^p spaces culminates in the decisive work of C. Fefferman and E.M. Stein [2]. Here we list chronologically the main steps in this process:

1) E.M. Stein and G. Weiss [1] introduce the notion of conjugate system of harmonic functions as a natural generalization of the concept of holomorphic function. As we see at the end of section 4, it is simply a solution of a system of partial differential equations which generalizes to higher dimensions the Cauchy-Riemann system. They define $H^p(\mathbb{R}^{n+1}_+)$ for $p > (n-1)/n$ as a space whose elements are conjugate systems of harmonic functions and extend to this context the basic theory of F. Riesz on boundary behaviour. We discuss these results in section 1 for $n = 1$ and in section 4 for the general case. Their main tool is that of harmonic majorization, possible after they proved lemma 4.14.

2) It was observed that some singular integral operators can be extended from L^p, $1 < p < \infty$ to H^p, $p \leq 1$ (see E.M. Stein [2], chap. VII for the early proof of one of these results). The proofs were rather unnatural because they depended on inequalities for some auxiliary functions, for example, on the H^p inequalities for the area function obtained by A.P. Calderón [2] and C. Segovia [1]. Section 7 contains a systematic study of these boundedness properties based upon the atomic description of the H^p spaces.

3) A major breakthrough was the theorem of Burkholder, Gundy and Silverstein [1] (theorem 3.10), proved originally by probabilistic methods (Brownian motion). This theorem opens the way for defining $H^p(\mathbb{R}^{n+1}_+)$ directly as a space of harmonic functions without appealing to any notion of conjugacy. This is what we do in section 4.

4) The next significant advance was the identification of the dual of H^1 by C. Fefferman [1] as the space B.M.O. This is a space of functions on the boundary, $\mathbb{R}^n$. This identification confirmed that it was appropriate to look at H^1 as a space of real functions in $\mathbb{R}^n$ and also provided a direct method to establish the boundedness of more singular integrals in H^1.

5) C. Fefferman and E.M. Stein [2] define $H^p(\mathbb{R}^{n+1}_+)$ as a space of harmonic functions $u(x,t)$ in $\mathbb{R}^{n+1}_+$ such that the vector $F(x,t)$, whose components are u and a number of conjugates depending on p, is uniformly in $L^p(\mathbb{R}^n)$, i.e.:

$$\int_{\mathbb{R}^n} |F(x,t)|^p dx \leq C \quad \text{independently of} \quad t > 0 .$$

The definition is the same as that of E.M. Stein and G. Weiss [1], except that, after Calderón and Zygmund [3], they are able to allow any $p > 0$ by just taking conjugate systems of increasing complexity (higher order gradients). Starting with this definition, they show that the following properties are equivalent for a tempered distribution f in $\mathbb{R}^n$:

a) $f = \lim_{t\to 0} u(\cdot,t)$ in the sense of tempered distributions, for some $u \in H^p$.

b) $\sup_{t>0} |f * \phi_t(x)|$ is in L^p for each $\phi \in S$

c) $\sup_{t>0} |f * \phi_t(x)|$ is in L^p for just one $\phi \in S$ having $\int_{\mathbb{R}^n} \phi(x)dx \neq 0$.

d) The non-tangential maximal function $P^*_\nabla(f)(x) = \sup_{|y-x|<t} |u(y,t)|$ is in L^p where $u(y,t) = P_t * f(y)$.

The equivalence between a) and d) is an extension to several variables of the theorem of Burkholder, Gundy and Silverstein. The equivalence of a) with b) or c) shows that the Poisson kernel does not play a significant role and may be replaced by any reasonable approximate identity. This clarifies the position of H^p as a space of real objects in $\mathbb{R}^n$, and helps to explain the permanence of H^p under many natural operators.

8.2.- The first three sections of the chapter are devoted to the study of $H^p(\mathbb{R}^2_+)$. These spaces were first studied by Krylov [1] by using conformal mapping from the disk. Our method is quite different. It uses the ideas of E.M. Stein and G. Weiss [1] and of C. Fefferman and E.M. Stein [2] plus a third ingredient, which will turn out to be the basic one for the whole chapter: atomic decompositions. The atomic decomposition for H^1 follows immediately from the duality theorem by using the Hahn-Banach theorem. Indeed, one sees that B.M.O. is the dual of the atomic space, just from the definition. R. Coifman [1] was able to obtain the atomic decomposition constructively for $\mathrm{Re}H^p(\mathbb{R})$, $0 < p \leq 1$. In particular this gives a different proof of the duality theorem. We adopt Coifman's method, based upon a smooth version of the Calderón-Zygmund decomposition (theorem 3.6), which was already used in Fefferman-Riviere-Sagher [1] for the purpose of interpolating.

The atomic decomposition of $H^p(\mathbb{R}^n)$ for $n > 1$ was achieved by Latter [1]. The technical details of Latter's proof were simplified by Uchiyama (see Latter and Uchiyama [1]). We have chosen a completely different approach to the atomic decomposition in higher dimensions in order to avoid the technical differences between the cases $n = 1$ and $n > 1$, and also because this approach, based upon a reproducing formula of A.P. Calderón [8], applies to many different situations (see J.M. Wilson [1]).

8.3.- It is important to note that in the definition of the atomic "norm" $N_{p,r}$ (immediately after the proof of theorem 3.7), it is not legitimate to take the infimum only over the finite linear combinations of atoms even if we know that f admits such a finite decomposition. The "norm" obtained in this way may be much larger, as the following example, due to Y.Meyer, shows (see Meyer, Taibleson and Weiss [1]):

Let U be an open dense subset of the interval $(0,1)$ with Lebesgue measure $|U| < \varepsilon$. Write U as a disjoint union of open intervals $U = \bigcup_j I_j$. To each I_j, associate a $(1,\infty)$-atom a_j supported on I_j, which equals $1/|I_j|$ on a half of I_j and $-1/|I_j|$ on the other half. Let $f = \sum_{j=1}^{\infty} |I_j| a_j$. Clearly $f \in H^1$ and $N_{1,\infty}(f) \leq \sum_{j=1}^{\infty} |I_j| = |U| < \varepsilon$. But f is also an atom supported on $[0,1]$, since $|f(x)| = \chi_U(x) \leq 1$. Now suppose that $f = \sum_{k=1}^{n} \lambda_k b_k$ where each b_k is a $(1,\infty)$-atom supported on the interval $J_k \subset (0,1)$. Then, if we call $s_k(x) = (1/|J_k|)\chi_{J_k}(x)$, we have

$$\chi_U(x) = |f(x)| \leq \sum_{k=1}^{n} |\lambda_k| |b_k(x)| \leq \sum_{k=1}^{n} |\lambda_k| s_k(x) = g(x).$$

The function g is continuous in all but finitely many points of $(0,1)$. Since $g \geq \chi_U$ and U is dense in $(0,1)$, it follows that $g(x) \geq 1$ for all but finitely many $x \in (0,1)$. Consequently

$$1 \leq \int_0^1 g(x)dx = \sum_{k=1}^{n} |\lambda_k| \quad .$$

We have that $\|f\|_{H^1} \leq C\varepsilon$ while the inf of $\Sigma|\lambda_k|$ over all finite atomic decompositions is necessarily ≥ 1. Since ε is as small as we please, this shows that the inf over the finite sums is not a norm equivalent to the H^1-norm. The same can be done for $p < 1$. Observe that what makes this example work is the difference between the Lebesgue and the Riemann integrals. Of course, the finite linear combinations of atoms are dense in H^p; but, as we have seen, in order to compute their H^p-"norm", it is not enough to look only at the finite decompositions.

8.4.- L. Carleson [5] has given a proof of the duality between H^1 and B.M.O. in $\mathbb{R}^n$, which differs both from the original proof of C. Fefferman and from the atomic proof. Carleson's proof is obtained by proving the following decomposition theorem for B.M.O.:

Let $h \in S(\mathbb{R}^n)$, *even and such that* $\int h \neq 0$. *Suppose* $\phi \in$ B.M.O.$(\mathbb{R}^n)$ *and has compact support. Then, there exists a number* B, *only depending on* n, *a sequence of functions* $b_j(x)$, *such that*

$$\sum_{j=0}^{\infty} \|b_j\|_\infty \leq B\|\phi\|_*$$

and $t_j(y) > 0$ *such that*

$$(*) \quad \phi(x) = \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} h_{t_j(y)}(x-y)b_j(y)dy + b_0(x) + \text{Constant}$$

Conversely, any function of this type belongs to B.M.O. *and* $\|\phi\|_* \leq C\sum_{j=0}^{\infty}\|b_j\|_\infty$.

After this decomposition, the characterization of $H^1(\mathbb{R}^n)$ given by b) in 8.1., leads immediately to the inequality:

$$\left| \int_{\mathbb{R}^n} f(x)\phi(x)dx \right| \leq C\|f\|_{H^1}\|\phi\|_*$$

and this proves the duality.

A. Uchiyama [3] has observed that, in the above decomposition theorem, one can take $b_j = 0$ for every $j \geqq 2$.

Carleson's paper [5] also contains another important observation, which has to do with an alternative generalization of $H^1(\mathbb{R})$ to several variables via homomorphic functions. Consider the subspace S of $L^1(\mathbb{R}^n)$ generated by those functions $f(x)$ which admit analytic extension to some poly-half-plane. The prototype of such a function is one whose Fourier transform vanishes unless all coordinates are $\geqq 0$. Then, the following holds:

The space S *coincides with* $H^1(\mathbb{R}^n)$.

To prove this, we decompose $\mathbb{R}^n$ in $n + 1$ open convex cones with vertex at the origin such that the union is $\mathbb{R}^n \setminus \{0\}$, and no Γ_j contains a straight line. Then take a partition of unity $\sum_{j=1}^{n+1} \phi_j = 1$ by means of functions $\phi_j \in C^\infty(\mathbb{R}^n \setminus \{0\})$, homogeneous of degree 0 and such that each ϕ_j has support in Γ_j^*, the cone cual to Γ_j. We have, if $f \in H^1$:

$$f = f_1 + f_2 + \dots + f_{n+1}, \quad \text{where} \quad f_j = (\phi_j \hat{f})^{\vee} \quad .$$

It follows from theorem 7.30 that each f_j is also in H^1 and its Fourier transform is supported in Γ_j^* . The result clearly extends to $p < 1$.

For the theory of H^p spaces defined in terms of holomorphic functions of several variables see Chapter III of Stein and Weiss [2].

8.5- As we mentioned in 8.1., the theorem of Burkholder, Gundy and Silverstein [1] was proved by probabilistic methods and was extended by C. Fefferman and E.M. Stein [2] to $n > 1$. The direct proof we give in 3.13 for $n = 1$ comes from P. Koosis [2]. There is an extension of this proof to higher dimensions also due to P. Koosis [3].

Probably, the best way to understand this theorem is through the area function $A(u)$ introduced in 7.14.

Burkholder and Gundy [1] showed the following:

Let $u(x,t)$ *be harmonic in* $\mathbb{R}^{n+1}_+$ *and let* $0 < p < \infty$. *Then*

$$(*) \qquad \int_{\mathbb{R}^n} (A(u)(x))^p dx \leq C \int_{\mathbb{R}^n} (m_u(x))^p dx$$

With C *independent of* u. *If the left hand side of* $(*)$ *is finite, then* $\lim_{t\to\infty} u(x,t)$ *exists and is finite and constant, for* $x \in \mathbb{R}^n$. *If* u *is normalized so that this limit is* 0, *then the converse inequality holds:*

$$(**) \qquad \int_{\mathbb{R}^n} (m_u(x))^p dx \leq C \int_{\mathbb{R}^n} (A(u)(x))^p dx$$

with C *independent of* u.

Actually their result is more general, because they allow a class of "norms" which includes the L^p "norms" as a particular case.

This theorem can be used to prove inequalities of the form

$$\int_{\mathbb{R}^n} (m_v(x))^p dx \leq C \int_{\mathbb{R}^n} (m_u(x))^p dx$$

where v, normalized to vanish at ∞, is a conjugate of u. This leads immediately to the Burkholder-Gundy-Silverstein theorem.

The above theorem of Burkholder and Gundy [1], also yields a characterization of $H^p(\mathbb{R}^{n+1}_+)$ in terms of the area function, namely:

The harmonic function $u(x,t)$ *belongs to* $H^p(\mathbb{R}^{n+1}_+)$ *if and only if* $u(x,t) \to 0$ *as* $t \to \infty$ *and* $A(u) \in L^p(\mathbb{R}^n)$.

This characterization is considered by C. Fefferman and E.M. Stein [2] and, prior to them, by A.P. Calderón [2], C. Segovia [1] and G. Gasper [1]. It was this characterization that allowed A.P. Calderón [2] to study his commutator operator, the first step in understanding the Cauchy operator associated to a Lipschitz curve (see note 7.12 in Chapter II).

The area function $A(u)$ can be replaced by the g function (its radial analogue)

$$g(u)(x) = \left(\int_0^\infty |\nabla u(x,t)|^2 t\,dt\right)^{1/2}$$

See Fefferman-Stein [2].

The way that Burkholder and Gundy [1] prove their theorem is by comparing the distribution functions of m_u and $A(u)$. They obtain a so-called "good-λ" inequality. This kind of inequalities has appeared already in this book. It is essentially a "good-λ" inequality that allowed us to prove theorem 3.6 in Chapter II.

The method of "good-λ" inequalities appeared for the first time in Burkholder and Gundy [2], also in connection with probability, more concretely with the theory of martingales. To understand the relation between martingales and harmonic functions, the reader is advised to look at Burkholder [4]. For other applications of "good-λ" inequalities see Coifman [3], Coifman and Fefferman [1] and Hunt and Young [1].

From their "good-λ" inequalities, Burkholder and Gundy [1] obtain also very simple proofs of the theorems of Calderón and Stein on the local behaviour at the boundary of harmonic functions (see note 7.14 in Chapter II).

8.6.- In his excellent expository lecture on H^p, C. Fefferman [5] made the conjecture that any non-degenerate system of very smooth singular integrals in $\mathbb{R}^n$ would characterize $H^1(\mathbb{R}^n)$. In more precise terms, his conjecture was this: Let $f \in L^1(\mathbb{R}^n)$ and let $K_1, K_2, \dots, K_N$ be a finite collection of singular integral kernels $K_j(x) = \Omega_j(x')|x|^{-n}$ where Ω_j has mean value 0 and is C^∞. Suppose that the system is non-degenerate - for example we could assume that it is elliptic, so that the $\hat{K}_j(\xi)$ have no common zeroes on $\mathbb{R}^n \setminus \{0\}$ - . Finally suppose that $K_j * f \in L^1$ for every j. Then f is in $H^1(\mathbb{R}^n)$.

J. García-Cuerva [1] found that the system formed by the Riesz transforms of order 2 (or any even order) is a counterexample to the above conjecture. For example, in $\mathbb{R}^2$, the Riesz transforms of order 2, given by the kernels

$$K_1(x_1,x_2) = \frac{x_1^2 - x_2^2}{|x|^4} \quad , \quad K_2(x_1,x_2) = \frac{-2x_1x_2}{|x|^4}$$

fail to characterize H^1. Actually, there is a radial function

$f \in L^1(\mathbb{R}^2)$ such that both $K_1 * f$ and $K_2 * f$ belong to $L^1(\mathbb{R}^2)$, whereas $f \notin H^1(\mathbb{R}^2)$. Note that $K_1(z) + iK_2(z) = z^{-2}$, a very simple kernel, whose associated multiplier is given, in polar coordinates, by $e^{i2\theta}$. See also Gandulfo, García-Cuerva and Taibleson [1] where this is analyzed together with the corresponding phenomenon in local fields.

The following theorem holds:

Let $m_1, m_2, \ldots, m_N$ *be homogeneous functions of degree* 0 *in* $\mathbb{R}^n$, C^∞ *in* Σ_{n-1}, *and let* T_j *be the corresponding multiplier operators given by* $(T_j f)\hat{}(\xi) = m_j(\xi)\hat{f}(\xi)$. *Then, the system* $\{T_1, T_2, \ldots, T_N\}$ *characterizes* $H^1(\mathbb{R}^n)$ *if and only if:*

(*) *For every* $\xi \neq 0$ *there is some* $j = 1, \ldots, N$ *such that* $m_j(\xi) \neq m_j(-\xi)$.

That $\{T_1, \ldots, T_N\}$ characterizes $H^1(\mathbb{R}^n)$ means, of course, that $f \in L^1(\mathbb{R}^n)$ is in $H^1(\mathbb{R}^n)$ if and only if $T_j f \in L^1(\mathbb{R}^n)$ for every $j = 1, \ldots, N$. Note that this is the same as saying that:

$$f \longmapsto \|f\|_1 + \sum_{j=1}^{N} \|T_j f\|_1$$

is equivalent to the H^1-norm.

That the condition (*) is necessary was proved by S.Janson [1] who also conjectured the sufficiency by analogy with the martingale setting. The sufficiency was finally established by A. Uchiyama [2] who showed that if

$$(**) \quad \operatorname{rank} \begin{pmatrix} m_1(\xi), \ldots, m_N(\xi) \\ m_1(-\xi), \ldots, m_N(-\xi) \end{pmatrix} = 2$$

for every $\xi \in \Sigma_{n-1}$, then every function $f \in B.M.O.(\mathbb{R}^n)$ with compact support, can be written as

$$f = \sum_{j=1}^{N} T_j g_j$$

with $g_j \in L^\infty$ and $\sum_{j=1}^{N} \|g_j\|_\infty \leq C\|f\|_{B.M.O.}$

As a corollary he obtains from (**) that

$$f \longmapsto \sum_{j=1}^{N} \|T_j f\|_1$$

is a norm equivalent to the H^1 norm. Note that (*) is the same as

$$\text{rank} \begin{pmatrix} 1 & m_1(\xi) & \dots & m_N(\xi) \\ 1 & m_1(-\xi) & \dots & m_N(-\xi) \end{pmatrix} \equiv 2$$

which is (**) for the system $\{I, T_1, \dots, T_N\}$ where I is the identity.

The paper of Uchiyama is a great achievement, because the decomposition $f = \sum_{j=1}^{N} T_j g_j$ is obtained constructively. For the Riesz transforms $R_1, \dots, R_n$, Fefferman and Stein obtain a decomposition of any $f \in$ B.M.O. in $\mathbb{R}^n$ as $f = g + R_1 g_1 + \dots + R_n g_n$ with $\|g\|_\infty + \sum_{j=1}^{n} \|g_j\|_\infty \leq C\|f\|_*$. However, they get this decomposition from the duality $(H^1)^* =$ B.M.O. by using the Hahn-Banach theorem, that is, in an essentially non-constructive way. In case $n = 1$, the Fefferman-Stein decomposition, which now takes the form $f = g + Hh$ with $\|g\|_\infty + \|h\|_\infty \leq C\|f\|_*$, had already been obtained constructively by P.Jones [4] by using complex function theory. For this case $n = 1$, we shall give a very simple constructive proof of the decomposition, and consequently, of the duality theorem, in section 5 of Chapter IV.

A. Uchiyama [4] has been even able to extend the theorem above to $H^p(\mathbb{R}^n)$ for a range of p's < 1.

8.7.- The Lipschitz spaces Λ_α have a long history. They appear in many different areas of Analysis. For example: in the convergence of Fourier series; in Approximation Theory and in Partial Differential Equations. For the classical theory developed by Hardy, Littlewood, Zygmund, etc., see Zygmund [1], Chapters II and III. Taibleson [1] and [2] made a systematic study of these spaces in higher dimensions by using the harmonic extension to a half-space. This point of view is also presented in Stein [1], Chapter V. The approach to the Lipschitz spaces via mean oscillation over cubes, which led us to the duality theorems, originates with the independent but simultaneous work of S. Campanato [1] and N. Meyers [1]. See also Campanato [2] and J. Peetre [1]. For an approach via mean oscillation to the general class of Besov spaces, see J. Dorronsoro [1].

The realization that the Lipschitz spaces Λ_α are the duals of the Hardy spaces H^p is due to Duren-Romberg-Shields [1] in the torus and to T. Walsh [1] in higher dimensions. It casts a new light on the classical results of Calderón-Zygmund about the preservation of the spaces Λ_α by singular integral operators (see Taibleson [2]).

8.8.- The classical approach to interpolation of operators between H^p spaces was via complex function theory. See, for example, the contributions of Calderón and Zygmund [4] and G. Weiss [1]. Our presentation, based upon the atomic decomposition, originates with Igari [1] and culminates in Riviere and Sagher [1], and in Fefferman, Riviere and Sagher [1]. Atoms are already used in the paper of Igari [1], as early as 1963. It was natural for the atoms to appear as a result of the effort to generalize the Marcinkiewicz interpolation theorem by looking closely at the Calderón-Zygmund decomposition.

The theorem of E.M. Stein [5] on interpolation of analytic families of operators which we mention in Chapter II, 7.5, was extended to H^p spaces by Stein and Weiss [3] in the classical case of the torus using complex methods and by E. Hernández [1] in several dimensions using the atomic approach.

8.9.- Coifman [2] gives a characterization of the Fourier transform $\hat{f}$ of a distribution f belonging to H^p in terms of functions of exponential type. What he does is to translate, via the Paley-Wiener theorem, the characterization of H^p obtained in Coifman [1]. Then, he uses the characterization of $\widehat{H^p}$ to obtain multiplier theorems. It is also in Coifman [2] that molecules appear explicitly for the first time. The molecular theory was further exploited in Coifman-Weiss [3] and fully developed in Taibleson-Weiss [1], which is the source of the multiplier theorem 7.30.

The results on Hardy kernels (theorems 7.44, 7.49, etc.) are due to Jodeit-Shaw [1]. So is the sharp inequality in lemma 7.38, which allows them to obtain theorem 7.54. Lemmas 7.39 and 7.40 and also theorem 7.55 come from García-Cuerva [1].

8.10.- Coifman, Rochberg and Weiss [1] have been able to obtain a weak version of the factorization theorem (3.4 in Chapter I) valid for $H^1(\mathbb{R}^n)$ with $n > 1$. In order to understand their approach, let us start with the classical factorization theorem, according to which any $F \in H^1(D)$ can be written as $F = G_1 \cdot G_2$ with $G_j \in H^2(D)$ and $\|G_j\|_{H^2} = \|F\|_{H^1}^{1/2}$ for $j = 1,2$. If we set $F = f + i\tilde{f}$ and $G_j = g_j + i\tilde{g}_j$ we obtain $\tilde{f} = g_1\tilde{g}_2 + \tilde{g}_1 g_2$. Thus, in terms of boundary functions, a function is the imaginary (or real) part of an H^1 function if and only if it is of the form $g_1\tilde{g}_2 + \tilde{g}_1 g_2$ for $g_1, g_2 \in L^2$ and, in that case, its H^1 norm is bounded by $C\|g_1\|_2\|g_2\|_2$. Now, the duality theorem implies that for $b \in$ B.M.O.:

$$(*) \qquad \left|\int_{-\pi}^{\pi} b(t)(g_1(t)\tilde{g}_2(t) + \tilde{g}_1(t)g_2(t))\,dt\right| \le C\|b\|_*\|g_1\|_2\|g_2\|_2$$

Note that the integral above can be rewritten as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g_1(t)\left(\int_{-\pi}^{\pi}(b(t)-b(s))\cot\left(\frac{t-s}{2}\right)g_2(s)ds\right)dt =$$

$$= \int_{-\pi}^{\pi} g_1(t)(b(t)\tilde{g}_2(t)-(b\cdot g_2)^{\sim}(t))dt$$

so that the estimate $(*)$ is equivalent to the L^2 boundedness of the operator $C_b = [M_b, \sim]$, the commutator operator of the multiplication operator $M_b f(x) = b(x)f(x)$ and the conjugate function operator

$$C_b f(x) = b(x)\tilde{f}(x) - (b\cdot f)^{\sim}(x) \quad .$$

Thus the factorization plus the duality imply the L^2 boundedness of C_b for $b \in$ B.M.O. But there are other possibilities:

a) From the factorization and the L^2 boundedness of C_b for $b \in$ B.M.O., the duality follows.

b) From the duality and the boundedness of the commutator, we get a weak form of the factorization.

What Coifman, Rochberg and Weiss do is to prove directly that if K is a regular singular integral kernel in $\mathbb{R}^n$ and b is in B.M.O.$(\mathbb{R}^n)$, then the commutator operator

$$[M_b, K](f)(x) = b(x)(K * f)(x) - K * (b\cdot f)(x)$$

is bounded in $L^2(\mathbb{R}^n)$ with operator norm dominated by $\|b\|_*$. They give two proofs of this fact. One of them is based on the theory of A_p weights and will be given in section 7 of Chapter IV. Actually for the Riesz transforms R_j, $j = 1,\dots,n$, we have equivalence between the fact that $b \in B.M.O.$ and the fact that each commutator $[M_b, R_j]$ is L^2-bounded.

Then, observation a) above leads to yet another proof of the duality theorem and observation b) allows them to give the following result, which contains a weak factorization theorem:

Let K *be as above and let* K^* *be the adjoint kernel. Then if* $g_1, g_2 \in L^2(\mathbb{R}^n)$, *the function* $f = g_1 K * g_2 - (K^* * g_1) g_2$ *is in* $H^1(\mathbb{R}^n)$ *with* $\|f\|_{H^1} \leq C \|g_1\|_2 \|g_2\|_2$. *Conversely every* $f \in H^1(\mathbb{R}^n)$ *can be written as*

$$f = \sum_{j=1}^{\infty} \sum_{i=1}^{n} (g_{ij} R_i(h_{ij}) + h_{ij} R_i(g_{ij}))$$

with $\sum_{i,j} \|g_{ij}\|_2 \|h_{ij}\|_2 \leq C \|f\|_{H^1}$.

8.11.- Coifman and Weiss [3] have developed a theory of Hardy spaces where the underlying space X is a space of homogeneous type, as defined in Chapter II, 7.11, that is, X is endowed with a quasi-distance d, which allows us to consider balls $B(y,r) = \{x \in X : d(x,y) < r\}$, and a doubling measure μ. If $0 < p < q$, and $p \leq 1 \leq q \leq \infty$, then a (p,q)-atom is, by definition, a real valued function $a(x)$, such that:

i) a is supported in a ball $B(y,r)$

ii) $\left(\frac{1}{\mu(B(y,r))} \int_{B(y,r)} |a(x)|^q d\mu(x)\right)^{1/q} \leq (\mu(B(y,r)))^{-(1/p)}$

iii) $\int_X a(x) d\mu(x) = 0$.

If $q = \infty$, the left hand side of ii) is interpreted as the essential supremum of a over the ball. In case $\mu(X) < \infty$, the constant function $\mu(X)^{-(1/p)}$ is counted as an atom. Actually, in this case, one can simply assume $\mu(X) = 1$.

In a context as general as this, one cannot consider higher moments. Consequently, the H^p spaces to be defined, will coincide with the classical ones only for a range of p's close to 1. Still, the point of view of Coifman and Weiss allows a very convenient unification of the ever growing family of Hardy spaces.

Since the H^p spaces for $p < 1$ will have to be spaces of "distributions", the next thing to do is to define the space of test functions. For $\alpha > 0$, we consider L_α, the space of functions f on X, for which

$$|f(x) - f(y)| \leq C(\mu(B))^\alpha$$

where B is any ball containing both x and y and C depends only on f. This space is normed by the smallest C above, or, in case $\mu(X) < \infty$, by the smallest C plus $|\int_X f(x)d\mu(x)|$. Next we observe that (p,q)-atoms act continuously on L_α for $\alpha = (1/p)-1$ if $0 < p < 1$. Then, for $0 < p < 1 \leq q$, $H^{p,q}(X)$ is defined as the subspace of the dual L_α^* of L_α where $\alpha = (1/p)-1$, consisting of those linear functionals admitting an atomic decomposition $h = \sum_{j=0}^{\infty}\lambda_j a_j$, where the a_j's are (p,q)-atoms and $\sum_0^\infty|\lambda_j|^p < \infty$. The infimum of $(\sum_0^\infty|\lambda_j|^p)^{1/p}$ over all such decompositions is denoted by $\|h\|_{p,q}$. It is clear that $\|h\|_{p,q}^p$ is a p-norm. For $p = 1$, $H^{1,q}(X)$ is defined as a subspace of $L^1(X)$.

Then, Coifman and Weiss show that $H^{p,q} = H^{p,\infty}$ whenever $p < q < \infty$, and that, moreover, the two metrics are equivalent. This completes the definition of H^p. They go on to show that, for $p < 1$, the dual $(H^p)^*$ of H^p, coincides with L_α, where $\alpha = (1/p)-1$. For $p = 1$, the dual is identified with B.M.O., which is defined in the natural way. Actually, different B.M.O.'s are introduced. B.M.O.q is defined by using the L^q norm of $f - f_B$ over each ball B. It is shown that $(H^{1,q})^* = $ B.M.O.q', from which all the B.M.O.'s are seen to coincide.

Coifman and Weiss [3] observe that Carleson's proof of the duality H^1-B.M.O. (see note 8.4), can be extended to the setting of a space of homogeneous type satisfying a mild geometric condition. They use this to obtain a maximal function characterization of $H^1(X)$. The maximal function characterization of $H^p(X)$ for $0 < p < 1$ follows after Macias-Segovia [1] and [2] and A. Uchiyama [5].

Here are some instances of spaces of homogeneous type for which the Hardy spaces of Coifman and Weiss are interesting objects connected with classical problems (see Coifman and Weiss [3] for details).
1) $\mathbb{R}^n$ with a nonisotropic distance and a doubling measure. This setting is appropriate for studying singular integrals with the cor-

responding homogeneity (see note 7.8. in Chapter II).
2) An interval of the line and a doubling measure naturally associated with the study of certain classical polynomial expansions (Jacobi series, etc.).
3) A compact Riemannian manifold with its natural distance and measure. For example a homogeneous space $X = G/K$ where K is a closed subgroup of the Lie group G. In particular the unit sphere Σ_{n-1} of $\mathbb{R}^n$ can be viewed as $\Sigma_{n-1} = SO(n)/SO(n-1)$, and then d = euclidean distance and μ = Lebesgue measure.
4) The boundary of a Lipschitz domain in $\mathbb{R}^n$ with euclidean distance and harmonic measure.
5) The boundary of a smooth and bounded pseudoconvex domain in $\mathbb{C}^n$ with Lebesgue measure and the nonisotropic quasidistance associated to its complex structure. In particular for Σ_{2n-1} this provides an example essentially different from that in 3).
6) The Heisenberg group and its generalizations (see Folland-Stein [2]).
7) A locally compact group with a system of open neighbourhoods of the identity and a measure satisfying certain natural properties. This was introduced as a reasonable setting for the study of singular integrals in Koranyi and Vagi [1] and N. Riviere [1].

8.12.- Even though the Coifman-Weiss theory briefly sketched in 8.11 covers many interesting spaces, still in many particular situations, a richer theory is obtained by considering atoms with an increasing number of vanishing moments. Here is a, necessarily incomplete, list of these richer but more particular theories going beyond the Coifman-Weiss frame.
1) The parabolic H^p spaces introduced by A. Calderón and Torchinsky [1] and [2] are associated to a space of homogeneous type which falls within point 1) in 8.11., the measure being Lebesgue measure. Initially these spaces were introduced by means of a maximal function. Then, the atomic decomposition (with higher moments) was obtained by A.P. Calderón [8] for the case when the parabolic metric is given by a diagonalizable matrix, and by Latter and Uchiyama [1] in general.
2) Two different kinds of weighted Hardy spaces were studied in J. García-Cuerva [1]. The underlying space is the line $\mathbb{R}$ with the usual distance and the measure $d\mu(x) = w(x)dx$ with w a weight belonging to the class A_∞ to be studied systematically in Chapter IV. One kind of $H^p(w)$ is characterized by the fact that

an appropriate (smooth) maximal function belongs to $L^p(w)$. Equivalently, atomic characterizations (involving higher moments with respect to Lebesgue measure) are obtained for $0 < p \leq 1$. The other kind of $H^p(w)$ is simply the one associated to the space of homogeneous type $(\mathbb{R},d,w(x)dx)$. There is a relation between these two kinds of spaces. For the homogeneous type $H^1(w)$ the following characterization is obtained $H^1(w) = \{f \cdot w^{-1} : f \in H^1(\mathbb{R})\}$. Also in García-Cuerva [1] it is shown that if f is defined in $[0,\infty]$, then $x \mapsto f(|x|)$ is in $H^1(\mathbb{R}^n)$ if and only if $r \mapsto f(|r|)|r|^{n-1}$ is in $H^1(\mathbb{R})$. This characterization of the radial functions in $H^1(\mathbb{R}^n)$, whose proof is based on theorem 7.55, is the source of the counterexample alluded to in 8.6.
General weighted Hardy spaces have been studied by Stromberg and Torchinsky (see for example their paper [1]).
3) C. Kenig ([1] and [2]) has given a theory of weighted Hardy spaces on Lipschitz domains of the plane. This extends the situation discussed in 2) and also the two different kinds of atoms appear. One thing that makes these spaces interesting is the fact that harmonic measure on the boundary of a Lipschitz domain in $\mathbb{R}^n$ is given by an A_∞ weight times surface measure. The theory of weighted Hardy spaces has been extended to Lipschitz domains of $\mathbb{R}^n$ for arbitrary n (see, for example Fabes, Kenig and Neri [1]).
4) Regarding the space of homogeneous type considered in point 5) of 8.11 over the unit sphere Σ_{2n-1} of $\mathbb{C}^n$, a richer theory is presented in Garnett-Latter [1], which also discusses the relation between the atomic space and the classical H^p space of holomorphic functions on the unit ball of $\mathbb{C}^n$. From this relation they obtain weak factorization theorems for the holomorphic H^p, extending the results obtained for holomorphic H^1 by Coifman, Rochberg and Weiss [1].
5) Atomic decompositions (with moments) for the Hardy spaces of the Heisenberg group, appear in Latter-Uchiyama [1]. A general theory for a wide class of generalizations of the Heisenberg group, is presented in Folland-Stein [2].
6) A particular case falling under point 7) in 8.11 is that of p-adic fields. The associated Hardy spaces are examples of Hardy spaces of martingales, but the theory of these latter spaces is much more general. It was in this context that the atomic decomposition was first explicitly noted by Carl Herz [1] (see also [2]).

8.13.- There are still some settings where the natural H^p theory is far removed from the Coifman-Weiss atomic theory. We are mainly thinking of the Hardy spaces associated to "product spaces", whose theory is finally beginnning to be understood, thanks mainly to the efforts of Chang and R. Fefferman (see [1], [2] and the survey [3]). See also Lin [1].

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CHAPTER IV

WEIGHTED NORM INEQUALITIES

The L^p inequalities obtained in the previous chapters for several kinds of operators remain true when Lebesgue measure dx is replaced by certain measures $w(x)dx$. This is not entirely new for us: In Chapter I we proved the Helson-Szegö theorem characterizing those $w(x)$ for which the Hilbert transform is bounded in $L^2(w)$, while the boundedness of the Hardy-Littlewood maximal function in $L^p(w)$ was established in Chapter I for every A_1 weight $w(x)$ and $1 < p < \infty$.

We shall devote this chapter to a systematic study of this type of inequalities. For the maximal function and regular singular operators, it is possible to give a very precise and satisfactory answer to the question of finding those w for which either

$$(*) \qquad \int |Tf(x)|^p\, w(x)dx \leq C_p(w) \int |f(x)|^p\, w(x)dx$$

or the corresponding weak type inequality hold. This answer is provided by the A_p theory, which has already reached a great level of perfection and has found applications in several branches of Analysis, from Complex Function Theory to Partial Differential Equations. The same problem for two weights is also considered (in sections 1 and 4), but here some unsolved questions remain which are probably far from being completely settled.

Why should one be interested in inequalities like (*)? We shall briefly sketch some answers

1) Conjugate functions, H^p spaces etc. can be defined in domains of the complex plane with a "reasonable" boundary ∂D. When estimating the L^p norms of operators appearing in this context, some of the problems that arise can be reduced, by a change of variables, to estimates for known operators on the line or on the torus, but with respect to a measure $w(x)dx$ for a certain w. These ideas were actually used in Calderón's original proof of his celebrated theorem on Cauchy integrals along Lipschitz Curves (see Chapter II, 7.12 for the precise statement).

2) Given a complete orthonormal system $\{\phi_k\}$ in $L^2(I)$, for some interval I, does the series

$$\sum_k c_k \, \phi_k(x) \qquad\qquad (c_k = \int_I f \, \overline{\phi}_k)$$

converge to $f(x)$ in the L^p norm, $p \neq 2$? For ordinary Fourier series, the partial sums are very closely related to the Hilbert transform, and M. Riesz theorem implies $\|S_n f - f\|_p \to 0$, $1 < p < \infty$. In some other interesting cases, the partial sums can be expressed in terms of the operator $f(x) \to k(x)H(f\, k^{-1})(x)$ for a certain $k(x)$, and then, convergence in L^p is equivalent to $(*)$ with $T = H$, $w(x) = |k(x)|^p$.

3) Inequalities like $(*)$ imply, when the structure of the weights satisfying them is sufficiently known, the following

$$(**) \qquad \int |Tf(x)|^2 \, u(x)dx \leq C \int |f(x)|^2 \, Nu(x)dx$$

for arbitrary $u(x) \geq 0$, where N is (in the most desirable case) some kind of "maximal operator" which we can control. An inequality like $(**)$ was proved in Chapter II, 2.12 for the Hardy-Littlewood operator, and a similar one for singular integrals will appear in corollary 3.8 of this chapter. Such inequalities are very easy to handle, and contain essentially all the relevant information about the boundedness properties of T (we shall come back to this point in Chapter VI).

4) There is an intimate connection between weights, BMO functions and C. Fefferman's duality theorem, which can be roughly described by saying that BMO consists of the logarithms of good weights. This connection is exploited in section 5, where we give a new proof of the duality theorem and obtain, by real variable methods, a satisfactory analogue in $\mathbb{R}^n$ of the Helson-Szegö theorem.

1. THE CONDITION A_p

By a weight on a given measure space, we shall always mean a measurable function w with values in $[0,\infty]$.

Our main problem is going to be the following

PROBLEM 1. *Given* p, $1 < p < \infty$, *determine those weights* w *on* $\mathbb{R}^n$ *for which the maximal operator* M *is of strong type* (p,p) *with respect to the measure* $w(x)dx$, *that is, for which we have an inequality:*

$$(1,1) \qquad \left(\int_{\mathbb{R}^n} (Mf(x))^p w(x)dx\right)^{1/p} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{1/p}$$

We can also pose this more general

PROBLEM 2. *Given* p, $1 < p < \infty$, *determine those pairs of weights on* $\mathbb{R}^n$ (u,w), *for which* M *is of strong type* (p,p) *with respect to the pair of measures* $(u(x)dx,\ w(x)dx)$, *that is, for which we have an inequality:*

$$(1.2) \qquad \left(\int_{\mathbb{R}^n} (Mf(x))^p u(x)dx\right)^{1/p} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{1/p}$$

We can pose similar problems substituting weak type for strong type in the two problems above. For example:

PROBLEM 3. *Given* p, $1 \leq p < \infty$, *determine those pairs of weights on* $\mathbb{R}^n$, (u,w), *for which* M *is of weak type* (p,p) *with respect to the pair of measures* $(u(x)dx,\ w(x)dx)$, *that is, for which we have the inequality:*

$$(1.3) \qquad u(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq Ct^{-p} \int_{\mathbb{R}^n} |f(x)|^p w(x)dx, \qquad t > 0$$

For a set E, $u(E)$ stands for $\int_E u(x)dx$. This notation has been used in (1.3) and it will be used systematically.

We shall keep the usual conventions for multiplication in $[0,\infty]$, namely $\infty . t = t . \infty = \infty$ for $0 < t \leqq \infty$ and $\infty . 0 = 0 . \infty = 0$. Also $\infty^{-1} = 0$ and $0^{-1} = \infty$ when we consider w^{-1} for a weight w.

Let us start by analizing Problem 3. Suppose that the pair of weights (u,w) is such that (1.3) holds for a given p, $1 \leqq p < \infty$, every function f and every $t > 0$. Let f be a function $\geqq 0$. Let Q be a cube such that the average $f_Q = \frac{1}{|Q|} \int_Q f(x)dx > 0$. Observe that $f_Q \leqq M(f.\chi_Q)(x)$ for every $x \in Q$. Then, for every t with $0 < t < f_Q$, $Q \subset E_t = \{x \in \mathbb{R}^n : M(f.\chi_Q)(x) > t\}$ so that, by (1.3):

$$u(Q) \leqq C\, t^{-p} \int_Q f(x)^p w(x)dx$$

It follows that

$$(f_Q)^p u(Q) \leqq C \int_Q f(x)^p w(x)dx \tag{1.4}$$

We can actually write this inequality in a seemingly stronger form. If S is a measurable subset of Q, we can replace f in (1.4) by $f.\chi_S$, obtaining

$$\left(\frac{1}{|Q|} \int_S f(x)dx\right)^p u(Q) \leqq C \int_S f(x)^p w(x)dx \tag{1.5}$$

Of course (1.5) is just equivalent to (1.4), but (1.5) is more readily applicable sometimes. For $f(x) \equiv 1$, (1.5) yields:

$$(|S|/|Q|)^p\, u(Q) \leqq C\, w(S) \tag{1.6}$$

From (1.6) we can draw some relevant information about the pair (u,w) : a) $w(x) > 0$ for a.e. $x \in \mathbb{R}^n$ (unless $u(x) = 0$ for a.e. $x \in \mathbb{R}^n$, trivial case which we shall exclude). b) u is locally integrable (unless $w(x) = \infty$ for a.e. $x \in \mathbb{R}^n$, again a trivial case which we shall also exclude). Let us prove a) and b). If $w(x) = 0$ on a set S with $|S| > 0$, a set S which we could assume to be bounded, (1.6) would imply that $u(Q) = 0$ for every cube Q containing S, and consequently, $u(x) = 0$ for almost every $x \in \mathbb{R}^n$. If $u(Q) = \infty$ for some cube Q and, consequently, for any cube containing Q, we would have $w(S) = \infty$ for any set S with $|S| > 0$, and this implies $w(x) = \infty$ for a.e. $x \in \mathbb{R}^n$.

We are about to derive a necessary condition on the pair (u,w) for

(1.3) to hold for every f and t. If $p = 1$, (1.6) can be written in the form:

$$(1.7) \qquad \frac{1}{|Q|} \int_Q u(x)dx \leqq C \frac{1}{|S|} \int_S w(x)dx$$

the inequality being valid for every cube Q and every set $S \subset Q$ with $|S| > 0$. Fix Q and let $a > \text{ess.}_Q\text{inf.}(w)$, the essential infimum of w over Q, which is defined as the $\inf\{t > 0 : |\{x \in Q : w(x) < t\}| > 0\}$. Then, $S_a = \{x \in Q : w(x) < a\}$ has $|S_a| > 0$, and (1.7) holds for $S = S_a$, from which we get: $u(Q)/|Q| \leqq Ca$. Since this is true for every $a > \text{ess}_Q\text{inf.}(w)$, we arrive finally at:

$$(1.8) \qquad \frac{1}{|Q|} \int_Q u(x)dx \leqq C \ \text{ess.}_Q\text{inf.}(w) \leqq C\, w(x), \quad \text{for a.e. } x \in Q$$

Observe that the fact that (1.8) holds for every Q is equivalent to:

$$(1.9) \qquad M(u)(x) \leqq C\, w(x) \qquad \text{for a.e.} \quad x \in \mathbb{R}^n$$

Indeed, it is clear that (1.9) implies (1.8) for every cube. Conversely if (1.8) holds for every Q, let us show (1.9) holds, that is, the set $\{x \in \mathbb{R}^n : M(u)(x) > C\, w(x)\}$ has measure 0. If $M(u)(x) > C\, w(x)$, it will be

$$\frac{1}{|Q|} \int_Q u(x)dx > C\, w(x)$$

for some cube Q containing x, and we can assume that Q has vertices with all coordinates rational. Then our set is contained in a denumerable union of sets of measure 0 and, consequently has measure 0.

Condition (1.9) is known as condition A_1 for the pair (u,w). When it holds, we also say that the pair (u,w) belongs to the class A_1, viewing A_1 as a collection of pairs of weights (u,w). We often speak of the A_1 constant for the pair (u,w) which is the smallest C for which (1.8), or equivalently (1.9), holds.

We have just seen that $(u,w) \in A_1$ is a necessary condition for M to be of weak type (1,1) with respect to the pair (u,w). It is very satisfactory to realize that this condition is actually sufficient. Indeed, let $(u,w) \in A_1$, so that (1.9) holds. Then, using inequality (2.14) in chapter II, we get

$$u(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leqq C\, t^{-1} \int_{\mathbb{R}^n} |f(x)| Mu(x) dx \leqq$$

$$\leqq C\, t^{-1} \int_{\mathbb{R}^n} |f(x)| w(x) dx.$$

Now we shall treat the case $1 < p < \infty$. We start by looking for a necessary condition. So far we know that if M is of weak type (p,p) with respect to (u,w), then (1.5) holds for every function $f \geqq 0$, every cube Q and every measurable set $S \subset Q$. Let us choose f such that $f(x) = f(x)^p w(x)$. This gives $f(x) = w(x)^{-1/(p-1)}$. A priori this function needs not be locally integrable. Fix a cube Q and take $S = S_j = \{x \in Q : w(x) > j^{-1}\}$ for $j=1,2,\ldots$ On every S_j our f is bounded, so that $\int_{S_j} f < \infty$. With our choice for f, (1.5) gives:

$$\left(\frac{1}{|Q|} \int_{S_j} w(x)^{-1/(p-1)} dx\right)^p \frac{u(Q)}{|Q|} \leqq C \frac{1}{|Q|} \int_{S_j} w(x)^{-1/(p-1)} dx$$

or, since the integrals are finite,

$$\left(\frac{1}{|Q|} \int_Q u(x) dx\right) \left(\frac{1}{|Q|} \int_{S_j} w(x)^{-1/(p-1)} dx\right)^{p-1} \leqq C$$

Now $S_1 \subset S_2 \subset \ldots$ and $\bigcup_{j=1}^{\infty} S_j = \{x \in Q : w(x) > 0\}$, whose complement in Q has measure 0, as was previously observed. Thus, letting $j \to \infty$, we get finally

$$(1.10) \qquad \left(\frac{1}{|Q|} \int_Q u(x) dx\right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx\right)^{p-1} \leqq C$$

We shall say that the pair (u,w) satisfies the condition A_p or that it belongs to the class A_p, if and only if there is a constant C such that (1.10) holds for every cube Q. The smallest such constant will be called the A_p constant for the pair (u,w). We have proved that $(u,w) \in A_p$ is necessary for M to be of weak type (p,p) with respect to the pair (u,w). Observe that $(u,w) \in A_p$ implies that both u and $w^{-1/(p-1)}$ are locally integrable. Indeed if one of the integrals in (1.10) were ∞, the same would happen for any cube containing Q, and that would force the other factor to be 0. This would imply either $u(x) = 0$ for a.e. $x \in \mathbb{R}^n$ or $w(x) = \infty$ for a.e. $x \in \mathbb{R}^n$. Both trivial situations have been excluded beforehand. Another observation that has to be made is that the condition A_1 can be viewed as a limit case of the conditions A_p for $p \downarrow 1$. Indeed, (1.8) can be written as

$$(1.11)\qquad \left(\frac{1}{|Q|}\int_Q u(x)dx\right).\mathrm{ess.}_Q\sup.(w^{-1}) \leqq C$$

while

$$\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} = \|w^{-1}\|_{L^{1/(p-1)}(Q,|Q|^{-1}dx)} \rightarrow$$

$$\rightarrow \|w^{-1}\|_{L^\infty(Q)} = \mathrm{ess.}_Q\sup.(w^{-1})$$

as $p \rightarrow 1$, i.e. $1/(p-1) \rightarrow \infty$.

Thus (1.11) is the right companion for (1.10) when $p = 1$. Also note that $(u,w) \in A_1$ implies that u is locally integrable and w^{-1} is locally bounded.

Our task will be now to show that, exactly as in the case $p = 1$, when $1 < p < \infty$, $(u,w) \in A_p$ is not only necessary, but also sufficient for M to be of weak type (p,p) with respect to the pair (u,w). We have obtained condition A_p from (1.4). The first step will be to show that, conversely, if $(u,w) \in A_p$, then (1.4) holds for every $f \geqq 0$ and every cube Q. This is actually true for $1 \leqq p < \infty$. If $p = 1$ and $(u,w) \in A_1$, we have, for every cube Q and every $f \geqq 0$:

$$\left(\frac{1}{|Q|}\int_Q f(x)dx\right)u(Q) = \int_Q f(x)dx\,\frac{u(Q)}{|Q|} \leqq C\int_Q f(x)w(x)dx,$$

which is (1.4) for $p = 1$. If $1 < p < \infty$ and $(u,w) \in A_p$, we have, for every cube Q and every $f \geqq 0$, using Hölder's inequality with p and its conjugate exponent $p' = p/(p-1)$,

$$f_Q = \frac{1}{|Q|}\int_Q f(x)w(x)^{1/p}w(x)^{-1/p}dx \leqq \left(\frac{1}{|Q|}\int_Q f(x)^p w(x)dx\right)^{1/p}$$
$$\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{\frac{p-1}{p}}.$$

Thus

$$(f_Q)^p u(Q) \leqq \frac{u(Q)}{|Q|}\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}.$$
$$.\int_Q f(x)^p w(x)dx \leqq C\int_Q f(x)^p w(x)dx$$

so that (1.4) holds. We have established the equivalence between (1.4) and A_p.

Now, suppose that (1.4) holds for every cube Q and every $f \geqq 0$. We shall obtain (1.3) with a possibly bigger C. Of course, we have (1.5) for every $f \geqq 0$, every cube Q and every set $S \subset Q$. Let $f \in L^p(w)$. We can obviously assume that $f \geqq 0$. Observe that $L^p_{loc}(w) \subset L^1_{loc}(\mathbf{R}^n)$ as follows from (1.4) using Q such that $u(Q) > 0$. Now we can also assume that $f \in L^1(\mathbf{R}^n)$. Indeed, we can always write $f = \lim_{k\to\infty} f_k$ where $f_k = f.\chi_{Q(0,k)}$. If we have (1.3) for every f_k in place of f, passing to the limit we obtain (1.3) for f. Thus, assuming f integrable, we want to estimate $u(E_t)$ where $E_t = \{x \in \mathbf{R}^n : Mf(x) > t\}$. We use theorem 1.6 from chapter II, to write $E_t \subset \bigcup_j Q_j^3$, where the Q_j's are non-overlapping cubes for which

$$\frac{t}{4^n} < \frac{1}{|Q_j|} \int_{Q_j} f(x)dx \leq \frac{t}{2^n}$$

Then,

$$u(E_t) \leqq \sum_j u(Q_j^3) \leqq C \sum_j \Big(\frac{1}{|Q_j^3|}\int_{Q_j} f(x)dx\Big)^{-p} \int_{Q_j} f(x)^p w(x)dx \leqq$$

$$\leqq C\, 3^{np}\, 4^{np}\, t^{-p} \sum_j \int_{Q_j} f(x)^p w(x)dx \leqq C\, t^{-p} \int_{\mathbf{R}^n} f(x)^p w(x)dx$$

where the second inequality resulted from applying (1.5) with $Q = Q_j^3$ and $S = Q_j$. We have a complete proof of the fact that the solution to Problem 3 is precisely the class A_p of pairs of weights. We can collect our findings in the following

THEOREM 1.12. *Let* u *and* w *be weights on* $\mathbf{R}^n$ *and let* $1 \leqq p < \infty$. *Then, the following conditions are equivalent:*

a) M *is of weak type* (p,p) *with respect to* (u,w), *that is:* M *takes* $L^p(w)$ *to* $L^p_*(u)$ *boundedly or, in other words, there is a constant* C *such that for every function* $f \in L^1_{loc}(\mathbf{R}^n)$ *and every* $t > 0$

$$u(\{x \in \mathbf{R}^n : Mf(x) > t\}) \leqq C\, t^{-p} \int_{\mathbf{R}^n} |f(x)|^p w(x)dx.$$

b) There is a constant C *such that for every function* $f \geqq 0$ *in* $\mathbf{R}^n$ *and for every cube* Q

$$\Big(\frac{1}{|Q|}\int_Q f(x)dx\Big)^p u(Q) \leqq C \int_Q f(x)^p w(x)dx$$

c) $(u,w) \in A_p$, *that is, there is a constant* C *such that for every cube* Q *we have, in case* $1 < p < \infty$,

$$\left(\frac{1}{|Q|}\int_Q u(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leq C$$

and, in case $p = 1$,

$$\left(\frac{1}{|Q|}\int_Q u(x)dx\right)\ \text{ess.}_Q\text{sup.}(w^{-1}) \leq C$$

Besides, the constants C *appearing in a), b) and c) are of the same order.*

COROLLARY 1.13. *Let* $(u,w) \in A_p$. *Then, for every* q *with* $p < q < \infty$, *the maximal operator* M *is bounded from* $L^q(w)$ *to* $L^q(u)$, *that is, there exists a constant* C *such that for every* $f \in L^1_{loc}(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} |Mf(x)|^q u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^q w(x)dx$$

Proof: We already known that M is of weak type (p,p) with respect to (u,w), that is, M takes $L^p(w)$ boundedly to $L^p_*(u)$. We shall see presently that M is also bounded from $L^\infty(w)$ to $L^\infty(u)$. This is a consequence of the chain of inequalities:

$$\|Mf\|_{\infty,u} \leq \|Mf\|_\infty \leq \|f\|_\infty \leq \|f\|_{\infty,w}$$

Since

$$\|Mf\|_{\infty,u} = \sup\{\alpha : u(\{x \in \mathbb{R}^n : Mf(x) > \alpha\}) > 0\}$$

the first inequality follows from the fact that $u(E) > 0$ implies $|E| > 0$. Likewise, the last inequality follows from the fact that $|E| > 0$ implies $w(E) > 0$, because $w(x) = 0$ only for the points in a set of measure 0. Once we know that M is bounded from $L^p(w)$ to $L^p_*(u)$ and from $L^\infty(w)$ to $L^\infty(u)$, we use Marcinkiewicz interpolation theorem to conclude that M is bounded from $L^q(w)$ to $L^q(u)$ provided $p < q < \infty$. □

A particular instance of corollary 1.13 is the inequality

$$\int_{\mathbb{R}^n} |Mf(x)|^p u(x)dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^p Mu(x)dx$$

valid for $1 < p < \infty$, which appeared in Chapter II, (2.13). It is contained in our corollary because $(u,Mu) \in A_1$.

The following theorem contains some simple basic facts about the classes A_p.

THEOREM 1.14. *a) Let* $1 < p < q < \infty$. *Then* $A_1 \subset A_p \subset A_q$

b) Let $1 \leqq p < \infty$, $0 < \varepsilon < 1$ *and* $(u,w) \in A_p$. *Then* $(u^\varepsilon, w^\varepsilon) \in A_{\varepsilon p+1-\varepsilon}$.

c) Let $1 < p < \infty$. *Then* $(u,w) \in A_p$ *if and only if* $(w^{-1/(p-1)}, u^{-1/(p-1)}) \in A_{p'}$, p' *being, as usual, the exponent conjugate to* p, *that is* $p' = p/(p-1)$.

Proof: a) We just need to observe that

$$\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(q-1)}dx\right)^{q-1} \leqq \left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leqq$$

$$\leqq \operatorname{ess.}_Q\sup.(w^{-1})$$

The first inequality follows from Jensen's or Hölder's inequality, since $(q-1)(p-1)^{-1} > 1$. The second one is obvious.

b) For $r = \varepsilon p+1-\varepsilon$ is $r-1 = \varepsilon(p-1)$. Then

$$\left(\frac{1}{|Q|}\int_Q u(x)^\varepsilon dx\right)\left(\frac{1}{|Q|}\int_Q (w(x)^\varepsilon)^{-1/(r-1)}dx\right)^{r-1} \leqq$$

$$\leqq \left(\frac{1}{|Q|}\int_Q u(x)dx\right)^\varepsilon\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{\varepsilon(p-1)} \leqq C^\varepsilon$$

if C is the A_p constant for the pair (u,w). Again Jensen's inequality has been used to deal with the integral containing u. The case $p = 1$ is included by interpreting the second factor properly.

c) Suppose $(u,w) \in A_p$. Thus:

$$\left(\frac{1}{|Q|}\int_Q u(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leqq C$$

Since $(p-1)(p'-1) = pp'-p-p'+1 = 1$, we can write the previous inequality as:

$$\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)\left(\frac{1}{|Q|}\int_Q (u(x)^{-1/(p-1)})^{-1/(p'-1)}dx\right)^{p'-1} \leqq$$

$$\leqq C^{p'-1}$$

and conclude that $(w^{-1/(p-1)}, u^{-1/(p-1)}) \in A_{p'}$. Actually we see that $(u,w) \in A_p$ is equivalent to $(w^{-1/(p-1)}, u^{-1/(p-1)}) \in A_{p'}$. $\square$

EXAMPLE 1.15. We shall give here an example which shows that corollary 1.13 can not be improved so as to include also the case $q = p$. We shall give weights u,w such that $(u,w) \in A_p$ and, however, M is not bounded from $L^p(w)$ to $L^p(u)$. If $p = 1$, we just need to take $u = w \equiv 1$. We already know that if $g \geq 0$ is integrable and is not 0 at a.e. x, then Mg is never integrable (see the remark following theorem 2.6 in Chapter II). This same fact leads to an example for $p > 1$. Let $g \geq 0$, integrable and non trivial, in such a way that $Mg \notin L^1$. Take g bounded so that you can guarantee that $Mg(x)$ is always finite. Then $(g, Mg) \in A_1 \subset A_{p'}$, and hence $((Mg)^{-1/(p'-1)}, g^{-1/(p'-1)}) = ((Mg)^{1-p}, g^{1-p}) \in A_p$. If we take $u = (Mg)^{1-p}$ and $w = g^{1-p}$, we have a pair $(u,w) \in A_p$ for which the inequality

$$\int |Mf(x)|^p u(x)dx \leq C \int |f(x)|^p w(x)dx$$

cannot hold, since for $f = g$ we have: $\int |Mf|^p u = \int Mg = \infty$ and $\int |f|^p w = \int g < \infty$.

In this way, we have seen that the condition $(u,w) \in A_p$ does not solve Problem 2. It is necessary for (1.2) to hold, since (1.2) implies (1.3). However, it is not sufficient.

2. THE REVERSE HÖLDER'S INEQUALITY AND THE CONDITION A_∞.

The theory developed in section 1 becomes particularly interesting for the case $u = w$. First of all, theorem 1.2 reads as follows in this situation:

THEOREM 2.1. *Let* w *be a weight on* $\mathbb{R}^n$, *and let* $1 \leq p < \infty$. *Then, the following conditions are equivalent:*

> *a)* M *is of weak type* (p,p) *with respect to* w, *i.e.*
>
> $$w(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq C\, t^{-p} \int_{\mathbb{R}^n} |f(x)|^p w(x)dx$$
>
> *b) There is a constant* C *such that, for every function*

$f \geq 0$ *and for every cube* Q

$$(f_Q)^p\, w(Q) \leq C \int_Q f(x)^p\, w(x)dx$$

c) $(w,w) \in A_p$, *that is, in case* $1 < p < \infty$

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leq C$$

for every cube Q, *and, in case* $p = 1$, $Mw(x) \leq C\, w(x)$ *a.e. The constants* C *appearing in a), b) and c) are of the same order.*

When w satisfies c), we say that w satisfies the condition A_p, and write $w \in A_p$. We also speak of the A_p constant for w, with the natural meaning. Notice that the class A_1 is the same which appeared in Chapter II (after theorem 2.12 and also in theorem 3.4).

We saw (in 1.15) that a pair of weights (u,w) may be in A_p and yet M may not be bounded from $L^p(w)$ to $L^p(u)$. In contrast to this situation, for $p > 1$, it suffices that w is in A_p for M to be bounded in $L^p(w)$.

This fact depends on a basic property enjoyed by the A_p weights: the reverse Hölder's inequality (R.H.I.) appearing in lemma 2.5 below. First we present a couple of simple properties of the A_p weights.

We start with an estimate for the w-measure of the dilated Q^λ of a cube Q.

LEMMA 2.2. *Let* w *be an* A_p *weight in* $\mathbb{R}^n$. *Then, for every cube* Q *and every* $\lambda > 1$

$$w(Q^\lambda) \leq C\, \lambda^{np}\, w(Q)$$

where C *is of the same order as the* A_p *constant for* w.

Proof: In b) of theorem 2.1, take $f = \chi_S$ with $S \subset Q$, and Q a cube. Then

$$(2.3) \qquad (|S|/|Q|)^p w(Q) \leq C\, w(S)$$

Using (2.3) with Q in place of S and Q^λ in place of Q we get $w(Q^\lambda) \leq C\, \lambda^{np}\, w(Q)$. □

In particular the lemma implies that for an A_p weight w, the measure μ given by $d\mu(x) = w(x)dx$ is a doubling measure.

Actually, what we have shown is that property b) in theorem 2.1 implies that μ is a doubling measure. Observe also that the same property b) implies that

$$Mf(x) \leqq C^{1/p} \{M_\mu(|f|^p)(x)\}^{1/p} \quad \text{a.e.}$$

where M_μ is the operator introduced in Chapter II (after theorem 1.12). We showed there that, for μ doubling, M_μ is of weak type (1,1) with respect to μ. We can rely upon this fact to prove that b) implies a) in theorem 2.1. Indeed

$$w(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leqq w(\{x \in \mathbb{R}^n : CM_\mu(|f|^p)(x) > t^p\}) \leqq$$
$$\leqq C\, t^{-p} \int_{\mathbb{R}^n} |f(x)|^p\, w(x)dx$$

where the last C contains the doubling constant and the constant in b).

The next lemma is a comparison between the measure $w(x)dx$ and Lebesgue measure

LEMMA 2.4. *Let* $w \in A_p$. *Then, for every positive* $\alpha < 1$, *there exists* $\beta < 1$ *depending on* α *such that, whenever* A *is a measurable set contained in a cube* Q *and satisfying* $|A| \leq \alpha|Q|$, *it follows that* $w(A) \leq \beta\, w(Q)$.

Proof: We start from (2.3) where, of course, it is always $C \geqq 1$ (set $S = Q$). If we use in (2.3) $S = Q\setminus A$ where $|A| \leqq \alpha|Q|$, we get:

$$(1-\alpha)^p w(Q) \leqq (1 - |A|/|Q|)^p w(Q) \leqq C(w(Q) - w(A))$$

Thus, $w(A) \leqq C^{-1}(C - (1-\alpha)^p)w(Q)$ and the lemma holds with $\beta = C^{-1}(C - (1-\alpha)^p)$. □

We shall use the previous lemma to establish our basic inequality

LEMMA 2.5. *Let* $w \in A_p$. *Then, there exists* $\varepsilon > 0$, *depending only on* p *and on the* A_p *constant for* w, *such that, for every cube* Q

$$\left(\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}dx\right)^{1/(1+\varepsilon)} \leqq \frac{C}{|Q|}\int_Q w(x)dx$$

with a constant C *not depending on* Q.

The opposite inequality holds, with $C = 1$, for every function w and is a particular case of Hölder's inequality. This is why the lemma is called the reverse Hölder's inequality (R.H.I.).

Proof: We shall fix the cube Q and we shall get the inequality with ε and C independent of Q. We take an increasing sequence $\lambda_o < \lambda_1 < \dots < \lambda_k < \dots$ with $\lambda_o = w_Q = \frac{1}{|Q|}\int_Q w(x)dx$ and, for each λ_k, we make the Calderón-Zygmund decomposition of Q for the function w and the value λ_k; that is, we consider those maximal dyadic subcubes of Q over which the average of w is $>\lambda_k$ (the dyadic subcubes of Q are the cubes resulting from dividing each side of Q in 2^N equal parts, $N = 0,1,2,\dots$). Let them be $\{Q_{k,j}\}_{j=1,2,\dots}$. It follows that, for each j is $\lambda_k < w_{Q_{k,j}} \leqq 2^n\lambda_k$, while for a.e. x not belonging to $\bigcup_j Q_{k,j} = D_k$ is $w(x) \leqq \lambda_k$. Since $\lambda_{k+1} > \lambda_k$, each $Q_{k+1,j}$ is contained in $Q_{k,i}$ for some i, in such a way that $D_{k+1} \subset D_k$. Let us see what portion of $Q_{k,i}$ can be covered by D_{k+1}. We know that:

$$2^n\lambda_k \geqq \frac{1}{|Q_{k,i}|}\int_{Q_{k,i}\cap D_{k+1}} w(x)dx =$$

$$= \frac{1}{|Q_{k,i}|}\sum_{Q_{k+1,j}\subset Q_{k,i}}\int_{Q_{k+1,j}} w(x)dx > \lambda_{k+1}\frac{|Q_{k,i}\cap D_{k+1}|}{|Q_{k,i}|}$$

Thus $\frac{|Q_{k,i}\cap D_{k+1}|}{|Q_{k,i}|} < \frac{2^n\lambda_k}{\lambda_{k+1}}$. Let us take this ratio equal to $\alpha < 1$, that is $\lambda_{k+1} = 2^n\alpha^{-1}\lambda_k$, $\lambda_k = (2^n\alpha^{-1})^k\lambda_o$. If we consider the β associated to α according to the previous lemma, we shall have $w(Q_{k,i}\cap D_{k+1}) \leqq \beta\, w(Q_{k,i})$ and, summing over i, $w(D_{k+1}) \leqq \beta w(D_k)$, which leads to $w(D_k) \leqq \beta^k w(D_o)$. Of course, we also have $|D_{k+1}| \leqq \alpha|D_k|$ and $|D_k| \leqq \alpha^k|D_o|$, which implies that $|\bigcap_{k=o}^{\infty} D_k| = \lim_{k\to\infty}|D_k| = 0$. Then:

$$\int_Q w(x)^{1+\varepsilon}dx = \int_{Q\setminus D_o} w(x)^{1+\varepsilon}dx + \sum_{k=o}^{\infty}\int_{D_k\setminus D_{k+1}} w(x)^{1+\varepsilon}dx \leqq$$

$$\leqq \lambda_o^{\varepsilon}\, w(Q\setminus D_o) + \sum_{k=o}^{\infty}\lambda_{k+1}^{\varepsilon}\, w(D_k\setminus D_{k+1}) \leqq$$

$$\leqq \lambda_o^\varepsilon\{w(Q\setminus D_o) + \sum_{k=o}^{\infty} (2^n\alpha^{-1})^{(k+1)\varepsilon} \beta^k w(D_o)\} =$$

$$= \lambda_o^\varepsilon\{w(Q\setminus D_o) + (2^n\alpha^{-1})^\varepsilon \sum_{k=o}^{\infty} ((2^n\alpha^{-1})^\varepsilon\beta)^k w(D_o)\}$$

If we take $\varepsilon > 0$ small enough to have $(2^n\alpha^{-1})^\varepsilon\beta < 1$, the series will have a finite sum and we shall get:

$$\int_Q w(x)^{1+\varepsilon}dx \leqq C\lambda_o^\varepsilon(w(Q\setminus D_o) + w(D_o)) = C\, w_Q^\varepsilon\, w(Q)$$

Thus

$$\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}dx \leqq C\, w_Q^{1+\varepsilon} = C\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)^{1+\varepsilon}. \qquad \square$$

Lemma 2.5 has far reaching consequences which we shall presently see

THEOREM 2.6. *If* $w \in A_p$ *with* $1 < p < \infty$, *then there is some* $q < p$ *such that* $w \in A_q$; *that is, for every* p, $1 < p < \infty$, *we have* $A_p = \bigcup_{q<p} A_q$.

Proof: Theorem 1.14(c) for the special case $u = w$ tells us that $w \in A_p$ implies $w^{-1/(p-1)} \in A_{p'}$. On the other hand, from lemma 2.5 for the weight $w^{-1/(p-1)}$ we know that there exist $\varepsilon > 0$, $C > 0$ such that, for every cube Q:

$$\Big(\frac{1}{|Q|}\int_Q w(x)^{-(1+\varepsilon)/(p-1)}dx\Big)^{1/(1+\varepsilon)} \leqq \frac{C}{|Q|}\int_Q w(x)^{-1/(p-1)}dx$$

But $\frac{1+\varepsilon}{p-1} > \frac{1}{p-1}$ implies $\frac{1+\varepsilon}{p-1} = \frac{1}{q-1}$ for some $1 < q < p$. Then

$$\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-1/(q-1)}dx\Big)^{q-1} =$$

$$= \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-(1+\varepsilon)/(p-1)}dx\Big)^{(p-1)/(1+\varepsilon)} \leqq$$

$$\leqq C^{p-1}\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\Big)^{p-1} \leqq C. \qquad \square$$

Actually, since w itself satisfies a R.H.I., we obtain the following stronger result.

THEOREM 2.7. *If* $w \in A_p$ *with* $1 \leqq p < \infty$, *then, there exists* $\varepsilon > 0$ *such that* $w^{1+\varepsilon} \in A_p$.

Proof: If $p = 1$ it suffices to observe that for $\varepsilon > 0$ small enough:

$$\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}dx \leqq \left(\frac{c}{|Q|}\int_Q w(x)dx\right)^{1+\varepsilon} \leqq c\, w(x)^{1+\varepsilon} \quad \text{for a.e. } x \in Q$$

in such a way that $w^{1+\varepsilon} \in A_1$.

If $p > 1$, it suffices to take $\varepsilon > 0$ small enough to have, at the same time:

$$\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}dx \leqq \left(\frac{C}{|Q|}\int_Q w(x)dx\right)^{1+\varepsilon}$$

and

$$\frac{1}{|Q|}\int_Q w(x)^{-(1+\varepsilon)/(p-1)}dx \leqq \left(\frac{c}{|Q|}\int_Q w(x)^{-1/(p-1)}\right)^{1+\varepsilon}. \qquad \square$$

Of course, theorem 2.7 combined with part b) of theorem 1.14, gives theorem 2.6.

Now, with the help of theorem 2.6, we can improve theorem 2.1 as anticipated, obtaining

THEOREM 2.8. *Let* w *be a weight on* $\mathbb{R}^n$ *and let* $1 < p < \infty$. *Then, the following conditions are equivalent:*

a) M *is of weak type* (p,p) *with respect to* w, *that is, there is a constant* C *such that for every function* $f \in L^1_{loc}(\mathbb{R}^n)$ *and every* $t > 0$

$$w(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leqq C\, t^{-p}\int_{\mathbb{R}^n} |f(x)|^p w(x)dx$$

b) There is a constant C *such that for every function* $f \geqq 0$ *in* $\mathbb{R}^n$ *and every cube* Q

$$\left(\frac{1}{|Q|}\int_Q f(x)dx\right)^p w(Q) \leqq C\int_Q f(x)^p w(x)dx$$

c) $w \in A_p$, *that is: there is a constant* C *such that for every cube* Q

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leqq C$$

d) M *is bounded in* $L^p(w)$, *that is, there is a constant* C *such that for every* $f \in L^p(w)$:

$$\int_{\mathbb{R}^n} (Mf(x))^p w(x)\,dx \leqq C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx.$$

Proof: All that remains to be proved is that c) implies d). Here is the proof. We have $w \in A_p$. Since $1 < p < \infty$, theorem 2.6 tells us that $w \in A_q$ for some $q < p$. Then M is of weak type (q,q) with respect to w and, since M is also bounded in $L^\infty(w) = L^\infty$ (this equality follows from the fact that $0 < w(x) < \infty$ for a.e. x), the Marcinkiewicz interpolation theorem allows us to conclude that M is bounded in $L^p(w)$. □

Also, the reverse Hölder's inequality allows us to give a more precise version of lemma 2.4.

THEOREM 2.9. *If* $w \in A_p$ *for some* $p \in [1,\infty)$, *then there exist* $\delta > 0$, $C > 0$ *such that, every time we have a measurable set* A *contained in a cube* Q, *the following inequality holds:*

$$\frac{w(A)}{w(Q)} \leqq C\left(\frac{|A|}{|Q|}\right)^{\delta} \tag{2.10}$$

Proof: The key fact is that w satisfies an inequality like the one appearing in lemma 2.5 for some $\varepsilon > 0$ (R.H.I.). We start by using Hölder's inequality with exponents $1+\varepsilon$ and its conjugate $(1+\varepsilon)/\varepsilon$, and then we apply the R.H.I. We get

$$\begin{aligned} w(A) &= \int_A w(x)\,dx \leqq \left(\int_A w(x)^{1+\varepsilon}dx\right)^{1/(1+\varepsilon)} |A|^{\varepsilon/(1+\varepsilon)} = \\ &= \left(\frac{1}{|Q|}\int_A w(x)^{1+\varepsilon}dx\right)^{1/(1+\varepsilon)} |Q|^{1/(1+\varepsilon)} |A|^{\varepsilon/(1+\varepsilon)} \leqq \\ &\leqq \frac{C}{|Q|}\int_Q w(x)\,dx\,|Q|^{1/(1+\varepsilon)} |A|^{\varepsilon/(1+\varepsilon)} = Cw(Q)(|A|/|Q|)^{\varepsilon/(1+\varepsilon)} \end{aligned}$$

which is (2.10) with $\delta = \varepsilon/(1+\varepsilon)$. □

Condition (2.10) is known as A_∞ for reasons which will appear very soon. We also speak of the class A_∞ which is, naturally, the class formed by those locally integrable weights w satisfying the A_∞ condition.

For the next result, μ_1 and μ_2 are going to be doubling measures, that is, both satisfy a doubling condition like (1.13) in Chapter II. For these measures, we give the following definition:

μ_1 *is comparable to* μ_2 *when there exist* $\alpha, \beta < 1$ *such that, every time we have a measurable subset* A *of a cube* Q *with* $\mu_2(A)/\mu_2(Q) \leq \alpha$, *it follows that* $\mu_1(A)/\mu_1(Q) \leq \beta$.

With this definition we can write

THEOREM 2.11. *The following conditions are equivalent*

a) There exist $\delta > 0$, $C > 0$ *such that for every measurable set* A *contained in a cube* Q

$$\frac{\mu_2(A)}{\mu_2(Q)} \leqq C\left(\frac{\mu_1(A)}{\mu_1(Q)}\right)^{\delta}$$

b) μ_2 *is comparable to* μ_1

c) μ_1 *is comparable to* μ_2

d) $d\mu_2(x) = w(x)d\mu_1(x)$ *with:*

$$\left(\frac{1}{\mu_1(Q)}\int_Q w(x)^{1+\varepsilon}d\mu_1(x)\right)^{1/(1+\varepsilon)} \leqq C\frac{1}{\mu_1(Q)}\int_Q w(x)d\mu_1(x)<\infty$$

for some $\varepsilon > 0$

Proof: a) $\Longrightarrow$ b) is clear. Indeed, if $\mu_1(A)/\mu_1(Q) \leqq \alpha$, it will be $\mu_2(A)/\mu_2(Q) \leqq C\,\alpha^{\delta}$. It suffices to start with some $\alpha > 0$ such that $C\alpha^{\delta} < 1$ and we obtain μ_2 comparable to μ_1 with constants α and $\beta = C\alpha^{\delta}$.

b) $\Rightarrow$ c). To say that $\mu_1(A)/\mu_1(Q) \leqq \alpha$ implies $\mu_2(A)/\mu_2(Q) \leqq \beta$ is equivalent to saying that $\mu_2(A)/\mu_2(Q) > \beta$ implies $\mu_1(A)/\mu_1(Q) > \alpha$. Then if $\mu_2(A)/\mu_2(Q) \leqq (1-\beta)/2 < 1-\beta$, it will be $\mu_2(Q\setminus A)/\mu_2(Q) > \beta$, in such a way that $\mu_1(Q\setminus A)/\mu_1(Q) > \alpha$ and, consequently $\mu_1(A)/\mu_1(Q) < 1-\alpha$. Thus, we have seen that μ_1 is comparable to μ_2 with constants $\alpha' = (1-\beta)/2$ and $\beta' = 1-\alpha$.

It becomes clear that, actually, b) and c) are equivalent. Let us see now that b) $\Rightarrow$ d). We start from the fact that μ_2 is comparable to μ_1 with constants α and β. We see, first of all, that μ_2 is absolutely continuous with respect to μ_1, that is: $\mu_1(E) = 0 \Rightarrow \mu_2(E) = 0$. Once this is proved, the Radon-Nikodym theorem guarantees that $d\mu_2(x) = w(x)d\mu_1(x)$ with w locally integrable with respect to μ_1. Let $\mu_1(E) = 0$ and suppose that $\mu_2(E) > 0$. Since the measures are regular, there will be an open set Ω such that $\Omega \supset E$ and $\mu_2(\Omega) < \beta^{-1}\mu_2(E)$. Let $\Omega = \bigcup_j Q_j$

where the Q_j's are non-overlapping cubes. Since for each j is $0 = \mu_1(Q_j \cap E) \leq \alpha\mu_1(Q_j)$, we shall have $\mu_2(Q_j \cap E) \leq \beta\mu_2(Q_j)$ and, adding in j: $\mu_2(E) \leq \beta\mu_2(\Omega)$, which contradicts the election of Ω. We have used the fact that the faces or edges of the cubes have measure μ_2 (or μ_1 for that matter) equal to 0. This follows easily from the doubling condition. Indeed, if μ is doubling, there is a constant $k < 1$ such that if Q is a cube and R is a half of Q (that is R is an interval contained in Q having one side equal to a half the corresponding side of Q and the other sides coinciding with those of Q) then $\mu(R) \leq k\mu(Q)$. The proof of this is essentially the same that the argument we used in chapter II (after theorem 1.12) to guarantee that the Calderón-Zygmund decomposition makes sense for a doubling measure. Viewing a face of Q as intersection of the R_j's resulting from repeatedly dividing by 2 a side of Q, we see that a face has μ measure 0 for μ doubling.

Let $d\mu_2(x) = w(x)d\mu_1(x)$. It remains to see that the inequality in d) holds. All we have to do is to repeat the proof of lemma 2.5. with μ_1 in place of Lebesgue measure. Observe that in the proof of lemma 2.5 we just used these two facts: $w(x)dx$ is comparable to Lebesgue measure and Lebesgue measure is doubling. These hypotheses still hold for $d\mu_2(x) = w(x)d\mu_1(x)$ and $d\mu_1(x)$. Thus, we obtain the inequality in d).

Finally we have to see that d) implies a). But this is done exactly as in the proof of theorem 2.9. □

COROLLARY 2.12. *The comparability of measures is an equivalence relation.*

Proof: The equivalence between b) and c) in 2.11 tells us that comparability is a symmetric relation. Transitivity is proved very simply by using the characterization given by a) in 2.11. □

COROLLARY 2.13. *Let* $w(x) \geq 0$ *be locally integrable in* $\mathbb{R}^n$. *The following conditions are equivalent:*

i) $w \in A_p$ *for some* $p \in [1,\infty)$

ii) There exist $\alpha, \beta < 1$ *such that* $|E| \leq \alpha|Q|$ *implies* $w(E) \leq \beta w(Q)$ *whenever* E *is a measurable subset of the cube* Q.

iii) There exist $\varepsilon > 0$ *and* $C > 0$ *such that for every cube* Q

$$\left(\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}dx\right)^{1/(1+\varepsilon)} \leq \frac{C}{|Q|}\int_Q w(x)dx$$

iv) $w \in A_\infty$.

Proof: All the implications i) $\Rightarrow$ ii) $\Rightarrow$ iii) $\Rightarrow$ iv) have already been proved. Observe that the proof of lemma 2.5 actually yields the fact that ii) $\Rightarrow$ iii). It only remains to see that iv) $\Rightarrow$ i). Let us see it. We know from theorem 2.11 that $w \in A_\infty$ is equivalent to saying that the measures dx and $w(x)dx$ are comparable and, taking into account that $dx = w(x)^{-1}w(x)dx$, the following R.H.I. must hold: There exist $\eta > 0$ and $C > 0$ such that for every cube Q

$$\left(\frac{1}{w(Q)}\int_Q w(x)^{-(1+\eta)}w(x)dx\right)^{1/(1+\eta)} \leq$$

$$\leq \frac{C}{w(Q)}\int_Q w(x)^{-1}w(x)dx = C\,\frac{|Q|}{w(Q)}$$

Hence

$$\left(\frac{1}{|Q|}\int_Q w(x)^{-\eta}\,dx\right)^{1/\eta} \leq C\,\frac{|Q|}{w(Q)}$$

Setting $\eta = \frac{1}{p-1}$, that is: $p = 1+\eta^{-1} > 1$, we obtain $w \in A_p$. □

Thus we have shown $A_\infty = \bigcup_{1\leq p<\infty} A_p$, which explains the name A_∞ given to condition (2.10).

Actually, the name A_∞ is just perfect, since, as we shall presently show, A_∞ coincides with the formal limit of condition A_p as p tends to ∞

$$\lim_{p\to\infty}\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} = \lim_{q\to 0}\|w^{-1}\|_{L^q(|Q|^{-1}dx)} =$$

$$= \exp\left(\frac{1}{|Q|}\int_Q \log\,(w(x)^{-1})dx\right)$$

The last identity above is a simple exercise in measure theory (see, for example Rudin [1], chapter III, problem 5).

Thus, the condition obtained by passing to the limit as p tends to ∞ in condition A_p is:

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\exp\left(\frac{1}{|Q|}\int_Q \log(w(x)^{-1})dx\right) \leq C$$

or, equivalently:

$$\frac{1}{|Q|}\int_Q w(x)dx \leqq C \exp\left(\frac{1}{|Q|}\int_Q \log w(x)dx\right) \tag{2.14}$$

The exponential in the right hand side of (2.14) is the geometric mean of w on Q, which is, of course, dominated by the arithmetic mean w_Q (Jensen's inequality). Thus (2.14) implies that the arithmetic and the geometric means of w on every cube, are equivalent. The equivalence between this condition and A_∞ is contained in the following

THEOREM 2.15. *Let* $w(x) \geqq 0$ *be locally integrable in* $\mathbb{R}^n$. *Then, the following conditions are equivalent:*

i) There exist $\alpha,\beta \in (0,1)$ *such that, for every cube* Q:

$$|\{x \in Q : w(x) \leqq \alpha w_Q\}| \leqq \beta|Q|$$

ii) $w \in A_\infty$

iii) There exists C, *such that, for every cube* Q:

$$\frac{1}{|Q|}\int_Q w(x)dx \leqq C \exp\left(\frac{1}{|Q|}\int_Q \log w(x)dx\right)$$

Proof. Suppose i) holds. Let us prove ii). After the proof of theorem 2.11, it will be enough to see that, for appropriately chosen $\gamma,\delta \in (0,1)$, the following property holds: If E is a subset of a cube Q such that $w(E)/w(Q) \leqq \gamma$, then $|E|/|Q| \leqq \delta$. To prove this property, assume $w(E)/w(Q) \leqq \gamma$, to be chosen later. Then we split $E = E_1 \cup E_2$, where $E_1 = \{x \in E : w(x) > \alpha\, w_Q\}$ and $E_2 = \{x \in E : w(x) \leqq \alpha\, w_Q\}$. For E_2, i) gives the estimate $|E_2| \leqq \beta|Q|$. For E_1 we use Chebichev's inequality to get:

$$|E_1| \leqq \frac{1}{\alpha\, w_Q}\int_E w(x)dx = \frac{|Q|}{\alpha}\frac{w(E)}{w(Q)} \leqq \frac{\gamma}{\alpha}|Q|$$

Adding up the two estimates, we have $|E| \leqq (\beta + \frac{\gamma}{\alpha})|Q|$. If we choose γ so small that $\beta + \frac{\gamma}{\alpha} < 1$, we get what we wanted with $\delta = \beta + (\gamma/\alpha)$.

To see that ii) implies iii) is quite easy. Indeed, if $w \in A_\infty$, it follows from corollary 2.13 and theorem 1.14 a), that there is a constant C such that, for all p large enough and for all cubes Q:

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \leqq C$$

Letting p tend to ∞ we obtain iii).

Finally, assuming iii), we are going to see that i) holds. Take a cube Q. Dividing w by an appropriate constant, we see that, without loss of generality, we may assume that $\int_Q \log w(x)dx = 0$ and, consequently, $w_Q \leqq C$.

Then, with $\lambda > 0$ still undetermined, we have:

$$|\{x \in Q : w(x) \leqq \lambda\}| = |\{x \in Q : \log(1 + w(x)^{-1}) \geqq \log(1+\lambda^{-1})\}| \leqq$$

$$\leqq \frac{1}{\log(1+\lambda^{-1})} \int_Q \log(1 + w(x)^{-1})dx =$$

$$= \frac{1}{\log(1+\lambda^{-1})} \int_Q \log \frac{1+w(x)}{w(x)} dx = \frac{1}{\log(1+\lambda^{-1})} \int_Q \log(1 + w(x))dx$$

since, by assumption, $\int_Q \log w(x)dx = 0$.

By using the simple inequality $\log(1+w) \leqq w$ and the hypothesis $w_Q \leqq C$, we get:

$$|\{x \in Q : w(x) \leqq \lambda\}| \leqq \frac{1}{\log(1+\lambda^{-1})} \int_Q w(x)dx \leqq$$

$$\leqq \frac{C}{\log(1+\lambda^{-1})} |Q| \leqq \frac{1}{2} |Q| \quad \text{if } \lambda \text{ is small enough.}$$

In particular:

$$|\{x \in Q : w(x) \leqq \alpha\, w_Q\}| \leqq |\{x \in Q : w(x) \leqq C\,\alpha\}| \leqq (1/2)|Q|$$

if α is small enough. We have obtained i) with $\beta = 1/2$. □

In chapter II (theorem 3.4) we gave examples of A_1 weights, namely those functions w of the form $w(x) = (M\mu(x))^\gamma$ where μ is a positive Borel measure such that $M\mu(x) < \infty$ for a.e. $x \in \mathbb{R}^n$ and $0 < \gamma < 1$. We used this result to show that $|x|^\alpha$ is an A_1 weight in $\mathbb{R}^n$ if and only if $-n < \alpha \leq 0$.

Starting with A_1 weights one can easily generate A_p weights for $1 < p < \infty$. Let $w_o, w_1 \in A_1$ in $\mathbb{R}^n$, and let $1 < p < \infty$. Then $w(x) = w_o(x).w_1(x)^{1-p}$ is and A_p weight. Indeed

$$\left(\frac{1}{|Q|} \int_Q w_o(x)w_1(x)^{1-p}dx\right)\left(\frac{1}{|Q|} \int_Q (w_o(x)w_1(x)^{1-p})^{-1/(p-1)}dx\right)^{p-1} \leqq$$

$$\leqq C\left(\frac{1}{|Q|} \int_Q w_1(x)dx\right)^{1-p}\left(\frac{1}{|Q|} \int_Q w_o(x)dx\right)\left(\frac{1}{|Q|} \int_Q w_o(x)dx\right)^{-1}$$

$$\left(\frac{1}{|Q|} \int_Q w_1(x)dx\right)^{p-1} = C.$$

We shall show in section 5 that every A_p weight w is actually of the form $w(x) = w_0(x)w_1(x)^{1-p}$ for some $w_0, w_1 \in A_1$ (factorization theorem). For the time being, we shall content ourselves with giving examples of A_p weights. If $-n < \alpha \leqq 0$ and $-n < \beta \leqq 0$, $|x|^\alpha |x|^{\beta(1-p)}$ is an A_p weight in $\mathbb{R}^n$. Hence $|x|^\alpha$ is an A_p weight in $\mathbb{R}^n$ if and only if $-n < \alpha < n(p-1)$ since $|x|^\alpha$ and $(|x|^\alpha)^{-1/(p-1)}$ have to be locally integrable.

By using the R.H.I. we get a converse of theorem 3.4 in Chapter II, giving the following characterization of A_1 weights:

THEOREM 2.16. *Let* $w(x)$ *be* ≥ 0 *and finite a.e. Then,* $w \in A_1$ *if and only if* $w(x) = k(x)(Mf(x))^\gamma$, *where* $k(x) \geq 0$ *is such that* $k, k^{-1} \in L^\infty$, f *is locally integrable and* $0 < \gamma < 1$.

Proof. Theorem 3.4 of chapter II implies that every function of the given form is an A_1 weight. Conversely, let $w \in A_1$. We know that w satisfies a R.H.I.:

$$\left(\frac{1}{|Q|}\int_Q w(x)^{1+\varepsilon}\,dx\right)^{1/(1+\varepsilon)} \leqq \frac{C}{|Q|}\int_Q w(x)dx \leqq C\, w(x) \quad \text{a.e.}$$

Thus $w(x) \leqq (M(w^{1+\varepsilon})(x))^{1/(1+\varepsilon)} \leqq C\, w(x)$. We can write $w(x) = k(x)(M(w^{1+\varepsilon})(x))^{1/(1+\varepsilon)}$ with $C^{-1} \leqq k(x) \leqq 1$ and we obtain the representation required with $f(x) = w(x)^{1+\varepsilon}$ and $\gamma = 1/(1+\varepsilon)$. $\square$

There is a relation between weights and B.M.O. functions. We have already seen in Chapter II (theorem 3.3) that the logarithm of an A_1 weight is a B.M.O. function. We shall see presently that the same is true for any A_∞ weight. Of course this follows trivially after the factorization theorem, but a simple proof can be given without appealing to that result which we have not proved yet. First of all, we give a characterization of A_p weights in terms of their logarithms.

THEOREM 2.17. a) *Let* ϕ *be a real locally integrable function on* $\mathbb{R}^n$ *and let* $1 < p < \infty$. *Then* $e^\phi \in A_p$ *if and only if the following two conditions are satisfied:*

i) $\displaystyle\frac{1}{|Q|}\int_Q e^{(\phi(x)-\phi_Q)}dx \leqq C$, *independent of the cube* Q

ii) $\displaystyle\frac{1}{|Q|}\int_Q e^{-(\phi(x)-\phi_Q)/(p-1)}dx \leqq C$, *independent of the cube* Q

b) *For* ϕ *as in a),* $e^\phi \in A_\infty$ *if and only if i) holds. Note* *that for* $p = \infty$, *condition ii) becomes empty, so that b) is just an extension of a) to* $p = \infty$

c) *It follows from a) and b) that* w *is in* A_p *if and only if both* w *and* $w^{-1/(p-1)}$ *are in* A_∞.

Proof. a) It is clear that the two conditions i) and ii) imply together that $e^{\phi} \in A_p$, since

$$\left(\frac{1}{|Q|}\int_Q e^{\phi(x)}\,dx\right)\left(\frac{1}{|Q|}\int_Q (e^{\phi(x)})^{-1/(p-1)}dx\right)^{p-1} =$$
$$= \left(\frac{1}{|Q|}\int_Q e^{\phi(x)-\phi_Q}\,dx\right)\left(\frac{1}{|Q|}\int_Q e^{-(\phi(x)-\phi_Q)/(p-1)}dx\right)^{p-1}$$

Conversely, suppose that $e^{\phi} \in A_p$. Then

$$\frac{1}{|Q|}\int_Q e^{\phi(x)-\phi_Q}\,dx = e^{-\phi_Q}\frac{1}{|Q|}\int_Q e^{\phi(x)}\,dx =$$
$$= (e^{-\phi_Q/(p-1)})^{p-1}\left(\frac{1}{|Q|}\int_Q e^{\phi(x)}\,dx\right) \leqq$$
$$\leqq \left(\frac{1}{|Q|}\int_Q e^{-\phi(x)/(p-1)}dx\right)^{p-1}\left(\frac{1}{|Q|}\int_Q e^{\phi(x)}\,dx\right) \leqq C$$

Also

$$\frac{1}{|Q|}\int_Q e^{-(\phi(x)-\phi_Q)/(p-1)}dx = \left(\frac{1}{|Q|}\int_Q e^{-\phi(x)/(p-1)}dx\right)(e^{\phi_Q})^{1/(p-1)} \leqq$$
$$\leqq \left(\frac{1}{|Q|}\int_Q e^{-\phi(x)/(p-1)}dx\right)\left(\frac{1}{|Q|}\int_Q e^{\phi(x)}dx\right)^{1/(p-1)} \leqq C.$$

b) Theorem 2.15 implies that $e^{\phi} \in A_\infty$ if and only if $\frac{1}{|Q|}\int_Q e^{\phi(x)}dx \leqq C\, e^{\phi_Q}$, which is equivalent to condition i).

c) It follows from b) that, for $w = e^{\phi}$, condition i) is equivalent to saying that $w \in A_\infty$ and condition ii) is equivalent to saying that $w^{-1/(p-1)} \in A_\infty$. □

In case $p = 2$, conditions i) and ii) become:

$$\frac{1}{|Q|}\int_Q e^{\phi(x)-\phi_Q}dx \leqq C \quad\text{and}\quad \frac{1}{|Q|}\int_Q e^{-(\phi(x)-\phi_Q)}dx \leqq C.$$

These two inequalities together are equivalent to

$$\frac{1}{|Q|}\int_Q e^{|\phi(x)-\phi_Q|}\,dx \leqq C$$

We can write:

COROLLARY 2.18. *Let ϕ be a real locally integrable function on $\mathbb{R}^n$. Then $e^{\phi} \in A_2$ if and only if there is a constant* C *such that for every cube* $Q \subset \mathbb{R}^n$

$$\frac{1}{|Q|}\int_Q e^{|\phi(x)-\phi_Q|}\,dx \leqq C.$$

The relation between weights and B.M.O. functions is now clear.

COROLLARY 2.19. $w \in A_\infty \Longrightarrow \log w \in$ B.M.O.

Proof. Let $w \in A_\infty$ and write $w = e^{\phi}$, that is: $\phi = \log w$. If $w \in A_2$, we know from corollary 2.18 that

$$\|\phi\|_* = \sup_Q \frac{1}{|Q|}\int_Q |\phi(x)-\phi_Q|\,dx \leqq \sup_Q \frac{1}{|Q|}\int_Q e^{|\phi(x)-\phi_Q|}\,dx \leqq C$$

so that $\phi = \log w \in$ B.M.O.

In general $w \in A_\infty \Longrightarrow w \in A_p$ for some $p \in [1,\infty)$. If $p \leqq 2$, we have $w \in A_p \subset A_2$ and, as we have just seen, $\log w \in$ B.M.O. If $p > 2$ we look at $w^{-1/(p-1)} \in A_{p'} \subset A_2$. It follows that $\log(w^{-1/(p-1)}) = -\frac{1}{p-1}\log w \in$ B.M.O. Thus, in any case, $\log w \in$ B.M.O. $\square$

Observe that, if $w \in A_p$, $\|\log w\|_*$ depends only on p and on the A_p constant for w.

If $\phi \in$ B.M.O., we know that $e^{\lambda\phi} \in A_2$ for $\lambda > 0$ small enough $(0 < \lambda < C_2/\|\phi\|_*$ with the notation used in Chapter II, 3.10). If we set $e^{\lambda\phi} = w$, we get $\phi = \lambda^{-1}\log w$. Thus

$$\text{B.M.O.} = \{\alpha \log w : \alpha \geq 0,\ w \in A_2\}$$

Actually, the same is true for any p with $1 < p \leq \infty$

$$\text{B.M.O.} = \{\alpha \log w : \alpha \geq 0,\ w \in A_p\}$$

We already know that this is true if $p \geq 2$. For $1 < p < 2$, if $\phi \in$ B.M.O., we can write $\phi = \alpha \log w$ with $\alpha \geqq 0$ and $w \in A_2$. But $\sigma = w^{p-1} \in A_p$ since $2(p-1)+1-(p-1) = p$ (see theorem 1.14 part b)). Therefore $\phi = \alpha \log w = \alpha \log(\sigma^{1/(p-1)}) = (\alpha/(p-1))\log\sigma$.

In contrast to this situation, we have

$$\{\alpha \log w : \alpha \geq 0,\ w \in A_1\} = \text{B.L.O.} \subsetneqq \text{B.M.O.}$$

In fact, we already know that $\alpha \log w \in$ B.L.O. when $\alpha \geq 0$, $w \in A_1$ (see the proof of theorem 3.3. in Chapter II). Conversely, let

$\phi \in$ B.L.O. Then, according to Corollary 3.10 (ii) in Chapter II, we have for $\varepsilon > 0$ small enough, every cube Q and given C:

$$C \geq \frac{1}{|Q|}\int_Q e^{\varepsilon|\phi(x)-\phi_Q|}\,dx \geq \frac{1}{|Q|}\int_Q e^{\varepsilon(\phi(x)-\phi_Q)}\,dx$$

which implies

$$\frac{1}{|Q|}\int_Q e^{\varepsilon\phi(x)}dx \leq C\exp(\varepsilon\phi_Q) \leq C\exp\{\varepsilon(C + \text{ess.}_Q\inf.\ \phi)\}$$
$$\leq C\ \text{ess.}_Q\inf.(e^{\varepsilon\phi})$$

It follows that $e^{\varepsilon\phi} \in A_1$. Thus, $\phi = \varepsilon^{-1}\log w$ with $w = e^{\varepsilon\phi} \in A_1$.

Combining this with theorem 2.16, which tells us that every $w \in A_1$ can be written as $w(x) = k(x)(Mf(x))^{\gamma}$, with $k(x) \geq 0$ such that $\log k \in L^{\infty}$ and $0 < \gamma < 1$, we are led to:

$$\text{B.L.O.} = \{h + \beta\log(Mf) : h \in L^{\infty},\ f \in L^1_{loc},\ \beta \geq 0\}$$

We finish this section by observing that the L^p inequality established in chapter II (theorem 3.6) between the Hardy-Littlewood maximal function Mf and the sharp maximal function $f^{\#}$, also holds when Lebesgue measure dx is replaced by the measure $w(x)dx$, where w is any A_{∞} weight.

The concrete statement is as follows

THEOREM 2.20. *Let* $w \in A_{\infty}$ *in* $\mathbb{R}^n$ *and let* f *be such that* $Mf \in L^{p_0}(w)$ *for some* p_0 *with* $0 < p_0 < \infty$. *Then, for every* p *such that* $p_0 \leq p < \infty$

$$\int_{\mathbb{R}^n}(Mf(x))^p w(x)dx \leq C\int_{\mathbb{R}^n}(f^{\#}(x))^p w(x)dx.$$

Proof. It is a repetition of the argument we used to prove theorem 3.6 in chapter II. First of all, the Calderón-Zygmund decomposition can be carried out for our f. Indeed if $Q_1 \subset Q_2 \subset \ldots$ is an increasing family of dyadic cubes for each of which $\frac{1}{|Q_k|}\int_{Q_k} f(y)dy > t$, then $w(Q_k)$ is bounded independently of k and, since $w(x)dx$ is a doubling measure, this implies that the sequence $\{Q_k\}$ is finite.

We consider as in theorem 3.6. of chapter II the family of cubes $\{Q_{t,j}\}$. Given $t > 0$ we fix $Q_0 = Q_{2^{-n-1}t,j_0}$ and take

$A > 0$. Proceed as in chapter II. Then after we have obtained the estimate

$$\sum_{j:Q_{t,j} \subset Q_o} |Q_{t,j}| \leqq 2A^{-1}|Q_o|$$

we use the fact tat $w \in A_\infty$ to conclude that, for some $\delta > 0$ and $C > 0$,

$$\sum_{j:Q_{t,j} \subset Q_o} w(Q_{t,j}) \leqq C(2A^{-1})^\delta\, w(Q_o)$$

Adding in all the possible Q_o's we obtain the estimate

$$\sum_j w(Q_{t,j}) \leqq w(\{x : f^\#(x) > t/A\}) + C(2A^{-1})^\delta \sum_k w(Q_{2^{-n-1}t,k})$$

After this, the proof continues practically unchanged. □

3. WEIGHTED NORM INEQUALITIES FOR SINGULAR INTEGRALS

Only for regular singular integrals (see 5.17 and 5.1 in chapter II for the definition), a satisfactory theory can be given in terms of the A_p classes. We start by using inequality 5.20 (i) from chapter II to prove the following

THEOREM 3.1. *Let* T *be a regular singular integral operator,* $1 < p < \infty$ *and* $w \in A_p$. *Then* T *is bounded in* $L^p(w)$. *More precisely, there is a constant* C *depending only on* T, w *and* p *such that, for every function* $f \in L_c^\infty$ *(that is:* f *is a bounded function with compact support), this inequality holds:*

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)dx \leqq C \int_{\mathbb{R}^n} |f(x)|^p\, w(x)dx$$

Proof. We shall show below that if $f \in L_c^\infty$, then $M(Tf) \in L^p(w)$. Once this is granted, we can use theorem 2.20 to write:

$$\int_{\mathbb{R}^n} |Tf(x)|^p\, w(x)dx \leqq \int_{\mathbb{R}^n} (M(Tf)(x))^p\, w(x)dx \leqq$$

$$\leqq C \int_{\mathbb{R}^n} ((Tf)^\#(x))^p\, w(x)dx \leqq C\, C_q^p \int_{\mathbb{R}^n} (M_q f(x))^p\, w(x)dx =$$

$$= C \int_{\mathbb{R}^n} (M(|f|^q)(x))^{p/q}\, w(x)dx \leqq C \int_{\mathbb{R}^n} |f(x)|^p\, w(x)dx.$$

We have used inequality 5.20 (i) from chapter II with a $q > 1$ small enough for having $w \in A_{p/q}$, $\frac{p}{q} > 1$ (this can be done according to theorem 2.6). Then the last inequality follows readily.

To finish the proof we just need to check that $M(Tf) \in L^p(w)$ for $f \in L^\infty_c$. It suffices to show that $Tf \in L^p(w)$. This can be done in the following way. Let $R > 0$ be such that $f(y) = 0$ for all y with $|y| \geqq R$. Then for $|x| > 2R$, the following estimate holds:

$$|Tf(x)| \leqq \int_{|y|<R} |K(x-y)||f(y)|dy \leqq \int_{|y|<R} \frac{B}{|x-y|^n} |f(y)|dy \leqq$$
$$\leqq \frac{C}{|x|^n}$$

since $|x-y| \geqq |x| - |y| > |x|/2$. Now we write

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)dx = \int_{|x|<2R} |Tf(x)|^p w(x)dx + $$
$$+ \int_{|x|\geqq 2R} |Tf(x)|^p w(x)dx$$

and see that both integrals in the right hand side are finite. For every $M > 0$

$$\int_{|x|<M} |Tf(x)|^p w(x)dx \leqq \left(\int_{\mathbb{R}^n} |Tf(x)|^{p(1+1/\varepsilon)} dx\right)^{\varepsilon/(1+\varepsilon)}$$
$$\left(\int_{|x|<M} w(x)^{1+\varepsilon} dx\right)^{1/(1+\varepsilon)}$$

The right hand side of this inequality is finite if $\varepsilon > 0$ is chosen in such a way that w satisfies a reverse Hölder's inequality with exponent $1+\varepsilon$ (see 2.5), since we know that T is bounded in $L^q(dx)$ for every q with $1 < q < \infty$, in particular for $q = p(1+1/\varepsilon)$.

Now, in view of the estimate obtained for $|Tf(x)|$ when $|x| > 2R$, we just need to observe that if $w \in A_p$ and $p > 1$, we have:

$$\int_{0<M<|x|} \frac{w(x)}{|x|^{np}} dx < \infty \tag{3.2}$$

Indeed

$$\int_{|x|>M} \frac{w(x)}{|x|^{np}} dx = \sum_{k=1}^{\infty} \int_{2^{k-1}M<|x|\leqq 2^k M} \frac{w(x)}{|x|^{np}} dx \leqq$$
$$\leqq C \sum_{k=1}^{\infty} (2^k M)^{-np} w(2^k B(0,M)) \leqq$$

$\leqq$ (taking $1 < q < p$ so that $w \in A_q$ and using lemma 2.2)

$$\leqq C \sum_{k=1}^{\infty} (2^k M)^{-np} 2^{knq} w(B(0,M)) = C\, w(B(0,M)) \sum_{k=1}^{\infty} (2^{n(q-p)})^k < \infty. \square$$

Since L_c^∞ is dense in $L^p(w)$, T can be extended by continuity to the whole space $L^p(w)$.

Theorem 3.1 says nothing about the case $p = 1$. Of course for $p = 1$, the estimate 5.20 (i) used in the proof, is totally useless, since it involves a $q > 1$. However, we are going to show that if $w \in A_1$, then T is of weak type $(1,1)$ with respect to w. The proof will be but a repetition of the Calderón-Zygmund proof given for Lebesgue measure in chapter II. We start by extending lemma 5.2 in chapter II to our situation.

LEMMA 3.3. *Let* K *be a regular singular integral kernel,* w *an* A_1 *weight and* $a \in L^1(w)$, *supported in a cube* Q *with* $\int_Q a(y)dy = 0$. *Then if we set, as usual,* $\tilde{Q} = Q^{2\sqrt{n}}$ *the following inequality holds:*

$$\int_{\mathbb{R}^n\setminus\tilde{Q}} |K*a(x)|w(x)dx \leqq C \int_{\mathbb{R}^n} |a(x)|w(x)dx$$

with C *depending only on* K *and* w.

Proof. We can assume, for simplicity, that Q is centered at the origin. Then, exactly as we did in chapter II (right before lemma 5.2), we can write

$$\int_{\mathbb{R}^n\setminus\tilde{Q}} |K*a(x)|w(x)dx \leqq \int_Q \int_{|x|>2|y|} |K(x-y)-K(x)|w(x)dx|a(y)|dy$$

(note that $a(x)$ is actually integrable and the convolution makes sense).

Then we just need to observe that

$$\int_{|x|>2|y|} |K(x-y)-K(x)|w(x)dx \leqq CMw(y) \tag{3.4}$$

This inequality is proved exactly as (5.21) in chapter II, choosing y instead of 0 to evaluate Mw.

Since $w \in A_1$, we have $Mw(y) \leqq Cw(y)$ a.e. and the inequality of the lemma is completely proved. $\square$

We are now ready to prove a substitute result of theorem 3.1 for $p = 1$.

THEOREM 3.5. *Let* T *be a regular singular integral operator, and* $w \in A_1$. *Then* T *is of weak type* $(1,1)$ *with respect to* w. *More precisely, there is a constant* C *dependding only on* T *and* w, *such that for every function* $f \in L^1(w)$ *and every* $t > 0$:

$$w(\{x \in \mathbf{R}^n : |Tf(x)| > t\}) \leqq Ct^{-1} \int_{\mathbf{R}^n} |f(x)|w(x)dx$$

Proof. We are assuming $w \in A_1$. This means that there is a constant C such that, for every cube Q, $\frac{w(Q)}{|Q|} \leqq C\, w(x)$ for a.e. $x \in Q$. Let $f \in L^1(w)$ and $t > 0$. We carry out the corresponding Calderón-Zygmund decomposition. If Q is a cube for which $t < \frac{1}{|Q|}\int_Q |f(x)|dx$, then also

$$t < \frac{1}{|Q|}\int_Q |f(x)|dx \leqq \frac{C}{w(Q)} \int_Q |f(x)|w(x)dx$$

Therefore, every dyadic cube over which the average of $|f|$ is $>t$ will be contained in a maximal one. These maximal cubes are the Calderón-Zygmund cubes $\{Q_j\}$. For each of them, $t < \frac{1}{|Q_j|}\int_{Q_j} |f(x)|dx \leqq 2^n t$, and outside of their union we know that $f(x) \leqq t$ a.e. Next, we obtain the Calderón-Zygmund decomposition of the function f. We write $f = g+b$ where g, the "good part", is given by: $g(x) = f(x)$ if $x \notin \Omega = \bigcup_j Q_j$, and $g(x) = \frac{1}{|Q_j|}\int_{Q_j} f(y)dy = f_{Q_j}$ if $x \in Q_j$, while b, the "bad part" can be split as $b(x) = \sum_j b_j(x)$ with $b_j(x) = (f(x)-f_{Q_j})\chi_{Q_j}(x)$.

In chapter II, the corresponding result for $w \equiv 1$ (inequality 5.9) was proved by using the following properties:

a) T is bounded in L^2

b) $\int_{\mathbf{R}^n} |g(x)|dx \leqq \int_{\mathbf{R}^n\setminus\Omega} |f(x)|dx + \sum_j |Q_j| \frac{1}{|Q_j|}\int_{Q_j} |f(y)|dy =$

$$= \int_{\mathbf{R}^n} |f(x)|dx$$

c) If we set $\tilde{Q}_j = Q_j^{2\sqrt{n}}$ and $\tilde{\Omega} = \bigcup_j \tilde{Q}_j$, then:

$$|\tilde{\Omega}| \leqq C\sum_j |Q_j| \leqq \frac{C}{t} \sum_j \int_{Q_j} |f(x)|dx \leqq \frac{C}{t}\int_{\mathbf{R}^n} |f(x)|dx$$

d) lemma 5.2 in chapter II.

It turns out that these four properties extend to the case of a general $w \in A_1$ in the following way:

a') Since $w \in A_1 \subset A_p$ for every $p > 1$, theorem 3.1 tells us that T is bounded in $L^p(w)$ for every $p > 1$.

b') $\int_{\mathbb{R}^n} |g(x)|w(x)dx \leq \int_{\mathbb{R}^n} |f(x)|w(x)dx +$

$+ \sum_j w(Q_j) \frac{1}{|Q_j|} \int_{Q_j} |f(y)|dy \leq \int_{\mathbb{R}^n\setminus\Omega} |f(x)|w(x)dx +$

$+ C \sum_j \int_{Q_j} |f(x)|w(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|w(x)dx$

c') $w(\tilde{\Omega}) \leq C \sum_j \frac{w(Q_j)}{|Q_j|} |Q_j| \leq \frac{C}{t} \sum_j \int_{Q_j} |f(x)|w(x)dx \leq$

$\leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|w(x)dx$

d') We have lemma 3.3.

Now we can write a proof entirely analogous to the one given in the unweighted case:

$w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq w(\{x \in \mathbb{R}^n : |Tg(x)| > t/2\}) +$

$+ w(\tilde{\Omega}) + w(\{x \notin \tilde{\Omega} : |Tb(x)| > t/2\})$

We estimate each of these three terms. For the first one we use properties a') (with some $p > 1$) and b') together with the fact that $|g(x)| \leq 2^n t$ a.e. We get:

$w(\{x \in \mathbb{R}^n : |Tg(x)| > t/2\}) \leq \frac{C}{t^p} \int_{\mathbb{R}^n} |Tg(x)|^p w(x)dx \leq$

$\leq \frac{C}{t^p} \int_{\mathbb{R}^n} |g(x)|^p w(x)dx = \frac{C}{t} \int_{\mathbb{R}^n} |g(x)| (\frac{|g(x)|}{t})^{p-1} w(x)dx \leq$

$\leq \frac{C}{t} \int_{\mathbb{R}^n} |g(x)|w(x)dx \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|w(x)dx$

The second term of the sum has been already estimated, with the same bound as the first, in c'). As for the third, we have:

$w(\{x \notin \tilde{\Omega} : |Tb(x)| > t/2\}) \leq \frac{C}{t} \int_{\mathbb{R}^n\setminus\tilde{\Omega}} |Tb(x)|w(x)dx \leq$

$\leq \frac{C}{t} \sum_j \int_{Q_j} |b(x)|w(x)dx \leq \frac{C}{t} \int_{\mathbb{R}^n} |f(x)|w(x)dx.$

The next to last inequality is a consequence of lemma 3.3 and the last one follows from b') since $b(x) = f(x) - g(x)$. □

If T is a regular singular integral operator with kernel K and $f \in L^p(w)$ with $w \in A_p$ and $1 < p < \infty$, then the truncated operators T_ε, $\varepsilon > 0$, can be defined acting directly on f by means of the formula

$$T_\varepsilon f(x) = \int_{|y-x|>\varepsilon} K(x-y)f(y)dy$$

Indeed, this integral converges absolutely, since

$$\int_{|y-x|>\varepsilon} |K(x-y)||f(y)|dy \leqq (\int_{R^n} |f(y)|^p w(y)dy)^{1/p}$$
$$(\int_{|y-x|>\varepsilon} |K(x-y)|^{p'} w(y)^{-p'/p} dy)^{1/p'}$$

and it is clear that the right hand side is finite, since the second factor is bounded by a constant times $(\int_{|y|>\varepsilon} w(x-y)^{-1/(p-1)} |y|^{-np'} dy)^{1/p'}$, which is finite because $w(x-y)^{-1/(p-1)}$ is, as a function of y, an $A_{p'}$ weight, and (3.2) holds for this function in place of w and p' in place of p.

Of course after theorem 3.1, Tf may be defined by density. But it is natural to ask whether a more explicit definition of Tf as limit of $T_\varepsilon f$ for $\varepsilon \to 0$ is possible.

We already know that the answer to this question rests upon the study of the boundedness properties of the maximal operator T^*, defined as $T^*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$, with respecto to the weight w. We can use inequality 5.20 (ii), from chapter II, obtaining the following result

THEOREM 3.6. *Let* T *be a regular singular integral operator,* $1 < p < \infty$ *and* $w \in A_p$. *Then the corresponding maximal operator* T^* *arising from the truncated operators* T_ε *is bounded in* $L^p(w)$. *More precisely, there is a constant* C *depending on* T *and* w, *such that for every* $f \in L^p(w)$:

$$\int_{R^n} (T^*f(x))^p w(x)dx \leqq C \int_{R^n} |f(x)|^p w(x)dx.$$

Proof. We observe that inequality 5.20 ii) in chapter II is valid for every $f \in L^p(w)$ provided q is close enough to 1. Indeed, we just need to take q such that $f \in L^q_{loc}(R^n)$, but $|f|^q \in L^{p/q}(w) \subset L^1_{loc}(R^n)$ as soon as $w \in A_{p/q}$, and we already know that this can be granted by taking $q > 1$ close enough to 1.

Now we can write:

$$\int_{R^n} (T^*f(x))^p w(x)dx \leqq C(\int_{R^n} (M_q f(x))^p w(x)dx +$$
$$+ \int_{R^n} (M(Tf)(x))^p w(x)dx) \leqq C \int_{R^n} |f(x)|^p w(x)dx$$

by 2.8 and 3.1, if q has been chosen so that $w \in A_{p/q}$. □

The weak type inequality, corresponding to the limit case, $p = 1$, of theorem 3.6 is also true. This will be proved in Chapter V, 4.11. It is now clear that corollary 5.22 of Chapter II remains valid for $f \in L^p(w)$ with $w \in A_p$, $1 \le p < \infty$.

We have just seen that regular singular integrals satisfy the same weighted inequalities as the Hardy-Littlewood maximal function. The question arises naturally whether or not the conditions imposed on w in theorems 3.1 and 3.5 are really necessary. We shall see next that the answer is positive.

THEOREM 3.7. *Let* w *be a weight in* $\mathbb{R}^n$ *and let* $1 \le p < \infty$. *Suppose that each of the* n *Riesz transformations* $R_1,\dots,R_n$ *is of weak type* (p,p), *with respect to* w. *Then, necessarily* $w \in A_p$.

Proof: Let Q be a cube, and let f be a non-negative function living in Q, with $f_Q > 0$. Denote by Q' the cube touching Q having the same side length as Q and such that $x_j \ge y_j$ for any $x \in Q'$, $y \in Q$ and $j=1,2,\dots,n$. Call $Rf(x) = \sum_{j=1}^{n} R_j f(x)$. Then, for every $x \in Q'$

$$Rf(x) = c_n \sum_{j=1}^{n} \int_Q (x_j-y_j)\ |x-y|^{-n-1}\ f(y)dy \ge$$

$$\ge C \int_Q \frac{f(y)dy}{|x-y|^n} \ge C\ f_Q$$

It follows that, for every $0 < t < C\ f_Q$, $Q' \subset \{x \in \mathbb{R}^n : |Rf(x)| > t\}$. Now, since R is of weak type (p,p) with respect to w, we shall have: $w(Q') \le C\ t^{-p} \int_Q f^p\ w$. We get $(f_Q)^p\ w(Q') \le C \int_Q (f(x))^p\ w(x)dx$. In particular, the choice $f = \chi_Q$ leads to $w(Q') \le C\ w(Q)$. By the same token we have $w(Q) \le C\ w(Q')$, from which we conclude that $(f_Q)^p\ w(Q) \le$ $\le C^2 \int_Q (f(x))^p\ w(x)dx$. In other words: $w \in A_p$. □

The information contained in our previous results on weighted inequalities for singular integral operators is very often applied in the following useful formulation:

COROLLARY 3.8. *Let* T *be a regular singular integral operator. Then, for every* $\varepsilon > 0$ *and* $1 < p < \infty$, *then inequality*

$$\int |Tf(x)|^p\ u(x)dx \le C_{\varepsilon,p} \int |f(x)|^p\ M_{1+\varepsilon}\ u(x)dx$$

holds for arbitrary functions f *and* $u \geq 0$ *for which the right hand side is finite. The same inequality is valid for the associated maximal operator* T^*.

Proof: We assume that $u(x)^{1+\varepsilon}$ is locally integrable, since otherwise $M_{1+\varepsilon} u(x) = +\infty$ at every point x. Then, by 2.16, $M_{1+\varepsilon} u \in A_1 \subset A_p$ with A_p constant depending only on ε, and theorem 3.1 gives

$$\int |Tf(x)|^p u(x)dx \leq \int |Tf(x)|^p M_{1+\varepsilon} u(x)dx \leq$$
$$\leq C_{\varepsilon,p} \int |f(x)|^p M_{1+\varepsilon} u(x)dx$$

The inequality for T^* follows in the same way from theorem 3.6. □

The novelty in the inequality just proved (which should be compared with 2.12 in Chapter II) stems from the fact that we try to estimate $\|Tf\|_{L^p(u)}$ for an arbitrary $u(x) \geq 0$. In spite of that, we are able to remove the T, the only price to pay for this being the change of u by another function, $M_{1+\varepsilon}u$, which is essentially of the same size (observe that $M_{1+\varepsilon}$ is bounded in L^p for $p > 1+\varepsilon$). A lot of information is contained in corollary 3.8; for example, by using it only when $p = 2$, all the L^p boundedness properties of T follow from Hölder's inequality plus duality. In this connection, see section V.6.

We shall end this section by giving a weighted version of the Hörmander-Mihlin multiplier theorem.

THEOREM 3.9. *Let* a *be an integer such that* $n/2 < a \leqq n$. *Suppose that* $m \in L^\infty(\mathbb{R}^n)$ *is of class* C^a *outside the origin and satisfies the condition:*

$$\left(R^{-1} \int_{R<|x|<2R} |D^\alpha m(\xi)|^2 \, d\xi\right)^{1/2} \leqq CR^{-|\alpha|}, \quad (0 < R < \infty)$$

for every multi-index α *such that* $|\alpha| \leqq a$. *Then:*

i) If $n/a < p < \infty$ *and* $w \in A_{pa/n}$, m *is a multiplier for* $L^p(w)$, *in other words: the operator* T_m *given by* $(T_m f)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$, *is bounded in* $L^p(w)$.

ii) If $1 < p < (n/a)'$ *and* $w^{-1/(p-1)} \in A_{p'a/n}$, T_m *is again bounded in* $L^p(w)$.

iii) If $w^{n/a} \in A_1$, T_m *is of weak type* $(1,1)$ *with respect to* w.

Proof: i) We look for an a priori inequality for $T_m f$ with $f \in S(\mathbb{R}^n)$. The notation will be as in section 6 of Chapter II. Observe that $T^N f(x) \to T_m f(x)$ uniformly as $N \to \infty$. Indeed:

$$\|T_m f - T^N f\|_\infty = \|((m-m^N)\hat{f})^\vee\|_\infty \leq \|(m-m^N)\hat{f}\|_1 \to 0$$

as $N \to \infty$, since $m^N(x) \to m(x)$ as $N \to \infty$ for every $x \neq 0$ and $\|m^N\|_\infty \leq C$ independently of N.

In general, we shall denote by $\|g\|_{p,w}$ the norm of the function g in $L^p(w)$. Since $T^N f(x) \to T_m f(x)$ uniformly as $N \to \infty$, it will be sufficient to obtain an estimate

$$\|T^N f\|_{p,w} \leq C \|f\|_{p,w} \tag{3.10}$$

with C independent of N.

In the proof of theorem 6.10 in Chapter II we obtained a pointwise estimate

$$(T^N f)^\#(x) \leq C_q M_q f(x) \tag{3.11}$$

valid for every $q > n/a$, with C_q independent of N. Even though this estimate was not written down explicitly, it was the key to prove the corresponding estimate for $(T_m f)^\#$ appearing in the theorem. From (3.11) and theorem 2.20 we shall obtain (3.10). First we shall make sure that $T^N f \in L^p(w)$, so that theorem 2.20 can actually be applied. This is very simple because $T^N f$ is a continuous function and we only have to check that it has the right behaviour at ∞. For $|x|$ large we have

$$|T^N f(x)| \leq \frac{C_N}{|x|^a}$$

since $(T^N f)^\wedge = m^N \hat{f}$ has derivatives in L^1 up to order a. Now it is enough to observe that

$$\int_{|x|>1} \frac{w(x)}{|x|^{ap}}\, dx = \int_{|x|>1} \frac{w(x)}{|x|^{n(pa/n)}} < \infty$$

as it follows from (3.2), since $w \in A_{pa/n}$.

Once we know that $T^N f \in L^p(w)$, we can write, using theorem 2.20 and (3.11) with some $n/a < q < p$

$$\|T^N f\|_{p,w} \leq C \|(T^N f)^\#\|_{p,w} \leq C \|M_q f\|_{p,w} =$$

$$= C\left(\int_{\mathbb{R}^n} (M(|f|^q)(x))^{p/q}\, w(x)dx\right)^{1/p}$$

Of course $p/q < pa/n$. However q can be chosen so close to n/a that $w \in A_{p/q}$. With this choice

$\|M_q f\|_{p,w} \leq C \|f\|_{p,w}$ and (3.10) follows.

ii) is obtained by duality. Writing $\sigma = w^{-1/(p-1)}$, we know that

$$\|T_m f\|_{p,w} = \sup_{\|g\|_{p',\sigma} \leq 1} \left| \int T_m f(x) \overline{g(x)}\, dx \right| =$$

$$= \sup_{\|g\|_{p',\sigma} \leq 1} \left| \int f(x) \overline{T_m g(x)}\, dx \right| \leq \|f\|_{p,w} \sup_{\|g\|_{p',\sigma} \leq 1} \|T_m g\|_{p',\sigma}$$

$$\leq C \|f\|_{p,w}$$

since $p < (\frac{n}{a})'$ implies that $p' > \frac{n}{a}$, and we can apply part i) to the weight $\sigma \in A_{p'a/n}$.

The proof of iii) is very similar to that of theorem 3.5. Let $f \in L^1(w)$. We are going to see that $w(\{x \in \mathbb{R}^n : |T^N f(x)| > t\}) \leq C\, t^{-1} \|f\|_{1,w}$ with C independent of f, $t > 0$ and N. This is clearly enough. Since $w \in A_1$, given $t > 0$, we can proceed exactly as in the proof of theorem 3.5, obtaining the Calderón-Zygmund cubes $\{Q_j\}$ for f at height t. The cubes $\{Q_j\}$ are disjoint and satisfy:

$$t < \frac{1}{|Q_j|} \int_{Q_j} |f(x)|dx \leq 2^n t \qquad \text{for every } j$$

and

$$|f(x)| \leq t \quad \text{for almost every } x \notin \bigcup_j Q_j .$$

Next we write $f = g+b$, where $g(x) = f(x)$ if $x \notin \Omega = \bigcup_j Q_j$, $g(x) = \frac{1}{|Q_j|} \int_{Q_j} f(y)dy = f_{Q_j}$ if $x \in Q_j$, and $b(x) = \sum_j b_j(x)$ with $b_j(x) = (f(x) - f_{Q_j})\, \chi_{Q_j}(x)$.

Now we just have to check that the four properties a') to d') in the proof of theorem 3.5 continue to hold for our w and the operator T^N

a') T^N is bounded in $L^p(w)$ provided $p > n/a$. This follows from i) since $w \in A_1 \subset A_{pa/n}$

b') and c') are proved as before using the fact that $w \in A_1$.

For d') we need to see that lemma 3.3 holds for our w and the kernel K^N of T^N. This is seen by showing that K^N satifies (3.4).

The proof of (3.4) for K^N and w such that $w^{n/a} \in A_1$ will be be based upon inequalities (6.11) of Chapter II with some $q > n/a$ such that $w^q \in A_1$. We know that such a q exists after theorem 2.7. With this choice of q we have:

$$\int_{|x|>2|y|} |K^N(x-y) - K^N(x)|w(x)dx \leq \sum_{j=1}^{\infty} \int_{2^j|y|<|x|<2^{j+1}|y|} |K^N(x-y)-K^N(x)|w(x)dx \leq$$

$$\leq \sum_{j=1}^{\infty} \left(\int_{2^j|y|<|x|<2^{j+1}|y|} |K^N(x-y)-K^N(x)|^{q'}dx\right)^{1/q'} \left(\int_{|x|<2^{j+1}|y|} w(x)^q dx\right)^{1/q} \leq$$

$$\leq C \sum_{j=1}^{\infty} (2^j|y|)^{-\varepsilon-n/q} |y|^{\varepsilon}(2^j|y|)^{n/q} (M(w^q)(y))^{1/q} \leq$$

$$\leq C \sum_{j=1}^{\infty} (2^{-\varepsilon})^j \, M\, w(y) = C\, M\, w(y)$$

since $M(w^q)(y) \leq \operatorname{ess\,inf}_{|x|\leq|y|} w(x)^q$

Once these four properties have been checked, the proof is identical to that of theorem 3.5. □

In the problem of convergence of Fourier series and integrals in one variable, one is interested in the multipliers χ_I, I an interval of $\mathbb{R}$, which give rise to the <u>partial sum operators</u> S_I, $(S_I f)^\wedge = \hat{f}\chi_I$. We know, see Chapter II (6.15), that S_I is a multiplier in $L^p(\mathbb{R})$ for all $1 < p < \infty$ (this fact is actually equivalent to M. Riesz theorem). Strictly speaking, S_I is neither a singular integral operator nor a multiplier operator satisfying the conditions of theorem 3.9, due to the lack of regularity of χ_I at the end points of the interval I. Nevertheless, the weighted inequalities for S_I are really an easy consequence of known estimates:

<u>COROLLARY 3.12</u>. *Let* $w(x) \geq 0$ *be a weight in* $\mathbb{R}$. *Then*

a) *If* $1 < p < \infty$, *the partial sum operators* S_I *are uniformly bounded (for all intervals* I*) in* $L^p(w)$ *if and only if* $w \in A_p$.

b) *The operators* S_I *are uniformly of weak type* $(1,1)$ *with respect to* w *if and only if* $w \in A_1$.

c) *For all* $f \in L^p(w)$, *with* $w \in A_p$ *and* $1 < p < \infty$,

$$\|S_{[-R,R]} f - f\|_{p,w} \to 0, \quad R \to \infty.$$

Proof: Let A be the analytic projection, defined by $(Af)^{\wedge}(\xi) = \hat{f}(\xi)\,\chi_{[0,\infty)}(\xi)$, or $Af(x) = f(x) + iHf(x)$, where H is the Hilbert transform. Denoting by M_a the isometry consisting in multiplying each function $f(x)$ by $e^{2\pi iax}$, we have

$$S_{[a,\infty)} = \frac{i}{2} M_a AM_{-a}$$

$$S_{[a,b)} = S_{[a,\infty)} - S_{[b,\infty)} = \frac{i}{2}(M_a AM_{-a} - M_b AM_{-b})$$

Since H is a regular singular integral operator, theorems 3.1 and 3.5 imply the sufficiency of the condition $w \in A_p$ in parts a) and b). For the necessity, we observe that $Hf = i(S_{(-\infty,0]}f - S_{[0,\infty)}f)$ and apply theorem 3.7 (since there is only one Riesz transform in $\mathbb{R}^1$, and it is precisely the Hilbert transform). Finally, c) follows from a) because the result is trivially true for the functions $f \in L^1(\mathbb{R})$ whose Fourier transform has compact support, and these are dense in $L^p(w)$. □

4. TWO-WEIGHT NORM INEQUALITIES FOR MAXIMAL OPERATORS

The main goal of this section is to give a solution of problem 2 (posed in section 1). This solution will be based on ideas of E. Sawyer [1] and B. Jawerth [1]. In order to introduce the main idea, and also for its own sake, we shall present first a general result of B. Jawerth, which contains B. Muckenhoupt's theorem 2.8 with a simple and elegant proof.

By a basis in $\mathbb{R}^n$ we shall simply mean a collection $\mathcal{B}$ of open sets in $\mathbb{R}^n$. For a basis $\mathcal{B}$ we shall consider weights w such that $w(B) < \infty$ for every $B \in \mathcal{B}$. Given $\mathcal{B}$ and w, we shall denote by $M_{\mathcal{B},w}$ the maximal operator sending the measurable function f into the function defined as follows: If $x \in \bigcup_{B\in\mathcal{B}} B$,

$$M_{\mathcal{B},w}f(x) = \sup \frac{1}{w(B)} \int_B |f(y)|\, w(y)dy$$

where the sup is taken over those $B \in \mathcal{B}$ such that $x \in B$ and $w(B) > 0$. Otherwise $M_{\mathcal{B},w}f(x) = 0$. When $w \equiv 1$, we simply write $M_{\mathcal{B}}$. Since the sets of $\mathcal{B}$ are open, we see that $M_{\mathcal{B}}f$ is a lower-semicontinuous function.

We shall say that the weight w is in the class $A_{p,\mathcal{B}}$, $1 \leq p < \infty$, if and only if, there is a constant C such that:

$$\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leq C$$

for every $B \in \mathcal{B}$ (with the usual interpretation of the second factor as $\operatorname{ess.sup}_B (w^{-1})$ when $p = 1$).

Jawerth's result is the following:

THEOREM 4.1. *Let* $\mathcal{B}$ *be a basis in* $\mathbb{R}^n$, w *a weight for* $\mathcal{B}$ *and* $1 < p < \infty$. *Call* $\sigma(x) = w(x)^{-1/(p-1)}$. *Then, the following conditions are equivalent:*

a) $M_{\mathcal{B}}$ *is bounded both in* $L^p(w)$ *and* $L^{p'}(\sigma)$

b) $w \in A_{p,\mathcal{B}}$; $M_{\mathcal{B},\sigma}$ *is bounded in* $L^p(\sigma)$ *and* $M_{\mathcal{B},w}$ *is bounded in* $L^{p'}(w)$.

Proof: We shall start by showing that b) implies a). Actually we just need to see that b) implies that $M_{\mathcal{B}}$ is bounded in $L^p(w)$. The reason is that $w \in A_{p,\mathcal{B}}$ if and only if $\sigma \in A_{p',\mathcal{B}}$. So, assuming b), we want to see that $M_{\mathcal{B}}$ is bounded in $L^p(w)$. Let $f \in L^p(w)$. We may assume that $M_{\mathcal{B}}f(x) < \infty$ except for a set of null w-measure (take, for instance, f bounded).

For every integer k, we shall consider the set

$$S_k = \{x \in \mathbb{R}^n : 2^k < M_{\mathcal{B}}f(x) \leq 2^{k+1}\}$$

From the definition of $M_{\mathcal{B}}$, $S_k \subset \bigcup_j B_{k,j}$, where $B_{k,j} \in \mathcal{B}$ satisfies

$$\frac{1}{|B_{k,j}|} \int_{B_{k,j}} |f(y)|dy > 2^k$$

Define $E_{k,1} = B_{k,1} \cap S_k$ and, for $j > 1$:

$$E_{k,j} = (B_{k,j} \setminus \bigcup_{s<j} B_{k,s}) \cap S_k.$$

The sets S_k form a disjoint collection and each S_k is the disjoint union of the sets $E_{k,j}$ for varying j.

$$\int_{\mathbb{R}^n} |M_{\mathcal{B}}f(x)|^p w(x)dx = \int_{\bigcup S_k} |M_{\mathcal{B}}f(x)|^p w(x)dx =$$

$$= \sum_{k,j} \int_{E_{k,j}} |M_{\mathcal{B}}f(x)|^p w(x)dx \leq 2^p \sum_{k,j} 2^{kp} w(E_{k,j}) \leq$$

$$\leq 2^p \sum_{k,j} w(E_{k,j}) \Big(\frac{1}{|B_{k,j}|} \int_{B_{k,j}} |f(y)|dy\Big)^p = 2^p \sum_{k,j} \mu_{k,j}\, g_{k,j},$$

where

$$\mu_{k,j} = w(E_{k,j})\left(\frac{\sigma(B_{k,j})}{|B_{k,j}|}\right)^p$$

$$g_{k,j} = \left(\frac{1}{\sigma(B_{k,j})}\int_{B_{k,j}} |f(y)|\sigma(y)^{-1}\sigma(y)dy\right)^p$$

We view the sum $\sum_{k,j} \mu_{k,j}\, g_{k,j}$ as an integral on a measure space (X,μ) built over the set $X = \{(k,j)\}$ by assigning to each (k,j) the measure $\mu_{k,j}$. For $\lambda > 0$, call

$$\Gamma(\lambda) = \{(k,j) \in X : g_{k,j} > \lambda\}$$

$$G(\lambda) = \bigcup_{(k,j)\in\Gamma(\lambda)} B_{k,j}$$

Then $\sum_{k,j} \mu_{k,j}\, g_{k,j} = \int_0^\infty \mu(\Gamma(\lambda))d\lambda$. Observe that $w \in A_{p,\mathcal{B}}$ is equivalent to saying that there is C such that, for every $B \in \mathcal{B}$:

$$\left(\frac{\sigma(B)}{|B|}\right)^p \leqq C\left(\frac{|B|}{w(B)}\right)^{p'}$$

We use this to estimate $\mu_{k,j}$

$$\mu_{k,j} \leqq C\, w(E_{k,j})\left(\frac{|B_{k,j}|}{w(B_{k,j})}\right)^{p'} \leqq$$

$$\leqq C\, w(E_{k,j})\left(\inf_{x\in B_{k,j}} M_{\mathcal{B},w}(\chi_{B_{k,j}}\, w^{-1})(x)\right)^{p'} \leqq$$

$$\leqq C\int_{E_{k,j}} |M_{\mathcal{B},w}(\chi_{B_{k,j}}\, w^{-1})(x)|^{p'}\, w(x)dx$$

Next we shall use the boundedness of $M_{\mathcal{B},w}$ to estimate $\mu(\Gamma(\lambda))$

$$\mu(\Gamma(\lambda)) = \sum_{(k,j)\in\Gamma(\lambda)} \mu_{k,j} \leqq C\sum_{(k,j)\in\Gamma(\lambda)}\int_{E_{k,j}} |M_{\mathcal{B},w}(\chi_{B_{k,j}}\, w^{-1})(x)|^{p'}w(x)dx \leqq$$

$$\leqq C\int_{\mathbb{R}^n} |M_{\mathcal{B},w}(\chi_{G(\lambda)}w^{-1})(x)|^{p'}w(x)dx \leqq C\int_{G(\lambda)} w(x)^{1-p'}\,dx =$$

$$= C\,\sigma(G(\lambda)) \leqq C\,\sigma(\{x \in \mathbb{R}^n : (M_{\mathcal{B},\sigma}\,(f/\sigma)(x))^p > \lambda\})$$

To conclude we shall use the boundedness of $M_{\mathcal{B},\sigma}$

$$\int_{\mathbb{R}^n} |M_{\mathcal{B}}f(x)|^p w(x)dx \leqq 2^p\int_0^\infty \mu(\Gamma(\lambda))d\lambda \leqq$$

$$\leqq C\int_0^\infty \sigma(\{x \in \mathbb{R}^n : (M_{\mathcal{B},\sigma}\,(f/\sigma)(x))^p > \lambda\})d\lambda =$$

$$= C \int_{\mathbb{R}^n} |M_{\mathcal{B},\sigma}(f/\sigma)(x)|^p \sigma(x)dx \leq C \int_{\mathbb{R}^n} |f(x)/\sigma(x)|^p \sigma(x)dx =$$

$$= C \int_{\mathbb{R}^n} |f(x)|^p \, w(x)dx$$

since $\sigma^{1-p} = w$. This finishes the proof of the fact that b) implies a).

Let us prove now that a) implies b). Fist the proof that the boundedness of $M_{\mathcal{B}}$ in $L^p(w)$ implies $w \in A_{p,\mathcal{B}}$, is exactly as the one given in section 1 for the basis of cubes. Then, observe that, by symmetry, it is enough to prove that $M_{\mathcal{B},\sigma}$ is bounded in $L^p(\sigma)$. The proof of this fact is very similar to the one given for the other half of the theorem. We take f nice in $L^p(\sigma)$ and define S_k, $B_{k,j}$ and $E_{k,j}$ with $M_{\mathcal{B},\sigma}$ in place of $M_{\mathcal{B}}$. Then

$$\int_{\mathbb{R}^n} |M_{\mathcal{B},\sigma} f(x)|^p \sigma(x)dx \leq C \sum_{k,j} \sigma(E_{k,j}) \Big(\frac{1}{\sigma(B_{k,j})} \int_{B_{k,j}} |f(y)|\sigma(y)dy\Big)^p$$

$$= C \sum_{k,j} \mu_{k,j} \, g_{k,j}$$

where now

$$\mu_{k,j} = \sigma(E_{k,j}) \Big(\frac{|B_{k,j}|}{\sigma(B_{k,j})}\Big)^p$$

$$g_{k,j} = \Big(\frac{1}{|B_{k,j}|} \int_{B_{k,j}} |f(y)|\sigma(y)dy\Big)^p$$

To estimate $\mu_{k,j}$ we use now Hölder's inequality

$$1 = \frac{1}{|B|} \int_B w(x)^{1/p} w^{-1/p}(x)dx \leq$$

$$= \Big(\frac{1}{|B|} \int_B w(x)dx\Big)^{1/p} \Big(\frac{1}{|B|} \int_B w(x)^{-p'/p} dx\Big)^{1/p'} = \Big(\frac{w(B)}{|B|}\Big)^{1/p} \Big(\frac{\sigma(B)}{|B|}\Big)^{1/p'}$$

We get

$$\mu_{k,j} \leq \sigma(E_{k,j}) \Big(\frac{w(B_{k,j})}{|B_{k,j}|}\Big)^{p'} \leq \int_{E_{k,j}} |M_{\mathcal{B}}(\chi_{B_{k,j}} w)(x)|^{p'} \sigma(x)dx$$

With $\Gamma(\lambda)$ and $G(\lambda)$ defined as before:

$$\mu(\Gamma(\lambda)) = \sum_{(k,j)\in\Gamma(\lambda)} \mu_{k,j} \leq \sum_{(k,j)\in\Gamma(\lambda)} \int_{E_{k,j}} |M_{\mathcal{B}}(\chi_{B_{k,j}} w)(x)|^{p'} \sigma(x)dx \leq$$

$$\leq \int_{\mathbb{R}^n} |M_{\mathcal{B}}(\chi_{G(\lambda)} w)(x)|^{p'} \sigma(x)dx \leq \int_{G(\lambda)} w(x)^{p'} \sigma(x)dx =$$

$$= w(G(\lambda)) \leq w(\{x \in \mathbb{R}^n : (M_{\mathcal{B}}(f\sigma)(x))^p > \lambda\})$$

Finally

$$\int_{\mathbb{R}^n} |M_{\mathcal{B},\sigma} f(x)|^p \sigma(x)dx \leqq C \int_0^\infty \mu(\Gamma(\lambda))d\lambda \leqq$$
$$\leqq C \int_0^\infty w(\{x \in \mathbb{R}^n : (M_{\mathcal{B}}(f\sigma)(x))^p > \lambda\})d\lambda =$$
$$= C \int_{\mathbb{R}^n} |M_{\mathcal{B}}(f\sigma)(x)|^p w(x)dx \leqq C \int_{\mathbb{R}^n} |f(x)|^p \sigma(x)^p w(x)dx =$$
$$= C \int_{\mathbb{R}^n} |f(x)|^p \sigma(x)dx. \qquad \square$$

COROLLARY 4.2. *Let* $1 < p < \infty$. *Suppose that the basis* $\mathcal{B}$ *is such that* $w \in A_{p,\mathcal{B}}$ *implies that* $M_{\mathcal{B},\sigma}$ *is bounded in* $L^p(\sigma)$ *and* $M_{\mathcal{B},w}$ *is bounded in* $L^{p'}(w)$. *Then* $M_{\mathcal{B}}$ *is bounded in* $L^p(w)$ *if and only if* $w \in A_{p,\mathcal{B}}$.

A particular case of corollary 4.2 in Muckenhoupt's theorem, which was proved in section 2 (theorem 2.8) and is derived here in a completely different way.

COROLLARY 4.3. *Let* $1 < p < \infty$ *and let* w *be a weight in* $\mathbb{R}^n$. *Then* M *is bounded in* $L^p(w)$ *if and only if* $w \in A_p$.

Proof: The reason is that the basis $\mathcal{Q}$ formed by all the open cubes in $\mathbb{R}^n$ satisfies the hypothesis of corollary 4.2. Indeed, if $w \in A_{p,\mathcal{Q}} = A_p$, then $\sigma \in A_{p'}$, and so, lemma 2.2 guarantees that both $w(x)dx$ and $\sigma(x)dx$ are doubling measures. Consequently, theorem 2.6 in Chapter II implies that $M_{\mathcal{Q},\sigma} = M_\sigma$ is bounded in $L^p(\sigma)$ and $M_{\mathcal{Q},w} = M_w$ is bounded in $L^{p'}(w)$. $\square$

Corollary 4.2 also applies to the basis $\mathcal{D}$ formed by the open dyadic cubes. (See Chapter II section 1 for the definition). Since the dyadic maximal functions will play a basic role in the sequel, it will be convenient to introduce short notations for them. We shall write $N = M_{\mathcal{D}}$ and, in general $N_w = M_{\mathcal{D},w}$. Also, for $R > 0$, we shall denote by $N^{(R)}$, $M^{(R)}$, $N_w^{(R)}$, $M_w^{(R)}$, the maximal operators obtained by taking in the corresponding definition just those cubes of the basis whose side length is $\leqq R$. Here is a simple but useful observation about the dyadic maximal functions.

LEMMA 4.4. *Let* $1 < p \leqq \infty$ *and let* w *be a locally integrable weight in* $\mathbb{R}^n$. *Then* N_w *is bounded in* $L^p(w)$.

Proof: The fact that w is locally integrable implies (as in the proof of corollary 1.13) that N_w is bounded in $L^\infty(w)$. Then, we just need to prove that N_w is of weak type (1,1) with respect to w and appeal to the Marcinkiewicz interpolation theorem to complete the proof of the lemma. Let us prove the weak type (1,1) inequality. Let $f \in L^1(w)$. Since $N_w f(x) = \lim_{R\to\infty} N_w^{(R)} f(x)$ and $N_w^{(R)} f(x)$ increases with R, it will be enough to prove the inequality for $N_w^{(R)}$ with constant independent of R. But, after fixing $R > 0$ and $\lambda > 0$, observe that $\{x \in \mathbb{R}^n : N_w^{(R)} f(x) > \lambda\} = \bigcup_j Q_j$, where Q_j are the maximal dyadic cubes of side length $\leqq R$ for which

$$\frac{1}{w(Q_j)} \int_{Q_j} |f(y)|w(y)dy > \lambda$$

These maximal dyadic cubes do exist because of the restriction on their size. Since they are disjoint, we get

$$w(\{x \in \mathbb{R}^n : N_w^{(R)} f(x) > \lambda\}) \leqq \sum_j w(Q_j) \leqq$$

$$\leqq \frac{1}{\lambda} \sum_j \int_{Q_j} |f(y)|w(y)dy \leqq \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(y)|w(y)dy$$

as we wanted to show. □

COROLLARY 4.5. *Let* $1 < p < \infty$, *and let* w *be a weight in* $\mathbb{R}^n$. *Then* N *is bounded in* $L^p(w)$ *if and only if* $w \in A_{p,\mathcal{D}}$, *that is, if* w *satisfies the* A_p *condition over dyadic cubes.*

Proof: Lemma 4.4 shows that corollary 4.2 applies to the basis $\mathcal{D}$. □

Jawerth's approach to Muckenhoupt's theorem provides a proof which is independent of the reverse Hölder's inequality. An even shorther proof has been given by M. Christ and R. Fefferman [1]. It is very similar to Jawerth's, but it uses Calderón-Zygmund cubes. Here is an account of it.

ALTERNATIVE PROOF OF COROLLARY 4.3.

Let $1 < p < \infty$ and let w be an A_p weight in $\mathbb{R}^n$. Take $f \in L^p(w)$. Choose a constant $a > 2^n$. Then for each integer k, let $\{Q_{k,j}\}$ be the Calderón-Zygmund cubes of f at height a^k. Call $E_{k,j} = Q_{k,j} \setminus \bigcup_{r>k} Q_{r,s}$. Then, calling as usual $\sigma = w^{-1/(p-1)}$

$$\int_{\mathbb{R}^n} |Mf(x)|^p\, w(x)dx \leqq a^p \sum_{k,j} \left(\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} f(y)dy\right)^p w(E_{k,j}) \leqq$$

$$\leq C \sum_{k,j} \left(\frac{1}{\sigma(Q_{k,j})} \int_{Q_{k,j}} f(y)\sigma(y)^{-1}\sigma(y)dy\right)^p \sigma(E_{k,j}) \left\{\frac{w(E_{k,j})}{\sigma(E_{k,j})} \left(\frac{\sigma(Q_{k,j})}{|Q_{k,j}|}\right)^p\right\}$$

Observe that $|E_{k,j}| > (1/4)|Q_{k,j}|$. But $\sigma \in A_{p'} \subset A_\infty$, and so $\sigma(E_{k,j}) \geq \alpha\sigma(Q_{k,j})$ for some $\alpha > 0$. Then

$$\frac{w(E_{k,j})}{\sigma(E_{k,j})} \left(\frac{\sigma(Q_{k,j})}{|Q_{k,j}|}\right)^p \leq \alpha^{-1} \frac{w(E_{k,j})}{|Q_{k,j}|} \left(\frac{\sigma(Q_{k,j})}{|Q_{k,j}|}\right)^{p-1} \leq C$$

since $w \in A_p$. Thus

$$\int_{\mathbb{R}^n} |Mf(x)|^p w(x)dx \leq C \int_{\mathbb{R}^n} |M_\sigma(f/\sigma)(x)|^p \sigma(x)dx \leq$$

$$\leq C \int_{\mathbb{R}^n} |f(x)|^p \sigma(x)^{1-p} dx = C \int_{\mathbb{R}^n} |f(x)|^p w(x)dx. \qquad \square$$

The following general result of Jawerth [1] will allow us to obtain a solution to problem 2 of section 1.

THEOREM 4.6. *Let* $\mathcal{B}$ *be a basis,* (u,w) *a couple of weights in* $\mathbb{R}^n$ *and* $1 < p < \infty$. *Suppose that* $\sigma(x) = w(x)^{-1/(p-1)}$ *is such that* $M_{\mathcal{B},\sigma}$ *is bounded in* $L^p(\sigma)$. *Then the following two conditions are equivalent:*

a) $M_{\mathcal{B}}$ *is bounded from* $L^p(w)$ *to* $L^p(u)$

b) *There is a constant* C *such that for every set* G *which is a union of sets in* $\mathcal{B}$, *this inequality holds:*

$$\int_G |M_{\mathcal{B}}(\sigma\, \chi_G)(x)|^p u(x)dx \leq C\, \sigma(G).$$

Proof: Implicit in the assumptions is the fact that $\sigma(B) < \infty$ for every $B \in \mathcal{B}$. That a) implies b) is almost immediate. If a) holds, we have an inequality

$$\int_{\mathbb{R}^n} |M_{\mathcal{B}}f(x)|^p u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x)dx$$

valid for every function f. If we apply this inequality to the function $f = \sigma\, \chi_G$, we obtain:

$$\int_{\mathbb{R}^n} |M_{\mathcal{B}}(\sigma\, \chi_G)(x)|^p u(x)dx \leq C \int_G \sigma(x)^p w(x)dx = C\, \sigma(G) \quad \text{and}$$

b) follows.

Next, assuming b), we shall prove a). Let f be a nice function in $L^p(w)$. We proceed exactly as in the proof of theorem 4.1. Just like there we define the sets S_k, $B_{k,j}$ and $E_{k,j}$ and write

$$\int_{\mathbb{R}^n} |M_{\mathcal{B}}f(x)|^p\, u(x)dx \leq C \sum_{k,j} u(E_{k,j})\Big(\frac{1}{|B_{k,j}|}\int_{B_{k,j}} |f(y)|dy\Big)^p =$$

$$= C \sum_{k,j} \mu_{k,j}\, g_{k,j}$$

where

$$\mu_{k,j} = u(E_{k,j})\Big(\frac{\sigma(B_{k,j})}{|B_{k,j}|}\Big)^p$$

$$g_{k,j} = \Big(\frac{1}{\sigma(B_{k,j})}\int_{B_{k,j}} |f(y)|\sigma(y)^{-1}\,\sigma(y)dy\Big)^p$$

The definitions of $\Gamma(\lambda)$ and $G(\lambda)$ for $\lambda > 0$ are exactly as in the proof of theorem 4.1. Then

$$\mu(\Gamma(\lambda)) = \sum_{(k,j)\in\Gamma(\lambda)} \mu_{k,j} \leq \sum_{(k,j)\in\Gamma(\lambda)} u(E_{k,j})\Big(\frac{\sigma(B_{k,j})}{|B_{k,j}|}\Big)^p \leq$$

$$\leq \sum_{(k,j)\in\Gamma(\lambda)} \int_{E_{k,j}} |M_{\mathcal{B}}(\sigma\, \chi_{B_{k,j}})(x)|^p\, u(x)dx \leq$$

$$\leq \int_{G(\lambda)} |M_{\mathcal{B}}(\sigma\, \chi_{G(\lambda)})(x)|^p\, u(x)dx \leq C\,\sigma(G(\lambda)) \leq$$

$$\leq C\,\sigma(\{x \in \mathbb{R}^n : (M_{\mathcal{B},\sigma}(f/\sigma)(x))^p > \lambda\})$$

where the next to last inequality is the result of applying condition b) to the set $G(\lambda) = \bigcup_{(k,j)\in\Gamma(\lambda)} B_{k,j}$. Now

$$\int_{\mathbb{R}^n} |M_{\mathcal{B}}f(x)|^p u(x)dx = C\int_0^\infty \mu(\Gamma(\lambda))d\lambda \leq$$

$$\leq C\int_0^\infty \sigma(\{x \in \mathbb{R}^n : (M_{\mathcal{B},\sigma}(f/\sigma)(x))^p > \lambda\})d\lambda =$$

$$= C\int_{\mathbb{R}^n} |M_{\mathcal{B},\sigma}(f/\sigma)(x)|^p\,\sigma(x)dx \leq C\int_{\mathbb{R}^n} |f(x)|^p\sigma(x)^{1-p}dx =$$

$= C\int_{\mathbb{R}^n} |f(x)|^p\, w(x)dx.$ We have obtained a). □

Theorem 4.6 is not immediately applicable to the basis of cubes $\mathcal{Q}$ because we can not conclude from b) that M_σ is bounded in $L^p(\sigma)$ (σ does not have to be doubling). However for the basis of dyadic cubes $\mathcal{D}$, the boundedness of N_σ is automatic, according to lemma 4.4. It will be through the dyadic case that we shall be able to solve problem 2 of section 1. Actually, for the basis $\mathcal{D}$, condition b) can be written in a simpler form. Here is the solution of the two weights problem for the dyadic maximal operator N.

THEOREM 4.7. *Let* $1 < p < \infty$ *and let* (u,w) *be a couple of weights in* $\mathbb{R}^n$. *Then, the following two conditions are equivalent:*

a) N *is bounded from* $L^p(w)$ *to* $L^p(u)$

b) *There is a constant* C *such that, for every dyadic cube* Q:

$$\int_Q |N(\sigma\chi_Q)(x)|^p\, u(x)dx \leq C\,\sigma(Q) < \infty$$

where $\sigma = w^{-1/(p-1)}$.

Proof: First, we have to see that a) implies $\sigma(Q) < \infty$ for every Q. Indeed if $\sigma(Q) = \infty$ we have $\int_Q (w(x)^{-1})^{p'} w(x)dx = \infty$. It follows that there exists $f \in L^p(w)$ such that $\int_Q w(x)^{-1} f(x)w(x)dx = = \int_Q f(x)dx = \infty$. But then $Mf(x) = \infty$ for every $x \in Q$, and it follows from a) that $u(Q) = 0$. Actually $\sigma(R) = \infty$ for every cube $R \supset Q$, so that $u \equiv 0$, a trivial case which we better exclude.

Now, to prove the theorem, we just need to see that condition b) extends from dyadic cubes to arbitrary open sets (i.e., unions of dyadic cubes). Suppose b) holds and let G be an open set. We shall prove an inequality

$$\int_G |N^{(R)}(\sigma\chi_G)(x)|^p\, u(x)dx \leq C\,\sigma(G)$$

with C independent of G and R. This is clearly sufficient. Let $\{Q_{k,j}\}$ be the maximal dyadic cubes contained in G and having side length $\leq R$ and such that

$$\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} \sigma(x)dx > 2^k.$$

Let $E_{k,j} = Q_{k,j} \setminus \bigcup_{r>k} Q_{r,s}$. Then

$$\int_G |N^{(R)}(\sigma\chi_G)(x)|^p\, u(x)dx \leq C \sum_{k,j} 2^{kp}\, u(E_{k,j}) \leq$$

$$\leq C \sum_{k,j} u(E_{k,j}) \left(\frac{\sigma(Q_{k,j})}{|Q_{k,j}|}\right)^p$$

Since all the cubes in the doubly indexed family $\{Q_{k,j}\}$ have side length $\leq R$, every cube will be contained in a maximal one. Let $\{Q_i\}$ be the subfamily formed by these maximal cubes. Then

$$\int_G |N^{(R)}(\sigma\chi_G)(x)|^p\, u(x)dx \leq C \sum_i \sum_{Q_{k,j}\subset Q_i} u(E_{k,j}) \left(\frac{\sigma(Q_{k,j})}{|Q_{k,j}|}\right)^p \leq$$

$$\leq C \sum_i \sum_{Q_{k,j} \subset Q_i} \int_{E_{k,j}} |M(\sigma \chi_{Q_i})(x)|^p u(x)dx \leq$$

$$\leq C \sum_i \int_{Q_i} |M(\sigma \chi_{Q_i})(x)|^p u(x)dx \leq C \sum_i \sigma(Q_i) \leq C\, \sigma(G). \qquad \square$$

What makes it possible to extend theorem 4.7 to the ordinary Hardy-Littlewood maximal operator M is the following estimate by an average of translations of the dyadic maximal operator N. The idea goes back to C. Fefferman and Stein [1].

LEMMA 4.8. *For every integer* k, *every locally integrable function* f *in* $\mathbb{R}^n$ *and every* $x \in \mathbb{R}^n$:

$$M^{(2^k)}f(x) \leq 2^{3n+1} \frac{1}{|Q(0,2^{k+2})|} \int_{Q(0,2^{k+2})} (\tau_{-t} \circ N \circ \tau_t) f(x)dt$$

where $\tau_t g(x) = g(x-t)$.

Proof:

$$(\tau_{-t} \circ N \circ \tau_t) f(x) = N(\tau_t f)\ (x+t) =$$

$$= \sup_{x+t \in Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q f(y-t)dy =$$

$$= \sup_{x \in Q-t,\, Q \in \mathcal{D}} \frac{1}{|Q|} \int_{Q-t} f(z)dz$$

Thus $\tau_{-t} \circ N \circ \tau_t$ is simply the operator $M_{\mathcal{D}-t}$ associated to the basis $\mathcal{D}-t$ formed by the cubes $Q-t$ with Q dyadic, that is, the translates by $-t$ of the dyadic cubes.

Given k, f and x, by definition of $M^{(2^k)}f(x)$, there will exist a cube R of side length $\leq 2^k$ such that $x \in R$ and

$$\frac{1}{2} M^{(2^k)}f(x) \leq \frac{1}{|R|} \int_R |f(y)|dy.$$

Let j be an integer such that $2^{j-1} <$ side length of $R \leq 2^j$. Of course $j \leq k$. Consider the set Ω consisting of those $t \in Q(0,2^{k+2})$ for which there is some $Q \in \mathcal{D}-t$ with side length equal to 2^{j+1} and such that $R \subset Q$. For every $t \in \Omega$, we have:

$$\frac{1}{2} M^{(2^k)}f(x) \leq \frac{1}{|R|} \int_R |f(y)|dy \leq \frac{2^{2n}}{|Q|} \int_Q |f(y)|dy \leq$$

$$\leq 2^{2n} (\tau_{-t} \circ N \circ \tau_t) f(x)$$

Now it is a simple geometrical observation that the measure of Ω

is at least $2^{(k+2)n} = \frac{|Q(0,2^{k+2})|}{2^n}$. Then

$$M^{(2^k)}f(x) \leqq \frac{2^{2n+1}}{|\Omega|}\int_\Omega (\tau_{-t}\circ N\circ\tau_t)f(x)dt \leqq$$

$$\leqq \frac{2^{3n+1}}{|Q(0,2^{k+2})|}\int_{Q(0,2^{k+2})}(\tau_{-t}\circ N\circ\tau_t)f(x)dt. \qquad \square$$

We are finally ready to give our promised solution to the two weights problem for M.

THEOREM 4.9. *Let* $1 < p < \infty$, *and let* (u,w) *be a couple of weights in* $\mathbb{R}^n$. *Then, the following two conditions are equivalent:*

a) M *is bounded from* $L^p(w)$ *to* $L^p(u)$

b) *There is a constant* C *such that, for every cube* Q:

$$\int_Q |M(\sigma\chi_Q)(x)|^p\, u(x)dx \leqq C\,\sigma(Q) < \infty$$

where $\sigma = w^{-1/(p-1)}$.

Proof: All we have to do is to show that b) implies a). But, after lemma 4.8, the boundedness of M is equivalent to the uniform boundedness of the operators $\tau_{-t}\circ N\circ\tau_t$ for $t \in \mathbb{R}^n$. Indeed, once this uniform boundedness has been established, we can use Minkowski's integral inequality and the monotone convergence theorem to get the boundedness of M. Now, since

$$\int_{\mathbb{R}^n} |(\tau_{-t}\circ N\circ\tau_t)f(x)|^p u(x)dx = \int_{\mathbb{R}^n} |N(\tau_t f)(y)|^p u(y-t)dy$$

and

$$\int_{\mathbb{R}^n} |f(x)|^p\, w(x)dx = \int_{\mathbb{R}^n} |\tau_t f(y)|^p\, w(y-t)dy,$$

we see that the uniform boundedness of the operators $\tau_{-t}\circ N\circ\tau_t$ between $L^p(w)$ and $L^p(u)$, is equivalent to the fact that the couples $(\tau_t u,\tau_t w)$ satisfy condition b) in theorem 4.7 with a constant independent of t. But this fact follows quite easily from our condition b). Indeed, for any $t \in \mathbb{R}^n$ and any dyadic cube Q, we have:

$$\int_Q |N((\tau_t\sigma)\chi_Q)(x)|^p(\tau_t u)(x)dx = \int_Q |N(\tau_t(\sigma\chi_{Q-t}))(x)|^p u(x-t)dx =$$

$$= \int_{Q-t} |(\tau_{-t}\circ N\circ\tau_t)(\sigma\,\chi_{Q-t})(y)|^p\, u(y)dy \leqq$$

$$\leqq \int_{Q-t} |M(\sigma\,\chi_{Q-t})(y)|^p u(y)dy \leqq C\,\sigma(Q-t) = C\int_{Q-t}\sigma(x)dx =$$

$= C(\tau_t\sigma)(Q)$. This finishes the proof. $\square$

Condition b) in theorem 4.9 will be called condition S_p on the couple (u,w). It is obvious that S_p implies A_p, since $\left(\frac{\sigma(Q)}{|Q|}\right)^p \frac{u(Q)}{|Q|} = \frac{1}{|Q|}\int_Q |M(\sigma\,\chi_Q)(x)|^p\,u(x)dx \leqq C\,\frac{\sigma(Q)}{|Q|}$ so that $\frac{u(Q)}{|Q|}\left(\frac{\sigma(Q)}{|Q|}\right)^{p-1} \leqq C$.

But, of course, S_p is stronger than A_p. However, for $u=w$, we know that S_p is equivalent to A_p. Actually, it is very easy to see directly that, in the case of equal weights, A_p implies S_p. This is an observation of Hunt, Kurtz and Neugebauer [1] which we shall presently explain. It is clear that, for a function f living in a cube Q, the Hardy-Littlewood maximal function at a point $x \in Q$ coincides with the supremum of the averages of $|f|$ over cubes contained in Q. Now, if R is a cube contained in Q and $w \in A_p$, we have

$$\frac{1}{|R|}\int_R \sigma(x)dx \leqq C\left(\frac{|R|}{w(R)}\right)^{1/(p-1)}$$

so that $|M(\sigma\,\chi_Q)(x)|^p \leqq C|M_w(w^{-1}\,\chi_Q)(x)|^{p'}$ for $x \in Q$, and, consequently, since $w(x)dx$ is "doubling"

$$\int_Q |M(\sigma\,\chi_Q)(x)|^p w(x)dx \leqq C\int_Q |M_w(w^{-1}\,\chi_Q)(x)|^{p'} w(x)dx \leqq$$

$$\leqq C\int_Q w(x)^{-p'} w(x)dx = C\,\sigma(Q)$$

and $(w,w) \in S_p$. This gives yet another proof of Muckenhoupt's theorem.

5. FACTORIZATION AND EXTRAPOLATION

We have already seen (immediately before theorem 2.16) that if we have two A_1 weights w_0 and w_1 and if $1 < p < \infty$, then $w(x) = w_0(x)\,w_1(x)^{1-p}$ is an A_p weight. Now we are going to show that, conversely, every weight $w \in A_p$ can be written in this form for certain $w_0, w_1 \in A_1$. This factorization theorem will have important consequences. The proof will be based on a simple lemma, which, as we shall see, provides a strikingly powerful method to deal with several problems about weights.

LEMMA 5.1. *Let* S *be a sublinear operator bounded in* $L^p(\mu)$, *where* $p \geq 1$ *and* μ *is an arbitrary positive measure on some measurable space. Suppose that* $Sf \geq 0$ *for every* $f \in L^p(\mu)$. *Then, for every* $u \geq 0$ *in* $L^p(\mu)$ *there is* $v \geq 0$ *in* $L^p(\mu)$ *such that:*

i) $u(x) \leq v(x)$ *for a.e.* x

ii) $\|v\|_p \leq 2\|u\|_p$

iii) $Sv(x) \leq C\, v(x)$ *for a.e.* x ($C = 2\|S\|$ *is enough*).

Proof: It suffices to take $v = \sum_{j=0}^{\infty} (2\|S\|)^{-j} S^j u$. This series converges clearly in norm, and $\|v\|_p \leq \sum_{j=0}^{\infty} (2\|S\|)^{-j}\|S\|^j\|u\|_p = 2\|u\|_p$, which is ii).

On the other hand, $u(x) \leq v(x)$ because all the partial sums are $\geq u$ as a consequence of the fact that $Sf \geq 0$ for every f. Finally, since S is sublinear, we have:

$$Sv \leq \sum_{j=0}^{\infty} (2\|S\|)^{-j} S^{j+1}u = 2\|S\| \sum_{j=0}^{\infty} (2\|S\|)^{-j-1} S^{j+1}u \leq 2\|S\|v. \qquad \square$$

Actually, with the help of this lemma, we can give a general factorization theorem which includes the one we were seeking for A_p weights.

THEOREM 5.2. *Let* T *be a positive sublinear operator acting on measurable functions on some measure space* (X,dx) *(this means that* $|T(f+g)| \leq |Tf| + |Tg|$ *and also that* $|f| \leq g$ *implies* $|Tf| \leq Tg$ *). For* $1 < p < \infty$, *let us call*
$W_p = \{w : 0 \leq w(x) < \infty$ *a.e. and* T *is bounded in* $L^p(w) = L^p(w(x)dx)\}$. *Also, we call*

$W_1 = \{w: 0 \leq w(x) < \infty$ *a.e. and* $Tw(x) \leq C\, w(x)$ *a.e. for some* C *independent of* $x\}$

Then, for every $1 < p < \infty$, *we have* $W_p \cap W_{p'}^{1-p} \subset W_1 . W_1^{1-p}$, *that is: if* $w \in W_p$ *and also* $w^{-1/(p-1)} \in W_{p'}$, *then there exist* $w_o, w_1 \in W_1$ *such that* $w = w_o w_1^{1-p}$. *Besides, the constants* C *for* w_o *and* w_1 *in the class* W_1 *depend only upon the constants for* w *and* $w^{-1/(p-1)}$ *in* W_p *and* $W_{p'}$ *respectively, that is, on the respective norms of* T *on* $L^p(w)$ *and* $L^{p'}(w^{-1/(p-1)})$.

Proof: We just need to consider the case $1 < p \leq 2$, since it follows from $W_p \cap W_{p'}^{1-p} \subset W_1 \cdot W_1^{1-p}$ by rising to the power $1-p'$, that

$W_{p'} \cap W_p^{1-p'} \subset W_1 \cdot W_1^{1-p'}$. So, let $1 < p \leqq 2$, and suppose that $w \in W_p \cap W_{p'}^{1-p}$, i.e., $w \in W_p$ and also $w^{-1/(p-1)} \in W_{p'}$. We want to see that $w = w_0 w_1^{1-p}$ with $w_0, w_1 \in W_1$. After writing $v^{-1} = w_1^{1-p}$, we see that this is equivalent to finding v such that:

a) $vw \in W_1$, that is: $T(vw) \leqq C\ vw$, and also

b) $v^{1/(p-1)} \in W_1$, that is: $T(v^{1/(p-1)}) \leqq C\ v^{1/(p-1)}$ or equivalently $(T(v^{1/(p-1)}))^{p-1} \leqq C\ v$.

Suppose that for every u in some L^q space we can find Su so that:

$$|T(u\ w)| \leqq S(u)w$$
$$(T(|u|^{1/(p-1)}))^{p-1} \leqq S(u).$$

If the operator S satisfies the hipotheses of lemma 5.1., we shall be able to find $v \geqq 0$ such that $S(v) \leqq C\ v$. This would suffice, because then we should have

$$T(v\ w) \leqq S(v)w \leqq C\ vw$$
$$T(v^{1/(p-1)})^{p-1} \leqq S(v) \leqq C\ v$$

All we have to do is to look for S and make sure that is satisfies the hypotheses of the lemma. The natural candidate for S is the operator sending the function u into Su given by

$$S(u) = |T(u\ w)|w^{-1} + T(|u|^{1/(p-1)})^{p-1}$$

First of all we observe that S is sublinear (for the first term in the sum, sublinearity is clear, and for the second it follows from the fact that, being $1 < p \leqq 2$, we have $1/(p-1) \geqq 1$, and it is a simple exercise to check that $u \mapsto T(|u|^\alpha)^{1/\alpha}$ is sublinear provided $\alpha \geqq 1$). Besides, S is bounded in $L^{p'}(w)$. Indeed:

$$\int |T(uw)w^{-1}|^{p'}\ w = \int |T(uw)|^{p'}\ w^{1-p'} \leq C \int |uw|^{p'}\ w^{1-p'} = C\int |u|^{p'}w$$

Since $w^{1-p'} = w^{-1/(p-1)} \in W_{p'}$, and also

$$\int |T(|u|^{1/(p-1)})|^{(p-1)p'}w = \int |T(|u|^{1/(p-1)})|^p w \leq C \int |u|^{p'}w$$

because $w \in W_p$. From the definition of S, it is evident that $Su \geq 0$ for every $u \in L^{p'}(w)$. Thus, S satisfies all the conditions required in lemma 5.1. Note that C in lemma 5.1(iii) depends only on the norm of S in $L^{p'}(w)$, and this depends only on the norms for T in $L^p(w)$ and in $L^{p'}(w^{-1/(p-1)})$. This finishes the proof. $\square$

COROLLARY 5.3. *(P. Jones' factorization theorem). For* $1 < p < \infty$, $A_p = A_1 . A_1^{1-p}$, *that is:* $w \in A_p$ *if and only if there exist* $w_o, w_1 \in A_1$ *such that* $w = w_o\, w_1^{1-p}$.

Proof: If we take $T = M$ = Hardy-Littlewood maximal operator in theorem 5.2, we know that $W_p = A_p$ and $W_1 = A_1$. Besides $W_{p'}^{1-p} = A_{p'}^{1-p} = A_p$ because $w \in A_p$ if and only if $w^{-1/(p-1)} \in A_{p'}$. Therefore, theorem 5.2 applies giving us $A_p \subset A_1 . A_1^{1-p}$. The inclusion $A_1 . A_1^{1-p} \subset A_p$ has been already established in section 2. □

By combining the factorization theorem with the characterization of A_1 weights given by theorem 2.16, we obtain a general expression for A_p weights in terms of maximal functions. This is the natural extension to $p > 1$ of theorem 2.16. Then, by using the John-Nirenberg theorem, this yields an expression for B.M.O. functions in terms of maximal functions:

COROLLARY 5.4. a) *Let* w *be a weight in* $\mathbb{R}^n$ *such that* $w(x) < \infty$ a.e. *Then,* $w \in A_p$ *if andonly if it can be written as*

$$w(x) = k(x)\ (Mf(x))^{\alpha}\ (Mg(x))^{\beta(1-p)}$$

with $f, g \in L^1_{loc}(\mathbb{R}^n)$, k *bounded away from* 0 *and* ∞ *and* $0 < \alpha, \beta < 1$. *In this representation,* k *can be taken between two positive bounds which depend only on the* A_p *constant for* w.

b) *There are constants* C_1 *and* C_2 *depending only on the dimension* n, *such that every* $\phi \in$ B.M.O. *in* $\mathbb{R}^n$ *can be written as:*

$$\phi(x) = b(x) + \gamma \log\ Mf(x) - \eta \log Mg(x)$$

with $f, g \in L^1$, $\gamma, \eta \geqq 0$ *and* $\|b\|_\infty + \gamma + \eta \leqq C_1 \|\phi\|_*$.

Conversely, every ϕ *which can be written as above, belongs to* B.M.O. *with* $\|\phi\|_* \leqq C_2(\|b\|_\infty + \gamma + \eta)$.

c) *We can write a statement like* b) *with* B.L.O. *in place of* B.M.O. *and* $\eta = 0$.

d) *As a consequence of* b) *and* c), *every* B.M.O. *function can be written as a difference of two* B.L.O. *functions. In short:* B.M.O. $\subset$ B.L.O. $-$ B.L.O.

Proof: a) The "if" part is clear, since it follows from theorem 2.16 that both $(Mf(x))^{\alpha}$ and $(Mg(x))^{\beta}$ are A_1 weights and, consequently $w \in A_p$.

Conversely if $w \in A_p$, corollary 5.3 implies that $w = w_0 . w_1^{1-p}$ with $w_0, w_1 \in A_1$. Then we just need to apply theorem 2.16 to obtain the desired representation. Observe that, in the proof of theorem 2.16 the lower bound obtained for the function k depends only upon the constant C in the reverse Hölder's inequality for the A_1 weight, and this, in turn, depends only upon its A_1 constant. The upper bound obtained for k in the proof of theorem 2.16 is just 1. In out present situation, the factorization theorem tells us that the A_1 constants for w_0 and w_1 depend only upon the A_p constant for w. Therefore in our representation for the A_p weight w, the function k is bounded away from 0 and ∞ with bounds depending only upon the A_p constant for w.

b) According to Corollary 3.5 of Chapter II, $\log Mf(x)$ is in B.M.O. with norm independent of f. Consequently, if ϕ has the representation exhibited in b), we have $\phi \in$ B.M.O. with $\|\phi\|_* \leqq C_2(\|b\|_\infty + \gamma + \eta)$ for some absolute constant C_2.

Conversely, if $\phi \in$ B.M.O., it follows from corollary 3.10 in Chapter II and corollary 2.18 that, taking $\lambda = C/\|\phi\|_*$ with C an absolute constant (say $C = C_2/2$ in the notation of corollary 3.10 Chapter II) the function $w(x) = e^{\lambda\phi(x)}$ is in A_2 with an A_2 constant independent of ϕ (look at the end of the proof of the forementioned corollary 3.10 and realize that, with the choice $\lambda = C_2/2$, the A_2 constant for w turns out to depend only on C_1 in the notation used there). Applying part a) to our w, we obtain

$$\log w(x) = \log k(x) + \alpha \log Mf(x) - \beta \log Mg(x)$$

Since $\phi = \lambda^{-1} \log w$, we get the desired decomposition with

$$b = \lambda^{-1} \log k, \qquad \gamma = \lambda^{-1} \alpha, \qquad \eta = \lambda^{-1} \beta$$

Observe that the L^∞ norm of $\log k$ does not depend on ϕ. Then, since $\lambda^{-1} = C^{-1}\|\phi\|_*$ and $0 \leqq \alpha < 1$, $0 \leqq \beta < 1$, we have

$$\|b\|_\infty + \gamma + \eta \leq \text{Constant } \|\phi\|_*$$

c) As we observed in the proof of theorem 3.3 in Chapter II, $\log Mf(x)$ is actually in B.L.O., so that any $\phi = b + \gamma \log Mf$ with $\|b\|_\infty < \infty$ and γ a real number $\geqq 0$, will also belong to B.L.O.

For the converse, the proof is very much like the one in part b). The difference is that, as we noted in the second remark following corollary 2.19, if $\phi \in$ B.L.O., the weight $w(x) = e^{\lambda\phi(x)}$ is actually in A_1, not just in A_2. Then we can use part a) as before, but now $p = 1$, so that we obtain the representation with $\eta = 0$.

Finally d) follows obviously from b) and c). $\square$

Next we are going to apply theorem 5.2 to study the following question: Let $\Omega \subset \mathbb{R}^n$ be a measurable set with $|\Omega| > 0$. Let f be a measurable function on Ω. We want to know how f has to be in order that there exists F defined on the whole space $\mathbb{R}^n$, belonging to B.M.O., and such that $F(x) = f(x)$ for a.e. $x \in \Omega$. In other words, we want to characterize the space consisting of the restrictions to Ω of the functions in B.M.O. We shall obtain the solution to this problem from the solution to the corresponding restriction problem for weights.

We shall use the operator M_Ω, defined on functions g integrable over bounded subsets of Ω, by

$$M_\Omega g(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_{Q\cap\Omega} |g(y)|dy$$

where the sup is taken over all the cubes of $\mathbb{R}^n$ containing x.

We shall say that $w(x) \geqq 0$, defined on Ω, belongs to the class $A_{p,\Omega}$, $1 \leqq p < \infty$, if and only if there is a constant C such that for every cube Q of $\mathbb{R}^n$

$$\left(\frac{1}{|Q|} \int_{Q\cap\Omega} w(x)dx\right)\left(\frac{1}{|Q|} \int_{Q\cap\Omega} w(x)^{-1/(p-1)}dx\right)^{p-1} \leqq C$$

(only the cubes Q intersecting Ω in a set of positive measure will be really relevant here; for $p = 1$, the second factor has the usual interpretation as the essential supremum over $Q\cap\Omega$ of $w(x)^{-1}$).

The following theorem is proved exactly as theorem 1.12.

THEOREM 5.5. *Let* $1 \leqq p < \infty$ *and let* (u,w) *be a couple of non-negative measurable functions on* Ω. *Then, the following three conditions are equivalent:*

a) M_Ω *is bounded from* $L^p(\Omega;w)$ *to* $L^p_*(\Omega,u)$, *that is, there is a constant* C *such that, for every* f *on* Ω *and every* $\lambda > 0$:

$$u(\{x \in \Omega : M_\Omega f(x) > \lambda\}) \leq C \lambda^{-p} \int_\Omega |f(x)|^p w(x)dx$$

b) *There exists* C *such that, for every* $f \geq 0$ *on* Ω *and for every cube* Q *of* $\mathbb{R}^n$:

$$\left(\frac{1}{|Q|}\int_{Q\cap\Omega} f(x)dx\right)^p u(Q\cap\Omega) \leq C\int_{Q\cap\Omega} f(x)^p w(x)dx$$

c) *There exists* C *such that, for every cube* Q *of* $\mathbb{R}^n$:

$$\left(\frac{1}{|Q|}\int_{Q\cap\Omega} u(x)dx\right)\left(\frac{1}{|Q|}\int_{Q\cap\Omega} w(x)^{-1/(p-1)}dx\right)^{p-1} \leq C$$

with the usual interpretation for the second factor in case $p = 1$.

In particular, M_Ω is of weak type (p,p) with respect to w, if and only if $w \in A_{p,\Omega}$.

Here is the solution to the restriction problem for weights.

THEOREM 5.6. *Let* $w \geq 0$ *on* Ω *and* $1 \leq p < \infty$. *Then, the following conditions are equivalent:*

i) *There exists* $W \in A_p$ *(*W *defined on* $\mathbb{R}^n$*), such that* $W(x) = w(x)$ *for a.e.* $x \in \Omega$.

ii) *There is* $\varepsilon > 0$ *such that* $w(x)^{1+\varepsilon} \in A_{p,\Omega}$.

Proof: It is clear that i) implies ii). Indeed, if w is the restriction to Ω of $W \in A_p$, then there will be $W^{1+\varepsilon} \in A_p$ for some $\varepsilon > 0$. On restricting to Ω, we shall have $w^{1+\varepsilon} \in A_{p,\Omega}$. Next we want to show that ii) implies i). Let $p > 1$. Suppose that $w^{1+\varepsilon} \in A_{p,\Omega}$ for some $\varepsilon > 0$. Then, as in 1.14 b), we can find δ, q with $0 < \delta < \varepsilon$, $1 < q < p$, such that $w^{1+\delta} \in A_{q,\Omega}$. It follows that M_Ω is of weak type (q,q) in Ω with respect to $w^{1+\delta}$. Since it is also of type (∞,∞) (sets of measure zero are exactly the same as those for Lebesgue measure), it follow by interpolation that M_Ω is bounded in $L^p(w^{1+\delta})$ or, in other words, that $w^{1+\delta}$ belongs to the class W_p corresponding to the operator M_Ω. On the other hand, since $(w^{1+\varepsilon})^{-1/(p-1)} \in A_{p',\Omega}$, we shall have $(w^{1+\delta})^{-1/(p-1)} \in A_{r,\Omega}$ for some $1 < r < p'$. Then M_Ω is of weak type (r,r) with respect to $(w^{1+\delta})^{-1/(p-1)}$ and, by interpolation, $(w^{1+\delta})^{-1/(p-1)} \in W_{p'}$. Theorem 5.2 can be applied, obtaining $w^{1+\delta} = w_0 w_1^{1-p}$ with $w_0, w_1 \in W_1$, that is: $M_\Omega w_j \leq C w_j$, $j = 0,1$. We extend each w_j making it 0 outside of Ω and consider the function $M_j = M(w_j)$ defined on $\mathbb{R}^n$. Of course, for a.e. $x \in \Omega$, we have $w_j(x) \leq M_j(x) = M w_j(x) = M_\Omega w_j(x) \leq C w_j(x)$. Therefore,

for a.e. $x \in \Omega$, $M_j(x) \simeq w_j(x)$, so that $w(x)^{1+\delta} = $
$= M_o(x)M_1(x)^{1-p}\, k(x)$, where k is a positive function defined on Ω and bounded away from 0 and ∞. We can write

$$w(x) = M_o(x)^{1/(1+\delta)}\, (M_1(x)^{1/(1+\delta)})^{1-p}\, k(x)^{1/(1+\delta)}$$

Observe that, according to theorem 3.4 in Chapter II, the functions defined on $\mathbb{R}^n$, $M_j^{1/(1+\delta)} = M(w_j)^{1/(1+\delta)}$ are A_1 weights. If we define $K(x) = k(x)^{1/(1+\delta)}$ for $x \in \Omega$ and $K(x) = 1$ for $x \notin \Omega$, and write, for $x \in \mathbb{R}^n$ $W(x) = M_o(x)^{1/(1+\delta)}(M_1(x)^{1/(1+\delta)})^{1-p}K(x)$, we see that $W \in A_p$ since K is obviously bounded away from 0 and ∞. Clearly $W(x) = w(x)$ for a.e. $x \in \Omega$.

In the case $p = 1$, the proof is direct, without need of factorization. □

<u>COROLLARY 5.7</u>. *<u>For</u>* f *<u>defined on</u>* Ω, *<u>the following conditions are equivalent</u>:*

i) *<u>There exists</u>* F *<u>defined on</u>* $\mathbb{R}^n$, *<u>such that</u>* $F \in$ B.M.O. *<u>and</u>* $F(x) = f(x)$ *<u>for a.e.</u>* $x \in \Omega$.

ii) *<u>There exist</u>* $\lambda > 0$, $C > 0$ *<u>and, for every cube</u>* Q *<u>of</u>* $\mathbb{R}^n$ *<u>a constant</u>* C_Q, *<u>such that</u>:*

$$\frac{1}{|Q|}\int_{Q\cap\Omega} e^{\lambda|f(x)-C_Q|}\, dx \leq C.$$

<u>Proof</u>: That i) implies ii) follows immediately from Chapter II, corollary 3.10 ii).

Let us see that ii) implies i). It follows from ii), exactly as in corollary 2.18, that $e^{\lambda f} \in A_{2,\Omega}$. Taking $0 < \lambda_o < \lambda$, we have $(e^{\lambda_o f})^{1+\varepsilon} \in A_{2,\Omega}$ for some $\varepsilon > 0$. According to theorem 5.6, there will exist $W \in A_2$ such that $e^{\lambda_o f(x)} = W(x)$ for a.e. $x \in \Omega$. If we write $W = e^G$, we have, by corollary 2.19, that $G \in$ B.M.O. Thus $f(x) = \lambda_o^{-1}G(x)$ for a.e. $x \in \Omega$. This is i) with $F = \lambda_o^{-1}G$. □

It seems natural to denote by B.M.O. (Ω) the space of functions f satisfying condition ii) in corollary 5.7.

We can give a version of theorem 5.2 valid for some linear, not necessarily positive operators, provided $p = 2$. The case of the

Hilbert transform H, is particularly interesting.

We know from theorems 3.1 and 3.7 that, for $1 < p < \infty$, the class $W_p(H)$ associated to H as in theorem 5.2, coincides with the class A_p. Now we give the following:

DEFINITION 5.8. *We shall denote by* $W_1(H)$ *the class formed by those functions* $w(x) \geq 0$ *on* $\mathbb{R}$ *satisfying the following conditions:*

a) $0 < \int_{-\infty}^{\infty} \frac{w(x)}{1+x^2}\, dx < \infty$

b) *There exists* ϕ *such that* $|\phi(x)| \leq C\, w(x)$ *for a.e.* x *with* C *independent of* x, *and the function* $F(x+it) = P_t * (w+i\phi)(x)$ *is holomorphic in* $\mathbb{R}^2_+$.

It follows from b) that $P_t*\phi(x)$ is a conjugate harmonic function for $P_t*w(x)$. It can be seen (as in theorem 1.20 of Chapter I) that $P_t*\phi(y) \to \phi(x)$ (as $y+it \xrightarrow{N.T.} x$) for a.e. x, so that ϕ would deserve to be called the Hilbert transform of w. Actually for w satisfying a), Hw is always well defined as an equivalence class modulo constants, since P_t*w always has a harmonic conjugate and this harmonic conjugate has non-tangential boundary values because the corresponding holomorphic function F does. (This follows from the fact that $\operatorname{Re} F \geq 0$. We just need to consider the holomorphic function $e^{-F} = G$ which satisfies $|G| \leq 1$ and use Fatou's theorem 4.19 in Chapter II). We can see b) above as a concrete form of the statement $|Hw| \leq C\, w$.

Here is the factorization theorem for the Hilbert transform.

THEOREM 5.9. *Every* $w \in A_2$ *in* $\mathbb{R}$ *can be written as* $w = w_0 w_1^{-1}$ *with* $w_0, w_1 \in W_1(H)$.

Proof: For $u \in L^2(w)$, define

$$Su(x) = |H(u.w)(x)|\, w(x)^{-1} + |Hu(x)|$$

This makes perfectly good sense, since we know (theorem 3.1) that H is bounded both in $L^2(w)$ and in $L^2(w^{-1})$, and we have $u \in L^2(w)$ and $u.w \in L^2(w^{-1})$. Also, clearly, S is bounded in $L^2(w)$. By appealing to lemma 5.1, we conclude that there exists $v \in L^2(w)$ such that $Sv(x) \leq C\, v(x)$ for a.e. x. This implies $|Hv(x)| \leq Cv(x)$ and also $|H(v.w)(x)| \leq C\, v(x)w(x)$ for a.e. x. Note also that we have $v \in L^2(w)$ and $vw \in L^2(w^{-1})$ and, since w and w^{-1}

are A_2 weights, it follows that both v and vw are integrable against $(1+x^2)^{-1}$. Actually, more is true: they are integrable against $(1+|x|^\alpha)^{-1}$ for some $0 < \alpha < 1$. Indeed, for example:

$$\int_{-\infty}^{\infty} \frac{v(x)}{1+|x|^\alpha}\, dx \leqq \left(\int_{-\infty}^{\infty} v(x)^2 w(x)\,dx\right)^{1/2}\left(\int_{-\infty}^{\infty} \frac{w(x)^{-1}}{(1+|x|^\alpha)^2}\, dx\right)^{1/2} < \infty$$

if we take α close enough to 1 (see 3.2).

If we write $w_0 = vw$ and $w_1 = v$, we have $w = w_0 w_1^{-1}$ and we can see that both w_0 and w_1 belong to $W_1(H)$. The reason is that we have $|Hw_j(x)| \leqq C\, w_j(x)$ a.e. for $j=0,1$ and $P_t*(w_j+iHw_j)(x)$ is analytic as can be seen from the identity $P_t*(w_j+iHw_j)(x) = (P_t+iQ_t)*w_j(x)$. In fact, this identity is obvious for nice (say, Schwartz) functions, and, in order to extend it to arbitrary $f \in L^2(w)$ (in particular to $f = w_j$) it suffices to observe that, for fixed $x \in \mathbb{R}$ and $t > 0$, both $f \mapsto P_t * Hf(x)$ and $f \mapsto Q_t * f(x)$ are bounded linear functionals on $L^2(w)$, because $\|P_t(x-.)\|_{2,w} < \infty$ and $\|Q_t(x-.)\|_{2,w} < \infty$. □

Next we shall see that the functions belonging to $W_1(H)$ satisfy the counterpart in the line of the Helson-Szegö condition appearing in theorem 8.14 of Chapter I.

THEOREM 5.10. *Every* $w \in W_1(H)$ *can be written in the form* $w(x) = e^{u(x)+H v(x)}$ *for some* u *and* v *satisfying* $\|u\|_\infty < \infty$ *and* $\|v\|_\infty < \pi/2$.

Proof: Let $w \in W_1(H)$. According to definition 5.8 this implies that there is a function $F(x+it)$, holomorphic in the upper half plane, such that $F(x+it) = P_t*(w+i\phi)(x)$ with $|\phi(x)| \leqq C\, w(x)$ for a.e. x. Now $|P_t*\phi(x)| \leqq C\, P_t*w(x)$. Observe also that always $P_t*w(x) > 0$, and $|\mathrm{Arg}\, F(x+it)| = |\mathrm{arc\ tg}\, (P_t*\phi(x))(P_t*w(x))^{-1}| \leqq$ $\leqq \mathrm{arc\ tg}\, C < \pi/2$. It follows that we can write $F(x+it) = e^{G(x+it)}$ with G holomorphic such that $|\mathrm{Im}\, G| \leqq \mathrm{arc\ tg}\, C < \pi/2$. This condition, allied to Fatou's theorem, guarantees as usual the existence of boundary values $G(x)$ for a.e. x. We shall have, modulo constants, $G(x) = Hv(x) - iv(x)$ with $\|v\|_\infty \leqq \mathrm{arc\ tg}\, C < \pi/2$. On the other hand, since $|F(x)| = |w(x) + i\phi(x)|$, we have: $w(x) \leqq |F(x)| \leqq w(x) + |\phi(x)| \leqq (1 + C)w(x)$, so that we can write $w(x) = k_0(x)|F(x)| = k(x)e^{Hv(x)}$ where k and k_0 are >0 and bounded away from 0 and ∞. After writing $u = \log k$, we get what we wanted. □

By combining theorem 5.9 with theorem 5.10. we get a weak form of the hard part of the Helson-Szegö theorem, namely:

THEOREM 5.11. *Every* $w \in A_2$ *in* $\mathbb{R}$ *can be written as* $w(x) = e^{u(x)+Hv(x)}$ *for some* u *and* v *satisfying* $\|u\|_\infty < \infty$ *and* $\|v\|_\infty < \pi$.

Proof: After theorem 5.9, we can write $w = w_0 w_1^{-1}$ with $w_0, w_1 \in W_1(H)$. Then, using theorem 5.10, we write

$$w_0(x) = e^{u_0(x)+Hv_0(x)}$$

and

$$w_1(x) = e^{u_1(x)+Hv_1(x)}$$

with $\|u_j\|_\infty < \infty$ and $\|v_j\|_\infty < \pi/2$ for $j=0,1$. It follows that $w = w_0 w_1^{-1} = e^{u_0-u_1+H(v_0-v_1)} = e^{u+Hv}$ with $\|u\|_\infty = \|u_0-u_1\|_\infty < \infty$ and $\|v\|_\infty = \|v_0-v_1\|_\infty \leqq \|v_0\|_\infty + \|v_1\|_\infty < \pi$. □

As we said, this is a weak form of the Helson-Szegö theorem because we are not able to get the bound $\|v\|_\infty < \pi/2$ and we have to content ourselves with the worse estimate $\|v\|_\infty < \pi$. However, we obtain as a consequence

COROLLARY 5.12. $\text{B.M.O.}(\mathbb{R}) = L^\infty(\mathbb{R}) + HL^\infty(\mathbb{R})$. *More concretely, every* $f \in \text{B.M.O.}(\mathbb{R})$ *can be written as* $f(x) = c + \phi(x) + H\psi(x)$ *with* c *a constant and* $\|\phi\|_\infty + \|\psi\|_\infty \leqq C\,\|f\|_*$ *where* C *is an absolute constant*.

Proof: Let $f \in \text{B.M.O.}(\mathbb{R})$. Then, as we explained in the proof of corollary 5.4.(b), taking $\lambda = C/\|f\|_*$ with C a certain absolute constant, the function $w(x) = e^{\lambda f(x)}$ is in A_2 with an A_2 constant independent of f. Then, keeping track of the constants in the proofs of theorems 5.9 and 5.10, we realize that we can write

$$w(x) = K\, e^{u(x) + Hv(x)}$$

with $K > 0$, $\|u\|_\infty \leqq C$, an absolute constant, and $\|v\|_\infty < \pi$. This gives

$$f(x) = c + \phi(x) + H\psi(x), \quad \text{where}$$

$c = (\log K)\,\|f\|_*/C$, $\phi(x) = u(x)\,\|f\|_*/C$ and $\psi(x) = v(x)\,\|f\|_*/C$. Clearly $\|\phi\|_\infty + \|\psi\|_\infty \leqq C\,\|f\|_*$ for some absolute constant C.

We already know (Proposition 5.15 in Chapter II) that, conversely, $L^\infty + HL^\infty \subset \text{B.M.O.}$ with the right estimate for the B.M.O.norm. □

We have established the identity between these two Banach spaces:

$$\text{B.M.O.} = (L^\infty + HL^\infty)/\mathbb{R}$$

the norm in the right hand side being the quotient norm corresponding to the seminorm

$$\inf\{\|\phi\|_\infty + \|\psi\|_\infty : f = \phi + H\psi\}$$

Now the space $(L^\infty(\mathbb{R}) + HL^\infty(\mathbb{R}))/\mathbb{R}$ is naturally identified with the dual of $H^1(\mathbb{R})$, by doing exactly the same that we did for the torus at the beginning of section 9 in Chapter I. Thus, corollary 5.12 gives yet another proof (the third one!) of the duality theorem $(H^1)^* = \text{B.M.O.}$ Observe that the proof of the decomposition in corollary 5.12 is a constructive one, whereas the two previous proofs of the duality theorem (sections I.9 and III.5), do not give an explicit construction of the decomposition.

COROLLARY 5.13. (*C. Fefferman's duality theorem*). $(H^1(\mathbb{R}))^* = \text{B.M.O.}(\mathbb{R})$.

In Chapter I we have proved the Helson-Szegö theorem only for the torus. However, the theorem also works for the real line substituting the Hilbert transform for the conjugate function. The proof is not essentially different, but some changes have to be made due to the unboundedness of the domain. By using this Helson-Szegö theorem, we can prove a converse of theorem 5.19, namely: if $\|v\|_\infty < \pi/2$ then $e^{Hv(x)} = k(x)w(x)$, where $k > 0$ is bounded away from 0 and ∞ and $w \in W_1(H)$. Indeed, the Helson-Szegö theorem implies that $e^{Hv} \in A_2$, so that it has the right integrability (condition a) in definition 5.8). Besides, $e^{Hv-iv} = e^{Hv}(\cos v - i \sin v)$ is the boundary function of a holomorphic function F in the upper half plane, and it can be shown that $F(x+it) = P_t*(e^{Hv-iv})(x)$. Thus, $F(x+it) = P_t*(w+i\phi)(x)$ where $w(x) = e^{Hv(x)} \cos v(x)$ and $\phi(x) = -e^{Hv(x)} \sin v(x)$. But, since $\|v\|_\infty < \pi/2$, we have: $0 < \cos \|v\|_\infty \leqq \cos v(x) \leqq 1$, so that if we write $k(x) = 1/\cos v(x)$, k is bounded away from 0 and ∞. Also $|\phi(x)| \leqq e^{Hv(x)} \leqq C\, w(x)$ with $C = 1/\cos \|v\|_\infty$. We have obtained $e^{Hv(x)} = k(x)w(x)$ with $k > 0$ bounded away from 0 and ∞ and $w \in W_1(H)$.

What we have is a characterization of $W_1(H)$. Using the Helson-Szegö theorem once more, we can write: $\exp(L^\infty).W_1(H) = \exp(L^\infty).A_2$. In other words, the functions belonging to $W_1(H)$ coincide with those in A_2 up to multiplication by functions bounded away from 0 and ∞. This can also be written as $W_1(H) \sim A_2$.

As a final application of the factorization theorem we shall give a theorem that relates the fact that w belongs to A_2 with the fact that $\log w$ is close to L^∞ in the metric of B.M.O.

THEOREM 5.14. *Let* ϕ *be a function on* $\mathbb{R}^n$. *Then, there exist constants* C_1 *and* C_2, *depending only on the dimension* n, *such that:*

a) $e^\phi \in A_2$ *provided* $\inf\{\|\phi-g\|_* : g \in L^\infty\} \leqq C_1$

b) $\inf\{\|\phi-g\|_* : g \in L^\infty\} \leqq C_2$ *whenever* $e^\phi \in A_2$.

Proof: a) If $\inf\{\|\phi-g\|_* : g \in L^\infty\} < C$, we shall be able to find $g \in L^\infty$, such that $\phi = g+h$ with $\|h\|_* < C$. Now if C is small enough, it follows from the John-Nirenberg theorem (corollary 3.10 in Chapter II) plus the characterization of A_2 weights in corollary 2.18, that $e^h \in A_2$. Then $e^\phi = e^g e^h \in A_2$.

If $n = 1$, part b) can be derived from the Helson-Szegö theorem. The weak version contained in theorem 5.11 suffices. Indeed, if $e^\phi \in A_2$, we shall be able to write $e^\phi = e^{u+Hv}$ with $\|u\|_\infty < \infty$ and $\|v\|_\infty < \pi$. Thus $\|\phi-u\|_* = \|Hv\|_* \leqq C\,\|v\|_\infty < C\pi$, so that $\inf\{\|\phi-g\|_* : g \in L^\infty\} \leqq C\pi = C_2$.

In the general case, part b) can be derived from corollary 5.4. (a). Indeed, if $e^\phi \in A_2$ we have: $\phi(x) = b(x) + h(x)$, where $b(x) = \log k(x)$ is a bounded function, and $h(x) = \alpha \log Mf(x) - \beta \log Mg(x)$ is a B.M.O. function with $\|h\|_* \leqq C$ independent of ϕ. Therefore

$$\inf\{\|\phi-g\|_* : g \in L^\infty\} \leqq \|h\|_* \leqq C = C_2. \qquad \square$$

Another way to formulate theorem 5.14 is the following:

COROLLARY 5.15. *(Garnett-Jones theorem). For* $f \in$ B.M.O., *we write* $A(f) = \sup\{\lambda > 0 : e^{\lambda f} \in A_2\}$. *Then*

$$\underset{\text{B.M.O.}}{\operatorname{dist}}(f,L^\infty) = \inf\{\|f-g\|_* : g \in L^\infty\} \sim A(f)^{-1}.$$

Proof: C_1 and C_2 will have the same meaning as in theorem 5.14, and "dist" will mean the distance in B.M.O.

a) Let $\lambda = C_1/\operatorname{dist}(f,L^\infty)$. Then $\operatorname{dist}(\lambda f,L^\infty) = \lambda \operatorname{dist}(f,L^\infty) = C_1$, so that, according to part a) of the theorem, $e^{\lambda f} \in A_2$ or, in the notation introduced in the statement: $\lambda \leqq A(f)$, which is the same as saying: $A(f)^{-1} \leqq C_1^{-1} \operatorname{dist}(f,L^\infty)$.

b) Let $0 < \lambda < A(f)$. Then $e^{\lambda f} \in A_2$, so that according to part b) of the theorem:

λ dist $(f, L^\infty) = \text{dist}(\lambda f, L^\infty) \leqq C_2$. It follows that
dist $(f, L^\infty) \leqq C_2\, A(f)^{-1}$. □

We shall finish the section by presenting an extrapolation theorem for the classes A_p. The key to this extrapolation theorem is the following.

LEMMA 5.16. *Suppose* $w \in A_p$ *for a given* $1 < p < \infty$, *and let* $0 < t \leqq 1$. *Consider the operator*

$$S(u) = (M(|u|^{1/t}\, w).w^{-1})^t$$

Then:

a) S *is bounded in* $L^{p'/t}(w)$ *with norm depending only upon the* A_p-*constant for* w.

b) *For every function* $u \geqq 0$ *belonging to* $L^{p'/t}(w)$, *the pair* $(uw, S(u)w)$ *belongs to the class* A_r *of pairs of weights for* $r = (1-t)p+t$, *with an* A_r-*constant for the pair which depends exclusively upon the* A_p-*constant for* w.

Proof:

a) $\int S(u)^{p'/t} w = \int M(|u|^{1/t} w)^{p'} w^{1-p'} \leqq C \int |u|^{p'/t} w$ since $w^{1-p'} = w^{-1/(p-1)} \in A_{p'}$ with $A_{p'}$-constant depending only upon the A_p-constant for w.

b) The case $t = 1$ is clear, since then $r = 1$ and $S(u)w = M(uw)$.

Now let $0 < t < 1$. Observe that $r-1 = (1-t)(p-1)$. For a cube Q, we have to estimate:

$$\left(\frac{1}{|Q|}\int_Q uw\right)\left(\frac{1}{|Q|}\int_Q (S(u)w)^{-1/(r-1)}\right)^{r-1} =$$

$$= \left(\frac{1}{|Q|}\int_Q uw\right)\left(\frac{1}{|Q|}\int_Q M(u^{1/t}w)^{-t/((1-t)(p-1))}\, w^{-1/(p-1)}\right)^{(1-t)(p-1)}$$

On the second factor we use the fact that

$$M(u^{1/t}w) \geqq \frac{1}{|Q|}\int_Q u^{1/t}\, w$$

On the first, after writing $w = w^t w^{1-t}$, we use Hölder's inequality with exponents $1/t$ and $(1/t)' = 1/(1-t)$. We get:

$$\left(\frac{1}{|Q|}\int_Q u\,w\right)\left(\frac{1}{|Q|}\int_Q (S(u)w)^{-1/(r-1)}\right)^{r-1} \leq$$

$$\leq \left(\frac{1}{|Q|}\int_Q u^{1/t}w\right)^t\left(\frac{1}{|Q|}\int_Q w\right)^{1-t}\left(\frac{1}{|Q|}\int_Q u^{1/t}w\right)^{-t}\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\right)^{(1-t)(p-1)} \leq$$

$$\leq C^{1-t}$$

if C is the A_p-constant for w. □

Combining lemma 5.16 with lemma 5.1, we obtain

LEMMA 5.17. *Let* w, p *ant* t *be as in lemma 5.16. Then, for every non-negative* $u \in L^{p'/t}(w)$, *there exists a non-negative* $v \in L^{p'/t}(w)$ *such that:*

a) $u(x) \leq v(x)$ *for a.e.* x

b) $\|v\|_{p'/t,w} \leq 2\,\|u\|_{p'/t,w}$

c) $v.w \in A_r$ *for* $r = (1-t)p+t$, *with an* A_r*-constant depending only on the* A_p*-constant for* w.

Proof: It is enough to apply lemma 5.1 to the operator S defined in lemma 5.16. Given $u \geq 0$ belonging to $L^{p'/t}(w)$, we have v satisfying a) and b) above and, besides, $Sv(x) \leq C\,v(x)$ with $C = 2\,\|S\|$. We know that $(vw, S(v)w) \in A_r$ with an A_r-constant that depends only upon the A_p-constant for w. Combining this with the fact that $Sv \leq Cv$, with C depending only upon the A_p-constant for w, we obtain $v.w \in A_r$ with an A_r-constant depending only on the A_p-constant for w. □

The extrapolation theorem will follow immediately from the next lemma, which is little more than a reformulation of lemma 5.17.

LEMMA 5.18. *Suppose* $1 < p < \infty$, $1 \leq r < \infty$, $r \neq p$. *Denote by* s *the exponent given by* $\frac{1}{s} = \left|1 - \frac{r}{p}\right|$. *Let* $w \in A_p$. *Then, for every* $u \geq 0$ *in* $L^s(w)$, *there exists a* $v \geq 0$ *in* $L^s(w)$, *such that:*

a) $u(x) \leq v(x)$ *for a.e.* x

b) $\|v\|_{s,w} \leq C\,\|u\|_{s,w}$, *with* C *an absolute constant*

c) *If* $r < p$, $vw \in A_r$, *and if* $p < r$, $v^{-1}w \in A_r$, *with* A_r*-constants that, in both cases, depend only on the* A_p*-constant for* w.

Proof: 1) If $r < p$, then $r = (1-t)p+t$ for some $0 < t \leq 1$, and $1/s = 1 - (r/p) = 1 - (1 - t+(t/p)) = t/p'$, so that $s = p'/t$. Therefore, in this case, the result coincides with lemma 5.17.

2) Let $p < r$. Then $1/s = (r/p) - 1 = (r-p)/p$, that is: $s = p/(r-p)$. Now $r' < p'$, and if we write $1/s^* = |1 - (r'/p')|$, we see that $s^* = s(r-1)$. Since $w \in A_p$, we have $w^{1-p'} \in A_{p'}$. We are in a position to apply the first part of the lemma already proved, with p' in place of p, r' in place of r, and $w^{1-p'}$ in place of w. If $u \geq 0$ belongs to $L^s(w)$, we shall have $u_o = u^{s/s^*} w^{p'/s^*} \in L^{s^*}(w^{1-p'})$. Therefore, there will exist $v_o \in L^{s^*}(w^{1-p'})$ with $u_o \leq v_o$, $\|v_o\| \leq C \|u_o\|$ and, besides, $v_o w^{1-p'} \in A_{r'}$. If we write $v_o = v^{s/s^*} w^{p'/s^*}$, that is, if we define $v = v_o^{s^*/s} w^{-p'/s}$, we see that $u \leq v$, $\|v\|_{s,w} \leq C\|u\|_{s,w}$, and besides:

$$v^{-1}w = v_o^{-s^*/s} w^{1+(p'/s)} = v_o^{1-r} w^{(r-1)/(p-1)} =$$

$$= (v_o w^{-1/(p-1)})^{1-r} = (v_o w^{1-p'})^{-1/(r'-1)} \in A_r. \quad \square$$

Here is, finally, the extrapolation theorem

THEOREM 5.19. *Let* T *be a sublinear operator. Let* $1 \leq r < \infty$, $1 < p < \infty$. *Suppose that* T *is bounded in* $L^r(w)$ *(respectively of weak type* (r,r) *with respect to* w*) for every weight* $w \in A_r$, *with a norm that depends only upon the* A_r*-constant for* w. *Then, for every* $w \in A_p$, T *is bounded in* $L^p(w)$ *(respectively,* T *is of weak-type* (p,p) *with respect to* w*), with a norm that depends only upon the* A_p*-constant for* w.

Proof: We start by dealing with the case in which T is assumed to be of strong type. There are two possibilities.

1) Let $r < p$. Take $w \in A_p$. We are going to apply lemma 5.18, observing that, with the notation used in that lemma, $1/s = 1-(r/p)$, so that $s = (p/r)'$. We have

$$\left(\int |Tf(x)|^p w(x)dx\right)^{r/p} = \| |Tf|^r \|_{s',w} = \int |Tf(x)|^r u(x)w(x)dx$$

for some $u \in L^s(w)$, such that $u \geq 0$ and $\|u\|_{s,w} = 1$. If we take the $v \in L^s(w)$ associated to u as in lemma 5.18, we shall have:

$$\left(\int |Tf(x)|^p w(x)dx\right)^{r/p} \leq \int |Tf(x)|^r v(x)w(x)dx \leq$$

$$\leq C \int |f(x)|^r \ v(x)w(x)dx \leq$$

$$\leq C(\int (|f(x)|^r)^{p/r} w(x)dx)^{r/p} (\int \ v(x)^{(p/r)'} w(x)dx)^{1/(p/r)'} =$$

$$= C \|v\|_{s,w} \ \|f\|^r_{p,w} \leq C' \ \|f\|^r_{p,w}$$

Observe that the constant C' depends only on the A_p-constant for w.

2) Now let $p < r$. Take $w \in A_p$. If $0 < \alpha < 1$, it follows from Hölder's inequality that $\int |f|^\alpha \leq (\int |f|u^{-1})^\alpha (\int u^{\alpha/(1-\alpha)})^{1-\alpha}$, and we know that, actually, the left hand side equals the minimum of the right hand side over all possible choices of u, that is:

$$\|f\|_\alpha = \inf \{\int |f|u^{-1} : \|u\|_{\alpha/(1-\alpha)} = 1\} = \int |f|u^{-1}$$

for a certain $u \geq 0$, such that $\|u\|_{\alpha/(1-\alpha)} = 1$.

In our case, we have

$$(\int |f(x)|^p w(x)dx)^{r/p} = \| |f|^r \|_{p/r,w} = \int |f(x)|^r u(x)^{-1} w(x)dx$$

for some $u \geq 0$ with $\|u\|_{p/(r-p),w} = 1$.

Observe that, with the notation of lemma 5.18 $1/s = (r/p)-1 = (r-p)/p$, so that $s = p/(r-p)$. Thus, given $f \in L^p(w)$, we have

$$\|f\|^r_{p,w} = \int |f(x)|^r \ u(x)^{-1} \ w(x)dx$$

for some $u \geq 0$ with $\|u\|_{s,w} = 1$. Now consider the function v associated to u as in lemma 5.18. We can write:

$$(\int |Tf(x)|^p w(x)dx)^{r/p} = \| |Tf|^r \|_{p/r,w} \leq$$

$$\leq \int |Tf(x)|^r v(x)^{-1} w(x)dx . \|v\|_{s,w} \leq C \int |f(x)|^r v(x)^{-1} w(x)dx \leq$$

$$\leq C \int |f(x)|^r u(x)^{-1} w(x)dx = C \|f\|^r_{p,w}$$

Observe that the constant C depends only on the A_p-constant for w.

When we just assume that T is of weak type, we only need a minor modification in the two arguments given above. For example, let $r < p$ and $w \in A_p$. Given $f \in L^p(w)$, write, for each $\lambda > 0$:

$$E_\lambda = \{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}$$

Then $(w(E_\lambda))^{r/p} = \|\chi_{E_\lambda}\|^r_{p,w} = \int \chi_{E_\lambda}(x)u(x)w(x)dx$

for some $u \in L^s(w)$ such that $u \geq 0$ and $\|u\|_{s,w} = 1$. Taking the corresponding v, we have:

$$w(E_\lambda)^{r/p} \leq \int \chi_{E_\lambda}(x)v(x)w(x)dx \leq C\lambda^{-r}\int |f(x)|^r v(x)w(x)dx \leq$$
$$\leq C\,\lambda^{-r}\,\|f\|_{p,w}^r$$

Thus $w(E_\lambda) \leq C\,\lambda^{-p}\,\|f\|_{p,w}^p$, and T is of weak-type (p,p) with respect to w as we wanted to prove.

The same modification applies when $p < r$. □

We may strengthen theorem 5.19 by combining it with the Marcinkiewicz interpolation theorem (theorem 2.11 in Chapter II). We get the following

COROLLARY 5.20. *Let* T *be a sublinear operator. Let* $1 \leq r < \infty$, $1 < p < \infty$. *Suppose that* T *is of weak-type* (r,r) *with respect to* w *for every weight* $w \in A_r$, *with a norm that depends only upon the* A_r*-constant for* w. *Then, for every* $w \in A_p$, T *is bounded in* $L^p(w)$, *with a norm that depends only upon the* A_p*-constant for* w.

Proof: Let $w \in A_p$. Then we know (theorem 2.6) that $w \in A_{p-\varepsilon}$ for some $\varepsilon > 0$. The ε and the $A_{p-\varepsilon}$-constant for w depend only upon the A_p-constant for w. Also $w \in A_q$ for every $q > p$. Now, applying theorem 5.19, we get that T is of weak-type $(p-\varepsilon,p-\varepsilon)$ with respect to w and also of weak type (q,q) with respect to w for each $q > p$. The norms depend only upon the A_p-constant for w. Then, by applying the Marcinkiewicz interpolation theorem, we conclude that T is of strong type (p,p) with respect to w (that is, bounded in $L^p(w)$), with norm depending only upon the A_p-constant for w. □

6. WEIGHTS IN PRODUCT SPACES

The operators considered so far in this chapter, as well as in previous chapters, were associated to the one-parameter family of dilations: $x \mapsto tx$, $t > 0$, in the sense that the conditions satisfied by them were not affected by such a change of scale in $\mathbb{R}^n$. In some cases, the operators themselves were invariant under these dilations, i.e.

$$T(f^t) = (T\,f)^t \quad \text{where} \quad f^t(x) = f(tx), \quad t > 0$$

(e.g., the Hardy-Littlewood maximal function or a homogeneous singular integral operator). The classes of weights associated to such operators were, accordingly, dilation invariant: If $w \in A_p$, then also $w^t \in A_p$ with the same A_p constant as w.

Now, we shall study operators and weights which are invariant under the n-parameter dilations in $\mathbb{R}^n$, defined by

$$x \mapsto \delta(x) = (\delta_1 x_1, \delta_2 x_2, \dots, \delta_n x_n)$$

with $\delta = (\delta_1, \delta_2, \dots, \delta_n) \in (\mathbb{R}_+)^n$. Here are the simplest examples of such operators:

i) The strong maximal function

$$M_S f(x) = \sup_{x \in R \in \mathcal{R}} \frac{1}{|R|} \int_R |f(y)| dy$$

where $\mathcal{R}$ denotes the family of bounded n-dimensional intervals $R = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$.

ii) The multiple Hilbert transform, defined by

$$\mathcal{H} f(x) = \lim_{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \to 0} \frac{1}{\pi^n} \int_{|x_j - y_j| > \varepsilon_j} \frac{f(y) dy}{(x_1 - y_1)(x_2 - y_2) \dots (x_n - y_n)}$$

or, in terms of Fourier transforms: $(\mathcal{H} f)\hat{}(\xi) =$
$= (-i)^n \operatorname{sign}(\xi_1 \xi_2 \dots \xi_n) \hat{f}(\xi)$.

iii) The partial sum operators

$$S_I f(x) = \int_I \hat{f}(\xi) e^{2\pi i x . \xi} d\xi$$

where I is a (not necessarily bounded) n-dimensional interval. These are the analogues in $\mathbb{R}^n$ of the partial sum operators in $\mathbb{R}$ studied in 3.12.

For an arbitrary function f and $\delta = (\delta_1, \delta_2, \dots, \delta_n) \in (\mathbb{R}_+)^n$, we shall denote by f^δ the function

$$f^\delta(x) = f(\delta(x)) = f(\delta_1 x_1, \delta_2 x_2, \dots, \delta_n x_n)$$

Then the following identities hold for the operators just described:

$$M_S(f^\delta) = (M_S f)^\delta; \quad \mathcal{H}(f^\delta) = (\mathcal{H} f)^\delta; \quad S_{\delta(I)}(f^\delta) = (S_I f)^\delta$$

The boundedness of these operators in $L^p(\mathbb{R}^n)$ follows by writing them as products of one-dimensional operators. Given an operator T acting on functions in $\mathbb{R}$, we denote by T^j, $j = 1, 2, \dots, n$, the operator defined on functions in $\mathbb{R}^n$ by letting T act on the j-th variable while keeping the remaining variables fixed, namely

$$T^j f(x) = T(f(x_1, x_2, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n))(x_j)$$

If $T = T_m$ is a multiplier operator, with $m \in L^\infty(\mathbb{R})$, it is easy to verify that

$$(T^j f)\hat{\ }(\xi_1,\xi_2,\dots,\xi_n) = \hat{f}(\xi_1,\xi_2,\dots,\xi_n)m(\xi_j)$$

In particular, we have

$$Hf(x) = H^1 \circ H^2 \circ \dots \circ H^n f(x)$$

$$S_I f(x) = S^1_{I_1} \circ S^2_{I_2} \circ \dots \circ S^n_{I_n} f(x)$$

where H is the (ordinary) Hilbert transform and $I = I_1 x I_2 x \dots x I_n$, with I_j intervals in $\mathbb{R}$. Also, if M denotes the one-dimensional Hardy-Littlewood operator, then

$$M^j f(x_1,x_2,\dots,x_n) = \sup_{a<x_j<b} \frac{1}{b-a}\int_a^b |f(x_1,x_2,\dots,x_{j-1},y,x_{j+1},\dots,x_n)|dy$$

and a moment's thought shows that

$$M_S f(x) \le M^1 \circ M^2 \circ \dots \circ M^n f(x)$$

On the other hand, we make the following simple observation based on Fubini's theorem: If T is a bounded operator in $L^p(\mathbb{R})$ with norm $\|T\|$, then

$$\int_{\mathbb{R}^n} |T^1 f(x)|^p dx = \int_{\mathbb{R}^{n-1}} \{\int_{\mathbb{R}} |T(f(.,x_2,\dots,x_n))(x_1)|^p\, dx_1\} dx_2 \dots$$
$$\dots dx_n \le \|T\|^p \int_{\mathbb{R}^n} |f(x)|^p dx$$

and similarly for $T^2,\dots,T^n$. From the identities for H, S_I and the inequality for M_S previously displayed, it is now evident that, for all $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$, we have

$$\|M_S f\|_p \le C_p \|f\|_p$$

$$\|Hf\|_p \le C_p \|f\|_p$$

$$\|S_I f\|_p \le C_p \|f\|_p \qquad (C_p \text{ independent of } I)$$

There is no weak type (1,1) inequality for these operators. For example, letting $f(x_1,x_2) = P_1(x_1)P_2(x_2) = \frac{1}{\pi^2(x_1^2+1)(x_2^2+1)}$, we have $Hf(x_1,x_2) = \frac{x_1\, x_2}{\pi^2(x_1^2+1)(x_2^2+1)}$, which is not in weak-$L^1(\mathbb{R}^2)$. By using the estimates near L^1 for the Hardy-Littlewood maximal function (see Chapter II, 2.3) one can prove that M_S is bounded from

$L(\log L)^{n-1}(\mathbb{R}^n)$ to weak-$L^1(\mathbb{R}^n)$. However, we are rather interested in L^p estimates, and our next objective will be the analogues in $L^p(w)$, for suitable weights $w(x)$ in $\mathbb{R}^n$, of the above inequalities. Our previous experience with A_p weights suggets that the condition similar to A_p with n-dimensional intervals instead of cubes must be necessary, while the method used in the unweighted case indicates that we can easily deal with weights $w(x)$ such that $w(x_1,x_2,\dots,x_{j-1},\dots,x_{j+1},\dots,x_n) \in A_p(\mathbb{R})$ uniformly in $(x_1,x_2,\dots,x_{j-1},x_{j+1},\dots,x_n)$. In order to give a precise meaning to this, we establish the following:

NOTATION: *Given* $v(x) \geq 0$ *in* $\mathbb{R}^n$, $n \geq 1$ *we denote by* $[v]_{A_p(\mathbb{R}^n)}$ *the* A_p*-constant for* v *(which can be* $+\infty$*), namely*

$$[v]_{A_p(\mathbb{R}^n)} = \sup_Q \{(\frac{1}{|Q|}\int_Q v(x)dx)(\frac{1}{|Q|}\int_Q v(x)^{-1/(p-1)})^{p-1}\}$$

if $1 < p < \infty$, *and*

$$[v]_{A_1(\mathbb{R}^n)} = \sup_Q \{(\frac{1}{|Q|}\int_Q v(x)dx)\ \|v^{-1}\chi_Q\|_\infty\} = \|\frac{Mv}{v}\|_\infty$$

$$[v]_{A_\infty(\mathbb{R}^n)} = \sup_Q \{(\frac{1}{|Q|}\int_Q v(x)dx)\ \exp(-\frac{1}{|Q|}\int_Q \log v(x)dx)\}$$

The last definition is, of course, motivated by (2.15). As usual, Q stands for an arbitrary cube (interval if $n = 1$) in $\mathbb{R}^n$.

DEFINITION 6.1. *Let* $w(x)$ *be a weight in* $\mathbb{R}^n$. *For* $1 \leq p \leq \infty$, *we say that* $w \in A_p^*$ *if* $w^\delta \in A_p$ *uniformly in* $\delta = (\delta_1,\delta_2,\dots,\delta_n) \in (\mathbb{R}_+)^n$, *i.e.*, $\sup_\delta [w^\delta]_{A_p(\mathbb{R}^n)} < \infty$.

Since dilations by $\delta = (\delta_1,\delta_2,\dots,\delta_n) \in (\mathbb{R}_+)^n$ transform cubes in arbitrary n-dimensional intervals, a change of variables proves that, for $1 < p < \infty$, $w \in A_p^*$ if and only if

$$(\frac{1}{|R|}\int_R w(x)dx)(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}dx)^{p-1} \leq C \qquad (R \in \mathcal{R})$$

and similarly for $p = 1$, $p = \infty$ (see theorems 6.5 and 6.7 below). Thus, A_p^* is the A_p class naturally associated with n-dimensional intervals.

THEOREM 6.2. *Given a weight* $w(x)$ *in* $\mathbb{R}^n$ *and* $1 < p < \infty$, *the following statements are equivalent:*

a) $w \in A_p^*$

b) *There exists* $C > 0$ *such that, for each* $j = 1,2,\dots,n$ *we have*

$$[w(x_1,x_2,\dots,x_{j-1},.,x_{j+1},\dots,x_n)]_{A_p(\mathbb{R})} \leq C$$

for almost every $(x_1,x_2,\dots,x_{j-1},x_{j+1},\dots,x_n) \in \mathbb{R}^{n-1}$

c) M_S *is a bounded operator in* $L^p(w)$

d) *The partial sum operators* (S_I) *are uniformly bounded in* $L^p(w)$

e) H *is a bounded operator in* $L^p(w)$

Proof: For notational simplicity, we shall prove the theorem for $n = 2$, since this case contains all the essential difficulties of the general situation. The mutual implications of statements a)-e) will be established according to the following diagram

$$\begin{tikzcd} & (a) \arrow[d, Rightarrow] & (e) \arrow[l, Rightarrow] \\ (c) \arrow[ur, Rightarrow] & (b) \arrow[l, Rightarrow] \arrow[r, Rightarrow] & (d) \arrow[u, Rightarrow] \end{tikzcd}$$

(a) implies (b): Given $w(x_1,x_2)$ in A_p^*, for each interval J in $\mathbb{R}$ we define

$$m_J(x_1) = \{\frac{1}{|J|}\int_J w(x_1,x_2)\,dx_2\}^{1/p}\{\frac{1}{|J|}\int_J w(x_1,x_2)^{-1/(p-1)}dx_2\}^{1/p'}$$

Now, given another interval I, we average $m_J(x_1)$ over I and apply Hölder's inequality

$$\frac{1}{|I|}\int_I m_J(x_1)\,dx_1 \leq (\frac{1}{|R|}\int_R w(x)\,dx)^{1/p}(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}dx)^{1/p'} \leq$$
$$\leq C^{1/p}$$

where $R = I \times J$ and C is the A_p^*-constant for w. Letting $I = (x_1-\delta,\ x_1+\delta)$ and $\delta \to 0$, we obtain

$$m_J(x_1)^p \leq C \qquad \text{for a.e. } x_1 \in \mathbb{R}$$

But $[w(x_1,.)]_{A_p(\mathbb{R})} = \sup_J m_J(x_1)^p$, and it is enough to consider the countable family of intervals J with rational endpoints. This proves $[w(x_1,.)]_{A_p} \leq C$ (a.e. $x_1 \in \mathbb{R}$), and similarly, $[w(.,x_2)]_{A_p(\mathbb{R})} \leq C$ (a.e. $x_2 \in \mathbb{R}$).

<u>(b) implies (c)</u>: By Muckenhoupt's theorem 2.8 applied to the maximal operator M in $\mathbb{R}$, the hypothesis implies

$$\int_{-\infty}^{\infty} (M^1 g(x_1,x_2))^p w(x_1,x_2)dx_1 \leq C \int_{-\infty}^{\infty} |g(x_1,x_2)|^p w(x_1,x_2)dx_1$$

and

$$\int_{-\infty}^{\infty} (M^2 f(x_1,x_2))^p w(x_1,x_2)dx_2 \leq C \int_{-\infty}^{\infty} |f(x_1,x_2)|^p w(x_1,x_2)dx_2$$

for almost every x_1 and x_2 in $\mathbb{R}$. Now, we only have to apply both inequalities consecutively with $g = M^2 f$, taking into account that $M_S f \leq M^1(M^2 f)$.

<u>(c) implies (a)</u>: For every function $f \geq 0$ and $R \in \mathcal{R}$, $M_S f \geq$ $\geq (\frac{1}{|R|}\int_R f)\chi_R$. Thus, from (c) we get

$$(\frac{1}{|R|}\int_R f(x)dx)^p \int_R w(x)dx \leq C \int_R |f(x)|^p w(x)dx \qquad (R \in \mathcal{R})$$

Taking $f = w^{-1/(p-1)}\chi_R$ (or an increasing sequence of bounded functions in R converging to $w^{-1/(p-1)}$) it follows that $w \in A_p^*$.

<u>(b) implies (d)</u>: It is proved exactly as (b) $\Longrightarrow$ (c). Given $I = I_1 \times I_2$, we know that $S_I f = S^1_{I_1}(S^2_{I_2} f)$, and the conditions on w together with Corollary 3.12 imply that both $S^1_{I_1}$ and $S^2_{I_2}$ are bounded in $L^p(w)$ with norms independent of I_1, I_2!

<u>(d) implies (e)</u>: Take $I = [0,\infty) \times [0,\infty)$ and $\tilde{I} = [0,\infty) \times (-\infty,0]$. Then

$$Hf = -(S_I f + S_{-I} f) + (S_{\tilde{I}} f + S_{-\tilde{I}} f)$$

and therefore $\|Hf\|_{p,w} \leq C \|f\|_{p,w}$ for all $f \in L^p(w)$.

<u>(e) implies (a)</u>: Given $R = [a,a+h] \times [b, b+\ell]$, consider the rectangle $\tilde{R} = [a-h,a] \times [b-\ell,b]$ which is symmetric to R with respect to the common vertex (a,b). Take $f \geq 0$ supported in R. Then, for every $x = (x_1,x_2) \in \tilde{R}$

$$Hf(x) = \frac{1}{\pi^2}\int_R \frac{f(y)dy}{(y_1-x_1)(y_2-x_2)} \geq \frac{1}{\pi^2}\int_R \frac{f(y)dy}{2h.2\ell} = \frac{1}{4\pi^2|R|}\int_R f$$

Using the fact that H is bounded in $L^p(w)$

$$(6.3) \qquad (\frac{1}{|R|}\int_R f)^p \int_{\tilde{R}} w \leq C \int_R |f|^p w$$

Now, we choose first $f(x) = w(x)^{-1/(p-1)}$ in (6.3). This gives

$$(6.4)\quad \frac{w(\tilde{R})}{|R|}\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}dx\right)^{p-1} \le C$$

But we can also interchange the roles of R and $\tilde{R}$ in (6.3), and then take $f(x) = \chi_{\tilde{R}}(x)$. This gives $w(R) \le C\, w(\tilde{R})$ which, inserted in (6.4), proves the condition A_p^* for $w(x)$. □

One of the consequences of the preceding theorem is the equivalence between A_p^* and the one-dimensional A_p conditions on each variable (uniformly in the remaining variables). It is natural to ask whether such an equivalence continues to hold in the extremal cases: $p = 1$, $p = \infty$. The next two theorems will give affirmative answers in both cases, while yielding some other equivalent characterizations of A_1^* and A_∞^* (remember that A_1^* and A_∞^* were already defined in (6.1)).

<u>THEOREM 6.5</u>. *<u>The following conditions for a weight</u>* $w(x)$ *<u>in</u>* $\mathbb{R}^n$ *<u>are equivalent</u>:*

i) $w \in A_1^*$

ii) $M_S\, w(x) \le C\, w(x)$ *<u>for a.e.</u>* $x \in \mathbb{R}^n$

iii) $M^j\, w(x) \le C\, w(x)$ *<u>for a.e.</u>* $x \in \mathbb{R}^n$ *<u>and</u>* $j = 1,2,\dots,n$

iv) *<u>The operators</u>* M^j, $j = 1,2,\dots,n$ *<u>are bounded from</u>* $L^1(w)$ *<u>to weak</u>*-$L^1(w)$.

Observe that iii) means $[w(x_1,x_2,\dots,x_{j-1},\cdot,x_{j+1},\dots,x_n)]_{A_1(\mathbb{R})} \le C$ for almost every $(x_1,x_2,\dots,x_{j-1},x_{j+1},\dots,x_n) \in \mathbb{R}^{n-1}$ and $j = 1,2,\dots,n$.

<u>Proof</u>: By a change of variables, the definition of A_1^* can be reformulated as

$$\frac{1}{|R|}\int_R w(x)dx \le C\,(\operatorname*{ess.\,inf.}_{x\in R} w(x)) \qquad \text{for all } R \in \mathcal{R}$$

The equivalence between this and ii) can be seen exactly as in the case of A_1 weights (see the argument after (1.8) and (1.9)).

To show that ii) and iii) are equivalent, we simply use the majorizations

$$M^j w(x) \le M_S w(x) \le M^1 \circ M^2 \circ \dots \circ M^n w(x) \qquad (\text{a.e. } x \in \mathbb{R}^n)$$

The second one is already known, and the first one follows by a step by step repetition of the argument we used to prove (a) $\Longrightarrow$ (b) in theorem 6.2, where we take now $p = 1$.

Let us now prove the equivalence of iii) and iv). We apply the inequality (2.14) of Chapter II to arbitrary functions $f(x_1,x_2,\dots,x_n)$ and weights $w(x_1,x_2,\dots,x_n)$ considered as functions of x_1 only, with the remaining variables fixed. By integrating then with respect to $x_2,x_3,\dots,x_n$, we get

$$w(\{x \in \mathbb{R}^n : M^1f(x) > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, M^1w(x)dx$$

and similarly for $M^2,M^3,\dots,M^n$. This shows that iii) implies iv). Conversely, if iv) holds, we apply the weak type inequality for M^1 to a function of the form $f(x) = g(x_1)\chi_E(\bar{x})$, with $g \geq 0$ and E a measurable subset of $\mathbb{R}^{n-1}$ (we are denoting $x = (x_1,\bar{x}) \in \mathbb{R}^n$, with $x_1 \in \mathbb{R}$, $\bar{x} \in \mathbb{R}^{n-1}$). Then, $M^1f(x) = \chi_E(\bar{x})\, Mg(x_1)$, and we get

$$\int_E \{\int_{\{x_1 \in \mathbb{R}: Mg(x_1)>\lambda\}} w(x_1,\bar{x})dx_1\}\, d\bar{x} \leq$$

$$\leq \frac{C}{\lambda} \int_E \{\int_{\mathbb{R}} g(x_1)w(x_1,\bar{x})dx_1\}\, d\bar{x}$$

Since E is arbitrary, this implies that, for each $g \geq 0$ and $\lambda > 0$, the inequality

$$(6.6) \quad \int_{\{Mg(x_1)\ \geq\lambda\}} w(x_1,\bar{x})dx_1 \leq \frac{C}{\lambda} \int_{\mathbb{R}} g(x_1)w(x_1,\bar{x})dx_1$$

holds for a.e. $\bar{x} \in \mathbb{R}^{n-1}$. Let A be the set of points $\bar{x} \in \mathbb{R}^{n-1}$ for which (6.6) holds for every function $g = \chi_I$, where I is any interval with rational endpoints, and every λ rational. Let

$$B = \{x = (x_1,\bar{x}) \in \mathbb{R}^n : \bar{x} \in A \text{ and } x_1 \text{ is a Lebesgue point of } w(.,\bar{x})\}.$$

Then, clearly, $|\mathbb{R}^n \setminus B| = 0$. Now, if $(x_1,\bar{x}) \in B$ and x_1 is an interior point of an interval I with rational endpoints, we take intervals $\{I_k\}_{k=1}^{\infty}$ with rational endpoints such that $x_1 \in I_k \subset I$ and $|I_k| \to 0$. Then we apply (6.6) to $g_k(y_1) = \chi_{I_k}(y_1)$. Since $Mg_k(y_1) \geq |I_k| \,/\, |I|$ for all $y_1 \in I$, we obtain

$$\frac{1}{|I|} \int_I w(y_1,\bar{x})dy_1 \leq \frac{C}{|I_k|} \int_{I_k} w(y_1,\bar{x})dy_1 \xrightarrow[k\to\infty]{} C\, w(x_1,\bar{x})$$

Taking the supremum for all intervals such as I

$$M^1 w(x_1,\bar{x}) \leq C\, w(x_1,\bar{x}) \qquad \text{for all} \qquad (x_1,\bar{x}) \in B$$

The same argument works for $M^2, M^3, \ldots, M^n$, and iii) is proved. □

Next, we shall establish several equivalent formulations of the condition A_∞^*:

THEOREM 6.7. *The following conditions for a weight* $w(x)$ *in* $\mathbb{R}^n$ *are equivalent:*

i) $w \in A_\infty^*$

ii) $\frac{1}{|R|}\int_R w(x)dx \leq C \exp(\frac{1}{|R|}\int_R \log w(x)dx)$ *for all* $R \in \mathcal{R}$.

iii) *For every* $R \in \mathcal{R}$ *and every measurable set* $A \subset R$

$$\frac{w(A)}{w(R)} \leq C\, (\frac{|A|}{|R|})^\varepsilon$$

for some fixed constants C, $\varepsilon > 0$ *depending only on* w.

iv) *There exist* α *and* β *with* $0 < \alpha$, $\beta < 1$, *such that, whenever we have* $A \subset R \in \mathcal{R}$ *and* $|A| \leq \alpha|R|$, *then* $w(A) \leq \beta\, w(R)$.

v) *There exist* α *and* β *with* $\alpha > 0$, $0 < \beta < 1$, *such that*

$$|\{x \in R : w(x) \leq \frac{\alpha}{|R|}\int_R w\}| \leq \beta|R| \qquad \textit{for all} \quad R \in \mathcal{R}$$

vi) w *satisfies a reverse Hölder's inequality over all n-dimensional intervals* R, i.e., *for some constants* C, $\varepsilon > 0$

$$\frac{1}{|R|}\int_R w(x)^{1+\varepsilon}\, dx \leq (\frac{C}{|R|}\int_R w(x)dx)^{1+\varepsilon}$$

vii) $w \in A_p^*$ *for some* $p < \infty$

viii) *For each* $j = 1,2,\ldots,n$, $[w(x_1,x_2,\ldots,x_{j-1},\cdot,x_{j+1},\ldots,x_n)]_{A_\infty(\mathbb{R})} \leq C$ *for a.e.* $(x_1,x_2,\ldots,x_{j-1},x_{j+1},\ldots,x_n) \in \mathbb{R}^{n-1}$.

Proof: By a change of variables, each one of the statements ii)-vi) is equivalent to saying that the weights $\{w^\delta\}_{\delta\in(\mathbb{R}_+)^n}$ satisfy the corresponding condition with cubes Q instead of intervals $R \in \mathcal{R}$, uniformly in δ. Taking into account the different characterizations of A_∞ (see 2.13 and 2.15) these statements are equivalent to

$$[w^\delta]_{A_\infty(\mathbb{R}^n)} \leq \text{Constant} \qquad (\text{for all } \delta \in (\mathbb{R}_+)^n)$$

which is the definitions of $w \in A_\infty^*$. Thus, all the conditions i)-vi) are equivalent. On the other hand, we know that $[w^\delta]_{A_\infty(\mathbb{R}^n)} \leq C$ implies $[w^\delta]_{A_p(\mathbb{R}^n)} \leq B$ for some $p = p(C,n) < \infty$ and $B = B(C,n)$. Therefore, i) implies vii), and the converse is obvious because $[\,\cdot\,]_{A_\infty(\mathbb{R}^n)} \leq [\,\cdot\,]_{A_p(\mathbb{R}^n)}$ by Jensen's inequality. Finally, we prove the equivalence between vii) and viii) by observing that each of them is equivalent to the following statement:

(*) *There exist* C *and* $p < \infty$ *such that*

$$[w(x_1,x_2,\dots,x_{j-1},\cdot,x_{j+1},\dots,x_n)]_{A_p(\mathbb{R})} \leq C$$

for a.e. $(x_1,x_2,\dots,x_{j-1},x_{j+1},\dots,x_n) \in \mathbb{R}^{n-1}$ *and* $j = 1,2,\dots,n$.

Indeed, vii) $\Longleftrightarrow$ (*) by theorem 6.2, while (*) $\Longleftrightarrow$ viii) by corollary 2.13 applied to weights in $\mathbb{R}$. □

A consequence of the characterization of A_∞^* provided by theorem 6.7.(vi) is that, if $1 \leq p \leq \infty$, $w \in A_p^*$ implies $w^{1+\varepsilon} \in A_p^*$ for some $\varepsilon > 0$. This follows also from the corresponding property for A_p weights and definition 6.1. We shall now use this fact in order to describe the classes A_p^* in terms of maximal functions. Given $u(x) > 0$ and $v(x) > 0$, we shall write: $u(x) \sim v(x)$ to indicate that both $\frac{u(x)}{v(x)}$ and $\frac{v(x)}{u(x)}$ are L^∞ functions.

THEOREM 6.8. *Let* $w(x) < \infty$ *a.e. Then*

a) $w \in A_1^*$ *if and only if there exist a locally integrable function* $g(x)$ *and a positive number* $\delta < 1$ *such that*

$$w(x) \sim (M^1g(x))^\delta \sim (M^2g(x))^\delta \sim \dots \sim (M^ng(x))^\delta$$

b) *For* $1 < p < \infty$ *we have*

$$A_p^* = A_1^*(A_1^*)^{1-p} = \{w_0\, w_1^{1-p} \mid w_0 \in A_1^* \text{ and } w_1 \in A_1^*\}$$

As a consequence, $w \in A_p^*$ *if and only if* $w(x) \sim (M^1g(x))^\delta (M^1h(x))^{\delta(1-p)}$ *for some* $0 < \delta < 1$ *and some locally integrable functions* $g(x)$ *and* $h(x)$ *such that*

$$M^1g(x) \sim M^2g(x) \sim \dots \sim M^ng(x)$$
$$M^1h(x) \sim M^2h(x) \sim \dots \sim M^nh(x)$$

Proof: a) By considering $(M^j g(x))^\delta$ as a function of x_j only and using (3.4) in Chapter II, it follows that $M^j((M^j g)^\delta)(x) \le \le C_\delta (M^j g(x))^\delta$ a.e. Thus, if $w(x) \sim (M^j g(x))^\delta$ for all $j=1,2,\dots,n$, then $M^j w(x) \le C\, w(x)$ a.e. $(j=1,2,\dots,n)$, which is equivalent to $w \in A_1^*$. Conversely, assume that $w \in A_1^*$. Then $w^{1+\varepsilon} \in A_1^*$ for some $\varepsilon > 0$, which means $M^j(w^{1+\varepsilon})(x) \sim w(x)^{1+\varepsilon}$ for all $j=1,2,\dots,n$. Letting $g(x) = w(x)^{1+\varepsilon}$ and $\delta = \frac{1}{1+\varepsilon}$ we have $w(x) \sim (M^j g(x))^\delta$ $(j=1,2,\dots,n)$ as desired.

b) We just need to prove the factorization result: $A_p^* = = A_1^*(A_1^*)^{1-p}$. But this is a particular case of theorem 5.2, since A_p^* are the classes of weights associated to the strong maximal operator, i.e. $A_p^* = W_p\,(M_S)$, $1 \le p < \infty$, (by 6.2 and 6.5) and $A_{p'}^* = (A_p^*)^{1-p'}$. □

The reader can easily check that the statement:

$$w(x) \sim (M^1 g_1(x))^\delta \sim (M^2 g_2(x))^\delta \sim \dots \sim (M^n g_n(x))^\delta$$

for some $0 < \delta < 1$ and some (different) locally integrable functions $g_1, g_2, \dots, g_n$, which is apparently weaker than the one in part a) of theorem 6.8, is also equivalent to $w \in A_1^*$. However, we do not know if $(M_S f)^\delta$ is, for every $f \in L^1_{loc}(\mathbb{R}^n)$ and $0 < \delta < 1$, a weight in A_1^* (the converse is true due to the reverse Hölder's inequality: $w \in A_1^*$ implies $w(x) \sim (M_S f(x))^\delta$ with $f = w^{1+\varepsilon}$, $\delta = \frac{1}{1+\varepsilon}$ and $\varepsilon > 0$). Thus, the preceding theorem does not give an easy method to produce "enough" A_1^* weights in $L^p(\mathbb{R}^n)$, but we can find such a method by imitating the argument used for the factorization theorem, namely:

Given $u \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, we take $q = \sqrt{p}$ and define

$$U(x) = \sum_{k=0}^{\infty} C^{-k}\, [M_S^k(u^q)]^{1/q}$$

where M_S^k denotes, as usual, the k-th iterate of M_S, and C equals twice the norm of M_S in L^q. Then, the series converges in L^p and

$$u(x) \le U(x)$$

$$\|U\|_p \le 2\|u\|_p$$

$$U \in A_1^* \text{ with } A_1^*\text{-constant depending only on } p$$

Likewise, the whole process that led to the extrapolation theorem applies mutatis mutandis to weights in A_p^*. We limit ourselves to state the final result

THEOREM 6.9. *Suppose that we replace* A_p *by* A_p^* *everywhere in the statements of theorem* 5.19 *and corollary* 5.20. *Then, the resulting statements are also true.*

The fact that, not only the extrapolation theorem, but also the lemmas we used in order to establish it can be extended to the product setting, will be important for our next result. What we shall need is the particular case of Lemma 5.18 corresponding to $r = 1$, which we now recall:

LEMMA 6.10. *Let* $w \in A_p^*$, $1 < p < \infty$. *Then, for every* $u \geq 0$ *in* $L^{p'}(w)$, *there exists* $v \geq 0$ *in* $L^{p'}(w)$ *such that*:

$$u(x) \leq v(x) \quad \text{a.e.}, \quad \|v\|_{p',w} \leq C\, \|u\|_{p',w} \quad \textit{and} \quad vw \in A_1^*$$

with A_1^*-*constant independent of* u.

The last objective in this section will be the study of the strong maximal function with respect to a regular Borel measure in $\mathbb{R}^n$, which is defined by

$$M_{S,\mu} f(x) = \sup_{x \in R \in \mathcal{R}} \frac{1}{\mu(R)} \int_R |f(y)|\, d\mu(y)$$

This is the analogue of the operator M_μ studied in Chapter II, where only cubes (instead of arbitrary n-dimensional intervals) were considered. In that case, M_μ was shown to be bounded in $L^p(\mu)$, $1 < p \leq \infty$, provided that μ is doubling: $\mu(Q^2) \leq C\,\mu(Q)$ for every cube Q. In our present situation, a trivial result is that $M_{S,\mu}$ is bounded in $L^p(\mu)$ for $1 < p \leq \infty$ when μ is a product measure: $\mu = \mu_1 \otimes \mu_2 \otimes \ldots \otimes \mu_n$ for some regular Borel measures $\mu_1, \mu_2, \ldots, \mu_n$ in $\mathbb{R}$. In fact, in this case we can repeat the argument given for Lebesgue measure, since

$$M_{S,\mu} f(x) \leq M^1_{\mu_1} \circ M^2_{\mu_2} \circ \ldots \circ M^n_{\mu_n} f(x)$$

and M_{μ_j} is bounded in $L^p(\mathbb{R};\mu_j)$, $1 < p \leq \infty$, for each $j=1,2,\ldots,n$. (This has been proved in Chapter II, theorem 2.6 when the μ_j's are doubling measures, but the doubling condition is not necessary for one-dimensional measures; see 7.4 in Chapter II).

A more significant result will be obtained for measures μ which are not necessarily products of measures in $\mathbb{R}$, namely:

$d\mu(x) = w(x)dx$ with $w \in A_\infty^*$. This will be based on covering arguments for the family $\mathcal{R}$. Since this technique is new for us, we shall establish first the general result which allows to pass from covering lemmas to boundedness properties of maximal functions:

LEMMA 6.11. *Given a basis* $\mathcal{B}$ *(i.e., a collection of open sets) in* $\mathbb{R}^n$, *and a measure* μ *such that* $0 < \mu(B) < \infty$ *for all* $B \in \mathcal{B}$, *suppose that, from any finite sequence* $\{B_j\} \subset \mathcal{B}$ *we can select a subsequence* $\{\tilde{B}_k\}$ *such that*

$$(6.12) \qquad \|\sum_k \chi_{\tilde{B}_k}\|_{L^q(\mu)} \leq C\, \mu(\bigcup_j B_j)^{1/q} \leq C'\, \mu(\bigcup_k \tilde{B}_k)^{1/q}$$

for some fixed constants $C,\ C' > 0$ *and* $1 < q \leq \infty$. *Then, the maximal operator*

$$M_{\mathcal{B},\mu} f(x) = \sup_{x \in B \in \mathcal{B}} \frac{1}{\mu(B)} \int_B |f|\, d\mu$$

is of weak type (q',q') *with respect to* μ *and bounded in* $L^p(\mu)$ *for* $q' < p \leq \infty$.

Proof: Given $f(x) \geq 0$ in $L^{q'}(\mu)$ and $\lambda > 0$, set

$$E_\lambda = \{x \in \mathbb{R}^n : M_{\mathcal{B},\mu} f(x) > \lambda\}$$

Every $x \in E_\lambda$ belongs to some $B \in \mathcal{B}$ such that $\int_B f\, d\mu > \lambda\mu(B)$, and E_λ will be covered by a countable family $\{B_j\}_{j\geq 1}$. Let $E_\lambda^N = \bigcup_{1\leq j\leq N} B_j$, and select from $\{B_j\}_{1\leq j\leq N}$ a subsequence $\{\tilde{B}_j\}$ satisfying (6.12). Then

$$\mu(E_\lambda^N) \leq C \sum_k \mu(\tilde{B}_k) \leq \frac{C}{\lambda} \sum_k \int_{\tilde{B}_k} f\, d\mu \leq$$

$$\leq \frac{C}{\lambda} \|f\|_{L^{q'}(\mu)} \|\sum_k \chi_{\tilde{B}_k}\|_{L^q(\mu)} \leq \frac{C}{\lambda} \|f\|_{L^{q'}(\mu)}\, \mu(E_\lambda^N)^{1/q}$$

Since $\mu(E_\lambda^N) < \infty$, we get $\mu(E_\lambda^N) \leq \text{Const.}\, \lambda^{-q'} \|f\|^{q'}_{L^{q'}(\mu)}$, and letting $N \to \infty$, the weak type inequality is proved. The strong type (p,p) for $p > q'$ follows from Marcinkiewicz interpolation theorem. □

It is remarkable that the converse of this lemma is also true (it was proved by Córdoba [2]), but we shall have no need of this fact. For results of this kind and generalizations we refer to M. de Guzmán [2], Ch. 6.

THEOREM 6.13. *Let* $d\mu(x) = w(x)dx$ *with* $w \in A_\infty^*$. *Then, the strong maximal operator with respect to* μ, $M_{S,\mu}$, *is bounded in* $L^p(\mu)$ *for all* $1 < p \leq \infty$.

Proof: We know that $w \in A_r^*$ for some $r < \infty$, and it will be enough to prove the inequalities (6.12) for $r \leq q < \infty$, since this will imply the weak type (q',q') of $M_{S,\mu}$ for q' arbitrarily close to 1. Given $\{R_j\}_{1\leq j\leq N}$ in $\mathcal{R}$, we take $\tilde{R}_1 = R_1$ and, once $\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_{k-1}$ have been selected, we choose $\tilde{R}_k$ to be the first interval in the given sequence (if any) such that at least half of it (in the sense of Lebesgue measure) is not covered by the intervals already selected, i.e.

$$|\tilde{R}_k \cap (\tilde{R}_1 \cup \tilde{R}_2 \cup \ldots \cup \tilde{R}_{k-1})| < \frac{1}{2} |\tilde{R}_k|$$

Now, we claim that

$$\bigcup_j R_j \subset \{x : M_S(\chi_{\bigcup_k \tilde{R}_k})(x) \geq \frac{1}{2}\}$$

Indeed, let f be the characteristic function of $\bigcup_k \tilde{R}_k$. If $x \in \bigcup_j R_j$, there are two possibilites: either x belongs to some $\tilde{R}_k$, in which case $M_S f(x) = 1$, or $x \in R_j$ for some R_j which has been discarded in our selection process, in which case we must have $|R_j \cap (\bigcup_k \tilde{R}_k)| \geq \frac{1}{2} |R_j|$, and therefore $M_S f(x) \geq \frac{1}{2}$. Now, since we know that M_S is bounded in $L^r(w)$

$$w(\bigcup_j R_j) \leq C_{r,w}\, 2^r \|\chi_{\bigcup \tilde{R}_k}\|_{r,w}^r = \text{Const. } w(\bigcup_k \tilde{R}_k)$$

which is the second inequality in (6.12). To prove the first inequality we use duality

$$\|\sum_k \chi_{\tilde{R}_k}\|_{q,w} = \sum_k \int_{\tilde{R}_k} u(x)\, w(x)dx$$

for some $u \in L_+^{q'}(w)$ of unit norm. Since $w \in A_r^* \subset A_q^*$, we apply lemma 6.10 to find $v(x) \geq u(x)$ such that $\|v\|_{q',w} \leq C_1$ and $vw \in A_1^*$ with A_1^*-constant $\leq C_2$ (independent of u). We denote $E_k = \tilde{R}_k \setminus (\tilde{R}_1 \cup \tilde{R}_2 \cup \ldots \cup \tilde{R}_{k-1})$, so that $\{E_k\}$ is a disjoint sequence and $|\tilde{R}_k| < 2\,|E_k|$. Then

$$\|\sum_k \chi_{\tilde{R}_k}\|_{q,w} \leq \sum_k \int_{\tilde{R}_k} v(x)w(x)dx \leq$$

$$\leq 2\, C_2 \sum_k |E_k| \operatorname*{ess.inf.}_{x\in\tilde{R}_k} (v(x)w(x)) \leq 2\, C_2 \sum_k \int_{E_k} v(x)w(x)dx =$$

$$= 2\, C_2 \int_{\bigcup_k \tilde{R}_k} v(x)\, w(x)dx \leq 2\, C_2\, C_1 \left[w(\bigcup_k \tilde{R}_k)\right]^{1/q}$$

and this completes the proof. □

The proof we have given of the last theorem is based on the boundedness properties of M_S in $L^p(w)$, $w \in A_p^*$. For an alternative (actually, the original) approach which does not use A_p^*-weights for $p < \infty$, see R. Fefferman [2]. We observe that, conversely, the fact that M_S is bounded in $L^p(w)$ for $w \in A_p^*$, $1 < p < \infty$, is a consequence of theorem 6.13 together with the implication: $w \in A_p^* \Longrightarrow w \in A_q^*$ for some $q < p$. Indeed, just as in the case of A_p weights, the condition $w \in A_q^*$ and Hölder's inequality imply

$$\frac{1}{|R|}\int_R |f| \leq C\left(\frac{1}{w(R)}\int_R |f|^q\, w\right)^{1/q} \qquad (R \in \mathcal{R})$$

that is, $M_S f(x) \leq C\,\{M_{S,w}(|f|^q)(x)\}^{1/q}$, and the right hand side is, according to 6.13, a bounded operator in $L^p(w)$, $p > q$.

7. NOTES AND FURTHER RESULTS

7.1.- Weighted inequalities for operators of the type considered here with weights of the form $|x|^\alpha$ were obtained in E.M. Stein [3]. Apart from that, the closest antecedents of A_p weights are probably Rosenblum [1] and the Fefferman - Stein inequality 2.12 of chapter II. The boundedness of the maximal function in $L^p(w)$ for $w \in A_p$ is due to Muckenhoupt [1]. The corresponding result for the Hilbert transform was established shortly afterwards in Hunt, Muchenhoupt and Wheeden [1]. The more systematic approach of Coifman and C. Fefferman [1] allowed them to deal with regular singular integral operators T in $\mathbb{R}^n$ by means of the inequality:

$$(*) \qquad \int |Tf(x)|^p \, w(x)dx \le C_{p,w} \int M f(x)^p \, w(x)dx$$

($w \in A_\infty$, $1 < p < \infty$). This approach is different from the one followed in section 3, which relies upon the sharp maximal function. More general weights for which (*) is true are found in E.T. Sawyer [2].

The fact that $A_\infty = \bigcup_{p>1} A_p$ is due independently to Muckenhoupt [3] and Coifman and C. Fefferman [1]. All other characterizations of A_∞ given in 2.13 and 2.15 are more or less implicit in both papers except for 2.15 (iii) (the equivalence of arithmetic and geometric means over cubes) which was found by the authors while writing this monograph and, independently, by Hruščev [1]. The characterization of A_1 given in 2.16 is due to Coifman (see Coifman and Rochberg [1] where it appears in connection with B.L.O.); Córdoba and C. Fefferman had previously proved that $(Mf)^\delta \in A_\infty$ if $\delta < 1$, a result which they used to obtain inequalities like corollary 3.8 (see Córdoba and C. Fefferman [1]).

7.2.- The aforementioned papers of Muckenhoupt and Coifman-Fefferman prompted a long series of results about weighted norm inequalities by many different authors. We cannot even attempt to give here a short account of these developments. As a sample of them, let us merely mention the following:

a) The last part of section 3 (specially theorem 3.9) is taken from Kurtz and Wheeden [1], where other related results are also obtained. Weighted multiplier inequalities beyond the A_p classes have been investigated in Muckenhoupt, Wheeden and W.-S. Young [1], [2].

b) Let $m_u(x)$ and $Au(x)$ denote, respectively, the nontangential maximal function and the area function (defined in 7.14 of Chapter II) of a harmonic function $u(x,t)$ in the upper half-space $\mathbb{R}^{n+1}_+$. The equivalence between the norms in $L^p(\mathbb{R}^n)$ of both functions, $0 < p < \infty$, which was stated in chapter III, 8.5, has the following extension: Let $w \in A_\infty$ and $0 < p < \infty$; if $Au \in L^p(w)$ and $u(x,t)$ is suitably normalized, then

$$c_p \int_{\mathbb{R}^n} (m_u(x))^p \, w(x)dx \le \int_{\mathbb{R}^n} (Au(x))^p \, w(x)dx \le C_p \int_{\mathbb{R}^n} (m_u(x))^p w(x)dx$$

See Gundy and Wheeden [1].

c) The following weighted version of the Carleson-Hunt theorem on pointwise convergence of Fourier series (see Carleson [3], Hunt [1]) was obtained in Hunt and W.-S. Young [1]: For a funtion f in $\mathbb{R}$, let $S^*f(x) = \sup_I |S_I f(x)|$; then S^* is a bounded operator on $L^p(w)$ if $w \in A_p$ and $1 < p < \infty$. This is clearly a strong improvement of corollary 3.12 (a).

7.3.- Analogues of the A_p theory have been developed in different contexts, such us:

a) Spaces of homogeneous type, as defined in chapter II, 7.11. For the maximal operator in this setting see A.P. Calderón [6] (from which the proof we have given of the R.H.I., lemma 2.5, was taken) and Jawerth [1]. For the Bergman projection in the unit ball of $\mathbb{C}^n$ and other domains, weighted inequalities are obtained in Bekollé [1].

b) Martingales. For regular martingales (an abstract generalization of the process of dividing each dyadic cube of $\mathbb{R}^n$ into 2^n dyadic subcubes) the R.H.I. can be obtained by essentially repeating the proof given for $\mathbb{R}^n$, and a complete analogy was found by several authors (e.g. Izumisawa and Kazamaki [1]). For general martingales, the R.H.I. may fail (Uchiyama [1]), but the boundedness of the maximal operator in $L^p(w)$ for $w \in A_p$ can nevertheless be proved by using the ideas of section 4; see Jawerth [1].

c) Ergodic theory. See Atencia and de la Torre [1], and also Martín Reyes [1].

7.4.- We shall state here some simple general facts about weighted inequalities for an arbitrary operator T. To simplify matters, suppose T is defined in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and is linear and (essentially) self-adjoint. Consider the class $W_p(T)$ consisting of those weights $w(x) \geq 0$ such that T is a bounded operator in $L^p(w)$, and denote by $N_p(w)$ the least constant such that

$$\left(\int |Tf(x)|^p \, w(x)dx\right)^{1/p} \leq N_p(w)\left(\int |f(x)|^p \, w(x)dx\right)^{1/p}$$

for all (good enough) f. Then:

a) W_p is a cone, and $N_p(.)$ satisfies the ultrametrical inequality: $N_p(u+v) \leq \max\,(N_p(u), N_p(w))$. Also, $N_p(\lambda u) = N_p(u)$ if $\lambda > 0$.

b) $W_{p'}(T) = W_p(T)^{1-p'}$, i.e., $w \in W_{p'}(T)$ if and only if

$w = u^{1-p'}$ with $u \in W_p(T)$ and, if this is the case, then $N_{p'}(w) = N_p(u)$.

c) $W_p(T)$ is a lattice: $u,v \in W_p(T)$ implies $\max(u,v) \in W_p(T)$ and $\min(u,v) \in W_p(T)$. Moreover

$$N_p(\max(u,v)) \leq 2^{1/p} \max(N_p(u), N_p(v))$$

To prove this, use (a) together with $\frac{u+v}{2} \leq \max(u,v) \leqq u+v$.
A similar inequality holds for $\min(u,v)$ with $2^{1/p'}$ instead of $2^{1/p}$ (use b)).

d) If T is translation invariant, so are W_p and $N_p(.)$. As a consequence, they are also invariant by convolution with a function $k(x) \geq 0$ (or a measure $\mu \geq 0$) in the following sense: If $u \in W_p$ and $k * u(x) < \infty$ a.e., then $k * u \in W_p$ and $N_p(k * u) \leq$ $\leq N_p(u)$.

The same properties hold for pairs of weights. As an application of property (c), when proving an inequality for A_p weights, it is enough to consider weights w which are bounded away from 0 and ∞ (i.e. w and w^{-1} belong to L^∞), provided that the constant in the resulting inequality depends only upon the A_p constant for w, and not upon $\|w\|_\infty$ or $\|w^{-1}\|_\infty$. This observation can be used for instance in theorem 3.1. Applications of property (d) will be given below.

7.5.- We showed in chapter I that if the conjugate function operator is bounded in $L^2(\mu)$, then the measure μ must be absolutely continuous with respect to Lebesgue measure. In this chapter, inequalities in $L^p(\mu)$ have been considered only for measures $d\mu(x) =$ $= w(x)dx$, so that the following question arises naturally: Let R_j, $j = 1,2,\dots,n$ denote the Riesz transforms in $\mathbb{R}^n$; for which regular measures μ in $\mathbb{R}^n$ do the inequalities

$$(*) \qquad \|R_j f\|_{L^p(\mu)} \leq C \|f\|_{L^p(\mu)} \qquad j = 1,2,\dots,n$$

(or their weak type analogues) hold? The answer is that (*) implies that μ is absolutely continuous (and then, of course, $\frac{d\mu}{dx} \in A_p$ as we know from section 3). The argument sketched below was indicated to us by B. Jawerth:

If (*) holds, we can apply the analogue of property (d) in 7.4 to show that the same inequality is satisfied by the weight functions

$u_\delta * \mu = w_\delta$, where u is ≥ 0 continuous with compact support, $\int u = 1$, and $u_\delta(x) = \delta^{-n}\, u(\frac{x}{\delta})$. By theorem 3.7, $w_\delta \in A_p \subset A_\infty$, with A_∞-constant independent of δ. Now, given a compact set $E \subset \mathbb{R}^n$ with $|E| = 0$, we fix a cube Q containing E in its interior, and take open sets V_j such that $|V_j| < \frac{1}{j}$ and $E \subset V_j \subset Q$. Since $w_\delta \to \mu$ $(\delta \to 0)$ in the weak-* topology of measures, Uryshon's lemma gives

$$\mu(E) \leq \liminf_{\delta \to 0} \; w_\delta(V_j) \leq \liminf_{\delta \to 0} \; C \left(\frac{|V_j|}{|Q|}\right)^{\varepsilon} w_\delta(Q)$$
$$\leq C\, \mu(Q^2)\left(\frac{|V_j|}{|Q|}\right)^{\varepsilon}$$

with C, $\varepsilon > 0$ fixed; therefore, $\mu(E) = 0$.

Another related question is: Can the Hardy-Littlewood maximal operator or a singular integral operator verify a strong type $(1,1)$ inequality with respect to a pair of regular measures in $\mathbb{R}^n$? For Lebesgue measure, we know that only the weak type $(1,1)$ inequality holds, and nothing better can hold for any other measures if the operator being considered is translation and dilation invariant. In fact, if $\|Rf\|_{L^1(\nu)} \leq C\, \|f\|_{L^1(\mu)}$ (where R is some Riesz transform), we can assume that $d\mu(x) = u(x)dx$ and $d\nu(x) = v(x)dx$ with u and v continuous and everywhere positive (just convolve with a positive integrable function). By dilation invariance, the same inequality holds for the weights $u^\delta(x) = u(\delta x)$ and $v^\delta(x) = v(\delta x)$, and letting $\delta \to 0$ we obtain that R is a bounded operator in $L^1(\mathbb{R}^n)$, but we know that this is absurd. This sort of argument will appear again in chapter VI, 2.8 and 2.9.

7.6.- The analogue of Problem 2 (in section 1) for the Fourier transform is probably one of the most interesting and hard problems concerning weighted inequalities. The observation made in the preceding note about the Riesz transforms no longer applies in this case, and one looks for inequalities of the form

$$(*) \qquad \left(\int_{\mathbb{R}^n} |\hat{f}(x)|^q \, d\mu(x)\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p \, d\nu(x)\right)^{1/p}$$

for some $p,q \geq 1$ and regular Borel measures $\mu,\nu \geq 0$. The best possible inequalities of this type with measures of the form $d\mu(x) = |x|^\alpha dx$, $d\nu(x) = |x|^\beta dx$ are known (see Stein [5] and, for the particular case $p = q$, corollary 7.4 in chapter VI). Extensions to more general absolutely continuous measures $d\mu(x) = u(x)dx$, $d\nu(x) = v(x)dx$ are in Muckenhoupt [4] (see also the references cited there), where a sufficient condition for (*) is given which, in the case $q = p'$, is specially simple:

$$\left(\int_{\{u(x)>Br\}} u(x)dx\right)\left(\int_{\{\sigma(x)<1/r\}} \sigma(x)dx\right) \leq C \quad \text{for all} \quad r > 0$$

where $\sigma(x) = v(x)^{-1/(p-1)}$, and B, C are constants independent of r.

For measures which are not absolutely continuous, the case more extensively studied corresponds to $d\nu(x) = dx$ and $d\mu(x)$ = rotation invariant measure supported in the unit sphere Σ_{n-1}. If (*) holds in this case, we say that the (L^p,L^q) restriction (to the unit sphere) holds for the Fourier transform. Observe that this means, in particular, that it makes sense to define $\hat{f}$ in Σ_{n-1}, which is a set of a measure 0, for every $f \in L^p(\mathbb{R}^n)$. The fact that this actually happens for some $p > 1$ was first noticed by E.M. Stein, and an argument of A. Knapp (based on the dilation properties of the Fourier transform) shows that necessary conditions for the (L^p,L^q) restriction in $\mathbb{R}^n$ are

$$1 \leq q \leq \left(\frac{n-1}{n+1}\right)p' \qquad \text{and} \qquad 1 \leq p < \frac{2n}{n+1}$$

The conjecture is that this is actually the best possible range. In $\mathbb{R}^2$, the conjecture was completely proved by Zygmund [3]. For $n \geq 3$, the conjecture is true in the case $q = 2$, i.e., (L^p,L^2) restriction holds if and only if $1 \leq p \leq 2\left(\frac{n+1}{n+3}\right)$; see Tomas [1]. This result is connected with the spherical summation multipliers discussed in 7.15 of chapter II; see C. Fefferman [4].

7.7.- The characterization of pairs of weights for which the Hardy-Littlewood maximal operator is of strong type (p,p) (theorem 4.9) was first obtained in E.T. Sawyer [1]. This can be used to solve, for the maximal operator M, the following weak version of the two weights problem: Find those $u(x) > 0$ (resp. $w(x) > 0$) for which M is bounded from $L^p(w)$ to $L^p(u)$ for some $w(x) > 0$ (resp. $u(x) > 0$); see E.T. Sawyer [3]. The precise statements can be found in chapter VI, where a different and systematic approach to this kind of problems will be followed.

For singular integral operators, and even for the Hilbert transform, the two weights problem (i.e., Problems 2 and 3 of section 1) remains completely open; see, however Cotlar and Sadosky [2] for some partial results in the spirit of the Helson-Szegö theorem.

The idea of relating Carleson measures and A_1 weights, which appears in section 2 of chapter II (following C. Fefferman and Stein [1]) can be further exploited, giving a unified approach to Carleson measures and the A_p or S_p conditions for weights. Consider the maximal operator $\mathcal{M}f(x,t)$ defined in chapter II, right before theorem 2.15, which maps functions $f(x)$ in $\mathbb{R}^n$ into functions defined in $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times [0,\infty)$. We try to find pairs (w,μ), where $w(x)$ is a weight in $\mathbb{R}^n$ and μ a positive measure in $\mathbb{R}^{n+1}_+$, such that $\mathcal{M}$ is bounded from $L^p(\mathbb{R}^n;\, w(x)dx)$ to $L^p(\mathbb{R}^{n+1}_+,\, \mu)$ or $L^p_*(\mathbb{R}^{n+1}_+,\, \mu)$. The strong type (p,p) holds if and only if

$$\text{(a)} \quad \int_{\tilde{Q}} (\mathcal{M}(\sigma\, \chi_Q)(x,t))^p d\mu(x,t) \leq C\, \sigma(Q)$$

while, for the weak type (p,p), the necessary and sufficient condition is

$$\text{(b)} \quad \frac{\mu(\tilde{Q})}{|Q|} \left(\frac{\sigma(Q)}{|Q|}\right)^{p-1} \leq C$$

where, as usual, $\sigma(x) = w(x)^{-1/(p-1)}$ and $\tilde{Q}$ denotes the cube $Q \times [0,\, \ell(Q)]$ in $\mathbb{R}^{n+1}_+$. When $w(x) \equiv 1$, both (a) and (b) are equivalent to the Carleson measure condition for μ. When μ is supported in $\mathbb{R}^n \times \{0\}$ and is given by $d\mu(x) = u(x)dx$, then (a) and (b) become, respectively, the S_p and A_p conditions for the pair (u,w). See Ruiz [1] and Ruiz and Torrea [1].

7.8.- This note is intended to illustrate the second motivation for the study of weighted inequalities given at the introduction to this chapter. Let $\{P_k(x)\}_{k=0}^{\infty}$ denote the orthonormal sequence of Legendre Polynomials. We want to know for which values of p is it true that

$$\sum_0^N \left(\int_{-1}^1 f\, P_k\right) P_k(x) \xrightarrow[N\to\infty]{} f(x) \qquad (\text{in } L^p \text{ norm})$$

for every $f \in L^p([-1,1])$. The N-th partial sum, $T_N f$, is an integral operator given by the Dirichlet kernel $D_N(x,y) = \sum_0^N P_k(x)P_k(y)$ which, by the Christoffel-Darboux formula, can be written as

$$D_N(x,y) = \frac{N+1}{2(x-y)} (P_{N+1}(x)\; P_N(y) - P_N(x)\; P_{N+1}(y)) =$$

$$= \frac{1}{x-y} \;(K_N(x)L_N(y) - K_N(y)L_N(x)) + \text{trivial error terms}$$

where $|K_N(t)| \leq C(1-t^2)^{-1/4}$ and $|L_N(t)| \leq C(1-t^2)^{1/4}$. Thus, for $-1 \leq x \leq 1$, and denoting by H the Hilbert transform

$$T_N f(x) = K_N(x)\ H(fL_N(x)) - L_N(x)H(fK_N)(x) + \text{error}$$

and a sufficient condition to have T_N uniformly bounded in L^p is $(1-t^2)^{\pm p/4} \in A_p$ (in the interval $[-1,1]$), i.e. $\frac{4}{3} < p < 4$. This can easily be shown to be the exact range of p's (see Pollard [1], Newman and Rudin [1]). By using the same argument together with the Hunt - W.S. Young result stated in 7.2 (c), it also follows that $T_* f(x) = \sup_{N>0} |T_N f(x)|$ is bounded in $L^p([-1,1])$ if $\frac{4}{3} < p < 4$, and therefore, the Legendre series of each $f \in L^p$, $p > \frac{4}{3}$, converges a.e. to $f(x)$.

For similar results concergining other orthogonal expansions we refer to Muckenhoupt and Stein [1], Askey and Wainger [1], Benedek, Murphy and Panzone [1].

7.9.- The first proof of the Ganett-Jones theorem (in J. Garnett and P. Jones [1]) was long and complicated. Further elaboration of the ideas in that proof led to a still more complicated stopping time argument, by means of which, the factorization theorem for A_p weights was originally proved (in P. Jones [1]). This result had been previously conjectured by Muckenhoupt. The approach followed here, based on the reiteration method of lemma 5.1, originates in Rubio de Francia [5], where the extrapolation theorem was also found. That approach was later clarified in Coifman, Jones and Rubio de Francia [1] (which contains also the application to the weak version of the Helson - Szegö theorem and the constructive decomposition of BMO) and García-Cuerva [2]. B. Jawerth [1] further exploited these ideas. obtaining general versions of the factorization and extrapolation theorems. The result about restrictions of A_p weights and B.M.O. functions to arbitrary sets (5.6 and 5.7) are due to T, Wolff [1]. Another application of lemma 5.1 appears in Neugebauer [1]: "*Given positive function* $u(x)$, $v(x)$ *in* R^n, *the necessary and sufficient condition for the existence of* $w \in A_p$ *such that* $u(x) \leq w(x) \leq v(x)$ *is that* $(u^{1+\varepsilon}, v^{1+\varepsilon}) \in A_p$ *for some* $\varepsilon > 0$". This result is interesting for the two weights problem for maximal and singular integral operators.

7.10.- The extension of the A_p theory to the product setting seems rather straightforward, but we can mention here two non-trivial problems:

I) If $f \in L^1_{loc}(\mathbb{R}^n)$ and $M_S f(x) < \infty$ a.e., is it true that $(M_S f)^\delta \in A_1^*$ for every $\delta < 1$?(+)

II) Does the analogue of the Fefferman-Stein inequality (2.12 in chapter II) hold for the strong maximal function, i.e.

$$(*) \qquad \int (M_S f(x))^p \, w(x) dx \leq \int |f(x)|^p M_S w(x) \, dx \qquad (1 < p)$$

for arbitrary $w(x) \geq 0$?

The best we know concerning the second problem is that (*) holds for all $p > 1$ if $w \in A_\infty^*$; see Lin [1].

Weights of product type can also be defined in $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$. The family of intervals to be considered in this case is $P = \{Q^{(1)} \times Q^{(2)} : Q^{(1)}$ and $Q^{(2)}$ cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively$\}$, the classes $A_p(P)$ are defined as expected, and the whole section 6 applies with the obvious modifications. In particular, if $T^{(1)}$ and $T^{(2)}$ are regular singular integral operators in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, then, $T = T^{(1)} \otimes T^{(2)}$ (defined as the composition of $T^{(1)}$ acting on the first n variables and $T^{(2)}$ acting on the last m variables) is bounded in $L^p(w)$ for every $w \in A_p(P)$, $1 < p < \infty$. There is, however, a more interesting extension of the theory of singular integrals to product domains, which was initiated in R. Fefferman and E.M. Stein [1]: One can easily write the cancellation and size conditions satisfied by the kernel of $T = T^{(1)} \otimes T^{(2)}$, and taking these as the starting hypothesis, without assuming that T is of product type, the L^p-boundedness of T as well as A_p-weighted inequalities can be proved. A far-reaching theory of singular integrals in product spaces (which we cannot describe here) has been built from this, including H^p and BMO spaces and non translation invariant operators. We refer to the survey by A. Chang and R. Fefferman [3].

The close connection between B.M.O. functions and A_p weights, as well as important consequences of it, became apparent in sections 2 and 5 of this chapter. Some other results emphasizing this connection are collected in the last four notes:

(+) This question has recently been answered in the negative by F. Soria.

7.11.- Let $h : \mathbb{R} \to \mathbb{R}$ be a homeomorphism of the real line. Suppose, for instance, that h is increasing. Then, h preserves B.M.O. (i.e. $f(h^{-1}(x))$ belongs to B.M.O. whenever $f(x)$ does) if and only if $h' \in A_\infty$. If this is the case, then also $(h^{-1})' \in A_\infty$ (by theorem 2.11) and $f \mapsto f\circ h^{-1}$ is actually an isomorphism of B.M.O. onto itself. See P. Jones [2]. The corresponding problem in $\mathbb{R}^n$ was solved by H.M. Reimann, who showed that h preserves B.M.O. $(\mathbb{R}^n)$ if and only if h is quasiconformal (Reimann [1]).

7.12.- Given a measurable function $b(x)$ in $\mathbb{R}$, we denote by M_b the operator of pointwise multiplication by $b(x)$. The commutator of this operator and the Hilbert transform H is

$$[M_b,H]f(x) = b(x)Hf(x) - H(bf)(x) = \text{p.v.}\ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{b(x)-b(y)}{x-y} f(y)dy$$

The following result was proved by Coifman, Rochberg and Weiss [1]

"*For any* p, $1 < p < \infty$, $[M_b, H]$ *is bounded in* $L^p(\mathbb{R})$ *if and only if* $b \in$ B.M.O.$(\mathbb{R})$".

We shall sketch the proof of the "if" part in the case $p = 2$. We suppose that $b(x)$ is real, and consider the power series in the complex variable $z = r\, e^{i\alpha}$

$$\sum_{n=o}^{\infty} \frac{z^n}{n!}\int_{-\infty}^{\infty} \frac{(b(x)-b(y))^n}{x-y} f(y)dy = \int_{-\infty}^{\infty} \frac{\exp(z(b(x)-b(y)))}{x-y} f(y)dy$$

$$= M_{\exp(zb)} \circ H \circ M_{\exp(-zb)}\, f(x) = T_z\, f(x)$$

By Cauchy's formula, the term corresponding to $n = 1$ is

$$\pi[M_b,H]\ f(x) = \frac{1}{2\pi i}\int_{|z|=r} z^{-2}\, T_z f(x)\, dz$$

But $\|T_z\| \leq \|T_r\|$ if $|z| = r$ ($\|.\|$ stands for the norm as an operator in L^2) and, for $r > 0$, T_r is bounded in L^2 if and only if $e^{2rb} \in A_2$, which is certainly the case for small enough r, since $b \in$ B.M.O. Therefore

$$\|[M_b,\ H]\| \leq \frac{1}{\pi r}\ \|T_r\| < \infty$$

The same proof applies to higher order commutators and to regular singular integral operators in $\mathbb{R}^n$.

The boundedness of $[M_b,\ R_j]$ for $b \in$ B.M.O.$(\mathbb{R}^n)$ where R_j are the Riesz transforms is then used by Coifman, Rochberg and Weiss to give a weak factorization theorem for $H^1(\mathbb{R}^n)$. See note 8.10 in chapter III.

7.13.- The following characterization of the closure of L^∞ in B.M.O. can be obtained from the Garnett-Jones theorem. Given a weight $w(x)$, we write $w \in$ R.H.I.(q) to indicate that w satisfies a reverse Hölder's inequality with exponent q (i.e., lemma 2.5 holds with $1+\varepsilon = q$). Then, for a real function $f \in$ B.M.O.$(\mathbb{R}^n)$, setting $w = e^f$, the following assertions are equivalent:

a) f belongs to the closure of L^∞ in B.M.O.

b) $w \in$ R.H.I.(q) and $w^{-1} \in$ R.H.I.(q) for all $q < \infty$

c) $w \in A_p$ for all $p > 1$ and $w \in$ R.H.I.(q) for all $q < \infty$

d) $w \in A_p$ and $w^{-1} \in A_p$ for all $p > 1$

e) $w^q \in A_\infty$ and $w^{-q} \in A_\infty$ for all $q < \infty$

Observe that every $f \in$ V.M.O.$(\mathbb{R}^n)$ (defined as in the case of the torus, see 11.18 in chapter I) satisfies a). This is used by Guadalupe and Rezola [1] to prove A_p-weighted estimates for the conjugate function operators in certain curves of the plane.

7.14.- The typical example of a function in B.M.O.$(\mathbb{R}^n)$ which is not bounded is $\log|x|$. The following extension of this fact has been noticed by E.M. Stein [11]: "*Let* $P(x)$ *be a polinomial of degree* k *in* $\mathbb{R}^n$; *then* $\log|P(x)| \in$ B.M.O. *and* $\|\log|P|\|_* \leq C_{k,n}$ (*constant depending only on* k *and* n)". Here is the simple proof of this result. It is enough to prove a uniform R.H.I., for instance

$$(*) \qquad \left(\frac{1}{|Q|}\int_Q |P(x)|^2\,dx\right)^{1/2} \leq B_{k,n}\left(\frac{1}{|Q|}\int_Q |P(x)|\,dx\right)$$

for all cubes Q and all polynomials of degree k. By translation and dilation invariance of the polynomials, it suffices to prove it for the unit cube $Q = [0,1]^n$. But then, both sides of (*) define norms on the finite-dimensional space of polynomials of degree k in $\mathbb{R}^n$, and therefore, they are equivalent.

CHAPTER V

VECTOR VALUED INEQUALITIES

The estimates for the operators that we have considered so far can be formulated in the context of Banach space valued functions. The first half of this chapter is devoted to describe the general methods which allow to obtain such estimates. There are at least two reasons which motivate this type of extension (apart from the search of greater generality for its own sake):

a) The non-linear operators which appear in Fourier Analysis can be viewed, in almost all cases, as linear operators whose range constists of vector valued functions. This will be the case for the maximal operators and square functions which are studied in sections 4 and 5 respectively.

b) There is an intimate connection between vector valued inequalities and weighted norm inequalities. We begin to explore this connection in the last section, but further insight will be gained with the theory to be developed in Chapter VI.

1. OPERATORS ACTING ON VECTOR VALUED FUNCTIONS: SOME BASIC FACTS

Let (X,m) be a σ-finite measure space. If B is a Banach space, a function $F : X \to B$ is said to be (strongly) measurable if the following two conditions hold:

i) There is a separable subspace B_o of B such that

$$F(x) \in B_o \quad \text{for a.e.} \quad x \in X$$

ii) For each $b' \in B^* =$ dual of B, the mapping: $x \mapsto \langle F(x), b'\rangle$ is measurable

A very easy consequence of this definition is that the positive function $\|F\|_B : x \mapsto \|F(x)\|_B$ is measurable. As in the scalarly valued case, we shall identify B-valued functions which coincide m-a.e. For $0 < p \leq \infty$, we define the Bochner-Lebesgue space $L^p_B(m)$ consisting of all measurable B-valued functions F for which

$$\|F\|_p = \|F\|_{L_B^p(m)} = \left(\int \|F(x)\|_B^p \, dm(x)\right)^{1/p} < \infty$$

(with the usual modification when $p = \infty$). Then $L_B^p(m)$ is a Banach (p-Banach if $0 < p < 1$) space. In the same way, we define the space $L_{*B}^p(m) = \text{weak} - L_B^p(m)$ which is "normed" by

$$\|F\|_{p*} = \sup_{t>0} \; t \, m(\{x : \|F(x)\| > t\})^{1/p}$$

The following facts are very simple to verify:

(1.1) *If* $f \in L^p$ *and* $b \in B$, *then* $(f.b)(x) = f(x)b$ *belongs to* L_B^p, *and* $\|f.b\|_{L_B^p} = \|f\|_p \|b\|_B$. *If* $0 < p < \infty$, *the subspace* $L^p \otimes B$ *formed by all finite linear combinations of functions* $f.b$ *is dense in* L_B^p.

Given $F = \sum_j f_j . b_j \in L^1 \otimes B$, we can define its integral (which will be an element of B) in the obvious way:

$$\int F(x) \, dm(x) = \sum_j \left(\int f_j(x) \, dm(x)\right) b_j$$

(1.2) *For every* $F_0 \in L^1 \otimes B$, $\left\|\int F_0 \, dm\right\|_B \leq \|F_0\|_{L_B^1}$. *Thus we can extend the mapping* $F \mapsto \int F \, dm$ *to all* L_B^1 *by continuity. If* $F \in L_B^1$, *then* $\int F \, dm$ *is the only element of* B *which satisfies*

$$\left\langle \int F \, dm, b' \right\rangle = \int \langle F(x), b' \rangle \, dm(x) \qquad (b' \in B^*)$$

If $F \in L_B^p$ with $1 \leq p \leq \infty$, and $G \in L_{B*}^{p'}$, then $\langle F,G\rangle(x) = \langle F(x), G(x)\rangle$ is trivially an integrable function, and we have

(1.3) *If* $1 \leq p \leq \infty$, *and* $G \in L_{B*}^{p'}$, *then*

$$\|G\|_{L_{B*}^{p'}} = \sup\left\{\left|\int \langle F(x), G(x)\rangle \, dm(x)\right| : \|F\|_{L_B^p} \leq 1\right\}$$

This means that $L_{B*}^{p'}$ *is (with the natural identification) isometrically contained into the dual of* L_B^p, *i.e.* $L_{B*}^{p'} \subset (L_B^p)^*$.

When B is reflexive one actually has $L_{B*}^{p'} = (L_B^p)^*$, $1 \leq p < \infty$, but we shall have no need of this fact.

In this chapter we shall deal with operators on spaces L^p_B. The following is a method of producing such operators which occurs naturally in many instances and which follows the same pattern used in (1.2) to define the integral: Suppose that T is a linear operator which maps L^p into L^q; then T can be extended to an operator $T^B = T \otimes Id_B$ on $L^p \otimes B$ by

$$(1.4) \qquad T^B(\sum_j f_j.b_j)(x) = \sum_j Tf_j(x)b_j$$

If T^B happens to be bounded, i.e.

$$\|T^B(\sum_j f_j.b_j)\|_{L^q_B} \leq C \, \|\sum_j f_j.b_j\|_{L^p_B} \qquad (f_j \in L^p, \; b_j \in B)$$

then it can be uniquely extended to a bounded operator from L^p_B into L^q_B which we still denote by T^B. In this case, we say that <u>T has a B-valued extension</u>, and the following identity holds:

$$(1.5) \qquad \langle T^B F(x), b'\rangle = T(\langle F(.), b'\rangle)(x) \qquad (F \in L^p_B, \; b' \in B^*)$$

(this is actually a characterization of T^B). The same can be done with an operator T mapping L^p into L^q_*.

<u>Examples 1.6.(a)</u>. Let $B = \ell^r$, $1 \leq r < \infty$. Then a measurable B-valued function is just a sequence $F(x) = (f_j(x))_{j\in\mathbb{N}}$ where each f_j is a (real or complex valued) measurable function, and $\sum_j |f_j(x)|^r < \infty$ a.e. The space $L^p_B = L^p(\ell^r)$ consists of all sequences (f_j) such that

$$\|(f_j)\|_{L^p(\ell^r)} = \|(\sum_j |f_j|^r)^{1/r}\|_p < \infty$$

Given a linear operator T of strong type (p,q), T^B is defined on functions $F = (f_j)_{j\in\mathbb{N}}$ with a finite number of nonvanishing components by

$$(1.7) \qquad T^B((f_j)_{j\in\mathbb{N}})(x) = (Tf_j(x))_{j\in\mathbb{N}}$$

Thus, T has a B-valued extension if and only if the following inequality holds

$$(1.8) \qquad \|(\sum_j |Tf_j|^r)^{1/r}\|_q \leq C \, \|(\sum_j |f_j|^r)^{1/r}\|_p$$

for all countable sequences of functions $f_j \in L^p$. If this is the case, T^B is defined by (1.7) over the whole space $L^p(\ell^r)$.

(b) As a more general example, take $B = L^r(\Omega,\mu)$, where (Ω,μ) is another σ-finite measure space. Then $L^p_B(m)$ can be identified with the mixed norm space $L^{r,p}(\Omega\times X,\ \mu\otimes m)$ which is formed by all $\mu\otimes m$-measurable functions $f(\omega,x)$ such that

$$\|f\|_{r,p} = \{\int(\int|f(\omega,x)|^r\, d\mu(\omega))^{p/r}\, dm(x)\}^{1/p} < \infty$$

The details of this identification are left to the reader, who is also referred to Benedek and Panzone [1]. If an operator T has a B-valued extension T^B, this is defined by the analogue of (1.7), namely

$$T^Bf(\omega,x) = T(f(\omega,.))(x) \tag{1.9}$$

(c) There are also interesting operators on vector valued functions which do not arise as extensions of operators in L^p. If A and B are Banach spaces, we denote by $L(A,B)$ the space of all bounded linear operators from A into B. Given a measurable $L(A,B)$-valued kernel $K(x,y)$ defined in $X\times X$ such that $K(x,.) \in L^{p'}_{L(A,B)}$ for a.e. x, then the operator

$$TF(x) = \int K(x,y).F(y)dm(y) \tag{1.10}$$

is well defined on $L^p_A(m)$, and one may try to investigate its continuity properties. If $A = B = \ell^r$, and if $K(x,y)$ is, for each (x,y), a diagonal operator with entries $(K_j(x,y))_{j\in\mathbb{N}}$, then T is given by

$$T((f_j)_{j\in\mathbb{N}})(x) = (T_jf_j(x))_{j\in\mathbb{N}} \tag{1.11}$$

where T_j is the linear operator defined on L^p by the kernel $K_j(x,y)$.

We shall now prove two elementary results. First of all, we present the simplest case in which the B-valued extension of an operator can be assured to exist

THEOREM 1.12. *Let* T *be a linear operator which is positive (i.e.* $g(x) \geq 0$ *implies* $Tg(x) \geq 0$*) and bounded, with norm* $\|T\|$, *as an operator from* $L^p(m)$ *to* $L^q(m)$ *(resp.* $L^q_*(m)$*). Then, for any Banach space* B, T *has a* B-*valued extension* T^B *which maps* $L^p_B(m)$ *into* $L^q_B(m)$ *(resp.* $L^q_{*B}(m)$*) with the same norm:* $\|T^B\| = \|T\|$.

Proof: It is enough to show that, for every $F \in L^p \otimes B$

$$|T^B F(x)|_B \leq T(|F|_B)(x)$$

Let $F(x) = \sum_{j=1}^{n} f_j(x) b_j$, and let $(b'_k)_{k\in\mathbb{N}}$ be a countable subset of the unit sphere in B^* such that $|s|_B = \sup_k |\langle s, b'_k\rangle|$, for every s in the span of $\{b_1, b_2, \ldots, b_n\}$. Then, we have

$$|T^B F(x)| = \sup_k |\langle T^B F(x), b'_k\rangle| =$$

$$= \sup_k |T(\langle F(.), b'_k\rangle)(x)| \leq T(|F|_B)(x)$$

where we have used the identity (1.5) and, for the last inequality, the positivity of T. □

In particular, for a positive linear operator T which is bounded in L^p, we have the inequalities

$$(1.13)\qquad \|(\sum_j |Tf_j|^r)^{1/r}\|_p \leq C\,\|(\sum_j |f_j|^r)^{1/r}\|_p \qquad (1 \leq r \leq \infty)$$

(when $r = \infty$ this must be understood in the usual way).

The second result deals with operators of the type described in the example 1.6.(c), and it generalizes the fact that convolution with a kernel in $L^1(\mathbb{R}^n)$ produces a bounded operator in $L^1(\mathbb{R}^n)$.

THEOREM 1.14. *Let* $K(x)$ *be a measurable* $L(A,B)$*-valued kernel defined in* $\mathbb{R}^n$ *and such that*

$$(1.15)\qquad \int |K(x).a|_B\, dx \leq C\, |a|_A$$

Then, for every $F \in L^1_A(\mathbb{R}^n)$, *the integral*

$$(1.16)\qquad TF(x) = \int K(x-y).F(y)dy$$

exists a.e., and the operator T *verifies:* $\|TF\|_{L^1_B} \leq C\,\|F\|_{L^1_A}$.

Proof: Given $F \in L^1_A(\mathbb{R}^n)$, Fubini's theorem and (1.15) give

$$\int\{\int |K(x-y).F(y)|_B\, dy\}dx =$$

$$= \int\{\int |K(x).F(y)|_B\, dx\}dy \leq \int C\,|F(y)|_A\, dy$$

and both assertions of the theorem follow immediately. □

Simple as it is, this result will nevertheless be useful in proving some Littlewood-Paley inequalities in section 2.

Remarks 1.17. If (1.15) is replaced by the stronger condition

$$\int |K(x)|_{L(A,B)}\, dx < \infty$$

then the operator T defined by (1.16) is bounded from L^p_A to L^p_B for all $1 \leq p \leq \infty$. More generally, Young's inequalities for convolution hold in this context if $K \in L^q_{L(A,B)}$. However, one cannot obtain from (1.15) L^p boundedness of T if $p > 1$, as the following example shows:

Given functions $k_j \in L^1(\mathbb{R}^n)$ $(j=1,2,\ldots)$ with $|k_j|_1 = 1$, define the kernel $K_\lambda(x) = (k_j^\lambda(x))_{j\in\mathbb{N}}$ with values in $L(\ell^1,\mathbb{C}) = \ell^\infty$, where $k_j^\lambda = k_j\chi_{\{x\,:\,|k_j(x)| \leq \lambda\}}$. The corresponding operator $T_\lambda : L^1(\ell^1) \to \ell^1$ is given by $T_\lambda((f_j))(x) = \sum_j k_j^\lambda * f_j(x)$, and it is clear that (1.15) holds with $C = 1$. Suppose for a moment that $T_\lambda : L^p(\ell^1) \to L^p$ is bounded for some $p > 1$ uniformly for all $\lambda > 0$. By duality (see (1.3)) and letting $\lambda \to \infty$, this would imply

$$\|\sup_j |k_j * f|\|_{p'} \leq c\, \|f\|_{p'} \qquad (f \in L^{p'}(\mathbb{R}^n))$$

which is known to be false if we take for instance $k_j = |R_j|^{-1}\chi_{R_j}$, where $\{R_j\}_{j\in\mathbb{N}}$ is a "dense" sequence in the family of all rectangles in $\mathbb{R}^2$ containing the origin (see de Guzmán [2]).

As it is well known, interpolation is a basic tool to deal with operators in L^p spaces, and this turns out to be also the case for operators acting on B-valued functions. Here we shall simply state the Marcinkiewicz and Riesz-Thorin theorems in the form that we shall need to use them. No proof will be given since both theorems are proved exactly as in the scalar case.

THEOREM 1.18. (Marcinkiewicz). *Let* A, B *be Banach spaces, and let* T *be an operator defined on* $L^{p_0}_A + L^{p_1}_A$ $(0 < p_0 < p_1 \leq \infty)$ *whose values are* B-*valued functions. If* T *is sublinear (with the natural definition of this) and of weak type* (p_0,p_0) *and* (p_1,p_1), *i.e.*

$$\|\chi_{\{x\,:\,|TF(x)|_B > \lambda\}}\|_{p_i} \leq C_i\, \lambda^{-1} |F|_{L^{p_i}_A} \qquad (i=0,1)$$

then, for all p *with* $p_0 < p < p_1$, *we have*

$$\|TF\|_{L^p_B} \le C_p \|F\|_{L^p_A} \qquad (F \in L^p_A)$$

where C_p *depends only on* C_0, C_1 *and* p.

THEOREM 1.19. (Riesz-Thorin). *Let* T *be a linear operator which is bounded in* $L^{p_0}(\ell^{r_0})$ *and in* $L^{p_1}(\ell^{r_1})$, *i.e.*

$$\|TF\|_{L^{p_i}(\ell^{r_i})} \le C_i \|F\|_{L^{p_i}(\ell^{r_i})} \qquad (i=0,1;\ F \in L^{p_i}(\ell^{r_i}))$$

with $1 \le p_0, p_1, r_0, r_1 \le \infty$. *If* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, *with* $0 < \theta < 1$, *then* T *is bounded in* $L^p(\ell^r)$, *and more precisely*

$$\|TF\|_{L^p(\ell^r)} \le C_0^{1-\theta} C_1^{\theta} \|F\|_{L^p(\ell^r)} \qquad (F \in L^p(\ell^r))$$

Further generalization of these results can be found in Bergh-Löfstrom [1]. In particular, the Riesz-Thorin theorem remains true if T is bounded from $L^{p_i}(\ell^{r_i}_A)$ into $L^{p_i}(\ell^{r_i}_B)$, $i=0,1$, where A, B are fixed Banach spaces, and

$$\ell^r_B = \{\bar{b} = (b_k)_{k\in\mathbb{N}} \mid b_k \in B,\ \|\bar{b}\|_{\ell^r_B} = (\sum_k \|b_k\|^r_B)^{1/r} < \infty\}$$

This observation is sometimes useful for the study of certain operators which we now describe:

DEFINITION 1.20. *An operator* T *defined in* $L^p(m)$ *is called linearizable if there exists a linear operator* U *defined in* $L^p(m)$ *whose values are* B*-valued functions (for some Banach space* B*) and such that*

$$|Tf(x)| = \|Uf(x)\|_B \qquad (f \in L^p(m))$$

It is clear that linear operators are linearizable, but there are some other interesting examples like

$$(1.21) \qquad Mf(x) = \sup_n |T_n f(x)|$$

$$(1.22) \qquad Gf(x) = (\sum_n |T_n f(x)|^2)^{1/2}$$

where $(T_n)_{n\in\mathbb{N}}$ is a sequence of linear operators (we take $B = \ell^\infty$

for the operator M and $B = \ell^2$ for G). Continuous analogues of the operators defined by (1.21) and (1.22) are also linearizable, and these include all kinds of maximal operators and g-functions of wide use in Fourier Analysis.

COROLLARY 1.23. *Let* T *be a linearizable operator which is bounded in* $L^p(m)$ *for some* $1 \le p < \infty$. *If* T *is positive (in the sense that:* $|f(x)| \le g(x)$ *a.e. implies* $|Tf(x)| \le Tg(x)$ *a.e.) then, the following inequalities hold:*

$$(1.24) \qquad \|(\sum_j |Tf_j|^r)^{1/r}\|_p \le C\,\|(\sum_j |f_j|^r)^{1/r}\|_p \qquad (p \le r \le \infty)$$

Proof: Observe that (1.24) is trivially true if $r = p$, and since

$$\sup_j |Tf_j(x)| \le T(\sup_j |f_j|)(x)$$

it is also true for $r = \infty$. Now (1.24) would follow for all $p \le r \le \infty$ if we were allowed to use interpolation. But we are, because $|Tf(x)| = \|Uf(x)\|_B$ for some linear operator U, and thus, (1.24) is equivalent to the boundedness of the linear operator:

$$\tilde{U} : L^p(\ell^r) \to L^p(\ell^r_B)$$

defined by $\tilde{U}((f_j)_{j\in\mathbb{N}})(x) = (Uf_j(x))_{j\in\mathbb{N}}$. $\square$

It may be interesting to remark that what we have just proved is a generalization of (1.13) (only for some values of r) to a wider class of operators. It is natural to ask whether (1.24) would be true for $1 < r \le \infty$. We shall present below some examples in which the answer is affirmative, but, for the general class of operators considered in the Corollary, (1.24) is best possible.

2. A THEOREM OF MARCINKIEWICZ AND ZYGMUND

The basic result to be proved in this section it that ℓ^2-valued extensions of linear operators in L^p spaces always exist. To begin with, we need to describe some elementary facts from Probability.

DEFINITION 2.1. *By a Gaussian sequence we shall mean a sequence* $(z_j)_{j\in\mathbb{N}}$ *of random variables (i.e. real valued measurable functions)* *in a probability space* (Ω, P) *which are independent and identically distributed with density function* $h(x) = e^{-\pi x^2}$, *i.e.*

$$(2.2) \qquad P(z_j \in A) = P(\{\omega \in \Omega : z_j(\omega) \in A\}) = \int_A e^{-\pi x^2} dx \qquad (j \in \mathbb{N})$$

for every Borel subset A *of* $\mathbb{R}$.

What (2.2) means is that, for every $j \in \mathbb{N}$, the image measure of P under the mapping $z_j : \Omega \to \mathbb{R}$ is $e^{-\pi x^2}dx$. An equivalent condition is

$$\int_\Omega \phi(z_j(\omega))dP(\omega) = \int_{\mathbb{R}} \phi(x)e^{-\pi x^2}dx$$

for every Borel function $\phi(x)$ on $\mathbb{R}$ for which the integral to the right exists. In particular, we see that $z_j \in L^r(\Omega)$ for all $r < \infty$, since

$$(2.3) \qquad \| z_j\|_r = \left(\int |z_j|^r dP\right)^{1/r} = \left(\int_{-\infty}^{\infty} |x|^r \, e^{-\pi x^2}dx\right)^{1/r} = b_r < \infty$$

What makes Gaussian sequences useful for us is the following property:

LEMMA 2.4. *Let* $(z_j)_{j\in\mathbb{N}}$ *be a Gaussian sequence in* (Ω,P), *and let* $(\lambda_j)_{j\in\mathbb{N}} \in \ell^2$. *Then, the series* $\sum_j \lambda_j z_j$ *converges in* $L^r(\Omega)$ *for all* $r < \infty$, *and the following identity holds*

$$(2.5) \qquad \|\sum_j \lambda_j z_j\|_r = b_r\left(\sum_j |\lambda_j|^2\right)^{1/2} \qquad (0 < r < \infty)$$

where b_r *is defined by (2.3) (thus, it depends only on* r*).*

Proof: It suffices to prove (2.5) for a finite sequence $(\lambda_j)_{j=1}^N$ such that $\sum_{j=1}^N |\lambda_j|^2 = 1$. Let $z = \sum_j \lambda_j z_j$, and denote by ρ an orthogonal transformation in $\mathbb{R}^N$ such that: $\rho(\lambda_1,\lambda_2,\ldots,\lambda_N) = (1,0,\ldots,0)$. Then, for every Borel subset A of $\mathbb{R}$:

$$P(z \in A) = \int_{\{x\in\mathbb{R}^N: \ \sum\lambda_j x_j \in A\}} h(x_1)h(x_2) \ldots h(x_N)dx =$$

$$= \int_{\{x\in\mathbb{R}^N:(\rho x)_1\in A\}} e^{-\pi|x|^2} dx = \int_{\{x\in\mathbb{R}^N:x_1\in A\}} e^{-\pi|x|^2} dx =$$

$$= \int_A e^{-\pi x_1^2} dx_1$$

Therefore, z is also distributed with the same density function:

$h(x) = e^{-\pi x^2}$, and $\|z\|_r = b_r$. $\square$

Remark 2.6. Gaussian sequences do exist. An explicit construction may be the following: Take $\Omega = \mathbb{R}^{\mathbb{N}} = \mathbb{R}\times\mathbb{R}\times\ldots\times\mathbb{R}\times\ldots$, and define P as the product of countably many copies of the measure $d\mu(x) = h(x)dx$; then, a Gaussian sequence $(z_j)_{j\in\mathbb{N}}$ in (Ω,P) is defined by

$$z_j(\omega) = \omega_j \qquad (\omega = (\omega_1,\omega_2,\ldots,\omega_n,\ldots) \in \Omega)$$

Now we are able to prove our main result. As always, (X,m) will be a fixed σ-finite measure space, and we shall write simply L^p instead of $L^p(X,m)$.

THEOREM 2.7. (Marcinkiewicz and Zygmund). *Let* $T : L^p \to L^q$ *be a bounded linear operator,* $0 < p,q < \infty$, *with norm* $\|T\|$. *Then,* T *has an* ℓ^2*-valued extension, and more precisely:*

$$\|(\sum_j |Tf_j|^2)^{1/2}\|_q \le C_{p,q} \|T\| \|(\sum_j |f_j|^2)^{1/2}\|_p \qquad (f_j \in L^p)$$

with $C_{p,q}$ *depending only on* p *and* q. *Moreover,* $C_{p,q} = 1$ *if* $p \le q$.

Proof: Consider first the case $q \le p$. If we take a Gaussian sequence $(z_j)_{j\in\mathbb{N}}$ in (Ω,P), Lemma 2.4 implies

$$\|(\sum_j |Tf_j|^2)^{1/2}\|_q^q = b_q^{-q} \int_X dm(x) \int_\Omega |\sum_j Tf_j(x).z_j(\omega)|^q dP(\omega) =$$

$$= b_q^{-q} \int_\Omega \|T(\sum_j z_j(\omega)f_j)\|_q^q \, dP(\omega) \le$$

$$\le b_q^{-q} \|T\|^q \int_\Omega \|\sum_j z_j(\omega)f_j\|_p^q \, dP(\omega) \le$$

$$\le b_q^{-q} \|T\|^q \{\int_\Omega \|\sum_j z_j(\omega)f_j\|_p^p \, dP(\omega)\}^{q/p} =$$

$$= (b_p/b_q)^q \|T\|^q \|(\sum_j |f_j|^2)^{1/2}\|_q^q$$

and the theorem is proved with $C_{p,q} = b_p/b_q$. In particular $C_{p,p} = 1$. Now we consider the case $p < q$, and denote $s = q/p$. For every $u(x) \ge 0$ with $\|u\|_{s'} \le 1$, the operator $T_u f = u^{1/p}.Tf$ satisfies

$$\|T_u f\|_p \le \|Tf\|_q \le \|T\| \|f\|_p$$

and by the case already proved, we have

$$\|(\sum_j |Tf_j|^2)^{1/2}\|_q = \sup_u \{\int (\sum_j |Tf_j|^2)^{p/2} u\}^{1/p} =$$

$$= \sup_u \|(\sum_j |T_u f_j|^2)^{1/2}\|_p \leq \|T\| \, \|(\sum_j |f_j|^2)^{1/2}\|_p . \quad \square$$

Next, we shall establish the analogue of the Marcinkiewicz-Zygmund theorem for weak type operators. The idea is to express the L^q_*-norm of a function as a supremum of L^r-norms, by means of the so called Kolmogorov condition, which can be stated as follows:

LEMMA 2.8. *Let* $0 < r < q < \infty$, *and for each function* $f(x) \geq 0$, *define*

$$\|f\|_{q^*} = \sup_{t>o} t \, m(\{x : |f(x)| > t\})^{1/q}$$

$$N_{q,r}(f) = \sup_E \frac{\|f \chi_E\|_r}{\|\chi_E\|_s} \qquad \left(\frac{1}{s} = \frac{1}{r} - \frac{1}{q}\right)$$

where the "sup" is taken for all measurable sets E *with* $0 < m(E) < \infty$. *Then we have the inequalities*

$$\|f\|_{q^*} \leq N_{q,r}(f) \leq \left(\frac{q}{q-r}\right)^{1/r} \|f\|_{q^*}$$

Observe that $\|.\|_{q^*}$ is the usual weak-L^q norm. On the other hand, if instead of χ_E we allow in the definition of $N_{q,r}(f)$ arbitrary functions $\phi \in L^s$, then we get exactly $\|f\|_q$ (by the converse of Hölder's inequality).

Proof: Given $t > 0$, for an arbitrary set $E \subset \{x : |f(x)| > t\}$ of finite measure, we have

$$m(E)^{1/r} \leq t^{-1} \|f\chi_E\|_r \leq t^{-1} m(E)^{1/s} N_{q,r}(f).$$

and the first inequality follows. To prove the second inequality, we can assume $\|f\|_{q^*} = 1$, so that the distribution function of f satisfies

$$\lambda_f(t) = m(\{x : |f(x)| > t\}) \leq t^{-q}$$

Then, for every set E of finite measure, it is clear that $\lambda_{f\chi_E}(t) \leq \inf(m(E), t^{-q})$, and we get

$$\|f\chi_E\|_r = \left(\int_0^\infty r t^{r-1} \lambda_{f\chi_E}(t)dt\right)^{1/r} \leq$$

$$\leq \{r\, m(E) \int_0^h t^{r-1}dt + r\int_h^\infty t^{r-q-1}\, dt\}^{1/r} =$$

$$= \left(h^r\, m(E) + \frac{rh^{r-q}}{q-r}\right)^{1/r}$$

Taking $h = m(E)^{-1/q}$ we obtain $\|f\chi_E\|_r \leq \left(\frac{q}{q-r}\right)^{1/r} m(E)^{1/s}$ as desired. □

THEOREM 2.9. *Let* $0 < p,q < \infty$, *and assume that* T *is a linear operator of weak type* (p,q), *i.e.*

$$m(\{x : |Tf(x)| > t\})^{1/q} \leq \frac{M}{t}\, \|f\|_p$$

Then T *has an* ℓ^2*-valued extension of weak type* (p,q). *More precisely, there exists* $C'_{p,q}$ *depending only on* p *and* q *such that*

$$m(\{x : (\sum_j |Tf_j(x)|^2)^{1/2} > t\})^{1/q} \leq C'_{p,q}\, \frac{M}{t}\, \|(\sum_j |f_j|^2)^{1/2}\|_p$$

Proof: Write $H = \ell^2$ and $F = (f_j)_{j\in\mathbb{N}} \in L^p_H$, so that $T^H F(x) = (Tf_j(x))_{j\in\mathbb{N}}$. Take $r < q$, and use the previous lemma to obtain

$$\|T_E f\|_r \leq \left(\frac{q}{q-r}\right)^{1/r} \|\chi_E\|_s\, M\, \|f\|_p \qquad (f \in L^p) \tag{2.10}$$

where E is an arbitrary set of finite measure, $1/s = 1/r - 1/q$, and $T_E f = (Tf)\chi_E$. Now, the Marcinkiewicz-Zygmund theorem applied to (2.10) gives

$$\|T_E^H F\|_{L^r_H} = \|(\|T^H F\|_H)\chi_E\|_r \leq C_{p,r} \left(\frac{q}{q-r}\right)^{1/r} \|\chi_E\|_s\, M\, \|F\|_{L^p_H}$$

and the proof is ended by invoking the first inequality in Lemma 2.8. □

We remark that theorems 2.7 and 2.9 are equally true for a linear operator T from $L^p(X,m)$ to $L^q(Y,\mu)$ (or $L^q_*(Y,\mu)$), where (Y,μ) is another measure space.

The following possible generalization of theorem 2.7 arises naturally: Given a family $\mathcal{T}$ of linear operators uniformly bounded in $L^p(m)$:

$$\|Tf\|_p \leq M\, \|f\|_p \qquad (f \in L^p,\ T \in \mathcal{T})$$

does the ℓ^2-valued inequality

$$(2.11) \qquad \|(\sum_j |T_j f_j|^2)^{1/2}\|_p \leq C \|(\sum_j |f_j|^2)^{1/2}\|_p \qquad (T_j \in T)$$

automatically hold?. The answer is trivally affirmative when $p = 2$, but only in this case, as the following examples show.

EXAMPLES 2.12. (a) Let us consider in $L^p(\mathbb{R})$ the translation operators $T_j f(x) = f(x-j)$, $j=1,2,3,\ldots$. If (2.11) is supposed to hold for some $p > 2$, we apply it to $f_j = \chi_{[-j,1-j]}$ $(j=1,2,\ldots,N)$ to obtain

$$\|(N\chi_{[0,1]})^{1/2}\|_p \leq C \|(\sum_j |f_j|^2)^{1/2}\|_p$$

i.e., $N^{1/2} \leq C N^{1/p}$, which is impossible for large N. On the other hand, if (2.11) holds for some $p < 2$, we take $f_1 = f_2 = \ldots = f_N = \chi_{[0,1]}$, so that $T_j f_j = \chi_{[j,j+1]}$ and we obtain $N^{1/p} \leq C N^{1/2}$ which again is absurd for large N.

(b) A much deeper and interesting counterexample is provided by the directional Hilbert transforms in $\mathbb{R}^n$, $n > 1$. These are defined by

$$(H_u f)\hat{}(\xi) = -i \operatorname{sign}(\xi.u)\hat{f}(\xi)$$

for each unit vector $u \in \Sigma_{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. If ρ is a rotation taking u into the vector $(1,0,\ldots,0)$, one easily shows that

$$H_u f(\rho x) = H(f\circ\rho(.,x_2,\ldots,x_n))(x_1)$$

where H is the Hilbert transform in $\mathbb{R}^1$. Thus, each H_u is a bounded operator in $L^p(\mathbb{R}^n)$ with norm equal to the norm of H as an operator in $L^p(\mathbb{R})$, $1 < p < \infty$. However, the inequality

$$\|(\sum_j |H_{u_j} f_j|^2)^{1/2}\|_p \leq C \|(\sum_j |f_j|^2)^{1/2}\|_p \qquad (f_j \in L^p;\ \ u_j \in \Sigma_{n-1})$$

is false for every $p \neq 2$. This was shown by C. Fefferman [2] in order to prove that the characteristic function of the unit ball is not a multiplier in $L^p(\mathbb{R}^n)$. We refer to M. de Guzmán [2] for this and related results.

Here is however an example in which inequality (2.11) does hold. By an interval I in $\mathbb{R}^n$ we shall mean the Cartesian product of n

intervals of $\mathbb{R}$. The "partial sum" operator S_I is then defined by the multiplier χ_I:

$$(S_I f)^\wedge(\xi) = \hat{f}(\xi)\chi_I(\xi)$$

COROLLARY 2.13. *Let* $\{I_j\}$ *be arbitrary intervals in* $\mathbb{R}^n$. *Then*

$$\|(\sum_j |S_{I_j} f_j|^2)^{1/2}\|_p \leq C_p \|(\sum_j |f_j|^2)^{1/2}\|_p \qquad (1<p<\infty) \tag{2.14}$$

In the case $n = 1$, *the following weak type inequality also holds:*

$$|\{x : (\sum_j |S_{I_j} f_j(x)|^2)^{1/2} > \lambda\}| \leq \frac{C}{\lambda} \|(\sum_j |f_j|^2)^{1/2}\|_1 \tag{2.15}$$

Proof: Matters are easily reduced to considering intervals of the form

$$I(a) = [a_1,\infty)\times[a_2,\infty) \times\ldots\times [a_n,\infty) \qquad (a \in \mathbb{R}^n)$$

and we write S_a instead of $S_{I(a)}$. Then

$$(S_0 f)^\wedge(\xi) = \hat{f}(\xi)\chi_P(\xi_1)\chi_P(\xi_2) \ldots \chi_P(\xi_n) \qquad (P = [0,\infty))$$

and we can write $S_0 = 2^{-n}(Id + iH_1)\circ(Id + iH_2) \circ\ldots\circ (Id + iH_n)$, where H_k denotes the Hilbert transform in the direction of $e_k = (0,0,\ldots,0,\overset{(k)}{1},0,\ldots,0)$. Therefore, S_0 is a bounded operator in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. On the other hand

$$S_a f(x) = e^{2\pi i a.x} S_0(e^{-2\pi i a.t} f(t))(x)$$

and Theorem 2.7 gives us

$$\|(\sum_j |S_{a_j} f_j|^2)^{1/2}\|_p = \|(\sum_j |S_0(e^{-2\pi i a_j\cdot} f_j)|^2)^{1/2}\|_p$$

$$\leq C_p \|(\sum_j |f_j|^2)^{1/2}\|_p$$

When $n = 1$, the operator $S_0 = \frac{1}{2}(Id + iH)$ is also of weak type (1,1), and the same argument combined with Theorem 2.9 gives (2.15). □

This corollary is very useful in Littlwood-Paley theory, as we shall see in section 4. We can already show how it works to produce some inequalities of Littlewood-Paley type which are simpler but less known than the usual ones.

THEOREM 2.16. *Let* I *be a bounded* n-*dimensional interval, and for each lattice point* $k \in \mathbf{Z}^n$ *consider the translated interval* $k+I$. *Then, for every* $f \in L^p(\mathbf{R}^n)$ *with* $2 \leq p < \infty$, *the following inequality holds:*

$$\|(\sum_k |S_{k+I}f|^2)^{1/2}\|_p \leq C_p \|f\|_p$$

Proof: Take a Schwartz function $\phi(x)$ in $\mathbf{R}^n$, and consider the finite-dimensional Hilbert space $B = \ell^2(F)$, where F is a finite subset of $\mathbf{Z}^n$. The operator

$$T((f_k)_{k\in F})(x) = \sum_{k\in F} \int e^{2\pi i k.y} \phi(y) f_k(x-y)\,dy$$
$$= \sum_{k\in F} \int \hat{f}_k(\xi)\hat{\phi}(\xi-k)e^{2\pi i x.\xi}\,d\xi$$

maps $L^q_B(\mathbf{R}^n)$ boundedly into $L^q(\mathbf{R}^n)$ for all $1 \leq q \leq 2$. In fact, for $q = 2$ this is a consequence of Plancherel's theorem, since

$$\sum_{k\in F} |\hat{\phi}(\xi-k)|^2 \leq C \qquad (\xi \in \mathbf{R}^n)$$

and for $q = 1$, it follows from Theorem 1.14, since the kernel

$$K(x) = (e^{2\pi i k.x} \phi(x))_{k\in F} \in B = L(B,\mathbf{C})$$

defining the operator T satisfies, for each $b = (b_k)_{k\in F}$

$$\int |K(x).b|\,dx = \int |(\sum_k b_k e^{2\pi i k.x})\phi(x)|\,dx$$
$$= \sum_{j\in\mathbf{Z}^n} \int_{j+Q} (\dots) \leq \|b\|_B \sum_{j\in\mathbf{Z}^n} (\int_{j+Q} |\phi(x)|^2 dx)^{1/2} =$$
$$= C\,\|b\|_B$$

where Q denotes the unit cube of $\mathbf{R}^n$, and $C < \infty$ because $\phi \in S(\mathbf{R}^n)$. By interpolation we obtain the result for $1 \leq q \leq 2$. The adjoint operator

$$T'f(x) = (T_k f(x))_{k\in F}$$

(where $(T_k f)^\wedge(\xi) = \hat{f}(\xi)\hat{\phi}(\xi-k))$ is therefore a bounded operator from $L^{q'}(\mathbf{R}^n)$ into $L^{q'}_B(\mathbf{R}^n)$ with norm independent of F, so that letting F increase to all $\mathbf{Z}^n$ we get

$$\|(\sum_{k\in\mathbf{Z}^n} |T_k f|^2)^{1/2}\|_p \leq C \|f\|_p \qquad (2 \leq p \leq \infty)$$

Now, assume that we take ϕ such that $\hat{\phi}(\xi) = 1$ for all $\xi \in I$. Then $S_{k+I}f = S_{k+I}(T_k f)$, and we can invoke Corollary 2.13 (applied to the sequence of functions $(T_k f)_{k\in\mathbb{Z}^n}$) together with the preceding inequality to complete the proof. $\square$

Remarks 2.17. (a) One can replace in Theorems 2.7 and 2.9 ℓ^2 by any other (separable) Hilbert space B. In particular, we can obtain a continuous version of both theorems by choosing $B = L^2(\mathbb{R}, \omega(t)dt)$, where $\omega(t) \geq 0$. If this is applied in the proof of 2.13, one obtains the corresponding continuous version of that Corollary, namely:

$$(2.14') \qquad \left\| \left(\int_{\mathbb{R}} |S_{I(t)} f_t|^2 \, \omega(t)dt\right)^{1/2} \right\|_p \leq$$

$$\leq C_p \left\| \left(\int_{\mathbb{R}} |f_t(.)|^2 \, \omega(t)dt\right)^{1/2} \right\|_p \qquad (1 < p < \infty)$$

Where $f_t(x)$ is a measurable function of $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, and $I(t)$ are intervals in $\mathbb{R}^n$ whose vertices are measurable functions of t. Of course, there is also the corresponding continuous version of the weak type inequality (2.15).

(b) By taking into account the known weighted norm inequalities for the Hilbert transform and the product operators arising from it (see Chapter IV, (6.2)) it follows immediately that inequality (2.14) holds in $L^p(w)$ provided that $w \in A_p^*$ and $1 < p < \infty$. Similarly, the weighted analogue of (2.15) (with a weight $w(x)$ in $\mathbb{R}$) holds if $w \in A_1$.

(c) When $n > 1$, we cannot replace the intervals $\{I_j\}$ in Corollary 2.13 by parallelepipeds in arbitrary directions. This fact is actually equivalent to the counterexample 2.12 (b) (see however 7.3 below).

(d) The inequality of Theorem 2.16 is false if $p < 2$. To see this, define f_k in $\mathbb{R}$ by $\hat{f}_k = \chi_{[k,k+1]}$, and $f = \sum_{k=0}^{N-1} f_k$ (so that $\hat{f} = \chi_{[0,N]}$). Then, $S_{k+I}f(x) = f_k(x) = e^{2\pi ikx} f_0(x)$, $k = 0,1,\ldots,N-1$, and $f(x) = Nf_0(Nx)$, so that the inequality of Theorem 2.16 implies:

$$\| N^{1/2} |f_0| \|_p \leq C_p \, N^{1/p'} \|f_0\|_p$$

which, when $N \to \infty$, forces $2 \leq p$.

3. VECTOR VALUED SINGULAR INTEGRALS

So far, we have proved two general results concerning the existence of B-valued extension for a linear operator T which is bounded in $L^p(m)$: The extension exists if T is positive (theorem 1.12) or if B is a Hilbert space (theorem 2.7). This is all that can be said in general. It is the study of particular cases what has to be done now (take an interesting non positive operator T appearing in Fourier Analysis, take some fixed Banach space, such as $B = \ell^r$ with $r \neq 2$, and try to know wheter T has a B-valued extension or not). This is, by far, the more interesting part, and the first natural question would concern the Hilbert transform: Does the inequality

$$(3.1) \qquad \left\| \left(\sum_j |Hf_j|^r\right)^{1/r} \right\|_p \leq C_{r,p} \left\| \left(\sum_j |f_j|^r\right)^{1/r} \right\|_p \qquad (f_j \in L^p(\mathbb{R}))$$

hold? An affirmative answer can be given if $r = 2$ and $1 < p < \infty$ by the theorem of Marcinkiewicz and Zygmund, and this can be slightly improved by interpolating with the obvious case: $1 < p = r < \infty$. However, as early as 1939, Boas and Bochner [1] proved that (3.1) does hold for all $1 < p,r < \infty$. They used a very clever argument involving complex function theory, but now, with the Calderón-Zygmund machinery available, the proof of (3.1) and its n-dimensional analogues becomes fairly easy.

We shall adopt a quite general viewpoint: Let A and B be Banach spaces, and consider a kernel $K(x)$ defined in $\mathbb{R}^n$ whose values are bounded linear operators from A to B : $K(x) \in \mathcal{L}(A,B)$. We assume that $K(x)$ is measurable and locally integrable away from the origin, so that the integral

$$(3.2) \qquad TF(x) = \int_{\mathbb{R}^n} K(x-y).F(y)dy$$

is well defined for all $F \in L^\infty_A(\mathbb{R}^n)$ with compact support and for all $x \notin \operatorname{supp}(F)$. To make the Calderón-Zygmund theory work in this context, some Hörmander type condition must be imposed on $K(x)$, like

$$(3.3) \qquad \int_{|x|>2|y|} \|K(x-y) - K(x)\|_{L(A,B)}\, dx \leq M \qquad (y \in \mathbb{R}^n)$$

Now, we can state the fundamental result, essentially taken from Benedek, Calderón and Panzone [1]:

THEOREM 3.4. *Let* T *be a bounded linear operator from* $L^r_A(\mathbb{R}^n)$ *to* $L^r_B(\mathbb{R}^n)$ *for some fixed* r, $1 \leq r \leq \infty$, *and assume that, when* F *is bounded and with compact support,* $TF(x)$ *is defined by (3.2) for every* $x \notin \operatorname{supp}(F)$, *with a kernel* $K(x)$ *satisfying (3.3). Then,* T *can be extended to an operator defined in* L^p_A, $1 \leq p < \infty$, *such that*

$$(3.5) \quad \|TF\|_{L^p_B} \leq C_p \|F\|_{L^p_A} \qquad (1 < p < \infty)$$

$$(3.6) \quad |\{x : \|TF(x)\|_B > \lambda\}| \leq C_1 \lambda^{-1} \|F\|_{L^1_A}$$

$$(3.7) \quad \|TF\|_{BMO(B)} \leq C_\infty \|F\|_{L^\infty_A} \qquad (f \in L^r_A \cap L^\infty_A)$$

$$(3.8) \quad \|Ta\|_{L^1_B} \leq C_1 \quad \textit{for every } A\text{-}\underline{atom}\ a(x)$$

The space $BMO(B)$ appearing in (3.7) is defined in a natural way: it consists of all B-valued functions $F(x)$ for which

$$\|F\|_{BMO(B)} = \sup_Q \frac{1}{|Q|} \int_Q \|F(x) - F_Q\|_B\, dx < \infty$$

Observe that $F \in BMO(B)$ implies $\|F\|_B \in BMO$, and

$$\|(\|F\|_B)\|_{BMO} \leq 2\, \|F\|_{BMO(B)}$$

On the other hand, A-atoms are also as expected, i.e., functions $a \in L^\infty_A$ supported in a cube Q and such that: $\int a(x)dx = 0$, $\|a(x)\|_A \leq |Q|^{-1}$.

Proof of Theorem 3.4: It is a straightforward generalization of Theorem 5.7 in Chapter II. One first proves (3.6) by using a Calderón-Zygmund decomposition of $\|F\|_A$, where $F \in L^1_A \cap L^\infty_A$, and estimating its "good" part by means of the L^r-boundedness of T (a slight modification is required here in the case $r = \infty$) and its "bad" part by means of (3.2) and (3.3). Interpolation (use Theorem 1.18) then gives the estimates (3.5) for $1 < p \leq r$.

The second step is (3.7), whose proof (as well as that of (3.8)) is exactly as in the scalar case. Then, the sublinear operator $SF(x) = (\|TF\|_B)^{\#}(x)$ satisfies

$$\|SF\|_p \leq C_p \|F\|_{L^p_A} \qquad (F \in L^r_A \cap L^p_A; \quad r \leq p \leq \infty)$$

because this is true for $p = r$, $p = \infty$ and we can apply again Theorem 1.18. Now, since

$$\|TF\|_{L^p_B} = \|(\|TF\|_B)\|_p \leq C_p \|(\|TF\|_B)^{\#}\|_p$$

(see Chapter II, 3.6) the remaining inequalities in (3.5) are also proved. □

As in the scalar case, the constants C_p, $1 \leq p \leq \infty$, depend only on p, M and on the (L^r_A, L^r_B)-norm of T.

There are certain remarks to be made now which will pave the way to some of the applications of vector valued singular integrals. The first one is the striking fact that the very statement of Theorem 3.4. leads to its immediate self-improvement in the following sense:

THEOREM 3.9. *Under the hypothesis of the preceding theorem, the following inequalities are also verified for* $1 < p,q < \infty$:

$$|\{x : (\sum_j \|TF_j(x)\|_B^q)^{1/q} > \lambda\}| \leq C_q \lambda^{-1} \int (\sum_j \|F_j(x)\|_A^q)^{1/q} dx$$

$$\|(\sum_j \|TF_j\|_B^q)^{1/q}\|_p \leq C_{p,q} \|(\sum_j \|F_j\|_A^q)^{1/q}\|_p$$

Proof: Denote by $\ell^q(A)$, $1 < q < \infty$, the Banach space of all sequences $(a_j)_{j\in\mathbb{N}}$ with $a_j \in A$ such that $\|(a_j)\| = (\sum_j \|a_j\|_A^q)^{1/q} < \infty$. The previous theorem tells us that $T : L^q_A \to L^q_B$ is a bounded operator. It is therefore trivial that the operator $\tilde{T} : (F_j)_{j\in\mathbb{N}} \longmapsto (TF_j)_{j\in\mathbb{N}}$ maps $L^q_{\ell^q(A)}$ boundedly into $L^q_{\ell^q(B)}$, and it is defined by the kernel $\tilde{K}(x) \in L(\ell^q(A), \ell^q(B))$ given by

$$\tilde{K}(x).((a_j)_{j\in\mathbb{N}}) = (K(x).a_j)_{j\in\mathbb{N}}$$

The operator norms of $\tilde{K}(x)$ and $K(x)$ coincide, and similarly

$$\|\tilde{K}(x-y) - \tilde{K}(x)\| = \|K(x-y) - K(x)\|$$

so that $\tilde{K}$ satisfies (3.3) and Theorem 3.4 applies again proving the desired inequalities. □

The second important fact to be noted is that, if the Banach space A is one-dimensional: $A = \mathbb{C}$, then $L(A,B) = B$. Moreover, in this case the L^1-estimate (3.8) has a more interesting meaning due to the atomic decomposition of $H^1(\mathbb{R}^n)$ described in Chapter III. We state explicitly the corresponding results as a Corollary.

COROLLARY 3.10. *Let* $T : L^r(\mathbb{R}^n) \to L^r_B(\mathbb{R}^n)$ *be a bounded linear operator for some fixed* r, $1 \le r \le \infty$, *and assume that*

$$Tf(x) = \int f(y)K(x-y)dy \qquad (x \notin \text{supp}(f))$$

for some B-*valued measurable function* $K(x)$ *which verifies Hörmander's condition (3.3) (with* $|\cdot|_{L(A,B)} = |\cdot|_B$). *Then* T *extends to a bounded operator from* $L^p(\mathbb{R}^n)$ *to* $L^p_B(\mathbb{R}^n)$, $1 < p < \infty$, *from* $H^1(\mathbb{R}^n)$ *to* $L^1_B(\mathbb{R}^n)$ *and from* $L^1_B(\mathbb{R}^n)$ *to weak*-$L^1_B(\mathbb{R}^n)$.

Of course, $H^1_A(\mathbb{R}^n)$ can be defined in terms of atoms for an arbitrary Banach space A, and if this is done, the estimate (3.8) can be read as follows: $T : H^1_A \to L^1_B$.

A final remark, which should have been observed by anyone trying to adapt the proof of Theorem 5.7 in Cahpter II to the vector valued case, is the following: In the proof of the inequalities (3.6), (3.8) and (3.5) for $1 < p \le r$, one can replace (3.3) by the weaker condition

$$(3.3') \qquad \int_{|x|>2|y|} |(K(x-y) - K(x)).a|_B dx \le M|a|_A$$

(for all $a \in A$, $y \in \mathbb{R}^n$). However, (3.7) and the remaining inequalities in (3.5) do not follow from (3.3'). We do not emphasize very much the interest of this weaker condition since, in most of the applications of vector valued singular integrals, (3.3) does hold.

Turning to the problems stated at the beginning of this section, we have the first immediate applications of our general result.

THEOREM 3.11. *Let* $(k_j)_{j\in\mathbb{N}}$ *be a sequence of tempered distributions in* $\mathbb{R}^n$ *with* $\sup_j |\hat{k}_j|_\infty < \infty$. *Assume that each* k_j *coincides away from the origin with a locally integrable function, and*

$$\int_{|x|>2|y|} \sup_j |k_j(x-y)-k_j(x)|dx \leq C \qquad (y \in \mathbb{R}^n)$$

Then, the following inequalities hold, provided the right hand side is finite:

(3.12) $$|\{x : (\sum_j |k_j*f_j(x)|^r)^{1/r} > \lambda\}| \leq$$
$$\leq C_r \lambda^{-1} \|(\sum_j |f_j|^r)^{1/r}\|_1 \qquad (1 < r < \infty)$$

(3.13) $$\|(\sum_j |k_j*f_j|^r)^{1/r}\|_p \leq C_{r,p} \|(\sum_j |f_j|^r)^{1/r}\|_p \qquad (1<r,p<\infty)$$

Proof: First of all, we must recall that, under the hypothesis, k_j*f can be defined for all $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and the resulting operators are uniformly bounded in L^p for all $1<p<\infty$. Now we fix $r > 1$ and consider as Banach spaces: $A = B = \ell^r$. The operator

$$TF(x) = (k_j*f_j(x))_{j\in\mathbb{N}} \qquad (F = (f_j)_{j\in\mathbb{N}} \in L^r_B)$$

is then bounded in L^r_B, and it is clear that

$$TF(x) = \int K(x-y).F(y)dy \qquad (x \notin \mathrm{supp}(F))$$

where $K(x) \in \mathcal{L}(B)$ is the diagonal operator whose entries are $k_j(x)$:

$$K(x).b = (k_j(x)b_j)_{j\in\mathbb{N}} \qquad (b = (b_j) \in B)$$

Since $\|K(x-y) - K(x)\|_{\mathcal{L}(B)} = \sup_j |k_j(x-y) - k_j(x)|$, (3.3) is verified, and Theorem 3.4 applies giving the desired vector valued inequalities. □

The hypothesis of this theorem are fulfilled in the following specially interesting cases:

a) When each $k_j(x)$ is C^1 outside the origin and

$$|\hat{k}_j(\xi)| \leq C, \qquad |\nabla k_j(x)| \leq C\,|x|^{-n-1} \qquad (j \in \mathbb{N})$$

b) When $k_j(x) = k(x)$ for all j, where $\hat{k} \in L^\infty(\mathbb{R}^n)$ and

$$\int_{|x|>2|y|} |k(x-y) - k(x)|dx \leq C \qquad (y \in \mathbb{R}^n)$$

The second case can also be obtained as an application of Theorem 3.9 which $A = B = \mathbb{C}$ (complex numbers). What this case means is that every singular integral operator has a bounded ℓ^r-valued extension if $1 < r < \infty$. In particular, we can take $k(x) = \text{p.v.}\frac{1}{\pi x}$ in $\mathbb{R}$, proving (3.1) for all $1 < p,r < \infty$. An application of this inequality is the following improvement of Corollary 2.13 (without modifying its proof):

COROLLARY 3.14. *Let* r *be fixed,* $1 < r < \infty$. *The inequalities (2.14) (2.15) remain true if we substitute everywhere* $(\sum_j |.|^2)^{1/2}$ *by* $(\sum_j |.|^r)^{1/r}$.

Not surprisingly, weighted estimates can also be obtained for vector singular integral operators, provided that we assume more regularity for the kernel $K(x)$, namely

$$(3.15) \qquad \|K(x-y)-K(x)\|_{L(A,B)} \leq C\,|y|\,|x|^{-n-1} \qquad (\text{if } |x|>2|y|>0)$$

We invite the reader to look for a complete analogy with the results for regular singular integral operators which, in the scalar case, were described in Chapter II, Theorem 5.20, and in section IV.3. Here, we merely state the result which we shall need to apply in the next section.

THEOREM 3.16. *Suppose that* T *is as in Theorem 3.4 wit a kernel* $K(x)$ *satisfying, instead of Hörmander's condition, the stronger assumption (3.15). Suppose also that, for some fixed* r, $1 < r \leq \infty$, *and for all weights* $w \in A_1$, *we have*

$$\|TF\|_{L^r_B(w)} \leq C(w)\,\|F\|_{L^r_A(w)}$$

with $C(w)$ *depending only on the* A_1*-constant of* w. *Then, we have*

$$(3.17) \qquad w(\{x : \|TF(x)\|_B > t\}) \leq C(w)\, t^{-1} \int \|F(x)\|_A^p\, w(x)dx$$

for every $w \in A_1$, *and also, for every* $w \in A_p$ *with* $1 < p < \infty$,

$$(3.18) \qquad \int \|TF(x)\|_B^p w(x)dx \leq C_p(w) \int \|F(x)\|_A^p\, w(x)dx.$$

The proof of (3.17) is again an exact repetition (except for a minor change in the estimation of "good part" when $r = \infty$) of the one given in Chapter IV, Theorem 3.5, for scalar functions (observe that the hypothesis $|K(x)| \leq C|x|^{-n}$ was not really needed there).

Once the weak type result is proved, (3.18) follows by the extrapolation theorem of Chapter IV, (5.20). □

4. APPLICATIONS: SOME MAXIMAL INEQUALITIES

Almost every nonlinear operator appearing in Fourier Analysis is linearizable (in the sense of Definition 1.20). It is sometimes useful to look at such operators as linear operators taking complex valued functions into B-valued functions, and if the kernel happens to satisfy the appropriate condition, the theory developed in the preceding section can be applied. In particular, we shall see that the Hardy-Littlewood maximal operator and some of its generalizations can be viewed as vector valued singular integrals.

Given $\phi \in L^1(\mathbb{R}^n)$, we define the maximal operator

$$M_\phi f(x) = \sup_{\delta>0} |f*\phi_\delta(x)| \qquad (\phi_\delta(x) = \delta^{-n}\phi(x/\delta))$$

which is certainly bounded in $L^\infty(\mathbb{R}^n)$. Denoting $B = \ell^\infty(\mathbb{R}_+)$, it is equivalent to consider the linear operator $T : L^\infty \to L^\infty_B$ defined (as in Corollary 3.10) by the B-valued kernel $K(x) = (\phi_\delta(x))_{\delta>0}$, since we have: $M_\phi f(x) = \|Tf(x)\|_B$. Now, Hörmander's condition for the kernel $K(x)$ amounts to the following:

$$(4.1) \qquad \int_{|x|>2|y|} \sup_{\delta>0} |\phi_\delta(x-y)-\phi_\delta(x)|\,dx \le C \qquad (y \in \mathbb{R}^n)$$

and we are led to

THEOREM 4.2. *Let* $\phi \in L^1(\mathbb{R}^n)$ *be a function satisfying (4.1). Then:*

i) M_ϕ *is an operator bounded in* $L^p(\mathbb{R}^n)$, $1 < p \le \infty$, *of weak type* $(1,1)$ *and mapping* $H^1(\mathbb{R}^n)$ *into* $L^1(\mathbb{R}^n)$.

ii) For every $f \in \bigcup_{1\le p<\infty} L^p(\mathbb{R}^n)$

$$\lim_{\delta\to 0} f*\phi_\delta(x) = \left(\int_{\mathbb{R}^n}\phi\right) f(x) \qquad \text{a.e.}$$

iii) The following vector valued inequalities hold for all $1 < p,q < \infty$:

$$|\{x : (\sum_j M_\phi f_j(x)^q)^{1/q} > t\}| \le C_q\, t^{-1} \|(\sum_j |f_j|^q)^{1/q}\|_1$$

$$\|(\sum_j (M_\phi f_j)^q)^{1/q}\|_p \le C_{p,q} \|(\sum_j |f_j|^q)^{1/q}\|_p$$

Proof: Since $f*\phi_\delta(x)$ depends continuously on δ, it suffices to obtain in (i) and (iii) uniform estimates for the operators

$$M_F f(x) = \sup_{\delta\in F} |f*\phi_\delta(x)| = \left\| \int K_F(x-y)f(y)dy \right\|_{\ell^\infty(F)}$$

where F is an arbitrary finite subset of $\mathbb{R}_+$ and

$$K_F(x) = (\phi_\delta(x))_{\delta\in F} \in \ell^\infty(F)$$

Now, we are under the hypothesis of Theorem 3.4 and 3.9 (with $r = \infty$, $A = \mathbb{C}$, $B = \ell^\infty(F)$), which give (i) and (iii) respectively. Finally, (ii) follows in the usual way (see Chapter II, (1.9), for instance) since it is trivially verified by all continuous f with compact support. □

A trivial but important observation is that, if ψ is such that $|\psi| \le |\phi|$, the whole theorem, except for the $H^1 \to L^1$ result, holds also for the operator M_ψ. In particular, we can take as ψ the characteristic function of the unit cube centered at the origin (thus, M_ψ = Hardy-Littlewood maximal operator) and as ϕ a non-negative Schwartz function such that $\phi(x) \ge 1$ when $x \in Q$. Then, we have

COROLLARY 4.3. *The Hardy-Littlewood maximal operator verifies the vector valued inequalities (iii) of the previous theorem.*

Next, we observe that, since $L^\infty(w) = L^\infty(\mathbb{R}^n)$ for every $w \in A_1$, the operator M_ϕ satisfies the estimate required in Theorem 3.16 with $r = \infty$, and we can therefore obtain weighted estimates for this kind of maximal operators:

THEOREM 4.4. *Suppose that* $\phi \in L^1(\mathbb{R}^n)$ *satisfies*

$$|\phi(x-y) - \phi(x)| \le C|y||x|^{-n-1} \qquad (|x|>2|y|>0)$$

Then, the maximal operator M_ϕ *is bounded in* $L^p(w)$ *if* $1 < p < \infty$ *and* $w \in A_p$, *and it is also of weak type* $(1,1)$ *with respect to the measure* $w(x)dx$ *if* $w \in A_1$

Proof: It suffices to apply 3.16 taking into account that the condition imposed on ϕ is dilation invariant, so that the same inequality is verified by ϕ_δ for all $\delta > 0$, i.e.

$$\sup_{\delta>0} |\phi_\delta(x-y) - \phi_\delta(x)| \leq C\,|y|\,|x|^{-n-1} \qquad (|x| > 2|y| > 0)$$

and this is exactly the condition (3.15) for the ℓ^∞-valued kernel $K(x) = (\phi_\delta(x))_{\delta>0}$. $\square$

One can consider more generally maximal operators defined by a family $\Phi = (\phi^i)_{i\in I}$, namely

$$M_\Phi f(x) = \sup_{i\in I} |\phi^i * f(x)|$$

where, either I is countable or $I = \mathbb{R}_+$ and $\phi^i(x)$ depends continuously on i. If (4.1) is verified by the family (ϕ^i), and if M_Φ happens to be bounded in $L^q(\mathbb{R}^n)$ for some q (this is certainly true with $q = \infty$ when $\sup_{i\in I} \|\phi^i\|_1 < \infty$), then, parts (i) and (iii) of Theorem 4.2 hold for M_Φ.

In particular, we can consider approximations of the identity (ϕ_{δ_k}) corresponding to a lacunary sequence (δ_k) of dilations, and we have

THEOREM 4.5. *Suppose that* ϕ *satisfies the following two conditions:*

$$(4.6) \quad \int_{\mathbb{R}^n} |\phi(x)|\, \log(2+|x|)\,dx < \infty$$

$$(4.7) \quad \int_0^1 \omega_1(t)\,\frac{dt}{t} < \infty, \quad \textit{where} \quad \omega_1(t) = \sup_{|h|\leq t} \int |\phi(x+h)-\phi(x)|\,dx$$

Then, the conclusions of Theorem 4.2 hold for the dyadic maximal operator

$$N_\phi f(x) = \sup_{k\in\mathbb{Z}} |f * \phi_{2^{-k}}(x)|$$

(replacing in part (ii) $\lim_{\delta\to 0}$ *by* $\lim_{k\to\infty}$).

Proof: Instead of proving (4.1) for the sequence $(2^{-k})_{-\infty}^{\infty}$, we shall obtain the stronger inequality

$$(4.8) \qquad \sum_{k=-\infty}^{\infty} \int_{|x|>2|y|} |\phi_{2^{-k}}(x-y)-\phi_{2^{-k}}(x)|\,dx \leq C \qquad (y \in \mathbb{R}^n)$$

Observe that the left hand side of this inequality remains unchanged if we substitute y by $2y$, so that we can assume: $\frac{1}{2} \le |y| \le 1$. Then, we study the sums corresponding to $k \ge 0$ and to $k < 0$ separately:

$$\sum_{k=0}^{\infty} (\dots) \le 2 \sum_{k=0}^{\infty} \int_{|x|>2^{k-1}} |\phi(x)|dx =$$
$$= 2 \int_{|x|>1/2} |\phi(x)| \sum_{k=0}^{\infty} \chi_{\{|x|>2^{k-1}\}}(x) \le$$
$$\le 2 \int_{|x|>1/2} |\phi(x)|(2 + \frac{\log |x|}{\log 2})dx = C_1 < \infty$$

due to (4.6). For the remaining sum we have to use (4.7):

$$\sum_{k=-\infty}^{-1} (\dots) \le \sum_{k=1}^{\infty} \omega_1(2^{-k}|y|) \le \sum_{k=1}^{\infty} \omega_1(2^{-k}) \le$$
$$\le 2 \int_0^1 \frac{\omega_1(t)}{t} dt = C_2 < \infty$$

and this completes the proof. $\square$

Examples 4.9. (a) Given a function $\Omega(x)$ in $\mathbb{R}^n$ which is positive, homogeneous of degree 0 and integrable over the unit sphere, we consider the following nonisotropic maximal function, which was suggested by E.M. Stein and studied by R. Fefferman [1]:

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\le r} |f(x-y)|\Omega(y')dy \qquad (y' = \frac{y}{|y|})$$

This is the maximal operator associated to $\phi(x) = \Omega(x')\chi_B(x)$, where B is the unit ball. Since $\phi \in L^1(\mathbb{R}^n)$ and has compact support, (4.6) is verified. Now, as we did in Chapter II for the homogeneous kernels of singular integrals, we define

$$\omega_1(\Omega,t) = \sup_{|h|\le t} \int_{|x'|=1} |\Omega(x'+h)-\Omega(x')|d\sigma(x')$$

and we try to see the relationship between $\omega_1(\Omega,t)$ and the L^1-modulus of continuity of ϕ, $\omega_1(\phi,t)$. If $|h| \le t \le 1$, we have

$$\int_{\mathbb{R}^n} |\phi(x+h)-\phi(x)|dx \le 2 \int_{\{|x|\le 2t\}\cup\{1-t\le|x|\le 1\}} |\phi(x)|dx$$
$$+ \int_t^1 r^{n-1}dr \int_{|x'|=1} |\Omega(x'+\frac{h}{r}) - \Omega(x')|d\sigma(x') \le$$
$$\le C\, t + \int_t^1 \omega_1(\Omega, \frac{t}{r})\, r^{n-1}dr$$

and therefore

$$\int_0^1 \omega_1(\phi,t)\,\frac{dt}{t} \le C + \frac{1}{n}\int_0^1 \omega_1(\Omega,t)\,\frac{dt}{t}$$

Thus, we can state our result:

"*If* $\int_0^1 \omega_1(\Omega,t)\,\frac{dt}{t} < \infty$, *then* M_Ω *is bounded in* $L^p(\mathbf{R}^n)$, $1 < p \le \infty$, *and of weak type* $(1,1)$".

In fact, this is true for the operator N_Ω defined by taking the supremum over all $r = 2^k$, $k \in \mathbf{Z}$ (due to 4.5), but, since $\phi_r(x) \le 2^n \phi_{2^k}(x)$ when $2^{k-1} \le r < 2^k$, we have $M_\Omega f(x) \le 2^n N_\Omega f(x)$ for all positive f, and the result for M_Ω follows.

(b) For every $\alpha > 0$, consider the kernel

$$\Phi^\alpha(x) = \frac{2}{\Gamma(\alpha)} \max(1 - |x|^2,\ 0)^{\alpha-1}$$

Then $\Phi^\alpha \in L^1(\mathbf{R}^n)$, has compact support and its L^1-modulus of continuity satisfies $\omega_1(\Phi^\alpha,t) \le C_\alpha t^\alpha$, so that the Dini condition (4.7) is verified. Therefore, the maximal operator

$$N^\alpha f(x) = \sup_{k\in\mathbf{Z}} \left|\int f(x-2^{-k})\Phi^\alpha(y)dy\right|$$

is bounded in $L^p(\mathbf{R}^n)$, $1 < p \le \infty$, and of weak type $(1,1)$. An easy computation shows that

$$\lim_{\alpha\to 0} \Phi^\alpha = \sigma \qquad \text{(in the distribution sense)}$$

where σ (Lebesgue measure in the unit sphere) is identified with a singular Borel measure in $\mathbf{R}^n$ supported in $\{x \in \mathbf{R}^n : |x| = 1\}$. Therefore, the limiting operator (corresponding to $\alpha = 0$) is given by

$$Nf(x) = N^0 f(x) = \sup_{k\in\mathbf{Z}} \left|\int_{|y'|=1} f(x-2^{-k}y')d\sigma(y')\right|$$

However, the constants obtained by application of Theorem 4.5 to N^α blow up when $\alpha \to 0$, and nothing can be said about the maximal operator N by the methods developped so far. Estimates for Nf will be obtained in 5.19 below.

Now, following Fefferman and Stein [1], we shall use the vector valued inequalities for the Hardy-Littlewood maximal operator in order to obtain some estimates for Marcinkiewicz integrals.

Given a closed set F whose complement has finite Lebesgue measure, the Marcinkiewicz integral of order $\lambda > 0$ corresponding to F is the function:

$$H_\lambda(F;x) = H_\lambda(x) = \int_{\mathbb{R}^n \setminus F} \frac{\delta(y)^\lambda}{|x-y|^{n+\lambda} + \delta(x)^{n+\lambda}}\, dy$$

where $\delta(y)$ denotes the distance of the point y from F.

On the other hand, given a family of disjoint balls (or cubes) $\{B_j\}$, the Marcinkiewicz integral of order λ corresponding to this family is defined as:

$$S_\lambda(x) = \sum_j \frac{d_j^{n+\lambda}}{|x-c_j|^{n+\lambda} + d_j^{n+\lambda}}$$

where c_j denotes the center of B_j and d_j its diameter.

COROLLARY 4.10. *Given* $\lambda > 0$, *we let* $q = 1 + \frac{\lambda}{n}$. *Then, the Marcinkiewicz integrals* H_λ *and* S_λ *defined above satisfy:*

$$\int_{\mathbb{R}^n} H_\lambda(x)^p dx \le C_{p,\lambda}\ |\mathbb{R}^n \setminus F| \qquad (\tfrac{1}{q} < p < \infty)$$

$$\int_{\mathbb{R}^n} S_\lambda(x)^p dx \le C_{p,\lambda}\ \sum_j |B_j| \qquad (\tfrac{1}{q} < p < \infty)$$

Moreover, we have the corresponding weak type inequalities for $p = \frac{1}{q}$:

$$|\{x : H_\lambda(x) > t\}| \le C_\lambda\ t^{-1/q}\ |\mathbb{R}^n \setminus F|$$

$$|\{x : S_\lambda(x) > t\}| \le C_\lambda\ t^{-1/q}\ \sum_j |B_j|$$

Proof: Let us consider first S_λ. If B denotes the unit ball centered at the origin, it is very easy to see that: $M(\chi_B)(x) \ge$ $\ge \mathrm{Const.}(1 + |x|^n)^{-1}$. (Here M stands for the Hardy-Littlewood maximal operator). By translation and dilation invariance of M, we have

$$M(\chi_{B_j})(x) \ge C \frac{d_j^n}{|x-c_j|^n + d_j^n}$$

and therefore

$$S_\lambda(x) \le C' \sum_j M(\chi_{B_j})(x)^q$$

The inequalities for S_λ result now from an application of Corollary 4.3 with $f_j = \chi_{B_j}$. It is clear that this argument works equally well for a family of cubes, instead of balls.

In order to prove the results for H_λ, we first observe the dilation invariance of H_λ in the following sense: $H_\lambda(tF;x) = H_\lambda(F;\frac{x}{t})$. This allows us to assume $|\mathbb{R}^n\setminus F| = 1$. Now, taking into account that $\delta(y) \leq \delta(x) + |x-y|$

$$H_\lambda(x) \leq C_\lambda \int \frac{\delta(y)^\lambda}{|x-y|^{n+\lambda} + \delta(y)^{n+\lambda}}\, dy = C_\lambda \sum_j \int_{Q_j}$$

where $\mathbb{R}^n\setminus F = \bigcup_j Q_j$ is a Whitney decomposition of $\mathbb{R}^n\setminus F$ (see Stein [1], p. 16), i.e., $\{Q_j\}$ are disjoint cubes (with center c_j and diameter d_j) and

$$A^{-1}d_j \leq \delta(x) \leq Ad_j \qquad (x \in Q_j)$$

for some absolute constant A. Therefore, letting S_λ be the Marcinkiewixz integral corresponding to the family $\{Q_j\}$

$$H_\lambda(x) \leq C'_\lambda \sum_j |Q_j| \frac{d_j^\lambda}{|x-c_j|^{n+\lambda} + d_j^{n+\lambda}} \leq C'_\lambda\, S_\lambda(x) \qquad (x \in F)$$

On the other hand, for $x \notin F$ we can make the trivial estimate

$$\int_{Q_j} \frac{\delta(y)^\lambda}{|x-y|^{n+\lambda} + \delta(y)^{n+\lambda}} \leq C\,|Q_j|\,\frac{d_j^\lambda}{d_j^{n+\lambda}} \leq C$$

and, since $|\mathbb{R}^n \setminus F| = 1$, the results for H_λ are now a consequence of the corresponding inequalities for S_λ. $\square$

We conclude this section by showing how vector valued singular integrals can be used to prove the weak type (1,1) for

$$T^*f(x) = \sup_{\varepsilon>0} \left|\int_{|y|>\varepsilon} K(y)f(x-y)dy\right|$$

where K is the kernel of a regular singular integral operator. This fact was stated without proof in Chapter II, Theorem 5.20, but we did prove there that T^* is bounded in L^p for all $1 < p < \infty$.

Instead of T^*, we shall deal with the maximal operator obtained from smooth truncations

$$\tilde{T}^*f(x) = \sup_{\varepsilon>0} \left|\int K(y)h(\tfrac{|y|}{\varepsilon})f(x-y)dy\right|$$

where $h(t)$ is a C^1 function such that $0 \le h(t) \le 1$, $h(t) = 0$ when $0 \le t \le 1$ and $h(t) = 1$ when $2 \le t < \infty$. The first thing to notice is that

$$|T^*f(x) - \tilde{T}^*f(x)| \le \sup_{\varepsilon>0} \int_{\varepsilon<|y|<2\varepsilon} |K(y)f(x-y)|dy$$

$$\le \sup_{\varepsilon>0} C\,\varepsilon^{-n} \int_{|y|<2\varepsilon} |f(x-y)|dy \le C\,Mf(x)$$

Thus, estimates for T^* or $\tilde{T}^*$ are equivalent. The second observation is that $|h(|x-y|) - h(|x|)| \le C|y|/|x|$ whenever $|x| > 2|y|$. Therefore, if $K^\varepsilon(x) = K(x)h(|x|/\varepsilon)$ and $|x| > 2|y|$, we have

$$|K^\varepsilon(x-y) - K^\varepsilon(x)| \le |K(x-y) - K(x)|\; h(\tfrac{|x-y|}{\varepsilon}) +$$

$$+ |K(x)|\; |h(\tfrac{|x-y|}{\varepsilon}) - h(\tfrac{|x|}{\varepsilon})| \le$$

$$\le B\;|y|\;|x|^{-n-1} + B|x|^{-n}\,C|y|\;|x|^{-1} \le C'|y|\;|x|^{-n-1}$$

with C' independent of ε. This implies the inequality (4.1) for the family $(K^\varepsilon)_{\varepsilon>0}$, and since $\tilde{T}^*$ is bounded in L^p, the remark preceding 4.5 applies and the weak type (1,1) for $\tilde{T}^*$ follows.

Moreover, since we have really proved that the ℓ^∞-valued kernel $(K^\varepsilon)_{\varepsilon>0}$ satisfies (3.15), and we know that T^* (and therefore also $\tilde{T}^*$) is bounded in $L^p(w)$ if $w \in A_1$ and $1 < p < \infty$ (see Chapter IV, 3.6) we can appeal to Theorem 3.16 and state the following

COROLLARY 4.11. *Let* T *be a regular singular integral operator, and let* T^* *be the associated maximal operator defined as above. Then, for every weight* $w \in A_1$

$$w(\{x : T^*f(x) > t\}) \le C_w\, t^{-1} \int |f(x)|w(x)dx$$

5. APPLICATIONS: SOME LITTLEWOOD-PALEY THEORY

In this section we continue the application of the theorems for vector valued singular integrals to some important linearizable operators. This program was carried out in section 4 for ℓ^∞-valued kernels, and here we shall undertake the case of Hilbert space valued kernels. This will lead us to various kinds of square functions, one of the typical ingredients of what is called Littlewood-Paley theory. We emphasize, however, that this theory is much deeper and richer than what we can afford to present here, which must be considered only as an introduction to some of its basic aspects.

Suppose we are given a sequence of kernels $k_j(x)$ in $\mathbb{R}^n$ and we form the square function

$$Gf(x) = \left(\sum_{j=-\infty}^{\infty} |k_j * f(x)|^2\right)^{1/2}$$

Then $Gf(x) = |Tf(x)|_B$, where $B = \ell^2$ and T is the linear operator taking $f(x)$ into the B-valued function $(k_j * f(x))_{j=-\infty}^{\infty}$. By Plancherel's theorem, G (equivalently T) is bounded in $L^2(\mathbb{R}^n)$ if and only if

$$(5.1) \qquad \sum_j |\hat{k}_j(\xi)|^2 \leq C \qquad (\xi \in \mathbb{R}^n)$$

If this is verified, Corollary 3.10 can be applied provided that we also have

$$(5.2) \qquad \int_{|x|>2|y|} \left(\sum_j |k_j(x-y) - k_j(x)|^2\right)^{1/2} dx \leq C \qquad (y \in \mathbb{R}^n)$$

Looking for some concrete examples satisfying (5.1) and (5.2) we find out the following:

<u>THEOREM 5.3</u>. *Let* $\phi(x)$ *be an integrable function in* $\mathbb{R}^n$ *such that*

$$\hat{\phi}(0) = \int \phi(x)dx = 0$$

and assume that, for some $\alpha > 0$, *it verifies*

$$(5.4) \qquad |\phi(x)| \leq C\,|x|^{-n-\alpha} \qquad (x \in \mathbb{R}^n)$$

and

$$(5.5) \qquad \int |\phi(x+h) - \phi(x)|dx \leq C\,|h|^{\alpha} \qquad (h \in \mathbb{R}^n)$$

(i.e. $\omega_1(t) \leq C\,t^{\alpha}$). *Then, the operators*

$$Gf(x) = \left(\sum_{j=-\infty}^{\infty} |\phi_{2^j} * f(x)|^2\right)^{1/2}$$

and

$$\Delta f(x) = \left(\int_0^{\infty} |\phi_t * f(x)|^2 \frac{dt}{t}\right)^{1/2}$$

are bounded in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, *and of weak type* $(1,1)$. *They are also bounded operators from* $H^1(\mathbb{R}^n)$ *to* $L^1(\mathbb{R}^n)$.

As usual, we have denoted by ϕ_t the dilation of ϕ : $\phi_t(x) = = t^{-n}\,\phi(\frac{x}{t})$. Observe that Δf is the continuous analogue of Gf (this is more evident if we make the change of variables $t = 2^s$ in the definition of Δf).

Proof of the Theorem: The results for Gf will follow from Corollary 3.10 once we show that the kernels $(\phi_{2^j})_{j=-\infty}^{\infty}$ verify (5.1) and (5.2). Similarly, for Δf we must verify the conditions:

$$(5.1') \qquad \int_0^\infty |\hat{\phi}(t\xi)|^2 \frac{dt}{t} \le C \qquad (\xi \in \mathbb{R}^n)$$

and

$$(5.2') \qquad \int_{|x|>2|y|} \left(\int_0^\infty |\phi_t(x-y)-\phi_t(x)|^2 \frac{dt}{t}\right)^{1/2} dx \le C \quad (y \in \mathbb{R}^n)$$

The properties (5.1) and (5.1') are derived from two different estimates for the Fourier transform of ϕ : For small $|\xi|$ we have

$$|\hat{\phi}(\xi)| = \left|\int \phi(x)\left[e^{-2\pi i x.\xi}-1\right]dx\right| \le \int_{|x|\le|\xi|^{-1/2}} +$$

$$+ \int_{|x|>|\xi|^{-1/2}} \le 2\pi\,\|\phi\|_1 |\xi|^{1/2} + C\int_{|\xi|^{-1/2}}^\infty t^{-1-\alpha}dt \le C\,|\xi|^\beta$$

with $\beta = \inf(\frac{1}{2}, \frac{\alpha}{2}) > 0$. The estimate for large $|\xi|$ is based on the fact that $\hat{\phi}(\xi)(e^{2\pi i h.\xi}-1)$ is the Fourier transform of $\phi(x+h)-\phi(x)$, and choosing $h = \xi/(2|\xi|^2)$ we get

$$2|\hat{\phi}(\xi)| \le \int|\phi(x+h)-\phi(x)|dx \le C\,|\xi|^{-\alpha}$$

Since the left hand side of (5.1') is invariant under dilations of ξ, it can be assumed that $|\xi| = 1$, and then

$$\int_0^\infty |\hat{\phi}(t\xi)|^2 \frac{dt}{t} \le \int_0^1 C\,t^{2\beta-1}\,dt + \int_1^\infty t^{-1-2\alpha}\,dt = C_1 < \infty$$

The proof of (5.1) is quite similar, and it is left to the reader.

Now, we observe that (5.4) and (5.5) are respectively stronger than the hypothesis of Theorem 4.5, under which, the inequality

$$\int_{|x|>2|y|} \sum_{-\infty}^{\infty} |\phi_{2^{-k}}(x-y)-\phi_{2^{-k}}(x)|dx \le C$$

(which is stronger than (5.2)) was obtained. Thus, only (5.2') remains to be proved, and again, by dilation invariance, we can assume that

$|y| = 1$. Then, Schwarz inequality and Fubini's theorem are used to majorize the left hand side by

$$\{\int_{|x|>2} |x|^{-n-\alpha/2}dx\}^{1/2}\{\int_{|x|>2}\int_0^\infty |\phi_t(x-y)-\phi_t(x)|^2|x|^{n+\alpha/2}\frac{dt}{t}dx\}^{1/2}$$

$$\leq C_\alpha \{\int_0^\infty t^{\alpha/2}(\int_{|x|>2/t}|\phi(x-\tfrac{y}{t})-\phi(x)|^2|x|^{n+\alpha/2}dx)\frac{dt}{t}\}^{1/2}$$

$$= C_\alpha \{\int_0^\infty I(t)t^{\alpha/2}\frac{dt}{t}\}^{1/2}$$

But $|\phi(x - \frac{y}{t}) - \phi(x)| \leq C|x|^{-n-\alpha}$ when $|x| > \frac{2}{t} = \frac{2|y|}{t}$ (due to (5.4)) and therefore

$$I(t) \leq \int_{|x|>2/t} C|x|^{-n-3\alpha/2}dx \leq C\, t^{3\alpha/2}$$

On the other hand, if we use (5.5) and the fact that $\phi(x)$ is bounded:

$$I(t) \leq C\int|\phi(x - \tfrac{y}{t}) - \phi(x)|dx \leq C\, t^{-\alpha}$$

and combining both estimates, we finally obtain

$$\int_0^\infty I(t)t^{\alpha/2}\frac{dt}{t} \leq C\{\int_0^1 t^{2\alpha-1}dt + \int_1^\infty t^{-1-\alpha/2}dt\} < \infty. \qquad \Box$$

Remarks 5.6. (a) The hypothesis (5.4) and (5.5) can be slightly relaxed, but they are general enough to cover all cases of interest. In particular, both of them are trivially verified if $\phi \in S(\mathbb{R}^n)$. However, the hypothesis $\hat{\phi}(0) = 0$ is absolutely necessary.

(b) It occurs quite often that one has the equivalence $\|Gf\|_p \sim \|f\|_p$, and not only the inequality $\|Gf\|_2 \leq C_p\|f\|_p$. In fact, if B is a Hilbert space and $T : L^2 \to L^2_B$ is a linear operator such that

$$\|Tf\|_{L^2_B} = k\,\|f\|_2 \qquad\qquad (f \in L^2)$$

then, an inequality of the form $\|Tf\|_{L^p_B} \leq C\,\|f\|_p$ $(f \in L^2 \cap L^p)$ for some $p \geq 1$, automatically implies $\|f\|_{p'} \leq C\,k^{-2}\|Tf\|_{L^{p'}_B}$. To see this, we apply polarization to the above identity:

$$k^2|\int fg| = |\int\langle Tf(x),\, Tg(x)\rangle dx| \leq \|Tf\|_{L^{p'}_B}\,\|Tg\|_{L^p_B}$$

and take the supremum over all g such that $\|g\|_p \leq 1$.

(c) Vector valued inequalities for the operator G are obtained by an application of Theorem 3.9. In particular, if this is done for $q = 2$, we obtain

$$\|(\sum_{j,k} |f_k * \phi_{2^j}|^2)^{1/2}\|_p \leq C_p \|(\sum_k |f_k|^2)^{1/2}\|_p \qquad (1 < p < \infty)$$

provided that ϕ satisfies the hypothesis of Theorem 5.3.

Our first application of Theorem 5.3 will be one of the classical results obtained (in the periodic setting) by Littlewood and Paley in their fundamental series of papers. In order to state it, same notation must be introduced: The dyadic intervals in $\mathbb{R}$ are those of the form:

$$I_j = [2^{j-1}, 2^j); \qquad -I_j = (-2^j, -2^{j-1}] \quad (j \in \mathbb{Z})$$

and they form a partition of $\mathbb{R} \setminus \{0\}$. The family of dyadic intervals in $\mathbb{R}^n$, denoted by $\Delta = \Delta(\mathbb{R}^n)$, consists of all n-dimensional intervals which are the Cartesian product of n 1-dimensional dyadic intervals. It is plain that the intervals in $\Delta(\mathbb{R}^n)$ are pairwise disjoint, and their union covers almost all $\mathbb{R}^n$. Thus, Plancherel's theorem implies

$$(5.7) \qquad \int |f(x)|^2 dx = \int \sum_{I \in \Delta} |S_I f(x)|^2 dx \qquad (f \in L^2(\mathbb{R}^n))$$

where S_I stands for the partial sum operator: $(S_I f)^\wedge = \hat{f}\chi_I$.

THEOREM 5.8. *There exist constants* $c_p, C_p > 0$ $(1 < p < \infty)$ *such that*

$$(5.9) \qquad c_p \|f\|_p \leq \|(\sum_{I \in \Delta} |S_I f|^2)^{1/2}\|_p \leq C_p \|f\|_p \qquad (f \in L^p(\mathbb{R}^n))$$

When $n = 1$, *we also have the weak type result:*

$$|\{x : (\sum_{I \in \Delta} |S_I f(x)|^2)^{1/2} > \lambda\}| \leq C \lambda^{-1} \|f\|_{H^1} \qquad (f \in H^1(\mathbb{R}))$$

Proof: By the observation made in 5.6.(b), and taking into account (5.7), it is enough to prove the second inequality in (5.9). We shall consider first the case $n = 1$. Take $\phi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\phi}(0) = 0$ and $\hat{\phi}(\xi) = 1$ for all $\xi \in I_0$. Then

$$[S_{I_j}(\phi_{2^{-j}} * f)]^\wedge(\xi) = \hat{f}(\xi)\hat{\phi}(2^{-j}\xi)\chi_{I_j}(\xi) = \hat{f}(\xi)\chi_{I_j}(\xi)$$

i.e., $S_{I_j} f = S_{I_j}(\phi_{2^{-j}} * f)$. Now we use inequality (2.14) and

Theorem 5.3 to get

$$\|(\sum_{j=-\infty}^{\infty} |S_{I_j} f|^2)^{1/2}\|_p = \|(\sum_j |S_{I_j}(\phi_{2^{-j}} * f)|^2)^{1/2}\|_p \leq$$
$$\leq C_p \|(\sum_j |\phi_{2^{-j}} * f|^2)^{1/2}\|_p \leq C'_p \|f\|_p$$

and the same inequality holds for the intervals $\{(-I_j)\}_{j=-\infty}^{\infty}$. Thus, (5.9) is proved in this case. The weak type result follows in the same way by using now (2.15), instead of (2.14):

$$|\{x : (\sum_j |S_{I_j} f(x)|^2)^{1/2} > \lambda\}| \leq C \lambda^{-1} \|Gf\|_1 \leq C' \lambda^{-1} \|f\|_{H^1}$$

where $Gf(x)$ is the square function defined in Theorem 5.3.

The proof for $\mathbb{R}^n$ will be by induction on n, so that we assume that (5.9) holds for $\mathbb{R}^{n-1}$. We shall write a point $x \in \mathbb{R}^n$ as $x = (x_1, x')$ with $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{n-1}$. Let Δ and Δ' denote respectively the families of dyadic intervals in $\mathbb{R}^n$ and in $\mathbb{R}^{n-1}$, so that $\Delta = \Delta_+ \cup \Delta_-$, where

$$\Delta_+ = \{I_j \times I'\}_{j\in\mathbf{Z}, I'\in\Delta'}; \qquad \Delta_- = \{(-I_j)\times I'\}_{j\in\mathbf{Z}, I'\in\Delta'}$$

and it suffices to prove the second inequality in (5.9) for the family Δ_+. Let $\phi \in S(\mathbb{R})$ be defined as in the first part of the proof, and define the operator T_j in $L^p(\mathbb{R}^n)$ as convolution with $\phi_{2^{-j}}$ in the first variable, letting the other variables fixed, i.e.

$$(T_j f)\hat{}\,(\xi_1, \xi') = \hat{f}(\xi_1, \xi')\hat{\phi}(2^{-j}\xi_1)$$

By Fubini's theorem, the remark 5.6.(c) can be applied to the operators T_j to yield

$$\|(\sum_{j,k} |T_j f_k|^2)^{1/2}\|_p \leq C_p \|(\sum_k |f_k|^2)^{1/2}\|_p \qquad (f_k \in L^p(\mathbb{R}^n))$$

and the induction hypothesis implies

$$\|(\sum_{I'\in\Delta'} |S_{\mathbb{R}\times I'} f|^2)^{1/2}\|_p \leq C_p \|f\|_p \qquad (f \in L^p(\mathbb{R}^n))$$

Combining both inequalities, we get

$$\|(\sum_{j\in\mathbf{Z}} \sum_{I'\in\Delta'} |T_j(S_{\mathbb{R}\times I'} f)|^2)^{1/2}\|_p \leq C_p \|f\|_p$$

Finally, for every interval $I = I_j \times I' \in \Delta_+$, one immediatly verifies (by computing Fourier transforms) that: $S_I f = S_I \circ T_j \circ S_{\mathbb{R}\times I'} f$. Then, the truncation argument (based on Corollary 2.13) already used in the case of $\mathbb{R}$ applies again, and this completes the proof. $\square$

One would like to have the Littlewood-Paley inequalities (5.9) for other different decompositions of $\mathbb{R}^n$. For instance, we can consider the decomposition into lacunary spherical shells:

$$\mathbb{R}^n \setminus \{0\} = \bigcup_{k=-\infty}^{\infty} D_k \qquad \text{with} \qquad D_k = \{x : 2^{k-1} \leq |x| < 2^k\}$$

Now, the proof of Theorem 5.8 presents basically two steps. First, the result is proved for a smooth modification of the desired decomposition, by using Theorem 5.3. This works in many other situations and, in particular, in the one we have in mind. Thus, we can state:

COROLLARY 5.10. *Let* $m(t)$ *be a* C^∞ *function with compact support in* $\mathbb{R}_+ = (0,\infty)$ *such that* $m(t) = 1$ *for all* $t \in [1/2,1]$ *and*

$$\sum_{k=-\infty}^{\infty} m(2^{-k}t) = 2 \qquad (0 < t < \infty)$$

For each $f \in L^2 \cap L^p(\mathbb{R}^n)$ *we define the square function*

$$Sf(x) = \Big(\sum_{k=-\infty}^{\infty} \Big|\int \hat{f}(\xi)m(2^{-k}|\xi|)e^{2\pi i x.\xi}\, d\xi\Big|^2\Big)^{1/2}$$

Then, $c_p \|f\|_p \leq \|Sf\|_p \leq C_p\|f\|_p$ *for all* $1 < p < \infty$.

The second step in the theorem just proved is the truncation of the smooth decomposition by means of Corollary 2.13. This part is more specific of n-dimensional intervals (though it also holds in other cases where the proof is harder, see 7.3 below). In particular, we should need in our case the inequality

$$\Big\|\Big(\sum_k |S_{D_k} f_k|^2\Big)^{1/2}\Big\|_p \leq C_p \Big\|\Big(\sum_k |f_k|^2\Big)^{1/2}\Big\|_p$$

$((S_D f)^\wedge = \hat{f}\chi_D)$, and this is certainly false for every $p \neq 2$, since the characteristic function of a sphericall shell is not a multiplier for $L^p(\mathbb{R}^n)$ if $n > 1$ and $p \neq 2$ (C. Fefferman [2]).

In the study of multiplier transformations: $(T_m f)^\wedge = \hat{f}m$, the equivalence of L^p-norms given by (5.9) can be used as a scissor theorem (we have borrowed this expression from A. Córdoba), since it allows to cut the multiplier into its dyadic pieces. To be precise, we have

COROLLARY 5.11. *Let* $m \in L^\infty(\mathbb{R}^n)$, *and let us decompose it as*

$m = \sum_{I \in \Delta} m_I$, *where, for each dyadic interval* I, *we denote* $m_I = m\chi_I$. *Given* $1 < p < \infty$, m *is an* L^p *multiplier, i.e.* $|T_m f|_p \leq C_p |f|_p$, *if and only if the inequality*

$$(5.12) \qquad |(\sum_{I \in \Delta} |T_{m_I} f_I|^2)^{1/2}|_p \leq C_p |(\sum_{I \in \Delta} |f_I|^2)^{1/2}|_p$$

holds for arbitrary functions $f_I \in L^2 \cap L^p(\mathbb{R}^n)$.

Proof: If T_m is bounded in $L^p(\mathbb{R}^n)$, then (5.12) follows from the consecutive application of Corollary 2.13 and the Marcinkiewicz-Zygmund theorem. On the other hand, if (5.12) holds, we take an arbitrary $f \in L^2 \cap L^p$ and apply Theorem 5.8:

$$|T_m f|_p \leq c_p^{-1} |(\sum_I |S_I T_m f|^2)^{1/2}|_p =$$

$$= c_p^{-1} |(\sum_I |T_{m_I} S_I f|^2)^{1/2}|_p \leq C_p |(\sum_I |S_I f|^2)^{1/2}|_p \leq C_p' |f|_p. \qquad \square$$

The corollary remains true if we use instead of Δ any other decomposition of $\mathbb{R}^n$ for which the conclusions of Theorem 5.8 hold; in particular we can take $\tilde{\Delta} = \{I \times \mathbb{R}^{n-k} \ / \ I \in \Delta(\mathbb{R}^k)\}$.

We shall now see how this works to produce significant multiplier theorems. For the sake of simplicity, we shall consider only the case of $\mathbb{R}$. The operators S_t defined by $(S_t f)^\wedge = \hat{f}\chi_{[t,\infty)}$ are uniformly bounded in $L^p(\mathbb{R})$, $1 < p < \infty$, and so is every "linear combination" of them

$$Tf(x) = \int_{\mathbb{R}} \lambda(t) S_t f(x)\, dt \qquad\qquad (\lambda \in L^1(\mathbb{R}))$$

But T is the operator corresponding to the multiplier

$$m(\xi) = \int_{\mathbb{R}} \lambda(t) \chi_{[t,\infty)}(\xi)\, dt = \int_{-\infty}^{\xi} \lambda(t)\, dt$$

and every bounded C^1 function $m(\xi)$ whose derivative is integrable can be written in this form. Thus, we have established a simple result: "*If* $m \in L^\infty(\mathbb{R})$ *is of class* C^1, *and* $m' \in L^1(\mathbb{R})$, *then* $|T_m f|_p \leq C_p |f|_p$, $1 < p < \infty$". The Littlewood-Paley theory provides the following improvement of this result:

COROLLARY 5.13. (Marcinkiewicz multiplier theorem). *Let* $m \in L^\infty(\mathbb{R})$ *be a function of class* C^1 *outside the origin and such that* $\int_I |m'(\xi)|\, d\xi \leq B$ *for every dyadic interval* I. *Then, the operator*

T_m *defined by* $(T_m f)^\wedge = \hat{f}m$ *is bounded in* $L^p(\mathbb{R})$, $1 < p < \infty$.

Proof: For every Schwartz function f, one easily verifies by taking Fourier transforms that

$$T_{m_I} f(x) = m(2^{j-1}) S_I f(x) + \int_I S_t S_I f(x) . m'(t) dt$$

where I is the dyadic interval: $I = [2^{j-1}, 2^j)$, and S_t is defined as above. Therefore, the hypothesis on m imply

$$|T_{m_I} f(x)| \leq C \{ |S_I f(x)| + (\int_I |S_t S_I f(x)|^2 |m'(t)| dt)^{1/2} \}$$

By Corollary 2.13, we have

$$\| (\sum_{I \in \Delta} |T_{m_I} f_I|^2)^{1/2} \|_p \leq C_p \| (\sum_{I \in \Delta} |f_I|^2)^{1/2} \|_p +$$

$$+ C \| (\int_{\mathbb{R}} |S_{J(t)} f_t(.)|^2 |m'(t)| dt)^{1/2} \|_p$$

where, given $t \in \mathbb{R}$, we define $J(t) = [t, \infty) \cap I$ and $f_t = f_I$, if I is the dyadic interval to which t belongs. The second term can be estimated by means of the continuous version of Corollary 2.13 stated in 2.17.(a) so that it is majorized by

$$C_p \| (\int_{\mathbb{R}} |f_t(.)|^2 |m'(t)| dt)^{1/2} \|_p \leq C_p \| (\sum_{I \in \Delta} B |f_I|^2)^{1/2} \|_p$$

Thus, we have proved that (5.12) holds, and the preceding corollary applies. □

The hypothesis of this corollary can be slightly weakened: It suffices to assume that $m \in L^\infty(\mathbb{R})$ and that m has uniformly bounded variation on each dyadic interval. As the reader may guess, there is a corresponding version of the Marcinkiewicz multiplier theorem for $\mathbb{R}^n$. For this we refer to Stein [1], Ch. IV.

We shall now turn to one of the early applications of quadratic expressions, namely, the majorization of maximal functions in order to establish results of pointwise convergence. The idea is roughly as follows: We need to obtain L^p estimates for the maximal operator

$$T^* f(x) = \sup_k |T_k f(x)|$$

and we take an appropriate Schwartz function $\phi(x)$ to make the trivial majorization

$$T^*f(x) \leq \sup_k |T_k f(x) - \phi_{2^{-k}}*f(x)| + \sup_k |\phi_{2^{-k}}*f(x)|$$

$$\leq (\sum_k |T_k f(x) - \phi_{2^{-k}}*f(x)|^2)^{1/2} + CMf(x)$$

where M denotes the Hardy-Littlewood maximal operator. Thus, matters are reduced to estimate the L^p norm of

$$Qf = (\sum_k |T_k f - \phi_{2^{-k}}*f|^2)^{1/2}$$

When $p = 2$, this is easily made by means of the Fourier transform, and the method dates back to Kolmogorov [1], who proved the a.e. convergence of lacunary partial sums of Fourier series for L^2 periodic functions. Now, a basic feature of Littlewood-Paley theory is that it allows to extend orthogonality arguments (which are simple in L^2) to L^p, $1 < p < \infty$. Not surprisingly, therefore, it was up to Littlewood and Paley to extend Kolmogorov's result to L^p functions, $1 < p < \infty$, as we shall now see.

THEOREM 5.14. a) *Let* $m \in L^\infty(\mathbf{R}^n)$ *be a function of class* C^1 *in a neighbourhood of the origin, with* $m(0) = 1$ *and* $|m(\xi)| \leq C|\xi|^{-\varepsilon}$ *for some* $\varepsilon > 0$. *In* $L^2(\mathbf{R}^n)$, *we define the multiplier transformations*

$$(T_k f)^\wedge(\xi) = \hat{f}(\xi)m(2^{-k}\xi) \qquad (k \in \mathbf{Z})$$

Then, the maximal operator $T^*f(x) = \sup_k |T_k f(x)|$ *is bounded in* $L^2(\mathbf{R}^n)$ *and*

$$(5.15) \qquad \lim_{k\to\infty} T_k f(x) = f(x) \quad \text{a.e.} \qquad (f \in L^2(\mathbf{R}^n))$$

b) *Take* $m = \chi_I$ *in part (a), where* I *is bounded open interval in* $\mathbf{R}^n$ *containing the origin. Then,* $T_k f = S_{2^k I} f$, *and the maximal operator* $S^*f(x) = \sup_k |S_{2^k I} f(x)|$ *is in this case bounded in* $L^p(\mathbf{R}^n)$ *for all* $1 < p < \infty$. *As a consecuencia, we have*

$$(5.16) \qquad \lim_{k\to\infty} \int_{2^k I} \hat{f}(\xi)e^{2\pi i x.\xi}d\xi = f(x) \quad \text{a.e.} \quad (f \in \bigcup_{1<p<\infty} L^p(\mathbf{R}^n))$$

c) *In the case* $n = 1$, *(5.16) holds for every* $f \in H^1(\mathbf{R})$, *and*

$$|\{x : S^*f(x) > \lambda\}| \leq C\,\lambda^{-1}\,\|f\|_{H^1} \qquad (f \in H^1(\mathbf{R}))$$

Proof: We shall limit ourselves to prove the maximal inequalities.

The corresponding pointwise convergence results are immediately derived from them, since $\lim T_k f(x) = f(x)$ for all $f \in S(\mathbb{R}^n)$ (due to the fact that $m(0) = 1$).

a) Take $\phi \in S(\mathbb{R}^n)$ such that $\hat{\phi}(0) = 1$. Then $|m(\xi) - \hat{\phi}(\xi)| \leq C|\xi|$ and $|m(\xi) - \hat{\phi}(\xi)| \leq C|\xi|^{-\varepsilon}$, and both inequalities together imply (as in the proof of Theorem 5.3) that

$$\sum_{k=-\infty}^{\infty} |m(2^{-k}\xi) - \hat{\phi}(2^{-k}\xi)|^2 \leq C \qquad (\xi \in \mathbb{R}^n)$$

If we define Qf as above, it follows from Plancherel's theorem that

$$\|Qf\|_2^2 = \int |\hat{f}(\xi)|^2 \sum_{k=-\infty}^{\infty} |m(2^{-k}\xi) - \hat{\phi}(2^{-k}\xi)|^2 d\xi \leq C \|f\|_2^2$$

and this proves: $\|T^* f\|_2 \leq \text{Const.} \|f\|_2$.

b) Take $\psi \in S(\mathbb{R}^n)$ such that $\hat{\psi}(0) = 0$ and $\hat{\psi}(\xi) = 1$ for all ξ in a neighbourhood of ∂I. Then, $(1-\hat{\psi})\chi_I \in C_0^\infty(\mathbb{R}^n)$, and there exists $\phi \in S(\mathbb{R}^n)$ such that $\hat{\phi} = (1-\hat{\psi})\chi_I$. With this ϕ, we define Qf as above, and observe that

$$S_{2^k I} f - \phi_{2^{-k}} * f = S_{2^k I}(\psi_{2^{-k}} * f)$$

Then, the same argument of Theorem 5.8 (i.e., inequality (2.14) plus Theorem 5.3) proves that $\|Qf\|_p \leq C_p \|f\|_p$, which was the only point to be proved.

c) It suffices to observe that, when $n = 1$, the operator Q defined in part (b) maps $H^1(\mathbb{R})$ boundedly into weak-$L^1(\mathbb{R})$ (the proof is again as in Theorem 5.8). $\square$

We have stated part (b) because of the simplicity of its proof. A much deeper result is true, however: If $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, then

$$(5.17) \qquad \lim_{t\to\infty} \int_{tI} \hat{f}(\xi) e^{2\pi i x.\xi} d\xi = f(x) \qquad \text{a.e.}$$

For $n = 1$, this is the theorem of Carleson [3] and Hunt [1]. The result for $\mathbb{R}^n$ can be easily reduced to the one-dimensional case (see C. Fefferman [3]). We must mention, however, that part (c) is sharp in two senses: First of all, (5.17), does not hold for every $f \in H^1(\mathbb{R})$ (one can actually have $\limsup_{t\to\infty} |S_{tI} f(x)| = +\infty$ a.e. for such f); secondly, one cannot replace $H^1(\mathbb{R})$ by $L^1(\mathbb{R})$ in the statement of 5.14.(c). Proofs of both facts, in the periodic setting, will be found in Zygmund [1], Ch. VIII.

An application of Theorem 5.14(a) with $m(\xi) = \chi_B(\xi)$ where B denotes the unit ball in $\mathbb{R}^n$, gives:

COROLLARY 5.18. *For every* $f \in L^2(\mathbb{R}^n)$

$$\lim_{k\to\infty} \int_{|\xi|\leq 2^k} \hat{f}(\xi)e^{2\pi i x.\xi}d\xi = f(x) \qquad \text{a.e.}$$

The corresponding result for $L^p(\mathbb{R}^n)$ is false if $p < 2$ and $n > 1$. This is a consequence of the negative result of C. Fefferman [2] for the ball multiplier together with some general principles to be studied in the next chapter (see VI.2.8(c)). Concerning L^2 functions, it is not known (and this is a beautiful open question) wheter dyadic partial sums can be replaced by arbitrary partial sums or not, i.e.: Does (5.17) hold for every $f \in L^2(\mathbb{R}^n)$ after replacing I by the unit ball B?.

Another application of Theorem 5.14(a) is the following: Remember the kernels introduced in Example 4.9(b)

$$\Phi^\alpha(x) = \frac{2}{\Gamma(\alpha)} \max(1 - |x|^2, 0)^{\alpha-1}$$

which we shall consider now defined for each complex number α with $\mathrm{Re}(\alpha) > 0$. By the usual formula for computing Fourier transforms of radial functions (see Stein-Weiss [1], Ch. IV) we have

$$(\Phi^\alpha)\hat{}\,(\xi) = m^\alpha(\xi) = 2\pi^{1-\alpha}|\xi|^{1-\alpha-n/2} J_{\alpha-1+n/2}(2\pi|\xi|)$$

where, for each $\beta > -\frac{1}{2}$, $J_\beta(t)$ denotes the corresponding Bessel function. The definition of m^α makes sense whenever $\mathrm{Re}(\alpha) > \frac{1-n}{2}$, and for every such α we introduce the maximal operator

$$N^\alpha f(x) = \sup_{k\in\mathbf{Z}} \left|\int \hat{f}(\xi)m^\alpha(2^{-k}\xi)e^{2\pi i x.\xi}d\xi\right| \qquad (f \in S(\mathbb{R}^n))$$

which coincides with the one defined in 4.9(b) when $\alpha \geq 0$ (for $\alpha = 0$, we identified Φ^0 with Lebesgue measure in the unit sphere, σ, and it turns out that $\hat{\sigma}(\xi) = m^0(\xi)$). We can now state.

COROLLARY 5.19. *The maximal operator (defined a priori in* $S(\mathbb{R}^n)$)

$$Nf(x) = \sup_{k\in\mathbf{Z}} \left|\int_{|y'|=1} f(x-2^{-k}y')\, d\sigma(y')\right|$$

is bounded in $L^p(\mathbb{R}^n)$ *if* $n \geq 2$ *and* $1 < p < \infty$.

Proof: In 4.9(b) we have obtained

$$\|N^\alpha f\|_p \leq C_{\alpha,p} \|f\|_p \qquad (1 < p < \infty; \quad \mathrm{Re}(\alpha) > 0)$$

On the other hand, since $t^{-\beta}J_\beta(t)$ is a C^∞ function in $[0,\infty)$ (whose value at $t = 0$ is $2^{-\beta}/\Gamma(1+\beta)$) and $J_\beta(t) \leq C_\beta t^{-1/2}$ $(t \to \infty)$, the multiplier m^α satisfies (except for a multiplicative constant depending on α) the hypothesis of Theorem 5.14(a), and we get

$$\|N^\alpha f\|_2 \leq C_\alpha \|f\|_2 \qquad (f \in S(\mathbb{R}^n), \quad \mathrm{Re}(\alpha) > \tfrac{1-n}{2})$$

We shall omit the technical details needed to conclude the proof, but the reader who is familiar with the interpolation of analytic families of operators (see Stein-Weiss [1], Ch. V) should have no difficulty in interpolating both estimates for the operators N^α, after a suitable linearization, to obtain

$$\|N^\alpha f\|_p \leq C_{\alpha,p} \|f\|_p \qquad (f \in S(\mathbb{R}^n))$$

for $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{2} + \frac{\mathrm{Re}(\alpha)}{n-1}$ and $\frac{1-n}{2} < \mathrm{Re}(\alpha) \leq 0$. The case $\alpha = 0$ is what we wanted to prove. □

To finish this section, we shall describe some results corresponding to continuous type square functions. We begin by the so called Littlewood-Paley g-function, which was first studied by complex methods, in the periodic setting and for $n = 1$, as a previous step to the inequalities of Theorem 5.8 (This approach is described in Zygmund [1], Ch. XIV). For a function $f(x)$ in $\mathbb{R}^n$, we define

$$g(f)(x) = \left(\int_0^\infty t|\nabla u(x,t)|^2 dt\right)^{1/2}$$

where $u(x,t) = P_t * f(x)$ denotes the Poisson integral of f. Then, we have

COROLLARY 5.20. *Let* $1 < p < \infty$. *Then* $f \in L^p(\mathbb{R}^n)$ *if and only if* $g(f) \in L^p(\mathbb{R}^n)$, *and there exists constants* c_p, $C_p > 0$ *such that*

$$c_p \|f\|_p \leq \|g(f)\|_p \leq C_p \|f\|_p$$

Proof: Let us first look at what happens for $p = 2$:

$$\|g(f)\|_2^2 = \int_0^\infty \int_{\mathbb{R}^n} t\{|\tfrac{\partial}{\partial t} u(x,t)|^2 + |\nabla_x u(x,t)|^2\} dxdt$$
$$= \int_0^\infty \int_{\mathbb{R}^n} t|\hat{f}(\xi)|^2 e^{-4\pi|\xi|t}\, 8\pi^2|\xi|^2\, d\xi dt$$
$$= \int_{\mathbb{R}^n} \tfrac{1}{2}\, |\hat{f}(\xi)|^2 d\xi = \tfrac{1}{2}\, \|f\|_2^2$$

Thus, we are in conditions of using the observation 5.6(b), and it will be enough to prove $\|g(f)\|_p \leq C_p \|f\|_p$, $1 < p < \infty$. This will be made separately for the operators

$$g_o(f) = (\int_0^\infty t|\tfrac{\partial}{\partial t} u(x,t)|^2 dt)^{1/2}$$

$$g_k(f) = (\int_0^\infty t|\tfrac{\partial}{\partial x_k} u(x,t)|^2 dt)^{1/2} \qquad (1 \leq k \leq n)$$

Take $\phi(x) = \frac{\partial}{\partial t} P_t(x)\Big|_{t=1}$. A simple computation shows that

$$|\phi(x)| \leq C(1 + |x|^2)^{-(n+1)/2}$$

$$|\nabla\phi(x)| \leq C\, |x|(1 + |x|^2)^{-(n+3)/2}$$

The first inequality shows that (5.4) holds, while the second easily implies (5.5). On the other hand, $\hat{\phi}(\xi) = 2\pi|\xi|e^{-2\pi|\xi|}$, so that $\hat{\phi}(0) = 0$, and we can apply Theorem 5.3 to the operator Δ defined by means of ϕ, which coincides with the operator g_o, because

$$\frac{\partial}{\partial t} u(x,t) = (\frac{\partial}{\partial t} P_t) * f(x) = t^{-1}\phi_t * f(x)$$

The inequalities for $g_k(f)$, $1 \leq k \leq n$, follow in the same way by an application of Theorem 5.3 to the function $\phi(x) = \frac{\partial}{\partial x_k} P_1(x)$. □

The next application is concerned with an operator introduced by Marcinkiewicz. For a function F in $\mathbb{R}$, we define

$$\mu(F)(x) = (\int_0^\infty t^{-3}\ |F(x+t)+F(x-t)-2F(x)|^2 dt)^{1/2}$$

The function $\mu(F)$ is closely connected with the differentiability properties of $F(x)$: At almost every point x at which $F'(x)$ exists, one has $\mu(F)(x) < \infty$ (see Zygmund [1]) and the converse is true if the derivative is taken to exist in the L^2 sense. The result to be proved here is:

COROLLARY 5.21. *Given* $f \in L^1_{loc}(\mathbb{R})$, *we denote by* F *the indefinite integral of* f, *i.e.* $F(x) = \int_0^x f$. *Then, the inequalities*

$$c_p \|f\|_p \leq \|\mu(F)\|_p \leq C_p \|f\|_p$$

hold for all $1 < p < \infty$.

Proof: Define $\phi = \chi_{[-1,0]} - \chi_{[0,1]}$. Then

$$\phi_t * f(x) = t^{-1} \int_0^t [f(x+y) - f(x-y)]dy = t^{-1}[F(x+t) + F(x-t) - 2F(x)]$$

Thus, with this choice of ϕ, $\mu(F)$ equals the function Δf of Theorem 5.3. It is clear that ϕ satisfies (5.4) and (5.5), and $\hat{\phi}(0) = 0$, so that: $\|\mu(F)\|_p \leq C_p \|f\|_p$, $1 < p < \infty$. On the other hand:

$$\int_0^\infty |\hat{\phi}(t\xi)|^2 \frac{dt}{t} = \pi^{-2} \int_0^\infty |1 - \cos 2\pi t|^2 t^{-3} = C < \infty \quad (\xi \neq 0)$$

and this proves the reverse inequality by a new appeal to 5.6(b). □

Finally, we shall mention the continuous analogue of the dyadic maximal operators considered in Theorem 5.14. Given a multiplier $m \in L^\infty(\mathbb{R}^n)$, we define the maximal operator

$$(5.22) \qquad \sup_{R>0} \left(\frac{1}{R} \int_0^R |T_s f(x)|^2 ds\right)^{1/2}$$

where $(T_s f)^\wedge(\xi) = \hat{f}(\xi)m(s\xi)$. As we did for the dyadic case, this maximal operator is trivially majorized by

$$c\, Mf(x) + \left(\int_0^\infty |T_s f(x) - \phi_s * f(x)|^2 \frac{ds}{s}\right)^{1/2}$$

where $\phi \in \mathcal{S}(\mathbb{R}^n)$, and we can state:

COROLLARY 5.23. *The results of Theorem 5.14 remain to hold after replacing* T^*f *by the maximal operator defined by (5.22).*

6. VECTOR VALUED INEQUALITIES AND A_p WEIGHTS

A new application of weighted norm inequalities will be given now: They can be used to obtain inequalities like (1.8) for a (not necessarily linear) operator. This fact was actually at the very source of the A_p theory, since the inequality (see Chapter II, 2.12)

$$\int (Mf(x))^p w(x)dx \leq C_p \int |f(x)|^p Mw(x) \qquad (1 < p < \infty)$$

which is the closest antecedent of A_p weights, was proved by Fefferman and Stein as a tool to obtain the result we stated as Corollary 4.3. This approach to vector valued inequalities is not the one we have followed here, but it is nevertheless interesting and we are going to describe it.

THEOREM 6.1. *Let $p > 0$ and $s \geq 1$ be fixed, and let (T_j) be a sequence of sublinear operators in $L^p(\mathbb{R}^n)$ such that, to each $u \in L^s_+(\mathbb{R}^n)$ we can associate $U \in L^s_+(\mathbb{R}^n)$ with $\|U\|_s \leq \|u\|_s$ and satisfying*

$$(6.2) \qquad \int |T_j f(x)|^p u(x)dx \leq C^p \int |f(x)|^p \, U(x)dx$$

for all $f \in L^p(U)$ and every j. Then, the operators (T_j) are uniformly bounded in $L^q(\mathbb{R}^n)$, where $q = ps'$, and moreover, the following vector valued inequality holds

$$(6.3) \qquad \|(\sum_j |T_j f_j|^p)^{1/p}\|_q \leq C\|(\sum_j |f_j|^p)^{1/p}\|_q \qquad (f_j \in L^q)$$

Proof: We shall verify the second assertion, which consists in a very simple application of Hölder's inequality. Given a sequence (f_j) in $L^q(\mathbb{R}^n)$, there exists $u(x) \geq 0$ such that $\|u\|_s = 1$ and

$$\|(\sum_j |T_j f_j|^p)^{1/p}\|_q = \|\sum_j |T_j f_j|^p\|_{s'}^{1/p} =$$

$$= (\int \sum_j |T_j f_j(x)|^p u(x)dx)^{1/p} \leq (C^p \sum_j \int |f_j(x)|^p \, U(x)dx)^{1/p} \leq$$

$$\leq (C^p \|\sum_j |f_j|^p\|_{s'} \|U\|_s)^{1/p} \leq C \, \|(\sum_j |f_j|^p)^{1/p}\|_q . \qquad \square$$

In the case of the Hardy-Littlewood maximal operator ($T_j = M$ for all j) we can apply the theorem with arbitrary $p > 1$ and $s > 1$ by choosing $U(x) = c_s \, Mu(x)$, where c_s denotes the inverse of the norm of M in $L^s(\mathbb{R}^n)$. Thus, we obtain

$$\|(\sum_j |Mf_j|^p)^{1/p}\|_q \leq C_{p,q}\|(\sum_j |f_j|^p)^{1/p}\|_q \qquad (1 < p \leq q < \infty)$$

which is the more difficult half of the strong type inequalities contained in 4.3. More generally, we can state the following

THEOREM 6.4. *Let (T_j) be a sequence of linearizable operators (in the sense of 1.20) and suppose that, for some fixed $r > 1$, these operators are uniformly bounded in $L^r(w)$ for every weight*

$w \in A_r$, *i.e.*

$$\int |T_j f(x)|^r w(x)dx \leq C_r(w) \int |f(x)|^r w(x)dx \qquad (w \in A_r)$$

with $C_r(w)$ *depending only on the* A_r *constant of* $w(x)$. *Then, the vector valued inequality (6.3) holds for all* $1 < p, \ q < \infty$ *(with* $C = C_{p,q}$).

Proof: It suffices to prove (6.3) for $p = r$ and $1 < q < \infty$, since the extrapolation theorem for A_p weights proved in Chapter IV shows that the hypothesis of the theorem are actually verified for every $r > 1$. If $r < q$ and $s = (q/r)'$, we take, for instance, $\sigma = \frac{s+1}{2}$, so that $1 < \sigma < s$, and observe that, given $u \in L^s_+(\mathbb{R}^n)$, the function $w(x) = \{M(u^\sigma)(x)\}^{1/\sigma}$ satisfies:

i) $u(x) \leq w(x)$
ii) $\|w\|_s \leq C_s \|u\|_s$
iii) $w \in A_1 \subset A_r$ with A_1 constant depending only on σ.

Therefore, the previous theorem can be applied with $U(x) = C_s^{-1} w(x)$, and the proof of this case is complete.

Now, we consider the case $r > q$. We recall that the operators T_j are linearizable, i.e.

$$|T_j f(x)| = \|U_j f(x)\|_B$$

for some linear operators U_j which are uniformly bounded from $L^r(w)$ to $L^r_B(w)$ whenever $w \in A_r$. Here B is a Banach space which, for notational simplicity, will be assumed to be the same for all j. Given a sequence $(f_j) \in L^q(\ell^r)$, there exists a sequence $(g_j) \in$ $\in L^{q'}(\ell^{r'})$ with unit norm and such that

$$\|(\sum_j |T_j f_j|^r)^{1/r}\|_q = \int \sum_j T_j f_j(x) . g_j(x)dx \leq$$

$$\leq \sum_j \int \|g_j(x) . U_j f_j(x)\|_B \, dx =$$

$$= \sup \sum_j \int \langle g_j(x) . U_j f_j(x), \ G_j(x) \rangle \, dx$$

where the supremum is taken over all $G_j \in L^\infty_{B^*}$ with $\|G_j(x)\|_{B^*} \leq 1$ a.e. (see 1.3). Now, if U_j' denotes the adjoint of U_j (which will map B^*-valued functions into complex valued functions):

$$\|(\sum_j |T_j f_j|^r)^{1/r}\|_q \leq \sup \int \sum_j f_j(x).U_j'(g_jG_j)(x)dx$$
$$\leq \|(\sum_j |f_j|^r)^{1/r}\|_q \ \|(\sum_j |U_j'(g_jG_j)|^{r'})^{1/r'}\|_{q'}$$

and matters are reduced to proving that the sequence (U_j') defines a bounded operator from $L^{q'}(\ell^{r'}_{B*})$ into $L^{q'}(\ell^{r'})$. But, since $r' < q'$, this in turn will be a consequence of the part already proved (which works equally well for vector valued functions) once the following weighted norm inequalities are obtained:

$$\int |U_j'F(x)|^{r'} w(x)dx \leq C_{r'}(w) \int \|F(x)\|^{r'}_{B*} w(x)dx \qquad (w \in A_{r'})$$

(with $C_{r'}(w)$ depending only on the $A_{r'}$ constant of w). Given $w \in A_{r'}$, we denote $v = w^{-r/r'} \in A_r$ (observe that the A_r constant of v is equal to the $A_{r'}$ constant of w raised to the power $\frac{r}{r'}$), and take an arbitrary $F \in L^{r'}_{B*}(w)$. There is some $g \in L^r(R^n)$ with $\|g\|_r = 1$ such that

$$(\int |U_j'F(x)|^{r'} w(x)dx)^{1/r'} = \int U_j'F(x) \ w(x)^{1/r'} g(x)dx =$$
$$= \int \langle U_j(w^{1/r'}g)(x),\ F(x)\rangle \, v(x)^{1/r} w(x)^{1/r'} dx \leq$$
$$\leq (\int \|F(x)\|^{r'}_{B*} w(x)dx)^{1/r'} (\int |T_j(v^{-1/r}g)(x)|^r v(x)dx)^{1/r}$$

and the last factor is bounded by $C_r(v)$ by the hypothesis of the theorem. □

As an application of this theorem, we obtain again the vector valued inequalities

$$\|(\sum_j |k_j * f_j|^p)^{1/p}\|_q \leq C_{p,q} \|(\sum_j |f_j|^p)^{1/p}\|_q \qquad (1 < p,q < \infty)$$

where $\|\hat{k}_j\|_\infty \leq C$, $|k_j(x)| \leq C\,|x|^{-n}$ and $|\nabla k_j(x)| \leq C|x|^{-n-1}$, which were proved in section 3.

<u>Remarks 6.5</u>. We mention here some extensions of Theorem 6.4:

a) The conclusion of the theorem can be strenghtened as follows: If $1 < p,q < \infty$ and $w \in A_q$, then the weighted vector valued inequality

$$(6.6) \qquad \|(\sum_j |T_j f_j|^p)^{1/p}\|_{L^q(w)} \leq C_{p,q}(w) \ \|(\sum_j |f_j|^p)^{1/p}\|_{L^q(w)}$$

holds. This can be easily proved by using the relationship between weights of different A_p classes, as in the proof of the extrapolation theorem. A direct proof of (6.6) for the Hardy-Littlewood maximal operator and for singular integrals has been given by Andersen and John [1].

b) The class A_r can be replaced in the statement of the theorem by the class A_r^*. (This relaxes the hypothesis, since $A_r^* \subset A_r$). In this case, the weighted vector valued inequalities of the previous remark still hold, with weights $w \in A_q^*$.

A final comment about Theorem 6.1 is in order. What this theorem (which can be stated for an arbitrary measure space (X,m) instead of $(\mathbb{R}^n,dx)$) shows is that vector valued inequalities for a family of operators can be obtained if one has enough information about the weights associated to these operators. In 6.4, this process was efectively carried out by using our information about A_p weights. What is perhaps more surprising, is that the process can be reversed, and some information about the weighted norm inequalities which an operator verifies can be obtained if one knows that certain vector valued inequalities hold (and the results in sections 1, 2 and 3 can be used to this end). Bringing to light the general principles on which this reversed process is based is the principal aim of the next chapter.

7. NOTES AND FURTHER RESULTS

7.1.- The only linear operators which admit B-valued extensions for every Banach space B are essentially the positive operators. To be precise: Let $T : L^p \to L^q$ be a linear operator having a bounded ℓ^1-valued extension; then, there exists a bounded linear operator $T_+ : L^p \to L^q$ which is positive and verifies: $|Tf(x)| \leq T_+(|f|)(x)$ for every $f \in L^p$. The same happens if ℓ^1 is replaced by ℓ^∞. See Virot [1].

7.2.- Theorem 2.7 was first proved by Paley [1] in the case $q = p$ and with a bigger constant $C_{p,p}$. The version presented here follows Marcinkiewicz and Zygmund [1] rather closely. For a general operator of weak or strong type (p,q), the values of r for which a bounded ℓ^r-valued extensions exists are described in Rubio de Francia and Torrea [1].

An interesting extension of theorem 2.7 which is based on Grothedieck's fundamental theorem is the following: Given Banach lattices A, B and a bounded linear operator $T : A \to B$, then, for arbitrary $f_1, f_2, \ldots, f_n \in A$:

$$\|(\sum_j |Tf_j|^2)^{1/2}\|_B \leq K_G \|T\| \|(\sum_j |f_j|^2)^{1/2}\|_A$$

where K_G is the so called Grothendieck's universal constant, whose exact value is still unknown (however, $1 < K_G < 2$). See Krivine [1].

7.3.- The following result is of interest in relation with the negative result of C. Fefferman described in 2.12(b): Take in $\mathbb{R}^2$ a lacunary sequence of directions $u_j \in \Sigma_1$ (for instance, $u_j = \exp(2^{-j}\pi i)$). Then, the ℓ^2-valued inequality

$$\|(\sum_j |H_{u_j} f_j|^2)^{1/2}\|_p \leq C_p \|(\sum_j |f_j|^2)^{1/2}\|_p \qquad (1 < p < \infty)$$

holds. A related result is that the maximal operator associated to the family of all rectangles having one of their sides parallel to some u_j is bounded in $L^p(\mathbb{R}^2)$, $1 < p \leq \infty$. For a geometric proof of this fact when $p > 2$, see Stromberg [1] and Córdoba and R. Fefferman [2]. The general case, $p > 1$, was proved by Nagel, Stein and Wainger [1].

7.4. The more usual form of the Marcinkiewicz integral corresponding to the closed set F is

$$J_\lambda(x) = \int \delta(y)^\lambda \, |x-y|^{-n-\lambda} dy \qquad (x \in F)$$

This is the natural extensions to $\mathbb{R}^n$ of Marcinkiewicz' original definition for the case $n = 1$, which he successfully applied to several problems of convergence and summability of Fourier series. However, $J_\lambda(x)$ is infinite for all $x \notin F$, and the modified definition, H_λ, has the advantage of being defined over all $\mathbb{R}^n$, while $H_\lambda(x) = J_\lambda(x)$ for every $x \in F$. On the other hand, the Marcinkiewicz integral S_1 corresponding to a family of intervals in $\mathbb{R}$ has been used by Carleson [3] in proving the pointwise convergence of Fourier series. We refer to the expository paper by Zygmund [2] for more details on this.

The integrals J_λ and H_λ have been extended to the parabolic setting (described in II.7) by M. Walias [1] and C. Calderón [2]. The results obtained are exactly as in 4.10, with n replaced by the homogeneous dimension.

7.5.- The statements (i) and (ii) of Theorem 4.2 are due to Zo [1], while 4.3 was proved by C. Fefferman and Stein [1] by the method described in section 6. The unified and simpler approach followed here originates in Rubio de Francia, Ruiz and Torrea [1].

7.6.- Given a Banach space B, if the Hilbert transform has a bounded B-valued extension H^B to $L^q_B(\mathbb{R})$ for some q, then H^B is also bounded in all $L^p_B(\mathbb{R})$, $1 < p < \infty$, and of weak type $(1,1)$. This follows from Theorem 3.4. The good spaces B for which this happens are characterized by a condition called ζ-convexity:

> B *is* ζ*-convex if there exists* $\zeta : B\times B \to \mathbb{R}$ *symmetric, convex in each variable and satisfying:*
>
> $$\zeta(0,0) > 0; \quad \zeta(x,y) \leq |x+y|_B \quad \textit{when} \quad |x|_B \leq 1 \leq \|y|_B.$$

See Burkholder [2] for the sufficiency and Bourgain [1] for the necessity.

The Riesz transform can be defined in L^p_B, $p \geq 1$, when B is ζ-convex, and then, the space $H^1_B(\mathbb{R}^n)$ defined in terms of B-atoms coincides with

$$H^1_B = \{f \in L^1_B \text{ such that } R^B_j f \in L^1_B, \quad j=1,2,\dots,n\}$$

Also, if B is a ζ-convex Banach lattice of sequences, then every regular singular integral operator has a bounded B-valued extension to $L^p_B(\mathbb{R}^n)$, $1 < p < \infty$, and the following extension of the Fefferman-Stein inequalities holds:

$$\int |(Mf_j(x))_{j\in\mathbb{N}}|_B{}^p\, dx \leq C_p \int |(f_j(x))_{j\in\mathbb{N}}|^p_B\, dx$$

$(1 < p < \infty)$. See Bourgain [2].

7.7.- The operator $N = N^o$ defined in 4.9(b) and 5.19 is the dyadic version of Stein's maximal spherical mean:

$$Mf(x) = \sup_{r>o} \left|\int_{|y'|=1} f(x-ry')\,d\sigma(y')\right| \qquad (f \in S(\mathbb{R}^n))$$

For $\operatorname{Re}(\alpha) > \frac{1-n}{2}$ are defines also M^{α}, whose dyadic version N^{α} was considered in 5.19 and (when $\alpha > 0$) in 4.9(b). Stein's theorem is the following:

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \qquad (n \geq 3;\ \frac{n}{n-1} < p \leq \infty)$$

and this is best possible when $n \geq 3$ (see Stein [2]), while the results for $\mathbb{R}^2$ remain unknown. The proof of this theorem is not far avay from the techniques developped in section 5, and we can sketch it: One applies analytic interpolation to the two estimates:

a) $\|M^{\alpha}f\|_p \leq C_{\alpha,p} \|f\|_p \qquad (\operatorname{Re}(\alpha) = 1;\ 1 < p < \infty)$

b) $\|M^{\alpha}f\|_2 \leq C_{\alpha} \|f\|_2 \qquad (\operatorname{Re}(\alpha) > 1 - \frac{n}{2})$

When $\operatorname{Re}(\alpha) = 1$, $|\Phi^{\alpha}(x)| = C_{\alpha}\chi_B(x)$, and M^{α} is essentially the Hardy-Littlewood maximal operator. For the second estimate, one first proves the following inequality which is based on the properties of Bessel functions

$$M^{\alpha}f(x) \leq C_{\alpha-\beta} \sup_{R>0} \left(\frac{1}{R}\int_0^R |T_s^{\beta}f(x)|^2 ds\right)^{1/2} \qquad (\alpha-\beta > \frac{1}{2})$$

(where $(T^{\beta}f)^{\wedge} = \hat{f}m^{\beta}$; see the discussion previous to 5.19), and observes that the maximal operator to the right is bounded in $L^2(\mathbb{R}^n)$ if $\beta > \frac{1-n}{2}$ (by Corollary 5.23). We refer to Stein and Wainger [1] for details.

The fact that the dyadic operator N is bounded in all the L^p spaces (i.e., 5.19) was observed by C. Calderón [1], and also by R. Coifman and G. Weiss.

7.8.- Let S_nf denote the n-th partial sum of the Fourier series of $f \in L^1(\mathbb{T})$. If $1 < p,q < \infty$ and $(\sum_j |f_j|^q)^{1/q} \in L^p$, then

$$\lim_{n\to\infty} \sum_j |S_nf_j(x) - f_j(x)|^q = 0 \qquad \text{a.e.}$$

This is a vector valued version of the Carleson-Hunt theorem which follows very easily from Theorem 6.4 and the weighted norm inequalities for $S^*f = \sup_n |S_nf|$ described in Chapter IV, (see Rubio de Francia [1]).

7.9. Vector valued inequalities can be used to obtain mixed norm estimates, i.e., estimates involving the Benedek and Panzone

$L^{q,p}$-norms defined in 1.6(b): "Let S be a linear operator in $\mathbb{R}^{n+m} = \mathbb{R}^n x \mathbb{R}^m$ which is translation invariant and satisfies

$$\|(\sum_j |Sf_j|^q)^{1/q}\|_p \leq C\|(\sum_j |f_j|^q)^{1/q}\|_p \qquad (f_j \in L^p(\mathbb{R}^{n+m}))$$

for some fixed $p,q \geq 1$. Then S is a bounded operator in $L^{q,p}(\mathbb{R}^n x \mathbb{R}^m)$". As a corollary, every classical singular integral operator in $\mathbb{R}^n$ is bounded in $L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)$ for arbitrary $p_1,p_2,p_n > 1$, and every multiplier operator in $L^p(\mathbb{R}^{n+m})$ is bounded in $L^{2,p}(\mathbb{R}^n x \mathbb{R}^m)$. Details and generalizations can be found in Benedek, Calderón and Panzone [1], Herz and Rivière [1], Rubio de Francia and Torrea [1]. Here we can sketch a proof in the compact case ($\mathbb{T}^{n+m}$ instead of $\mathbb{R}^{n+m}$): First of all, it is not difficult to replace ℓ^q by $L^q(T^n)$ in the vector valued inequality, so that:

$$\int_{\mathbb{T}^n x \mathbb{T}^m} \left(\int_{\mathbb{T}^n} |Sf_u(x,y)|^q du\right)^{p/q} dxdy \leq$$
$$\leq C \int_{\mathbb{T}^n x \mathbb{T}^m} \left(\int_{\mathbb{T}^n} |f_u(x,y)|^q du\right)^{p/q} dxdy$$

Then, given $g \in L^{q,p}$, we define $f_u(x,y) = g(u+x,y)$, and the result is proved because Lebesgue measure and the operator S are translation invariant.

7.10.- A generalization of theorem 2.16 has been recently found by J.L. Rubio de Francia (unpublished). For an arbitrary sequence $\{I_j\}$ of disjoint intervals in $\mathbb{R}$, the inequality

$$\|(\sum_j |S_{I_j} f|^2)^{1/2}\|_p \leq C_p \|f\|_p \qquad (2 \leq p < \infty)$$

holds. The proof is a combination of the ideas in 2.16 and 3.8.

CHAPTER VI

FACTORIZATION THEOREMS AND WEIGHTED NORM INEQUALITIES

A chief objective of this chapter consists in establishing a general principle of equivalence between certain types of weighted and vector valued inequalities. Such a result appears in section 5, and it is essentially a translation into a language more accessible to Fourier analysts of results due to B. Maurey which are obtained in the previous section.

One can actually prove directly the theorems of section 5, with no reference to factorization results (see Rubio de Francia [3], where this is done), but the ideas leading to them form a long road whose milestones would be placed at Kolmogorov [2], Calderón (see Zygmund [1], XIII.1.22), Stein [6], Burkholder [3], Sawyer [1], Nikishin [1], [2] and Maurey [1], [2]. Thus, we felt more reasonable to give a natural context to such theorems by starting from Nikishin's theorem (proved in section 2), wich is also an important general principle in view of its applications, and by giving some results in its double formulation: as factorization of operators and as weighted norm inequalities. For a rather close look at the historical path up to Nikishin's theorem we refer to de Guzmán [2], Ch. II.

We have followed Maurey [1], [2] for the proofs of the main results in the first four sections. The organization of the material in these sections also owes very much to the viewpoints of J. Gilbert and G. Pisier which both generously shared with us. Sections 6, 7 and 8 present a variety of applications of the general principle just obtained.

1. FACTORIZATION THROUGH WEAK-L^p

We must begin by establishing some notation and basic facts. All through this chapter, B will denote a Banach (or r-Banach) space, and (X,m) a σ-finite measure space. By $L^0(X,m) = L^0(m)$ we shall indicate the space of all measurable functions provided with the topology of local convergence in measure:

$$f_j \to 0 \quad (\text{in } L^0(m)) \quad \text{iff} \quad m(E \cap \{x : |f_j(x)| > \lambda\}) \to 0$$

for every $\lambda > 0$ and $E \subset X$ with $m(E) < \infty$. The nonincreasing rearrangement of a function f is the only nonincreasing function $f^* : \mathbb{R}_+ \to \mathbb{R}_+$ which is equimeasurable with f, and it is defined by

$$f^*(t) = \inf\{\lambda > 0 : m(\{x : |f(x)| > \lambda\}) \leq t\}$$

The following facts are easily verified, and the reader is urged to supply a proof for them:

(1.1) *When* $m(X) < \infty$, *a set of functions* $\Phi \subset L^0(m)$ *is bounded (for the convergence in measure) if and only if the function*:

$$C(\lambda) = \sup_{f\in\Phi} m(\{x : |f(x)| > \lambda\})$$

satisfies $\lim_{\lambda\to\infty} C(\lambda) = 0$. *Equivalently*, Φ *is bounded in* $L^0(m)$ *if and only if*

$$\sup_{f\in\Phi} f^*(t) < \infty \qquad \textit{for every} \quad t > 0$$

(1.2) *When* $m(X) < \infty$, *a sublinear operator* $T : B \to L^0(m)$ *is con-continuous (or bounded) in measure if and only if any of the following equivalent conditions hold:*

a) There exists $C(\lambda)$ *with* $\lim_{\lambda\to\infty} C(\lambda) = 0$ *such that*

$$m(\{x : |Tf(x)| > \lambda\|f\|_B\}) \leq C(\lambda) \qquad (\lambda > 0; \quad f \in B)$$

b) For every $t > 0$, *there exists* $K(t) < \infty$ *such that:*

$$(Tf)^*(t) \leq K(t)\|f\|_B \qquad (f \in B).$$

In many applications, X is $\mathbb{R}^n$ and has not finite measure, but we can still apply the preceding results together with:

(1.3) *Let* $X = \bigcup_{j=1}^{\infty} X_j$ *with* $m(X_j) < \infty$. *Then,* $\Phi \subset L^0(m)$ *is bounded if and only if* $\Phi_j = \{f\chi_{X_j} : f \in \Phi\}$ *is bounded in* $L^0(X_j,m)$ *for every* j. *Similarly,* $T : B \to L^0(m)$ *is continuous in measure if and only if so are the operators* $T_j f = (Tf)\chi_{X_j}$ *for all* j.

In V.1, the spaces L^p_B were defined for every $p > 0$. Just in the same way, $L^0_B(m)$ is defined as the space of all strongly measurable B-valued functions, and $F_j \to 0$ in $L^0_B(m)$ simply means that $\|F_j\|_B \to 0$ in $L^0(m)$. The three results we have just stated remain true if B-valued functions are considered instead of complex valued ones. On the other hand, if B is a Banach space, $L^0_B(m)$ is a *complete* metric linear space, so that such general principles as the Banach-Steinhauss or the closed graph theorems can be applied. In particular, we can state the so called Banach continuity principle:

PROPOSITION 1.4. *Let* $T_j : B \to L^0(m)$, $j \in \mathbb{N}$, *be a sequence of continuous linear operators. If*

$$T^*f(x) = \sup_j |T_j f(x)| < \infty \quad \text{a.e.} \qquad (f \in B)$$

Then T^* *is a continuous (sublinear) operator from* B *to* $L^0(m)$, *i.e., for every subset* $E \subset X$ *of finite measure:*

$$m(\{x \in E : T^*f(x) > \lambda\|f\|_B\}) \leq C_E(\lambda) \qquad (f \in B)$$

with $\lim_{\lambda\to\infty} C_E(\lambda) = 0$.

Proof: Let $Uf = (T_j f)_{j=1}^{\infty}$. Then U is a well defined linear operator from B to $L^0_{\ell^\infty}(m)$, and by the closed graph theorem, it is continuous. Since $T^*f(x) = \|Uf(x)\|_{\ell^\infty}$, T^* is continuous. □

Remark 1.5. In the situation described by the last proposition, suppose that

i) The space B is contained in $L^1_{loc}(\mathbb{R}^n)$

ii) The operators T_j are positive: $|T_j f| \leq T_j(|f|)$

iii) B as well as the operators T_j are translation invariant, i.e.

$f \in B$ implies $f_h \in B$, $\|f_h\|_B = \|f\|_B$ and $T_j(f_h) = (T_j f)_h$

for all $j \in \mathbb{N}$ and $h \in R^n$, where $f_h(x) = f(x-h)$. Then, the following alternative holds: Either

a) $T^*f(x) = \sup_j |T_j f(x)|$ is a continuous operator from B to $L^o(R^n)$ or

b) there exists $f \in B$ such that $T^*f(x) = \infty$ a.e.

In fact, if (a) fails to occur, there are $g \in B$ and $E_o \subset R^n$ such that $|E_o| > 0$ and $T^*f(x) = \infty$ for all $x \in E_o$. Then, denoting by $\{q_k\}$ and enumeration of Q^n (the points of R^n with rational coordinates) we define

$$f(x) = \sum_k 2^{-k} |g(x-q_k)|$$

so that $f \in B$ and $T^*f(x) \geq \sup_k 2^{-k} T^*g(x-q_k) = \infty$ for all $x \in E$, where

$$E = \bigcup_k (q_k + E_o) = Q^n + E_o$$

But E has positive measure and a dense subgroup of periods: $q+E = E$ for all $q \in Q^n$, and it follows that $|R^n - E| = 0$.

By a more careful argument (see Katznelson [1], II.3 and Stein [6]) the positivity assumption (ii) can be eliminated, and the alternative still stands.

Now, we shall introduce our (restricted) concept of factorization of operators:

Definition 1.6. *A continuous operator* $T : B \to L^o(m)$ *factors through the Banach (or* r-*Banach) space* $L \subset L^o(m)$ *if there exists a measurable function* $g(x) > 0$ *a.e. and a continuous operator* $T_o : B \to L$ *which make the following diagram commute*

$$\begin{tikzcd} B \arrow[rr, "T"] \arrow[dr, "T_o"'] & & L^o(m) \\ & L \arrow[ur, "M_g"'] & \end{tikzcd}$$

Where M_g *stands for the multiplication operator:* $M_g f = gf$.

We shall be interested in the cases $L = L^p(m)$ and $L = L^p_*(m) =$ = weak - $L^p(m)$.

In order to state our first important result, given an operator $T : B \to L^0(m)$, we shall denote by $\tilde{T}$ its extension mapping sequences in B into sequences in $L^0(m)$:

$$(f_j)_{j=1}^{\infty} \subset B, \qquad \tilde{T}((f_j)_{j=1}^{\infty}) = (Tf_j)_{j=1}^{\infty}$$

THEOREM 1.7. *Let* $T : B \to L^0(m)$ *be a continuous sublinear operator, and let* $0 < p < \infty$. *The following conditions are equivalent:*

a) T *factors through* $L^p_*(m)$

b) There exists a measurable function $w(x) > 0$ *a.e. such that*

$$\int_{\{x:|Tf(x)|>\lambda\}} w(x)dm(x) \leq \left(\frac{\|f\|_B}{\lambda}\right)^p \qquad (f \in B;\ \lambda > 0)$$

c) (only when $m(X) < \infty$*). For every* $\varepsilon > 0$, *there exist* $C_\varepsilon > 0$ *and* $E(\varepsilon) \subset X$ *with* $m(X-E(\varepsilon)) < \varepsilon$ *such that*

$$m(\{x \in E(\varepsilon) : |Tf(x)| > \lambda\}) \leq C_\varepsilon\left(\frac{\|f\|_B}{\lambda}\right)^p \qquad (f\in B;\quad \lambda>0)$$

d) $\tilde{T}$ *is a bounded operator from* $\ell^p(B)$ *to* $L^0_{\ell^\infty}(m)$

Before proving this theorem, a few remarks must be pointed out: First of all, the equivalence of (a), (b) and (c) is rather trivial, and the interest of the theorem lies in the fact that any of them is implied by (d). Observe that, when $B = L^q(u\,dm)$ for some $u(x)>0$, condition (b) is a weighted weak type inequality. Thus, the difference between (this kind of) factorization results and weighted norm inequalities is just a matter of language. Condition (c) was first considered by Nikishin [1], [2], and is sometimes expressed by saying that T is almost bounded from B to $L^p_*(m)$. Finally, to clarify condition (d), we state it more explicitly in the case $m(X) < \infty$: There exists $C(\lambda)$ such that $\lim_{\lambda\to\infty} C(\lambda) = 0$ and

$$(1.8) \qquad \sum_j \|f_j\|_B^p \leq 1 \quad \text{implies} \quad m(\{x : \sup_j |Tf_j(x)| > \lambda\}) < C(\lambda).$$

Proof of Theorem 1.7: In view of (1.3), it can be assumed that $m(X) = 1$ in all cases. Then, we shall prove the following chain of implications:

$$(c) \implies (b) \implies (a) \implies (d) \implies (c)$$

(c) $\implies$ (b): This is very easy. It suffices to take

$$w(x) = \sum_{j=1}^{\infty} k_j \chi_{E(1/j)}(x)$$

where $k_j > 0$ and $\sum_{j=1}^{\infty} k_j C_{1/j} = 1$.

(b) $\implies$ (a). Given $w(x) > 0$ a.e., we define the sets

$$E_j = \{x : \tfrac{1}{j} \le w(x) < \tfrac{1}{j-1}\} \qquad (j=1,2,3,\ldots)$$

which are disjoint and cover all X. Set $g = \sum_j 2^j \chi_{E_j}$, and let us prove that $T_0 f(x) = g(x)^{-1} Tf(x)$ is a bounded operator from B to $L_*^p(m)$:

$$m(\{x : |T_0 f(x)| > \lambda\}) = \sum_{j=1}^{\infty} m(\{x \in E_j : |Tf(x)| > 2^j \lambda\})$$

$$\le \sum_{j=1}^{\infty} j \int_{\{x:|Tf(x)|>2^j\lambda\}} w(x)\,dm(x) \le$$

$$\le \sum_{j=1}^{\infty} j(2^j\lambda)^{-p} \|f\|_B^p = C(\|f\|_B / \lambda)^p.$$

(a) $\implies$ (d): Now, $Tf(x) = g(x) T_0 f(x)$ and we know that T_0 is a bounded operator from B to $L_*^p(m)$. Thus, given $(f_j)_{j\ge1} \in \ell^p(B)$, we have

$$m(\{x : \sup_j |Tf_j(x)| > \lambda\}) \le$$

$$\le m(\{x : |g(x)| > \lambda^{1/2}\}) \sum_j m(\{x : |T_0 f_j(x)| > \lambda^{1/2}\})$$

$$\le C(\lambda^{1/2}) \|T_0\|^p \lambda^{-p/2} \sum_j \|f_j\|_B^p$$

where $C(t) \to 0$ $(t \to \infty)$. Therefore, (1.8) is verified.

(d) $\implies$ (c): Let $C(\lambda)$ be the function appearing in (1.8) and, given $\varepsilon > 0$, take R large enough so that $C(R) < \varepsilon$. We are going to consider the family of measurable subsets $F \subset X$ such that

$$(1.9)\qquad \begin{cases} m(F)\ |Tf(x)|^p > R^p & (\text{a.e. } x \in F) \\ \text{for some } f \in B \text{ with } \|f\|_B \leq 1 \end{cases}$$

If there is some set F satisfying (1.9), let $\mathcal{F}$ be the family of all countable sequences of disjoint sets which verify it. We order $\mathcal{F}$ in the usual way: $(F_j) \prec (G_j)$ when (F_j) is a subsequence of (G_j). By Zorn's lemma, there is a maximal sequence (F_j) in $\mathcal{F}$, and denoting by (f_j) the corresponding elements of B, and also $c_j = m(F_j)^{1/p}$, $F = \bigcup_j F_j$, we have

$$\sup_j |T(c_j f_j)(x)| > R \qquad (\text{a.e. } x \in F)$$

and

$$\sum_j \|c_j f_j\|_B^p \leq \sum_j m(F_j) = m(F) \leq 1$$

Both inequalities together imply $m(F) < C(R) < \varepsilon$, so that the set $E = E(\varepsilon) = X-F$ verifies $m(X-E) < \varepsilon$ and

$$(1.10)\qquad m(\{x \in E : |Tf(x)| > \lambda\}) \leq \left(\frac{R\|f\|_B}{\lambda}\right)^p \qquad (f \in B;\ \lambda > 0)$$

since otherwise, we could choose f with $\|f\|_B = 1$ such that $E' = E \cap \{x : |Tf(x)| > \lambda\}$ satisfies (1.9), thus contradicting the maximality of (F_j). Therefore, (c) is proved in this case. It remains to consider the possibility that no measurable set F satisfied (1.9). But, if this is the case, (1.10) holds with $E = X$ by the same previous argument. $\square$

2. NIKISHIN'S THEOREM

We have already obtained a necessary and sufficiente condition (1.7 (d)) for the factorization of an operator through weak-L^p. However, this condition is not easy to apply on specific examples, and our next step will consist in showing that, when the space B has a certain structure, 1.7(d) is automatically verified (no matter how good the operator T is). Moreover, it is specially simple to check whether the space $B = L^q$ has the required structure or not, and this will lead us to formulate Nikishin's theorem and obtain some applications.

Denote by $(r_j(t))_{j\geq 1}$ the sequence of Rademacher functions, which are defined in the interval $[0,1)$ by

$$r_j(t) = \sum_{k=1}^{2^j} (-1)^{k+1} I_j^k(t)$$

where I_j^k stands for the characteristic function of the interval $[(k-1)2^{-j}, k2^{-j})$. The Rademacher functions form a (not complete) orthonormal system in $L^2([0,1))$ having many interesting and well known properties, among them, the following two which we shall need:

(2.1) *The Rademacher functions form a symmetric sequence in the following sense: Given a measurable function* $F(x_1,x_2,\dots,x_n)$ *in* $\mathbb{R}^n$ *and numbers* $\varepsilon_j = \pm 1$ $(j=1,2,\dots,n)$, *the functions* $F(r_1(t),r_2(t),\dots,r_n(t))$ *and* $F(\varepsilon_1 r_1(t),\varepsilon_2 r_2(t),\dots,\varepsilon_n r_n(t))$ *are equidistributed.*

(2.2) *If* $0 < p < \infty$, *then there exist constants* k_p, $K_p > 0$ *such that Kintchine's inequalities hold:*

$$k_p(\sum_j |\alpha_j|^2)^{1/2} \leq \|\sum_j \alpha_j r_j\|_{L^p([0,1))} \leq K_p(\sum_j |\alpha_j|^2)^{1/2}$$

for every finite sequence (α_j) *of complex numbers.*

Detailed proofs of both properties will be given in Appendix A.1 for completeness. Since $p \leq 2 \leq q$ implies $L^q([0,1)) \subset L^2([0,1) \subset L^p([0,1))$ and $\ell^p \subset \ell^2 \subset \ell^q$ (with continuous inclussions), we observe that Kintchine's inequalities imply

$$\text{(2.3)} \qquad \|\sum_j \alpha_j r_j\|_p \leq K_p(\sum_j |\alpha_j|^p)^{1/p}, \qquad 0 < p \leq 2$$

and

$$\text{(2.3')} \qquad (\sum_j |\alpha_j|^q)^{1/q} \leq k_q^{-1}\|\sum_j \alpha_j r_j\|_q, \qquad 2 \leq q < \infty$$

Both (2.3) and (2.3') hold only for the specified ranges of p's and q's. For instance, taking $\alpha_1 = \alpha_2 = \dots = \alpha_N = 1$ in (2.3) and using Kintchine's inequalities, it follows that $N^{1/2} \leq k_p^{-1}K_p N^{1/p}$, which, for large N, forces $p \leq 2$.

Now, we come to our main result in this section:

THEOREM 2.4. *Let* $0 < p \leq 2$, *and suppose that the space* B *is of (Rademacher) type* p, *which means the following: There is a constant*

$C > 0$ *such that, for every finite sequence* $(f_j) \subset B$.

$$(2.5) \qquad \int_0^1 \|\sum_j r_j(t) f_j\|_B^p \, dt \leq C \sum_j \|f_j\|_B^p$$

Then, every continuous sublinear operator $T : B \to L^0(m)$ *factors through* $L_*^p(m)$.

Proof: It can be assumed with no loss of generality that $m(X) = 1$. Our task is then reduced to showing that (1.8) is verified, and by a simple limit argument, it suffices to show it for a finite sequence $f_1, f_2, \ldots, f_N \in B$ such that $\sum_j \|f_j\|_B^p \leq 1$. For every $t \in [0,1]$, define

$$g_t = \sum_j r_j(t) f_j \in B$$

Then

$$2|Tf_k(x)| \leq |Tg_t(x)| + |T(2r_k(t)f_k - g_t)(x)|$$

and, for each fixed $x \in X$, both terms on the right hand side are equimeasurable as functions of t (since the second one is obtained from the first one by changing $r_j(t)$ into $-r_j(t)$ for all $j \neq k$). Therefore

$$|\{t : |Tg_t(x)| > |Tf_k(x)|\}| \geq \frac{1}{2}$$

($|.|$ stands for Lebesgue measure in $[0,1]$). Since x is fixed, and this holds for $k = 1, 2, \ldots, N$, we can replace $|Tf_k(x)|$ by $\sup_j |Tf_j(x)|$, and thus

$$m(\{x : \sup_j |Tf_j(x)| > \lambda\}) \leq 2\int_X |\{t : |Tg_t(x)| > \lambda\}| dm(x)$$

Let $E = \{t \in [0,1] : \|g_t\|_B > \lambda^{1/2}\}$, and denote by $C(\lambda)$ the function appearing in 1.2(a) which exists by the continuity of T. Then, we apply (5) and Fubini's theorem to get

$$m(\{x : \sup_j |Tf_j(x)| > \lambda\}) \leq 2|E| +$$
$$+ 2\int_0^1 m(\{x : |Tg_t(x)| > \lambda^{1/2}\|g_t\|_B) dt \leq$$
$$\leq 2\, C\, \lambda^{-p/2} + 2\, C(\lambda^{1/2}) = \tilde{C}(\lambda)$$

Since $\tilde{C}(\lambda)$ is independent of N and $\lim_{\lambda \to \infty} \tilde{C}(\lambda) = 0$, we have verified (1.8), and Theorem 1.7 applies. $\square$

According to the terminology introduced in this theorem, (2.3) means that $\mathbb{C}$ = {complex numbers} is of type 2, and hence, of type p for all $0 < p \leq 2$ (in general, type p_0 implies type p for all $0 < p \leq p_0$). We observe also that there is no Banach space of type $p > 2$, due to the fact that (2.3) is best possible.

Examples 2.6(a) Every Banach space is of type 1. More generally, every r-Banach space is of type r.

(b) $B = L^p(\mu)$ is of type $\inf(p, 2)$, $0 < p < \infty$. When $p \leq 2$, this is a trivial consequence of (2.3). When $p > 2$, we use Kintchine's inequalities:

$$\int \|\sum_j r_j(t) f_j\|_B^2 \, dt \leq \{\int\int |\sum_j r_j(t) f_j(x)|^p \, dt \, d\mu(x)\}^{2/p}$$

$$\leq K_p^2 \, \|(\sum_j |f_j|^2)\|_{L^{p/2}(\mu)} \leq K_p^2 \sum_j \|f_j\|_B^2$$

More generally, if B is a Banach space of type p_0, then $L_B^p(\mu)$ is of type $\inf(p, p_0)$.

Some general information about the type of Banach spaces is contained in 9.2. For our purposes, example (b) is specially interesting, since it gives the following immediate aplication:

COROLLARY 2.7. (Nikishin's theorem). *Let* (Y, μ) *be an arbitrary measure space, and let* $T : L^p(\mu) \to L^0(m)$ *be a continuous sublinear operator, with* $0 < p < \infty$. *Then, there exists* $w(x) > 0$ a.e. *such that*

$$\int_{\{x:|Tf(x)|>\lambda\}} w(x) dm(x) \leq (\|f\|_p/\lambda)^q \quad (f \in L^p(\mu);\ \lambda > 0)$$

where $q = \inf(p, 2)$. *Moreover, if* T *is positive (i.e.* $|f(x)| \leq |g(x)|$ *implies* $|Tf(x)| \leq |Tg(x)|$ *a.e.) then we can take* $q = p$.

Proof: Since $L^p(\mu)$ is of type q, the first assertion follows from Theorem 2.4. On the other hand, if T is positive, we shall verify the condition 1.7(d) directly: Given $(f_j)_{j=1}^\infty$ in $L^p(\mu)$ such that $\sum_j \|f_j\|_p^p \leq 1$, we define $f = (\sum_j |f_j|^p)^{1/p}$. Then $\|f\|_p \leq 1$ and $\sup_j |Tf_j(x)| \leq |Tf(x)|$, and therefore

$$m(\{x : \sup_j |Tf_j(x)| > \lambda\}) \leq m(\{x : |Tf(x)| > \lambda\}) \leq C(\lambda)$$

where $C(\lambda)$ is as in 1.2(a) (we have assumed, as always, $m(X) < \infty$). □

The weak point of Nikishin's theorem is that it says nothing about the weight $w(x)$, but no information could really be expected under so general hypothesis. However, the more we know about the operator T and its invariance properties, the more $w(x)$ becomes determined. The next two corollaries will be enough to illustrate this principle.

COROLLARY 2.8. (Stein's theorem). *Let G be a locally compact group with left Haar measure m, and let $T : L^p(G) \to L^0(G)$, $0 < p < \infty$, be continuous in measure, sublinear and invariant under left translations:*

$$T(f_y) = (Tf)_y \qquad (y \in G;\ \ f \in L^p(G))$$

(where $f_y(x) = f(yx)$). Then, for every compact set $K \subset G$, there exists $C_K > 0$ such that

$$m(\{x \in K : |Tf(x)| > \lambda\}) \leq C_K \left(\frac{\|f\|_p}{\lambda}\right)^q$$

with $q = \inf(p,2)$ (or $q = p$ if T is positive). In particular, if G is a compact group, then T is an operator of weak type (p,q).

Proof: Starting from the weighted weak type inequality of Corollary 2.7, and using the left invariance of m and T, we obtain

$$\int_{\{|Tf|>\lambda\}} w(y^{-1}x)\,dm(x) \leq (\|f\|_p/\lambda)^q \qquad (y \in G)$$

Take $h \in L^1_+(G)$ with $\|h\|_1 = 1$, and integrate both sides of the inequality with respect to $h(y)dm(y)$ to get

$$\int_{\{|Tf|>\lambda\}} w*h(x)\,dm(x) \leq (\|f\|_p/\lambda)^q$$

We can assume from the beginning that $w \in L^\infty(G)$. Then, $w*h(x)$ turns out to be continuous and everywhere positive, and it suffices to take $C_K = (\inf_{x\in K} w*h(x))^{-1}$. □

A global weak type result can be proved in the non-compact case by assuming further invariance of the operator T:

COROLLARY 2.9. *Let* $0 < p \leq 2$. *Every sublinear operator* $T : L^p(\mathbb{R}^n) \to L^0(\mathbb{R}^n)$ *which is continuous in measure and invariant under translations and dilations, is of weak type* (p,p):

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq C(\|f\|_p/\lambda)^p$$

If T *is positive, the result is valid for all* $0 < p < \infty$.

Proof: Arguing as in the previous corollary, we obtain

$$\int_{\{x:|Tf(x)|>\lambda\}} w(x)dx \leq (\|f\|_p/\lambda)^p$$

with $w(x)$ continuous and everywhere positive. Now, we replace $f(x)$ by $f^\delta(x) = f(\delta x)$ and use the dilation invariance of T (which means: $T(f^\delta) = (Tf)^\delta$ for every $\delta > 0$) to get the above inequality with $w(x)$ replaced by $w(\delta^{-1}x)$. Letting $\delta \to \infty$, Fatou's lemma gives the desired weak type inequality with $C = w(0)^{-1}$. □

The applications presented below are intended to illustrate the power of Nikishin's and Stein's theorems, but we have limited ourselves to examples which do not require a heavy background nor a long discussion.

EXAMPLES 2.10.(a) Let $\mathbf{T} = [0,1)$ denote the torus. For each $f \in L^1(\mathbf{T})$, the conjugate function was defined in Ch.I as

$$\tilde{f}(x) = \lim_{r\uparrow 1} \tilde{f}_r(e^{2\pi ix}) = \lim_{r\uparrow 1} \sum_k - i \operatorname{sgn}(k)\, r^{|k|}\hat{f}(k)e^{2\pi ikx}$$

where the limit was proved to exist by a short argument from complex function theory. Since $f \to \tilde{f}_r$ is, for each fixed $r < 1$, a continuous operator in $L^1(\mathbf{T})$, we conclude from Banach's continuity principle that the operator: $f \to \tilde{f}_* = \sup_r |\tilde{f}_r|$ is continuous in measure and translation invariant. Thus, 2.8 applies giving Kolmogorov's inequality

$$|\{x : \tilde{f}_*(x) > \lambda\}| \leq C\,\lambda^{-1}\,\|f\|_1$$

Therefore, the basic inequality for the conjugate function is really a consequence of the mere existence of the operator at almost every point. The same can be done for the Hilbert transform in $L^1(\mathbb{R})$ by using 2.9.

(b) Kolmogorov's example of a divergent Fourier series can be modified to produce a function $f \in H^1(\mathbf{T})$ such that

$$S_N f(x) = \sum_{-N}^{N} \hat{f}(k)\, e^{2\pi ikx}$$

diverges a.e. (see Zygmund [1], Ch. VIII). Something more can be said in this direction: "*Given a translation invariant Banach lattice* $B \subset L^1(\mathbf{T})$ *such that the trigonometric polynomials form a dense subspace of* B, *if there exists* $f \in B$ *such that* $\limsup_{N\to\infty} |S_N f(x)| = \infty$ a.e., *then, the same is true for some* $f \in B$ *of analytic type:* $f(x) \sim \sum_{0}^{\infty} \hat{f}(k) e^{2\pi ikx}$". In other words, for the problem of almost everywhere convergence of Fourier series, there is no difference between the spaces B and

$$H_B = \{\text{closure in } B \text{ of analytic polynomials: } \sum_{0}^{N} c_k e^{2\pi ikx}\}$$

In fact, by translation invariance, it is easy to prove (see 1.5) the following alternative for the maximal operator

$$S^*f(x) = \sup_N |S_N f(x)|$$

Either S^* is continuous in measure in H_B, or $S^*f(x) = \infty$ a.e. for some $f \in H_B$. Let us suppose that the first possibility happens. Since every Banach space is of type 1, we must have

$$\int_{\{x:S^*f(x)>\lambda\}} w(x)dx \le \lambda^{-1} \|f\|_B \qquad (f \in H_B)$$

and translation invariance can be put into play to show, as in Corollary 2.8, that we can take $w(x) = c$ (constant.). Now, given a trigonometric polynomial

$$g(x) = \sum_{-N}^{N} c_k\, e^{2\pi ikx}$$

we form another polynomial of analytic type: $h(x) = e^{2\pi iNx} g(x)$, and observe that

$$S_n\, g(x) = e^{-2\pi iNx}\{S_{N+n}\, h(x) - S_{N-n}\, h(x)\}$$

$(0 \le n \le N)$. Therefore:

$$|\{x : S^*g(x) > \lambda\}| \le |\{x : S^*h(x) > \tfrac{\lambda}{2}\}| \le$$
$$\le \frac{2}{c\,\lambda} \|h\|_B = \frac{2}{c\,\lambda} \|g\|_B$$

By density, the weak type inequality holds for all $g \in B$, and then, we should obtain a.e. convergence of Fourier series for all functions in B, which we assumed to be false.

(c) We try to study the almost everywhere convergence of the spherical sums

$$S_R f(x) = \int_{|\xi| \le R} \hat{f}(\xi) e^{2\pi i x.\xi}\, d\xi \qquad (f \in L^p(\mathbb{R}^n))$$

in the case $n > 1$. *If* $1 \le p < 2$, *there exists* $f \in L^p(\mathbb{R}^n)$ *such that* $\limsup_{R \to \infty} |S_R f(x)| = \infty$ a.e. In fact, if this were not the case, we should have $S^* f(x) = \sup_{R>0} |S_R f(x)| < \infty$ a.e. for every $f \in L^p(\mathbb{R}^n)$ (by the same alternative stated in (b)), and by Corollary 2.9, S^* would be an operator of weak type (p,p). A fortiori, the ball multiplier operator S_1 would be of weak type (p,p), and interpolating with the obvious L^2 result:

$$\|S_1 f\|_q \le C_q \|f\|_q \qquad (p < q \le 2)$$

which contradicts Fefferman's negative result (Fefferman [2]). The argument can be modified to prove the same divergence result in L^p, $p < 2$, for every fixed sequence $R_j \to \infty$.

(d) Let (Y,μ) be an arbitrary measure space, and suppose that $T_j : L^p(\mu) \to L^0(m)$ is a sequence of continuous linear operators such that the limit

$$Tf(x) = \lim_j T_j f(x)$$

exists a.e. for every $f \in L^p(\mu)$. Then, the ℓ^2-valued version of this property is equally true: "*For every sequence* (f_k) *such that* $(\sum_k |f_k|^2)^{1/2} \in L^p(\mu)$, *we have*

$$\lim_j \Big(\sum_k |T_j f_k(x) - Tf_k(x)|^2\Big)^{1/2} = 0 \qquad \textit{a.e.}"$$

To prove this, we start from Nikishin's theorem applied to $T^* f(x) = \sup_j |T_j f(x)|$, which gives

$$(2.11) \qquad \tilde{m}(\{x : T^* f(x) > \lambda\}) \le (\|f\|_p/\lambda)^q$$

for some measure $d\tilde{m}(x) = w(x)dm(x)$ with $w(x) > 0$ a.e., and $q = \inf(p,2)$. Now, we linearize T^* by means of the operators

$$T_\nu f(x) = T_{\nu(x)} f(x) = \sum_j T_j f(x) \chi_{E_j}(x) \qquad (E_j = \nu^{-1}(j))$$

where $\nu(x)$ is an integer-valued measurable function. It is clear that (2.11) remains true if we replace $T^* f(x)$ by $|T_\nu f(x)|$, for arbitrary ν, and then, we can invoke the weak type version of the Marcinkiewicz-Zygmund theorem (V. 2.9) to get

$$(2.12) \qquad \tilde{m}(\{x : \|T_\nu F(x)\|_B > \lambda\}) \le C_{p,q} \lambda^{-q} \left(\int \|F(y)\|_B^p d\mu(y)\right)^{q/p}$$

where $B = \ell^2$, $F = (f_k)_{k=1}^\infty \in L_B^p(\mu)$ and $T_\nu F = (T_\nu f_k)_{k=1}^\infty$. To finish the proof, we observe that, once F is fixed, $\nu(x)$ can be chosen so that

$$\sup_j \|T_j F(x)\|_B \le 2\|T_\nu F(x)\|_B$$

with $T_j F = (T_j f_k)_{k=1}^\infty$. Therefore (2.12) gives raise to a weak type inequality for the maximal operator: $F \to \sup_j \|T_j F\|_B$. Since $\|T_j F(x) - TF(x)\|_B \to 0$ a.e. for every $F = (f_k)_{k=1}^\infty$ with a finite number of nonvanishing components, and these ones form a dense subspace of $L_B^p(\mu)$, the same is true for a general $F \in L_B^p(\mu)$.

(e) The fact that, in the theorems of Nikishin and Stein, we do not obtain a bounded operator from L^p to weak-L^p when $p > 2$, deserves a word of explanation. Consider a multiplier operator in $L^2(\mathbf{T}^n)$: $T_m f(x) \sim \sum_k \hat{f}(k)\, m_k e^{2\pi i k.x}$, where $m = (m_k)_{k\in Z^n}$ is a bounded sequence. T_m is bounded in $L^2(\mathbf{T}^n)$, and a fortiori

$$\|T_m f\|_2 \le C\|f\|_2 \le C\|f\|_p \qquad (2 < p \le \infty)$$

Should T_m be of weak type (p,p) for some $p > 2$, it would be bounded in L^q for all $2 < q < p$ (by interpolation), but this is certainly not true for an arbitrary bounded sequence (m_k). The simplest way to see this is as follows: Pick a function $f \in L^2(\mathbf{T}^n)$ such that, for every $q > 2$, $f \notin L^q(\mathbf{T}^n)$, and consider the operator T_ε associated to the multiplier sequence $\varepsilon = (\varepsilon_k)_{k\in Z^n}$ with $\varepsilon_k = \pm 1$. For some choice of signs ε (actually, for almost every ε), we have $T_\varepsilon f \in L^q(T^n)$ for all $q < \infty$ (this can be easily proved by means of Kintchine's inequalities (2.2); see Zygmund [1]; Ch. V). Since $T_\varepsilon(T_\varepsilon f) = f$, it is clear that T_ε cannot be a bounded operator in L^q if $q > 2$.

The maximal Bochner-Riesz operator in $\mathbf{T}^n$

$$S_*^\alpha f(x) = \sup_{R>0} \Big| \sum_{|k|\leq R} \hat{f}(k)(1 - |k|^2/R^2)^\alpha e^{2\pi i k.x} \Big|$$

(see Stein-Weiss [2], Ch. VII) provide another interesting example of the same situation. For every $\alpha > 0$:

$$\|S_*^\alpha f\|_2 \leq C_\alpha \|f\|_2 \leq C_\alpha \|f\|_p \qquad (2 \leq p \leq \infty)$$

but S_*^α is not an operator of weak type (q,q) if $q > \frac{2n}{n-1-2\alpha}$ and $\alpha < (n-1)/2$.

3. FACTORIZATION THROUGH L^p

In this section, we try to carry out a program for the problem of factorization though L^p similar to that of sections 1 and 2 for the factorization though weak-L^p. We begin with an auxiliar result which is interesting in itself.

PROPOSITION 3.1. *Assume that* $m(X) = 1$, *and let* A *be a convex subset of* $L_+^0(m) = \{$*non-negative functions in* $L^0(m)\}$. *Then, for every* $\varepsilon > 0$, *there is a measurable subset* $S(\varepsilon) \subset X$ *such that* $m(X - S(\varepsilon)) < \varepsilon$ *and*

$$\sup_{f\in A} \int_{S(\varepsilon)} f(x)\,dm(x) \leq 2 \sup_{f\in A} f^*(\tfrac{\varepsilon}{2})$$

To clarify the meaning of this statement, observe that, for a single function $f \in L_+^0(m)$, one can define $S(\varepsilon) = \{x : f(x) \leq f^*(\varepsilon)\}$, and then:

$$m(X - S(\varepsilon)) < \varepsilon \quad \text{and} \quad \int_{S(\varepsilon)} f\,dm \leq f^*(\varepsilon)$$

The aim of the proposition is obtaining essentially the same for all the functions in a convex subset of $L_+^0(m)$ simultaneously.

Just as in Theorem 1.7 a certain form of the axiom of choice was used, here we shall need to apply the following lemma (whose proof we give in Appendix A.2) in an essentially nonconstructive way.

MINIMAX LEMMA 3.2. *Let* A *and* B *be convex subsets of certain vec-*

tor spaces, and assume that a topology is given in B *for which it is compact. If the function* $\Phi : A\times B \to \mathbb{R} \cup \{+\infty\}$ *satisfies:*

i) $\Phi(.,b)$ *is a concave function on* A *for each* $b \in B$

ii) $\Phi(a,.)$ *is a convex function on* B *for each* $a \in A$

iii) $\Phi(a,.)$ *is lower semicontinuous on* B *for each* $a \in A$

then, the following identity holds:

$$\min_{b\in B} \sup_{a\in A} \Phi(a,b) = \sup_{a\in A} \min_{b\in B} \Phi(a,b)$$

Proof of 3.1: Define

$$B = \{g \in L^0(m) : 0 \le g(x) \le 1, \quad \int g\, dm \ge 1 - \tfrac{\varepsilon}{2}\}$$

Then, B is a convex $*$-weakly compact subset of $L^\infty(m)$, and the function

$$\Phi(f,g) = \int fg\, dm \qquad (f \in A, \quad g \in B)$$

satisfies the conditions of the minimax lemma. In fact, to verify condition (iii) put $E_t(f) = \{x : f(x) \le t\}$, so that

$$\Phi(f,g) = \sup_{t>0} \Phi_t(f,g), \quad \text{where} \quad \Phi_t(f,g) = \int_{E_t(f)} fg\, dm$$

and each function $\Phi_t(f,.)$ is $*$-weakly continuous. Let us denote $M = \sup_{f\in A} f^*(\frac{\varepsilon}{2})$, and assume that $M < \infty$, since otherwise there is nothing to prove. Then, we observe the following: For every $f \in A$

$$1 - \int \chi_{E_M(f)}\, dm = m(\{x : |f(x)| > M\}) \le$$

$$\le m(\{x : |f(x)| > f^*(\tfrac{\varepsilon}{2})\}) \le \tfrac{\varepsilon}{2}$$

i.e., $f \in A$ implies $\chi_{E_M(f)} \in B$, and therefore

$$\min_{g\in B} \sup_{f\in A} \int fg\, dm = \sup_{f\in A} \min_{g\in B} \int fg\, dm \le$$

$$\le \sup_{f\in A} \int_{E_M(f)} f\, dm \le M$$

If the minimum is attained at $g_0 \in B$, we define $S(\varepsilon) = \{x : g_0(x) \ge \frac{1}{2}\}$, so that

$$\sup_{f\in A} \int_{S(\varepsilon)} f\, dm \le 2 \sup_{f\in A} \int f g_o\, dm \le 2M$$

and

$$m(X-S(\varepsilon)) < 2\int (1 - g_o(x))\,dm(x) \le \varepsilon \qquad \text{q.e.d.} \quad \square$$

Now, we can state the analogue of Theorem 1.7 for our present problem:

THEOREM 3.3. *Let* $T : B \to L^o(m)$ *be a continuous sublinear operator, and let* $0 < p < \infty$. *The following conditions are equivalent:*

a) T *factors through* $L^p(m)$

b) There exists a measurable function $w(x) > 0$ *a.e. such that*

$$\int |Tf(x)|^p w(x)\,dm(x) \le \|f\|_B^p \qquad (f \in B)$$

c) (only when $m(X) < \infty$*). For every* $\varepsilon > 0$, *there exist* $C_\varepsilon > 0$ *and* $E(\varepsilon) \subset X$ *such that* $m(X-E(\varepsilon)) < \varepsilon$ *and*

$$\int_{E(\varepsilon)} |Tf(x)|^p dm(x) \le C_\varepsilon \|f\|_B^p \qquad (f \in B)$$

d) $\tilde{T}$ *is a bounded operator from* $\ell^p(B)$ *to* $L^o_{\ell^p}(m)$

As in the case of factorization through weak-L^p, the interesting part of the theorem is that (d) implies the other three conditions. A more explicit formulation of (d) is: There exists $C(\lambda)$ such that $\lim_{\lambda\to\infty} C(\lambda) = 0$ and

$$(3.4) \qquad \sum_j \|f_j\|_B^p \le 1 \text{ implies } m(\{x : (\sum_j |Tf_j(x)|^p)^{1/p} > \lambda\}) \le \le C(\lambda)$$

Proof of Theorem 3.3: The proof of the implications

$$(c) \implies (b) \implies (a) \implies (d)$$

is quite similar to (and sometimes simpler than) the corresponding implications in Theorem 1.7. Thus, we shall limit ourselves to prove that (d) implies (c). We assume as always $m(X) = 1$. Since $(g^r)^*(t) = g^*(t)^r$ for every $r > 0$, the continuity of $\tilde{T}$ can be expressed as follows:

$$(\sum_j |Tf_j|^p)^*(t) \leq K(t)^p \sum_j \|f_j\|_B^p \qquad (f_j \in B)$$

with $K(t) < \infty$ for each $t > 0$ (see (1.2) and the remarks after (1.3)). Now, we apply our Proposition 3.1 to the convex set

$$A = \{\sum_j |Tf_j|^p : f_j \in B \quad \text{and} \quad \Sigma\|f_j\|_B^p \leq 1\}$$

and we find a set $S(\varepsilon) \subset X$ such that $m(X-S(\varepsilon)) < \varepsilon$ and

$$\int_{S(\varepsilon)} |Tf(x)|^p dm(x) \leq 2\ K(\tfrac{\varepsilon}{2})^p$$

for every $f \in B$ with $\|f\|_B \leq 1$. But this is precisely what (c) asserts. □

In view of Nikishin's theorem, there is now a question which arise in a natural way: Is there a condition on the space B which ensures that every operator defined on B and continuous in measure factors through L^p? Here is all that we can say for $p \neq 2$:

COROLLARY 3.5. (of Nikishin's theorem). *If* $0 < p < 2$ *and* B *is a space of (Rademacher) type* $p+\varepsilon$ *for some* $\varepsilon > 0$, *then, every sub-linear operator defined on* B *and continuous in measure factors through* L^p.

This is essentially contained in Theorem 2.4, since, for finite measure spaces, we have $L_*^{p+\varepsilon} \subset L^p$ (see V.2.8). It is only in the case $p = 2$ that we can partially improve Nikishin's theorem:

THEOREM 3.6. *Let* B *be a Banach space of (Rademacher) type* 2. *Then, every continuous linear operator* $T : B \to L^0(m)$ *factors through* $L^2(m)$.

Proof: The first idea consists in replacing the Rademacher functions by a Gaussian sequence in the definition of type 2. To be precise, we shall prove that, if B is of type 2, then

$$(3.7) \qquad \int_\Omega \|\sum_j z_j(\omega) f_j\|_B^2\, dP(\omega) \leq C \sum_j \|f_j\|_B^2 \qquad (f_j \in B)$$

for some (and hence for all) Gaussian sequence (see Definition V.2.1) $\{z_j(\omega)\}$ in the probability space (Ω,P).

Take momentarily (3.7) for granted, assume as always that $m(X) = 1$

and fix $f_1, f_2, \ldots, f_n \in B$ with $\sum_j \|f_j\|_B^2 \leq 1$. For every $\lambda > 0$, we set

$$Y_\lambda = \{x \in X : (\sum_j |Tf_j(x)|^2)^{1/2} > \lambda\}$$

and, as in the proof of Theorem 2.4, we denote

$$g_\omega = \sum_j z_j(\omega) f_j \in B; \qquad E = \{\omega \in \Omega : \|g_\omega\|_B > \lambda^{1/2}\}$$

Now, from the proof of V.2.4 we recall that, given complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $\sum_j |\alpha_j|^2 = 1$:

$$P(\{\omega : |\Sigma \alpha_j \, z_j(\omega)| > t\}) = 1 - \int_{-t}^{t} e^{-\pi x^2} \, dx$$

In particular, for every $x \in Y_\lambda$, we have

$$P(\{\omega : |T(g_\omega)(x)| > \lambda\}) \geq 1 - \int_{-1}^{1} e^{-\pi x^2} \, dx = k > 0$$

(here we have used the fact that T is <u>linear</u>). Let us put everything together:

$$m(Y_\lambda) \leq k^{-1} \int_\Omega m(\{x \in X : |Tg_\omega(x)| > \lambda\}) dP(\omega) \leq$$

$$\leq k^{-1} \{P(E) + \int_\Omega m(\{x : |Tg_\omega(x)| > \lambda^{1/2} \|g_\omega\|_B\}) dP(\omega)\}$$

$$\leq k^{-1} \{\lambda^{-1} \int \|g_\omega\|_B^2 \, dP(\omega) + C(\lambda^{1/2})\} \leq$$

$$\leq k^{-1} \{C \, \lambda^{-1} + C(\lambda^{1/2})\}$$

where $C(.)$ denotes the function expressing the continuity in measure of T (see (1.2)), and C is the constant in (3.7). Thus, we have obtained condition 3.4 as desired, and it only remains to prove (3.7). We start with a Gaussian sequence $\{y_j(\omega')\}_{j \in \mathbb{N}}$ in a probability space (Ω', P'), and form the sequence $\{z_j(\omega)\}_{j \in \mathbb{N}}$ defined in $\Omega = \Omega' \times [0,1]$ by

$$z_j(\omega) = y_j(\omega') r_j(t) \qquad (\omega = (\omega', t) \in \Omega)$$

The probability we consider in Ω is the product of P' and Lebesgue measure. Since the Rademacher functions are independent random variables, it is a trivial matter to verify that $\{z_j\}_{j \in \mathbb{N}}$ is again a Gaussian sequence. Now, since B is of (Rademacher) type 2:

$$\int_0^1 \|\sum_j r_j(t) y_j(\omega') f_j\|^2 dt \leq C \sum_j |y_j(\omega')|^2 \, \|f_j\|_B^2$$

$(f_j \in B;\ \omega' \in \Omega')$. If we integrate this inequality with respect to $dP'(\omega')$ taking into account that

$$\int |y_j(\omega')|^2 \, dP'(\omega') = \int_{-\infty}^{\infty} x^2 e^{-\pi x^2} dx = c < \infty \quad (j \in \mathbb{N})$$

we obtain (3.7). □

COROLLARY 3.8. *Let* (Y,μ) *be an arbitrary measure space, and let* $T : L^p(\mu) \to L^o(m)$ *be linear and continuous in measure*

a) If $2 \le p < \infty$, *then there exists* $w(x) > 0$ *a.e. such that*

$$\int |Tf(x)|^2 w(x) dm(x) \le \|f\|_p^2 \qquad (f \in L^p(\mu))$$

b) If T *is positive and* $1 \le p < \infty$, *then*

$$\int |Tf(x)|^p w(x) dm(x) \le \int |f(y)|^p d\mu(y) \qquad (f \in L^p(\mu))$$

for some $w(x) > 0$ a.e.

Proof: Part (a) follows directly from the previous theorem. To prove (b) we shall verify 3.4 directly. Given $(f_j) \subset L^p(\mu)$ with $\sum_j \|f_j\|_p^p \le 1$, we define $f = (\sum_j |f_j|^p)^{1/p}$, so that $\|f\|_p \le 1$. Since T is linear and positive:

$$(\sum_j |Tf_j(x)|^p)^{1/p} = \sup_\alpha \sum_j \alpha_j |Tf_j(x)| \le$$

$$\le \sup_\alpha T(\sum_j \alpha_j |f_j|)(x) \le Tf(x)$$

where $\alpha = (\alpha_j)$ lies in the unit ball of $\ell^{p'}$ and $\alpha_j \ge 0$. Therefore

$$m(\{x : (\sum_j |Tf_j(x)|^p)^{1/p} > \lambda\}) \le m(\{x : |Tf(x)| > \lambda\}) \le$$
$$\le C(\lambda)$$

with $C(\lambda)$ as in (1.2). □

EXAMPLES 3.9. (a) Let G be a locally compact group with left Haar measure m, and let $T : L^p(G) \to L^o(G)$ be linear, continuous in measure and invariant under left translations. If $2 \le p < \infty$, then, for every compact $K \subset G$ there exists $C_K > 0$ such that

$$(3.10) \qquad \int_K |Tf(x)|^2 dm(x) \le C_K^2 \|f\|_p^2 \qquad (f \in L^p(G))$$

The proof is as in Corollary 2.8. Observe that assuming the linearity of T allows to improve the weak type inequality of that corollary to the strong type result (3.10).

(b) Let $f \to \tilde{f}$ denote the conjugate function operator on the torus $\mathbf{T} \cong [-\frac{1}{2}, \frac{1}{2})$, and consider the weighted L^2-inequality

$$(3.11) \qquad \int |\tilde{f}(x)|^2 u(x)dx \leq \int |f(x)|^2 v(x)dx$$

The question, raised for the first time by Muckenhoupt [2], is the following: Find all $v(x)$ (resp. all $u(x)$) such that (3.11) holds for some $u(x)$ (resp. some $v(x)$). Although a more systematic study of this kind of problem will be given in section 6, we can already give here the following simple answer due to Koosis [4]:

(3.11) *holds for some non-trivial* $u(x)$ *if and only if* $v^{-1} \in L^1(T)$, *and it holds for some nontrivial* $v(x)$ *if and only if* $u \in L^1(T)$

By a simple duality argument (which we urge the reader to supply) both assertions are actually equivalent, so that we shall only prove the first one. First of all, by translation invariance, we can assume $u(x) > 0$ and $v(x) > 0$ for all x (argue as in Corollary 2.8), and then

$$c = \{\int_0^{1/4} |\cot \pi(x+1/4)|^2 u(x)dx\}^{1/2} > 0$$

Now, if (3.11) holds and we write $f_1 = |f|\chi_{[-1/4,0)}$, we have for every $x \in [0,1/4)$

$$\tilde{f}_1(x) = \int_{-1/4}^{0} |f(y)| \operatorname{cotg} \pi(x-y)dy \geq \operatorname{cotg} \pi(x+\tfrac{1}{4})\int_{-1/4}^{0}|f|$$

and therefore,

$$\|f\|_{L^2(v)} \geq \|f_1\|_{L^2(v)} \geq \|\tilde{f}_1\|_{L^2(u)} \geq c\int_{-1/4}^{0} |f|$$

A similar inequality holds for the intervals $[-1/2, -1/4)$, $[0,1/4)$ and $[1/4, 1/2)$. Thus, the mapping

$$f \to \int f(x)dx = \int f(x)v(x)^{-1}v(x)dx$$

is a continuous linear functional in $L^2(v)$, which means $v^{-1} \in L^2(v)$, i.e. $v^{-1} \in L^1(T)$.

The converse follows from Theorem 3.6., because $v^{-1} \in L^1(T)$

implies that $L^2(v)$ is continuously embedded in $L^1(T)$:

$$|f|_1 \leq |f|_{L^2(v)} \{\int v(x)^{-1}dx\}^{1/2}$$

and therefore, the conjugate function is a continuous linear operator from $L^2(v)$ to $L^1_*(T)$ (and, a fortiori, to $L^o(T)$).

(c) We have a similar result for one of Littlewood-Paley's operators: Let Δ denote the family of dyadic intervals in $\mathbb{R}$, and let H and S_I stand for the Hilbert transform and the partial sum operator corresponding to I, respectively. If $v^{-1} \in L^1(\mathbb{R})$, then the inequality

$$\int_{\mathbb{R}} \sum_{I\in\Delta} |S_I f(x)|^2 u(x)dx \leq \int (|f(x)|^2+|Hf(x)|^2)v(x)dx$$

holds form some $u(x) > 0$. The reader will have no difficulty in filling in the details (use Theorem V.5.8). We point out that the term Hf in the right hand side cannot be omitted.

4. FACTORIZATION OF OPERATORS WITH VALUES IN L^q. THE DUAL PROBLEM

In this section, we are going to consider operators $T : B \to L^q(X,m)$, for some $q > 0$. For such an operator, factorization through $L^p(m)$ means

$$(4.1) \qquad T : B \xrightarrow{T_o} L^p(m) \xrightarrow{M_g} L^q(m)$$

where T_o and M_g are continuous operators. This case, presents some differences which we are going to comment:

a) Since T factors through $L^p(m)$ if and only if T maps B into $L^p(wdm)$ for some $w(x) > 0$, when $0 < p \leq q$ factorization through $L^p(m)$ always occur, because we can take $w \in L^1 \cap L^\infty$ so that $L^q(m) \subset L^p(wdm)$. Thus, only factorization through $L^p(m)$ with $p > q$ is interesting now.

b) In the case $0 < q < p$, an arbitrary measurable function, $g(x) > 0$, does not define a continuous mapping $M_g : L^p(m) \to L^q(m)$. It is necessary that $g \in L^r(m)$, where $1/q = 1/p + 1/r$. In other words, if T factors through $L^p(m)$, then the range of T is contained in $L^p(w\,dm)$ with $w^{-1/p} = g \in L^r(m)$.

Here is the factorization theorem corresponding to this situation

THEOREM 4.2. *Let* $T : B \to L^q(m)$ *be a sublinear operator. If* $0 < q < p < \infty$ *and* $1/q = 1/p + 1/r$, *then the following statements are equivalent:*

(a) T *admits the factorization (4.1) with* $g(x) > 0$, $\|g\|_r \leq 1$ *and* $\|T_o\| \leq C$.

(b) There exists a measurable function $w(x)>0$ *a.e. such that* $\|w^{-1}\|_{r/p} \leq 1$ *and*

$$\int |Tf(x)|^p w(x)\, dm(x) \leq C^p \|f\|_B^p \qquad (f \in B)$$

(c) The following vector valued inequality holds

$$\|(\sum_j |Tf_j|^p)^{1/p}\|_q \leq C(\sum_j \|f_j\|_B^p)^{1/p} \qquad (f_j \in B)$$

In order to emphasize the similarity with our previous factorization theorems (see 1.7 and 3.3), we can state the equivalence between (a) and (c) as follows:

> *A continuous sublinear operator* $T : B \to L^q(m)$ *factors through* $L^p(m)$ *if and only if* $\tilde{T}$ *is a bounded operator from* $\ell^p(B)$ *into* $L^q_{\ell^p}(m)$.

The theorem is somewhat more precise, since it also gives the equality of $\|\tilde{T}\|$ and the factorization norm of T (which is defined as the product of the norms of the operators T_o and M_g). In proving the theorem, we shall need a simple technical result:

LEMMA 4.3. *Given a measurable function* $a(x) \geq 0$ *and numbers* $\alpha > 0$, $s > 1$, *the function*

$$\Phi(b) = \int_X a(x) b(x)^{-\alpha} dm(x) \qquad (b \in L^s_+(m))$$

is convex and weakly lower semicontinuous on $L^s_+(m)$.

Proof: Since the function: $t \to t^{-\alpha}$ is convex in $[0,\infty)$, the first assertion is immediate. Given $k \in \mathbb{R}$, we have to show that

$$A = \{b \in L^s_+(m) : \Phi(b) \leq k\}$$

is weakly closed. But A is convex, and therefore, it suffices to see that A is closed in the norm topology of $L^s(m)$. Now, if

$$b_n \in A, \qquad \|b_n - b\|_s \to 0$$

then, for some subsequence we have $\lim_j b_{n_j}(x) = b(x)$ a.e., and Fatou's lemma implies that $b \in A$. $\square$

<u>Proof of Theorem 4.2</u>. If (a) holds, then (b) holds with the some constant C and $w(x) = g(x)^{-p}$, and conversely. Thus, we only have to prove the equivalence between (b) and (c).

Suppose first that (b) is satisfied, and let $f_j \in B$, $j \in \mathbb{N}$, be such that $\sum_j \|f_j\|_B^p \leq 1$. Then

$$\|(\sum_j |Tf_j|^p)^{1/p}\|_q = \{\int (\sum_j |Tf_j|^p)^{q/p} w^{q/p}\, w^{-q/p} dm\}^{1/q} \leq$$

$$\leq \{\int \sum_j |Tf_j|^p w\, dm\}^{1/p} \{\int w^{-r/p}\, dm\}^{1/r} \leq C$$

Now, if we assume that (c) holds, the minimax lemma 3.2 can be applied as follows: Let $\alpha = p/q > 1$, and define

$$A = \{\sum_j |Tf_j|^p : f_j \in B, \ \sum_j \|f_j\|_B^p \leq 1\}$$

$$B = \{b \in L^{\alpha'}(m) : b(x) \geq 0, \ \|b\|_{\alpha'} \leq 1\}$$

which are both convex sets, and B is also weakly compact in $L^{\alpha'}(m)$. The function $\Phi(a,b)$ is defined in $A \times B$ as

$$\Phi(a,b) = \int a(x) b(x)^{-\alpha} dm(x) =$$

$$= \int \sum_j |Tf_j(x)|^p b(x)^{-\alpha}\, dm(x)$$

where $a = \sum |Tf_j|^p \in A$ and $b \in B$. Observe that, for each $b \in B$, $\Phi(.,b)$ is a linear (therefore concave) function on A, while the previous lemma says that, for each $a \in A$, $\Phi(a,.)$ is convex and lower semicontinuous on B. Now, given $a = \sum |Tf_j|^p \in A$, the inequality:

$$\int a(x)^{1/\alpha}dm(x) \leq \{\int a(x)b(x)^{-\alpha}dm(x)\}^{1/\alpha}\{\int b(x)^{\alpha'}dm(x)\}^{1/\alpha'}$$
$$= \Phi(a,b)^{1/\alpha}$$

holds for all $b \in B$, but for a certain $b \in B$, it becomes equality, according to the reciprocal of Hölder's inequality, and thus

$$\min_{b\in B} \Phi(a,b) = \{\int(\sum_j |Tf_j|^p)^{q/p}\}^{p/q} \leq C^p$$

This being true for every $a \in A$, the minimax lemma gives

$$\min_{b\in B} \sup_{a\in A} \Phi(a,b) \leq C^p$$

i.e., there exists $b \in B$ such that

$$\int |Tf(x)|^p\, b(x)^{-\alpha}\, dm(x) \leq C^p$$

for all $f \in B$ with $\|f\|_B \leq 1$. Then, $w(x) = b(x)^{-\alpha}$ satisfies condition (b), because $\|w^{-1}\|_{r/p} = \|b\|_{\alpha'}^{\alpha-1} \leq 1$. □

We shall now study the dual factorization problem, which consists in giving conditions for the factorization through $L^p(m)$ of an operator $T : L^q(m) \to B$, i.e., we try to decompose T in the form

$$(4.4) \qquad T : L^q(m) \xrightarrow{M_g} L^p(m) \xrightarrow{T_o} B$$

By the same considerations made at the beginning of this section, this problem is only interesting when $0 < p < q$, and the condition $g \in L^r(m)$, with $1/p = 1/q + 1/r$, is then required. In order to find a reasonable guess for a theorem solving this problem, we can start assuming in Theorem 4.2 that B and $L^q(m)$ are Banach spaces and T is a linear operator. Then, the adjoint of T is an operator $T^* : L^{q'}(m) \to B^*$, and one can easily translate (a), (b) and (c) into equivalent statements in terms of T^*. In this way, a theorem is found which turns out to be true without any restriction on B, q or T:

THEOREM 4.5. *Let the operator* $T : L^q(m) \to B$ *be sublinear in the following sense:*

$$\|T(f+g)\|_B \leq \|Tf\|_B + \|Tg\|_B; \quad \|T(\lambda f)\|_B = |\lambda|\,\|Tf\|_B \qquad (\lambda \in \mathbb{C})$$

If $0 < p < q < \infty$ *and* $1/p = 1/q + 1/r$, *then the following statements are equivalent:*

(a') T *admits the factorization (4.4) with* $g(x) > 0$, $\|g\|_r \le 1$ *and* $\|T_o\| \le C$.

(b') There exists a measurable function $w(x) > 0$ *a.e. such that* $\|w\|_{r/p} \le 1$ *and*

$$\|Tf\|_B^p \le C^p \int |f(x)|^p w(x)dx \qquad (f \in L^q(m))$$

(c') The following vector valued inequality holds

$$(\sum_j \|Tf_j\|_B^p)^{1/p} \le C\|(\sum_j |f_j|^p)^{1/p}\|_q \quad (f_j \in L^q(m))$$

Proof: The equivalence of (a') and (b') is again obvious, with $w(x) = g(x)^p$, and the implication: (b') $\Rightarrow$ (c') follows directly from Hölder's inequality. Thus, once more, the important part of the theorem is the sufficiency of (c'), and this will be proved by a new appeal to the minimax lemma 3.2. Let $\alpha = q/p > 1$, so that $\alpha' = r/p$, and define the convex sets

$$A = \{\sum_j |f_j|^p : f_j \in L^q(m), \quad \sum_j \|Tf_j\|_B^p \le 1\}$$

$$B = \{b \in L^{\alpha'}(m) : b(x) \ge 0, \quad \|b\|_{\alpha'} \le 1\}$$

and the function $\Phi(a,b)$ in AxB:

$$\Phi(a,b) = -\int a(x)b(x)dm(x) = -\int \sum_j |f_j(x)|^p b(x)dm(x)$$

Observe that Φ is bilinear and weakly continuous with respect to b, so that we can use the identity

$$\min_{b\in B} \sup_{a\in A} \Phi(a,b) = \sup_{a\in A} \min_{b\in B} \Phi(a,b) = \sup_{a\in A} (-\|a\|_{q/p})$$

Thus, there exists $b \in B$ such that, for every $a = \sum_j |f_j|^p \in A$ we have the inequality

$$\sum_j \|Tf_j\|_B^p \le C^p\|a\|_{q/p} \le -C^p\ \Phi(a,b)$$

This implies, in particular, that (b') holds, with $w(x) = b(x)$, for all f such that $\|Tf\|_B \le 1$, and, by homogeneity, the same is true for an arbitrary $f \in L^p(m)$. □

The achievements of both factorization theorems can be summarized as follows:

In the factorization theorem 4.2 we improve the range of the operator by showing that it is smaller than what was initially assumed: range $(T) \subset L^p(w) \subset L^q$ (with $p > q$). On the other hand, the dual factorization theorem (4.5) aims at enlarging the domain of the operator, while mantaining its range: $\mathrm{dom}(T) = L^p(w) \supset L^q$ (with $p < q$). In fact, it follows from condition (b') that T has a unique extension to a bounded operator from $L^p(w)$ to B.

As in the previous sections, conditions on the space B can be given ensuring that factorization through L^p always occur for operators $T : B \to L^q$, but we shall not pursue this matter any further.

For the applications we have in mind, a slightly stronger form of the factorization theorems is needed, which consists in taking a family $\mathcal{T}$ of operators and studying their simultaneous factorization through L^p (i.e., factorization with the same operator M_g for all $T \in \mathcal{T}$). Since the proofs go exactly the same, we merely state the results:

THEOREM 4.2'. *If* $\mathcal{T}$ *is a family of sublinear operators* $T : B \to L^q(m)$ *and* $1/r = 1/q - 1/p > 0$, *then, the inequality*

$$\|(\sum_j |T_j f_j|^p)^{1/p}\|_q \le C(\sum_j \|f_j\|_B^p)^{1/p} \qquad (f_j \in B;\quad T_j \in \mathcal{T})$$

holds if and only if there exists $w(x) > 0$ *such that* $\|w^{-1}\|_{r/p} \le 1$ *and*

$$\|Tf\|_{L^p(w)} \le C\,\|f\|_B \qquad (f \in B;\quad T \in \mathcal{T})$$

THEOREM 4.5'. *If* $\mathcal{T}$ *is a family of sublinear operators* $T : L^q(m) \to B$ *and* $1/r = 1/p - 1/q > 0$, *then, the inequality*

$$(\sum_j \|T_j f_j\|_B^p)^{1/p} \le C\|(\sum_j |f_j|^p)^{1/p}\|_q \qquad (f_j \in L^q;\quad T_j \in \mathcal{T})$$

holds if and only if there exists $w(x) > 0$ *such that* $\|w\|_{r/p} \le 1$ *and*

$$\|Tf\|_B \le C\,\|f\|_{L^p(w)} \qquad (f \in L^q;\quad T \in \mathcal{T})$$

5. WEIGHTED NORM INEQUALITIES AND VECTOR VALUED INEQUALITIES

The norm of $\ell^p(B)$, which appears in the vector valued inequalities (c) and (c') of the previous section, is not easy to handle in general. However, it becomes almost trivial when B is precisely an L^p-space, and then, the factorization theorems have significant consequences. In particular, we shall derive from them the promised general principle of equivalence between weighted and vector valued inequalities.

Let $\mathcal{T}$ be a family of sublinear operator from $L^q(\mathbf{R}^n)$ to $L^s(\mathbf{R}^n)$, and consider the following vector valued inequality, which may or may not hold:

$$(5.1) \qquad \|(\sum_j |T_j f_j|^p)^{1/p}\|_s \leq C\|(\sum_j |f_j|^p)^{1/p}\|_q$$

for all $f_j \in L^q(\mathbf{R}^n)$ and all $T_j \in \mathcal{T}$. Here, the exponents p, q and s are fixed and such that, either $p < \min(s,q)$ or $\max(s,q) < p$. The general principle mentioned above can be stated as follows:

THEOREM 5.2. *Let* $0 < p,q,s < \infty$, *and define* α *and* β *by*

$$\frac{1}{\alpha} = |1 - \frac{p}{s}| , \qquad \frac{1}{\beta} = |1 - \frac{p}{q}|$$

i) *If* $p < \min(s,q)$, *then (5.1) holds if and only if, to each* $u \in L_+^\alpha(\mathbf{R}^n)$, *we can associate* $U \in L_+^\beta(\mathbf{R}^n)$ *such that* $\|U\|_\beta \leq \|U\|_\alpha$ *and*

$$\int |Tf(x)|^p u(x)dx \leq C^p \int |f(x)|^p U(x)dx \qquad (T \in \mathcal{T})$$

ii) *If* $\max(s,q) < p$, *then (5.1) holds if and only if, to each* $u \in L_+^\beta(\mathbf{R}^n)$, *we can associate* $U \in L_+^\alpha(\mathbf{R}^n)$ *such that* $\|U\|_\alpha \leq \|u\|_\beta$ *and*

$$\int |Tf(x)|^p U(x)^{-1}dx \leq C^p \int |f(x)|^p u(x)^{-1}dx \qquad (T \in \mathcal{T})$$

Proof: Consider first the case $p < \min(s,q)$. It was proved in Theorem V.6.1 that the weighted norm inequality implies (5.1) (it was assumed there that $q = s$, but this makes no real difference). Conversely, if (5.1) is verified, let $u \in L_+^\alpha(\mathbf{R}^n)$ be given with $\|u\|_\alpha \leq 1$, and consider the space $B = L^p(u) = L^p(\mathbf{R}^n;\, u(x)dx)$. By

Hölder's inequality, $L^s(\mathbb{R}^n) \subset B$ and $\|\cdot\|_B \le \|\cdot\|_s$, so that, if $T_j \in \mathcal{T}$:

$$(\sum_j \|T_j f_j\|_B^p)^{1/p} = \|(\sum_j |T_j f_j|^p)^{1/p}\|_B \le$$
$$\le C \|(\sum_j |f_j|^p)^{1/p}\|_q$$

Now, Theorem 4.5' applies, and there exists $U \in L_+^\beta$, with $\|U\|_\beta \le 1$, such that

$$\int |Tf|^p u dx = \|Tf\|_B^p \le C \int |f|^p U\, dx \qquad (T \in \mathcal{T})$$

Suppose now that $\max(s,q) < p$, and let us prove first the easy part. Given functions $f_j \in L^q$, we can find $u \in L_+^\beta(\mathbb{R}^n)$ of unit norm such that

$$\|(\sum_j |f_j|^p)^{1/p}\|_q = \|(\sum_j |f_j|^p)^{1/p} u^{-1/p}\|_p \|u^{1/p}\|_{\beta p}$$
$$= \{\sum_j \int |f_j|^p u^{-1}\}^{1/p}$$

(this is due to the reciprocal of Hölder's inequality, since $\frac{1}{q} = \frac{1}{p} + \frac{1}{\beta p}$). If the weighted norm inequality corresponding to this case is verified, then we take $U(x) > 0$ associated to $u(x)$, so that $\|U\|_\alpha \le 1$ and

$$\|(\sum_j |f_j|^p)^{1/p}\|_q \ge C^{-1} \{\int \sum_j |T_j f_j|^p U^{-1}\}^{1/p} \ge$$
$$\ge C^{-1} \|U\|_\alpha \|(\sum_j |T_j f_j|^p)^{1/p}\|_s$$

which is the vector valued inequality (5.1).

Finally, if (5.1) holds, $\max(s,q) < p$, and $u(x) > 0$ is given with $\|u\|_\beta \le 1$, we define $B = L^p(u^{-1})$. Then, the choice of β and Hölder's inequality imply

$$B \subset L^q(\mathbb{R}^n) \qquad \text{and} \qquad \|\cdot\|_q \le \|\cdot\|_B$$

so that the operators of $\mathcal{T}$ are defined in B and verify

$$\|(\sum_j |T_j f_j|^p)^{1/p}\|_s \le C \|(\sum_j |f_j|^p)^{1/p}\|_B =$$
$$= C (\sum_j \|f_j\|_B^p)^{1/p}$$

Now, Theorem 4.2' can be applied with $\frac{1}{r} = \frac{1}{s} - \frac{1}{p}$ (i.e. $\alpha = \frac{r}{p}$), and, denoting $U(x) = w(x)^{-1}$, this gives exactly the desired weighted norm inequality. □

The iteration argument used in Chapter IV for the factorization and extrapolation of A_p weights, can be applied here in order to unify the weights appearing on both sides of the inequalities obtained in Theorem 5.2.:

COROLLARY 5.3. *Let* $\mathcal{T}$ *be a family of sublinear operators satisfying the vector valued inequality (5.1) with* $s = q > p$, *and let* $\alpha = (\frac{q}{p})'$. *Then, to every* $u \in L^{\alpha}_{+}(\mathbb{R}^n)$, *we can associate* $w(x)$ *such that:*

i) $u(x) \leq w(x)$ *for every* $x \in \mathbb{R}^n$

ii) $\|w\|_{\alpha} \leq 2\|u\|_{\alpha}$

iii) $\|Tf\|_{L^p(w)} \leq 2^{1/p}\, C\, \|f\|_{L^p(w)} \qquad (T \in \mathcal{T})$

Proof: Given $u(x)$, we define a sequence $(u_j)_{j=0}^{\infty}$ in $L^{\alpha}(\mathbb{R}^n)$ inductively: $u_0 = u$, and $u_{j+1} = U_j$ for every $j \geq 0$ (where U_j denotes the weight associated to u_j according to 5.2(i)). Then,

$$w(x) = \sum_{j=0}^{\infty} 2^{-j}\, u_j(x)$$

satisfies (i), and since $\|u_{j+1}\|_{\alpha} \leq \|u_j\|_{\alpha} \leq \dots \leq \|u\|_{\alpha}$ for every j, the series converges in $L^{\alpha}(\mathbb{R}^n)$ and (ii) is also verified. Finally, for each $T \in \mathcal{T}$, the inequalities

$$\int |Tf(x)|^p\, 2^{-j} u_j(x)\,dx \leq 2\, C^p \int |f(x)|^p\, 2^{-(j+1)} u_{j+1}(x)\,dx$$

can be summed up to yield (iii). □

There is, of course, a converse of Corollary 5.3: If its conclussion is verified, then (5.1) holds with only a trivial loss in the constant (we obtain $8^{1/p}C$ instead of C).

The unifying weight argument does not seem to work in the case $s = q < p$. The same result can be obtained, however, provided that the operators $T \in \mathcal{T}$ are linearizable (in the sense of V.1.20) and $1 < q < p < \infty$. This follows from the case already proved by an obvious duality argument which the reader will easily supply.

A consequence of our general principle together with the theorem of Marcinkiewicz and Zygmund (V.2.7) is that almost all the information that one may wish concerning the boundedness properties of a linear operator, is contained in the weighted-L^2 inequalities that this operator satisfies. Just as a sample of this fact, we give the following

COROLLARY 5.4. *Let* $1 < p < \infty$ *and* $\frac{1}{\alpha} = |1 - \frac{2}{p}|$. *A linear operator* T *is bounded in* $L^p(\mathbb{R}^n)$ *if and only if, for every* $u \in L^\alpha_+(\mathbb{R}^n)$, *we can find* w *such that:* $u(x) \leq w(x)$, $\|w\|_\alpha \leq 2\|u\|_\alpha$ *and* T *is bounded in* $L^2(w^\sigma)$ *(where* $\sigma = 1$ *when* $2 \leq p$ *and* $\sigma = -1$ *when* $p < 2$*) with norm independent of* u.

Proof: The "if" part is, in both cases, a trivial consequence of Hölder's inequality. To prove the "only if" statement, consider first the case $2 < p$. Since T is bounded in $L^p(\mathbb{R}^n)$, we know (see V.2.7) that

$$\|(\sum_j |Tf_j|^2)^{1/2}\|_p \leq \|T\| \; \|(\sum_j |f_j|^2)^{1/2}\|_p$$

and we only have to apply Corollary 5.3. Now, if $p < 2$, we consider the adjoint operator $T^* : L^{p'}(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ and apply the previous case taking into account that $\frac{1}{\alpha} = |1 - \frac{2}{p}| = |1 - \frac{2}{p'}|$ and that the inequalities

$$\int |T^*f(x)|^2 \, w(x)dx \leq C \int |f(x)|^2 \, w(x)dx$$

and

$$\int |Tf(x)|^2 \, w(x)^{-1}dx \leq C \int |f(x)|^2 \, w(x)^{-1}dx$$

are equivalent. □

We invite the reader to search for more general formulations of our last corollary, involving, for instance, weighted inequalities in $L^p(\ell^r)$ for the operator T.

6. THE TWO WEIGHTS PROBLEM FOR CLASSICAL OPERATORS

Let T be a singular integral operator in $\mathbb{R}^n$, i.e., $Tf = K*f$, where the tempered distribution K has bounded Fourier transform and coincides away from the origin with a locally integrable function satisfying Hörmander's condition. The two-weights problem for this operator consists in finding all pairs (u,v) of positive functions for which the inequality

$$(6.1) \qquad \int |Tf(x)|^p u(x)\,dx \leq C_p(u,v) \int |f(x)|^p\, v(x)\,dx$$

$(f \in L^p(v))$ holds true. For regular singular integrals operators, it was proved in Chapter IV that this inequality is verified if $u = v \in A_p$ and $1 < p < \infty$. The general problem, however, remains unsolved, and we shall only deal here with the following weak variant which was already studied in 3.9(b) in a particular case:

Find conditions on $v(x)$ *(resp.* $u(x)$*) such that (6.1) is satisfied by some* $u(x)$ *(resp.* $v(x)$*).*

This is really a factorization problem, and it can be easily solved by means of the results of the previous sections. We begin with a lemma which is a consequence of Kolmogorov's condition and the estimates for vector valued singular integrals

LEMMA 6.2. *Let* T *be a singular integral operator whose kernel satisfies*

$$(6.3) \qquad |K(x)| \leq C|x|^{-n} \qquad (x \in \mathbb{R}^n)$$

and let $0 < r < 1 < p$. *Then, for every* $k = 0,1,2,\dots$

$$\|(\sum_j |Tf_j|^p)^{1/p}\|_{L^r(S_k)} \leq C_{r,p}\, |S_k|^{1/r} \int (\sum_j |f_j(x)|^p)^{1/p} (1 + |x|)^{-n} dx$$

where S_0 *is the unit ball and each,* S_k, $k \geq 1$, *is a spherical shell:* $S_k = \{x : 2^{k-1} \leq |x| < 2^k\}$.

Proof: Given $k \geq 0$, we decompose each function as $f = f' + f''$, where $f' = f\chi_{B_k}$, $f'' = f\chi_{\mathbb{R}^n - B_k}$ and $B_k = \{x : |x| < 2^{k+1}\}$. A consequence of (6.3) is that $|K(x-y)| \leq C|x|^{-n}$ when $|x| > 2|y|$, and therefore, for all $x \in S_k$

$$|Tf''(x)| = \left|\int_{|y| \geq 2^{k+1} > 2|x|} K(x-y)\ f(y)\,dy\right| \leq$$

$$\leq C \int_{|y| > 2^{k+1}} |f(y)|\ |y|^{-n} dy \ \leq C\, \|f\|_{L^1(w)}$$

with $w(x) = (1 + |x|)^{-n}$. Thus, Minkowski's inequality gives

$$\sup_{x\in S_k} \left(\sum_j |Tf''_j(x)|^p\right)^{1/p} \leq C \left\| \left(\sum_j |f_j|^p\right)^{1/p} \right\|_{L^1(w)}$$

and the inequality for the functions (f''_j) follows immediately.

On the other hand, we use V.2.8 and V.3.11. to obtain:

$$\left\| \left(\sum_j |Tf'_j|^p\right)^{1/p} \right\|_{L^r(S_k)} \leq C_r\ |S_k|^{1/r-1} \left\| \left(\sum_j |Tf'_j|^p\right)^{1/p} \right\|_{1*}$$

$$\leq C_{r,p}\ |S_k|^{1/r-1} \int_{|x|<2^{k+1}} \left(\sum_j |f_j(x)|^p\right)^{1/p} dx \leq$$

$$\leq C_{r,p}\ |S_k|^{1/r} \int_{\mathbb{R}^n} \left(\sum_j |f_j(x)|^p\right)^{1/p} w(x)\,dx$$

(we have denoted by $\|\cdot\|_{1*}$ the "norm" in weak-$L^1(\mathbb{R}^n)$). This completes the proof. $\square$

THEOREM 6.4. *Let* $1 < p < \infty$, *and suppose that the kernel of the singular integral, operator* T *satisfies (6.3).*

(i) In order that (6.1) holds for some $u(x) > 0$ *a.e., it is sufficient that*

$$(6.5) \qquad I = \int_{\mathbb{R}^n} v(x)^{1-p'} (1 + |x|)^{-np'} dx < \infty$$

In this case, u *can be found such that* $u(x)^{-\alpha}(1 + |x|)^{-np'}$ *is integrable, provided that* $\alpha < p' - 1$.

(ii) In order that (6.1) holds for some $v(x) < \infty$ *a.e., it is sufficient that*

$$(6.6) \qquad I = \int_{\mathbb{R}^n} u(x)(1 + |x|)^{-np} dx < \infty$$

In this case, v *can be found such that* $u(x)^{\alpha}(1 + |x|)^{-np}$ *is integrable, provided that* $\alpha < 1$.

(iii) Both conditions (6.5) and (6.6) are also necessary in order that (6.1) holds for the Riesz transforms: $T = R_1, R_2, \ldots, R_n$.

Proof: Since T is essentially self-adjoint, a simple duality argument shows that the pair $(u(x), v(x))$ satisfies (6.1) for the exponent p if and only if the pair $(v(x)^{1-p'}, u(x)^{1-p'})$ satisfies the same inequality with exponent p'. Thus, parts (i) and (ii) of the theorem are actually equivalent statements, and we limit ourselves to prove (i).

Given $\alpha < p' - 1$, take $r < 1$ such that $\frac{1}{\alpha} = \frac{p}{r} - 1$, and let $\frac{1}{\beta} = p-1$. For each $k \geq 0$, the operator

$$T_k\, g(x) = T(g(y)\,(1+|y|)^n)(x)\,\chi_{S_k}(x)$$

has, according to the previous lemma, a vector valued extension mapping $L^1(\ell^p)$ into $L^r(\ell^p)$ with norm $\leq C_r 2^{kn/r}$. Then, Theorem 5.2(ii) can be applied, and to each $w \in L_+^\beta(\mathbb{R}^n)$ we associate $W_k \in L_+^\alpha(S_k)$ with $\|W_k\|_\alpha \leq \|w\|_\beta$ and

$$\int_{S_k} |Tf(x)|^p\, W_k(x)^{-1} dx \leq$$

$$\leq C_{r,p}^p\, 2^{knp/r} \int_{\mathbb{R}^n} |f(x)|^p\, w(x)^{-1} (1+|x|)^{-np}\, dx$$

Now, we simply take $w(x) = v(x)^{-1}(1+|x|)^{-np}$, so that $\|w\|_\beta^\beta = I < \infty$, and (6.1) is verified if we define

$$u(x) = \sum_{k=0}^{\infty} 2^{-\varepsilon k}\, 2^{-knp/r}\, W_k(x)^{-1}\chi_{S_k}(x)$$

with $\varepsilon > 0$. To check that $u(x)$ has the correct size, observe that

$$\int_{\mathbb{R}^n} u(x)^{-\alpha}(1+|x|)^{-np'} dx \leq C \sum_{k=0}^{\infty} 2^{-k(np'-\varepsilon\alpha-\alpha np/r)}$$

and since $\frac{\alpha p}{r} = \frac{p}{p-r} < p'$, the series converges provided that a small enough ε is chosen.

Now, we shall prove (iii). Suppose that $u(x) > 0$ and $v(x) < \infty$ are fixed and such that (6.1) is verified by each one of the Riesz transforms. Let

$$E_j = \{x \in \mathbb{R}^n \mid \max(|x_1|,|x_2|,\ldots,|x_n|) = x_j\} \qquad (j=1,2,\ldots,n)$$

so that $\mathbb{R}^n = E_1 \cup (-E_1) \cup E_2 \cup (-E_2) \cup \ldots \cup E_n \cup (-E_n)$. Take a bounded subset $A \subset -E_j$ of positive Lebesgue measure and such that $v(A) = \int_A v(x)dx < \infty$. Then, for every $x \in E_j$:

$$T(\chi_A)(x) = c_n \int_A (x_j - y_j)|x-y|^{-n-1}\, dy \geq$$

$$\geq \text{Const.} \int_A |x-y|^{-n}\, dy \geq \text{Const.}\ (1+|x|)^{-n}$$

and therefore, (6.1) implies

$$\int_{E_j} (1 + |x|)^{-np} u(x)\,dx \le C\, v(A) < \infty$$

The same can be proved for E_j and for all $j=1,2,\dots,n$, by using in each case the corresponding Riesz transform, and thus, (6.6) is verified. The necessity of (6.5) follows by the duality argument mentioned at the beginning of the proof. □

For regular singular integral operators, we always have the inequality (see IV.3.8).

$$\int |Tf(x)|^p u(x)\,dx \le C_{p,s} \int |f(x)|^p M_s u(x)\,dx \qquad (1<p,\ s<\infty)$$

with $u(x) \ge 0$ arbitrary, and $M_s u = M(u^s)^{1/s}$. Observe that, if $q > 1$, the mapping: $u \to M_s u$ is bounded in $L^q(\mathbb{R}^n)$ provided that we chose $s < q$, and this is what makes the above inequality specially useful. It is therefore important to know that this fact remains true for the most general kind of singular integral operators, namely:

THEOREM 6.7. *Let* T *be an arbitrary singular integral operator: Then, given* $q > 1$ *and* $u \in L^q_+(\mathbb{R}^n)$, *we can find* $U \in L^q_+(\mathbb{R}^n)$ *such that* $\|U\|_q \le \|u\|_q$ *and*

$$\int |Tf(x)|^p u(x)\,dx \le C_{p,q}(T) \int |f(x)|^p U(x)\,dx \qquad (1 < p < \infty)$$

Moreover, if $u \in L^1_+(\mathbb{R}^n)$ *and* $\alpha < 1$ *is given, then the same inequality holds for some* $U \in L^\alpha_{loc}(\mathbb{R}^n)$.

Proof: The argument is basically as in the preceding theorem, the details being even simpler here. The first statement follows from Theorem 5.2 applied to the vector valued inequality (see V.3.11)

$$\Big\| \Big(\sum_j |Tf_j|^p\Big)^{1/p} \Big\|_s \le C_{p,s} \Big\| \Big(\sum_j |f_j|^p\Big)^{1/p} \Big\|_s$$

with $s = pq'$. The second assertion is equivalent, by duality, to finding for each $v \in L^{p'-1}_+$ some $V \in L^{\alpha(p'-1)}_{loc}$ such that

$$\int |Tf(x)|^p V(x)^{-1}\,dx \le C \int |f(x)|^p\, v(x)^{-1}\,dx$$

and this is again a consequence of Theorem 5.2 applied to the weak type vector valued inequality

$$\Big|\Big\{x : \Big(\sum_j |Tf_j(x)|^p\Big)^{1/p} > \lambda\Big\}\Big| \le C\, \lambda^{-1} \int \Big(\sum_j |f_j(x)|^p\Big)^{1/p} dx. \ □$$

The preceding results remain true for vector valued singular integrals. To be precise, we consider an operator $T : L^r_A(\mathbb{R}^n) \to L^r_B(\mathbb{R}^n)$ which is bounded for some fixed $r > 1$, and is given by convolution with a kernel $K(x) \in \mathcal{L}(A,B)$ satisfying Hörmander's condition. Then, T can be extended to a bounded operator from $L^1_{\ell^p(A)} \to$ weak-$L^1_{\ell^p(B)}$ (see V.3.4 and V.3.9), and therefore, Theorem 6.7 holds for inequalities of the form

$$\int |Tf(x)|^p_B \, u(x)dx \leq C \int |f(x)|^p_B \, v(x)dx$$

Likewise, Lemma 6.2 and Theorem 6.4(i) are also verified in this context, provided that $|K(x)|_{\mathcal{L}(A,B)} \leq C|x|^{-n}$, and since the adjoint $T^* : L^{r'}_{B^*} \to L^{r'}_{A^*}$ is an operator of the same kind, part (ii) of Theorem 6.4 also holds.

This seemingly trivial remark will be useful to deal with the two-weights problem for the Hardy-Littlewood maximal operator, M. As before, we look for inequalities of the form

$$(6.8) \qquad \int Mf(x)^p \, u(x)dx \leq C_p(u,v) \int |f(x)|^p v(x)dx$$

and our first step is an analogue of Lemma 6.2 with the space $L^1((1 + |x|)^{-n}dx)$, replaced by the largest Banach space on which M can be defined, namely:

$$L = \{f \in L^1_{loc}(\mathbb{R}^n) : |f|_L = \sup_{R\geq 1} R^{-n} \int_{|x|\leq R} |f(x)|dx < \infty\}$$

<u>LEMMA 6.9</u>. *If* $0 < r < 1 < p$, *then*

$$|(\sum_j (Mf_j)^p)^{1/p}|_{L^r(S_k)} \leq C_{r,p} |S_k|^{1/r} |(\sum_j |f_j|^p)^{1/p}|_L$$

$k=0,1,2,\ldots$, *where* S_k *are the subsets of* $\mathbb{R}^n$ *introduced in 6.2*.

The proof follows the same pattern of 6.2: Given $k \geq 0$, we decompose $f = f' + f''$, and we have the trivial estimate

$$Mf''(x) \leq C \, |f|_L \qquad (x \in S_k)$$

while the vector valued inequality for the sequence (f'_j) follows from Fefferman-Stein's inequalities (V.4.3).

<u>THEOREM 6.10</u>. *Suppose that* $1 < p < \infty$.

(i) In order that (6.8) holds for some $u(x)>0$ *it is necessary and sufficient that* $v(x)$ *satisfies*

$$(6.11) \qquad \sup_{R\geq 1} R^{-np'} \int_{|x|\leq R} v(x)^{1-p'}dx < \infty$$

(ii) In order that (6.8) holds for some $v(x) < \infty$, *it is necessary and sufficient that* $u(x)$ *satisfies (6.6)*

(iii) Given $\alpha < 1$, $u(x)$ *can be found in (i) such that* $u(x)^{\alpha(1-p')}(1 + |x|)^{-np'}$ *is integrable, and* $v(x)$ *can be found in (ii) such that* $v(x)^{\alpha}(1 + |x|)^{-np}$ *is integrable.*

Proof: (i) The necessity is easy, because (6.8) implies $(u,v) \in A_p$ and, in particular

$$\sup_{R\geq 1} \{R^{-n} \int_{|x|\leq R} u(x)dx\} \{R^{-n} \int_{|x|\leq R} v(x)^{1-p'}dx\}^{p-1} = c < \infty$$

so that the left hand side of (6.11) is majorized by $c^{p'-1} (\int_{|x|\leq 1} u)^{1-p'}$. To prove the sufficiency, we observe that (6.11) means exactly that $L^p(v) \subset L$ and $\|f\|_L \leq C\|f\|_{L^p(v)}$ (by Hölder's inequality). Thus, the operator $M_kf(x) = Mf(x)\chi_{S_k}(x)$ is well defined in $L^p(v)$ and satisfies

$$\|(\sum_j (M_kf_j)^p)^{1/p}\|_r \leq C_{r,p} 2^{kn/r} (\sum_j \|f_j\|^p_{L^p(v)})^{1/p}$$

by Lemma 6.9. In this case, we apply directly the factorization theorem 4.2 (the Banach space being $B = L^p(v)$), and we find $u_k(x) > 0$ in S_k such that

$$\int_{S_k} u_k(x)^{-r/(p-r)}dx \leq 1$$

and

$$\int_{S_k} Mf(x)^p u_k(x)dx \leq C^p_{r,p} 2^{knp/r} \int |f(x)|^p v(x)dx$$

Now, we simply define $u(x) = \sum_{k=0}^{\infty} 2^{-\varepsilon k} 2^{-knp/r} u_k(x) \chi_{S_k}(x)$ with $\varepsilon > 0$ small enough, so that (6.8) holds and the first assertion of (iii) is also verified.

(ii) Consider the vector valued singular integral $T : L^p(R^n) \to L^p_B(R^n)$, $1 < p \leq \infty$, defined by

$$Tf(x) = (k_\delta * f(x))_{\delta \in Q_+}, \qquad B = \ell^\infty$$

where $k(x)$ is a positive Schwartz function such that $k(x) \geq 1$ when $|x| \leq 1$ (see section V.4). Then, 6.4 (ii) applies to T, and since $Mf(x) \leq \|Tf(x)\|_B$ when $f \geq 0$, this proves the sufficiency part as well as the second assertion in (iii).

The necessity is proved essentially as in Theorem 6.4. Take a bounded subset $A \subset R^n$ of positive Lebesgue measure such that $v(A) < \infty$. Then $M(\chi_A)(x) \geq C(1 + |x|)^{-n}$, and (6.8) implies:

$$\int (1 + |x|)^{-np} u(x)dx \leq C^{-p} \int M(\chi_A(x)^p u(x)dx$$
$$\leq \text{Const.} \int_A v(x)dx < \infty. \qquad \square$$

Remarks 6.12. (a) So far, we have only used the general principle of section 5 for families consisting of a single operator. Considering a huge family, one is allowed to have the same weights for all the operators in the family. Let us consider for instance regular singular operators $Tf = K*f$, and denote by $C(T)$ the least constant C such that the inequalities (6.3), $\|\hat{K}\|_\infty \leq C$ and

$$|K(x-y) - K(x)| \leq C|y| \; |x|^{-n-1} \qquad (|x| \geq 2|y|)$$

hold. Then, given $v(x)$ satisfying (6.5), we can find $u(x)$ such that

$$\int |Tf(x)|^p u(x)dx \leq C(T)^p \int |f(x)|^p v(x)dx$$

for every regular singular integral operator T. The similar statement corresponding to 6.4(ii) is also true. To show this, one can argue as above taking into account that, for each sequence (T_j) with (say) $C(T_j) \leq 1$, the vector valued inequalities of Theorem V.3.11 apply.

(b) A comparison between Theorems 6.4 and 6.10 shows that the maximal operator M behaves only slightly better than a singular integral operator. In fact, condition (6.5) implies (6.11), but this in turn implies that (6.5) holds for $p-\varepsilon$ with arbitrarily small $\varepsilon > 0$. In particular, the limiting condition of both (6.5) and (6.11) when $p \to 1$ is the same, namely:

$$v(x) \geq C(1 + |x|)^{-n}$$

and this turns out to be necessary and sufficient for the existence of some $u(x) > 0$ such that the weak type inequality

$$\int_{\{|Tf(x)|>\lambda\}} u(x)dx \leq C\ \lambda^{-1} \int |f(x)|v(x)dx$$

holds, either for $Tf(x) = Mf(x)$, or for an arbitrary singular integral operator T of the type considered in 6.4. The sufficiency can be proved in both cases by appealing to Nikishin's theorem.

7. WEIGHTS OF THE FORM $|x|^a$ FOR HOMOGENEOUS OPERATORS

We intend to show here how the general factorization theorems can be used in order to obtain weighted norm inequalities with concrete weights, provided that there is enough invariance in the operators under consideration. We have to introduce some notation:

Given an operator T acting on function in $\mathbb{R}^n$, the rotated operator T_ρ defined by a rotation $\rho \in SO(n)$ is

$$T_\rho f(x) = T(f \circ \rho^{-1})(\rho x)$$

and the dilation of T corresponding to some $\delta > 0$ is the operator T_δ given by

$$T_\delta f(x) = T(f^\delta)(\tfrac{x}{\delta}) \qquad \text{(with } f^\delta(y) = f(\delta y))$$

A family $\mathcal{T}$ of operators is rotation (resp. dilation) invariant if, for every $T \in \mathcal{T}$ and every $\rho \in SO(n)$ (resp. $\delta > 0$), the operator T_ρ (resp. T_δ) belongs to $\mathcal{T}$.

We can now state the main result of this section

THEOREM 7.1. *Given* p, q *with* $1 \leq p < \infty$ *and* $1 < q < \infty$, *let* $\mathcal{T}$ *be a family of linearizable operators (see V.1.20) in* $L^q(\mathbb{R}^n)$ *which is rotation and dilation invariant, and suppose that the vector valued inequality*

$$\|(\sum_j |T_j f_j|^p)^{1/p}\|_q \leq C\ \|(\sum_j |f_j|^p)^{1/p}\|_q$$

is verified for all $T_j \in \mathcal{T}$ *and* $f_j \in L^q(\mathbb{R}^n)$. *Then*

$$\int |Tf(x)|^p |x|^a dx \leq C^p \int |f(x)|^p |x|^a dx \qquad (T \in \mathcal{T})$$

where $a = n(\frac{p}{q} - 1)$.

Only for the proof of this theorem, the symbol $*$ will denote convolution in the multiplicative group $\mathbb{R}_+ = (0,\infty)$:

$$h * k(s) = \int_0^\infty h(\tfrac{s}{t})\; k(t)\; \frac{dt}{t} \qquad (s > 0)$$

The spaces $L^\alpha(\mathbb{R}_+)$ are defined with respect to the invariant measure on $\mathbb{R}_+$: $\frac{dt}{t}$. We shall need the following technical result:

<u>LEMMA 7.2</u>. <u>*Given*</u> α <u>*and*</u> ε <u>*with*</u> $0 < \varepsilon < 1 < \alpha < \infty$, <u>*we can find*</u> $h \in L^\alpha(\mathbb{R}_+)$ <u>*and*</u> $k \in L^{\alpha'}(\mathbb{R}_+)$ <u>*such that*</u> $\|h\|_\alpha = \|k\|_{\alpha'} = 1$ <u>*and*</u>

$$h * k(s) \geq 1-\varepsilon \quad \underline{\textit{for every}} \quad s \in [\varepsilon, 1/\varepsilon]$$

<u>Proof</u>: Take $N > \frac{1}{\varepsilon}$ and define

$$h(t) = (2N + 2\log N)^{-1/\alpha} \chi_{[N^{-1}e^{-N},\; Ne^N]}(t)$$

$$k(t) = (2N)^{-1/\alpha'}\, \chi_{[e^{-N},\; e^N]}(t)$$

The normalizing constants are chosen so that $\|h\|_\alpha = \|k\|_{\alpha'} = 1$. On the other hand, for every $s \in [N^{-1}, N]$, we have

$$h*k(s) = (2N + 2\log N)^{-1/\alpha}(2N)^{-1/\alpha'} \int_{e^{-N}}^{e^N} \frac{ds}{s} =$$

$$= \left(\frac{2N}{2N + 2\log N}\right)^{1/\alpha} > 1-\varepsilon$$

(the last inequality holds if we take N large enough). □

A few words will perhaps clarify what this lemma really achieves. First of all, Hölder's inequality imply: $h*k(s) \leq 1$ for every s, and moreover, it is well known that $h*k \in C_0(\mathbb{R}_+)$ (i.e., $h*k$ is continuous and $h*k(s) \to 0$ when $s \to 0$ or $s \to \infty$). Thus, nothing essentially better than what has been proved can be hoped for. On the other hand, we observe that the lemma is trivial for $\alpha = 1$, since we can take $k(t) \equiv 1$ obtaining $h*k(s) = 1$ for every s.

<u>Proof of Theorem 7.1</u>. Consider first the case $p \leq q$, and denote $\alpha = (\frac{q}{p})'$, so that Theorem 5.2 gives us the inequalities

$$(7.3) \qquad \int |Tf(x)|^p u(x)\,dx \leq C^p \int |f(x)|^p U(x)\,dx \qquad (T \in \mathcal{T})$$

where $u \in L_+^\alpha(\mathbb{R}^n)$ can be chosen arbitrarily, and then, U can be found such that $\|U\|_\alpha \leq \|u\|_\alpha$. Suppose that we choose $u(x)$

radial, $u(x) = u_0(|x|)$. By rotation invariance, we can then replace $U(x)$ by $U(\rho x)$ in the right hand side of (7.3), and integrating with respect to normalized Haar measure, $d\rho$, in $SO(n)$, we obtain (7.3) with the radial function

$$U_0(|x|) = \int_{SO(n)} U(\rho x)d\rho =$$

$$= \frac{1}{2}\Gamma(\tfrac{n}{2})\pi^{-n/2}\int_{|x'|=1} U(|x|x')d\sigma(x')$$

instead of U. Now, we write $v(r) = u_0(r)r^{n/\alpha}$ and $V(r) = U_0(r)r^{n/\alpha}$, so that $v, V \in L^\alpha(\mathbb{R}_+)$, v being at our disposal and V being associated to it with $\|V\|_\alpha \le \|v\|_\alpha$. Since T is also dilation invariant, we can apply (7.3) to T_δ, and it becomes

$$\int |Tf(x)|^p v(\delta|x|)|x|^{-n/\alpha}dx \le C^p \int |f(x)|^p V(\delta|x|)|x|^{-n/\alpha}dx$$

Observe that $-\frac{n}{\alpha} = a$ in this case. The next step consists in taking $k \in L^{\alpha'}(\mathbb{R}_+)$, multiplying both sides of the last inequality by $k(\delta^{-1})$ and integrating with respect to $\frac{d\delta}{\delta}$. Thus, we are led to

$$\int |Tf(x)|^p v*k(|x|)|x|^a dx \le C^p \int |f(x)|^p V*k(|x|)|x|^a dx$$

Finally, if we choose v and k as in the Lemma with $\|v\|_\alpha = \|k\|_{\alpha'} = 1$, and we use in the right hand side the simple majorization $V*k(|x|) \le 1$, it follows that

$$(1-\varepsilon)\int_{\varepsilon\le|x|\le 1/\varepsilon} |Tf(x)|^p|x|^a dx \le C^p \int |f(x)|^p|x|^a dx$$

and we let $\varepsilon \to 0$.

The case $q < p$ will be obtained from the previous one by duality, and it is to make this argument work that the hypothesis of being linearizable the operators of T was posed. In fact, associated with each $T \in T$, there is a linear operator $U : L^q(\mathbb{R}^n) \to L^q_B(\mathbb{R}^n)$ such that $|Tf(x)| = \|Uf(x)\|_B$, and we shall denote by $\mathcal{U}$ the family of all these operators U. The Banach space B could also depend on T, but we shall suppose for simplicity that it is the same for all the operators. The adjoint operators $U^* : L^{q'}_{B^*} \to L^{q'}$ satisfy the inequalities

$$\|(\sum_j |U_j^* f_j|^{p'})^{1/p'}\|_{q'} \le C\,\|(\sum_j \|f_j\|^{p'}_{B^*})^{1/p'}\|_{q'}$$

for all $f_j \in L^{q'}_{B^*}$ and $U_j \in \mathcal{U}$. Since $p' < q'$, the previous case can be applied now (the fact that each U^* acts on B^*-valued functions makes no difference at all), and we obtain

$$\int |U^*f(x)|^{p'} |x|^b dx \leq C^{p'} \int \|f(x)\|_{B^*}^{p'} |x|^b dx$$

$(U \in \mathcal{U})$ with $b = n(\frac{p'}{q'} - 1)$. By duality, this is equivalent to

$$\int \|Uf(x)\|_B^p |x|^{b(1-p)} dx \leq C^p \int |f(x)|^p |x|^{b(1-p)} dx$$

$(U \in \mathcal{U})$ and this is the end of the proof, because $b(1-p) = n(\frac{p}{q} - 1) = a$ $\square$

Our first application is a simple proof of certain well known inequalities for the Fourier transform.

COROLLARY 7.4. (Pitt's inequalities). *Let* $1 < p < \infty$. *For functions* $f(x)$ *in* $\mathbb{R}^n$, *the inequality*

$$\int |\hat{f}(\xi)|^p |\xi|^{-nb} d\xi \leq C_{p,b} \int |f(x)|^p |x|^{na} dx \tag{7.5}$$

holds if $0 \leq b < 1$ *and* $0 \leq a = b+p-2$.

Proof: We shall consider only the case $1 < p \leq 2$, since the case $p \geq 2$ is equivalent to it by duality. Let $x \to x^*$ denote the inversion in $\mathbb{R}^n$ leaving the unit sphere fixed, i.e. $x^* = x/|x|^2$. By using polar coordinates one easily establishes the identity

$$\int_{\mathbb{R}^n} g(x)dx = \int_{\mathbb{R}^n} g(x^*)|x|^{-2n} dx \tag{7.6}$$

for every $g(x) \geq 0$. Therefore, the operator

$$Tf(x) = \hat{f}(x^*)|x|^{-n}$$

is an isometry in $L^2(\mathbb{R}^n)$, and since $|Tf(x)| \leq \|f\|_1 |x|^{-n}$, it is also of weak type (1,1). By interpolation, and by the theorem of Marcinkiewicz and Zygmund (V.2.7), T has a vector valued extension

$$\tilde{T} : L^q_{\ell^p}(\mathbb{R}^n) \to L^q_{\ell^p}(\mathbb{R}^n); \qquad 1 < q \leq p \leq 2$$

On the other hand, T is clearly invariant under rotations and dilations, so that Theorem 7.1 applies and, together with (7.6), gives

$$\begin{aligned}\int |\hat{f}(x)|^p |x|^{-nb} dx &= \int |Tf(x)|^p |x|^{n(b+p-2)} dx \\ &\leq C \int |f(x)|^p |x|^{n(b+p-2)} dx\end{aligned}$$

where $b+p-2 = \frac{p}{q} - 1$ and q can be chosen with $1 < q \leq p$, i.e. $2-p \leq b < 1$. □

It can be proved that (7.5) only holds for the specified values of a and b. In particular, the relation $a = b+p-2$ is forced by the behaviour of the Fourier transform with respect to dilations: $(f^{\delta})^{\wedge}(\xi) = \delta^{-n}\hat{f}(\delta^{-1}\xi)$. There are also estimates from $L^p(|x|^a dx)$ to $L^q(|x|^b dx)$, with $p \neq q$, which can be obtained by a suitable modification of the method shown here. We refer to Stein [5] for the precise statements.

Let us now see how Theorem 7.1 works for maximal operators.

COROLLARY 7.7. *Let $\phi \in L^1(\mathbb{R}^n)$ be a radial function satisgying F. Zo's condition (see V.4)*

$$\int_{|x|>2|y|} \sup_{\delta>0} |\phi_\delta(x-y)-\phi_\delta(x)|\,dx \leq C \qquad (y \in \mathbb{R}^n)$$

Then, the following inequalities hold for the maximal operator $M_\phi f(x) = \sup_{\delta>0} |\phi_\delta * f(x)|$ associated to ϕ:

$$(7.8) \qquad \int M_\phi f(x)^p |x|^a dx \leq C_{p,a} \int |f(x)|^p |x|^a dx$$

$(1 < p < \infty;\ \ -n < a < n(p-1))$.

Proof: By its own definition, the operator M_ϕ is dilation invariant, and since ϕ is radial, M_ϕ is also rotation invariant. On the other hand, we know (see V.4.2) that the inequalities

$$\|(\sum_j (M_\phi f_j)^p)^{1/p}\|_q \leq C_{p,q} \|(\sum_j |f_j|^p)^{1/p}\|_q \qquad (1 < p,\ q < \infty)$$

hold. Thus, it suffices to apply Theorem 7.1. □

For the Hardy-Littlewood maximal function, (7.8) was already known (see Chapter IV), and in this cases, the range $-n < a < n(p-1)$ is best possible. What we have just proved is that these inequalities are also true for a more general kind of maximal operators, M_ϕ. This, in turn, is only a particular case of operators of the form

$$M_\mu f(x) = \sup_{t>0;\ \rho\in SO(n)} \left|\int f(x - t\rho y)\,d\mu(y)\right|$$

where μ is a positive Borel measure in $\mathbb{R}^n$.

COROLLARY 7.9. *Suppose that* M_μ *is bounded in* $L^q(\mathbf{R}^n)$ *for some* $q < \infty$. *Then, the inequalities:*

$$\int M_\mu f(x)^p |x|^a dx \leq C_{p,a} \int |f(x)|^p |x|^a dx$$

hold for $q \leq p < \infty$ *and* $0 \leq a < n(\frac{p}{q} - 1)$.

Proof: We apply Theorem 7.1 to the inequalities

$$\|(\sum_j (M_\mu f_j)^p)^{1/p}\|_q \leq C \|(\sum_j |f_j|^p)^{1/p}\|_q \qquad (q \leq p < \infty)$$

which hold because M_μ is a positive linearizable operator (see V.1.23). □

An interesting example arises when μ is Lebesgue measure in the unit sphere Σ_{n-1} (which can be considered as a singular Borel measure in $\mathbf{R}^n$ supported in Σ_{n-1}). Then, M_μ is Stein's maximal spherical mean

$$Mf(x) = \sup_{t>0} \left| \int_{|y'|=1} f(x-ty')d\sigma(y') \right|$$

which was considered in V.7.7. The last corollary gives in this case:

$$\int_{\mathbf{R}^n} Mf(x)^p |x|^a dx \leq C_{p,a} \int_{\mathbf{R}^n} |f(x)|^p |x|^a dx$$

for $n \geq 3$ and $0 \leq a < p(n-1)-n$. This inequality can be shown to be best possible for positive a.

8. DIRECTIONAL HILBERT TRANSFORMS

As a final application of the equivalence between weighted and ℓ^q-valued inequalities, we shall push a little bit further some ideas of A. Córdoba and R. Fefferman [1] relating the study of certain multipliers to vector valued inequalities for directional Hilbert transforms and estimates for maximal functions.

We fix a countable set of unit vectors $V = \{v_j\}_{j\geq 1}$ in $\mathbf{R}^2$, and write H_j for the Hilbert transform along v_j, i.e., H_j is a bounded operator in $L^p(\mathbf{R}^2)$, $1 < p < \infty$, such that

$$(H_j f)^\wedge(\xi) = -i\ \mathrm{sign}(\xi . v_j)\hat{f}(\xi) \qquad (f \in L^2 \cap L^p)$$

The family of all rectangles in $\mathbf{R}^2$ one of whose sides is parallel to some v_j will be denoted by $\mathcal{R}_V$.

Finally, a polygonal region $E = E(V)$ with infinitely many sides can be defined in terms of the given set of directions. We assume for simplicity that each v_j points upwards: $v_j = (v_{j,1}, v_{j,2})$ with $v_{j,2} > 0$. Then, we set $E = \bigcup\limits_{j \geq 1} E_j$, where E_j is contained in the dyadic vertical strip: $2^{j-1} \leq x_1 < 2^j$, and limited from below by a straight line segment perpendicular to v_j, i.e.

$$E_j = \{x = (x_1, x_2) \in \mathbb{R}^2 : 2^{j-1} \leq x_1 < 2^j \quad \text{and} \quad x.v_j \geq c_j\}$$

for certain constants $c_j \in \mathbb{R}$ (see Figure VI.8.1).

All through this section, the notation S_A will be used to represent the partial sum operator corresponding to the multiplier χ_A, where A is any measurable subset of $\mathbb{R}^2$: $(S_A f)\hat{}(\xi) = \hat{f}(\xi)\chi_A(\xi)$.

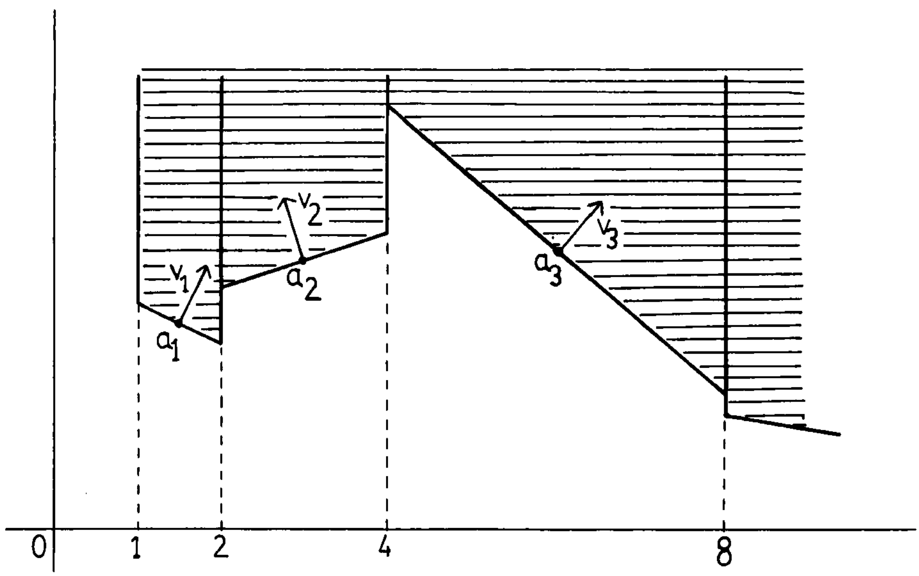

Figure VI.8.1.

THEOREM 8.1. *With the notation just defined, given* p *with* $1 < p < \infty$, *the following assertions are equivalent:*

(a) $\|(\sum\limits_j |H_j f_j|^2)^{1/2}\|_p \leq C_p \|(\sum\limits_j |f_j|^2)^{1/2}\|_p \qquad (f_j \in L^p)$

(b) S_E *is a bounded operator in* $L^p(\mathbb{R}^2)$

(c) *For every* $u \in L_+^q(\mathbb{R}^n)$, *with* $\frac{1}{q} = |1 - \frac{2}{p}|$, *we can find* w *such that:* $u(x) \le w(x)$, $\|w\|_q \le 2\|u\|_q$ *and*

$$\left\{\frac{1}{|R|}\int_R w(x)\,dx\right\}\left\{\frac{1}{|R|}\int_R w(x)^{-1}\,dx\right\} \le C \qquad (R \in \mathcal{R}_V) \tag{8.2}$$

Moreover, the set of p's *for which these statements are true, either reduces to* $\{2\}$ *or is an open interval:* $p_0 < p < p_0'$.

Proof: We shall prove that (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) in the case $p > 2$. This is enough, because no one of the statements is changed if we replace p by p'.

(a) $\Rightarrow$ (c). Let H_j^* denote the Hilbert transform in the direction orthogonal to v_j. Then, (8.2) is equivalent to

$$\int\{|H_jf(x)|^2+|H_j^*f(x)|^2\}w(x)\,dx \le C'\int|f(x)|^2w(x)\,dx \qquad (j \ge 1)$$

(with C' depending only on C and viceversa), because $w \in A_2^*$ (see IV.6.2) if and only if the Hilbert transforms in the directions $e_1 = (1,0)$ and $e_2 = (0,1)$ are bounded operators in $L^2(w)$, and (8.2) simply means that $w\circ\rho_j \in A_2^*$ uniformly in j, where ρ_j is the rotation taking e_1 into v_j. Now, by a trivial change of variables, (a) implies that the same vector valued inequality is verified by (H_j^*), and then, (c) follows immediately from Corollary 5.3.

(c) $\Rightarrow$ (b). Let I_j be the strip

$$I_j = \{\xi = (\xi_1,\xi_2) \in \mathbb{R}^2 : 2^{j-1} \le \xi_1 < 2^j\} \qquad (j \ge 1)$$

By Littlewood-Paley theory in 1 variable (see Theorem V.5.8) plus Fubini's theorem, it follows that

$$c_p\,\|S_If\|_p \le \|(\sum_{j\ge1}|S_{I_j}f|^2)^{1/2}\|_p \le C_p\,\|f\|_p \qquad (f \in L^p(\mathbb{R}^2))$$

where $I = \bigcup I_j = \{\xi : \xi_1 \ge 1\}$. In particular, we have

$$\|S_Ef\|_p^p \le c_p^{-p}\int(\sum_{j\ge1}|S_{I_j\cap E}f(x)|^2)^{p/2}dx$$
$$= c_p^{-p}\{\sum_{j\ge1}\int|S_{I_j\cap E}f(x)|^2u(x)\,dx\}^{p/2}$$

for some $u(x) \ge 0$ such that $\|u\|_q = 1$.

Now, let P_j denote the half-plane $P_j = \{\xi : \xi.v_j \ge 0\}$, so that

$I_j \cap E = (a_j + P_j) \cap I_j$, where a_j is the middle point of the j-th side of ∂E (see Figure VI.8.1). Then

$$S_{a_j+P_j}\, g(x) = \frac{1}{2}\, e^{2\pi i a_j . x}\,(Id+iH_j)(e^{-2\pi i a_j . y}\, g(y))(x)$$

and therefore, the operators $\{S_{a_j+P_j}\}_{j\geq 1}$ are uniformly bounded in $L^2(w)$ if w satisfies (8.2). Thus, w being the weight associated to u according to our present hypothesis, we have

$$\| S_E\, f\|_p^2 \leq c_p^{-2} \sum_{j\geq 1} \int |S_{a_j+P_j}\, (S_{I_j} f)(x)|^2\, w(x)dx \leq$$

$$\leq c_p^{-2}\, C \int \sum_{j\geq 1} |S_{I_j} f(x)|^2 w(x)dx \leq c_p^{-2}\, C\, C_p^2\, \|f\|_p^2\, \|w\|_q$$

and we are done, because $\|w\|_q \leq 2$.

<u>(b) $\Longrightarrow$ (a)</u>. This is based on a well known idea due to Y. Meyer which was crucial for the negative result of C. Fefferman [2] for the ball multiplier (see also de Guzmán [2]). By the theorem of Marcinkiewicz and Zygmund applied to the linear operator S_{rE}, (b) implies

$$(8.3)\qquad \| (\sum_{j\geq 1} |S_{r(E-a_j)} f_j|^2)^{1/2}\|_p \leq C\, \| (\sum_{j\geq 1} |f_j|^2)^{1/2}\|_p$$

for arbitrary functions $f_j \in L^p$, points $a_j \in \mathbb{R}^2$ and constant $r > 0$. If the points a_j are chosen as above, the set $r(E-a_j)$ tends to P_j when $r \to \infty$ and, at least for Schwartz functions f_j, (8.3) yields

$$\| (\sum_{j\geq 1} |S_{P_j} f_j|^2)^{1/2}\|_p \leq C\, \| (\sum_{j\geq 1} |f_j|^2)^{1/2}\|_p$$

which is equivalent to (a) because $S_{P_j} = \frac{1}{2}\,(Id + iH_j)$.

The last assertion of the theorem is a consequence of the reverse Hölder's inequality. In fact, if $w(x)$ satisfies (8.3), then $w(\rho_j x)$ belongs to A_2^* with constant C for the rotation ρ_j taking e_1 into v_j, and therefore, there exist $r > 1$ and $C_1 < \infty$ (both depending only on C) such that $w(x)^r$ satisfies (8.3) with constant C_1. But this means that, once we know that (c) is true for some $q < \infty$, it is also true for some $q_1 = \frac{q}{r} < q$, and this is exactly what we had to prove. □

It is not difficult to figure out which is the maximal function related to the multiplier operator S_E:

$$M_V f(x) = \sup_{x \in R \in R_V} \frac{1}{|R|} \int_R |f|$$

To state the connection between M_V and S_E we need to impose a weak additional hypothesis

[*] *"there exists no set $A \subset \mathbb{R}^2$ such that $|A| < \infty$ and $M_V(\chi_A)(x) = 1$ a.e."*

THEOREM 8.4. *Suppose that [*] is true. Then, the statements (a), (b) and (c) of the previous theorem are also equivalent to*

(d) M_V is a bounded operator in $L^q(\mathbb{R}^2)$

where p and q are related as before: $\frac{1}{q} = |1 - \frac{2}{p}|$.

Proof: We shall show the equivalence of (c) and (d). First, if (d) holds, we can associate to each $u \in L^q_+(\mathbb{R}^2)$ the weight

$$w(x) = \sum_{k=0}^{\infty} (2C)^{-k} M_V^k u(x)$$

where C is the norm of M_V as an operator in $L^q(\mathbb{R}^2)$ and $M_V^0 = \mathrm{Id.}$, $M_V^k u = M_V(M_V^{k-1}u)$. Then, $u(x) \leq w(x)$, $\|w\|_q \leq 2\|u\|_q$ and $M_V w(x) \leq 2\,C\,w(x)$. The last inequality means that w is an A_1 weight with respect to the family of rectangles R_V, and this certainly implies that (8.2) holds with constant $2C$.

In order to establish the converse, we begin with a simple observation: If $w(x)$ satisfies (8.2) and $A = \{x : w(x) > 1\}$, then

$$M_V(\chi_A)(x) \geq 1 - C\{M_V\, w(x)\}^{-1} \qquad (x \in \mathbb{R}^n) \tag{8.5}$$

In fact, given $x \in R \in R_V$, we have

$$\frac{1}{|R|} \int_R w \leq C\Big(\frac{1}{|R|} \int_R w^{-1}\Big)^{-1} \leq C\Big(\frac{|R-A|}{|R|}\Big)^{-1}$$

which can also be written as

$$\frac{|A \cap R|}{|R|} \geq 1 - C\Big(\frac{1}{|R|} \int_R w\Big)^{-1}$$

and taking the supremum for all possible R, we obtain (8.5).

Now, it suffices to prove that (c) implies the weak type (q,q) of M_V, since (c) also holds for some $q_1 < q$ and therefore, interpolating by means of Marcinkiewicz theorem, we get the strong type (q,q) for M_V. We argue by contradiction. Since M_V is positive and invariant under translations and dilations, if it were not of weak type (q,q), there would exist some $u \in L^q(\mathbb{R}^n)$ such that $M_V u(x) = \infty$ a.e. (this is a consequence of Nikishin's theorem; see 2.9. and 1.5). Let $w(x)$ be the weight which, according to (c), we can associate to $u(x)$, and put $A = \{x : w(x) > 1\}$. Then $|A| < \infty$, because $w \in L^q$, while (8.5) gives

$$M_V(\chi_A)(x) \geq 1 - C\{M_V\, w(x)\}^{-1} \geq 1 - C\{M_V u(x)\}^{-1} = 1 \quad \text{a.e.}$$

contradicting our hypothesis [*]. □

<u>Remarks 8.6</u>. (a) The same results can be obtained for bounded polygonal regions. It suffices to make the same construction of $E(V)$ but considering instead vertical strips $I_j = \{x : 2^{j-1} < x_1 \leq 2^j\}$ for all $j \leq 0$, and then cutting each strip above and below by straight lines perpendicular to some $v_j \in V$.

(b) The implication: (d) $\Rightarrow$ (a), (b) and (c) does not require the additional hypothesis [*], as the proof itself shows. It also follows from the proof that, if (d) holds for some $q < \infty$, then it holds for an open interval: $q_0 < q \leq \infty$. This is actually true for every maximal operator associated to a family of rectangles invariant under translations and dyadic dilations, due to the fact that A_1-weights with respect to such a family satisfy a reverse Hölder's inequality.

(c) In order to get some feeling about the hypothesis [*], let us explain what the failure of such a condition would mean. If [*] fails to hold, we can find a set A of <u>finite Lebesgue</u> measure and a familify of rectangles $\{R_i\}_{i\in I}$ in $\mathcal{R}_V$ covering almost all $\mathbb{R}^2$, while each R_i has 99.999 per cent of its area inside A. However difficult this seems to accomplish, there are some wild maximal functions for which it actually happens. For instance, [*] is false for the maximal function corresponding to the family of all rectangles in $\mathbb{R}^2$ (we are indebted to B. López-Melero for pointing this out to us). On the other hand, if a huge family of rectangles does not satisfy [*], it is not likely that either (a), (b) or (c) are verified, so that one would expect that condition [*] could be avoided in theorem 8.2.

9. NOTES AND FURTHER RESULTS

9.1.- In the first version of 2.7, given by Nikishin [1], it is only proved that T maps $L^p(\mu)$ into $L^{p-\varepsilon}(w\,dm)$ if $1 \leq p \leq 2$. The (more precise) weak type result appears in Nikishin [2]. Nikishin's proofs, however, are somewhat more complicated, and apply to linearizable operators only. The fact that the result holds for sublinear operators seems to be observed for the first time in Maurey [2].

Theorems 3.3, 3.6 and 4.2 form part of the general theory developped by Maurey [1]. Many applications of this theory different from those considered here can be seen in Gilbert [1].

9.2.- There is a condition for Banach spaces which is in some sense dual of having type p: B has (Rademacher) cotype q, where $2 \leq q < \infty$, iff

$$\sum_j \|f_j\|_B^q \leq C \int_0^1 \|\sum_j r_j(t) f_j\|_B^q dt \qquad (f_j \in B)$$

Every Banach space has cotype ∞ (with the obvious meaning of this) and in general, cotype q_0 implies cotype q for all $q \geq q_0$. Some interesting facts about type and cotype are listed below:

i) $L^p(\mu)$ has cotype sup $(p,2)$, $1 \leq p \leq \infty$. The space $H^1(\mathbb{R}^n)$ behaves in this sense exactly as $L^1(\mathbb{R}^n)$, and thus, it has the best possible cotype: 2.

ii) If B has type $p \geq 1$, its dual B^* has cotype p'. This is immediate, but the corresponding assertion interchanging the roles of type and cotype is false: L^1 has cotype 2, while $L^\infty = (L^1)^*$ has only the trivial type: 1.

iii) There is, however, a duality formula for type and cotype. Let

$$p(B) = \sup\{p : B \text{ has type } p\}$$

$$q(B) = \inf\{q : B \text{ has cotype } q\}$$

If $p(B) > 1$, then $\frac{1}{p(B)} + \frac{1}{q(B^*)} = 1$. This very deep result was proved by Pisier [1].

iv) A Banach space B and its bidual B^{**} always have the same type and cotype. This is due to the so-called "local reflexivity principle": Every finite-dimensional subspace of B^{**} is almost isometrically isomorphic to a subspace of B.

v) A Hilbert space has type 2 and cotype 2. The converse is also true, but not obvious (see Kwapien [1]).

9.3.- One may ask wheter "B of type p" is the weakest possible condition on B under which theorem 2.4 holds. In fact, it is not: For p fixed, $1 \leq p \leq 2$, the following two conditions are equivalent for a Banach space B

(a) Every continuous operator $T : B \to L^0(m)$ factors through $L^p_*(m)$

(b) B^* has cotype p' (this is slightly weaker than "B has type p")

A similar result holds for factorization through L^p : If $1 \leq p < 2$, the conditions

(a') Every continuous operator $T : B \to L^0(m)$ factors through $L^p(m)$ and

(b') B^* has cotype q for some $q < p'$

are equivalent. The proofs of both statements can be found in Maurey [3].

If B has a predual B_o (i.e., $B_o^* = B$), then condition (b) is also equivalent to "B_o has cotype p'". This is due to the property iv) of the preceding Note. An interesting application results by taking $B_o = L^1(\mathbb{R}^n)$ or $B_o = H^1(\mathbb{R}^n)$:

COROLLARY: *Let* $B = L^\infty(\mathbb{R}^n)$ *or* $B = BMO(\mathbb{R}^n)$. *Then, for every continuous linearizable operator* $T : B \to L^0(m)$, *there exists* $w(x) > 0$ *such that*

$$\int_{\{x:|Tf(x)|>\lambda\}} w(x)\, dm(x) \leq C\, (\|f\|_B/\lambda)^2 \qquad (f \in B)$$

This means that Nikishin's theorem applies to L^∞ and BMO with q=2.

9.4.- The inequality (3.10) can be improved in case the group G is amenable (a class which includes both compact and Abelian groups).

In that case, the operator T is automatically bounded in $L^2(G)$, see Cowling [1]. There are, however, examples of non-amenable groups G and operators T of weak type (2,2) and translation invariant, which are not bounded in $L^2(G)$ (N. Lohué [1], R. Szwarc [1]).

On the other hand, for every $1 \leq p < 2$, there are examples of translation invariant linear operators in the torus $\mathbf{T}^n$, which are of weak type (p,p) but not of strong type (p,p) (see Zafran [1]). This shows that, even for linear operators, one cannot obtain anything better than Corollary 2.8 for $p < 2$, and it also proves indirectly that the statement analogous to 3.6 for spaces of type $p < 2$ is false.

9.5.- The approach followed to establish the results of section 6 was intended to illustrate the applicability of factorization theorems. There are, however, constructive proofs of theorems 6.2 and 6.5. These are due, for 6.2 and 6.5(i), to L. Carleson and P. Jones [1], and for 6.5(ii) to Gatto and Gutiérrez [1], E.T. Sawyer (oral communication) and W.-S. Young [1]. It is also possible to obtain 6.5(i) from E.T. Sawyer's characterization of the pairs of weights associated to the maximal function (see Chapter IV, 7.7).

A different approach (in the spirit of the Helson-Szegö theorem) to the two-weights problem for the Hilbert transform H is followed by Cotlar and Sadosky [2]. They characterize the pairs $(u(x),v(x))$ such that H is bounded from $L^p(v)$ to $L^2(u)$, $2 \leq p < \infty$.

9.6.- Let $M^{(n)}$ denote the Hardy-Littlewood maximal operator in $\mathbf{R}^n$ defined by means of centered balls. As we mentioned in Chapter II, 7.3, $M^{(n)}$ is bounded in $L^p(\mathbf{R}^n)$, $1 < p < \infty$, with a norm independent of n. Stein's argument in [7] can be adapted to show, moreover, that

$$\Big\| \Big(\sum_j (M^{(n)} f_j)^q\Big)^{1/q} \Big\|_{L^p(\mathbf{R}^n)} \leq C_{p,q} \Big\| \Big(\sum_j |f_j|^q\Big)^{1/q} \Big\|_{L^p(\mathbf{R}^n)}$$

with $1 < p,q < \infty$ and $C_{p,q}$ independent of n. If we apply theorem 7.1 to this inequality, it follows that, given p, a such that $1 < p < \infty$, $-1 < a < p-1$, there exists $C_{p,a} > 0$ for which

$$\int_{\mathbf{R}^n} (M^{(n)} f(x))^p |x|^{an} dx \leq C_{p,a} \int_{\mathbf{R}^n} |f(x)|^p \, |x|^{an} dx$$

for every $n \in \mathbb{N}$.

9.7.- The results proved in section 8 are of a conditional nature. They simply establish the equivalence between certain statements, without giving any clue about the truth or falsity of all of them. The interesting question is: Given the vectors $\{v_j\}$, for which values of p (resp. q) do (a) and (b) (resp. (c) and (d)) hold? The answers we know so far correspond to extremal cases:

i) If the closure of $\{v_j\}$ in the unit circle has positive length, then the statements of both theorems are only true in the trivial cases: $p = 2$ and $q = \infty$. The same happens, for instance, if $\{v_j\} = \{(\cos \frac{\pi}{2j}, \text{ sen } \frac{\pi}{2j})\}_{j=1}^{\infty}$. See de Guzmán [2].

ii) If the sequence $\{v_j\}$ converges lacunarly to some vector of unit lenght, then all four statements are true for the largest range: $1 < p < \infty$ and $1 < q \leq \infty$. See Nagel, Stein and Wainger [1].

APPENDIX

We shall establish here several results concerning Rademacher functions and minimax lemmas which were stated in sections 2 and 3 of Chapter VI, and whose proofs were postponed.

A.1. RADEMACHER FUNCTIONS

The Rademacher functions $(r_j(t))_{j\geq 1}$ were defined in section VI.2. The first fact about them stated there was the following

Symmetry of the Rademacher Sequence: *Given a Borel function* $F(x_1,x_2,\dots,x_n)$ *in* $\mathbb{R}^n$, *and numbers* $\varepsilon_j = \pm 1$ $(j=1,2,\dots,n)$, *the functions* $F(r_1(t), r_2(t),\dots,r_n(t))$ *and* $F(\varepsilon_1 r_1(t), \varepsilon_2 r_2(t),\dots,\varepsilon_n r_n(t))$ *are equidistributed.*

Proof: Denote $r(t) = (r_1(t),r_2(t),\dots,r_n(t))$ and $r_\varepsilon(t) = (\varepsilon_1 r_1(t),\varepsilon_2 r_2(t),\dots,\varepsilon_n r_n(t))$. Then, we have to determine the Borel measures in $\mathbb{R}^n$

$$\mu(E) = |\{t \in [0,1) : r(t) \in E\}|$$

$$\mu_\varepsilon(E) = |\{t \in [0,1) : r_\varepsilon(t) \in E\}|$$

and show that they are equal. This is enough because, for any Borel subset B of $\mathbb{R}$, $|\{t \in [0,1) : F(r(t)) \in B\}| = \mu(F^{-1}(B))$. First of all, it is clear that both μ and μ_ε are supported in a subset of $\mathbb{R}^n$ consisting of 2^n points, namely

$$S = \{+1, -1\}^n = \{s \in \mathbb{R}^n : |s_j| = 1,\ j=1,2,\dots,n\}$$

Moreover, for each $s \in S$, there is an interval $I_n^k = [(k-1)2^{-n},k2^{-n})$ (for some $k = k(s) \in \{1,2,\dots,2^n\}$) such that $r(t) = s$ for all $t \in I_n^k$. Since there are as many points in S as intervals $\{I_n^k\}_{1\leq k\leq 2^n}$, this gives a one-to-one correspondece between both sets. Thus

$$\mu(\{s\}) = |\{t \in [0,1) : r(t) = s\}| = |I_n^{k(s)}| = 2^{-n}$$

for all $s \in S$. On the other hand, $r_\varepsilon(t) = s$ if and only if

$r(t) = s_\varepsilon = (\varepsilon_1 s_1, \varepsilon_2 s_2, \ldots, \varepsilon_n s_n)$, and therefore, $\mu_\varepsilon(\{s\}) = \mu(\{s_\varepsilon\}) = 2^{-n}$ for all $s \in S$. This shows that $\mu = \mu_\varepsilon$, completing the proof. □

Observe that we have also shown that the measure μ in $\mathbb{R}^n$ (which is the joint probability distribution of the random variables $r_1(t), r_2(t), \ldots, r_n(t)$) is the product of n measures in $\mathbb{R}$, each of them equal to $\frac{1}{2}(\delta_1 + \delta_{-1})$. As a consequence, we have

$$(1.1) \qquad \int_0^1 \prod_{j=1}^n F_j(r_j(t))dt = \prod_{j=1}^n \int_0^1 F_j(r_j(t))dt$$

for arbitrary Borel functions $F_1, F_2, \ldots, F_n$. What (1.1) means is that the Rademacher functions are independent random variables. In particular, they are orthonormal in $L^2([0,1),dt)$, because all of them have zero integral.

Let us now prove the second fact, stated in VI.2.2, about Rademacher functions. We denote by $\|\cdot\|_p$ the norm in $L^p([0,1),dt)$

Kintchine's Inequalities: *If* $0 < p < \infty$, *there exist constants* k_p, $K_p > 0$ *such that*

$$(1.2) \qquad k_p(\sum_j |\alpha_j|^2)^{1/2} \leq \|\sum_j \alpha_j r_j\|_p$$

and

$$(1.3) \qquad \|\sum_j \alpha_j r_j\|_p \leq K_p (\sum_j |\alpha_j|^2)^{1/2}$$

for every finite sequence (α_j) *of complex numbers.*

Proof: For $p = 2$, we have equality $(k_2 = K_2 = 1)$ due to the orthogonality of $(r_j(t))_{j\geq 1}$. Therefore, (1.2) needs to be proved only for $p < 2$, and this will be a consequence of (1.3) because, letting t be such that $\frac{1}{2} = \frac{1-t}{p} + \frac{t}{4}$ and $f = \sum_j \alpha_j r_j$, we have

$$\|f\|_2 \leq \|f\|_p^{1-t} \|f\|_4^t \leq \|f\|_p^{1-t} (K_4 \|f\|_2)^t$$

and, since all the norms are finite, (1.2) follows with $k_p = K_4^{-t/(1-t)}$.

Thus, it suffices to prove (1.3) with p an integer > 2 (since $\|\cdot\|_p$ is an increasing function of p), the α_j's real and $\sum_j |\alpha_j|^2 = 1$. Under these assumptions, we use the inequalities $|x|^p \leq p!\, e^{|x|} \leq p!\, (e^x + e^{-x})$, $x \in \mathbb{R}$, and (1.1) to obtain

$$\int_0^1 |\sum_j \alpha_j r_j(t)|^p\, dt \leq p! \int_0^1 \{\exp(\sum_j \alpha_j r_j(t)) +$$

$$+ \exp(-\sum_j \alpha_j r_j(t))\}dt = 2p! \prod_{j=1}^{n} \int_0^1 \exp(\alpha_j r_j(t))dt =$$

$$= 2p! \prod_{j=1}^{n} \left(\frac{e^{\alpha_j} + e^{-\alpha_j}}{2}\right) \leq 2p! \prod_{j=1}^{n} e^{\alpha_j^2} = 2ep! \quad \square$$

The estimate we have obtained for K_p in (1.3) is of the order $K(p) = O(p)$, $p \to \infty$, but a slightly more careful proof shows that $K(p) = O(p^{1/2})$. On the other hand, the best possible value of k_1 in (1.2) is known to be $k_1 = \frac{1}{\sqrt{2}}$ (Szarek [1]). We mention also that J.-P. Kahane has extended Kintchine's inequalities to the case where $(\alpha_j)_{j=1}^n$ are elements of an arbitrary Banach space (see Lindenstrauss-Tzafriri [1], vol. II).

A.2. THE MINIMAX LEMMA

The setting for lemma 3.2 in Chapter VI was as follows: We are given convex subsets A and B of certain vector spaces, a topology is defined in B for which it is a compact (Hausdorff) space, and we have a function $\Phi : A \times B \to \mathbb{R} \cup \{+\infty\}$. Then, the statement to be proved is

MINIMAX LEMMA: *If* Φ *satisfies*

i) $\Phi(.,b)$ *is a concave function on* A *for each* $b \in B$

ii) $\Phi(a,.)$ *is a convex function on* B *for each* $a \in A$

iii) $\Phi(a,.)$ *is lower semicontinuous on* B *for each* $a \in A$

then, the following identity holds

$$\min_{b\in B} \sup_{a\in A} \Phi(a,b) = \sup_{a\in A} \min_{b\in B} \Phi(a,b) \tag{2.1}$$

Results of this kind are used in the Theory of Games, and have their origin in the well known paper of Von Neumann [1]. The proofs are usually based on techniques of Convex Analysis, an example of which is the next proposition containing the most difficult step in the proof of (2.1). We have followed H. Kneser [1] and K. Fan [1] rather closely (the second paper contains several generalizations of the minimax lemma).

PROPOSITION 2.2. *Let* B *be as above, and suppose that* $g_j : B \to \mathbb{R} \cup \{+\infty\}$, $j=1,2,\dots,n$ *are convex and lower semicontinuous. If*

$$\max_{1\le j\le n} g_j(b) > 0 \quad \textit{for all} \quad b \in B$$

then there exist nonnegative numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ *such that*

$$\lambda_1 g_1(b) + \lambda_2 g_2(b) + \dots + \lambda_n g_n(b) > 0 \quad \textit{for all} \quad b \in B$$

Proof: Let us consider first the case $n = 2$. The subsets of B

$$B_j = \{b \in B : g_j(b) \le 0\}, \qquad j=1,2$$

are closed (hence, compact) and can be assumed to be nonempty, because, if $B_1 = \emptyset$, we simply take $\lambda_1 = 1$ and $\lambda_2 = 0$. The hypothesis of the proposition implies that $g_2(b) > 0 \ge g_1(b)$ for all $b \in B_1$, so that the function $-g_1(b) / g_2(b)$ is well defined and upper semicontinuous on B_1, and attains its maximum

$$\mu_1 = \max_{b\in B_1} \frac{-g_1(b)}{g_2(b)} \ge 0 \tag{2.3}$$

Similarly, we define

$$\mu_2 = \max_{b\in B_2} \frac{-g_2(b)}{g_1(b)} \ge 0 \tag{2.4}$$

Now, we wish to find $\lambda > 0$ such that $\lambda g_1(b) + g_2(b) > 0$ for all $b \in B$. This is obviously satisfied if $b \notin B_1 \cup B_2$, while for $b \in B_1 \cup B_2$ we have

$$\lambda\, g_1(b) + g_2(b) \ge (1-\lambda\mu_1)\, g_2(b) \qquad (\text{if } b \in B_1)$$

$$\lambda\, g_1(b) + g_2(b) \ge (\lambda - \mu_2)\, g_1(b) \qquad (\text{if } b \in B_2)$$

Thus, it suffices to find $\lambda > 0$ such that $1-\lambda\mu_1 > 0$ and $\lambda-\mu_2 > 0$. Such a λ exists if and only if $\mu_1\mu_2 < 1$. Let us prove that this inequality holds. We can obviously assume that $\mu_1 \ne 0$ and $\mu_2 \ne 0$. Then, we take $b_1 \in B_1$ and $b_2 \in B_2$ for which the corresponding maximum in (2.3) and (2.4) are attained, i.e.

$$\begin{cases} g_1(b_1) + \mu_1\, g_2(b_1) = 0 \\ g_1(b_2) + \dfrac{1}{\mu_2}\, g_2(b_2) = 0 \end{cases} \tag{2.5}$$

But $g_1(b_1) < 0 < g_1(b_2)$, so that we can take $b_t = tb_1 + (1-t)b_2$

with $0 < t < 1$ such that

$$g_1(b_t) \leq tg_1(b_1) + (1-t)g_1(b_2) = 0$$

By considering the same convex combination of both equations in (2.5), it follows that

$$\mu_1\mu_2 \; t \; g_2(b_1) + (1-t) \; g_2(b_2) = 0$$

On the other hand, the hypothesis of the proposition implies that $g_2(b_t) > 0$ and, by the convexity of g_2

$$t \; g_2(b_1) + (1-t)g_2(b_2) > 0$$

Therefore, $\mu_1\mu_2 \; g_2(b_1) < g_2(b_1)$, which forces $\mu_1\mu_2 < 1$ (because $g_2(b_1) > 0$).

To prove the proposition for arbitrary n, we use induction. We assume that the result has been proved for $n-1$ functions, and take the subset of B

$$B_n = \{b \in B : g_n(b) \leq 0\}$$

If $B_n = \emptyset$, we choose $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = 0$ and $\lambda_n = 1$. Otherwise, we restrict $g_1, g_2, \ldots, g_{n-1}$ to B_n (which is compact and convex) and use the induction hypothesis to find $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \geq 0$ such that

$$g_o(b) = \lambda_1 g_1(b) + \lambda_2 g_2(b) + \ldots + \lambda_{n-1} g_{n-1}(b) > 0$$

for all $b \in B_n$. Then, g_o and g_n are convex lower semicontinuous functions on B, and $\max\,(g_o(b), g_n(b)) > 0$ for all $b \in B$. Since the proposition has been proved in the case of two functions, we find $\lambda_o, \lambda_n \geq 0$ such that, for all $b \in B$.

$$0 < \lambda_o g_o(b) + \lambda_n g_n(b) = \lambda_o\lambda_1 g_1(b) + \lambda_o\lambda_2 g_2(b) + \ldots +$$

$$+ \lambda_o\lambda_{n-1} g_{n-1}(b) + \lambda_n g_n(b). \quad \square$$

<u>Proof of the Minimax Lemma</u>: We have to establish the identity (2.1). The inequality $\geq$ is obvious, and to prove $\leq$ we can assume that the right hand side is finite. Since nothing is changed by substracting a finite constant from $\Phi(a,b)$, we can also assume that

$$(2.6) \qquad \sup_{a \in A} \min_{b \in B} \Phi(a,b) = 0$$

Then, by the hypothesis iii), the subsets of B

$$B_a = \{b \in B : \Phi(a,b) \leq 0\} \qquad (a \in A)$$

are closed and nonempty, and we shall see that they verify the finite intersection property. Indeed, suppose that we have

$$B_{a_1} \cap B_{a_2} \cap \ldots \cap B_{a_n} = \emptyset$$

for some $a_1, a_2, \ldots, a_n \in A$. Writing $g_j(b) = \Phi(a_j, b)$, $j=1,2,\ldots,n$, we are under the conditions of Proposition 2.2, and we find $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ such that

$$\lambda_1\ \Phi(a_1,b) + \lambda_2\ \Phi(a_2,b) + \ldots + \lambda_n\ \Phi(a_n,b) > 0 \quad (\text{for all} \quad b \in B)$$

We can obviously normalize the λ_j's so that they satisfy $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$. If we set $a_o = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n$, the concavity hypothesis i) gives

$$\Phi(a_o, b) > 0 \qquad (\text{for all} \quad b \in B)$$

contradicting (2.6). Therefore, $\{B_a\}_{a \in A}$ is a family of closed subsets of B with the finite intersection property, and the compactness of B implies $\bigcap_{a \in A} B_a \neq \emptyset$. Take $b_o \in \bigcap_{a \in A} B_a$. Then, $\Phi(a, b_o) \leq 0$ for every $a \in A$, and

$$\min_{b \in B}\ \sup_{a \in A}\ \Phi(a,b) \leq \sup_{a \in A}\ \Phi(a, b_o) \leq 0$$

as we wanted to prove. □

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