Visualizing Integer Partitions

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Presentation Outline

We study congruence properties of the partition function.

1. What are partitions? Objects from combinatorics, number theory, representation theory and physics.
2. Why do we care? **Generating functions, useful analytic identities, building blocks for counting discrete objects**.
3. How do we visualize them? `vispart`
4. Future work.

Integer Partitions

Fix $n \in \mathbb{N}_0$ an integer. We say that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{N}^k$ is a *partition* of n with $k \in \mathbb{N}_0$ parts if

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$$

and

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k = n.$$

We let $p(n)$ denote the total number of partitions of n, where we use the convention $p(0) = 1$ and $p(n) = 0$ for n negative.

The partitions of 4

We note that $p(4) = 5$ by enumerating the partitions of 4:

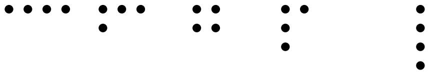
$$\begin{aligned} 4 &= 4 \\ &= 3 + 1 \\ &= 2 + 2 \\ &= 2 + 1 + 1 \\ &= 1 + 1 + 1 + 1 \end{aligned}$$

We see that there is only one partition with a single part: (4), two partitions with two parts: (3,1) and (2,2), one partition with three parts: (2,1,1), and a single partition with four parts: (1,1,1,1).

Ferrers Diagrams

We can represents integer partitions geometrically using Ferrers diagrams,

$$(4) \qquad (3,1) \quad (2,2) \quad (2,1,1) \quad (1,1,1,1)$$



Or using Young diagrams
TODO: Add diagrams here
An alternative way of thinking of partitions is using an algebraic tool.

Formal power series

In general, we say that a sequence $(a_n)_{n\geq 0}$ has generating series $A(q) \in \mathbb{N}[[q]]$ if

$$A(q) = \sum_{n\geq 0} a_n q^n,$$

where the variable x is variable only in the formal sense (usual algebraic properties hold, nothing analytic).

Generating series for integer partitions

If $P(q)$ is the generating series for integer partitions, then $P(q)$ has the closed form

$$P(q) = \sum_{n \geq 0} p(n) q^n = \prod_{i \geq 1} \frac{1}{1 - q^i}.$$

We can think of the right hand side as building blocks for each type of part in a partition. That is,

$$\frac{1}{1 - q^i} = \sum_{j \geq 0} q^{ij} = 1 + q^i + q^{2i} + q^{3i} + \cdots$$

represents the number of ways we could pick no parts of size i, the number of ways we could pick exactly one part of size i, exactly two parts of size i, three parts of size i,...

Applications

The benefits of being able to visualize integer partitions are apparent when considering how many theorems can be proved using them.

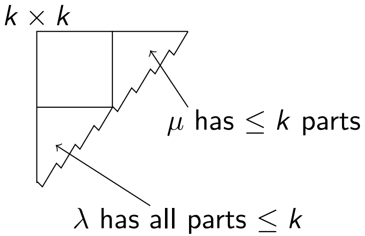
Dufree squares

Theorem

The generating series for integer partitions is equivalent to

$$P(q) = \prod_{i\geq 1} \frac{1}{1-q^i} = \sum_{k\geq 1} x^{k^2} \prod_{i=1}^{k} \frac{1}{(1-q^i)^2}.$$

Proof.



Self-conjugate partitions

Definition

We say that a partition is self-conjugate partition if its reflection in $y = -x$ line is itself.

Theorem

The set of self-conjugate partitions is in bijection with the set of all partitions with distinct odd parts. In particular, if $s(n)$ is the number of self conjugate partitions of n, then

$$\sum_{n \geq 0} s(n) q^n = \prod_{i \geq 0} (1 + q^{2i+1}).$$

Self-conjugate partitions

TODO: Add examples from our package of the partitions. Ideal examples: (5,3,1), (3). We should colour partitions so that bijection is clear.

`vispart`

Our dependencies are

- `partitions` - to list all partitions up to a particular value.
- `grid` - in order to draw the partitions in a way that scales up.

Our package allows one to

- Draw any integer partition.
  - Using either Ferrers diagrams or Young tableaux
  - Colour specific boxes; helps communicate specific properties of partitions.
  - Conjugation mapping is builtin.
- Draw all integer partitions of n for a fixed n.

TODO: Demonstration with partitions of 4

Benefits of `vispart`

The benefits of `vispart` are as follows:

- LaTeX's difficult to use for creating multiple partitions diagrams at once as LaTeX is generally not used for computation.
- To our knowledge, there is no implementation of the operation of conjugation within LaTeX.
- LaTeX's implementation of the partition diagrams does not do input handling.
- Maple, Matlab, Mathematica do not have built-in functionality for visualizing them to the best of our knowledge.

Future work

In the study of symmetric functions, interest lies in assigning numbers within the boxes of a Young tableau.

- This helps communicate concepts from the theory of irreducible representations of the symmetric group S_n.
- Provides an interpretation for **Littlewood-Richardson rule** which describes how to take a product of symmetric functions.

Improvements to the integer partition list generation.

- Currently, `partitions` uses a recursive algorithm which relies on the recurrence:

$$p(n, m) = \sum_{k=1}^{m} p(n-k, k).$$