Chapter 1

The Littlewood-Paley Theory and Multipliers

Note 1. In this book notation, the Fourier transform of f is $\int f(x)e^{2\pi it\cdot x}dx$ instead of $\int f(x)e^{-2\pi it\cdot x}dx$. And the inverse Fourier transform in this book is $\int \hat{f}(x)e^{-2\pi it\cdot x}dx$. But we will still use the common notation.

There are three main topic originally in the Littlewood-Paley theory:

1. The auxiliary g-function
2. Partial sum operators and the dyadic decomposition
3. Marcinkiewicz multiplier theorem

All these three topics are related to multipliers. The auxiliary g-function helps us prove Mihlin multiplier theorem (section 3) and further some multipliers in M_p for specific p. Partial sum operator (section 4) shows some indicate functions of convex sets are multipliers. Also with help of partial sum operator, we can prove Marcinkiewicz multiplier theorem (section 6). This theorem and Mihlin multiplier theorem overlap and neither includes the other.

1.1 The first tool: The Littlewood-Paley g-function

For a function $f \in L^p(\mathbb{R}^n)$, we define its Poisson integral is:

$$u(x, y) = \int_{\mathbb{R}^n} P_y(t)f(x - t)dt$$

where $P_y(t) = \int_{\mathbb{R}^n} e^{-2\pi|x|y}e^{-2\pi it\cdot x}dx = \frac{c_n y}{(|t|^2+y^2)^{\frac{n+1}{2}}}$. The g-function of f is:

$$g(f)(x) = \left(\int_0^\infty |\nabla u(x, y)|^2 \, ydy\right)^{\frac{1}{2}}$$

One application of g-function is it can control the norm of f:

Theorem 1.1.1. *Suppose $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then $g(f) \in L^p(\mathbb{R}^n)$ and*

$$A'_p \|f\|_p \leq \|g(f)\|_p \leq A_p \|f\|_p$$

For $p = 2$, we have identity $\|g(f)\|_2 = 2^{-\frac{1}{2}} \|f\|_2$. This identity is important in dyadic decomposition.

The part $\|g(f)\|_p \leq A_p \|f\|_p$ is from vector-valued analogues of singular integral.

Define

$$\mathscr{H}_2^0 = \left\{ f : |f|^2 = \int_0^\infty |f(y)|^2 \, y dy < \infty \right\}$$

and $\mathscr{H}_2$ be the direct sum of n+1 copies of $\mathscr{H}_2^0$.

The kernel of singular integral here is $K_\epsilon(x) = (\frac{\partial P_{y+\epsilon(x)}}{\partial y}, \frac{\partial P_{y+\epsilon(x)}}{\partial x_1}, \ldots, \frac{\partial P_{y+\epsilon(x)}}{\partial x_k})$. Notice $|\nabla u(x, y+\epsilon)|^2 \leq |\nabla u(x,y)|^2$. We have $|T_\epsilon(f)(x)| = \left| \int_{\mathbb{R}^n} K_\epsilon(t) f(x-t) dt \right| \leq g(f)(x)$ ($T_\epsilon(f)(x)$ is in Hilbert space) and $T_\epsilon(f)(x)$ converges to $g(f)(x)$ pointwise.

The converse part $A'_p \|f\|_p \leq \|g(f)\|_p$ is by polarization to the identity, Holder inequality and dual of L^p.

1.2 The function g_λ^*

First section is relied on singular integral. In this section we give the same result based on characteristic properties of harmonic functions. The ideas here are useful when singular integral is not applicable.

Author shows how to avoid singular integral method to prove the following inequality

$$\|g(f)\|_p \leq A_p \|f\|_p \quad (1 < p \leq 2) \tag{1.2.1}$$

The case $2 < p < \infty$ is not shown here.

In the rest of this section, we talk about the positive function g_λ^*

$$(g_\lambda^*(f)(x))^2 = \int_0^\infty \int_{t \in \mathbb{R}^n} (\frac{y}{|t|+y})^{\lambda n} |\nabla u(x-t,y)|^2 \, y^{1-n} dt dy$$

The first important inequality is:

$$g(f)(x) \leq CS(f)(x) \leq C_\lambda g_\lambda^*(f)(x) \tag{1.2.2}$$

where

$$(S(f)(x))^2 = \int_{\Gamma(x)} |\nabla u(t,y)|^2 \, y^{1-n} dy dt = \int_\Gamma |\nabla u(x-t,y)|^2 \, y^{1-n} dy dt$$

where $\Gamma = \left\{ (t,y) \in \mathbb{R}_+^{n+1} : |t| < y, y > 0 \right\}$ and $\Gamma(x)$ is cone Γ with vertex at x.

The second important inequality is:

$$\|g_\lambda^*(f)(x)\|_p \le A_{p,\lambda}\|f\|_p \quad (1 < p < \infty, p > \frac{2}{\lambda}). \tag{1.2.3}$$

The case $p \ge 2$ is by proving $\|g_\lambda^*(f)\|_p \le A_\lambda \|g(f)\|_p$ and inequality (1.2.1). It is interesting that the p-norm of $g(f)$, $g_\lambda^*(f)$ can control each other when $p \ge 2$.

The proof for the case $p < 2$ is like the proof for inequality (1.2.1).

We conclude that the norm of $g_\lambda^*(f)$, $g(f)$, $S(f)$, f can control each other.

1.3 Multipliers (first version)

Most of results in this section can be found in section 6, chapter 2 in Rubio's *Weighted Norm inequalities and Related topics*. And proofs of results are in my note of that book.

Multiplier m is related to a linear transformation T_m defined as:

$$(T_m f)^\wedge(x) = m(x)\hat{f}(x)$$

and we shall say that m is a multiplier for L^p if:

$$\|T_m f\|_p \le \|f\|_p$$

Multipliers has some general properties:

1. M_p is a Banach algebra under pointwise multiplication.
2. M_2 is identical with the bounded measurable function L^∞.
3. M_1 is identical with the finite Borel measures $\mathscr{B}$ (Refer Theorem 2.5.8 in *Classical Fourier Analysis*).
4. Suppose $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p, q \le \infty$, then $M_p = M_q$ with identity of norms.
5. M_∞ is identical with the finite Borel measures $\mathscr{B}$ (By 3 and 4).

Then we prove two important multiplier theorems. The first is Mihlin multiplier theorem:

Theorem 1.3.1. *Suppose that $m(x)$ is of class C^k in the complement of the origin of $\mathbb{R}^n$, where k is an integer greater than $\frac{n}{2}$. Assume also that for every differential monomial $(\frac{\partial}{\partial x})^\alpha$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, we have:*

$$\left|\frac{\partial}{\partial x}^\alpha m(x)\right| \le B\,|x|^{-|\alpha|}, \quad |\alpha| \le k \tag{1.3.1}$$

Then $m \in M_p$, $1 < p < \infty$

And the second is Hörmander Mihlin multiplier theorem:

Theorem 1.3.2. *The assumption* (1.3.1) *can be replaced by the weaker assumptions:*

$$|m(x)| \leq B$$

$$\sup_{0<R<\infty} R^{2|\alpha|+n} \int_{R\leq|x|\leq 2R} \left| \frac{\partial}{\partial x}^{\alpha} m(x) \right|^2 dx \leq B, \quad |\alpha| \leq k$$

The proof of Mihlin multiplier theorem can be done with help of g-functions and their variants:

Lemma 1.3.3. *Under the assumption* (1.3.1)*, let us set for each* $f \in L^2$*:*

$$F(x) = (T_m f)(x)$$

Then

$$g_1(F, x) \leq B_\lambda g_\lambda^*(f, x), \quad \lambda = \frac{2k}{n}$$

By above lemma, we have:

$$\|T_m f\|_p = \|F\|_p \leq A_1 \|g_1(F, x)\|_p \leq A_2 \|g_\lambda^*(f, x)\|_p \leq A_3 \|f\|_p$$

I can not understand the proof of corollary in the end of this section.

1.4 The second tool: partial sums operators

Given a rectangle ρ, we define he partial sum operator $S_\rho(f)$ to be:

$$S_\rho(f)^\wedge = \chi_\rho \hat{f}$$

and for this operator we have:

$$\|S_\rho(f)\|_p \leq A_p \|f\|_p \quad 1 < p < \infty \tag{1.4.1}$$

This bounded linear operator has an extended version on function space $L^p(\mathbb{R}^n, \mathscr{H})$, where $\mathscr{H}$ is the sequence Hilbert space: $\mathscr{H} = \{(c_j)_{j=1}^\infty : (\sum_j |c_j|^2)^{\frac{1}{2}} < \infty\}$.

Given a sequence of rectangle $\mathscr{R} = \{\rho_j\}_{j=1}^\infty$, we define

$$S_\mathscr{R}(f) = (S_{\rho_1}(f_1), \ldots, S_{\rho_j}(f_j), \ldots)$$

where

$$f = (f_1, \ldots, f_j, \ldots)$$

The general version of inequality (1.4.1) is

$$\|S_\mathscr{R}(f)\|_p \leq A_p \|f\|_p \quad 1 < p < \infty \tag{1.4.2}$$

To prove this theorem, we use vector-valued analogues of Hilbert transform. But first we consider the one dimensional and single valued case. We have following identity for operator $S_{(-\infty,0)}$:

$$S_{(-\infty,0)} = \frac{I + iH}{2} \tag{1.4.3}$$

For one dimensional and vector-valued case, by property of vector-valued singular integral, we have:

$$\|\tilde{H}f\|_p \leq A_p\|f\|_p$$

where $\tilde{H}f = (Hf_1, \ldots, Hf_j, \ldots)$. Combining with equation (1.4.3), we have inequality (1.4.2) holds when f is vector-valued with single variable and $\mathscr{R}$ is collection of rectangles $(-\infty, 0)$.

By translation, we can show:

$$S_{(-\infty,a_j)}f_j(x) = \frac{f_j + ie^{2\pi ix\cdot a_j}H(e^{-2\pi ix\cdot a_j}f_j)}{2}$$

Thus inequality (1.4.2) holds when f is vector-valued with single variable and $\mathscr{R}$ is collection of rectangles $(-\infty, a_j)$.

To prove n dimensional case, we claim:

Lemma 1.4.1 (Linear span is dense in section 4.2.3)**.** *The linear span of form $f'(x_1)f''(x_2, \ldots, x_n)$ is dense in $L^2(\mathbb{R}^n)$.*

This lemma is true since we can choose f' and f'' as orthogonal basis of $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^{n-1})$. By this lemma, we can consider first variable x_1 separately. Thus inequality (1.4.2) holds when f is vector-valued with n variables and $\mathscr{R}$ is collection of half space $\{x : x_1 < a_j\}_{j=1}^{\infty}$.

Finally since finite rectangle is the intersection of 2n half-spaces, a 2n-fold of previous result proves the inequality (1.4.2) where $\mathscr{R}$ contains finite rectangles. The result is not depended on number of rectangles. Thus using limit argument the inequality (1.4.2) holds for infinite number of rectangles.

The author describes two problems:

1. Let B be the unit ball in $\mathbb{R}^n$. Can we replace the rectangle ρ by the ball B in inequality (1.4.1)?.

2. Can the rectangles of inequality (1.4.2) be replaced by rectangles that are each arbitrarily rotated?

We know the the answer of first problems can be affirmative only in the range $\frac{2n}{n+1} < p < \frac{2n}{n-1}$. The solution of first problem implies the resolution of the second problem for the same p. And the answer of the second problem is in the negative for p outside the interval $\frac{2n}{n+1} \leq p \leq \frac{2n}{n-1}$.

Also there is a continuous analogue of inequality (1.4.2). Let $(\Gamma, d\gamma)$ be an abstract measure space. We can replace the sequence Hilbert space by square integrable functions space $L^2(\Gamma, d\gamma)$. And $\rho_\gamma = \rho(\gamma)$ is a measurable function from Γ to rectangles in $\mathbb{R}^n$.

1.5 The dyadic decomposition

First we introduce the dyadic decomposition of $\mathbb{R}$. We decompose $\mathbb{R} \setminus \{0\}$ as $(\cup_{k=\infty}^{\infty}[2^k, 2^{k+1}]) \cup (\cup_{k=\infty}^{\infty}[-2^{k+1}, -2^k])$. Then we consider the product of intervals in decomposition of $\mathbb{R}$. This is the dyadic decomposition of $\mathbb{R}^n$ and we denote as Δ.

Recall the partial sum operator, we have:

$$\sum_{\rho\in\Delta} S_\rho = I$$

and since the rectangles in decomposition are disjoint, by Plancherel's theorem we have:

$$\sum_{\rho\in\Delta} \|S_\rho f\|_2^2 = \|f\|_2^2 \tag{1.5.1}$$

The special property of partial sum operator on dyadic decomposition is we can control the norm of f by this partial sum operator:

Theorem 1.5.1. *Suppose $f \in L^p$, $1 < p < \infty$. Then $\left(\sum_{\rho\in\Delta} |S_\rho f(x)|^2\right)^{\frac{1}{2}} \in L^p$ and*

$$A_p\|f\|_p \leq \|(\sum_{\rho\in\Delta} |S_\rho f(x)|^2)^{\frac{1}{2}}\|_p \leq B_p\|f\|_p$$

To prove above theorem we need Rademacher functions $\{r_i(t)\}_{i=1}$ on integral $(0,1)$. $r_0(t)$ is defined to be:

$$r_0(t) = \begin{cases} 1 & 0 < t \leq \dfrac{1}{2} \\ -1 & \dfrac{1}{2} \leq t < 1 \end{cases}$$

and $r_m(t)$ is defined to be $r_0(2^m t)$ and they are extended outside by periodicity. The important of Rademacher functions is all p-norms of linear combination of Rademacher functions are comparable:

Theorem 1.5.2. *Suppose $\sum |a_m|^2 < \infty$ and set $F(t) = \sum a_m r_m(t)$. Then $F(t) \in L^p$ for all $p < \infty$ and*

$$A_p\|F\|_p \leq \|F\|_2 = \left(\sum_{m=0}^{\infty} |a_m|^2\right)^{\frac{1}{2}} \leq B_p\|F\|_p$$

By equation (1.5.1), we only need to proof one inequality in theorem 1.5.1:

$$\|(\sum_{\rho\in\Delta} |S_\rho f(x)|^2)^{\frac{1}{2}}\|_p \leq B_p\|f\|_p$$

The other inequality is by polarization and dual of L^p.

1.6 The Marcinkiewicz multiplier theorem

The Marcinkiewicz multiplier theorem is one of the most important results of the Littlewood-Paley theory

Theorem 1.6.1. *Let m be a bounded function on $\mathbb{R}^n$ of the type described. Suppose also*

1. $|m| \leq B$
2. *for each* $0 < k \leq n$

$$\sup_{x_{k+1},\ldots,x_n} \int_\rho \left| \frac{\partial^k m}{\partial x_1 \ldots \partial x_k} \right| dx_1 \ldots dx_k \leq B$$

3. *The condition analogous to 2. is valid for every on of the $n!$ permutations of the variables $x_1, \ldots, x_n$.*

Then $m \in M_p$, $1 < p < \infty$; and more precisely, if $f \in L^2 \cap L^p$, $\|T_m f\|_p \leq A_p \|f\|_p$ where A_p depends only on B, p and n.

The difference between Hörmander multiplier theorem and Marcinkiewicz multiplier theorem can be illustrated by invariance considerations. The class of multipliers in Hömander multiplier theorem is invariant under dilations, $m(x) \mapsto m(\epsilon x)$ and rotations $m(x) \mapsto m(\rho^{-1}x)$. But the class of multipliers in Marcinkiewicz multiplier theorem is invariant under a larger group of dilations, $m(x) \mapsto m(\epsilon \circ x)$, where $\epsilon \circ x = (\epsilon_1 x_1, \ldots, \epsilon_n x_n)$. But it not invariant under rotations.

Let Δ denote the dyadic decomposition and $f \in L^2 \cap L^p$, and write $F = T_m f$. To prove this theorem, we only need to show:

$$\|(\sum_{\rho\in\Delta} |S_\rho F|^2)^{\frac{1}{2}}\|_p \leq C_p \|(\sum_{\rho\in\Delta} |S_\rho f|^2)^{\frac{1}{2}}\|_p$$

Using Theorem 1.5.1 we conclude $m \in M_p$.

1.7 Details of proof and errata, section 1

Note 2 (section 1.2)**.** We have identity:

$$u(x,y) = \int_{\mathbb{R}^n} \hat{f}(t) e^{2\pi i t\cdot x} e^{-2\pi|t|y} dt$$

Then:

$$\frac{\partial u}{\partial y} = \int_{\mathbb{R}^n} -2\pi\,|t|\,\hat{f}(t) e^{2\pi i t\cdot x} e^{-2\pi|t|y} dt$$

and

$$\frac{\partial u}{\partial x_j} = \int_{\mathbb{R}^n} 2\pi i t_j \hat{f}(t) e^{2\pi i t\cdot x} e^{-2\pi|t|y} dt$$

Notice $(\frac{\partial u}{\partial y})^\wedge = -2\pi |t| \hat{f}(t)e^{-2\pi|t|y}$ and $(\frac{\partial u}{\partial x_j})^\wedge = 2\pi i t_j \hat{f}(t)e^{-2\pi|t|y}$. Thus by Plancherel's theorem:

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla u(x,y)|^2\, dx &= \int_{\mathbb{R}^n} \left|\frac{\partial u}{\partial y}\right|^2 dx + \sum_{j=1}^{n}\int_{\mathbb{R}^n}\left|\frac{\partial u}{\partial x_j}\right|^2 dx \\ &= \int_{\mathbb{R}^n} \left|(\frac{\partial u}{\partial y})^\wedge\right|^2 dt + \sum_{j=1}^{n}\int_{\mathbb{R}^n}\left|(\frac{\partial u}{\partial x_j})^\wedge\right|^2 dt \\ &= \int_{\mathbb{R}^n} 4\pi^2(|t|^2 + \sum_{j=1}^{n}|t_j|^2)|\hat{f(t)}|^2 e^{-4\pi|t|y} dt \\ &= \int_{\mathbb{R}^n} 8\pi^2 |t|^2 |\hat{f(t)}|^2 e^{-4\pi|t|y} dt\end{aligned}$$

Errata 1 (P83)**.** Missing dy in bracket:

$$\int_{\mathbb{R}^n} |\hat{f}(t)|^2 \left\{8\pi^2 |t|^2 \int_0^\infty e^{-4\pi|t|y} y dy\right\} dt$$

Note 3 (section 1.3)**.** We have following estimations:

$$\frac{\partial P_y}{\partial y} = \frac{c_n}{(|x|^2+y^2)^{\frac{n+1}{2}}} + \frac{2c_n y^2}{(|x|^2+y^2)^{\frac{n+3}{2}}} \le \frac{c_n}{(|x|^2+y^2)^{\frac{n+1}{2}}} + \frac{2c_n(|x|^2+y^2)}{(|x|^2+y^2)^{\frac{n+3}{2}}} \le \frac{A}{(|x|^2+y^2)^{\frac{n+1}{2}}}$$

$$\frac{\partial^2 P_y}{\partial y \partial x_j} = \frac{2c_n |x_j|}{(|x|^2+y^2)^{\frac{n+3}{2}}} + \frac{4c_n y^2 |x_j|}{(|x|^2+y^2)^{\frac{n+5}{2}}} \le \frac{A|x_j|}{(|x|^2+y^2)^{\frac{n+3}{2}}}$$

$$\frac{\partial P_y}{\partial x_i} = \frac{2c_n y |x_i|}{(|x|^2+y^2)^{\frac{n+3}{2}}} \le \frac{c_n(y^2+|x_i|^2)}{(|x|^2+y^2)^{\frac{n+3}{2}}} \le \frac{A}{(|x|^2+y^2)^{\frac{n+1}{2}}}$$

$$\frac{\partial^2 P_y}{\partial x_i \partial x_j} = \frac{4c_n y |x_i||x_j|}{(|x|^2+y^2)^{\frac{n+5}{2}}} \le \frac{A|x_j|}{(|x|^2+y^2)^{\frac{n+3}{2}}} \quad (i \ne j)$$

$$\frac{\partial^2 P_y}{\partial x_i^2} = \frac{\pm 2c_n y}{(|x|^2+y^2)^{\frac{n+3}{2}}} + \frac{4c_n y |x_i||x_j|^2}{(|x|^2+y^2)^{\frac{n+5}{2}}} \le \frac{A(|x_j|+y)}{(|x|^2+y^2)^{\frac{n+3}{2}}}$$

Thus

$$\left|\frac{\partial K_\epsilon(x)}{\partial x_j}\right|^2 \le \frac{A|x_j|^2}{(|x|^2+y^2)^{n+3}} + \frac{(n-1)A|x_j|^2}{(|x|^2+y^2)^{n+3}} + \frac{A(|x_j|+y)^2}{(|x|^2+y^2)^{n+3}} \le \frac{A}{(|x|^2+y^2)^{n+2}}$$

Note 4 (section 1.4)**.**

Theorem 1.7.1 (Polarization to the identity)**.**

$$< x, y >= \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2$$

Notice $g_1(f)$ is not an linear map. Denote $u(f)(x) = \int_{\mathbb{R}^n} P_y(t)f(x-t)dt$. Using the fact that $\mathscr{H}_2^0$ is a Hilbert space:

$$\int_{\mathbb{R}^n} f_1\bar{f}_2 = < f_1, f_2 >= \frac{1}{4}\sum_{k=0}^{3} i^k \|f_1 + i^k f_2\|_2^2 = \sum_{k=0}^{3} i^k \|g_1(f_1 + i^k f_2)\|_2^2$$
$$= \int_{\mathbb{R}^n} (\sum_{k=0}^{3} \int_0^\infty i^k \left|\frac{\partial u(f_1 + i^k f_2)}{\partial y}\right|^2 ydy)dx$$
$$= \int_{\mathbb{R}^n} (\sum_{k=0}^{3} i^k \int_0^\infty \left|\frac{\partial u(f_1)}{\partial y} + i^k \frac{\partial u(f_2)}{\partial y}\right|^2 ydy)dx$$
$$=4\int_{\mathbb{R}^n}\int_0^\infty \frac{\partial u(f_1)}{\partial y}\overline{\frac{\partial u(f_2)}{\partial y}} ydydx$$

Using Holder inequality:

$$\left|\int_{\mathbb{R}^n} f_1\bar{f}_2\right| \leq 4\int_{\mathbb{R}^n}\int_0^\infty \left|\frac{\partial u(f_1)}{\partial y}\overline{\frac{\partial u(f_2)}{\partial y}}\right| ydydx$$
$$\leq 4\int_{\mathbb{R}^n}\left(\int_0^\infty \left|\frac{\partial u(f_1)}{\partial y}\right|^2 ydy\right)^{\frac{1}{2}}\left(\int_0^\infty \left|\frac{\partial u(f_2)}{\partial y}\right|^2 ydy\right)^{\frac{1}{2}} dx$$
$$\leq 4\int_{\mathbb{R}^n} g_1(f_1)(x)g_1(f_2)(x)dx$$

Errata 2 (P85).

$$\left|\int_{\mathbb{R}^n} f_1\bar{f}_2\right| \leq 4\int_{\mathbb{R}^n} g_1(f_1)(x)g_1(f_2)(x)dx$$

Note 5 (section 1.5)**.**

$$
\begin{aligned}
(g_k(f,x))^2 =& \int_0^\infty \left|\frac{\partial^k u(x,y)}{\partial y^k}\right|^2 y^{2k-1}dy \\
\leq& \int_0^\infty \left(\int_y^\infty \left|\frac{\partial^{k+1} u(x,s)}{\partial s^{k+1}}\right|^2 s^{2k}ds \left(\int_y^\infty s^{-2k}ds\right)\right) y^{2k-1}dy \\
=& \frac{1}{2k-1}\int_0^\infty \left(\int_y^\infty \left|\frac{\partial^{k+1} u(x,s)}{\partial s^{k+1}}\right|^2 s^{2k}ds\ y^{-2k+1}\right) y^{2k-1}dy \\
=& \frac{1}{2k-1}\int_0^\infty \int_y^\infty \left|\frac{\partial^{k+1} u(x,s)}{\partial s^{k+1}}\right|^2 s^{2k}dsdy \\
=& \frac{1}{2k-1}\int_0^\infty \int_0^s \left|\frac{\partial^{k+1} u(x,s)}{\partial s^{k+1}}\right|^2 s^{2k}dyds \\
=& \frac{1}{2k-1}\int_0^\infty \left|\frac{\partial^{k+1} u(x,s)}{\partial s^{k+1}}\right|^2 s^{2k+1}ds \\
=& (g_{k+1}(f,x))^2
\end{aligned}
$$

Errata 3 (P86)**.** Lower index of $\int_y^\infty s^{-2k}ds$ is y instead of 1.

$$
\int_y^\infty \left|\frac{\partial^{k+1} u(x,s)}{\partial s^{k+1}}\right|^2 s^{2k}ds \left(\int_y^\infty s^{-2k}ds\right)
$$

1.8 Details of proof and errata, section 2

Note 6 (Lemma 1 in section 2.1)**.** The Δ is applied on u^p. Since $\frac{\partial^2 u^p}{\partial x_i^2} = p((p-1)u^{p-2}(\frac{\partial u}{\partial x_i})^2 + u^{p-1}\frac{\partial^2 u}{\partial x_i^2})$ and u is harmonic, we have $\Delta(u^p) = p((p-1)u^{p-2}\left|\nabla u\right|^2 + u^{p-1}\Delta u) = p(p-1)u^{p-2}\left|\nabla u\right|^2$.

Errata 4 (P89)**.** In equation (18) is $\int_\Gamma \left|\Delta u(x-t,y)\right|^2 y^{1-n}dydt$

Note 7 ($g(f)(x) \leq CS(f)(x)$ in section 2.3)**.** Recall:

$$
g(f)(x) = \left(\int_0^\infty \left|\nabla u(x,y)\right|^2 ydy\right)^{\frac{1}{2}}
$$

and

$$
(S(f)(x))^2 = \int_{\Gamma(x)} \left|\nabla u(t,y)\right|^2 y^{1-n}dydt = \int_\Gamma \left|\nabla u(x-t,y)\right|^2 y^{1-n}dydt
$$

The proof here using mean-value theorem. Thus we only need to prove the case $x = 0$. The general case is proved by moving the center of ball B_y to x.

From figure 1.1, it is clear that $(x, s) \in B_y$ implies $s \in ((1-\frac{1}{\sqrt{2}})y, (1+\frac{1}{\sqrt{2}})y)$. Thus $y \in ((\frac{\sqrt{2}}{\sqrt{2}+1})s, (\frac{\sqrt{2}}{\sqrt{2}-1})s)$. The inequality at the end of page 90 is by y and s controlling each other.

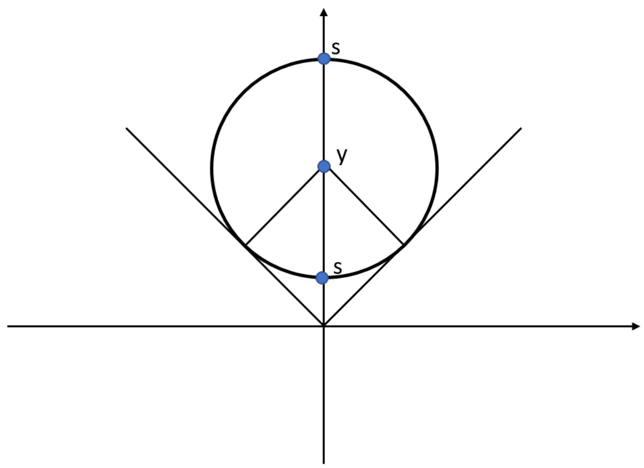

Figure 1.1: Ball B_y and s

Note 8 (Proof of theorem 2 in section 2.4)**.** $\lambda > 1$ guarantees that for $\phi(x) = (1 + |x|)^{-\lambda n}$, $(\phi_\epsilon(x))$ are approximations to the identity.

Note 9 (Proof in section 2.5)**.** Since Poisson kernel is homogeneous of degree -n, the convolution $P * f$ is homogeneous of degree 0.

$Q_t(x) \leq c_n$ for $|x| \leq 2\,|t|$ and $Q_t(x) \leq A'(1+|x|^2)^{\frac{-n-1}{2}}$ for $|x| > 2\,|t|$. Thus:

$$\begin{aligned}
\int_{\mathbb{R}^n} Q_t(x)dx &= \int_{|x|\leq 2|t|} Q_t(x)dx + \int_{|x|\leq 2|t|} Q_t(x)dx \\
&\leq \int_{|x|\leq 2|t|} c_n dx + \int_{|x|\leq 2|t|} A'(1+|x|^2)^{\frac{-n-1}{2}} dx \\
&\leq \int_0^{2|t|} c_n r^{n-1} dr + \int_{2|t|}^{\infty} A'(1+r^2)^{\frac{-n-1}{2}} r^{n-1} dr \\
&\leq \frac{c_n 2^n}{n} |t|^n + \int_0^{\infty} A'(1+r^2)^{\frac{-n-1}{2}} r^{n-1} dr \\
&\leq \frac{c_n 2^n}{n} |t|^n + \int_0^{\frac{\pi}{2}} A'(\sin u)^{n-1} du \\
&\leq \frac{c_n 2^n}{n} |t|^n + C \\
&\leq A(1+|t|)^n
\end{aligned}$$

Estimation for $\int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^{\lambda' n}}$

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^{\lambda' n}} &= C \int_0^{\infty} \frac{r^{n-1}dr}{(1+r)^{\lambda' n}} \\
&= C \int_1^{\infty} \frac{(r-1)^{n-1}dr}{r^{\lambda' n}} \\
&\leq C \int_1^{\infty} \frac{1}{r^{\lambda' n}} C_1 \sum_{k=0}^{n-1} r^k dr \\
&\leq C \sum_{k=0}^{n-1} \int_1^{\infty} r^{k-\lambda' n} dr \\
&\leq C \sum_{k=0}^{n-1} \frac{1}{k-\lambda' n+1} \\
&< \infty
\end{aligned}$$

1.9 Details of proof and errata, section 3

Note 10 (proof in section 3.3.1)**.** Inequality (34) does not need hypothesis (30).

By Leibniz's rule:

$$\begin{aligned}
\left|(\frac{\partial}{\partial x})^\alpha(|x|^k\, m(x))\right| \leq& C \sum_{\alpha=\beta+\gamma}\left|(\frac{\partial}{\partial x})^\beta\, |x|^k\right|\left|(\frac{\partial}{\partial x})^\gamma m(x)\right| \\
\leq& C \sum_{\alpha=\beta+\gamma}(|x|^{k-2|\beta|}\prod_\beta |x_j|^{\beta_j})B\,|x|^{-|\gamma|} \\
\leq& C \sum_{\alpha=\beta+\gamma}(|x|^{k-|\beta|})B\,|x|^{-|\gamma|} \\
\leq& B'\,|x|^{k-|\alpha|}
\end{aligned}$$

To estimate $\left|(\frac{\partial}{\partial x})^\alpha(|x|^k\, m(x)e^{-2\pi|x|y})\right|$, we need following:

$$\left|(\frac{\partial}{\partial x_i})e^{-2\pi|x|y}\right| = 2\pi y e^{-2\pi|x|y}\frac{|x_i|}{|x|}$$

$$\begin{aligned}
\left|(\frac{\partial^2}{\partial x_i x_j})e^{-2\pi|x|y}\right| =& 2\pi y(\frac{\partial}{\partial x_j})(e^{-2\pi|x|y}\frac{|x_i|}{|x|}) \\
\leq& 2\pi y\left((\frac{\partial}{\partial x_j})e^{-2\pi|x|y}\right)\frac{|x_i|}{|x|} + 2\pi y e^{-2\pi|x|y}\left((\frac{\partial}{\partial x_j})\frac{|x_i|}{|x|}\right) \\
\leq& 4\pi^2 y^2 e^{-2\pi|x|y}\frac{|x_i x_j|}{|x|^2} + 2\pi y e^{-2\pi|x|y}\frac{|x_i x_j|}{|x|^3}
\end{aligned}$$

Thus by induction:

$$\begin{aligned}
\left|(\frac{\partial}{\partial x})^\gamma e^{-2\pi|x|y}\right| \leq& e^{-2\pi|x|y}\sum_{k=1}^{|\gamma|}(2\pi y)^k\frac{|x^\gamma|}{|x|^{2|\gamma|-k}} \\
\leq& e^{-2\pi|x|y}\sum_{k=1}^{|\gamma|}(2\pi y)^k\,|x|^{k-|\gamma|} \\
\leq& \frac{e^{-2\pi|x|y}}{|x|^{|\gamma|}}\sum_{k=1}^{|\gamma|}(2\pi y)^k\,|x|^k
\end{aligned}$$

So we have following estimation:

$$\begin{aligned}
&\left|(\frac{\partial}{\partial x})^{\alpha}(|x|^{k} m(x)e^{-2\pi|x|y})\right| \\
\leq &C \sum_{\alpha=\beta+\gamma} \left|(\frac{\partial}{\partial x})^{\beta} |x|^{k} m(x)\right| \left|(\frac{\partial}{\partial x})^{\gamma} e^{-2\pi|x|y}\right| \\
\leq &C \sum_{\alpha=\beta+\gamma} B' |x|^{k-|\beta|} \frac{e^{-2\pi|x|y}}{|x|^{|\gamma|}} \sum_{l=1}^{|\gamma|}(2\pi y)^{l} |x|^{l} \\
\leq &C \sum_{\alpha=\beta+\gamma} B' |x|^{k-|\alpha|} e^{-2\pi|x|y} \sum_{l=1}^{|\gamma|}(2\pi y)^{l} |x|^{l} \\
\leq &C' |x|^{k-|\alpha|} e^{-2\pi|x|y} \sum_{\alpha=\beta+\gamma} \sum_{l=1}^{|\gamma|}(2\pi y)^{l} |x|^{l}
\end{aligned}$$

Let $k = \alpha$ and use inequality in book, we have:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|(\frac{\partial}{\partial x})^{\alpha}(|x|^{k} m(x)e^{-2\pi|x|y})\right|^{2} dx \\
\leq &C' \sum_{r=1}^{m} \int_{\mathbb{R}^n} (2\pi y)^{r} |x|^{r} e^{-4\pi|x|y} dx \\
\leq &C' \sum_{r=1}^{m} C y^{-n}
\end{aligned}$$

where m is a finite value. Thus $\int_{\mathbb{R}^n} \left|(\frac{\partial}{\partial x})^{\alpha}(|x|^{k} m(x)e^{-2\pi|x|y})\right|^{2} dx \leq B' y^{-n}$.

Note 11 (proof in section 3.3.2)**.**

$$
\begin{aligned}
\left|U^{(k+1)(x,y)}\right|^2 &= \left(\int_{\mathbb{R}^n} M^{(k)}(t,\frac{y}{2})u^{(1)}(x-t,\frac{y}{2})dt\right)^2 \\
&\leq \left(\int_{|t|\leq\frac{y}{2}} M^{(k)}(t,\frac{y}{2})u^{(1)}(x-t,\frac{y}{2})dt\right)^2 + \left(\int_{|t|>\frac{y}{2}} M^{(k)}(t,\frac{y}{2})u^{(1)}(x-t,\frac{y}{2})dt\right)^2 \\
&\leq \left(\int_{|t|\leq\frac{y}{2}} \left|M^{(k)}(t,\frac{y}{2})\right|^2 dt\right)\left(\int_{|t|\leq\frac{y}{2}} \left|u^{(1)}(x-t,\frac{y}{2})\right|^2 dt\right) \\
&\quad + \left(\int_{|t|>\frac{y}{2}} |t|^{2k}\left|M^{(k)}(t,\frac{y}{2})\right|^2 dt\right)\left(\int_{|t|>\frac{y}{2}} \frac{\left|u^{(1)}(x-t,\frac{y}{2})\right|^2 dt}{|t|^{2k}}\right) \\
&\leq \left(\int_{|t|\leq\frac{y}{2}} B'y^{-2n-2k}dt\right)\left(\int_{|t|\leq\frac{y}{2}} \left|u^{(1)}(x-t,\frac{y}{2})\right|^2 dt\right) \\
&\quad + B'y^{-n}\left(\int_{|t|>\frac{y}{2}} \frac{\left|u^{(1)}(x-t,\frac{y}{2})\right|^2 dt}{|t|^{2k}}\right) \\
&\leq B'y^{-2n-2k}y^n\int_{|t|\leq\frac{y}{2}} \left|u^{(1)}(x-t,\frac{y}{2})\right|^2 dt + B'y^{-n}\int_{|t|>\frac{y}{2}} \frac{\left|u^{(1)}(x-t,\frac{y}{2})\right|^2 dt}{|t|^{2k}} \\
&\leq B'y^{-n-2k}\int_{|t|\leq\frac{y}{2}} \left|u^{(1)}(x-t,\frac{y}{2})\right|^2 dt + B'y^{-n}\int_{|t|>\frac{y}{2}} \frac{\left|u^{(1)}(x-t,\frac{y}{2})\right|^2 dt}{|t|^{2k}}
\end{aligned}
$$

1.10 Details of proof and errata, section 4 and 5

Note 12 (Dual in section 5.3.1)**.**

$$
\begin{aligned}
\int f\bar{g}dx = <f,g> &= \frac{1}{4}\sum_{k=0}^{3} i^k\|f+i^kg\|_2^2 = \frac{1}{4}\sum_{k=0}^{3} i^k\sum_{\rho\in\Delta}\|S_\rho(f+i^kg)\|_2^2 \\
&= \frac{1}{4}\sum_{k=0}^{3} i^k\sum_{\rho\in\Delta}\|S_\rho(f)+i^kS_\rho(g)\|_2^2 = \frac{1}{4}\sum_{\rho\in\Delta}\sum_{k=0}^{3} i^k\|S_\rho(f)+i^kS_\rho(g)\|_2^2 \\
&= \sum_{\rho\in\Delta}\int S_\rho(f)\overline{S_\rho(g)}dx
\end{aligned}
$$

Note 13 (equation (50))**.** Assume $|\phi'(x)| \leq B$. Since $\phi'_{I_m}(x) = 2^{-m}\phi'(2^{-m}x)$ and $x \in [2^m, 2^{m+1}]$, we have:

$$\left|\phi'_{I_m}(x)\right| = 2^{-m}\left|\phi'(2^{-m}x)\right| \leq \frac{B}{|x|}$$

When $I=[-2^{m+1},-2^m]$ the inequality also holds.

Note 14 (inequality (53))**.** You can refer the proof in section 6.1.3 in *Classical Fourier Analysis*.

Since $S_I\tilde{S}_I=S_I$, we have:

$$\|(\sum_m |S_{I_m}(f)|^2)^{\frac{1}{2}}\|_p = \|(\sum_m |S_{I_m}\tilde{S}_{I_m}(f)|^2)^{\frac{1}{2}}\|_p$$

We want to prove:

$$\|(\sum_m |S_{I_m}\tilde{S}_{I_m}(f)|^2)^{\frac{1}{2}}\|_p \le \|(\sum_m |\tilde{S}_{I_m}(f)|^2)^{\frac{1}{2}}\|_p$$

Let F be:

$$F=(\tilde{S}_{I_1}(f),\tilde{S}_{I_2}(f),\tilde{S}_{I_3}(f),\dots)$$

So

$$|F|=(\sum_m \left|\tilde{S}_{I_m}(f)\right|^2)^{\frac{1}{2}}$$

Let $\mathscr{R}$ be Δ_1, we have

$$S_{\mathscr{R}}(F)=(S_{I_1}\tilde{S}_{I_1}(f),S_{I_2}\tilde{S}_{I_2}(f),S_{I_3}\tilde{S}_{I_3}(f),\dots)$$

By Theorem 4' in book, we have:

$$\|(\sum_m |S_{I_m}\tilde{S}_{I_m}(f)|^2)^{\frac{1}{2}}\|_p = \|S_{\mathscr{R}}(F)\|_p \le \|F\|_p = \|(\sum_m |\tilde{S}_{I_m}(f)|^2)^{\frac{1}{2}}\|_p$$

Note 15 (inequality (54))**.** By (46)

$$B_p\|T_t^N f\|_p \le \|(\sum_\rho \left|S_\rho(T_t^N f)\right|^2)^{\frac{1}{2}}\|_p$$

Notice $S_{I_m}S_{I_m}f=S_{I_m}f$ and $S_{I_m}S_{I_n}f=0$ if $n\neq m$, we have:

$$\sum_\rho \left|S_\rho(T_t^N f)\right|^2 = \sum_{m=0}^N |r_m(t)S_{I_m}f|^2 \le \sum_{m=0}^N |S_{I_m}f|^2$$

Thus

$$B_p\|T_t^N f\|_p \le \|(\sum_\rho \left|S_\rho(T_t^N f)\right|^2)^{\frac{1}{2}}\|_p \le \|(\sum_{m=0}^N |S_{I_m}f|^2)^{\frac{1}{2}}\|_p \le C_p\|f\|_p$$