Visualizing Integer Partitions

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Presentation Outline

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4. Future work.

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Integer Partitions

Fix $n \in \mathbb{N}_0$ an integer. We say that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{N}^k$ is a *partition* of n with $k \in \mathbb{N}_0$ parts if

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We let $p(n)$ denote the total number of partitions of n, where we use the convention $p(0) = 1$ and $p(n) = 0$ for n negative.

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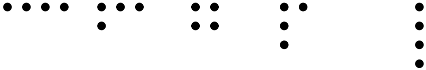
Alternatively, we could visualize integer partitions.

Visualizing integer partitions

We can represents integer partitions geometrically using Ferrers diagrams,

$$(4) \qquad (3,1) \qquad (2,2) \qquad (2,1,1) \qquad (1,1,1,1)$$



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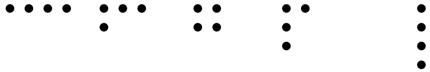

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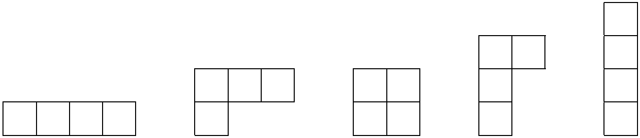

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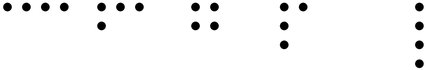

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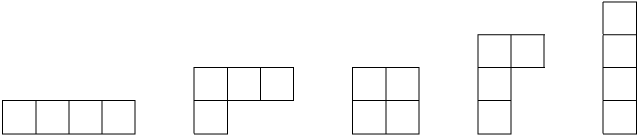

An alternative way of thinking of partitions is algebraic via formal power series.

Formal power series

In general, we say that a sequence $(a_n)_{n\geq 0}$ has generating series $A(q) \in \mathbb{N}[[q]]$ if

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- addition,
- multiplication,
- formal derivative and integral.

But we may not substitute without being careful.

Generating series for integer partitions

If $P(q)$ is the generating series for integer partitions, then $P(q)$ has the closed form

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We can think of the right hand side as building blocks for each type of part in a partition. That is,

$$\frac{1}{1 - q^i} = \sum_{j \geq 0} q^{ij} = 1 + q^i + q^{2i} + q^{3i} + \cdots$$

represents the number of ways we could pick no parts of size i, the number of ways we could pick exactly one part of size i, exactly two parts of size i, three parts of size i,...

Applications

The benefits of being able to visualize integer partitions are apparent when considering how many theorems can be proved by visualization.

Dufree squares

Theorem

The generating series for integer partitions is equivalent to

$$P(q) = \prod_{i \geq 1} \frac{1}{1-q^i} = 1 + \sum_{k \geq 1} q^{k^2} \prod_{i=1}^{k} \frac{1}{(1-q^i)^2}.$$

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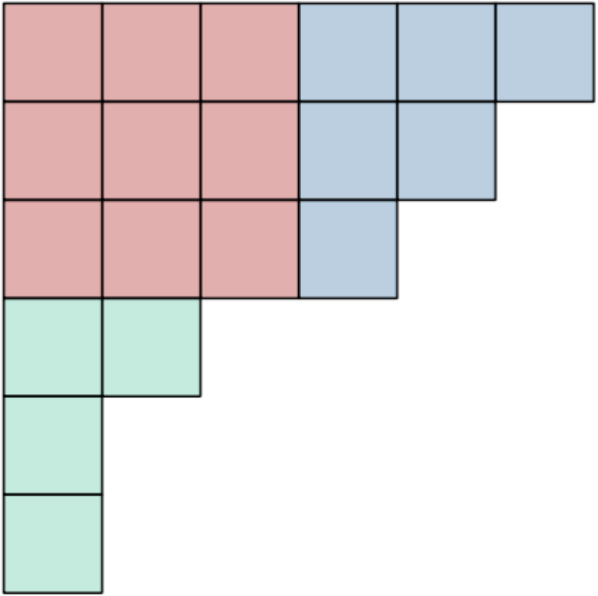
Proof by picture

Dufree square example



Self-conjugate partitions

Definition

We say that a partition is self-conjugate partition if its reflection in $y = -x$ line is itself.

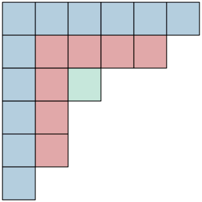
Self-conjugate partitions

Theorem

The set of self-conjugate partitions is in bijection with the set of all partitions with distinct odd parts. In particular, if $s(n)$ is the number of self conjugate partitions of n, then

$$S(q) := \sum_{n\geq 0} s(n)q^n = \prod_{i\geq 0}(1+q^{2i+1}).$$

Self-conjugate partitions becomes odd with distinct parts



(a) $(6, 5, 3, 2, 2, 1)$

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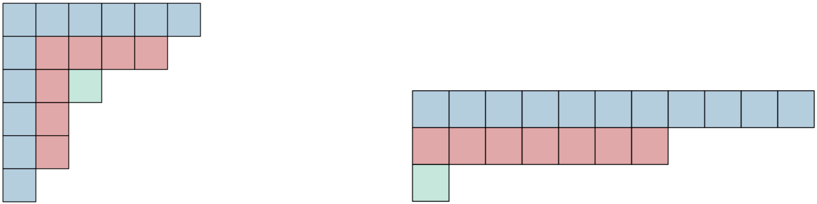

(a) $(6, 5, 3, 2, 2, 1)$ (b) $(11, 7, 1)$

Figure: A self-conjugate partition with its corresponding image under the bijection.

`vispart`

Our dependencies are

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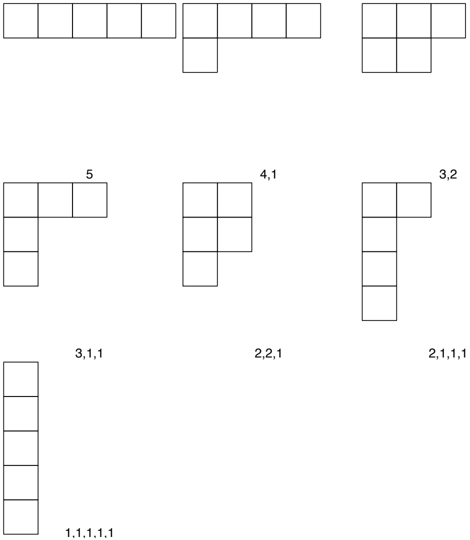
  - Conjugation mapping is builtin.
- Draw all integer partitions of n for a fixed n.

The partitions of 5



Alternatives to `vispart`

In contrast to `vispart`,

- LaTeX's implementation of the partition diagrams does not do input handling.
- LaTeX is difficult to use for creating multiple partitions diagrams at once as LaTeX is generally not used for computation.
- To our knowledge, there is no implementation of the operation of conjugation within LaTeX.
- Maple, Matlab, Mathematica do not have built-in functionality for visualizing them to the best of our knowledge.

Future work

Improvements to the integer partition list generation.

- Currently, `partitions` uses a recursive algorithm which relies on the recurrence:

$$p(n, m) = \sum_{k=1}^{m} p(n-k, k),$$

  where $p(n, m)$ is the number of partitions of n in which all parts are of size at most m.

- This algorithm is known to have $O(n \log n)$ time complexity and $O(n^2)$ space complexity.

Future work - math

In the study of symmetric functions, interest lies in assigning numbers within the boxes of a Young diagram in a way that satisfy a particular rule. Such configurations are known as **Young tableaux**.

- This helps communicate concepts from the theory of irreducible representations of the symmetric group S_n.
- Provides an interpretation for **Littlewood-Richardson rule** which describes how to take a product of two symmetric functions (known as Schur functions).

That's all folks!