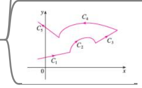
Line Integrals

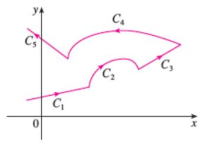
Introduction
- We knew how to do douple and multiple integration on SCALAR functions of several varibales
- Line Integrals arise when you try to integrate a vector function/field along a path or a curve

Curves
- What is a curve ? — It's the range of a vector function of real variable r(t)
- If an oriented curve from a to b is given as the range of a vector function then we say r(t) is the **PARAMETRIZATION** of the curve C
- a curve is **smooth** if it is parametrized by differentiable vector function r(t) that is r'(t) != 0
  - it can be piecewise smooth as well
  - made of finite smooth curves



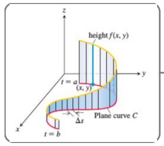
- a curve is **simple** if it doesn't cross itself
- A **closed** curve starts and ends at the same point; r(a)=r(b); a=b
- Parametrization
  - a line segment
    - The oriented line segment is the curve $\mathbf{r}(t) = \overrightarrow{OP} + t\overrightarrow{PQ}$ where $0 \le t \le 1$.
  - A curve
    - you just draw the graph of the functions and find the intersection
      - you can use **Polar** coordinates, **Cylindrical**, or even **Spherical**
  - How to invert the direction of that parametrization ?
    - look at the worksheet
  - **NOTE!** a parametrization is a **realtion** and a **direction**(**orientation**)

General Notes
- Any parametrization of the same curve yields the same answer
  - $\int_C \mathbf{F}.d\mathbf{r} = \int_C \mathbf{F}.d\mathbf{r}^*$ for any two parametizations $\mathbf{r}(t)$, $a \le t \le b$ and $\mathbf{r}^*(t)$, $c \le t \le d$ of C.
  - $\int_C f(x,y,z)ds = \int_C f(x,y,z)ds^*$ for any two parametizations $\mathbf{r}(t)$, $a \le t \le b$ and $\mathbf{r}^*(t)$, $c \le t \le d$ of C.
- if a curve is a union of some piecwise smooth curves then



  - $\int_C f(x,y,z)ds = \int_{C_1} f(x,y,z)ds + \cdots + \int_{C_n} f(x,y,z)ds$
  - $\int_C \mathbf{F}.d\mathbf{r} = \int_{C_1} \mathbf{F}.d\mathbf{r} + \int_{C_2} \mathbf{F}.d\mathbf{r} + \cdots + \int_{C_n} \mathbf{F}.d\mathbf{r}$

Line Integrals of **Scalar** Functions
- $$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\,|\mathbf{r}'(t)|dt = \int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt$$
  - $ds \approx |d\mathbf{r}(t)| \approx |\mathbf{r}'(t)|dt$
    - think of s as the distance so that r'(t) = ds/dt and ds = r' dt
- In order to compute a line integral of a scalar function you need to know
  - the **parametrization** of the curve
  - the **derivative** of that r(t)
  - the **magnitude** of that derivative
- Interpretations
  - Physical
    - Think of f(x,y,z) as the linear **density** of a wire
    - then f(x,y,z)ds is the **mass** of an **infintisimal** portion of that wire
    - and \int{f(x,y,z)ds} = the total **mass** of the wire
  - Geometrical
    - Think of f as a two varibale function and C as a plane curve (a 2D one)
    - as shown in the figure and from the definition of scalar line integral; \int f(x,y)ds
    - f(x,y) is basically the z component of the graph which is the height



    - ds is simply the width of an infintisimal rectabgle
    - the line integral is the **area of the cyllindrical surface**
      - حيب مساحة بطانية واقفة

Line Integrals of **Vector** Functions
- just the same as scalar ones but with vectors
- this is a dot product
- $\int_C \mathbf{F}.d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)).\mathbf{r}'(t)dt.$
  - $\int_C \mathbf{F}.d\mathbf{r} = \int_C Mdx + Ndy + Pdz = \int_a^b \left(Mx' + Ny' + Pz'\right) dt$
- The **Differential** form — **Mdx+Ndy+Pdz**
- Interpretation
  - Physical
    - Let F be the force vector field. Then **F.dr** = **the work done** when an object is moved(displaced) along the curve **C** by a displacment vector **dr**
    - The line integral then is the **Total Work** of displacing an object from point **r(a)** to point **r(b)**