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Finite line of charge

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A finite line of uniform charge is either ignored or handled incompletely in most textbooks. New simple and practical expressions are presented for the electric potential and electric field for this charge distribution. The equipotentials are shown to be prolate ellipsoids and the electric field lines follow hyperboloids confocal to the ellipsoids. The common foci are shown to be the endpoints of the charged line segment. These geometries are demonstrated with simple VPython programs.

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I. INTRODUCTION

When presented with the equations for the electric field and potential of an infinite line of charge, some students become curious about the finite length case. After obtaining an analytical expression for the potential, I still couldn't guess the shape of the equipotentials, although the 3D plots generated by VPython[1] looked surprisingly like ellipsoids.

It turns out that the equipotential surfaces are ellipsoids and the foci are the endpoints of the line segment. Visually, even beginning students can see this geometry as a natural result of stretching a point into a line segment. This problem was solved long ago. Ernst Weber treated it in the 1965 revision[2] of his book, referencing earlier work of Abraham and Becker[3] among others. The latter introduced elliptic coordinates to show that the equipotentials are prolate ellipsoids. Weber continued from this result to obtain a complicated expression for the electric field. The correct potential for a line segment is also given in the classic texts by Morse and Feshbach[4] and Landau and Lifshitz,[5] although both books use elliptic coordinates in an even more cryptic manner.

Some authors cover specific cases, such as the potential in a perpendicular plane through a segment endpoint,[6,7] the electric field on the axis of the segment,[6] the electric field in a perpendicular plane through a segment endpoint,[8] or the electric field in a perpendicular plane through the middle of the segment.[8–10] Other authors have given the multipole expansion[11,12] for the potential, although these expansions are valid only outside the sphere containing the line segment. Some recent articles[13,14] have shown the correct results in passing. As Jackson[15] explains, it would be easy to overlook whether Maxwell[16] covered the problem and I haven't been able to determine whether he did or not.

Most textbooks[17–29] on electrostatics do not discuss this particular problem. The main point of this paper is that the problem follows naturally after studying the point charge, can be presented in an accessible way, and should be presented before the usual coverage of the dipole and infinite line of charge. I derive a simpler form for the electric potential, whose equipotentials are obviously ellipsoids at first glance, and a much simpler expression for the electric field. These results and their VPython visualizations should be accessible even to first year physics students.

II. ELECTRIC POTENTIAL AND EQUIPOTENTIAL SURFACES

A. The electric potential

We make the conventional choice that the electric potential is zero at infinity, so that we can use the expression $dV = k(dq)/r$. However, this choice implies that our result for a finite line of charge will *not* transform into the standard formula for the infinite line of charge by letting the length of the line of charge $L \to \infty$. That problem will be remedied in Sec. II C.

Figure 1 will be used later for a coordinate-free approach. Assume the rectangular coordinate origin is at the center of the line segment, let dq on the segment be at $(s,0)$, and let the field point be at (x,y,z):

$$V(x,y,z) = \int \frac{kdq}{r_{dq}} = \int_{-L/2}^{L/2} \frac{k\lambda ds}{\sqrt{(x-s)^2+y^2+z^2}}$$
$$= k\lambda \ln\left|\sqrt{(x-s)^2+y^2+z^2} - (x-s)\right|_{-L/2}^{L/2}$$
$$= k\lambda \ln\left(\frac{\sqrt{(x-L/2)^2+y^2+z^2}-(x-L/2)}{\sqrt{(x+L/2)^2+y^2+z^2}-(x+L/2)}\right). \quad (1)$$

Equation (1) is equivalent to Eq. 9 in Ref. 3 and was used in the 3D equipotential plots[30] shown in Fig. 2. Those plots avoid field points on the line segment axis. The right-hand side of Eq. (1) is undefined on the axis to the right of the line segment. For these field points, we can use the alternative expression

$$V(x,0,0) = \int_{-L/2}^{L/2} \frac{k\lambda ds}{x-s} = k\lambda \ln\left(\frac{x+L/2}{x-L/2}\right). \quad (2)$$

In Fig. 1, $\mathbf{L} \equiv L\mathbf{e}$ points from endpoint $s1$ of the line segment to endpoint $s2$, $\mathbf{r}$ points from $s2$ to the field point, and $\mathbf{r}_+ \equiv \mathbf{r} + \mathbf{L}$ is directed from $s1$ to the field point. The endpoints $s1$ and $s2$ will turn out to be the focal points of the equipotential ellipsoids and $\mathbf{r}$ will turn out to be the radius used in the common polar equation of an ellipse.

Instead of redoing the integral in Eq. (1), we can cast it in a coordinate-free form

$$V(\mathbf{r}) = k\lambda \ln\left(\frac{r - \mathbf{r}\cdot\mathbf{e}}{r_+ - \mathbf{r}_+\cdot\mathbf{e}}\right). \quad (3)$$

Equation (3) looks simpler than Eq. (1), but still suffers from being undefined over half of the line segment axis. We will use the fact that the component of $\mathbf{r}_+$ perpendicular to $\mathbf{L}$

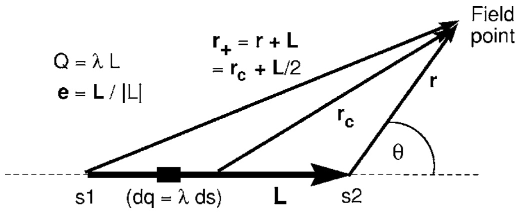

Fig. 1. Some defining relationships for the charged line segment (λ C/m, length L).

equals the component of $\mathbf{r}$ perpendicular to $\mathbf{L}$:

$$r_+^2 - (\mathbf{r}_+ \cdot \mathbf{e})^2 = r^2 - (\mathbf{r} \cdot \mathbf{e})^2 \Leftrightarrow \frac{r_+ + \mathbf{r}_+ \cdot \mathbf{e}}{r + \mathbf{r} \cdot \mathbf{e}} = \frac{r - \mathbf{r} \cdot \mathbf{e}}{r_+ - \mathbf{r}_+ \cdot \mathbf{e}}. \quad (4)$$

A useful algebra trick shown in a footnote in Ref. 3 is

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d} = \frac{a+c}{b+d}. \quad (5)$$

If we use Eq. (5) in Eq. (4) and substitute the result into Eq. (3), we find

$$V(\mathbf{r}) = k\lambda \ln\left(\frac{r_+ + r + L}{r_+ + r - L}\right). \quad (6)$$

This form of the potential is simple, symmetric, and coordinate-free, and is given in terms of only the charge density λ, line segment $\mathbf{L}$, and field point $\mathbf{r}$ ($\mathbf{r}_+ = \mathbf{r} + \mathbf{L}$). Unlike Eqs. (1) and (3), $V(\mathbf{r})$ in Eq. (6) is well-defined everywhere, including on the line axis, except (appropriately) on the charge itself. Note that the equipotentials occur whenever $r_+ + r$=constant, which is the classical equation for a prolate ellipsoid, and we'll see that the gradient of the potential, $-\nabla V$, gives the electric field in a straightforward manner.

B. The equipotentials

As we have seen from Eq. (6), the equipotentials are ellipsoids. It remains only to relate the electrical parameters to the parameters of an ellipse (see Fig. 3). We can group several constants together and rename them using the symbol ε:

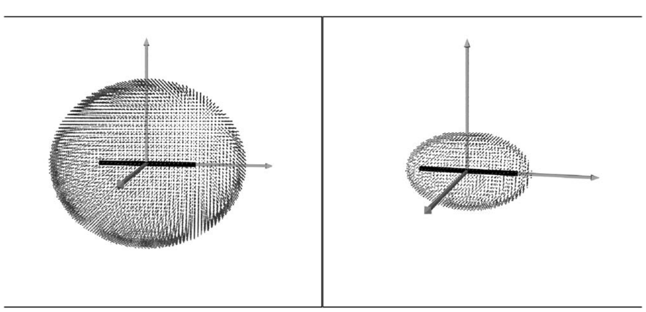

Fig. 2. VPython screen shots showing equipotentials for a finite line of charge. The line segment here went from $(-2,0,0)$ to $(2,0,0)$. The charge density λ was 5 nC/m. When the field point was $(3,-2,0)$, the potential was $V = 52.47$ V (left). The 100 V ellipsoid (right) is more eccentric and closer to the line segment.

$$[0 < V < \infty] \Rightarrow \left[0 < \epsilon \equiv \tanh\left(\frac{V}{2k\lambda}\right) < 1\right] \quad (7)$$

$$\Rightarrow \left[1 < \frac{1+\epsilon}{1-\epsilon} = e^{\left(\frac{V}{k\lambda}\right)} < \infty\right].$$

We see that if V is a constant, then ε is a constant and will turn out to be the eccentricity of the ellipsoid (see Eq. (12)). The substitution of Eq. (7) into Eq. (6) leads to the classical equation of an ellipse with respect to both foci (in agreement with Fig. 3):

$$\frac{1+\epsilon}{1-\epsilon} = \frac{r_+ + r + L}{r_+ + r - L} \Rightarrow r_+ + r = L/\epsilon. \quad (8)$$

Equation (8) and Eqs. (3) and (4) lead to the standard polar equation for the ellipse, Eq. (12), shown in Fig. 3:

$$\frac{1+\epsilon}{1-\epsilon} = \frac{r - \mathbf{r} \cdot \mathbf{e}}{r_+ - \mathbf{r}_+ \cdot \mathbf{e}} = \frac{r_+ + \mathbf{r}_+ \cdot \mathbf{e}}{r + \mathbf{r} \cdot \mathbf{e}}. \quad (9)$$

$$\frac{1+\epsilon}{1-\epsilon} = \frac{(L/\epsilon - r) + L + \mathbf{r} \cdot \mathbf{e}}{r + \mathbf{r} \cdot \mathbf{e}} = \frac{(1+\epsilon)L/\epsilon - (r - \mathbf{r} \cdot \mathbf{e})}{r + \mathbf{r} \cdot \mathbf{e}}. \quad (10)$$

$$(1+\epsilon)r(1+\cos(\theta)) + (1-\epsilon)r(1-\cos(\theta)) = (1-\epsilon^2)L/\epsilon. \quad (11)$$

$$r = \frac{(1-\epsilon^2)L/(2\epsilon)}{1+\epsilon\cos(\theta)} \equiv \frac{\ell}{1+\epsilon\hat{\mathbf{r}} \cdot \mathbf{e}} \equiv \frac{\ell}{1+\hat{\mathbf{r}} \cdot \boldsymbol{\epsilon}}, \quad (12)$$

where we have defined the eccentricity vector $\boldsymbol{\epsilon}$ and the semi-latus rectum ℓ:

$$0 < \ell \equiv \frac{1-\epsilon^2}{\epsilon}\frac{L}{2} < \infty. \quad (13)$$

C. Limiting cases

Problem 1. (a) What happens to Eq. (6) as $L \to 0$ while keeping the total charge constant? When L becomes very small, $r_+ = \sqrt{r^2 + 2\mathbf{r} \cdot \mathbf{L} + L^2}$ approaches r. Start with

$$V = k\lambda \ln\left(\frac{1 + \frac{L}{r_+ + r}}{1 - \frac{L}{r_+ + r}}\right) \xrightarrow{r_+ \to r} k\lambda \ln\left(1 + \frac{L}{2r}\right) - k\lambda \ln\left(1 - \frac{L}{2r}\right). \quad (14)$$

Use the Taylor series expansions for $\ln(1+x)$ and $\ln(1-x)$ and keep only the first order terms. Then let $\lambda L = Q$, the total charge, to show that $V = kQ/r$. (b) We have used the boundary condition $V = 0$ at $r \to \infty$ so that we can use $dV = k(dq)/r$. Now that we have Eq. (6) for $V(\mathbf{r})$ and the form of the equipotentials, we can redefine the zero of the potential. Choose any finite $\mathbf{r} \equiv \mathbf{r}_{GND}$ not on the line charge. This choice defines a potential $V_{GND} \equiv V(\mathbf{r}_{GND})$ according to Eq. (6), and an ellipsoidal ground equipotential surface. Then we can define a new form for the potential at all points not on the line charge

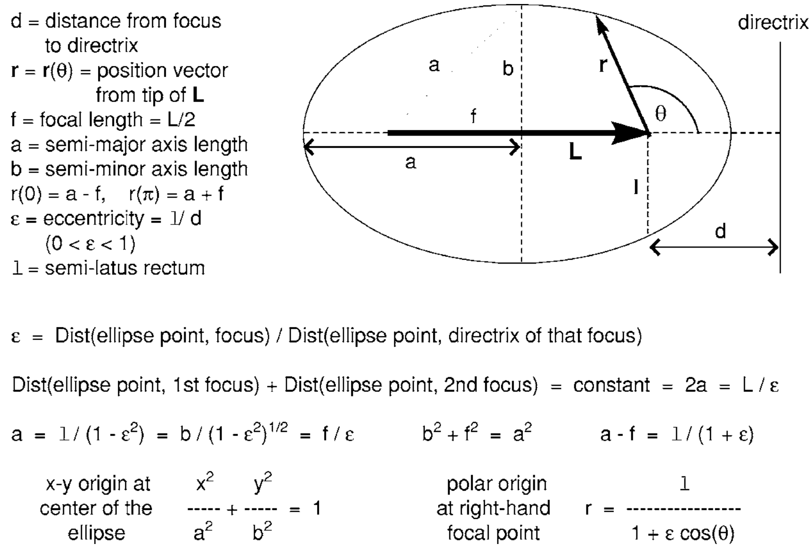

Fig. 3. Standard ellipse definitions and relationships.

$$V_{new}(\mathbf{r}) \equiv V(\mathbf{r}) - V(\mathbf{r}_{GND}) = k\lambda \ln\left(\frac{r_+ + r + L}{r_+ + r - L}\right) - V_{GND}, \tag{15}$$

$$V_{new}(\mathbf{r}) = k\lambda \ln\left(\frac{r_+ + r + L}{r_+ + r - L}\frac{r_{GND+} + r_{GND} - L}{r_{GND+} + r_{GND} + L}\right), \tag{16}$$

$$\begin{aligned} r_{GND+} + r_{GND} &= \frac{L}{\epsilon_{GND}} = L\coth\left(\frac{V_{GND}}{2k\lambda}\right) = 2a_{GND} \\ &= \sqrt{4b_{GND}^2 + L^2}. \end{aligned} \tag{17}$$

Equation (15) makes the ground redefinition simple. For example, in Fig. 2 we take $V(\mathbf{r}) = V([1,-2,0]) = 52.47$. So, if we choose the $V=50$ equipotential as the new ground surface, then $V_{new}(\mathbf{r}) = V([1,-2,0]) - 50 = 2.47$ (in volts). Equation (17) reminds us that we can choose any ellipsoid parameter as the designator of our ground ellipsoid and then calculate the value of any other parameter.

By combining the last term in Eq. (17) with Eq. (16), we obtain an expression that allows us to see what happens as we let $L \to \infty$, while holding λ and b (the perpendicular distance from the line segment) constant

$$\begin{aligned} V_{new}(\mathbf{r}) &= k\lambda \ln\left(\frac{\sqrt{4b^2 + L^2} + L}{\sqrt{4b^2 + L^2} - L}\frac{\sqrt{4b_{GND}^2 + L^2} - L}{\sqrt{4b_{GND}^2 + L^2} + L}\right) \\ &\underset{\lambda, b = constant}{\overset{L\to\infty}{\longrightarrow}} 2k\lambda \ln\left(\frac{b_{GND}}{b}\right). \end{aligned} \tag{18}$$

So, the limit in Eq. (18) recovers the usual expression for the potential at a perpendicular distance b from an infinitely long charged line, whose equipotentials are cylindrically shaped.

Problem 2. Rewrite Eq. (6) in terms of $\mathbf{r}_C$ from Fig. 1. It will also be convenient to introduce the parameter $\rho \equiv (L/2)/r_C$. Then

$$r_+^2 = (\mathbf{r}_C + \mathbf{L}/2)^2 = r_C^2(1 + \rho^2 + 2\rho\cos(\theta_C)), \tag{19a}$$

$$r^2 = (\mathbf{r}_C - \mathbf{L}/2)^2 = r_C^2(1 + \rho^2 - 2\rho\cos(\theta_C)), \tag{19b}$$

$$V = k\lambda \ln\left(\frac{\sqrt{1 + \rho^2 + 2\rho\cos(\theta_C)} + \sqrt{1 + \rho^2 - 2\rho\cos(\theta_C)} + 2\rho}{\sqrt{1 + \rho^2 + 2\rho\cos(\theta_C)} + \sqrt{1 + \rho^2 - 2\rho\cos(\theta_C)} - 2\rho}\right). \tag{20}$$

Use a symbolic manipulation program to expand Eq. (20) in a Taylor series about $\rho=0$, corresponding to $r_C \to \infty$. The result is

$$\begin{aligned} V &= 2k\lambda\rho + 2k\lambda\left(\frac{3\cos^2(\theta_C) - 1}{2}\right)\frac{\rho^3}{3} \\ &\quad + 2k\lambda\left(\frac{35\cos^4(\theta_C) - 30\cos^2(\theta_C) + 3}{8}\right)\frac{\rho^5}{5} + \cdots \end{aligned} \tag{21}$$

If we factor out $2k\lambda\rho = 2k\lambda(L/2)/r_C = kQ/r_C$, we obtain the multipole expansion for the potential of the finite line charge in terms of Legendre polynomials

$$V = \frac{kQ}{r_C}\left[1 + \mathrm{P}_2(\cos(\theta_C))\frac{\rho^2}{3} + \mathrm{P}_4(\cos(\theta_C))\frac{\rho^4}{5} + \cdots\right]. \tag{22}$$

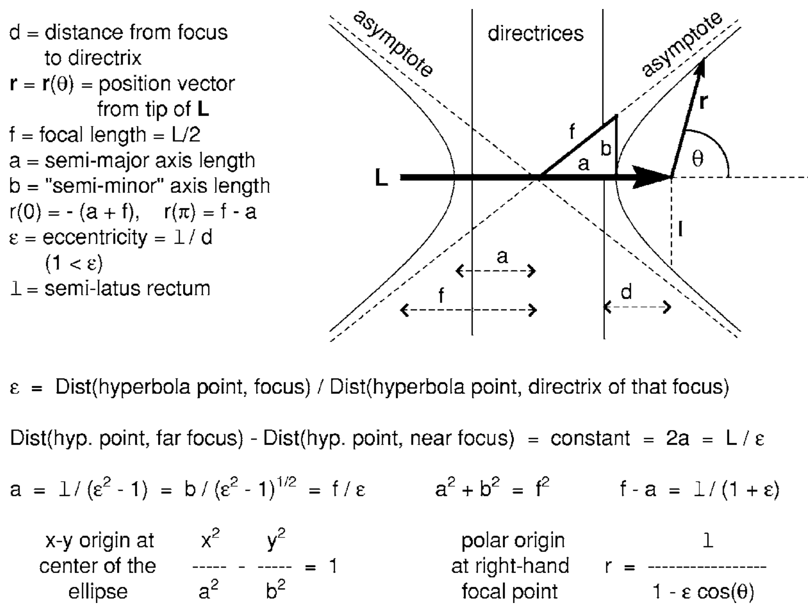

Fig. 4. Standard hyperbola definitions and relationships.

III. ELECTRIC FIELD AND FLUX LINES

A. The electric field

The symmetrical form of Eq. (6) makes it easy to calculate the electric field using $-\nabla V$. The result, Eq. (25), seems to be new, and is more elegant than previous results. We will use the theorems shown in Eq. (23):

$$\nabla r = \hat{\mathbf{r}}, \quad \nabla r_+ = \nabla\sqrt{(\mathbf{r}+\mathbf{L})^2} = \hat{\mathbf{r}}_+. \tag{23}$$

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -\nabla V(\mathbf{r}) = -\nabla\left[k\lambda \ln\left(\frac{r_+ + r + L}{r_+ + r - L}\right)\right] \\ &= -k\lambda\left(\frac{r_+ + r - L}{r_+ + r + L}\right) \\ &\quad\times\frac{(r_+ + r - L)(\hat{\mathbf{r}}_+ + \hat{\mathbf{r}}) - (r_+ + r + L)(\hat{\mathbf{r}}_+ + \hat{\mathbf{r}})}{(r_+ + r - L)^2}.\end{aligned} \tag{24}$$

$$\mathbf{E}(\mathbf{r}) = k\lambda\left[\frac{2L}{(r_+ + r)^2 - L^2}\right](\hat{\mathbf{r}}_+ + \hat{\mathbf{r}}) = \frac{k\lambda}{d}(\hat{\mathbf{r}}_+ + \hat{\mathbf{r}}). \tag{25}$$

Equation (25) is valid everywhere except on the charge itself. Note that the sum of the unit directions from the foci toward a point on the ellipse is normal to the curve at that point. As a corollary, the difference in these unit vectors gives a vector parallel to the curve.

We introduced the parameter d in Eq. (25) to simplify its form. The choice of this notation was intentional. Remember that the choice of the field point $\mathbf{r}$ also defines $\mathbf{r}_+$ and an equipotential ellipsoid at that point for which we can calculate any of the geometrical parameters, such as the eccentricity and the semi-latus rectum. This d is the distance from an ellipse focus to its corresponding directrix, as shown in Fig. 3. As an example of how easy it is to use Eq. (25), consider the data for the 52.47 V equipotential of Fig. 2, and calculate $\mathbf{E}(3,-2,0)$. We will use the bracket notation for vectors to obtain $\mathbf{E}$ in V/m:

$$k\lambda = 45, \quad \mathbf{r} = [1,-2,0], \quad \hat{\mathbf{r}} = [1,-2,0]/\sqrt{5}, \quad L = 4,$$

$$\mathbf{r}_+ = \mathbf{r} + \mathbf{L} = [5,-2,0],$$

$$\hat{\mathbf{r}}_+ = [5,-2,0]/\sqrt{29}, \quad \epsilon = \frac{L}{r_+ + r} = \frac{4}{\sqrt{29}+\sqrt{5}} = 0.525,$$

$$d = \frac{(r_+ + r)^2 - L^2}{2L} = \left(\frac{1}{\epsilon^2} - 1\right)\frac{L}{2} = 5.260,$$

$$\mathbf{E} = \frac{45}{5.26}(\hat{\mathbf{r}}_+ + \hat{\mathbf{r}}) = [11.77,-10.83,0].$$

B. Shape of the flux lines

VPython can be used to draw electric flux lines using Eq. (25). For each desired flux line, choose a starting position $\mathbf{r}$ near the charged line segment. Then use Eq. (25) to find a unit vector in the direction of $\mathbf{E}(\mathbf{r})$, multiply by a suitably small distance increment, and add that product to the previous $\mathbf{r}$ to obtain the next $\mathbf{r}$.

Alternatively, we could use the fact that the family of all ellipses sharing a common set of foci and the family of all hyperbolas sharing those same foci constitute *orthogonal confocal families*. All of the hyperbolas cross all the ellipses perpendicularly. Figure 4 shows standard hyperbola relations similar to the ellipse relations shown in Fig. 3.

If we choose any position $\mathbf{r}$ not on the axis, then $\mathbf{r}$ and $\mathbf{L}$ determine a plane, and Eq. (6) gives $V(\mathbf{r})$. Then Eqs. (7) and (13) lead to the polar equation, Eq. (12), for the equipotential ellipse. Equation (25) gives $\mathbf{E}(\mathbf{r})$. We can use the following definitions and relations to obtain the polar equation, Eq. (29), for the hyperbolic electric flux line passing through $\mathbf{r}$:

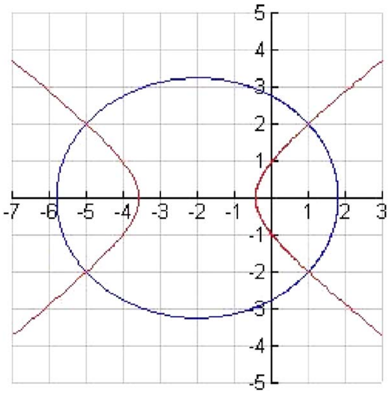

Fig. 5. Sample plots performed with a standard graphing calculator.

$$r = \frac{\ell}{1+\epsilon\cos(\theta)}(\text{ellipse}) = \frac{\ell'}{1-\epsilon'\cos(\theta)}(\text{hyperbola}). \tag{26}$$

We replace ℓ by $(1-\epsilon^2)f/\epsilon$ and ℓ' by $((\epsilon')^2-1)f/\epsilon'$, and then solve for ϵ':

$$\epsilon' = \frac{1+\epsilon\cos(\theta)}{\epsilon+\cos(\theta)}, \tag{27}$$

$$\ell' = \left(\frac{(\epsilon')^2-1}{\epsilon'}\right)f = \left(\frac{(\epsilon')^2-1}{\epsilon'}\right)\frac{L}{2}, \tag{28}$$

$$r_{\text{hyperbola}} = \frac{\ell'}{1-\epsilon'\cos(\theta)}. \tag{29}$$

For a point $\mathbf{r}$, Eq. (6) gives $V(\mathbf{r})$, Eq. (25) gives $\mathbf{E}(\mathbf{r})$, and Eqs. (7) and (13) give ϵ and ℓ. We then can calculate ϵ' and ℓ' from Eqs. (27) and (28). Figure 5 shows plots of the polar equations of the equipotential and electric field line running through $\mathbf{r}=[1,-2,0]$, for the 52.47 V equipotential of Fig. 2 using a graphing calculator.

As mentioned, the family of surfaces orthogonal to confocal ellipsoids consists of confocal hyperboloids (typically shown using elliptic coordinates). To derive this result we start with the electric field formula, Eq. (25), and show that the electric field lines in each plane are hyperbolas.

I prefer to use geometric algebra (GA) when seeking coordinate-free expressions, although this use is not necessary. Maxwell's equations could be written as eight scalar equations or as one GA equation. Most physicists would consider one vector equation to be far more elegant than its three associated scalar equations. An increasing number of physicists consider GA to be more elegant than ordinary vector analysis. Once students make the giant leap from scalar mathematics to the directed line segments of vector analysis and learn what dot and cross products mean, it is a much smaller step for them to grasp plane segments or volume elements, especially because there is nothing more to learn as intimidating as cross products. And it makes much more sense to use an extended vector algebra that allows for division as well as multiplication. For a good introduction to GA (with many references), see Refs. 31 and 32.

Start with Eq. (25) for the electric field from the charged line segment, which tells us that $\mathbf{E}$ is in the plane of $\mathbf{r}$ and $\mathbf{L}$, so the curve $\mathbf{r}(\theta)$ for the flux line is planar. Let's call the unit bivector for that plane $\mathbf{i}$ ($\mathbf{e}$ is the line segment unit vector), so we know the space curve must have the form

$$\mathbf{r}(\theta) = r(\theta)\angle\,\theta = r(\theta)\mathbf{e}e^{\mathbf{i}\theta} = r(\theta)\hat{\mathbf{r}}(\theta), \tag{30}$$

$$\frac{d\mathbf{r}}{d\theta} = r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}' = r'\mathbf{e}e^{\mathbf{i}\theta} + r\mathbf{e}e^{\mathbf{i}\theta}\mathbf{i} = r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}\mathbf{i}, \tag{31}$$

$$\left|\frac{d\mathbf{r}}{d\theta}\right|^2 = (r'\hat{\mathbf{r}} + r\hat{\mathbf{r}}\mathbf{i})^2 = (r' - r\mathbf{i})\hat{\mathbf{r}}\hat{\mathbf{r}}(r' + r\mathbf{i}) = r'^2 + r^2, \tag{32}$$

$$\mathbf{T} = \frac{r'\hat{\mathbf{r}} + r(\hat{\mathbf{r}}\mathbf{i})}{\sqrt{r'^2+r^2}}. \tag{33}$$

$\mathbf{T}$ is the required form for the unit tangent vector to any planar space curve, expressed in Eq. (33) in terms of its components parallel and perpendicular to $\mathbf{r}$. From Eq. (25) we can obtain the unit vector in the direction of $\mathbf{E}$. By equating these two unit vectors, we obtain a differential equation for $r(\theta)$, which will lead us to the form of Eq. (29):

$$\begin{aligned}
\frac{r'\hat{\mathbf{r}} + r(\hat{\mathbf{r}}\mathbf{i})}{\sqrt{r'^2+r^2}} = \mathbf{T} = \pm\hat{\mathbf{E}} &= \pm\frac{1}{\sqrt{2}}\frac{\hat{\mathbf{r}}_+ + \hat{\mathbf{r}}}{\sqrt{1+\hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}}}} \\
&= \pm\frac{1}{\sqrt{2}}\frac{\hat{\mathbf{r}}_+\hat{\mathbf{r}}\hat{\mathbf{r}} + \hat{\mathbf{r}}}{\sqrt{1+\hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}}}} \\
&= \pm\frac{1}{\sqrt{2}}\frac{(1+\hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} + (\mathbf{i}\hat{\mathbf{r}}_+\wedge\hat{\mathbf{r}})(\hat{\mathbf{r}}\mathbf{i})}{\sqrt{1+\hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}}}}.
\end{aligned} \tag{34}$$

To obtain the desired differential equation, divide the component parallel to $\mathbf{r}$ by the component perpendicular to $\mathbf{r}$ on each side of Eq. (34):

$$\frac{r'}{r} = \frac{1+\hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}}}{\mathbf{i}\hat{\mathbf{r}}_+\wedge\hat{\mathbf{r}}} = \frac{\mathbf{i}\hat{\mathbf{r}}_+\wedge\hat{\mathbf{r}}}{1-\hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}}}. \tag{35}$$

The last equality in Eq. (35) can be checked by cross multiplying, because

$$\begin{aligned}
1 = \hat{\mathbf{r}}_+\hat{\mathbf{r}}\hat{\mathbf{r}}\hat{\mathbf{r}}_+ &= (\hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}} + \hat{\mathbf{r}}_+\wedge\hat{\mathbf{r}})(\hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}} - \hat{\mathbf{r}}_+\wedge\hat{\mathbf{r}}) \\
&= (\hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}})^2 - (\hat{\mathbf{r}}_+\wedge\hat{\mathbf{r}})^2 \Rightarrow 1 - (\hat{\mathbf{r}}_+\cdot\hat{\mathbf{r}})^2 \\
&= (\mathbf{i}\hat{r}_+\wedge\hat{\mathbf{r}})(\mathbf{i}\hat{\mathbf{r}}_+\wedge\hat{\mathbf{r}}).
\end{aligned} \tag{36}$$

We multiply the numerator and denominator of the right side of Eq. (35) by r_+ and find

$$\begin{aligned}
\frac{r'}{r} &= \frac{\mathbf{i}(\mathbf{r}+\mathbf{L})\wedge\hat{\mathbf{r}}}{r_+ - (\mathbf{r}+\mathbf{L})\cdot\hat{\mathbf{r}}} = \frac{\mathbf{i}\mathbf{L}\wedge\hat{\mathbf{r}}}{r_+ - r - \mathbf{L}\cdot\hat{\mathbf{r}}} = \frac{\mathbf{L}\cdot(\hat{\mathbf{r}}\mathbf{i})}{r_+ - r - \mathbf{L}\cdot\hat{\mathbf{r}}} \\
&= \frac{\mathbf{L}\cdot\hat{\mathbf{r}}'}{r_+ - r - \mathbf{L}\cdot\hat{\mathbf{r}}}.
\end{aligned} \tag{37}$$

By rearranging Eq. (37), we can show that $r_+ - r$ is constant (confirming that the space curve should be a hyperbola)

$$\begin{aligned}
r'(r_+ - r) &= \mathbf{L}\cdot\hat{\mathbf{r}}'r + \mathbf{L}\cdot\hat{\mathbf{r}}r' = \frac{d}{d\theta}(\mathbf{L}\cdot\mathbf{r}) = \frac{d}{d\theta}(\mathbf{L}\cdot\mathbf{r}_+) \\
&= \frac{1}{2}\frac{d}{d\theta}[(\mathbf{r}_+ - \mathbf{r})\cdot(\mathbf{r}_+ + \mathbf{r})] = \frac{d}{d\theta}\left[\frac{1}{2}(r_+^2 - r^2)\right],
\end{aligned} \tag{38}$$

$$0 = \frac{d}{d\theta}\left[\frac{1}{2}(r_+ - r)(r_+ + r)\right] - \frac{d}{d\theta}[r(r_+ - r)] + r\frac{d}{d\theta}(r_+ - r)$$
$$= \frac{d}{d\theta}\left[\frac{1}{2}(r_+ - r)(r_+ - r)\right] + r\frac{d}{d\theta}(r_+ - r) = r_+\frac{d}{d\theta}(r_+ - r). \tag{39}$$

Because $r_+ \neq 0$, we have

$$r_+ - r = \text{constant} \equiv \frac{L}{\epsilon}. \tag{40}$$

If we substitute this form into Eq. (37), we find

$$\frac{dr}{d\theta} = \frac{r\mathbf{L}\cdot\hat{\mathbf{r}}'}{L/\epsilon - \mathbf{L}\cdot\hat{\mathbf{r}}} = \frac{\epsilon r\mathbf{e}\cdot\hat{\mathbf{r}}'}{1-\epsilon\mathbf{e}\cdot\hat{\mathbf{r}}}, \tag{41}$$

$$0 = \left(\frac{dr}{d\theta}\right)(1-\epsilon\mathbf{e}\cdot\hat{\mathbf{r}}) - \epsilon r\mathbf{e}\cdot\hat{\mathbf{r}}' = \frac{d}{d\theta}[r(1-\epsilon\mathbf{e}\cdot\hat{\mathbf{r}})]. \tag{42}$$

If we set the integration constant equal to ℓ, we find the form of Eq. (29) as desired.

IV. SUMMARY

We have derived elegant, coordinate-free expressions for the electric potential, Eq. (6), and the electric field, Eq. (25), for a finite charged line segment. The expressions for $V(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ are simple to visualize and to program on a calculator or using software such as VPython. The equipotentials are ellipsoids whose foci are the endpoints of the charged line segment. The electric field lines follow hyperboloids perpendicular to the ellipsoids and share the same foci. The formulas and resulting geometrical shapes are presentable even to first-year students, and the derivations can be presented in a junior/senior electricity and magnetism course.

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