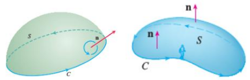
Big Integration Theorems

Notions on Surfaces



A positively oriented surface is the one that defined using the right hand rule

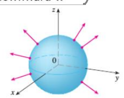
The region they enclose is always to your left

How to define a normal ?

Positive orientation w.r.t upward **n**

Positive orientation w.r.t downward **n**

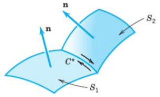
Closed Surfaces



Positive orientation is w.r.t. outward **n**

positive orientation is always outward and hence negative is inward

Piecewise smooth surfaces



A smooth surface is the one that has a unique tangent plane at each point

piecewise surfaces is the union of smooth surfaces

Surface integrals of vector fields

So far we know how to get the flux across a curve C

What if we want the flux across a surface S ?

We use the formula of surface integrals along with flux's one

$$\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|dA$$

$\iint_S \mathbf{F}.\mathbf{n}dS$ is called the flux of $\mathbf{F}$ across S (in the **direction** of $\mathbf{n}$).

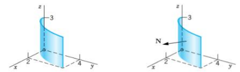
$$\iint_S \mathbf{F}.\mathbf{n}dS = \iint_D \mathbf{F}(\mathbf{r}(u,v)).\mathbf{N}\,dA$$

small n is unit normal

Some problems can be solved using the unit normal while others using the normal.Look at the following two examples

Examples

Compute the flux of water through the parabolic cylinder S: $y = x^2$, $0 \le x \le 2, 0 \le z \le 3$, shown in the figure, if the velocity vector is $\mathbf{F} = (3z^2, 6, 6xz)$, speed being measured in meters/sec. (Water density is $\rho = 1\text{ g/cm}^3 = 1\text{ ton/m}^3$).



The flux of water through S is given by $\iint_S \mathbf{F}.\mathbf{n}\,dS$.

A parametrization of the given parabolic cylinder is $\mathbf{r}(x,z) = (x, x^2, z)$ with $0 \le x \le 2, 0 \le z \le 3$. Thus, a normal vector to S is

$$\mathbf{N} = \mathbf{r}_x \times \mathbf{r}_z = (1, 2x, 0) \times (0,0,1) = (2x, -1, 0).$$

and $\mathbf{F}.\mathbf{N} = 6xz^2 - 6$.

$$\text{Flux} = \iint_S \mathbf{F}.d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)).\mathbf{N}\,dA = \int_0^2\int_0^3 6xz^2 - 6dzdx = 72\text{ m}^3/\text{sec}$$

Therefore, the **net rate of flow** across S (in the direction of $\mathbf{N}$) is

$$72\text{ m}^3/\text{sec} \times 1\text{ ton/ m}^3 = 72\text{ ton/ sec}.$$

Suppose an electric charge Q is located at the origin and consider the electric field of Q, $\mathbf{E}(\mathbf{r}) = \frac{\varepsilon Q}{|\mathbf{r}|^3}\mathbf{r}$. where $\mathbf{r} = (x,y,z)$ is a position vector. Show that the (outward) electric flux of $\mathbf{E}$ through any sphere S centered at the origin is $\iint_S \mathbf{E}.d\mathbf{S} = 4\pi\varepsilon Q$.

Solution

Notice that a unit normal to the sphere at the point $\mathbf{r} = (x,y,z)$ is $\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|}$. Thus, $\mathbf{E}.\mathbf{n} = \frac{\varepsilon Q}{|\mathbf{r}|^2}$. If the radius of the sphere is R then

$$\iint_S \mathbf{E}.d\mathbf{S} = \iint_S \frac{\varepsilon Q}{|\mathbf{r}|^2}dS = \iint_S \frac{\varepsilon Q}{R^2}dS = \frac{\varepsilon Q}{R^2}\iint_S dS \qquad (*)$$

However, using the parametrization of the sphere

$$\mathbf{r}(\theta,\phi) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi)$$

where, $0 \le \theta \le 2\pi, 0 \le \phi \le \pi$, it is easy to show that

$$\text{Area of sphere} = \iint_S dS = \int_0^{2\pi}\int_0^{\pi} \underbrace{|\mathbf{r}_\theta \times \mathbf{r}_\phi|}_{=R^2\sin\phi}\,d\phi d\theta = 4\pi R^2.$$

Substitution in $(*)$ yields the result.

Divergence Thm.

known as Gauss Thm

Converting the flux to a triple integral

Let $\mathbf{F}$ be a vector field whose components have continuous firs partial derivatives, and let S be a piecewise smooth oriented closed surface.

The flux of $\mathbf{F}$ across S in the direction of the outward unit normal field $\mathbf{n}$ equals to triple integral of the divergence of $\mathbf{F}$ over the region D enclosed by the surface:

$$\iiint_D \text{div } \mathbf{F}dV = \iint_S \mathbf{F}.d\mathbf{S}$$

Please memorize the integrals and the limits to know what is needed

For Ex. The problem may request \iint limits_S F.dS this is the divergence thm. without thinking

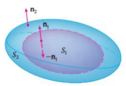
Remarks

Divergence Theorem (also known as Gauss' Theorem) is the three dimensional version of Green's Theorem in its normal form which we recall

$$\oint_C \mathbf{F}.\mathbf{n}ds = \oint_C Mdy - Ndx = \iint_D M_x - (-N)_y = \iint_D \text{div } \mathbf{F}dA,$$

Extended Div. Thm

Consider the region E that lies between the closed surfaces S_1 and S_2, where S_1 lies inside S_2. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be outward normals of S_1 and S_2. Then the boundary surface of E is $S = S_1 \cup S_2$ and its outward normal $\mathbf{n}$ is given by $\mathbf{n} = -\mathbf{n}_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2



The Divergence Theorem holds on S, namely,

$$\iiint_E \text{div } \mathbf{F} = \iint_S \mathbf{F}.d\mathbf{S}$$

Remark

If all flux is with respect to outward pointing normal, the flux across S is related to the flux across S_1 and S_2 as follows

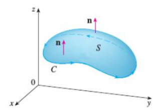
Flux across S = **Flux across** S_2 − **Flux across** S_1

This relationship is obtained as follows:

$$\iint_S \mathbf{F}.d\mathbf{S} = \iint_{S_1\cup S_2}\mathbf{F}.d\mathbf{S} = \iint_{S_1}\mathbf{F}.\mathbf{n}dS + \iint_{S_2}\mathbf{F}.\mathbf{n}dS = \iint_{S_2}\mathbf{F}.\mathbf{n}_2dS - \iint_{S_1}\mathbf{F}.\mathbf{n}_1dS$$

Stoke's Thm.

It's Basically Green's but for 3D



Let F be a vector field whose components have cont. partial derivatives

S is an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation according to n

$$\oint_C \mathbf{F}.d\mathbf{r} = \iint_S \text{curl } \mathbf{F}.\mathbf{n}dS = \iint_S \text{curl } \mathbf{F}.d\mathbf{S}$$

dS is surface area

Remeber Green's Thm ?

$$\oint_C \mathbf{F}.d\mathbf{r} = \iint_D \text{curl } \mathbf{F}.\mathbf{k}dA,$$

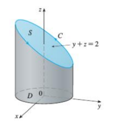
dA is area

Ex.

Evaluate $\oint_C \mathbf{F}.d\mathbf{r}$, where $\mathbf{F}(x,y,z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ and C is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$ (orientation shown).

Solution

We may, of course, parametrize the ellipse C and compute a line integral but instead we illustrate the use of Stokes' theorem.



S may be parametized as follows:

$$\mathbf{r}(x,y) = (x, y, 2-y),\ (x,y) \in D = \{(x,y)|x^2+y^2 \le 1\}$$

A normal to S is $\mathbf{N} = \mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} \times (\mathbf{j} - \mathbf{k}) = \mathbf{j} + \mathbf{k}$ (which is compatible with the orientation of C since $\mathbf{N}$ points above the plane $y + z = 2$). Is is easily shown that curl $\mathbf{F} = (1 + 2y)\mathbf{k}$. By Stokes' theorem,

$$\oint_C \mathbf{F}.d\mathbf{r} = \iint_S \text{curl } \mathbf{F}.\mathbf{n}dS = \iint_D \text{curl } \mathbf{F}.\mathbf{N}dA = \iint_D (1+2y)dA = \int_0^{2\pi}\int_0^1 (1 + 2r\sin\theta)rdrd\theta = \cdots = \pi.$$