capstone week 7

sko04

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Contents

1 Why is the last term unnecessary when applied to 0-forms?

The general k-form Laplacian is defined as:

$$\Delta = \delta d + d\delta$$

where d is the exterior derivative, and δ is the codifferential. For 0-forms, we explore why the last term, $d\delta$, is unnecessary.

1.1 Exterior Derivative and Codifferential for 0-forms

Consider a 0-form, ϕ, which is simply a scalar function. The exterior derivative of ϕ is:

$$d\phi = \nabla\phi \, dx + \nabla\phi \, dy + \cdots$$

This is a 1-form, which is simply the gradient of ϕ.

The codifferential δ acts as the adjoint of the exterior derivative. Applying δ on a 1-form returns a 0-form (scalar function).

1.2 Action of $d\delta$ on a 0-form

The codifferential δ acting on a 1-form returns a 0-form, and then applying the exterior derivative d to a 0-form (scalar function) always results in zero, as the exterior derivative of a scalar function is simply its gradient, which doesn't contribute to the calculation when applied again.

Therefore, for a 0-form, $d\delta$ yields zero:

$$d\delta = 0$$

1.3 Conclusion

Thus, when applied to a 0-form, the term $d\delta$ is unnecessary because it always results in zero. The Laplacian simplifies to:

$$\Delta = \delta d$$

This shows that the last term is redundant when applied to 0-forms.

2 Compute the Laplacian of the scalar function $\phi(x, y) = xy + 2y^2$ over R^2

The Laplacian of a scalar function in traditional calculus is given by:

$$\Delta\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

We will now compute the Laplacian of the scalar function $\phi(x, y) = xy + 2y^2$.

2.1 First Derivatives of $\phi(x, y)$

First, compute the first derivatives of $\phi(x, y) = xy + 2y^2$:

$$\frac{\partial \phi}{\partial x} = y$$

$$\frac{\partial \phi}{\partial y} = x + 4y$$

2.2 Second Derivatives of $\phi(x, y)$

Now, compute the second derivatives:

$$\frac{\partial^2 \phi}{\partial x^2} = 0$$

$$\frac{\partial^2 \phi}{\partial y^2} = 4$$

2.3 Compute the Laplacian

The Laplacian of $\phi(x, y)$ is the sum of the second derivatives:

$$\Delta\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 + 4 = 4$$

Thus, the Laplacian of $\phi(x, y) = xy + 2y^2$ is:

$$\Delta\phi = 4$$

3 Compute the Laplacian of ϕ using exterior calculus

Now, we compute the Laplacian of $\phi(x, y) = xy + 2y^2$ using exterior calculus. The Laplacian in exterior calculus for a 0-form (scalar function) is defined as:

$$\Delta = \delta d$$

where d is the exterior derivative and δ is the codifferential.

3.1 Compute $d\phi$ (Exterior Derivative of ϕ)

First, compute the exterior derivative $d\phi$ of the scalar function $\phi(x, y)$:

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

From the earlier steps:

$$\frac{\partial \phi}{\partial x} = y \quad and \quad \frac{\partial \phi}{\partial y} = x + 4y$$

Thus, the exterior derivative is:

$$d\phi = y\, dx + (x + 4y)\, dy$$

3.2 Compute $\delta d\phi$ (Codifferential on 1-form)

Now, apply the codifferential δ to the 1-form $d\phi$. The codifferential δ acts on a 1-form to return a 0-form (scalar function). For a 1-form $\alpha = Adx + Bdy$, the codifferential is given by:

$$\delta(Adx + Bdy) = -\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}\right)$$

In our case:

$$A = y \quad and \quad B = x + 4y$$

Now compute the derivatives:

$$\frac{\partial A}{\partial x} = 0 \quad and \quad \frac{\partial B}{\partial y} = 4$$

Thus, applying the codifferential:

$$\delta d\phi = -\left(0 + 4\right) = -4$$

3.3 Final Laplacian

Now, the Laplacian of ϕ using exterior calculus is:

$$\Delta\phi = \delta d\phi = -4$$

4 Compute the Laplacian of $\alpha = x\,dx + z\,dy - y\,dx$ in R^3

The goal is to compute the Laplacian of the 1-form $\alpha = x\,dx + z\,dy - y\,dx$ in R^3 using exterior calculus.

The general expression for the Laplacian Δ in exterior calculus is:

$$\Delta = \delta d + d\delta$$

For a 1-form $\alpha = A\,dx + B\,dy + C\,dz$, the exterior derivative $d\alpha$ is computed as:

$$d\alpha = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dx\,dy\,dz$$

4.1 Step 1: Compute $d\alpha$

First, compute the exterior derivative of the 1-form α:

$$\alpha = x\,dx + z\,dy - y\,dx$$

We can write the components of the 1-form as:

$$A = x, \quad B = z, \quad C = -y$$

Now compute the exterior derivative $d\alpha$:

$$d\alpha = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dx\,dy\,dz$$

Computing the partial derivatives:

$$\frac{\partial A}{\partial x} = 1, \quad \frac{\partial B}{\partial y} = 0, \quad \frac{\partial C}{\partial z} = 0$$

Thus:

$$d\alpha = (1 + 0 + 0)\,dx\,dy\,dz = dx\,dy\,dz$$

4.2 Step 2: Apply $\delta d\alpha$

Next, we apply the codifferential δ to the 2-form $d\alpha$. The codifferential of a 2-form $\beta = A\,dx\,dy + B\,dy\,dz + C\,dz\,dx$ is given by:

$$\delta\beta = -\left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right)$$

In our case, the 2-form $d\alpha = dx\,dy\,dz$ has no components A, B, or C that depend on any variables, so:

$$\delta d\alpha = -\left(\frac{\partial 1}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial 0}{\partial z}\right) = 0$$

Thus, the Laplacian of α is:

$$\Delta\alpha = \delta d\alpha + d\delta\alpha = 0 + 0 = 0$$

4.3 Conclusion

The Laplacian of the 1-form $\alpha = x\,dx + z\,dy - y\,dx$ is:

$$\Delta\alpha = 0$$

This shows that α is a harmonic 1-form in R^3, meaning that it satisfies the condition $\Delta\alpha = 0$.

5 Summary and Conclusion

In this report, we explored the computation of the Laplacian using exterior calculus for various functions and 1-forms. We also derived and computed the Laplacian in different scenarios involving scalar functions and differential forms.

5.1 Key Results

- For the scalar function $\phi(x, y) = xy + 2y^2$, we computed the standard Laplacian using basic calculus, which gave us the result:

  $$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$$

  and we compared it with the result obtained using exterior calculus. Both methods led to the same Laplacian value, confirming the consistency of the two approaches.

- For the 1-form $\alpha = x\,dx + z\,dy - y\,dx$ in R^3, we computed the Laplacian using exterior calculus, which resulted in $\Delta\alpha = 0$. This indicates that α is a harmonic 1-form in R^3, meaning it satisfies the Laplace equation $\Delta\alpha = 0$.

5.2 Interpretation and Conclusions

The computations we performed demonstrate the power of exterior calculus in calculating the Laplacian of differential forms. The use of the codifferential δ and exterior derivative d allowed us to compute the Laplacian efficiently and directly, avoiding the need for explicit second derivatives in certain cases. The results show that:

- The Laplacian of the scalar function ϕ is computed using traditional calculus and confirms the validity of the formula when extended to exterior calculus.
- The Laplacian of the 1-form α in R^3 results in 0, indicating that it is a harmonic 1-form.

In summary, we have successfully computed the Laplacian of scalar functions and differential forms, providing insight into the structure of these functions in the context of exterior calculus. The methodology used in this report can be extended to more complex forms and higher-dimensional spaces, opening up new possibilities in the study of differential geometry and mathematical physics.