Introduction to Quantum Field Theory

Notes re-written from lessons' attendance, 2022
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Corso di Laurea Magistrale in Fisica - Sapienza

Chapter 1

Ottobre

Appunti delle lezioni del Prof. Polosa relative al mese di ottobre 2022.

1.1 The rod

Given a 1-dimensional rod composed by N-particles, linked each others with a "spring", the hamiltonian density is

$$\mathcal{H} = \frac{1}{2}\sum_{n=1}^{N}\Big[P_n^2 + \Omega^2(q_n - q_{n+1})^2 + \Omega_0^2 q_n^2\Big]$$

where the last term $\Omega_0^2 q_n^2$ is relative to the equilibrium position of the n-particle. The *periodic boundaries conditions* to $N \to \infty$ and $a \to 0$.

On the other side we can write the Newtonian equation as

$$\begin{aligned} H &= \frac{1}{2}\int_0^L dx\Big[p^2(x) + v^2\Big(\frac{\partial q(x)}{\partial x}\Big)\Big] \\ p(x) &= \dot{q}(x) \\ \ddot{q}(x) &= v^2\frac{\partial^2 q(x)}{\partial x^2} \end{aligned}$$

the solution inside the boundaries is

$$\ddot{q}_n = \Omega^2\Big(q_{n+1} + q_{n-1} - 2q_n\Big)$$

Normal modes or normal frequencies

$$\begin{aligned} q_n &= \sum_j e^{ijn}\frac{Q_j}{\sqrt{N}} \\ q(x) &= \frac{1}{\sqrt{a}}\sum_n e^{\frac{2\pi l}{Na}(na)}\frac{Q_j}{\sqrt{N}} = \frac{1}{\sqrt{a}}\sum_k e^{ikx}\frac{Q_k}{\sqrt{N}} \\ k &= \frac{2\pi l}{L} \\ \Rightarrow \quad q(x) &= \sum_k e^{ikx}\frac{Q_k}{\sqrt{Na}} = \sum_k e^{ikx}\frac{Q_k}{\sqrt{L}} \end{aligned}$$

Considering now the Newtonian equation, $p^2(x) = \dot{q}^2(x)$, $\sum_{n=1}^{N} e^{in(j-j')} = \delta_{j,j'}$ where $j = \frac{2\pi l}{N}$, we can move from the sum to the integral using the following relation $\sum_{n=1}^{N} \to \frac{1}{a}\int_0^L dx$ and this leads to $\int_0^L dx e^{i(k-k')x} = L\delta_{k,k'}$.

Somehow we may land on this following expression:

$$\frac{1}{L}\sum_{k,k'} L\,\delta_{k,k'}Q_k\dot{Q}_{k'} = \sum_k Q_k\dot{Q}_k = \sum_k |\dot{Q}_k|^2$$

To finally get a *total classical description*: a discrete sum on a numerable set, as follow

$$H = \frac{1}{2} \sum_k |\dot{Q}_k|^2 + k^2 v^2 |Q_k|^2$$

As before, notice that the sum $\sum_{n=1}^{N}$ for $L \to \infty$ became $\frac{L}{2\pi} \int dk$ and it admits waves. Extending this to 3-dimensional space, it became

$$\sum_{\vec{k}} (\dots) \quad \text{(when } L \to \infty) \quad \frac{V}{(2\pi)^3} \int d^3k$$

Quantum system: let's consider now a quantum system, a quantum description.
Postulate the followings:

$$\begin{aligned} &\Big[q_l, p_n\Big] = i\,\delta_{l\,n} \qquad &&\Big[Q_l, P_n\Big] = i\,\delta_{ln} \\ &\Big[q_l, q_n\Big] = 0 &&\Big[Q_l, Q_n\Big] = 0 \\ &\Big[p_l, p_n\Big] = 0 &&\Big[P_l, P_n\Big] = 0 \end{aligned}$$

Where natural units are applied:

$h = 1$

$c = 1$

$$\Rightarrow \qquad q_n^\dagger = q_n \quad , \quad Q_{-j} = Q_j^\dagger \quad , \quad P_{-j} = P_j^\dagger$$

$$\text{e.g.} \quad q_n^\dagger = \Big(\sum_n e^{inj} \frac{Q_j}{\sqrt{N}}\Big)^\dagger = \sum_j e^{-inj} \frac{Q_j^\dagger}{\sqrt{N}} = q_n$$

From the hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_j \Big[P_j\, P_j^\dagger + \omega_j^2\, Q_j\, Q_j^\dagger\Big]$$

and given the following operators, we find Q_j and P_j:

$$\begin{aligned} a_j &= \frac{1}{\sqrt{2\omega_j}} \Big(\omega_j Q_j + i\, P_j^\dagger\Big) \\ a_j &= \frac{1}{\sqrt{2\omega_j}} \Big(\omega_j Q_j^\dagger - i\, P_j\Big) \end{aligned} \quad \Rightarrow \quad \begin{aligned} Q_j &= \frac{1}{\sqrt{2\omega_j}} \Big(a_j + a_{-j}^\dagger\Big) \\ P_j &= -i \Big(\frac{\omega_j}{2}\Big)^{\frac{1}{2}} \Big(a_{-j} - a_j^\dagger\Big) \end{aligned}$$

and keep in mind

$$\Big[a_j, a_{j'}\Big] = \delta_{j\,j'}$$

$$\begin{aligned} Q_j Q_j^\dagger &= \frac{1}{2\omega_j} \Big(a_j a_j^\dagger + a_j a_{-j} + a_{-j}^\dagger a_j^\dagger + a_{-j}^\dagger a_{-j}\Big) \\ P_j P_j^\dagger &= \Big(\frac{\omega_j}{2}\Big)^{\frac{1}{4}} \Big(a_{-j} a_{-j}^\dagger - a_{-j} a_j - a_j^\dagger a_{-j}^\dagger + a_j^\dagger a_j\Big) \end{aligned}$$

With these lasts results we may write the $\mathcal{H}$ as

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \sum_j \Big[P_j\, P_j^\dagger + \omega_j^2\, Q_j\, Q_j^\dagger\Big] = \frac{1}{2} \sum_j \omega_j \Big(a_j a_j^\dagger + a_j^\dagger a_j\Big) \\ &= \frac{1}{2} \sum_j \omega_j \Big(2\, a_j^\dagger a_j + 1\Big) = \sum_j \omega_j \Big(a_j^\dagger a_j + \frac{1}{2}\Big) \end{aligned}$$

Phonons description Phonons are bosons, they're used to describe the quantum problem of the rod. Phonons are like photons but in the world of sound instead of light. A n-particles system is defined with

$$|n_1, n_2, n_3, \dots\rangle = (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} (a_3^\dagger)^{n_3} \dots |0\rangle$$

and for the 1-d oscillator, with energy E_n, is as follows

$$\begin{aligned} |n\rangle &= (a^\dagger)^n\, |0\rangle \\ E_n &= \hbar\omega(n + \frac{1}{2}) \stackrel{nu}{=} \omega(n + \frac{1}{2}) \end{aligned}$$

For the phonons is easy to *understand* which is the medium that make the transmission possible, but what about light? For the light, photons, the medium may also be the *vacuum*.

Filosofeggiamo un po' ora:

Particles are the excitation of the field

If you don't touch the piano it stays quiet, but if you play it it makes music ... song's particles.

The field is permanent.

Particles are not fixed, they live and die.

You cannot touch or see the field that you're studying, but you can see/detect the particle that pop out from the field.

Fields are NOT real but mathematical description of the world.

When you measure an energy it's always relative to an offset, a ground-state. Because you want the *vacuum* to be Lorentz invariant.

$$\Rightarrow \quad \left(\mathcal{H} - E_0\right)|0\rangle = 0$$

Chapter 2

Novembre

Appunti delle lezioni del Prof. Polosa relative al mese di novembre 2022.

Chapter 3

Dicembre

Appunti delle lezioni del Prof. Polosa relative al mese di dicembre 2022.