Green's Theorem

Circulation density at a point

Top:

$\mathbf{F}(x, y+\Delta y).(-\mathbf{i})\Delta x = -M(x, y+\Delta y)\Delta x$

Bottom:

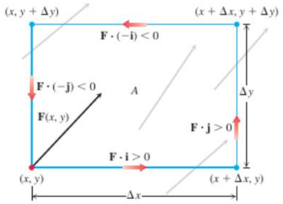
$\mathbf{F}(x, y).\mathbf{i}\Delta x = M(x, y)\Delta x$

Right:

$\mathbf{F}(x+\Delta x, y).\mathbf{j}\Delta y = N(x+\Delta x, y)\Delta y$

Left:

$\mathbf{F}(x, y).(-\mathbf{j})\Delta y = -N(x, y)\Delta y$



- Circulation = \int F.T ds
- Same as we did with the flux
- Total Circulation

$$[N(x+\Delta x, y) - N(x, y)]\,\Delta y - [M(x, y+\Delta y) - M(x, y)]\,\Delta x$$
$$\approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y - \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$
$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x\Delta y \approx (\text{curl } \mathbf{F}.\mathbf{k})\,\Delta x\Delta y$$

- **The Circulation and Curl**
  - curl F.k = circ. per unit area
  - curl F.K = circ. / area
  - circ = curl F.K . area

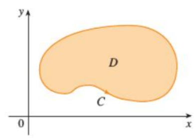
Green's Theorem

Circulation and Curl

The value of a line integral around a loop can be obtained as a double integral over the region enclosed by the loop

Let C be piecewise smooth, simple closed curve which is oriented positively (that is, counterclock wise) and let D be the region enclosed by C.

positively oriented = the area is to my left always when i'm going through



\iint curl . K . da

$$\oint_C Mdx + Ndy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} \Longrightarrow \text{curl } \mathbf{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

$$\oint_C \mathbf{F}.d\mathbf{r} = \iint_D \text{curl } \mathbf{F}.\mathbf{k}dA$$

Flux and Divergence

Green's Theorem also gives the flux of $\mathbf{F}$ across C as the integral of the flux density over the region enclosed by C:

$$\oint_C Mdy - Ndx = \iint_D \left[\frac{\partial M}{\partial x} - \frac{\partial}{\partial y}(-N)\right] dA = \iint_D \text{div } \mathbf{F} dA$$

Areas As line integrals

Area as a line integral: $A = \dfrac{1}{2}\oint_C (xdy - ydx).$

(i) $M = 0$ and $N = x$, we get $A = \iint_D dA = \oint_C xdy$

(ii) $M = -y$ and $N = 0$, we get $A = \iint_D dA = -\oint_C ydx$

Area of an ellipse

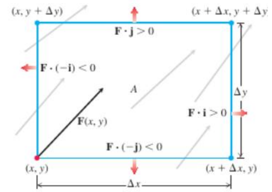
Find the area enclosed by the ellipse C: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

It is straight forward to verify that the point $(a\cos t, b\sin t)$ is on the ellipse for any t. The point traces the entire ellipse as t takes its values in the interval between 0 and 2π. Thus, a parametrization of the ellipse (assuming positive orientation) is

$$x(t) = a\cos t, \quad y(t) = b\sin t, \quad 0 \le t \le 2\pi.$$

$$\begin{aligned} Area &= \frac{1}{2}\oint_C (xdy - ydx) \\ &= \frac{1}{2}\int_0^{2\pi} [(a\cos t)(b\cos t) - (b\sin t)(-a\sin t)]\,dt \\ &= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab. \end{aligned}$$

Flux density at a point



Flux across top edge:

$\mathbf{F}(x, y+\Delta y).\mathbf{j}\Delta x = N(x, y+\Delta y)\Delta x$

Flux across bottom edge:

$\mathbf{F}(x, y).(-\mathbf{j})\Delta x = -N(x, y)\Delta x$

Flux across right edge:

$\mathbf{F}(x+\Delta x, y).\mathbf{i}\Delta y = M(x+\Delta x, y)\Delta y$

Flux across left edge:

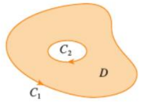
$\mathbf{F}(x, y).(-\mathbf{i})\Delta y = -M(x, y)\Delta y$

- We know that the flux = \int F . n ds
- We will do so for the above rectangle (for each side)
- and then collect all the terms together
- Total Flux

$$[N(x, y+\Delta y) - N(x, y)]\,\Delta x + [M(x+\Delta x, y) - M(x, y)]\,\Delta y$$
$$\approx \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x + \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$$
$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x\Delta y = (\text{div } \mathbf{F})\,\Delta x\Delta y$$

- **The Divergence and Flux**
  - div = flux per unit area
  - div = flux / area
  - flux = div . area

Extended Version of Green's Thm



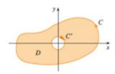
works when C = C1 union C2

Show that for any positively oriented simple closed path C which encloses the origin,

$$\oint_C -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 2\pi.$$

Green's Theorem cannot be directly applied to the region enclosed by the curve C since $M = -\frac{y}{x^2+y^2}$ and $N = \frac{x}{x^2+y^2}$ are undefined at the origin. **Thus, it is a good idea to avoid the origin by isolating it in a circle C' centered at the origin and of some (small) radius a.**

Now, one has no more trouble to apply Green's Theorem in its extended version to the region D bounded by $C \cup C'$, oriented positvely according to Green's theorem assumptions.



$$\Rightarrow \oint_{C\cup C'} Mdx + Ndy = \oint_C Mdx + Ndy + \oint_{C'} Mdx + Ndy = 0$$

Why zero ? However, $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$

Therefore, $\oint_C Mdx + Ndy = -\oint_{C'} Mdx + Ndy.$

However, C' may be parametrized as $x(t) = a\cos t$, $y(t) = -a\sin t$, $0 \le t \le 2\pi$. Thus,

$$\oint_{C'} Mdx + Ndy = 1/a^2 \int_0^{2\pi} \left(-a^2\sin^2 t - a^2\cos^2 t\right) dt = -2\pi.$$

Hence, $\oint_C Mdx + Ndy = -\oint_{C'} Mdx + Ndy = -(-2\pi) = 2\pi.$

So we basically calculated green's thm. for a circle rather than an entire region