0.1 Some Classical Inequalities

In this section we study two classical Inequalities: Hardy's inequality and Fejer-Riesz inequality. The first inequality is an example of why H^p is a natural replacement of L^p for $p \leq 1$. The second inequality shows some geometry properties of conformal mappings.

0.1.1 Hardy's inequality

Theorem 0.1.1.1 (Hardy's inequality)**.** *Let* $F(z) = \sum_{j=0}^{\infty} a_j z^j$ *be in* H^1*. Then:*

$$\sum_{j=0}^{\infty} \frac{|a_j|}{j+1} \leq C \|F\|_{H^1}$$

with a constant C *independent of* F*.*

Remark (notes on proof of theorem 0.1.1.1)**.** *We know the principal branch of the logarithm* $\log z = \log r + i\theta$ *where* $z = re^{i\theta}$ *with* $|\theta| < \pi$*. Thus* $\operatorname{Im} \log 1 - z = \arg 1 - z$*. It is easy to see* $-\frac{\pi}{2} < \arg 1 - z < \frac{\pi}{2}$*.*

$$\begin{aligned} F(re^{it})u(re^{it}) =& (\sum_{j=0}^{\infty} a_j (re^{it})^j)(\frac{i}{2}\sum_{j\neq 0} j^{-1} r^{|j|} e^{ijt}) \\ =& (\sum_{j=0}^{\infty} a_j r^j e^{ijt})(\frac{i}{2}\sum_{k\neq 0} k^{-1} r^{|k|} e^{ikt}) \end{aligned}$$

After taking integral, only $j + k = 0$ *term does not vanish, thus:*

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} F(re^{it})u(re^{it})dt =& (\frac{i}{2}\sum_{j+k=0} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_j r^j e^{ijt} k^{-1} r^{|k|} e^{ikt} dt) \\ =& \frac{i}{2}\sum_{j+k=0} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_j r^{j+|k|} e^{i(j+k)t} k^{-1} dt \\ =& \frac{i}{2}\sum_{j=1}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} a_j r^{2j} (-j)^{-1} dt \\ =& \frac{i}{2}\sum_{j=1}^{\infty} a_j r^{2j} (-j)^{-1} \\ =& -\frac{i}{2}\sum_{j=1}^{\infty} a_j j^{-1} r^{2j} \end{aligned}$$

The corollary 4.2 in book shows that if $F(e^{it})$ is absolutely continuous (equivalent to $F' \in H^1$), then $(\hat{F}(n))_n \in \ell^1$. But the converse is not true. $(\hat{F}(n))_n \in \ell^1$ only implies F extends to a continuous function on $\bar{D}$

Remark (Errata of $\operatorname{Re} H^1$). *Let $g(t)$ be $\operatorname{Re} F(e^{it}) = \sum_{j\geq 0} a_j e^{ijt}$. Then*

$$\begin{aligned} g(t) =& \frac{F(e^{it}) + \overline{F(e^{it})}}{2} \\ =& \frac{a_0 + \bar{a}_0}{2} + \sum_{j\geq 0} \frac{a_j}{2} e^{ijt} + \sum_{j\geq 0} \frac{\bar{a}_j}{2} e^{-ijt} \\ =& \frac{a_0 + \bar{a}_0}{2} + \sum_{j>0} \frac{a_j}{2} e^{ijt} + \sum_{j<0} \frac{\overline{a_{-j}}}{2} e^{ijt} \\ =& \frac{a_0 + \bar{a}_0}{2} + \sum_{j\neq 0} \hat{g}(j) e^{ijt} \end{aligned}$$

where $\hat{g}(j) = \frac{a_j}{2}$ for $j > 0$, $\hat{g}(j) = \frac{\overline{a_{-j}}}{2}$ for $j > 0$ and $\hat{g}(j) = \operatorname{Re} a_0$. Thus $|a_j| = |\hat{g}(j)| + |\hat{g}(-j)|$. Substitute $|a_j|$ to $\sum_{j=1}^{\infty} \frac{|a_j|}{j} \leq \pi \|F\|_{H^1}$. We have $\sum_{j\neq 0} \left| \frac{\hat{f}(j)}{j} \right| \leq \pi \|f\|_{\operatorname{Re} H^1}$

We have $\operatorname{Re} H^1$ is a proper subspace of $\operatorname{Re} L^1$. And Hardy's inequality may be considered an extension to $p = 1$ of Paley's inequality which says that for $f \in L^p$ with $1 < p \leq 2$:

$$\sum_{j=-\infty}^{\infty} \frac{\left|\hat{f}(j)\right|^p}{|j|^{p-2}} \leq C_p \|F\|_p^p$$

Later we will see in $\mathbb{R}^n$ this inequality can be extended to H^p for $p < 1$. And H^p for $p \leq 1$ are natural substitutes of Lebesgue spaces L^p.

0.1.2 Fejer-Riesz inequality

Recall the final corollary in last section. Let F be a conformal map from D to interior domain bounded by a Jordan curve Γ. Then Γ is rectifiable if and only if $F' \in H^1$.

Theorem 0.1.2.1 (Fejer-Riesz inequality). *Let $F \in H^p$, $0 < p < \infty$, then*

$$\int_{-1}^{1} |F(x)|^p \, dx \leq \frac{1}{2} \int_{-\pi}^{\pi} \left|F(e^{it})\right|^p dt$$

To prove this theorem, we first prove the $p = 2$ case. Then for $p \neq 2$ case, we factorize $F(z) = B(z)H(z)$ and let $|G(z)|^2 = |H(z)|^p$ to reduce this case to $p = 2$.

Here is a direct application of this inequality. Let F be the conformal map from D to interior domain bounded by a Jordan curve Γ. Then image of diameter of D has length at most half of length of Γ (corollary 4.6 in book).

Remark (notes on proof of corollary 4.6 in book)**.** *To prove that* $\frac{1}{2}$ *is the best constant in corollary 4.6 in book, we only need to show there is a conformal map from* D *to interior domain bounded by a rectifiable Jordan curve* Γ*, the constant* $\frac{1}{2}$ *can not be smaller. Let* $F(z)$ *is a conformal map from* D *to rectangle* $\{x+iy : |x| < 1, |y| < \epsilon\}$ *and* F *maps segment* $(-1,1)$ *in* D *to segment* $(-1,1)$ *in rectangle. It is easy to construct this map. The constant has to be at least* $\frac{2}{4+4\epsilon}$. *Let* $\epsilon \to 0$ *we conclude* $\frac{1}{2}$ *is the best constant.*

Another usage of conformal mapping $F' \in H^1$ is following: F can be extended on $\bar{D}$ and F is still conformal. More precisely, Let F be a conformal mapping from D to interior domain bounded by a rectifiable Jordan curve Γ. F is also conformal at almost every boundary point (corollary 4.7 in book).

Remark (notes on proof of corollary 4.7 in book)**.** *The step:*

$$\frac{F(e^{it_0}) - F(z)}{e^{it_0} - z} - F'(e^{it_0}) = \frac{1}{e^{it_0} - z} \int_z^{e^{it_0}} (F'(\xi) - F'(e^{it_0}))d\xi \to 0$$

as $z \to e^{it_0}$ *N.T. is by mean value theorem of integration.*

I don't know why the tangent to Γ *at the point* $F(e^{it_0})$ *happens for a.e. boundary point* e^{it_0}.

The angle between γ *and boundary in* D *is* $\lim \arg z - e^{it_0} - t_0 - \frac{\pi}{2}$ *and The angle between* $F(\gamma)$ *and boundary in* $F(D)$ *is* $\lim \arg F(z) - F(e^{it_0}) - t_0 - \arg(\frac{d}{dt}(F(e^{it}))|_{t=t_0})$. *Since* F *is conformal in* D*, to prove* F *is conformal in* $\bar{D}$*, we only need to prove the conformal map preserves angle on boundary. That is:*

$$\lim_{z \to e^{it_0}} \arg(z - e^{it_0}) - t_0 - \frac{\pi}{2} = \lim_{z \to e^{it_0}} \arg(F(z) - F(e^{it_0})) - \arg(\frac{d}{dt}(F(e^{it}))|_{t=t_0})$$

We have $\frac{d}{dt}F(e^{it})|_{t=t_0} = ie^{it_0}F'(e^{it_0})$. *Thus* $\arg(\frac{d}{dt}(F(e^{it}))|_{t=t_0}) = \frac{\pi}{2} + t_0 + \arg F'(e^{it_0})$. *So the equality is the same as:*

$$\lim_{z \to e^{it_0}} \arg(z - e^{it_0}) = \lim_{z \to e^{it_0}} \arg(F(z) - F(e^{it_0})) - \arg F'(e^{it_0})$$

which is clearly if we take $\arg$ *in both sides in* $\lim_{z \to e^{it_0}} \frac{F(e^{it_0}) - F(z)}{e^{it_0} - z} = F'(e^{it_0})$. *We use* $t_0 + \frac{\pi}{2}$ *instead of* $t_0 - \frac{\pi}{2}$ *match to* $\arg(\frac{d}{dt}(F(e^{it}))|_{t=t_0})$ *since they are in the same direction.*