Surface Integrals

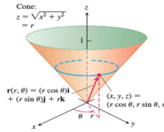
Parametrization

Any graph function can be a parametrization

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k} = (x, y, f(x,y)) \qquad (x,y) \in D.$$

Cone Parametrization

Consider the cone of equation $z = \sqrt{x^2+y^2}$.



The cone is a graph and may be readily parametrized as

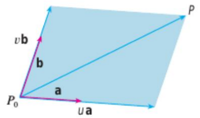
$$\mathbf{r}(x,y) = (x, y, \sqrt{x^2+y^2}), \qquad (x,y) \in \mathbf{R}^2.$$

OR

$$\mathbf{r}(r,\theta) = (r\cos\theta, r\sin\theta, r), \qquad r \ge 0,\ 0 \le \theta \le 2\pi.$$

the upper limit is changed per problem

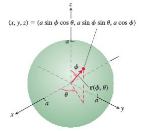
Plane Parametrization



$$\mathbf{r}(u,v) = \overrightarrow{OP_0} + u\mathbf{a} + v\mathbf{b},$$

where $u,\ v \in \mathbb{R}$.

Sphere Parametrization

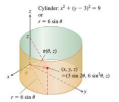


$$\mathbf{r}(\theta,\phi) = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi)$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.

Cylinder Parametrization

Consider the cylinder of equation $x^2 + (y-3)^2 = 9$.



$$x^2 + (y-3)^2 = 9 \iff x^2 + y^2 - 6y = 0 \iff r = 6\sin\theta$$

$x = r\cos\theta = 6\sin\theta\cos\theta = 3\sin 2\theta$

$y = r\sin\theta = 6\sin^2\theta, \quad 0 \le \theta \le \pi$

$z = z$, and a parametrization, therefore, is

$$\mathbf{r}(\theta,z) = (3\sin 2\theta, 6\sin^2\theta, z), \quad 0 \le \theta \le \pi, \quad z \in \mathbb{R}$$

Find a tangent plane to a parametrized surface

EASY

you find a normal vector using the previous thread

and BAM you have a tangent plane

a(x-x0) + b(y-y0) + c (z-z0) = 0 ????????

- a,b,c are the comp. of the normal vect
- x0 y0 z0 are any point that belongs to that plane

Find a noraml to a parametrized surface

Recall that a normal to a level surface is obtained as follows

$F(x,y,z) = k$

$\nabla F(a,b,c) = (F_x(a,b,c), F_y(a,b,c), F_z(a,b,c))$

Or if we thought of the level surface as a function of two variables

$z = f(x,y)$

$\mathbf{N} = (-f_x(a,b), -f_y(a,b), 1)$

as if you differentiated the follwing point (x,y,f(x,y))

FAKESSSS

How ? — Normal

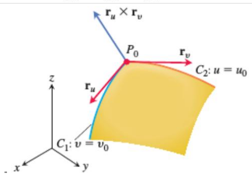
$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v.$

Holding $v = v_0$ constant, produces a curve $\mathbf{r}(u, v_0)$ on the surface whose tangent vector is $\dfrac{\partial \mathbf{r}}{\partial u}(u, v_0)$ denoted $\mathbf{r}_u$.

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

Holding $u = u_0$ constant, produces a curve $\mathbf{r}(u_0, v)$ on the surface whose tangent vector is $\dfrac{\partial \mathbf{r}}{\partial v}(u_0, v)$ denoted $\mathbf{r}_v$.

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$



Unit Normal

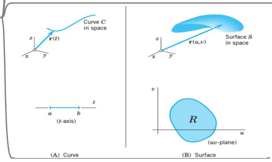
$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

Notions on surfaces

A surface can be obtained as the graph of a function of two variables
Z = f (x , y) is a surface

Also it can be a level curve for a function of three variables.
F(x,y,z) = k , is a surface

Not all surfaces can be obtained as graphs or level curves

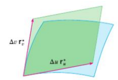


Surface Integrals

Let S be a surface parametrized by r(u,v)

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA.$$

Why cross product ?



$$\Delta S_{ij} \approx |\mathbf{r}_u^* \times \mathbf{r}_v^*|\Delta u \Delta v$$

$$\iint_S f(x,y,z)\,dS = \lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{i=1}^{n}\sum_{j=1}^{m} f(\mathbf{r}(u_i^*, v_j^*))|\mathbf{r}_u^* \times \mathbf{r}_v^*|\Delta u \Delta v.$$

Where u,v belongs to D

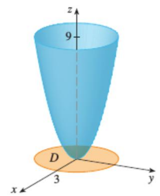
Surface Area ?

It's simply a surface integral while holding the function to be f(x,y,z) = 1

$$\text{Area}(S) = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

VERRRRYY IMP EXAMPLE

Example
Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.



A parametrization of the surface is

$$\mathbf{r}(x,y) = (x, y, x^2+y^2)$$

where $(x,y) \in D = \{(x,y) | x^2 + y^2 \le 9\}$.

$\mathbf{r}_x = (1, 0, 2x)$ and $\mathbf{r}_y = (0, 1, 2y)$

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$$

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + 4x^2 + 4y^2}$$

$$\text{Area} = \iint_D \sqrt{1 + 4x^2 + 4y^2}\,dydx = \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\,rdrd\theta = \frac{\pi}{6}(37\sqrt{37} - 1).$$

What if I used another Parametrization ?

- r(r,theta)
- **THEN the dA will be drd0 and not rdrd0**