Introduction to Quantum Field Theory

Notes re-written from lessons' attendance, 2022
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Corso di Laurea Magistrale in Fisica - Sapienza

Chapter 1

Ottobre

Appunti delle lezioni del Prof. Polosa relative al mese di ottobre 2022.

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The rod Given a 1-dimensional rod composed by N-particles, linked each others with a "spring", the hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\sum_{n=1}^{N}\Big[P_n^2 + \Omega^2(q_n - q_{n+1})^2 + \Omega_0^2 q_n^2\Big]$$

where the last term $\Omega_0^2 q_n^2$ is relative to the equilibrium position of the n-particle. The *periodic boundaries conditions* to $N \to \infty$ and $a \to 0$.

On the other side we can write the Newtonian equation as

$$\begin{aligned} H &= \frac{1}{2}\int_0^L dx\Big[p^2(x) + v^2\Big(\frac{\partial q(x)}{\partial x}\Big)\Big] \\ p(x) &= \dot{q}(x) \\ \ddot{q}(x) &= v^2\frac{\partial^2 q(x)}{\partial x^2} \end{aligned}$$

the solution inside the boundaries is

$$\ddot{q}_n = \Omega^2\Big(q_{n+1} + q_{n-1} - 2q_n\Big)$$

Normal modes or normal frequencies

$$\begin{aligned} q_n &= \sum_j e^{ijn}\frac{Q_j}{\sqrt{N}} \\ q(x) &= \frac{1}{\sqrt{a}}\sum_n e^{\frac{2\pi l}{Na}(na)}\frac{Q_j}{\sqrt{N}} = \frac{1}{\sqrt{a}}\sum_k e^{ikx}\frac{Q_k}{\sqrt{N}} \\ k &= \frac{2\pi l}{L} \\ \Rightarrow \quad q(x) &= \sum_k e^{ikx}\frac{Q_k}{\sqrt{Na}} = \sum_k e^{ikx}\frac{Q_k}{\sqrt{L}} \end{aligned}$$

Considering now the Newtonian equation, $p^2(x) = \dot{q}^2(x)$, $\sum_{n=1}^N e^{in(j-j')} = \delta_{j,j'}$ where $j = \frac{2\pi l}{N}$, we can move from the sum to the integral using the following relation $\sum_{n=1}^N \to \frac{1}{a}\int_0^L dx$ and this leads to $\int_0^L dx e^{i(k-k')x} = L\delta_{k,k'}$.

Somehow we may land on this following expression:

$$\frac{1}{L}\sum_{k,k'} L\,\delta_{k,k'}Q_k\dot{Q}_{k'} = \sum_k Q_k\dot{Q}_k = \sum_k |\dot{Q}_k|^2$$

To finally get a *total classical description*: a discrete sum on a numerable set, as follow

$$H = \frac{1}{2}\sum_k |\dot{Q}_k|^2 + k^2 v^2 |Q_k|^2$$

As before, notice that the sum $\sum_{n=1}^{N}$ for $L \to \infty$ became $\frac{L}{2\pi}\int dk$ and it admits waves. Extending this to 3-dimensional space, it became

$$\sum_{\vec{k}}(\dots) \quad (\text{when } L \to \infty) \quad \frac{V}{(2\pi)^3}\int d^3k$$

Quantum system: let's consider now a quantum system, a quantum description.
Postulate the followings:

$$\Big[q_l, p_n\Big] = i\,\delta_{l\,n} \qquad \Big[Q_l, P_n\Big] = i\,\delta_{ln}$$
$$\Big[q_l, q_n\Big] = 0 \qquad \Big[Q_l, Q_n\Big] = 0$$
$$\Big[p_l, p_n\Big] = 0 \qquad \Big[P_l, P_n\Big] = 0$$

Where natural units are applied:

$h = 1$

$c = 1$

$$\Rightarrow \qquad q_n^\dagger = q_n \quad , \quad Q_{-j} = Q_j^\dagger \quad , \quad P_{-j} = P_j^\dagger$$
$$\text{e.g.} \quad q_n^\dagger = \Big(\sum_n e^{inj}\frac{Q_j}{\sqrt{N}}\Big)^\dagger = \sum_j e^{-inj}\frac{Q_j^\dagger}{\sqrt{N}} = q_n$$

From the hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_j \Big[P_j\,P_j^\dagger + \omega_j^2\,Q_j\,Q_j^\dagger\Big]$$

and given the following operators, we find Q_j and P_j:

$$a_j = \frac{1}{\sqrt{2\omega_j}}\Big(\omega_j Q_j + i\,P_j^\dagger\Big) \qquad \qquad Q_j = \frac{1}{\sqrt{2\omega_j}}\Big(a_j + a_{-j}^\dagger\Big)$$
$$\Rightarrow$$
$$a_j = \frac{1}{\sqrt{2\omega_j}}\Big(\omega_j Q_j^\dagger - i\,P_j\Big) \qquad \qquad P_j = -i\Big(\frac{\omega_j}{2}\Big)^{\frac{1}{2}}\Big(a_{-j} - a_j^\dagger\Big)$$

and

keep in mind

$$\Big[a_j, a_{j'}\Big] = \delta_{j\,j'}$$

$$Q_j Q_j^\dagger = \frac{1}{2\omega_j}\Big(a_j a_j^\dagger + a_j a_{-j} + a_{-j}^\dagger a_j^\dagger + a_{-j}^\dagger a_{-j}\Big)$$
$$P_j P_j^\dagger = \Big(\frac{\omega_j}{2}\Big)^{\frac{1}{4}}\Big(a_{-j}a_{-j}^\dagger - a_{-j}a_j - a_j^\dagger a_{-j}^\dagger + a_j^\dagger a_j\Big)$$

With these lasts results we may write the $\mathcal{H}$ as

$$\mathcal{H} = \frac{1}{2}\sum_j \Big[P_j\,P_j^\dagger + \omega_j^2\,Q_j\,Q_j^\dagger\Big] = \frac{1}{2}\sum_j \omega_j\Big(a_j a_j^\dagger + a_j^\dagger a_j\Big)$$
$$= \frac{1}{2}\sum_j \omega_j\Big(2\,a_j^\dagger a_j + 1\Big) = \sum_j \omega_j\Big(a_j^\dagger a_j + \frac{1}{2}\Big)$$

Phonons description Phonons are bosons, they're used to describe the quantum problem of the rod. Phonons are like photons but in the world of sound instead of light. A n-particles system is defined with

$$|n_1, n_2, n_3, \dots\rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3}\dots|0\rangle$$

and for the 1-d oscillator, with energy E_n, is as follows

$$|n\rangle = (a^\dagger)^n\,|0\rangle$$
$$E_n = \hbar\omega(n + \frac{1}{2}) \overset{nu}{=} \omega(n + \frac{1}{2})$$

For the phonons is easy to *understand* which is the medium that make the transmission possible, but what about light? For the light, photons, the medium may also be the *vacuum*.

Filosofeggiamo un po' ora:

Particles are the excitation of the field

If you don't touch the piano it stays quiet, but if you play it it makes music ... song's particles.

The field is permanent.

Particles are not fixed, they live and die.

You cannot touch or see the field that you're studying, but you can see/detect the particle that pop out from the field.

Fields are NOT real but mathematical description of the world.

When you measure an energy it's always relative to an offset, a ground-state. Because you want the *vacuum* to be Lorentz invariant.

$$\Rightarrow \quad \Big(\mathcal{H} - E_0\Big) |0\rangle = 0$$

Given a general operator $\Theta(t)$ and its derivate $\dot{\Theta} = i\Big[\mathcal{H}, \Theta(t)\Big]$ so that:

$$\dot{a}(t) = i\Big[\mathcal{H}, a(t)\Big] = -i\,\omega a(t) \qquad \text{where} \quad \begin{aligned} \Big[a, \mathcal{H}\Big] &= \omega a \\ \Big[\mathcal{H}, a^\dagger\Big] &= \omega a^\dagger \end{aligned} \qquad \Rightarrow \quad a(t) = e^{-i\omega t} a(0)$$

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$$\begin{aligned}\phi(x) &= \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k V}}\Big(a_{\vec{k}}e^{ikx} + a^\dagger_{\vec{k}}e^{-ikx}\Big) \\ &= \sum_{\vec{k}} \frac{i}{2V}\Big[e^{i\,\vec{k}(\vec{x}\cdot\vec{y})} + e^{-i\,\vec{k}(\vec{x}\cdot\vec{y})}\Big]\end{aligned} \qquad \text{considering} \quad \begin{aligned}&\Big[a_k, a^\dagger_{\vec{k}}\Big] = \delta_{\bar{k},\bar{k}'} \\ &\Big[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)\Big] = i\,V\delta^3(\vec{x}-\vec{y})\end{aligned}$$

Da capire che senso ha
e contestualizzarlo $\Rightarrow$

$$\begin{aligned}k_\mu &= (\vec{k}, i\omega_{\vec{k}}) \\ x_\mu &= (\vec{x}, it) \\ k_\mu x_\mu &= \vec{k}\cdot\vec{x} = k_\mu k_\nu \delta_{\mu\,\nu} \\ k_\mu x_\mu &= \vec{k}\cdot\vec{x} - \omega_{\vec{k}}t\end{aligned}$$

When things go to infinity $\sum_{\vec{k}} \to \frac{V}{(2\pi)^3}\int d^3\vec{k}$ and remember that "*if things doesn't work there will be some volume V somewhere*". Creation and destruction operators are contained into the description of the field. The energy levels' order are given from the term $n\omega$ and you can forgot about the $\frac{1}{2}$.

Classical problem Given the coordinates $q_i(t)$ time-dependents, where $i = 1, 2, 3, \ldots, 3N$, we can write the system of the 2^o order derivate as follow

$$\begin{aligned}F_i &= m\ddot{q}_i \\ F_i &= -\frac{dV}{dq_i}\end{aligned} \qquad \text{given the initial conditions} \quad \begin{aligned}&q_i(t_0) \\ &\dot{q}_i(t_0)\end{aligned} \qquad \text{or given the boundary conditions} \quad \begin{aligned}&q_i(t_1)\,,\, q_i(t_2) \\ &\dot{q}_i(t_1)\,,\, \dot{q}_i(t_2)\end{aligned}$$

Action Functional The *Action Functional S*

$$S = \int_{t_1}^{t_2} dt\, L\Big(q_i(t), \dot{q}_i(t)\Big)$$

is defined such that a variation on the trajectory leads to a variation on S. So we can make a variables' transformation such that the new coordinates are the same as before plus a variational term

$$\begin{aligned}q_i(t) &\to q_i(t) + \delta q_i(t) \\ \dot{q}_i(t) &\to \dot{q}_i(t) + \delta\dot{q}_i(t) = \dot{q}_i(t) + \frac{d}{dt}\delta q_i(t)\end{aligned}$$

hence the action became

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2} L\Big(q_i(t) + \delta q_i(t), \dot{q}_i(t) + \frac{d}{dt}\delta q_i(t)\Big)dt - \int_{t_1}^{t_2} dt\, L\Big(q_i(t), \dot{q}_i(t)\Big)dt = \\ &= \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q_i}\delta q_i(t), \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i(t)\Big)dt = \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\Big)\delta q_i\, dt\end{aligned}$$

in the last step we used the boundary condition at t_1 and t_2, so that $\delta q_i(t_1) = \delta q_i(t_2) = 0$. The last step leads directly to the *lagrangian equation*, that referred to the following generic L in 3-D is:

$$\begin{aligned}\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) &= \frac{\partial L}{\partial q_i} \\ L &= \frac{1}{2}m\dot{\vec{q}}^2 - V(\vec{q})\end{aligned}$$

this leads to the Newton equation for the *free motion*:

$$\begin{aligned}F &= 0 \\ m\ddot{q}_i &= -\frac{\partial V}{\partial q_i} = F\end{aligned} \quad \Rightarrow \quad \begin{aligned}\ddot{\vec{q}} &= 0 \\ \dot{\vec{q}} &= \vec{w} \\ \vec{q} &= \vec{w}t + \vec{r}\end{aligned} \quad \Rightarrow \quad \begin{aligned}\vec{q}_1 &= \vec{w}t_1 + \vec{r} \\ \vec{q}_2 &= \vec{w}t_2 + \vec{r}\end{aligned}$$

so now we can find the Lagrangian depending on q_1, q_2 and t_1, t_2

$$\begin{aligned}\vec{w} &= \frac{\vec{q}_1\cdot\vec{q}_2}{t_1 - t_2} \\ \vec{q} &= \frac{\vec{q}_1\cdot\vec{q}_2}{t_1 - t_2}t + \vec{r}\end{aligned} \quad \Rightarrow \quad \begin{aligned}(t = t_1) &: \ (t_1 - t_2)\vec{q}_1 = (\vec{q}_1\cdot\vec{q}_2)t_1 + \vec{r}(t_1 - t_2) \\ (t = t_2) &: \ (t_1 - t_2)\vec{q}_2 = (\vec{q}_1\cdot\vec{q}_2)t_2 + \vec{r}(t_1 - t_2)\end{aligned} \quad \Rightarrow \quad \begin{aligned}\vec{q} &= \Big(\frac{\vec{q}_1\cdot\vec{q}_2}{t_1 - t_2}\Big)t + \frac{\vec{q}_2 t_1 - \vec{q}_1 t_2}{t_1 - t_2} \\ \dot{\vec{q}} &= \frac{\vec{q}_1\cdot\vec{q}_2}{t_1 - t_2}\end{aligned}$$

where the last two equations explicit the boundary conditions. The Lagrangian of a free motion became

$$L = \frac{1}{2}m\dot{\vec{q}}^2 = \frac{1}{2}m\Big(\frac{\vec{q}_1 \cdot \vec{q}_2}{t_1 - t_2}\Big)^2$$

And the minimal Action is written as follow and represents "the true trajectory"

$$\int_{t_1}^{t_2} dtL = \frac{1}{2}m\Big(\frac{\vec{q}_1 \cdot \vec{q}_2}{t_1 - t_2}\Big)^2 (t_1 - t_2) = S_{min}$$

"*The real motion is given by the minimum action.*"

Hamiltonian equation What happen with the generic lagrangian instead substituting the free motion path? Starting from the lagrangian equation:

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} &= 0 \qquad \Big[\text{where} \quad \dot{q}_i = p_i\Big] \\
\dot{q}_i \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \dot{q}_i \frac{\partial L}{\partial q_i} &= 0 \\
\frac{d}{dt}\Big(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\Big) - \underbrace{\ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} - \dot{q}_i \frac{\partial L}{\partial q_i}} &= 0 \qquad \Rightarrow \qquad \frac{d}{dt}\Big(\dot{q}_i p_i - L(q_i, \dot{q}_i)\Big) = 0 \\
\frac{d}{dt}\Big(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{d}{dt}\Big(L(q_i, \dot{q}_i)\Big) &= 0
\end{aligned}$$

Given the lagrangian as $L(q_i, \dot{q}_i)$ let's check its *invariance* when the coordinates change: the lagrangian must not change. For small ε

$$\begin{aligned}
q_i &\to q_i + \varepsilon_i f_i(q_1, \dots, q_i, \dots, q_N) \\
\dot{q}_i &\to \dot{q}_i + \varepsilon_i \frac{d}{dt} f_i(q_1, \dots, q_i, \dots, q_N)
\end{aligned}$$

We can write the variation of the lagrangian as

$$\begin{aligned}
&\sum \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = 0 \\
&\sum \underbrace{\frac{\partial L}{\partial q_i}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} f_i + \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{\tau_i} \frac{d}{dt} f_i = 0 \qquad \longrightarrow \qquad \frac{d}{dt}\sum_i p_i f_i = 0
\end{aligned}$$

Since the lagrangian is *invariant* for transformation, as we just saw, this leads to a *conservation* $\leftrightarrow$ *symmetry*. This concept is also known as *Noether theorem.*

Now ...something here not so clear...
about changing formalism from Classical Mechanic to Quantum Mechanic.

Hamiltonian and lagrangian density Since the beginning of the course we introduce, without saying, and use the hamiltonian density $\mathcal{H}$ instead of the hamiltonian H, that is defined as:

$$H = \int_0^L dx\, \mathcal{H}$$

as well as the lagrangian density $\mathcal{L}$ instead of the lagrangian L:

$$L = \int_0^L dx\, \mathcal{L} \qquad (\text{ ! controllare se è corretto ! })$$

Back to the *rod* using now in hamiltonian formalism

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}v^2(\partial_x\phi) = \frac{1}{2}p^2 + \frac{1}{2}v^2(\partial_x\phi)$$

$$\mathcal{L} = p\,\dot{q} - \mathcal{H} = \dot{\phi}\dot{\phi} - \Big(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}v^2(\partial_x\phi)^2\Big) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}v^2(\partial_x\phi)^2 \qquad \text{where} \quad p = \dot{\phi}$$

when the velocity is set as $v = \Omega a$ it depends strictly on the material of the medium. So when moving in the vacuum, and not along a rod, what happens? The velocity will be set to $v = c = 1$, as natural units are used, and "forget the mass" such that $m = 1$. The lagrangian density became consistent with this

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}v^2(\partial_x\phi)^2 \tag{1.1}$$

The definition of *field* goes with the definition of a proper *energy density.* The action became:

$$S = \int_{t_1}^{t_2} dt\, L = \int_{t_1}^{t_2} dt \int_0^L dx\, \mathcal{L}$$

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Continuing from the lagrangian 1.1

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}v^2(\partial_x\phi)^2 = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}v^2(\partial_x\phi)^2$$

we see the lagrangian as function of these variables $\mathcal{L} = \mathcal{L}(\phi,\, \partial_t\phi,\, \partial_x\phi)$ and we write the variation and the variational principle as follow

$$S = \int \mathrm{d}t \int \mathrm{d}x\, \mathcal{L} = \int \mathrm{d}t L \qquad\qquad \delta S = 0$$

$$\begin{aligned}
\delta S &= \int \mathrm{d}t \int \mathrm{d}x\, \mathcal{L}(\phi + \delta\phi\,, \dot{\phi} + \frac{\partial}{\partial t}\,\delta\phi\,, \partial_x\phi + \partial_x\,\delta\phi) - \int \mathrm{d}t \int \mathrm{d}x\, \mathcal{L}(\phi,\, \dot{\phi},\, \partial_x\phi) \\
&= \int \mathrm{d}t \int \mathrm{d}x \left\{ \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi - \frac{\partial\mathcal{L}}{\partial(\partial_t\,\phi)}\partial_t\,\delta\phi - \frac{\partial\mathcal{L}}{\partial(\partial_x\,\phi)}\partial_x\,\delta\phi \right\} \\
&= \int \mathrm{d}t \int \mathrm{d}x \left\{ \frac{\partial\mathcal{L}}{\partial\phi} - \partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\,\phi)} - \partial_x\frac{\partial\mathcal{L}}{\partial(\partial_x\,\phi)} \right\}\delta\phi
\end{aligned}$$

Hence we may set some boundary conditions at times t_1 and t_2, for the rod problem

$$\delta\phi\,(\vec{x}, t_1) = 0$$
$$\delta\phi\,(\vec{x}, t_2) = 0$$

Notice that the partial derivative of $\mathcal{L}$ with respect to ϕ is the following, and trough which we find the *Oiler-Lagrange equation* in *Quantum Field Theory*

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} + \partial_x\frac{\partial\mathcal{L}}{\partial(\partial_x\phi)} + \partial_y\frac{\partial\mathcal{L}}{\partial(\partial_y\phi)} + \partial_z\frac{\partial\mathcal{L}}{\partial(\partial_z\phi)} \qquad \Rightarrow \qquad \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial(\partial_\mu\phi)} = \frac{\partial\mathcal{L}}{\partial\phi}$$

with $\quad \partial_\mu = (\partial_1, \partial_2, \partial_3, \partial_4) = (\partial_t, \partial_x, \partial_y, \partial_z)$

The lagrangian density may be generalized in 3-D:

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_x\phi)^2 - \frac{1}{2}(\partial_y\phi)^2 - \frac{1}{2}(\partial_z\phi)^2 \\
=& \frac{1}{2}(\partial_\mu\phi)^2
\end{aligned} \qquad\qquad \text{where} \quad (\partial_\mu\phi)^2 = \partial_\mu\phi\,\partial_\mu\phi$$

This shorter notation implicitly implement the relativity, with $v = 1$ and natural units. If you want to describe a physical system, like the rod, you must instead keep it explicit with $v \neq 0$ and the Lagrangian will need one term more, such that

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$$

Solve the motion equation starting from the lagrangian above, we derive it w.r.t. ϕ and w.r.t. $\partial_\mu\phi$

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$$
$$\begin{cases} \frac{\partial\mathcal{L}}{\partial\phi} & = -m^2\phi \\ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} & = -(\partial_\mu\phi) \end{cases}$$

Using the *laplacian* we find

$$\begin{aligned}
\partial_\mu\,\partial_\mu &= \partial_1\,\partial_1 + \partial_2\,\partial_2 + \partial_3\,\partial_3 - \partial_t\,\partial_t \\
&= \nabla_1\,\nabla_1 + \nabla_2\,\nabla_2 + \nabla_3\,\nabla_3 - \nabla_t\,\nabla_t \\
&= \nabla^2 - \partial_t^2 = \Box
\end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \partial_\mu(-\partial_\mu\phi) &= -m^2\phi \\ -\Box\,\phi &= -m^2\phi \end{aligned} \qquad \Rightarrow \qquad (\Box - m^2)\,\phi = 0$$

obtaining the *Klein-Gordon equation.*

$$\phi = e^{i\,(\vec{p}\cdot\vec{x} - \phi_0\,t)} = e^{i\,P_\mu x_\mu} \quad \Rightarrow \quad \Big(-\vec{P}^2 + P_0^2 - m^2\Big) = 0 \quad \Rightarrow \quad P_0^2 = \vec{P}^2 + m^2$$

that describes the relativistic motion of a particle of mass m. Then the *free particle* may have a positive or negative energy value P_0

$$P_0 = \pm\sqrt{\vec{P}^2 + m^2}$$

Time evolution If we also want to consider the time dependency as QM time evolution we may write

$$e^{-i\,E\,t} = e^{-i\,(m+K_E)} \quad \Rightarrow \quad \phi = e^{-i\,m\,t}\psi \quad \Rightarrow \quad \partial_t\phi = \dot{\phi} = \Big((-i\,m)e^{-i\,m\,t}\psi + e^{-i\,m\,t}\dot{\psi}\Big)$$

and also the second time derivative

$$\begin{aligned}\partial_t^2\phi = \ddot{\phi} =& \Big((-i\,m)(-i\,m)e^{-i\,m\,t}\psi + (-i\,m)e^{-i\,m\,t}\dot{\psi} + (-i\,m)e^{-i\,m\,t}\dot{\psi} + \underbrace{e^{-i\,m\,t}\ddot{\psi}}\Big)\\ =& \Big(-m^2e^{-i\,m\,t}\psi - 2i\,me^{-i\,m\,t}\dot{\psi}\Big)\end{aligned}$$

With the approximation to the first order derivative $e^{-i\,m\,t}\ddot{\psi} = 0$. Finally we find the solution that's something familiar to us

$$\begin{aligned}& e^{-i\,m\,t}\Big(\nabla^2\psi + 2i\,m\dot{\psi} + m^2\psi - m^2\psi\Big) = 0 \\ & \Rightarrow \quad i\dot{\psi} = -\frac{i}{2m}\nabla^2\psi\end{aligned} \qquad \text{where} \quad \nabla^2\phi = e^{-i\,m\,t}\nabla^2\psi$$

that's the *Schrodinger equation* with $\hbar = 1$ that is the non-relativistic limit of the *Klein-Gordon equation* that is a relativistic equation.

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Chapter 2

Novembre

Appunti delle lezioni del Prof. Polosa relative al mese di novembre 2022.

Chapter 3

Dicembre

Appunti delle lezioni del Prof. Polosa relative al mese di dicembre 2022.