

# Introduction to Quantum Field Theory

*Notes re-written from lessons' attendance, 2022*

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# Chapter 1

## Settembre e Ottobre

Appunti delle lezioni del Prof. Polosa relative al mese di settembre e ottobre 2022.

**1.1    22 settembre**

**1.2    23 settembre**

**1.3    27 settembre**

## 1.4 4 ottobre

**The rod** Given a 1-dimensional rod composed by N-particles, linked each others with a "spring", the hamiltonian density is

$$\mathcal{H} = \frac{1}{2} \sum_{n=1}^N \left[ P_n^2 + \Omega^2 (q_n - q_{n+1})^2 + \Omega_0^2 q_n^2 \right]$$

where the last term  $\Omega_0^2 q_n^2$  is relative to the equilibrium position of the n-particle. The *periodic boundaries conditions* to  $N \rightarrow \infty$  and  $a \rightarrow 0$ .

On the other side we can write the Newtonian equation as

$$\begin{aligned} H &= \frac{1}{2} \int_0^L dx \left[ p^2(x) + v^2 \left( \frac{\partial q(x)}{\partial x} \right)^2 \right] \\ p(x) &= \dot{q}(x) \\ \ddot{q}(x) &= v^2 \frac{\partial^2 q(x)}{\partial x^2} \end{aligned}$$

the solution inside the boundaries is

$$\ddot{q}_n = \Omega^2 (q_{n+1} + q_{n-1} - 2q_n)$$

**Normal modes** or normal frequencies

$$\begin{aligned} q_n &= \sum_j e^{ijn} \frac{Q_j}{\sqrt{N}} \\ q(x) &= \frac{1}{\sqrt{a}} \sum_n e^{\frac{2\pi i}{Na}(na)} \frac{Q_j}{\sqrt{N}} = \frac{1}{\sqrt{a}} \sum_k e^{ikx} \frac{Q_k}{\sqrt{N}} \\ k &= \frac{2\pi l}{L} \\ \Rightarrow q(x) &= \sum_k e^{ikx} \frac{Q_k}{\sqrt{Na}} = \sum_k e^{ikx} \frac{Q_k}{\sqrt{L}} \end{aligned}$$

Considering now the Newtonian equation,  $p^2(x) = \dot{q}^2(x)$ ,  $\sum_{n=1}^N e^{in(j-j')} = \delta_{j,j'}$  where  $j = \frac{2\pi l}{N}$ , we can move from the sum to the integral using the following relation  $\sum_{n=1}^N \rightarrow \frac{1}{a} \int_0^L dx$  and this leads to  $\int_0^L dx e^{i(k-k')x} = L \delta_{k,k'}$ .

Somehow we may land on this following expression:

$$\frac{1}{L} \sum_{k,k'} L \delta_{k,k'} Q_k \dot{Q}_{k'} = \sum_k Q_k \dot{Q}_k = \sum_k |\dot{Q}_k|^2$$

To finally get a *total classical description*: a discrete sum on a numerable set, as follow

$$H = \frac{1}{2} \sum_k |\dot{Q}_k|^2 + k^2 v^2 |Q_k|^2$$

As before, notice that the sum  $\sum_{n=1}^N$  for  $L \rightarrow \infty$  became  $\frac{L}{2\pi} \int dk$  and it admits waves. Extending this to 3-dimensional space, it became

$$\sum_{\vec{k}} (\dots) \quad (\text{when } L \rightarrow \infty) \quad \frac{V}{(2\pi)^3} \int d^3k$$

**Quantum system:** let's consider now a quantum system, a quantum description.

*Postulate* the followings:

$$\begin{array}{lll} [q_l, p_n] = i \delta_{ln} & [Q_l, P_n] = i \delta_{ln} & \text{Where natural units are applied:} \\ [q_l, q_n] = 0 & [Q_l, Q_n] = 0 & h = 1 \\ [p_l, p_n] = 0 & [P_l, P_n] = 0 & c = 1 \end{array}$$

$$\Rightarrow \quad q_n^\dagger = q_n \quad , \quad Q_{-j} = Q_j^\dagger \quad , \quad P_{-j} = P_j^\dagger$$

$$\text{e.g.} \quad q_n^\dagger = \left( \sum_n e^{inj} \frac{Q_j}{\sqrt{N}} \right)^\dagger = \sum_j e^{-inj} \frac{Q_j^\dagger}{\sqrt{N}} = q_n$$

From the hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_j \left[ P_j P_j^\dagger + \omega_j^2 Q_j Q_j^\dagger \right]$$

and given the following operators, we find  $Q_j$  and  $P_j$ :

$$\begin{aligned} a_j &= \frac{1}{\sqrt{2\omega_j}} (\omega_j Q_j + i P_j^\dagger) & \Rightarrow \quad Q_j &= \frac{1}{\sqrt{2\omega_j}} (a_j + a_{-j}^\dagger) \\ a_j &= \frac{1}{\sqrt{2\omega_j}} (\omega_j Q_j^\dagger - i P_j) & P_j &= -i \left( \frac{\omega_j}{2} \right)^{\frac{1}{2}} (a_{-j} - a_j^\dagger) \end{aligned} \quad \text{and} \quad \begin{array}{l} \text{keep in mind} \\ [a_j, a_{j'}] = \delta_{jj'} \end{array}$$

$$\begin{aligned} Q_j Q_j^\dagger &= \frac{1}{2\omega_j} (a_j a_j^\dagger + a_j a_{-j} + a_{-j}^\dagger a_j^\dagger + a_{-j}^\dagger a_{-j}) \\ P_j P_j^\dagger &= \left( \frac{\omega_j}{2} \right)^{\frac{1}{2}} (a_{-j} a_{-j}^\dagger - a_{-j} a_j - a_j^\dagger a_{-j}^\dagger + a_j^\dagger a_j) \end{aligned}$$

With these last results we may write the  $\mathcal{H}$  as

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \sum_j \left[ P_j P_j^\dagger + \omega_j^2 Q_j Q_j^\dagger \right] = \frac{1}{2} \sum_j \omega_j (a_j a_j^\dagger + a_j^\dagger a_j) \\ &= \frac{1}{2} \sum_j \omega_j (2 a_j^\dagger a_j + 1) = \sum_j \omega_j \left( a_j^\dagger a_j + \frac{1}{2} \right) \end{aligned}$$

**Phonons description** Phonons are bosons, they're used to describe the quantum problem of the rod. Phonons are like photons but in the world of sound instead of light. A n-particles system is defined with

$$|n_1, n_2, n_3, \dots\rangle = (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} (a_3^\dagger)^{n_3} \dots |0\rangle$$

and for the 1-d oscillator, with energy  $E_n$ , is as follows

$$\begin{aligned} |n\rangle &= (a^\dagger)^n |0\rangle \\ E_n &= \hbar\omega \left( n + \frac{1}{2} \right) \stackrel{nu}{=} \omega \left( n + \frac{1}{2} \right) \end{aligned}$$

For the phonons is easy to *understand* which is the medium that make the transmission possible, but what about light? For the light, photons, the medium may also be the *vacuum*.

Filosofeggiamo un po' ora:

*Particles are the excitation of the field*

If you don't touch the piano it stays quiet, but if you play it it makes music ... song's particles.

*The field is permanent.*

*Particles are not fixed, they live and die.*

You cannot touch or see the field that you're studying, but you can see/detect the particle that pop out from the field.

Fields are NOT real but mathematical description of the world.

When you measure an energy it's always relative to an offset, a ground-state. Because you want the *vacuum* to be Lorentz invariant.

$$\Rightarrow \quad (\mathcal{H} - E_0) |0\rangle = 0$$

Given a general operator  $\Theta(t)$  and its derivate  $\dot{\Theta} = i[\mathcal{H}, \Theta(t)]$  so that:

$$\dot{a}(t) = i[\mathcal{H}, a(t)] = -i\omega a(t) \quad \text{where} \quad \begin{array}{l} [a, \mathcal{H}] = \omega a \\ [\mathcal{H}, a^\dagger] = \omega a^\dagger \end{array} \quad \Rightarrow \quad a(t) = e^{-i\omega t} a(0)$$

(Finire lezione del 4 ottobre manca mezza pagina di esercizio - chiedere appunti)

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$$\begin{aligned}\phi(x) &= \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k V}} \left( a_{\vec{k}} e^{i\vec{k}x} + a_{\vec{k}}^\dagger e^{-i\vec{k}x} \right) \\ &= \sum_{\vec{k}} \frac{i}{2V} \left[ e^{i\vec{k}(\vec{x}-\vec{y})} + e^{-i\vec{k}(\vec{x}-\vec{y})} \right]\end{aligned}$$

considering  $\begin{bmatrix} a_{\vec{k}}, a_{\vec{k}}^\dagger \end{bmatrix} = \delta_{\vec{k}, \vec{k}'}$   
 $\begin{bmatrix} \phi(\vec{x}, t), \dot{\phi}(\vec{y}, t) \end{bmatrix} = i V \delta^3(\vec{x} - \vec{y})$

Da capire che senso ha  
e contestualizzarlo  $\Rightarrow$

$$\begin{aligned}k_\mu &= (\vec{k}, i\omega_{\vec{k}}) \\ x_\mu &= (\vec{x}, it) \\ k_\mu x_\mu &= \vec{k} \cdot \vec{x} = k_\mu k_\nu \delta_{\mu\nu} \\ k_\mu x_\mu &= \vec{k} \cdot \vec{x} - \omega_{\vec{k}} t\end{aligned}$$

When things go to infinity  $\sum_{\vec{k}} \rightarrow \frac{V}{(2\pi)^3} \int d^3\vec{k}$  and remember that “if things doesn't work there will be some volume  $V$  somewhere”. Creation and destruction operators are contained into the description of the field. The energy levels' order are given from the term  $n\omega$  and you can forget about the  $\frac{1}{2}$ .

**Classical problem** Given the coordinates  $q_i(t)$  time-dependents, where  $i = 1, 2, 3, \dots, 3N$ , we can write the system of the 2° order derivate as follow

$$\begin{aligned}F_i &= m\ddot{q}_i \\ F_i &= -\frac{dV}{dq_i}\end{aligned}$$

given the initial conditions  $q_i(t_0)$  or given the boundary conditions  $q_i(t_1), q_i(t_2)$   
 $\dot{q}_i(t_0)$   $\dot{q}_i(t_1), \dot{q}_i(t_2)$

**Action Functional** The *Action Functional*  $S$

$$S = \int_{t_1}^{t_2} dt L(q_i(t), \dot{q}_i(t))$$

is defined such that a variation on the trajectory leads to a variation on  $S$ . So we can make a variables' transformation such that the new coordinates are the same as before plus a variational term

$$\begin{aligned}q_i(t) &\rightarrow q_i(t) + \delta q_i(t) \\ \dot{q}_i(t) &\rightarrow \dot{q}_i(t) + \delta \dot{q}_i(t) = \dot{q}_i(t) + \frac{d}{dt} \delta q_i(t)\end{aligned}$$

hence the action became

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} L(q_i(t) + \delta q_i(t), \dot{q}_i(t) + \frac{d}{dt} \delta q_i(t)) dt - \int_{t_1}^{t_2} dt L(q_i(t), \dot{q}_i(t)) dt = \\ &= \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q_i} \delta q_i(t), \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i(t) \right) dt = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt\end{aligned}$$

in the last step we used the boundary condition at  $t_1$  and  $t_2$ , so that  $\delta q_i(t_1) = \delta q_i(t_2) = 0$ . The last step leads directly to the *lagrangian equation*, that referred to the following generic  $L$  in 3-D is:

$$\begin{aligned}\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) &= \frac{\partial L}{\partial q_i} \\ L &= \frac{1}{2} m \dot{\vec{q}}^2 - V(\vec{q})\end{aligned}$$

this leads to the Newton equation for the *free motion*:

$$\begin{aligned}F &= 0 \\ m\ddot{q}_i &= -\frac{\partial V}{\partial q_i} = F\end{aligned}$$

$\Rightarrow \begin{aligned} \ddot{\vec{q}} &= 0 \\ \dot{\vec{q}} &= \vec{w} \\ \vec{q} &= \vec{w}t + \vec{r} \end{aligned} \Rightarrow \begin{aligned} \vec{q}_1 &= \vec{w}t_1 + \vec{r} \\ \vec{q}_2 &= \vec{w}t_2 + \vec{r} \end{aligned}$

so now we can find the Lagrangian depending on  $q_1, q_2$  and  $t_1, t_2$

$$\begin{aligned}\vec{w} &= \frac{\vec{q}_1 \cdot \vec{q}_2}{t_1 - t_2} \\ \vec{q} &= \frac{\vec{q}_1 \cdot \vec{q}_2}{t_1 - t_2} t + \vec{r}\end{aligned}$$

$\Rightarrow \begin{aligned} (t = t_1) : (t_1 - t_2) \vec{q}_1 &= (\vec{q}_1 \cdot \vec{q}_2) t_1 + \vec{r}(t_1 - t_2) \\ (t = t_2) : (t_1 - t_2) \vec{q}_2 &= (\vec{q}_1 \cdot \vec{q}_2) t_2 + \vec{r}(t_1 - t_2) \end{aligned} \Rightarrow \begin{aligned} \vec{q} &= \left( \frac{\vec{q}_1 \cdot \vec{q}_2}{t_1 - t_2} \right) t + \frac{\vec{q}_2 t_1 - \vec{q}_1 t_2}{t_1 - t_2} \\ \dot{\vec{q}} &= \frac{\vec{q}_1 \cdot \vec{q}_2}{t_1 - t_2} \end{aligned}$

where the last two equations explicit the boundary conditions. The Lagrangian of a free motion became

$$L = \frac{1}{2} m \dot{\vec{q}}^2 = \frac{1}{2} m \left( \frac{\vec{q}_1 \cdot \vec{q}_2}{t_1 - t_2} \right)^2$$

And the minimal Action is written as follow and represents “the true trajectory”

$$\int_{t_1}^{t_2} dt L = \frac{1}{2} m \left( \frac{\vec{q}_1 \cdot \vec{q}_2}{t_1 - t_2} \right)^2 (t_1 - t_2) = S_{min}$$

“The real motion is given by the minimum action.”

**Hamiltonian equation** What happen with the generic lagrangian instead substituting the free motion path? Starting from the lagrangian equation:

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} &= 0 \quad \left[ \text{where } \dot{q}_i = p_i \right] \\ \dot{q}_i \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \dot{q}_i \frac{\partial L}{\partial q_i} &= 0 \\ \frac{d}{dt} \left( \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - \underbrace{\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - \dot{q}_i \frac{\partial L}{\partial q_i}}_{\tau_i} &= 0 \\ \frac{d}{dt} \left( \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{d}{dt} (L(q_i, \dot{q}_i)) &= 0 \end{aligned} \quad \Rightarrow \quad \frac{d}{dt} (\dot{q}_i p_i - L(q_i, \dot{q}_i)) = 0$$

Given the lagrangian as  $L(q_i, \dot{q}_i)$  let's check its *invariance* when the coordinates change: the lagrangian must not change. For small  $\varepsilon$

$$\begin{aligned} q_i &\rightarrow q_i + \varepsilon_i f_i(q_1, \dots, q_i, \dots, q_N) \\ \dot{q}_i &\rightarrow \dot{q}_i + \varepsilon_i \frac{d}{dt} f_i(q_1, \dots, q_i, \dots, q_N) \end{aligned}$$

We can write the variation of the lagrangian as

$$\begin{aligned} \sum \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i &= 0 \\ \sum \underbrace{\frac{\partial L}{\partial q_i} f_i}_{\tau_i} + \underbrace{\frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} f_i}_{\tau_i} &= 0 \quad \longrightarrow \quad \frac{d}{dt} \sum_i p_i f_i = 0 \end{aligned}$$

Since the lagrangian is *invariant* for transformation, as we just saw, this leads to a *conservation*  $\leftrightarrow$  *symmetry*. This concept is also known as *Noether theorem*.

Now ...something here not so clear...

about changing formalism from Classical Mechanic to Quantum Mechanic.

**Hamiltonian and lagrangian density** Since the beginning of the course we introduce, without saying, and use the hamiltonian density  $\mathcal{H}$  instead of the hamiltonian  $H$ , that is defined as:

$$H = \int_0^L dx \mathcal{H}$$

as well as the lagrangian density  $\mathcal{L}$  instead of the lagrangian  $L$ :

$$L = \int_0^L dx \mathcal{L} \quad ( ! \text{ controllare se è corretto} ! )$$

**Back to the rod** using now in hamiltonian formalism

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}v^2(\partial_x\phi)^2 = \frac{1}{2}p^2 + \frac{1}{2}v^2(\partial_x\phi)^2 \\ \mathcal{L} = p\dot{\phi} - \mathcal{H} &= \dot{\phi}\dot{\phi} - \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}v^2(\partial_x\phi)^2\right) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}v^2(\partial_x\phi)^2\end{aligned}\quad \text{where } p = \dot{\phi}$$

when the velocity is set as  $v = \Omega a$  it depends strictly on the material of the medium. So when moving in the vacuum, and not along a rod, what happens? The velocity will be set to  $v = c = 1$ , as natural units are used, and “forget the mass” such that  $m = 1$ . The lagrangian density became consistent with this

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}v^2(\partial_x\phi)^2 \tag{1.1}$$

The definition of *field* goes with the definition of a proper *energy density*. The action became:

$$S = \int_{t_1}^{t_2} dt L = \int_{t_1}^{t_2} dt \int_0^L dx \mathcal{L}$$

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Continuing from the lagrangian 1.1

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}v^2(\partial_x\phi)^2 = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}v^2(\partial_x\phi)^2$$

we see the lagrangian as function of these variables  $\mathcal{L} = \mathcal{L}(\phi, \partial_t\phi, \partial_x\phi)$  and we write the variation and the variational principle as follow

$$\begin{aligned} S &= \int dt \int dx \mathcal{L} = \int dt L & \delta S &= 0 \\ \delta S &= \int dt \int dx \mathcal{L}(\phi + \delta\phi, \dot{\phi} + \frac{\partial}{\partial t}\delta\phi, \partial_x\phi + \partial_x\delta\phi) - \int dt \int dx \mathcal{L}(\phi, \dot{\phi}, \partial_x\phi) \\ &= \int dt \int dx \left\{ \frac{\partial\mathcal{L}}{\partial\phi} \delta\phi - \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} \partial_t\delta\phi - \frac{\partial\mathcal{L}}{\partial(\partial_x\phi)} \partial_x\delta\phi \right\} \\ &= \int dt \int dx \left\{ \frac{\partial\mathcal{L}}{\partial\phi} - \partial_t \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} - \partial_x \frac{\partial\mathcal{L}}{\partial(\partial_x\phi)} \right\} \delta\phi \end{aligned}$$

Hence we may set some boundary conditions at times  $t_1$  and  $t_2$ , for the rod problem

$$\begin{aligned} \delta\phi(\vec{x}, t_1) &= 0 \\ \delta\phi(\vec{x}, t_2) &= 0 \end{aligned}$$

Notice that the partial derivative of  $\mathcal{L}$  with respect to  $\phi$  is the following, and through which we find the *Euler-Lagrange equation in Quantum Field Theory*

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\phi} &= \partial_t \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} + \partial_x \frac{\partial\mathcal{L}}{\partial(\partial_x\phi)} + \partial_y \frac{\partial\mathcal{L}}{\partial(\partial_y\phi)} + \partial_z \frac{\partial\mathcal{L}}{\partial(\partial_z\phi)} & \Rightarrow & \quad \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial\mathcal{L}}{\partial\phi} \\ \text{with } \partial_\mu &= (\partial_1, \partial_2, \partial_3, \partial_4) = (\partial_t, \partial_x, \partial_y, \partial_z) \end{aligned}$$

The lagrangian density may be generalized in 3-D:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - \frac{1}{2}(\partial_y\phi)^2 - \frac{1}{2}(\partial_z\phi)^2 \\ &= \frac{1}{2}(\partial_\mu\phi)^2 \end{aligned} \quad \text{where } (\partial_\mu\phi)^2 = \partial_\mu\phi \partial_\mu\phi$$

This shorter notation implicitly implement the relativity, with  $v = 1$  and natural units. If you want to describe a physical system, like the rod, you must instead keep it explicit with  $v \neq 0$  and the Lagrangian will need one term more, such that

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$$

**Solve the motion equation** starting from the lagrangian above, we derive it w.r.t.  $\phi$  and w.r.t.  $\partial_\mu\phi$

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \\ \begin{cases} \frac{\partial\mathcal{L}}{\partial\phi} &= -m^2\phi \\ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} &= -(\partial_\mu\phi) \end{cases} \end{aligned}$$

Using the *laplacian* we find

$$\begin{aligned} \partial_\mu \partial_\mu &= \partial_1 \partial_1 + \partial_2 \partial_2 + \partial_3 \partial_3 - \partial_t \partial_t \\ &= \nabla_1 \nabla_1 + \nabla_2 \nabla_2 + \nabla_3 \nabla_3 - \nabla_t \nabla_t & \Rightarrow & \quad \partial_\mu(-\partial_\mu\phi) = -m^2\phi & \Rightarrow & \quad (\square - m^2)\phi = 0 \\ &= \nabla^2 - \partial_t^2 = \square & & \quad -\square\phi = -m^2\phi \end{aligned}$$

obtaining the *Klein-Gordon equation*.

$$\phi = e^{i(\vec{p}\cdot\vec{x} - \phi_0 t)} = e^{i P_\mu x_\mu} \quad \Rightarrow \quad (-\vec{P}^2 + P_0^2 - m^2) = 0 \quad \Rightarrow \quad P_0^2 = \vec{P}^2 + m^2$$

that describes the relativistic motion of a particle of mass  $m$ . Then the *free particle* may have a positive or negative energy value  $P_0$

$$P_0 = \pm \sqrt{\vec{P}^2 + m^2}$$



**Time evolution** If we also want to consider the time dependency as QM time evolution we may write

$$e^{-iEt} = e^{-i(m+K_E)t} \Rightarrow \phi = e^{-iEt}\psi \Rightarrow \partial_t \phi = \dot{\phi} = \left( (-im)e^{-iEt}\psi + e^{-iEt}\dot{\psi} \right)$$

and also the second time derivative

$$\begin{aligned} \partial_t^2 \phi = \ddot{\phi} &= \left( (-im)(-im)e^{-iEt}\psi + (-im)e^{-iEt}\dot{\psi} + (-im)e^{-iEt}\dot{\psi} + \underbrace{e^{-iEt}\ddot{\psi}} \right) \\ &= \left( -m^2 e^{-iEt}\psi - 2im e^{-iEt}\dot{\psi} \right) \end{aligned}$$

With the approximation to the first order derivative  $e^{-iEt}\ddot{\psi} = 0$ . Finally we find the solution that's something familiar to us

$$\begin{aligned} e^{-iEt} \left( \nabla^2 \psi + 2im\dot{\psi} + m^2\psi - m^2\psi \right) &= 0 \\ \Rightarrow i\dot{\psi} &= -\frac{i}{2m} \nabla^2 \psi \end{aligned} \quad \text{where} \quad \nabla^2 \phi = e^{-iEt} \nabla^2 \psi$$

that's the *Schrodinger equation* with  $\hbar = 1$  that is the non-relativistic limit of the *Klein-Gordon equation* that is a relativistic equation.

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The electromagnetic field theory approach starts with a postulate: *it is possible to derive the Maxwell's equations starting from a lagrangian*. From this postulate, we now ask to ourself which is that lagrangian that solves the Euler-Lagrangian equation (1.7)?

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} \quad \text{where} \quad \phi = \begin{cases} \phi & \text{scalar potential} \\ \vec{A} & \text{vector potential} \end{cases}$$

To do so, we start postulating that it may be

$$\mathcal{L} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) - \rho \phi + \vec{J} \cdot \vec{A} \quad \text{where} \quad \begin{cases} \phi & \mapsto \vec{E} = -\vec{\nabla} \phi - \frac{\partial \vec{A}}{\partial t} \\ \vec{A} & \mapsto \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$$

Where the magnetic field's components may be also written as  $B_i = \epsilon_{ijk} \partial_j A_k$ , using the Levi-Civita symbol. We can define a 4-vector that transforms following the Lorentz transformations as  $A_\mu = (\vec{A}, i\phi)$ .

### Maxwell's equations regression

$$\left\{ \begin{array}{l} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} \\ \vec{\nabla} \cdot \vec{E} = \rho \\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \end{array} \right. \quad \begin{array}{l} \text{current density - source} \\ \text{charge density - source} \end{array} \quad \Rightarrow \quad \text{continuity equation} \quad \frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$$

Now, similarly as before, we define a 4-vector that transforms following the Lorentz transformations as  $J_\mu = (\vec{J}, i\rho)$ . Which leads to the following lagrangian

$$\mathcal{L} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) - \rho \phi + J_\mu A_\mu$$

What's left is to check if our new postulates and definitions are compatible with what we already know.

(... calculations ...)

**Electromagnetic tensor** One we've our new lagrangian we may now introduce the *electromagnetic tensor*  $F_{\mu\nu}$ , as follow:

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix} \Rightarrow \text{antisymmetrical tensor}$$

and also

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \iff \begin{cases} B_i = \epsilon_{ijk} \partial_j A_k = (\vec{\nabla} \times \vec{A})_i \\ B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} \quad (\text{inverse relation}) \\ E_i = F_{i4} = -F_{4i} \end{cases}$$

$$\text{examples : } \begin{cases} F_{14} = \partial_1 A_4 - \partial_4 A_1 = i \partial_1 \phi - i \frac{\partial A_1}{\partial t} = i(-E_1) = -iE_1 \\ F_{12} = \partial_1 A_2 - \partial_2 A_1 = (\vec{\nabla} \times \vec{A})_3 = \begin{vmatrix} \partial_1 & \partial_2 \\ A_1 & A_2 \end{vmatrix}_3 = (\partial_1 A_2 - \partial_2 A_1)_k = B_3 \end{cases}$$

This leads to a compact way to write the Maxwell's equations, in only one tensorial equation:

$$\begin{cases} \partial_\mu F_{i\mu} = J_i \\ \partial_i F_{4i} = J_4 \end{cases} \Rightarrow \partial_\mu F_{\nu\mu} = J_\nu$$

**The electromagnetic tensor and its dual** Starting from the definition 1.7 of the electromagnetic tensor, we may see

$$F_{\mu\nu}F_{\mu\nu} = -F_{\mu\nu}F_{\nu\mu} = -T_\mu(F \cdot F) = 2(\vec{B}^2 - \vec{E}^2)$$

from which  $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + J_\mu A_\mu$

Trough the *antisymmetric tensor*  $T_{\mu\nu\lambda}$  it's possible to write the following identity:

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

The *dual tensor* of  $F$  is as well invariant for *cage transformations*

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho} = \begin{bmatrix} 0 & -i E_3 & i E_2 & B_1 \\ i E_3 & 0 & -i E_1 & B_2 \\ -i E_2 & i E_1 & 0 & B_3 \\ -B_1 & -B_2 & -B_3 & 0 \end{bmatrix}$$

Levi-Civita symbols:  
 3-D  $\epsilon_{\mu\nu\lambda} \rightarrow \epsilon_{123} = 1$   
 4-D  $\epsilon_{\mu\nu\lambda\rho} \rightarrow \epsilon_{1234} = -1$

$\rightsquigarrow$  An interesting exercise, calculate:  $F_{\mu\nu}\tilde{F}_{\mu\nu} = -4i \vec{E} \cdot \vec{B}$  and verify the equivalence.

Confusione con gli indici? Il Prof. considera, al fine del corso, equivalente scrivere:

$$A_\mu B^\mu \longleftrightarrow A_\mu B_\mu$$

quindi attenzione, soprattutto rispetto alla notazione usata nel corso di *General Relativity*.

We know want to demonstrate that the tensor  $F$  is *cage invariant*, so let's introduce the scalar quantity  $\Lambda$  and write the transformation:

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \longrightarrow F_{\mu\nu} = \partial_\mu A_\nu + \partial_\mu \partial_\nu \Lambda - \partial_\nu A_\mu - \partial_\nu \partial_\mu \Lambda$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu$$

We may say that  $F_{\mu\nu}$  “has the right combination of the magnetic and electron field”, to make two terms cancelled each other.

A generic vector changes under Lorentz transformations as

$$C_\mu \longrightarrow L_{\mu\nu} C_\nu$$

How does the  $F_{\mu\nu}$  tensor changes under Lorentz transformations?

$$F_{\mu\nu} \longrightarrow L_{\mu\mu'} L_{\nu\nu'} F_{\mu'\nu'}$$

**1.8 13 ottobre**

## 1.9 14 ottobre

1.10 18 ottobre

**1.11 20 ottobre**

**1.12 21 ottobre**



**1.13 25 ottobre**

**1.14 27 ottobre**

1.15 28 ottobre

## Chapter 2

# Novembre

Appunti delle lezioni del Prof. Polosa relative al mese di novembre 2022.

## Chapter 3

# Dicembre

Appunti delle lezioni del Prof. Polosa relative al mese di dicembre 2022.