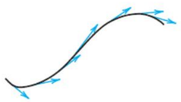
Vectors

Vector Functions

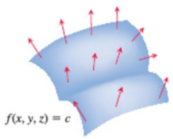
- a vector function r of a real varible t is a rule that assigns a vector to each t in the domain
  - so it a assigns a point (vector) to each value t
- if the codomain of r is (R2) or (R3) then as t takes all the values in the domain (D) . The range is a curve
  - either in 2D
  - or in space

Vector Fields

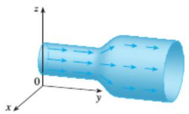
- a vector field is a function F that asigns a vector to each point in its domain D
- Examples
  - 4 The field of tangent vectors to a curve is shown



  - 5 The field of gradient vectors ∇f on a surface of equation $f(x, y, z) = c$

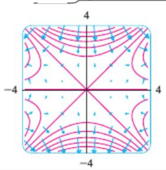


  - 6 Velocity Field $\mathbf{V}(x, y, z)$ in fluid flow



Gradient Vector Fields

- if F is a multivaribale function we can get it's grad. vector field from the derivatives of it
- grad. vect. field = nabla F
  - ∇f
- Example



  - note that the grad field is orthagonal to the level curves of the 2var function
  - Also The length/magnitude of the grad field is bigger when the level curves are closer together
  - closer level curves indicate a steeper graph; faster rate of increase

Conservative vector field

- A vector field is conservative if it can be written as a gradient vector field
- Or if there exist a potential function f that if you get its grad vector field it'll be equal to that field
- $\mathbf{F} = \nabla f.$

Curl of a vector field

- it measuers the convergence or stability of a vector field
- When the curl = 0 -> The field is conservative (has no curls)
  - **Proof**

$$\operatorname{curl} \nabla f = \nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i}$$

$$+ \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k}$$

$$= \mathbf{0} \qquad \text{by Clairaut's Theorem}$$

  - A zero curl means also that if you a point in anywhere in the field it will move in the same direction
- $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
- Curl is a VECTOR FIELD!
- Insights
  - Consider F to be the velocity vector field of a fluid flowing
    - The curl(F(a,b,c)) is associated with the rotation of the particles near the point (a,b,c) around an axis points in the direction of curl F
    - The magnitude of F = how quickly the particles orbit the axis
    - So, If curl=0 the fluid is free from rotation
    - **Irrotaional**

Divergence of a vector field

- The result of the symbolic dot product
  - $\operatorname{div} \mathbf{F} = \nabla . \mathbf{F}$
- Make a comparison between grad vect field of a function and a the deiv of a vect field
- $\operatorname{div} \operatorname{curl} \mathbf{F} = 0.$
  - We use this assumption to prove that some vector field can't be a curl of another vector field

$$\begin{aligned} \operatorname{div} \operatorname{curl} \mathbf{F} &= \nabla \cdot (\nabla \times \mathbf{F}) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\ &= \frac{\partial^2 R}{\partial x\, \partial y} - \frac{\partial^2 Q}{\partial x\, \partial z} + \frac{\partial^2 P}{\partial y\, \partial z} - \frac{\partial^2 R}{\partial y\, \partial x} + \frac{\partial^2 Q}{\partial z\, \partial x} - \frac{\partial^2 P}{\partial z\, \partial y} \\ &= 0 \end{aligned}$$

  the terms cancel in pairs by Clairaut's Theorem.

  In the preceding example, it has been shown that $\operatorname{div} \mathbf{F} = z + xz \neq 0$. If $\mathbf{F} = \operatorname{curl} \mathbf{G}$ then by the preceding question, $\operatorname{div} \mathbf{F} = \operatorname{div} \operatorname{curl} \mathbf{G} = 0$, a contradiction. Therefore, $\mathbf{F}$ is not the curl of another vector field.

    - Note that AxA.B = 0 (cross product of two parallel vects =0)
- Insights
  - Consider F to be the velocity vector field of a fluid flowing
    - The div(F(a,b,c)) is the net rate of change of the mass (w.r.t. time) of the fluid at location (a,b,c) per unit volume
    - div Measuers the tendency of the fluid to diverge from the point (a,b,c)
    - So, If div=0 the fluid can't be compressed
    - **Incompressible**