0.1 Subharmonic Functions

This section is about a new concept: subharmonic function. Subharmonic function can be considered as a generalization of harmonic function, as it preserves some important property of harmonic function such as maximum principle. On the other hand, we will see why we call it "sub" harmonic: subharmonic function can be controlled by harmonic function. Also, by some operations like composition and taking absolute value, subharmonic function can still be subharmonic, but harmonic function can not. Finally we will begin our study of zeroes of holomorphic function.

Now we give the definition of subharmonic function.

Definition. *A subharmonic function on an open set $\Omega \subset \mathbb{R}^n$ is a function v defined on Ω, with values $-\infty \le v(x) < \infty$ and satisfying the following two conditions:*

1. *v is upper semicontinuous in Ω.*
2. *For every $x_0 \in \Omega$, there is a ball $B(x_0, r(x_0)) \subset \Omega$, $r(x_0) > 0$, such that for every r with $0 < r < r(x_0)$*

$$v(x_0) \le \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} v(x_0 + r\sigma) d\sigma \tag{1}$$

0.1.1 Upper semicontinuous

There is two equivalence definition of v being upper semicontinuous in Ω:

1. For every $t \in \mathbb{R}$, the set $\{x \in \Omega : v(x) < t\}$ is open.
2. For every $x_0 \in \Omega$:

$$\limsup_{x \to x_0 \text{ in } \Omega} v(x) \le v(x_0) \tag{2}$$

   This is equivalent to that for every $y > f(x_0)$, there exists a neighborhood U of x_0 such that $f(x) < y$ for all $x \in U$.

Remark (Proof of equivalence)**.**
We prove by contradiction, Suppose that $\limsup v(x) > v(x_0)$, we can find x_k, $v(x_0) < v(x_k) < \limsup v(x)$. Since $v^{-1}([-\infty, v(x_k))$ is open and $v(x_0) < v(x_k)$, $v(x_0) \in v^{-1}([-\infty, v(x_k))$. Thus there is a neighborhood $U \in \mathscr{N}(x_0)$, $U \subset v^{-1}([-\infty, v(x_k))$. Now we can find another $x_n \in U$ s.t. $v(x_k) < v(x_n) < \limsup v(x)$. $v(x_n) > v(x_k)$ means $v(x_n) \notin v^{-1}([-\infty, v(x_k))$, contradicts to $v(x_n) \in U \subset v^{-1}([-\infty, v(x_k))$.

We prove the converse by contradiction. Suppose there is a number $t_0 \in \mathbb{R}$, $v^{-1}([-\infty, t_0))$ is not open. So there is $x_0 \in v^{-1}([-\infty, t_0))$, such that $\forall U_k \in \mathscr{N}(x_0)$, there is $x_k \in U_k$, $x_k \notin v^{-1}([-\infty, t_0))$, which is equivalent to $v(x_k) \ge t$. This contradicts to $\limsup v(x_k) \le v(x_0) < t_0$.

If v is subharmonic, inequality (1) implies another direction of inequality (2). Thus we actually have equality in (2).

An important and frequently used tool is following characterization of upper simicontinuity.

Proposition 1. *v is upper semicontinuous in Ω if and only if for every compact $K \subset \Omega$, v is the limit over K of a decreasing sequence of continuous function.*

This proposition is important tool in proof of some following theorems.

Remark (Notes on proof of proposition 1)**.** *The converse part, by using partition of the unity, we construct a sequence of decreasing function (u_k). We need to prove v is the limit of (u_k).*

For any $x_0 \in K$, there is a sequence of balls $(B(x_{n,i}, \epsilon_n))$, $\epsilon_n \to 0$ s.t. $x_0 \in B(x_{n,i}, \epsilon_n)$ for all n. For each n, $B(x_{n,i}, \epsilon_n)$ is in finite open cover $(B(x_{n,i}, \epsilon_n))_i$ of K. Since $m_{n,i} = \sup_{B(x_{n,i},\epsilon_n)} v$, $u_n(x_0) \geq v(x_0)$ for all n. By definition of upper semicontinuous, for any $y > v(x_0)$, there is a neighborhood $U \in \mathscr{N}(x_0)$, $v(x) < y$ for all $x \in U$. Let $B(x_{n,i}, \epsilon_n) \subset U$, $m_{n,i} < y$. Thus $u_n(x_0) < y$ for all large enough n. Since y is any number larger than $v(x_0)$, $\limsup u(x) \leq v(x_0)$. This shows $u(x_0) = v(x_0)$.

0.1.2 Property of subharmonic function

First, subharmonic function satisfies maximum principle.

Remark (Notes on proof of maximum principle)**.** *Like proof of maximum principle for harmonic function, but we need to take care of semicontinuous. Assume $v(x_0) = M$, the maximum value. Choose r to satisfy inequality (1). If for some $x \in \partial B(x_0, r)$, $v(x) = m < M$, by semicontinuous, $\limsup v(x_k) \leq v(x) < m + \epsilon < M$. Thus there is a neighborhood $U \in \mathscr{N}(x)$, $\sup_{x_k \in U} v(x_k) < M$. Then $\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0 + r\sigma)d\sigma < M = v(x_0)$. This contradicts to inequality (1). Then the following is same as proof for maximum principle for harmonic function.*

The best reason why we use name 'subharmonic' is following: v is subharmonic function if and only if when v less or equal to a harmonic function u on boundary of region, $v \leq u$ in entire region. We remind the reader that proposition 1 appears as an important step in proof.

There are two examples of using proposition 1 to detail with subharmonic function v. One is if v is not identically equal to $-\infty$, then

$$\frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0 + r\sigma)d\sigma > -\infty$$

for every $\overline{B(x_0, r)} \subset \Omega$. In proof of this statement we also use Poisson representation of harmonic function and little topological trick. Another example is

$$m(r) = \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(r\sigma)d\sigma \tag{3}$$

is an increasing function.

There is another necessary and sufficient condition for v to be harmonic using Laplace operator. It says v is subharmonic if and only if $\Delta v \geq 0$.

Remark (Proof of proposition 2.10 in book)**.** *Author says we need to show that* $v(x_0) \leq \frac{1}{|\Sigma_{n-1}|}\int_{\Sigma_{n-1}} v(x_0 + r\sigma)d\sigma$*. But I think this is obvious since we consider* x_0 *which* $v(x_0) = 0$ *and* $v(x) \geq 0$ *on* Ω*. And this inequality is not used in the following part of proof.*

0.1.3 Estimation for zeroes of holomorphic function

We first state that if v is subharmonic, ϕ is increasing and convex function. Then $\phi \circ v$ is also subharmonic. This is useful when we need to connect holomorphic function with subharmonic function.

Now here comes our first theorem about zero points of holomorphic: Jensen's formula.

Theorem 0.1.3.1 (Jensen's formula)**.** *Let* F *be holomorphic in* $D(0, R)$ *and suppose that* $F(0) \neq 0$*. Let* $0 < r < R$ *and call* $z_1, z_2, \cdots, z_n$ *the zeroes of* F *in* $\overline{D(0, r)}$ *listed according to their multiplicities. Then:*

$$\log|F(0)| + \sum_{j=1}^{n} \log\frac{r}{|z_j|} = \frac{1}{\pi}\int_{-\pi}^{\pi} \log\left|F(re^{it})\right|dt. \tag{4}$$

The proof in book need a lemma: $\int_{-\pi}^{\pi} \log\left|1 - e^{it}\right|dt = 0$. You can also refer section 1 in Chapter 6 of Stein's *Complex Analysis*. The proof there is very different to the one in this book.

Remark (Proof of lemma (Lemma 2.12 in book))**.** *There is an inequality: for* $|t| < \frac{\pi}{3}$, $\log\frac{1}{|\sin t|} \leq \frac{C_\alpha}{|t|^\alpha}$*. I can prove it using elementary calculus, but I think it is an easy observation.*

To continue our explorer of zeroes of holomorphic function, we show some connection between holomorphic function and subharmonic function. More precisely, If F is holomorphic, not identically 0, then $\log|F(z)|$, $\log^+|F(z)| = \max(\log|F(z)|, 0)$ and $|F(z)|^a$ for any $0 < a < \infty$, are all subharmonic. Then we give definition of Hardy space on D. We define for $f \in H(D)$ (F is holomorphic in D) :

- $m_0(F, r) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+\left|F(re^{it})\right|dt\right)$
- $m_p(F, r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F(re^{it})\right|^p dt\right)^{\frac{1}{p}}$
- $m_\infty(F, r) = \sup_t \left|F(re^{it})\right|$

This function is an increasing function of r in $[0, 1)$ (Hardy convex theorem), see equation (3) for case $0 \leq p < \infty$. $m_\infty(F, r)$ is also an increasing function but it uses a different method (Hadamard three-circle theorem).

Now we define Hardy space H^p:

Definition. *For $0 < p \leq \infty$, we define $H^p(D)$:*

$$H^p(D) = \{F \in H(D) : \|F\|_{H^p} = \sup_{0 \leq r < 1} m_p(F, r) < \infty\}$$

For $p = 0$, we have the Nevanlinna class N, defined by:

$$N = \{F \in H(D) : \sup_{0 \leq r < 1} m_0(F, r) < \infty\}$$

If $0 < p < q < \infty$, we have $H^\infty \subset H^q \subset H^p \subset N$

Remark. *The first two inclusions are as the same as the inclusion for L^p, the last inclusion is by:*

$$(m_0(r))^p = \exp{(p \int_{-\pi}^{\pi} \log^+ |F| \frac{dt}{2\pi})} \leq \int_{-\pi}^{\pi} \exp{(p \log^+ |F|)} \frac{dt}{2\pi}$$

Notice that:

$$\int_{-\pi}^{\pi} \exp{(p \log^+ |F|)} \frac{dt}{2\pi} = \int_{\substack{t \in [-\pi, \pi] \\ |F| > 1}} |F|^p \frac{dt}{2\pi} + \int_{\substack{t \in [-\pi, \pi] \\ |F| \leq 1}} 1 \frac{dt}{2\pi}$$

Thus:

$$(m_0(r))^p \leq \int |F|^p \frac{dt}{2\pi} + 1$$

There left three theorems in this section. I interpret it shortly and informally. First one is for $F \in N$, the zeroes (z_j) of F cannot be too far from the boundary, or $\sum_j (1 - |z_j|) < \infty$. The second one is that if $\sum_j (1 - |z_j|) < \infty$ holds, the "Blaschke product"

$$B(z) = z^k \prod_{j=1}^{\infty} \frac{z_j - z}{1 - z\bar{z}_j} \frac{|z_j|}{z_j}$$

converges uniformly on each compact subset to a function H^∞ and they have the same zeroes.

Remark. *If f is holomorphic in an open disc that vanishes on a sequence of distinct points with a limit point in the disc. Then f is identically 0 (Theorem 4.8 in chapter 2. Stein's Complex Analysis). However in Blaschke product case, there can be infinitely zeroes, since it can have limit points on boundary. Thus $B(z)$ can be not identically 0. However, for any $r < 1$, $B(z)$ can only have finite zeroes in $\overline{D(0, r)}$.*

Here is a convention. If for some function F in D, the non-tangential boundary value of F is known to exist at e^{it}, we shall denote it by $F(e^{it})$.

The third theorem is the Blaschke product has properties: $\left|B(e^{it})\right| = 1$ for a.e. t and

$$\lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left|B(re^{it})\right| dt = 0.$$

If $F \in H^p$ with $p \geq 1$, which means F is holomorphic F and can be write as Poisson (or Poisson-Stieltjes) integral. By Fatou Theorem, we know that $F(e^{it})$ exists a.e.. In the next section we shall extend this result to any $p > 0$. The above three theorems play important roles in proving extension.

Remark (Notes on proof of three theorems (theorem 2.19, 2.21 and 2.22))**.** *In proof of theorem 2.19, we assume* $F(0) = 0$*, since we can use function* $\frac{F(z)}{z^k}$ *if* $F(z)$ *has zero of order* k *in* $z = 0$*. And this modification does not affect the sum* $\sum_j (1 - |z_j|)$*.*

The step

$$\sum_1^n \log \frac{1}{|z_j|} \leq M - n \log r - \log |F(0)|$$

to

$$\sum_1^\infty \log \frac{1}{|z_j|} \leq M - \log |F(0)|$$

is not clear. I think we can not first let $r \to 1$ *then* $n \to \infty$*. We can not control taking limit for which one first.*

In proof of theorem 2.21, the final step is:

$$\begin{aligned}
\left|1 - \frac{z_j - z}{1 - z\bar{z}_j} \frac{|z_j|}{z_j}\right| &= \left|1 - \frac{z_j |z_j| - z|z_j|}{z_j - z|z_j|^2}\right| = \left|\frac{z_j - z|z_j|^2 - z_j|z_j| + z|z_j|}{z_j - z|z_j|^2}\right| \\
&= (1 - |z_j|)\left|\frac{z_j + z|z_j|}{z_j - z|z_j|^2}\right| = (1 - |z_j|)|z_j|\left|\frac{e^{it} + z}{e^{it} - z|z_j|}\right| \\
&= (1 - |z_j|)|z_j|\left|\frac{z' + 1}{1 - z'|z_j|}\right|
\end{aligned}$$

where $z' = z \cdot e^{-it}$*. Since* $|z_j| < 1$*,* $|z'| = |z| \leq r$*, we have* $|z' + 1| \leq |z'| + 1 \leq r + 1$*,* $|1 - z'|z_j|| \geq 1 - |z'|z_j|| = 1 - |z'||z_j| \geq 1 - r$*. Thus*

$$\left|1 - \frac{z_j - z}{1 - z\bar{z}_j} \frac{|z_j|}{z_j}\right| \leq (1 - |z_j|)\frac{1 + r}{1 - r}$$

In proof of theorem 2.22, we know if $z\bar{w} \neq 1$*, then Blaschke factors:*

$$\left|\frac{w - z}{1 - \bar{w}z}\right| = 1 \; if \;\; |z| = 1 \; or \;\; |w| = 1$$

Since in $B_n(z)$ $\left|e^{it}\right| = 1$*, I think* $B_n(e^{it}) = 1$ *everywhere, not a.e..*

$\left|B_n(re^{it})\right| \to 1$ *uniformly as* $r \to 1$*, since* B_n *is holomorphic in a neighborhood of* $\bar{D}$*. This is easy if we choose* $D(0, 1 + \epsilon)$*, s.t.* $z\bar{z}_j \neq 1$ *in* $D(0, 1 + \epsilon)$*.*