0.1 H^p as a Linear Space

In this section we look at H^p as a topological vector space. By considering distance $d(F,G) = \|F-G\|_{H^p}$ for $p \geq 1$ and $d(F,G) = \|F-G\|_{H^p}^p$ for $p < 1$, H^p is a metric space. By considering mapping $F(z) \mapsto F(e^{it})$, H^p is isometric to subspace of L^p. The main topic in this section is dual of H^p.

0.1.1 H^p is not Locally convex for $0 < p < 1$

If a space is locally convex, there is a convex neighborhood V contained in ball $B(0,1)$. Since V is a neighborhood, there is a ball $B(0,\epsilon)$ contained in V. Thus by contrapositive, If for all $\epsilon > 0$, there is a convex combination of F in ball $B(0,\epsilon)$ is out of $B(0,1)$, then the space is not locally convex.

It is easy to prove for $0 < p < 1$, L^p is not locally convex by using triangle wave function. To prove the same fact for H^p, we use trigonometric polynomials to approximate triangle wave function. And conclude these polynomials are in ball $B(0,\epsilon)$ and their convex combination is out of ball $B(0,1)$.

Remark (notes on proof of theorem 6.2 in book)**.** *For $1 < p < 1$, $(a+b)^p \leq a^p + b^p$ by $(a+b)^p \leq \frac{(2a)^p+(2b)^p}{2} = 2^{p-1}a^p + 2^{p-1}b^p \leq a^p + b^p$*

Remark (Algebric dual space and topological dual space from wikipedia: dual space)**.** *Given any vector space V over a field $\mathbb{F}$, the algebraic dual space V^* is defined as the set of all linear functionals $\phi : V \to \mathbb{F}$.*

When dealing with topological vector spaces, one is typically only interested in the continuous linear functionals $\phi : V \to \mathbb{F}$. This gives rise to the notion of the "continuous dual space" or "topological dual" which is a linear subspace of the algebraic dual space. For any finite-dimensional normed vector space or topological vector space, such as Euclidean n-space, the continuous dual and the algebraic dual coincide. This is however false for any infinite-dimensional normed space.

Being non locally convex has a great deal of continuous linear functionals. The topological dual or continuous linear functional on H^p is zero. First we prove the only convex neighborhood of 0 is the whole space. By using the proof of non locally convex reversely, Given a convex and open set $V \subset H^p$ and $0 \in V$, we can show for any $F \in H^p$, there is a combination $\sum_j \lambda_j F_j = F$, s.t. $\sum_j \lambda_j = 1$ and $F_j \in V$. Thus $F \in V$ by V convex and we have $V = H^p$.

Then we consider the continuous linear functionals on H^p. Assume $\phi : H^p \to \mathbb{F}$ is a continuous linear functional. Let $\mathscr{B}$ be a locally convex base for $\mathbb{F}$. For any $W \in \mathscr{B}$, we have $\phi^{-1}(W)$ is convex and open hence is H^p. $\phi(H^p) \subset W$ for all $W \in \mathscr{B}$. We conclude that $\phi(F) = 0$ for all $F \in H^p$. Thus all continuous linear functionals on H^p are zero (This part is following the section 1.47 in Rudin, 1991).

Using inequality:

$$|F(z)| \leq \frac{1}{(1-|z|)^{\frac{1}{p}}} \|F\|_{H^p}$$

for $F \in H^p$ with $0 < p < \infty$, we can prove H^p is a complete space. Thus H^p is closed subspace of L^p in isometry sense.

Remark. *I don't know why H^p is the minimal closed subspace which contains $\{e^{ijt} : j = 0, 1, \cdots\}$. The author give the reason as follows: If $F(z) = \sum_0^\infty a_j z^j$ is in H^p, $F(re^{it}) \to F(e^{it})$ in L^p as $r \to 1$. And for r fixed, $\sum_0^n a_j r^j e^{ijt} \to F(re^{it})$ uniformly as $n \to \infty$.*

0.1.2 Dual of H^p

In subsection 0.1.1, we show for $0 < p < 1$, the dual of H^p is zero. We investigate case $1 \le p \le \infty$ in this subsection.

In section 5, we show for $1 < p < \infty$, $H^p = \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^p, c \in \mathbb{R}\}$ and for $p = 1$, $H^1 = \{f + i\tilde{f} + ic : f \in \operatorname{Re} L^1, \tilde{f} \in L^1, c \in \mathbb{R}\}$. Thus H^p is a proper subspace of L^p.

By dual of L^p, any continuous linear functional $\phi(g)$ for $g \in L^p$ can be written as $\phi_f(g) = \int gf$ with $f \in L^{p'}$. We consider the restriction of ϕ to H^p. This is the continuous linear functional $\phi_f(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})f(t)dt$ on H^p. If we consider this mapping is as from $L^{p'} \to (H^p)^*$, we have:

$$
\begin{aligned}
\|\phi_f\| &= \sup_{\|f\|_{p'}=1} \frac{\|\phi_f\|_{(H^p)^*}}{\|f\|_{p'}} \\
&= \sup_{\|f\|_{p'}=1} \sup_{\|F\|_{H^p}=1} \frac{\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})f(t)dt\right|}{\|F\|_{H^p}} \\
&\le \sup_{\|f\|_{p'}=1} \sup_{\|F\|_{H^p}=1} \frac{\frac{1}{2\pi}(\int_{-\pi}^{\pi} \left|F(e^{it})\right|^p dt)^p (\int_{-\pi}^{\pi} |f(t)|^p dt)^{p'}}{\|F\|_{H^p}} \\
&= \sup_{\|f\|_{p'}=1} \sup_{\|F\|_{H^p}=1} \frac{\|F\|_{H^p}\|f\|_{p'}}{\|F\|_{H^p}} \\
&= 1
\end{aligned}
$$

This the mapping ϕ from $L^{p'} \to (H^p)^*$, $f \mapsto \phi_f$ is a continuous linear mapping. The Hahn-Banach theorem tells us that every $\Lambda \in (H^p)^*$ is of the form $\Lambda = \phi_f$ for some f with $\|f\|_{p'} \le \|\Lambda\|$. More precisely, any continuous linear functional $\Lambda \in (H^p)^*$ can be extended to all of L^p. Thus we get $f \in L^{p'}$ with $\phi_f(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})f(t)dt$ restricted back to H^p.

The kernel of mapping ϕ is $f \in L^{p'}$ for which $\phi_f = 0$. This is equivalent $\phi_f(F) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{it})f(t)dt = 0$ for all $F \in H^p$, clearly,

$$\ker\phi = \{f \in L^{p'} : \hat{f}(-j) = \int_{-\pi}^{\pi} e^{ijt} f(t)\frac{dt}{2\pi} = 0, j = 0, 1, \cdots\}$$

$\hat{f}(j)$ is zero for non-positive frequency j is equivalent to $f \in H^p$ and $\hat{f}(0) = 0$.

Thus

$$\{f \in L^{p'} : \hat{f}(-j) = \int_{-\pi}^{\pi} e^{ijt} f(t) \frac{dt}{2\pi} = 0, j = 0, 1, \cdots\}$$
$$= \{f \in H^{p'} : \int_{-\pi}^{\pi} f(t)dt = 0\}$$

We denote this space by $H^{p'}(0)$ and obtain an isometry

$$L^{p'}/H^{p'}(0) \cong (H^p)^* \tag{1}$$

Now we consider the continuous linear functionals on $H^p(0)$, We consider the kernel of mapping $L^{p'} \to (H^p(0))^*$:

$$\{f \in L^{p'} : \hat{f}(-j) = \int_{-\pi}^{\pi} e^{ijt} f(t) \frac{dt}{2\pi} = 0, j = 1, 2, \cdots\} = H^{p'}$$

Thus we obtain an isometry

$$(H^{p'}(0))^* \cong L^{p'}/H^{p'} \tag{2}$$

Remark (Topological complement)**.** *Two vector subspace X and Y are algebraic complement of each other if $X + Y = E$ and $X \cap Y = \{0\}$. We can write $X \oplus Y = E$.*

Two vector subspace X and Y are topological complement of each other if they are algebraic complement of each other and P_X (Projection from E to X) is continuous. If E is Banach space, another equivalent condition for topological complement is they are algebraic complement of each other and X and Y are closed.

X and Y are topological complement of each other means $E = X \oplus Y$ and $E \cong X \oplus Y$ in isometry sense.

For $1 < p < \infty$, $H^{p'}(0)$ has a topological complement in $L^{p'}$. Let us see how to construct this. Consider $f \in L^{p'}$, $f = \sum_{-\infty}^{\infty} a_j e^{ijt}$. Set

$$A(f) = \frac{1}{2}(f + \tilde{f} - \hat{f}(0)) = \sum_{j>0} a_j e^{ijt}$$

Then A is the projection of $L^{p'}$ onto $H^{p'}(0)$. $f - A(f) = \sum_{j \le 0} a_j e^{ijt} = \sum_{j \ge 0} a_{-j} e^{-ijt}$. If we write $F(z) = \sum_{j \ge 0} a_{-j} z^j$, we have $F \in H^{p'}$ and $f(t) = Af(t) + F(e^{-it})$. If we write $G(z) = \sum_{j \ge 0} \overline{a_{-j}} z^j$. $G(e^{it}) = F(e^{-it}) \in H^{p'}$. Thus we have:

$$f(t) = Af(t) + \overline{G(e^{it})}$$

Using notation $\overline{H^{p'}} = \overline{h(t)} : h \in H^{p'}$ and $(H^{p'})^- = h(-t) : h \in H^{p'}$. We have:

$$L^{p'} = H^{p'}(0) \oplus \overline{H^{p'}} = H^{p'}(0) \oplus (H^{p'})^-$$

If we consider $B(f) = \frac{1}{2}(f + \tilde{f} + \hat{f}(0)) = \sum_{j\geq 0} a_j e^{ijt}$. We have:

$$L^{p'} = H^{p'} \oplus \overline{H^{p'}(0)} = H^{p'} \oplus (H^{p'}(0))^{-}$$

By topological direct sum we have topological isomorphisms:

$$L^{p'}/H^{p'}(0) \cong H^{p'}$$

$$L^{p'}/H^{p'} \cong H^{p'}(0)$$

By equation (1) and (2), we derive a conclusion for H^p similar with dual of L^p.

$$(H^p)^* \cong H^{p'}$$

with the pairing

$$< G, F >= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{it}) G(e^{-it}) dt$$

$F \in H^p$, $G \in H^{p'}$. or

$$< G, F >= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{it}) \overline{G}(e^{it}) dt$$

$F \in H^p$, $G \in H^{p'}$. Besides, we also have $(H^p(0))^* \cong H^{p'}(0)$. We can not use same argument for $p = 1$ case since for $f \in L^{1'} = L^\infty$, $\hat{f}$ no longer in L^∞. We will study $(H^1)^*$ in section 9. Now we only have $(H^1)^* \cong L^\infty/H^\infty(0)$ by equation (1).