Chapter 1

Classical Theory of Hardy Space

1.1 Harmonic Functions, Poisson Representation

In plane case, we can rewrite harmonic condition: $\Delta F = 0$ as $\Delta F = (\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})F = 0$. Notices the equation $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 0$ is equivalent to the Cauchy-Riemann equations for function $F = u + iv$.

Remark (holomorphic indicates harmonic)**.** *If function F satisfies C-R equations, then F satisfies Laplace equation for $\mathbb{C}$. And if F satisfies $\Delta = 0$, so does $\bar{F}$. Thus F and $\bar{F}$ are all harmonic functions. Besides, the real and imaginary part of F, u and v also satisfy $\Delta = 0$, means u and v are harmonic functions.*

Remark (harmonic indicates holomorphic)**.** *Assume that u satisfies $\Delta = 0$. If we let $v_x = -u_y$ and $v_y = u_x$, then we have $v_{xy} = v_{yx}$, which indicates exists of v (equivalent of Fubini's Theorem and the equality of the mixed partial derivatives). And $v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$ indicates v is also harmonic. v is determined up to an additive constant, and $F = u + iv$ is holomorphic.*

Remark. *Later we will see holomorphic function is the special case of harmonic function.*

If u is a real harmonic function, by above remark, we know $u = \operatorname{Re} F$ for some holomorphic function $F(z) = \sum_{k=0}^{\infty} c_k z^k$. Then we can derive the series representation of u:

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta} \tag{1.1}$$

and it converges uniformly on compact subsets of $D(0, R)$.

Remark (uniformly converge on compact subsets)**.** *Using mean value property, sequence of holomorphic functions which uniformly converge on compact subsets, the limit is a holomorphic function.*

The partial sum of $u_n(re^{i\theta}) = \sum_{k=-n}^{n} a_k r^{|k|} e^{ik\theta}$ is obviously harmonic, and it converges uniformly on compact subsets of $D(0, R)$. Thus $u(re^{i\theta})$ is harmonic.

1.1.1 Harmonic function to Poisson integral (or Poisson representation of harmonic function)

Assumed u is harmonic in $D(0, R)$. If $R > 1$, we can represent a_k in equation (1.1) by $u(e^{it})$ using Fourier analysis. Finally we derive the Poisson kernel and the Poisson representation:

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) u(e^{it}) dt \tag{1.2}$$

If u is only harmonic on $D(0, 1)$, the equation (1.2) still can be valid in some sense once we add some restriction on u.

Theorem 1.1.1.1 (theorem 1.3 in book). *Let u be a harmonic function in D such that*

$$\sup_{0 \leq r < 1} \int_{-\pi}^{\pi} \left| u(re^{it}) \right|^p dt < \infty \tag{1.3}$$

for some $p > 1$. Then there is a function $f \in L^p$ such that

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(t) dt \tag{1.4}$$

This theorem still holds for $p = \infty$ if we replace inequality (1.3) by $\sup_{0 \leq r < 1} \left| u(re^{it}) \right| < \infty$.

In later section, we will see left side of inequality (1.3) is H^p norm. The proof of Theorem 1.1.1.1 shows that how dual space, w*-topology and Banach-Alaoglu theorem perform in integral representation. The tricky part is using representation $u(r_n re^{i\theta})$. In the proof we also need $P_r(\theta) \in L^{p'}$ which is easy since $\|P_r(\theta)\|_\infty = \frac{1+r}{1-r}$ and $L^\infty([-\pi, \pi]) \subset L^{p'}([-\pi, \pi])$.

Remark (some details of proof of theorem 1.1.1.1). *In metrizable space, compact and sequentially compact are equivalent.*

Let $f_n(t) = u(r_n e^{it})$, $(f_n) \subset L^{p'*}$ is in a close ball w.r.t p-norm and Banach-Alaoglu theorem indicates this ball is w* compact. Since w*-compact is equivalent to w*-sequentially compact if this close ball is metrizable in w*-topology, we need to prove this close ball is metrizable in w*-topology.

Here is a direct proof. Since $L^{p'}$ is separable there is a countable dense set $(g_n) \in L^{p'}$. For every $g \in L^{p'}$, let $\hat{g}(f) = f(g)$ where $f \in L^{p'*}$. Any pair of points $f_1, f_2 \in L^{p'*}$ can be separate by some $g_i \in L^{p'}$ (since every $\hat{g_n}$ is w*-continuous, density of (g_n) ensures f_1, f_2 can be separate), Thus $(\hat{g_n})$ is a countable family of continuous functions that separates points in $L^{p'*}$. Then we can use metric: $d(f_1, f_2) = \sum_{n=0}^{\infty} 2^{-n} |f_1(g_n) - f_2(g_n)|$.

We have proved that $L^{p'*}$ is metrizable in w*-topology. Here is a topological thought. By Nagata-Smirnov metrization theorem, $L^{p'*}$ is regular. we don't

know how to shown this property directly. Also, $L^{p'*}$ a basis countably locally finite (by Nagata-Smirnov metrization theorem), we also want to show this property directly. Besides, consider Urysohn metrization theorem, if we have shown $L^{p'*}$ is regular, $L^{p'*}$ is metrizable once we show there is a countable basis for $L^{p'*}$. We don't know if it is possible.

Now (f_n) is w* sequentially compact in $L^{p'*}$. This means for any $g \in L^{p'}$, there is a subsequence (f_{n_k}) and $f \in L^{p'*}$ such that $\int g f_{n_k} \to \int g f$.

Remark. $L^{p'*} \cong L^p$. *Thus $L^{p'*}$ is a normed space and $L^{p'*}$ is metrizable.*

The $p = 1$ case is failed since we can not use w*-topology. L^1 is not a separate dual space of some space (Assume $L^1 = X^*$. L^1 is separable in w*-topology implies X is separable). One method showing L^1 is not a separate dual space of some space by showing L^1 does not have the Radon-Nikodym property (Radon-Nikodym theorem is valid).

Remark. *If X is a Banach space, then X^* has the Radon-Nikodym property (RNP) if (and only if) every separable, linear subspace of X has a separable dual (Charles Stegall: The Radon-Nikodym property in Conjugate Banach Space. II).*

Thus for $p = 1$ case, we can not use same argument as Theorem 1.1.1.1. However there is a relative result. L^1 can be isometrically imbedded in to M, the space of Borel measures with bounded variation, which is dual of continuous function with compact support space C. Thus we can use same argument to M and C and introduce Poisson-Stieltjes integral.

When we use Borel measures case in Theorem 1.1.1.1, we get *Poisson integral of positive measure for harmonic functions.*

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{\pi}^{\pi} P_r(\theta - t)d\mu(t) \tag{1.5}$$

Here $d\mu(t)$ is the w*-limit of $u(r_n e^{it})dt$. The difference between (1.2) and (1.5) is that in (1.2), u needs to be harmonic in a little larger disk $D(0, R), R > 1$, but in (1.5) u needs not. However, this is not hold for all harmonic functions since we need constraints.

1.1.2 Poisson integral indicates harmonic

Using Fourier series, we show that given a function $f \in L^p$, $1 \le p \le \infty$ (or a complex Borel measure μ), Poisson integral u=$P(f)$ (or u=$P(\mu)$) is real part of a holomorphic function. Thus it is harmonic (Theorem 1.11 (or 1.14) in book). And we give bound of norm of u by norm of f (or measure μ):

- $\int_{-\pi}^{\pi} \left|u(re^{it})\right|^p dt \le \int_{-\pi}^{\pi} |f(t)|^p dt$ *for $p < \infty$*

- $|u(z)| \le \|f(t)\|_\infty$ *for $p = \infty$*

- $\int_{-\pi}^{\pi} \left|u(re^{it})\right| dt \le \int_{-\pi}^{\pi} d\left|\mu\right|(t)$

For $f \in L^p$, $p \le \infty$ case, the equality holds when $r \to 1$ or take sup on right hand side. We prove this in section 3.

1.1.3 Boundary behavior of Poisson integral (or norm convergence and pointwise convergence)

Definition (Dirichlet problem). *Given a continuous function f on ∂D, we want to find a continuous function on $\bar{D}$, which is harmonic in D and coincides with f on ∂D.*

From section 1.1.2, we can see $u(re^{it}) = P(f)$ is harmonic function. Roughly speaking, if $u(e^{it}) = f$ (Actually the domain of u is D, so $u(e^{it})$ is not defined), the classical Dirichlet problem is solved. Thus we need to study the boundary behavior of Poisson integral. *The key idea is approximate identity.* Using this idea, we prove the following theorem.

Theorem 1.1.3.1 (Theorem 1.16 in book). *Let ϕ_α be an approximate identity on the torus T. Then:*

1. *If $f \in L^p([-\pi, \pi])$ with $1 \le p < \infty$ and f_α stands for convolution*

$$f_\alpha(\theta) = (f * \phi_\alpha)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta - t)\phi_\alpha(t)dt$$

   it follows that $f_\alpha \to f$ in L^p, i.e.:

$$\int_{-\pi}^{\pi} |f_\alpha(t) - f(t)|^p \, dt \to 0$$

2. *If f is a continuous 2π-periodic function, we have $f_\alpha \to f$ uniformly on T.*

Theorem 1.1.3.1 shows that for $f \in L^p$, $1 \le p < \infty$ or $f \in C$, Poisson integral converges to boundary in norm. Following the proof of theorem 1.1.3.1, we can show another two case: $p = \infty$ and $f \in M$. The convergence becomes w* convergence (Corollary 1.19 in book).

Here comes another topic: What about pointwise convergence for $P(\mu)$ on boundary if $P(\mu)$ is Poisson-Stieltjes integral? We need a new concept: non-tangentially converge. We show that $P(\mu)(z) \to F'(\theta_1)$ as $z \to e^{i\theta_1}$ N.T., where $F(\theta) = \int_0^\theta d\mu(t)$ and $F'(\theta_1)$ exists and finite (Theorem 1.20 in book).

You can omit the following remark if you choose not to check the proof in book.

Remark (Proof of Theorem 1.20). *First we take $c > 0$, since we need proof for any $c > 0$. This c decides the approach region.*
Then given $\epsilon > 0$, we take δ small enough s.t. $|F(t)| < \epsilon |t|$ when $|t| < \delta$. This δ can be taken since $F(0) = 0$ and $F'(0) = 0$.
If we take r large enough and $re^{i\theta}$ in the region. then $|\theta|$ can be less than $\frac{\delta}{4}$.

The estimation of $u(re^{i\theta})$ is (we use $F(t) = \int_{-\pi}^{\pi} d\mu(t)$ in the third line):

$$
\left| u(re^{i\theta}) \right| = \left| \frac{1}{2\pi} \int_{\delta < |t| \leq \pi} P_r(\theta - t) d\mu(t) + \frac{1}{2\pi} \int_{-\delta}^{\delta} P_r(\theta - t) d\mu(t) \right|
$$

$$
\leq \left| \frac{1}{2\pi} \int_{\delta < |t| \leq \pi} P_r(\theta - t) d\mu(t) \right| + \left| \frac{1}{2\pi} \int_{-\delta}^{\delta} P_r(\theta - t) d\mu(t) \right|
$$

$$
\leq (\sup_{|t| > \delta} P_r(\theta - t)) \cdot \frac{1}{2\pi} \int_{\delta < |t| \leq \pi} d|\mu|(t)
$$

$$
+ \left| \left(P_r(\theta - t) \cdot \frac{1}{2\pi} F(t) \right) \Big|_{-\delta}^{\delta} + \int_{-\delta}^{\delta} P_r'(\theta - t) \frac{1}{2\pi} F(t) dt \right|
$$

$$
\leq (\sup_{\frac{3\delta}{4} < |t| < \pi + \frac{\delta}{4}} P_r(t)) \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} d|\mu|(t)
$$

$$
+ \left| \left(P_r(\theta - t) \cdot \frac{1}{2\pi} F(t) \right) \Big|_{-\delta}^{\delta} \right| + \left| \int_{-\delta}^{\delta} (P_r'(\theta - t) \frac{1}{2\pi} F(t)) dt \right|
$$

$$
\leq (\sup_{|t| > \frac{3\delta}{4}} P_r(t)) \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} d|\mu|(t)
$$

$$
+ (\sup_{|t| > \frac{3\delta}{4}} P_r(t)) \cdot \frac{1}{2\pi} \int_{-\delta}^{\delta} d|\mu|(t) + \left| \frac{1}{2\pi} \int_{-\delta}^{\delta} P_r'(\theta - t) F(t) dt \right|
$$

Lebesgue-Stieltjes integral is related to bounded increasing function. The integral has decomposition $d\mu(t) = f(t)dt + d\sigma(t)$, $f \in L^1$. $\int_0^\theta d\sigma(t)$ is a jump function. Thus $F'(\theta) = f(\theta)$ a.e.. Considering $P(f)$, $f \in L^p$, $1 \leq p \leq \infty$, or even $P(\mu)$, the integral $P(f)$ or $P(\mu)$ is L-S integral. Thus N.T. convergence holds a.e..

By section 1.1.1, Harmonic function with constraints (bounded in some sense) can be written as Poisson integral $P(f)$ or $P(\mu)$ (Corollary 1.10 in book). Thus the theorem of Fatou holds: *Any function holomorphic and bounded in D has non-tangential boundary values a.e..*

The difference between $p = 1$ and $p > 1$ is the starting point of the theory of Hardy spaces.

1.1.4 Harmonic function in higher dimension

A continuous function is harmonic in region Ω if and only if it satisfies mean value property. For mean value property to harmonic, we need an approximate identity.

You can omit the following remark if you choose not to check the proof in book.

Remark (Proof of Theorem 1.22). *In converse part, we choose region Ω_ϵ is to make integral $\int_{\mathbb{R}^n} u(x_0 + r\sigma - y)\phi_\epsilon(y) dy$ and $\int_{\mathbb{R}^n} u(x_0 - y)\phi_\epsilon(y) dy$ defined. In other words, if $x_0 + r\sigma - y$ or $x_0 - y$ out of Ω, then $\phi_\epsilon(y) = 0$.*

A consequence of mean value property is maximum principle. The proof can be given by topological technic: Assumed u attains maximum value m in Ω. Let $A = \{x : u(x) = m\}$. Using mean value property, we show every x is interior point of A. Thus A is open. However, $\Omega \setminus A = \{x : u(x) < m\}$ is open since u is continuous. Ω is connected and A is not empty. $\Omega \setminus A$ is empty. Thus u is constant.

By maximum principle and minimum principle, we can show u is unique in Ω if u is harmonic in Ω and u is continuous on $\partial\Omega$.

We introduce the Poisson kernel $P(x, s) = \frac{1-|x|^2}{|x-s|^n}$ for Dirichlet problem in n dimension. The solution is $\frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} P(x, s) f(s) ds$.

There is a weaken form of mean value property (or called discrete mean value property). u is harmonic if mean value property is satisfied only on a sequence of $r_n \to 0$ (Theorem 1.30 in book).

You can omit the following remark if you choose not to check the proof in book.

Remark (Proof of theorem 1.30). *The proof here is not well understand. The set $K = \{u(x) - v(x) = m : x \in \overline{B(x_0, R)}\}$ is compact in $\overline{B(x_0, R)}$ since K is closed in compact set $\overline{B(x_0, R)}$. We can use finite open cover to prove K is also compact in $B(x_0, R)$. Let $f : K \to \mathbb{R}$, $f(x) = d(x, x_0)$, f is continuous function on compact set K thus it attains maximum value of f. Let x_1 be a point of the maximum value. I just can image that for sphere $\partial B(x_1, r)$, only half of the sphere is in K otherwise $f(x_1)$ is not the maximum value. Here r need to be small enough to ensure $\partial B(x_1, r) \subset B(x_0, R)$. But then $u(x_1) - v(x_1) < m$ if we use integral with $r_j < r$. Thus it is a contradiction.*

I don't know how the sequence (r_n) plays in this proof.

Then we go to the reflection principle and Liouville theorem.

Finally we go to Dirichlet problem on unbounded domain, in particular $\mathbb{R}_+^{n+1}$. This problem cannot have a unique solution but can have a unique bounded solution. And we give Poisson kernel for $\mathbb{R}_+^{n+1}$ by Fourier transform method.