0.1 Multipliers

In this section, we keep talk about the convergence property and norm estimate of convolution operator. In last section, we know some convolution operator can not be defined as usual Lebesgue integral but a limit process called singular integral. The stimulation of studying the convolution operator is that the convolution operator is equivalent to translation invariant operator.

Suppose $Tg = \int f(x-y)g(y)dy$ is a convolution operator. Then by Fourier transform, we have $(Tf)^\wedge = \hat{f}\hat{g}$. The concept of multipliers give us another approach to study the convolution operator T:

Definition 0.1.0.1. *Let $1 \leq p < \infty$. Given $m \in L^\infty$, we say that m is a (Fourier) multiplier for L^p if the operator T_m, initially defined in L^2 by the relation:*

$$(T_m f)^\wedge(\xi) = m(\xi)\hat{f}(\xi)$$

satisfies the inequality

$$\|T_m f\|_p \leq C\|f\|_p \quad (f \in L^2 \cap L^p)$$

> **Remark 0.1.0.1.** Given a sequence of kernels $(k_N)_N$ in L^1, if $T_N f = k_N * f$ are uniformly bounded from $\mathscr{S}$ to L^p, it is easy to see $(T_N)_N$ is a Cauchy sequence of in bounded linear functional space $\mathscr{B}(\mathscr{S}, L^p)$ ($\mathscr{B}(\mathscr{S}, L^p)$ is complete by assigning L^p norm on $\mathscr{S}$ and completeness of L^p), then the limit of T_N exists. Indeed, under hypothesis, proposition 5.5 in book gives $T_N f$ is a singular integral for $f \in \mathscr{S}(\mathbb{R}^n)$ and T_N is uniformly bounded. Since $\mathscr{S}(\mathbb{R}^n)$ is dense in L^p, T can be uniquely extends to L^p.
>
> Notice that the limit process agrees the definition of singular integral.

0.1.1 Hormander-Mihlin multiplier theorem

We know every $m \in L^\infty$ is a multiplier for L^2 by Plancherel's theorem. The Hormander-Mihlin multiplier theorem gives a sufficient condition for $m \in L^\infty$ is a multiplier for L^p.

Theorem 0.1.1.1 (Hormander-Mihlin multiplier theorem)**.** *Let $a = [\frac{n}{2}] + 1$ be the first integer greater than $\frac{n}{2}$. If $m \in L^\infty(\mathbb{R}^n)$ is of class C^a outside the origin and satisfies:*

$$(R^{-n} \int_{R<|x|<2R} |D^\alpha m(\xi)|^2 \, d\xi)^{\frac{1}{2}} \leq CR^{-|\alpha|} \quad (0 < R < \infty) \tag{1}$$

for every multi-index α such that $|\alpha| \leq a$, then m is a multiplier for L^p, $1 < p < \infty$.

This theorem is an improvement of Mihlin multiplier theorem. Mihlin multiplier theorem supposes a stronger condition:

$$|D^\alpha m(\xi)| \leq CR^{-|\alpha|} \quad (|\alpha| \leq a) \tag{2}$$

In other words, Hormander weaken the decreasing speed as α increasing. The decreasing of uniform boundedness of $|D^\alpha m(\xi)|$ is weakened to the decreasing of L^2 norm of $|D^\alpha m(\xi)|$.

Inequality (2) is satisfied by every function $m(\xi)$ of class C^a outside the origin of degree ib. Indeed by chain rule: $\frac{\partial}{\partial x_i}(f_i(tx)) = f_i(tx)t$. And $\frac{\partial}{\partial x_i}(t^p f_i(x)) = t^p f_i(x) \leq C|\xi|^{-1}$. We have $f_i(tx)t = t^p f_i(x)$. Thus $|m_i(\xi)| = \left||\xi|^{ib-1} m_i(x')\right| = \frac{|m_i(x')|}{|\xi|}$ with $|x'| = 1$. By induction, $|D^\alpha m(\xi)| \leq \frac{|D^\alpha m(x')|}{\xi^{-\alpha}}$. Let

$$C = \max_{|\alpha|\leq a} \sup_{|x'|=1} |D^\alpha m(x')|$$

. We have $|D^\alpha m(\xi)| \leq C|\xi|^{-\alpha}$

There are two standard techniques in proof of the Hormander-Mihlin multiplier theorem. The first one is the smooth cutting of the multiplier into dyadic pieces.

Lemma 0.1.1.2 (Lemma 6.5 in book)**.** *There is a non negative function $\phi \in C^\infty$ supported in the spherical shell $\{\xi : \frac{1}{2}|\xi| < 2\}$ such that*

$$\sum_{j\in\mathbb{Z}} \phi(2^{-j}\xi) = 1 \quad (\xi \neq 0)$$

The second is an explicit formulation of the well known fact that the regularity of the multiplier is translated into control of the size of kernel.

Lemma 0.1.1.3 (Lemma 6.6 in book)**.** *Let $a = [\frac{n}{2}] + 1$, and let s be such that $a = \frac{n}{s} + \frac{1}{2}$ (so that $s \leq 2$). If $k \in L^2$ is such that $\hat{k}$ is of class C^a, then,*

$$\int_{|x|>t} |k(x)|\,dx \leq C_n t^{-\frac{1}{2}} \max_{|\alpha|=a} \|D^\alpha \hat{k}\|_s \quad (0 < t < \infty)$$

The details of proof of two lemmas and Hormander-Mihlin multiplier theorem is in book and the subsection 0.1.4.

0.1.2 More precise estimation for more regular multiplier

If we consider Mihlin multiplier theorem, the stronger condition (2) gives the smoothness of multipliers. And we have more precise estimation like Theorem 5.20 (i) in book.

Theorem 0.1.2.1 (theorem 6.10 in book)**.** *Let a be an integer such that $\frac{n}{2} < a \leq n$, and suppose that the multiplier $m(\xi)$ satisfies (2) for all $|\alpha| \leq a$. Then, for every $q > \frac{n}{a}$, the operator T_m satisfies*

$$(T_m f)^\#(x) \leq C_q M_q f(x) \quad (f \in \cup_{1<p<\infty} L^p)$$

0.1.3 Some properties of multipliers

Theorem 0.1.3.1. *m is a multiplier for L^p if and only if it is a multiplier for $L^{p'}$. And the norm of operator are identical.*

Proof. I want to use:

$$\begin{aligned}
\int T_m f(x)\overline{g(x)}dx &= \int (\int m(y)\hat{f}(y)e^{2\pi i x\cdot y}dy)\overline{g(x)}dx \\
&= \iint m(y)\hat{f}(y)e^{2\pi i x\cdot y}\overline{g(x)}dydx \\
&= \iint m(y)(\int f(z)e^{-2\pi i y\cdot z}dz)e^{2\pi i x\cdot y}\overline{g(x)}dydx \\
&= \iiint m(y)f(z)e^{-2\pi i y\cdot z}e^{2\pi i x\cdot y}\overline{g(x)}dzdydx \\
&= \iint m(y)f(z)e^{-2\pi i y\cdot z}\overline{(\int e^{-2\pi i x\cdot y}g(x)dx)}dzdy \\
&= \iint m(y)f(z)e^{-2\pi i y\cdot z}\overline{\hat{g}(y)}dzdy \\
&= \int f(z)\overline{(\int \overline{m(y)}\hat{g}(y)e^{2\pi i y\cdot z}dy)}dz \\
&= \int f(z)\overline{T_{\bar{m}}g(z)}dz
\end{aligned}$$

By dual of L^p and Hahn-Banach theorem,

$$\begin{aligned}
\|T_m\| &= \sup_{\|f\|_p\le 1}\|T_m f\|_p \\
&= \sup_{\|f\|_p\le 1}\sup_{\|g\|_q\le 1}\int T_m f\bar{g} = \sup_{\|f\|_p\le 1}\sup_{\|g\|_q\le 1}\int f\overline{T_{\bar{m}}g} = \|T_{\bar{m}}\|
\end{aligned}$$

Thus m is a multiplier for L^p if and only if $\bar{m}$ is a multiplier for $L^{p'}$. And the norm of operator are identical. And $\bar{m}$ is a multiplier for $L^{p'}$ implies m is a multiplier for $L^{p'}$. And their norms are equal. ☐

Theorem 0.1.3.2. *If m is a multiplier for L^p, then m is a multiplier for L^q with $p' < q < p$ or $p < q < p'$.*

This theorem is by theorem 0.1.3.1 and Marcinkiewicz' interpolation theorem. If we use Riesz-Thorin interpolation theorem, we can get estimation $\|T_m\|_{p,p} \geq \|m\|_\infty$.

Theorem 0.1.3.3. *The multipliers for L^p form a subalgebra of $L^\infty(\mathbb{R}^n)$ which is invariant under translations, rotations and dilations.*

0.1.4 Details of proof and errata

Note 1 (proof of lemma 6.1)**.** Since the constant C_p depends only on p, n and on the constants $\|\hat{K}\|_\infty$ and B_K of the kernel, and $\|\hat{K}\|_\infty$ and B_K is bounded by hypothesis a) and b), $T_N f = k_N * f$ are uniformly bounded.

Errata 1 (P209)**.** $k_N = k\chi_{\{x: \frac{1}{N} \le |x| \le N\}}$

Note 2 (proof of norm of T_m as an operator from L^2 to L^2 is $\|m\|_\infty$)**.** $T_m \le \|m\|_\infty$ is easy. To prove $T_m \ge \|m\|_\infty$, by Plancherel's theorem, this is equivalent to $\|\hat{Tf}\|_2 \ge \|m\|_\infty \|\hat{f}\|_2$ for some $\hat{f}$. For any $\epsilon > 0$, let $A_\epsilon = \{x \in [0,1] : m(x) > \|m\|_\infty - \epsilon\}$, $\hat{f}(\xi) = \chi_{A_\epsilon}$. $\hat{Tf} = m\hat{f} > (\|m\|_\infty - \epsilon)\hat{f}$ hence $\|\hat{Tf}\|_2 > (\|m\|_\infty - \epsilon)\|\hat{f}\|_2$. Thus $\|\hat{Tf}\|_2 \ge \|m\|_\infty \|\hat{f}\|_2$.

Note 3 (proof of lemma 6.6 in book)**.** The proof of lemma 6.6 is bad formatted. We give the complete proof of lemma 6.6.

First we have $|x|^a \le (\sqrt{n \max_i x_i{}^2})^a \le n^{\frac{a}{2}} max_i |x_i|^a \le n^{\frac{a}{2}} (\sum_i |x_i|^a)$.

By $|x|^a \le n^{\frac{a}{2}} (\sum_i |x_i|^a)$, Holder inequality and Minkowski inequality for $s' \ge 2$:

$$
\begin{aligned}
\int_{|x|>t} |k(x)|\,dx &= \int_{|x|>t} \frac{\sum_{j=1}^n (|x_j|^a)\,|k(x)|}{\sum_{j=1}^n (|x_j|^a)} dx \\
&\le \int_{|x|>t} \frac{C\sum_{j=1}^n (|x_j|^a)\,|k(x)|}{|x|^a} dx \\
&\le C\Big(\int (\sum_{j=1}^n (\big|x_j^a k(x)\big|)^{s'})dx\Big)^{\frac{1}{s'}} \Big(\int_{|x|>t} \frac{1}{|x|^{as}} dx\Big)^{\frac{1}{s}} \\
&\le C \sum_{j=1}^n \Big(\int \big|x_j^a k(x)\big|^{s'} dx\Big)^{\frac{1}{s'}} \Big(\int_{|x|>t} \frac{1}{|x|^{as}} dx\Big)^{\frac{1}{s}}
\end{aligned}
$$

Notice $\frac{2}{3} \le s \le 2$, $n + \frac{1}{3} \le as \le n+1$. Thus $\int_{|x|>t} \frac{1}{|x|^{as}} dx$ converges and $(\int_{|x|>t} \frac{1}{|x|^{as}} dx)^{\frac{1}{s}} = Ct^{-\frac{1}{2}}$.

Let $1 \le p \le 2$ and q is the dual exponent if p, the Hausdorff-Young inequality is:

$$(\sum |a_n|^q)^{\frac{1}{q}} \le (\frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^p \,d\theta)^{\frac{1}{p}}$$

and its dual:

$$(\frac{1}{2\pi} \int_0^{2\pi} |f(\theta)|^q \,d\theta)^{\frac{1}{q}} \le (\sum |a_n|^p)^{\frac{1}{p}}$$

But here it uses the analog for the Fourier transform (Corollary 2.6 in Chapter 2 in stein's *Functional Analysis*):

Theorem 0.1.4.1. *If $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the Fourier transform T has a unique extension to a bounded map from L^p to L^q, with $\|T(f)\|_q \le \|f\|_p$*

By Theorem 0.1.4.1 and property of Fourier transform.

$$
\begin{aligned}
(\int \left|(x_j^a k(x))\right|^{s'} dx)^{\frac{1}{s'}} &\le (\int \left|(x_j^a k(x))^{\wedge}\right|^{s} dx)^{\frac{1}{s}} \\
&= (\int \left|(-2\pi i)^{-a} D_j^a \hat{k}(\xi)\right|^{s} dx)^{\frac{1}{s}} = C\|D_j^a \hat{k}\|_s
\end{aligned}
$$

Thus

$$\int_{|x|>t} |k(x)|\, dx \le C t^{-\frac{1}{2}} \sum_{j=1}^{n} \|D_j^a \hat{k}\|_s$$

Observe that $\sum_{j=1}^{n} \|D_j^a \hat{k}\|_s \le n \max_j \|D_j^a \hat{k}\|_s \le n \max_{|\alpha|=a} \|D^\alpha \hat{k}\|_s$. We finally prove that:

$$\int_{|x|>t} |k(x)|\, dx \le C t^{-\frac{1}{2}} \max_{|\alpha|=a} \|D^\alpha \hat{k}\|_s$$

Errata 2 (P212)**.** By Leibnitz rule

$$D^\alpha m_j(\xi) = \sum_{\alpha=\beta+\gamma} \binom{|\alpha|}{|\beta|} D^\beta m(\xi) 2^{-j|\gamma|} D^\gamma \phi(2^{-j}\xi)$$

with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$

Errata 3 (P212)**.** $m = \sum_j m_j$ and at most two m_j can be non zero at any point. We have $\sum_j \left|\hat{k_j}(\xi)\right| = \sum_j |m_j(\xi)| \le 2\|m\|_\infty$

Errata 4 (P214)**.** $|(2\pi y)^\gamma| \le C\,|y|^{|\gamma|}$

Note 4 (proof of theorem 6.3 in book)**.** By Leibnitz rule we have

$$D^\alpha m_j(\xi) = \sum_{\alpha=\beta+\gamma} \binom{|\alpha|}{|\beta|} D^\beta m(\xi) 2^{-j|\gamma|} D^\gamma \phi(2^{-j}\xi)$$

Since for each γ, $\left|D^\gamma \phi(2^{-j}\xi)\right|$ is uniformly bounded and there are finite choice of γ, there is a constant C with $\left|D^\gamma \phi(2^{-j}\xi)\right| \le C$ for all ξ and γ. Notice $\binom{|\alpha|}{|\beta|}$ is also bounded. Thus

$$|D^\alpha m_j(\xi)| \le C' \sum_{|\beta|\le|\alpha|} \left|D^\beta m(\xi) 2^{-j|\gamma|}\right| = C' 2^{-j|\alpha|} \sum_{|\beta|\le|\alpha|} \left|D^\beta m(\xi) 2^{j|\beta|}\right|$$

Now we estimate the norm of $D^\alpha m_j(\xi)$. Notice $\text{supp}(m_j) \subset \{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. Thus the support of $D^\alpha m_j(\xi)$ is also in $\{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. Thus only need to integrate each $D^\beta m(\xi)$ on $\{\xi : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$. By separate the region of integral:

$$\begin{aligned}&(\int_{2^{j-1}\leq|\xi|\leq 2^{j+1}} \left|D^\beta m(\xi)\right|^s d\xi)^{\frac{1}{s}} \\ \leq &(\int_{2^{j-1}\leq|\xi|\leq 2^{j}} \left|D^\beta m(\xi)\right|^s d\xi)^{\frac{1}{s}} + (\int_{2^{j}\leq|\xi|\leq 2^{j+1}} \left|D^\beta m(\xi)\right|^s d\xi)^{\frac{1}{s}}\end{aligned}$$

By Holder inequality and hypothesis (1), the first integral on right hand side is:

$$\begin{aligned}&(\int_{2^{j-1}\leq|\xi|\leq 2^{j}} \left|D^\beta m(\xi)\right|^s d\xi)^{\frac{1}{s}} \\ \leq &(\int_{2^{j-1}\leq|\xi|\leq 2^{j}} \left|D^\beta m(\xi)\right|^2 d\xi)^{\frac{1}{2}}(\int_{2^{j-1}\leq|\xi|\leq 2^{j}} 1 d\xi)^{\frac{1}{s}-\frac{1}{2}} \\ \leq &C_1(2^{j-1})^{-|\beta|}(2^{j-1})^{\frac{n}{2}}C_2(2^{jn}-2^{(j-1)n})^{\frac{1}{s}-\frac{1}{2}} \\ = &C_1C_2(2^{j-1})^{\frac{n}{2}-|\beta|}(2^{jn}(1-2^{-n}))^{\frac{1}{s}-\frac{1}{2}} \\ = &C(1-2^{-n})^{\frac{1}{s}-\frac{1}{2}}(2^{-1})^{\frac{n}{2}-|\beta|}(2^j)^{\frac{n}{2}-|\beta|}(2^{jn})^{\frac{1}{s}-\frac{1}{2}} \\ \leq &C'(2^j)^{\frac{n}{2}-|\beta|}(2^{jn})^{\frac{1}{s}-\frac{1}{2}}\end{aligned}$$

Using the same argument for the second integral on right hand side:

$$\begin{aligned}(\int_{2^{j}\leq|\xi|\leq 2^{j+1}} \left|D^\beta m(\xi)\right|^s d\xi)^{\frac{1}{s}} &\leq C_1(2^j)^{-|\beta|}(2^j)^{\frac{n}{2}}C_2(2^{(j+1)n}-2^{jn})^{\frac{1}{s}-\frac{1}{2}} \\ &= C(2^n-1)^{\frac{1}{s}-\frac{1}{2}}(2^j)^{\frac{n}{2}-|\beta|}(2^{jn})^{\frac{1}{s}-\frac{1}{2}} \\ &\leq C'(2^j)^{\frac{n}{2}-|\beta|}(2^{jn})^{\frac{1}{s}-\frac{1}{2}}\end{aligned}$$

Thus

$$(\int_{2^{j-1}\leq|\xi|\leq 2^{j+1}} \left|D^\beta m(\xi)\right|^s d\xi)^{\frac{1}{s}} \leq C'(2^j)^{\frac{n}{2}-|\beta|}(2^{jn})^{\frac{1}{s}-\frac{1}{2}} = C'(2^j)^{-|\beta|+\frac{n}{s}}$$

Note 5 (proof of theorem 6.3 in book)**.** By Plancherel's theorem:

$$\|T^N f - T_m f\|_2 = \|(m-m^N)\hat{f}\|_2 = \|(m-m^N)\|_2\|\hat{f}\|_2$$

Thus $T^N f \to T_m f$ in L^2 for every $f \in L^2$.

Generally, L^p convergence does not implies pointwise convergence. But there is a subsequence converges pointwise. Thus we have:

$$\liminf_{N\to\infty}\left|T^N f(x) - T_m f(x)\right| = 0 \quad a.e.$$

By triangular inequality:

$$\begin{aligned}(\int |T_m f(x)|^p)^{\frac{1}{p}} \leq &(\int \left|T^N f(x) - T_m f(x)\right|^p)^{\frac{1}{p}} + \int \left|T^N f(x)\right|^p \\ \leq &(\int \left|T^N f(x) - T_m f(x)\right|^p)^{\frac{1}{p}} + C_p\|f\|_p\end{aligned}$$

Take lim inf on both side and use Fatou's Lemma:

$$\begin{aligned}
(\int |T_m f(x)|^p)^{\frac{1}{p}} \leq & \liminf (\int \left|T^N f(x) - T_m f(x)\right|^p)^{\frac{1}{p}} + C_p \|f\|_p \\
\leq & (\int \liminf \left|T^N f(x) - T_m f(x)\right|^p)^{\frac{1}{p}} + C_p \|f\|_p \\
= & C_p \|f\|_p
\end{aligned}$$

Note 6 (proof of theorem 6.3 in book)**.** By Leibnitz rule:

$$\begin{aligned}
|D^\alpha \tilde{m_j}(\xi)| &= \left| \sum_{\alpha=\beta+\gamma} \binom{|\alpha|}{|\beta|} D^\beta m_j(\xi) D^\gamma (e^{-2\pi i y\cdot\xi} - 1) \right| \\
&\leq C \sum_{\alpha=\beta+\gamma} \left|D^\beta m_j(\xi)\right| \left|D^\gamma (e^{-2\pi i y\cdot\xi} - 1)\right| \\
&\leq C' \sum_{\alpha=\beta+\gamma} |y|\, 2^{j(1-|\gamma|)} \left|D^\beta m_j(\xi)\right| \\
&\leq C' \sum_{\alpha=\beta+\gamma} |y|\, 2^{j(1-|\gamma|)} C'' 2^{-j|\beta|} \sum_{|\beta'|\leq|\beta|} \left|D^{\beta'} m(\xi) 2^{j\left|\beta'\right|}\right| \\
&\leq C \sum_{\alpha=\beta+\gamma} |y|\, 2^{j(1-|\alpha|)} \sum_{|\beta'|\leq|\beta|} \left|D^{\beta'} m(\xi) 2^{j\left|\beta'\right|}\right| \\
&\leq C \sum_{|\beta|\leq a} |y|\, 2^{j(1-a)} \sum_{|\beta'|\leq|\beta|} \left|D^{\beta'} m(\xi) 2^{j\left|\beta'\right|}\right| \\
&\leq C \sum_{|\beta|\leq a} |y|\, 2^{j(1-a)} \sum_{|\beta'|\leq a} \left|D^{\beta'} m(\xi) 2^{j\left|\beta'\right|}\right| \\
&= Ca\, |y|\, 2^{j(1-a)} \sum_{|\beta'|\leq a} \left|D^{\beta'} m(\xi) 2^{j\left|\beta'\right|}\right|
\end{aligned}$$

Thus

$$\begin{aligned}
\sup_{|\alpha|=a} \|D^\alpha \tilde{m_j}\|_s &\leq C\, |y|\, 2^{j(1-a)} \sum_{|\beta'|\leq a} 2^{j\left|\beta'\right|} \|D^{\beta'} m(\xi)\|_s \\
&\leq C\, |y|\, 2^{j(1-a)} \sum_{|\beta'|\leq a} 2^{j\left|\beta'\right|} C 2^{j(-\left|\beta'\right| + \frac{n}{s})} \\
&\leq C'\, |y|\, 2^{j(1-a)} \sum_{|\beta'|\leq a} 2^{j\frac{n}{s}} \\
&\leq C' a\, |y|\, 2^{j(1-a+\frac{n}{s})}
\end{aligned}$$

Note 7 (proof of theorem 6.10 in book)**.** We denote $t_j = 2^j |y|$

$$
\begin{aligned}
&\int_{|x|>2|y|} \left|k^N(x-y) - k^N(x)\right| |f(x)|\, dx \\
=&\sum_{j=1}^{\infty} \int_{t_j<|x|\le 2t_j} \left|k^N(x-y) - k^N(x)\right| |f(x)|\, dx \\
\le&\sum_{j=1}^{\infty} (\int_{t_j<|x|\le 2t_j} \left|k^N(x-y) - k^N(x)\right|^{q'} dx)^{\frac{1}{q'}} (\int_{t_j<|x|\le 2t_j} |f(x)|^q\, dx)^{\frac{1}{q}} \\
\le&\sum_{j=1}^{\infty} (\int_{t_j<|x|} \left|k^N(x-y) - k^N(x)\right|^{q'} dx)^{\frac{1}{q'}} (\int_{t_j<|x|\le 2t_j} |f(x)|^q\, dx)^{\frac{1}{q}} \\
\le&(\sum_{j=1}^{\infty} C t_j^{-\epsilon-\frac{n}{q}} |y|^{\epsilon} |B(0,2t_j)|^{\frac{1}{q}}) \sup_j (\frac{1}{|B(0,2t_j)|} \int_{|x|\le 2t_j} |f(x)|^q\, dx)^{\frac{1}{q}} \\
\le&C(\sum_{j=1}^{\infty} t_j^{-\epsilon-\frac{n}{q}} |y|^{\epsilon} t_j^{\frac{n}{q}}) M_q f(0) \\
=&C|y|^{\epsilon} (\sum_{j=1}^{\infty} t_j^{-\epsilon}) M_q f(0) \\
=&C|y|^{\epsilon} |y|^{-\epsilon} (\sum_{j=1}^{\infty} 2^{-j\epsilon}) M_q f(0) \\
=&C' M_q f(0)
\end{aligned}
$$

Since we prove that $(T_m f)^{\#}(0) \le A_n \sup_{\epsilon>0} \epsilon^{-n} \int_{|x|\le\frac{\epsilon}{2}} |f(y) - I_\epsilon|\, dy$ and using lemma 5.11 in book:

$$
\begin{aligned}
&\epsilon^{-n} \int_{|x|<\frac{\epsilon}{2}} |Tf(x) - I_\epsilon|\, dx \\
\le&C_q M_q f(0) + \epsilon^{-n} \iint_{2|x|<\epsilon<|y|} |K(x-y) - K(-y)|\, |f(y)|\, dxdy \\
\le&C_q M_q f(0) + \epsilon^{-n} \int_{2|x|<\epsilon} (\int_{2|x|<|y|} |K(y-x) - K(y)|\, |f(y)|\, dy) dx \\
\le&C_q M_q f(0) + \epsilon^{-n} \int_{2|x|<\epsilon} (C' M_q f(0)) dx \\
\le&C_q M_q f(0) + C'' M_q f(0) \\
\le&C M_q f(0)
\end{aligned}
$$

Thus $(T^N f)^{\#}(0) \le C M_q f(0)$.

Note 8 (proof of (6.11) in book)**.** First we prove a variant of Lemma 6.6:

$$(\int_{|x|>t} |k(x)|^{q'}\, dx)^{\frac{1}{q'}} \le Ct^{\epsilon-\frac{n}{q}} \max_{|\alpha|=a} \|D^\alpha \hat{k}\|_s \quad (0<t<\infty) \tag{3}$$

The proof is entirely similar with lemma 6.6, but Holder's inequality is applied in the form: $\|\cdot\|_{q'} \le \|\cdot\|_r \|\cdot\|_{s'}$, with $\frac{1}{r}=\frac{1}{s}-\frac{1}{q}$. Notice $-ar+n-1<-1$ by $a-\frac{n}{s}=\epsilon>-\frac{1}{q}$, the second integral on right hand side converges.

$$\begin{aligned}(\int_{|x|>t} |k(x)|^{q'}\, dx)^{\frac{1}{q'}} &\le C(\int(\sum_{j=1}^{n}(\left|x_j^a k(x)\right|)^{s'})dx)^{\frac{1}{s'}}(\int_{|x|>t}\frac{1}{|x|^{ar}}dx)^{\frac{1}{r}} \\ &\le C\sum_{j=1}^{n}(\int\left|x_j^a k(x)\right|^{s'}dx)^{\frac{1}{s'}}(t^{n-ar})^{\frac{1}{r}} \\ &\le C\max_{|\alpha|=a}\|D^\alpha\hat{k}\|_s(t^{\frac{n}{r}-a})\end{aligned}$$

By $\frac{n}{r}-a=\frac{n}{s}-\frac{n}{q}-a=-\epsilon-\frac{n}{q}$, we have proof inequality (3).

Now we prove:

$$(\int_{|x|>2t}|k_j(x-y)-k_j(x)|^{q'}dx)^{\frac{1}{q'}} \le \begin{cases} Ct^{-\epsilon-\frac{n}{q}}2^{-j\epsilon} & (2^j|y|\ge 1) \\ Ct^{-\epsilon-\frac{n}{q}}|y|\,2^{j(1-\epsilon)} & (2^j|y|<1)\end{cases}$$

The first inequality is by inequality (3) and inequality (6.7) in book:

$$\|D^\alpha m_j\|_s \le C2^{j(-|\alpha|+\frac{n}{s})} \quad (|\alpha|\le a; 1\le s\le 2)$$

we have

$$(\int_{|x|>t}|k_j(x)|^{q'}\,dx)^{\frac{1}{q'}} \le Ct^{-\epsilon-\frac{n}{q}}2^{-j\epsilon} \tag{4}$$

The proof of the second inequality is similar with (6.9) in book. By the same argument as above, we have

$$(\int_{|x|>t}|\tilde{k_j}(x)|^{q'}dx)^{\frac{1}{q'}} \le Ct^{-\epsilon-\frac{n}{q}}\max_{|\alpha|=a}\|D^\alpha\hat{\tilde{k_j}}\|_s$$

By $\hat{\tilde{k}}=\tilde{m}$ and inequality (6.9) in book

$$\sup_{|\alpha|=a}\|D^\alpha\tilde{m_j}\|_s \le C|y|\,2^{j(1-a+\frac{n}{s})}=C|y|\,2^{j(1-\epsilon)} \quad (1\le s\le 2; 2^j|y|<1)$$

, we have:

$$(\int_{|x|>t}|\tilde{k_j}(x)|^{q'}dx)^{\frac{1}{q'}} \le Ct^{-\epsilon-\frac{n}{q}}|y|\,2^{j(1-\epsilon)} \quad (1\le s\le 2; 2^j|y|<1) \tag{5}$$

Now by inequality (4) and inequality (5) we have:

$$(\int_{|x|>2t} |k_j(x-y) - k_j(x)|^{q'} dx)^{\frac{1}{q'}} \leq \begin{cases} Ct^{-\epsilon-\frac{n}{q}} 2^{-j\epsilon} & (2^j\,|y| \geq 1) \\ Ct^{-\epsilon-\frac{n}{q}}\,|y|\,2^{j(1-\epsilon)} & (2^j\,|y| < 1) \end{cases}$$

Finally, by summing with different part of index, (6.11) is an easy consequence of above inequality.

Note 9 (proof of (6.14) in book)**.** By definition of subalgebra, we only need to check that $T_{am_1+bm_2}$, $T_{m_1m_2}$ are bounded. These are easy since T_m is linear on m and $T_{m_1m_2} = T_{m_1}T_{m_2}$. Suppose the affine transformations $A(x) = h + L(x)$ where L is linear transformation. I don't know why the following holds:

$$(f \circ A)^\wedge(\xi) = e^{2\pi i h\cdot\tilde{L}(\xi)}\,|\det L|^{-1}\,\hat{f}(\tilde{L}(\xi))$$

with $\tilde{L} = (L^*)^{-1}$ and how this implies multipliers are invariant under transformation A.