Contents

Lectures 18,19,20

Math301

Fall 2020

1 Power Series REV

it's a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + ...$$

it's called a power series about $x = a$

1. The series converges only when x =a
2. the series converges for all x
3. convergent on an interval called interval of convergence such that it's convergent when $|x-a| < R$ and divergent when $|x-a| > R$

1.1 How to get R

R or the **radius of convergence** can be calculated using the following formulas:

$$R = \frac{1}{\lim_{n\to\infty} \frac{c_{n+1}}{c_n}} \qquad \text{or} \qquad R = \frac{1}{\lim_{n\to\infty} \sqrt{|c_n|}} \tag{1}$$

1.2 Interval of convergence

the power series is said to be convergent on the following interval : $[-R+a, R+a]$ and the interval can either be closed or open in any ending so we have to test all the sitiuations

Rule 1. if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

then $c_n = \frac{f^{(n)}(a)}{n!}$

so;

1. if $\sum = 0 \to c_n = 0$
2. if $\sum a_n(x-a)^n = \sum b_n(x-1)^n \to a_n = b_n$

1.3 Term by term summation

$$\sum_{n=0}^{\infty} a_n(x-a)^n + \sum_{n=0}^{\infty} b_n(x-1)^n = \sum_{n=0}^{\infty}[a_n+b_n](x-a)^n \tag{2}$$

this is applicable if

1. the two $\sum$s start and end with the same values

1.4 term by term differentiation

this is basically: the derivative on the $\sum$s is the $\sum$ of the derivatives

Note 1. The Radii of convergence of both $f(x), f'(x)$ are the same

1.5 series expansions of basic functions

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

1.6 Re-indexing sums

if you drop down in on side you rise up in the other one

$$\sum_{n=k}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+k} \tag{3}$$

$$\sum_{n=k}^{\infty} a_n = \sum_{n=k+h}^{\infty} a_{n-h} \tag{4}$$

2 Solving ODEs using power series method

the method is so simple, you assume that $y = \sum_{n=0}^{\infty} a_n x^n$ and this leads to having $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ **note the n in all sums**
and then we solve them together. to solve and sum them; they all must have the same n and all Xs must have the same power. In order to achieve that do the following :

1. try to change powers of x in all sums to be the same
2. then change the n of them by either method of the following
   (a) re-indexing
   (b) expanding some terms of the series

all we want to have to solve the ODE is the a_n (the coeffiecients in the series) and we can get them by substitution. Of course we can't get theme explicitly. You will eventually get a recurrence relation to solve. it can be solved either by substuting in it multiple times to have the final coeffiecients or by notice.

$$C_{n+2} = -\frac{(2n-1)}{(n+2)(n+1)} C_n \rightarrow n = 0, 1, 2, 3, 4, ... \tag{5}$$

Put $n = 0$: $\quad c_2 = \frac{-1}{1 \cdot 2} c_0$

Put $n = 1$: $\quad c_3 = \frac{1}{2 \cdot 3} c_1$

Put $n = 2$: $\quad c_4 = \frac{3}{3 \cdot 4} c_2 = -\frac{3}{1 \cdot 2 \cdot 3 \cdot 4} c_0 = -\frac{3}{4!} c_0$

Put $n = 3$: $\quad c_5 = \frac{5}{4 \cdot 5} c_3 = \frac{1 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 5} c_1 = \frac{1 \cdot 5}{5!} c_1$

Put $n = 4$: $\quad c_6 = \frac{7}{5 \cdot 6} c_4 = -\frac{3 \cdot 7}{4! \, 5 \cdot 6} c_0 = -\frac{3 \cdot 7}{6!} c_0$

Put $n = 5$: $\quad c_7 = \frac{9}{6 \cdot 7} c_5 = \frac{1 \cdot 5 \cdot 9}{5! \, 6 \cdot 7} c_1 = \frac{1 \cdot 5 \cdot 9}{7!} c_1$

Put $n = 6$: $\quad c_8 = \frac{11}{7 \cdot 8} c_6 = -\frac{3 \cdot 7 \cdot 11}{8!} c_0$

Put $n = 7$: $\quad c_9 = \frac{13}{8 \cdot 9} c_7 = \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!} c_1$

the pattern can be detected, as we have two patterns one for the even part and one for the odd

If the formula of c_n in terms of n cannot be discerned from the pattern, notice that,

- If n is **even**, applying the recurrence relation repeatedly (each time decreasing n by 2) gives

$$\begin{aligned} c_n &= \frac{(2n-5)}{n(n-1)} \frac{(2n-9)}{(n-2)(n-3)} c_{n-4} \\ &= \frac{(2n-5)}{n(n-1)} \frac{(2n-9)}{(n-2)(n-3)} \frac{(2n-13)}{(n-4)(n-5)} c_{n-6} \\ &\vdots \\ &= \frac{(2n-5)(2n-9).\dots.3.(-1)}{n(n-1).\dots.2.1} c_0, \quad (c_2 = -c_0/2!) \end{aligned}$$

- If n is **odd**, arguing as above, we obtain

$$c_n = \frac{(2n-5)(2n-9).\dots.5.1}{n(n-1).\dots.3.2} c_1, \quad (c_3 = c_1/3!)$$

Therefore,

- If $n \geq 2$ is even, $c_n = -\frac{(2n-5)(2n-9).\dots.3.1}{n!} c_0$
- If $n \geq 3$ is odd, $c_n = \frac{(2n-5)(2n-9).\dots.5.1}{n!} c_1$

The preceding formulas for c_n imply that, for all $k \geq 1$

- $c_{2k} = -\frac{1.3.....(4k-5)}{(2k)!} c_0$
- $c_{2k+1} = \frac{1.5.....(4k-3)}{(2k+1)!} c_1$

Therefore,

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=0}^{\infty} c_{2n} x^{2n} + \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1} \\
&= \left(c_0 + c_2 x^2 + \sum_{n=2}^{\infty} c_{2n} x^{2n} \right) + \left(c_1 x + \sum_{n=1}^{\infty} c_{2n+1} x^{2n+1} \right) \\
&= c_0 \left(1 - \frac{1}{2!} x^2 - \sum_{n=2}^{\infty} \frac{1.3.\ldots.(4n-5)}{(2n)!} x^{2n} \right) + c_1 \left(x + \sum_{n=1}^{\infty} \frac{1.5.\ldots.(4k-3)}{(2n+1)!} x^{2n+1} \right) \\
&= c_0 y_0 + c_1 y_1
\end{aligned}$$

and every choice of c_0 and c_1 contributes a different solution of the ODE.

Note that the functions y_0 and y_1 form a basis for the solution of the ODE (E)

$$y_0 = 1 - \tfrac{1}{2!} x^2 - \tfrac{1.3}{4!} x^4 - \tfrac{1.3.7}{6!} x^6 - \ldots \quad \text{and} \quad y_1 = x + \tfrac{1}{3!} x^3 + \tfrac{1.5}{5!} x^5 + \tfrac{1.5.9}{7!} x^7 + \ldots$$

cannot be readily expressed in terms of elementary functions.

Note 2. Sometimes the functions y_1, y_2 can be basic functions so you have to recall the power series expansions of them to have the solution in the standard form.

Note 3. in the last example we expanded the series because we can't substitute in the value of the coeffiecients unless the series starts from n=1 or n= any number that is indicated above in the question.

Exercise 1. Solve the Following ODE using power series method.

$$y' - x^2 y = 0 \tag{6}$$

Solution 1.

$$let \tag{7}$$

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \tag{8}$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} - x^2 (\sum_{n=0}^{\infty}) = 0 \tag{9}$$

$$\tag{10}$$