0.1 The Conjugate Function

In section 1, given a integrable function f, $f \in L^1$, we know the harmonic function $u = P(f)$ determines a holomorphic function F up to a constant c if we consider u as the real part of F. Let $v(re^{it}) = F(re^{it})$ with $v(0) = 0$. Then we can define the conjugate function of f to be:

$$\tilde{f}(t) = \lim_{t \to 1} v(re^{it})$$

The next theorem ensures the above limit exists for a.e. t:

Theorem 0.1.0.1 (theorem 5.2 in book)**.** *Let $F \in H(D)$ be such that $\operatorname{Re} F(z) \geq 0$ for every $z \in D$. Then F has N.T. limits at almost every boundary point.*

Remark (notes on proof of theorem 5.2 in book)**.** *I don't know why $\lim G(z) = \lim \frac{1}{1+F(z)}$ as $z \to e^{it}$ N.T. is different from 0 a.e. t.*

0.1.1 Estimate conjugate function $\tilde{f}$ for $f \in L^p$, $1 < p < \infty$

The following theorem is key to estimate $\|\tilde{f}\|_p$ for $1 < p < \infty$:

Theorem 0.1.1.1 ((theorem 5.3 in book))**.** *For every p with $1 < p \leq 2$, there is a constant C_p, s.t. for every $F(z) = u(z) + iv(z)$, holomorphic in D, with $u(z) > 0$ on D, v real valued and $v(0) = 0$, the inequality:*

$$\int_{-\pi}^{\pi} \left|v(re^{it})\right|^p dt \leq C_p \int_{-\pi}^{\pi} \left|u(re^{it})\right|^p dt$$

holds for every $0 < r < 1$.

Remark (notes on proof of theorem 0.1.1.1)**.** *We need inequality $|\sin\theta|^p \leq C_p |\cos\theta|^p - D_p \cos(p\theta)$ for $|\theta| < \frac{\pi}{2}$. To use this inequality, we need $\psi(z) < \frac{\pi}{2}$ where $\psi(z) = \arg F(z)$. $u(z) > 0$ ensures this inequality hold.*

However, this inequality actually holds if $u \geq 0$ since $u \geq 0$ implies $u > 0$ by maximum principle.

Now we give the estimation of $\|\tilde{f}\|_p$ for $1 < p < \infty$:

Corollary 0.1.1.2 (Marcel Riesz inequality (corollary 5.5 in book))**.** *For every p with $1 < p < \infty$, there is a constant C_p, s.t. for each $f \in L^p$:*

$$\int_{-\pi}^{\pi} |\tilde{f}(t)|^p dt \leq B_p \int_{-\pi}^{\pi} |f(t)|^p \, dt \tag{1}$$

Remark (notes on proof of corollary 0.1.1.2)**.** *First consider $1 < p \leq 2$ case. Let $f^+ = \max(f, 0)$, $v_1 = \tilde{f^+}$, $f^- = \max(-f, 0)$ and $v_2 = \tilde{f^-}$. Notice $u_1 = P(f^+) = 0$ and $v_1 = 0$ if $f^+ = 0$, $u_2 = P(f^-) = 0$ and $v_2 = 0$ if $f^- = 0$. Thus we can write:*

$$\int_{-\pi}^{\pi} |v(re^{it})|^p dt = \int_{-\pi}^{\pi} \left|v_1(re^{it})\right|^p + \left|v_2(re^{it})\right|^p dt$$

$$\int_{-\pi}^{\pi} |u(re^{it})|^p dt = \int_{-\pi}^{\pi} \left|u_1(re^{it})\right|^p + \left|u_2(re^{it})\right|^p dt$$

Since $\int \left|v_1(re^{it})\right| \leq C_p \int \left|u_1(re^{it})\right|$ *and* $\int \left|v_2(re^{it})\right| \leq C_p \int \left|u_2(re^{it})\right|$ *by theorem 0.1.1.1, we have* $\int \left|v(re^{it})\right| \leq C_p \int \left|u(re^{it})\right|$. *Fatou lemma yields inequality* (1).

For $2 < p < \infty$ *case, we use* $\|v(re^{i\cdot})\|_p = \sup_{\|g\|_{p'} \leq 1} \int v(re^{it})g(t)dt$ *by Hahn-Banach theorem.*

Let $h = P(g)$ *and* $w = \tilde{h}$ *with* $w(0) = 0$. *Since* $h + iw \in H^{p'}$, $h + iw$ *can be written as Poisson integral of boundary function,* $h(re^{it}) + iw(re^{it}) = P(g + i\tilde{g})$. *Thus we have* $w = P(\tilde{g})$.

By Holder inequality, We have:

$$\begin{aligned}
&\int \left|(u(rz) + iv(rz))(h(z) + iw(z)) - (u(re^{it}) + iv(re^{it}))(g(t) + i\tilde{g}(t))\right| \\
\leq &\int |(u(rz) + iv(rz))(h(z) + iw(z)) - (u(rz) + iv(rz))(g(t) + i\tilde{g}(t)) \\
&+ (u(rz) + iv(rz))(g(t) + i\tilde{g}(t))(u(re^{it}) + iv(re^{it}))(g(t) + i\tilde{g}(t))| \\
\leq &\int |u(rz) + iv(rz)| \, |((h(z) + iw(z)) - (g(t) + i\tilde{g}(t))| \\
&+ \int \left|((u(rz) + iv(rz)) - (u(re^{it}) + iv(re^{it})))\right| |(g(t) + i\tilde{g}(t))| \\
\leq &(\int |u(rz) + iv(rz)|^p)^{\frac{1}{p}} (\int |((h(z) + iw(z)) - (g(t) + i\tilde{g}(t)))|^{p'})^{\frac{1}{p'}} \\
&+ (\int \left|((u(rz) + iv(rz)) - (u(re^{it}) + iv(re^{it})))\right|^p)^{\frac{1}{p}} (\int |(g(t) + i\tilde{g}(t))|^{p'})^{\frac{1}{p'}}
\end{aligned}$$

We have $(\int |((h(z) + iw(z)) - (g(t) + i\tilde{g}(t)))|^{p'})^{\frac{1}{p'}} \to 0$ *since* $h + iw \in H^{p'}$.
We have$(\int \left|(u(rz) + iv(rz)) - (u(re^{it}) + iv(re^{it}))\right|^p)^{\frac{1}{p}} \to 0$ *since* $\left|re^{it}\right| < 1$.

Thus $(u(rz) + iv(rz))(h(z) + iw(z)) \to (u(re^{it}) + iv(re^{it}))(g(t) + i\tilde{g}(t))$ *in* L^1.

Our final result is estimation of imaginary part of holomorphic function:

Corollary 0.1.1.3 (corollary 5.8 in book)**.** *If* $F \in H(D)$, *then for every* $1 < p < \infty$ *and every* $0 \leq r < 1$*:*

$$(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\operatorname{Im} F(re^{it})\right|^p dt)^{\frac{1}{p}} \leq B_p^{\frac{1}{p}} (\frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\operatorname{Re} F(re^{it})\right|^p dt)^{\frac{1}{p}} + |\operatorname{Im} F(0)|$$

0.1.2 Estimate conjugate function $\tilde{f}$ for $f \in L^1$

The corollary 0.1.1.2 does not hold for $p = 1$. For example, Poisson kernel $P_r(t)$ is in L^1, $\|P_r(t)\|_1 = 2\pi$ but conjugate Poisson kernel $Q_r(t)$ is not, $\int |Q_r(t)| = 4 \log \frac{1+r}{1-r}$.

Remark. *By Hahn-Banach theorem and duality of* L^p, *for* $v \in L^1$, $\int |v| = \sup_{\|g\|_\infty \leq 1} \left|\int vg\right|$. *I don't know how the author concludes conjugate function operator is not bounded in* L^∞.

For $f \in L^1$, conjugate operator is of weak type $(1,1)$ (theorem 5.9 in book) and type $(1,p)$ for $0 < p < 1$ (corollary 5.10 in book).

Remark (notes on proof of theorem 5.9 in book). *$v(0) = 0$ is a necessary condition for conjugate operator is linear operator.*

The function $f(z) = \frac{z-i\lambda}{z+i\lambda} = \frac{|z|^2-\lambda^2-2i\lambda \operatorname{Re} z}{|z|^2+\lambda^2+2\lambda \operatorname{Im} z}$ maps $\operatorname{Re} z > 0$ to $\operatorname{Im} z < 0$ since $1 \mapsto \frac{1-\lambda^2-2i\lambda}{1+\lambda^2}$.

If $|z| = \lambda$, $f(z) = \frac{-i \operatorname{Re} z}{\lambda + \operatorname{Im} z}$ and $\arg f(z) = -\frac{\pi}{2}$. Thus $h_\lambda(z) = \frac{1}{2}$. If $k \neq \frac{1}{2}$, the level lines $h_\lambda(z) = k$ when $\tan \arg f(z) = \frac{-2\lambda \operatorname{Re} z}{|z|^2-\lambda^2} = \tan \pi(k-1)$. Which is $-2\lambda x = c(k)(x^2+y^2-\lambda^2)$. This is a circle passing through $i\lambda$ and $-i\lambda$.

Let $f(x) = \frac{1}{x} + \arg\tan x - \frac{\pi}{2}$. $f'(x) = -\frac{1}{x^2} + \frac{1}{1+x^2} < 0$ and $\lim_{x\to\infty} f(x) = 0$. Thus $f(x) > 0$. The inequality $\frac{\pi}{2} - \arg\tan \lambda \leq \frac{1}{\lambda}$ holds.

u is harmonic and F is holomorphic, then $u \circ F$ is holomorphic. u is $\operatorname{Re} G$ with G holomorphic. $G \circ F$ is holomorphic. Thus $u \circ F = \operatorname{Re} G \circ F$ is harmonic.

We should pay attention to this statement: Since $\tilde{f}(t) = \lim_{r\to 1} v(re^{it})$, then

$$\begin{aligned} |E_\lambda| &= \left| \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} \{t : |v(r_j e^{it})| > \lambda\} \right| \\ &= \left| \lim_{n\to\infty} \bigcap_{j=n}^{\infty} \{t : |v(r_j e^{it})| > \lambda\} \right| \\ &= \lim_{n\to\infty} \left| \bigcap_{j=n}^{\infty} \{t : |v(r_j e^{it})| > \lambda\} \right| \end{aligned}$$

The last equality is by continuity of measure.

I don't know why author choose $C = \frac{64}{\pi}$ finally.

0.1.3 Conjugate function and H^p space

We now describe H^p for $1 \leq q \leq \infty$. Suppose $F \in H^p$, we know $F = P(f)$ for some $f \in L^1$. $f \in L^1$ implies $\tilde{f}$ is well defined. We can write $F = f + \tilde{f} + i \operatorname{Im} F(0)$ for boundary F. $F(z) \in H^p$ implies $f \in \operatorname{Re} L^p$ (theorem 3.6 in book). Thus:

$$H^p \subset \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^p, c \in \mathbb{R}\}$$

By writing $\tilde{f} = F - f - \operatorname{Im} F(0)$, we know $\tilde{f} \in L^p$ for $1 \leq p \leq \infty$ (conclusion in book).

When $1 < p < \infty$, $f \in L^p$ guarantees $\tilde{f} \in L^p$. $(f + i\tilde{f} + ic) \in L^p$ implies $P(f + i\tilde{f} + ic) \in H^p$. Thus for $1 < p < \infty$:

$$H^p \supset \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^p, c \in \mathbb{R}\}$$

Here we take $f \in \operatorname{Re} L^p$ to make $\tilde{f}$ meaningful. We also have $\operatorname{Re} H^p = \operatorname{Re} L^p$ in this case.

For $p=1$, $f \in L^p$ no longer guarantees $\tilde{f} \in L^p$. But we know if $f, \tilde{f} \in L^1$, then $F = P(f + i\tilde{f} + ic) \in H^1$. Thus:

$$H^1 \supset \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^1, \tilde{f} \in L^1, c \in \mathbb{R}\}$$

Here we take $f \in \operatorname{Re} L^p$ to make $\tilde{f}$ meaningful. We also have $\operatorname{Re} H^1 = \{f \in \operatorname{Re} L^1 : \tilde{f} \in L^1\} \subsetneqq \operatorname{Re} L^1$ by counterexample Poisson kernel.

Similarly we have:

$$H^\infty \supset \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^\infty, \tilde{f} \in L^\infty, c \in \mathbb{R}\}$$

and

$$\operatorname{Re} H^\infty = \{f \in \operatorname{Re} L^\infty : \tilde{f} \in L^\infty\} \subsetneqq \operatorname{Re} L^\infty$$

Remark. *Author suppose $F(e^{it}) \in H^p$, but if $F(re^{it}) \in H^p$ we only know $F(e^{it}) \in L^p$. In other words, $F(e^{it}) \in H^p$ is not clear.*

0.1.4 Conjugate operator

Remark.

$$\frac{r \sin t}{1 + r^2 - 2r\cos t} \to \frac{1}{2\tan\frac{t}{2}}$$

as $r \to 1$ fails if $t = 0$.

$$\begin{aligned}
\left|\frac{r\sin t}{1+r^2-2r\cos t}\right| &= \left|\frac{2r\sin\frac{t}{2}\cos\frac{t}{2}}{1+r^2-2r(2(\cos\frac{t}{2})^2-1)}\right| \\
&= \left|\frac{2r\sin\frac{t}{2}\cos\frac{t}{2}}{(1+r)^2-4r(\cos\frac{t}{2})^2}\right| \\
&= \left|\frac{2r\sin\frac{t}{2}\cos\frac{t}{2}}{(1-r)^2+4r(\sin\frac{t}{2})^2}\right| \\
&\leq \left|\frac{2r\sin\frac{t}{2}\cos\frac{t}{2}}{4r(\sin\frac{t}{2})^2}\right| \\
&= \left|\frac{1}{2\tan\frac{t}{2}}\right| \\
&= \frac{1}{2\tan\left|\frac{t}{2}\right|} \\
&\leq \frac{1}{|t|}
\end{aligned}$$

The following theorem shows we can define conjugate function without getting inside the disk, by singular integral:

Theorem 0.1.4.1 (theorem 5.14 in book)**.** *If $f \in L^1$, then*

$$\tilde{f}(\theta) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{0<\epsilon<|t|<\pi} \frac{1}{2\tan\frac{t}{2}} f(\theta - t)dt$$

for every θ in the Lebesgue set of f and, consequently for a.e. θ.

Remark. *I don't know why*

$$\frac{1}{\pi}\int_{1-r<|t|<\pi}(\frac{r\sin t}{1+r^2-2r\cos t}-\frac{1}{2\tan\frac{t}{2}})f(\theta-t)dt$$

is bounded by

$$C(1-r)^2\int_{1-r<|t|<\pi}\frac{|f(\theta-t)-f(\theta)|}{|t|^3}dt$$

as $r\to 1$.

For upper half plane, we know $u(x,t)=P_t*f(x)$ is harmonic function. We have Hilbert transform as counterpart of conjugate function:

$$Hf(x)=\lim_{\epsilon\to 0}\int_{|y|>\epsilon}\frac{f(x-y)}{y}dy$$

for a.e. x.