Formulas for `multilayer_surface_plasmon`

1 Background

The layers are parallel to the $x - y$ plane. "Up" is defined as the direction of increasing z. I am calculating SPPs propagating along the x-axis, and uniform in the y-direction ($k_y = 0$). The N layers are numbered $0, 1, \ldots, (N-1)$, where layers 0 and $N-1$ are infinitely thick. Layer 0 is on the bottom ($z \ll 0$) and layer $(N-1)$ is on top ($z \gg 0$). Layer m has (AC) permittivity ϵ_{xm} in the x-direction and ϵ_{zm} in the z-direction, and likewise permeability μ_{ym}. (Although we're not assuming the permittivity and permeability are isotropic, we *are* assuming that the off-diagonal elements like ϵ_{xz} are zero.)

This whole document is in SI units. The permeabilities and permittivities are unitless (relative to ϵ_0 or μ_0), except for ϵ_0 and μ_0 themselves. All wavenumbers and wavevectors are angular wavenumbers and angular wavevectors.

2 Formulas

k_x is the complex in-plane wavevector. We don't know a priori what it is; we need to figure that out.

The wavevectors k_{zm} are defined in terms of k_x by:

$$k_{zm} = \pm\sqrt{\omega^2 \mu_{ym} \epsilon_{xm}/c^2 - (\epsilon_{xm}/\epsilon_{zm}) k_x^2} \text{ (choose the root with nonnegative imaginary part)}$$

[What if k_{zm} is real? Then choose one arbitrarily, it doesn't matter. The only place where this would matter is the semi-infinite layers, in which case k_{zm} should never be real or else the wave is not confined.]

When I write down a formula for $\vec{E}(z)$ or $\vec{H}(z)$, it is implicit that you should multiply it by $e^{ik_x x - i\omega t}$ and then take the real part.

I'll base the discussion on mainly around the H field, because it's a scalar (points in the y-direction), unlike the electric field which has two components. I may occasionally use $H(z)$ as a synonym of $H_y(z)$. Layer m has:

$$H_y(z) = H_{m\uparrow} e^{ik_{zm}(z - z_{\text{bottom of m}})} + H_{m\downarrow} e^{ik_{zm}(z_{\text{top of m}} - z)}$$

$$E_x(z) = E_{xm\uparrow} e^{ik_{zm}(z - z_{\text{bottom of m}})} + E_{xm\downarrow} e^{ik_{zm}(z_{\text{top of m}} - z)}$$
$$E_z(z) = E_{zm\uparrow} e^{ik_{zm}(z - z_{\text{bottom of m}})} + E_{zm\downarrow} e^{ik_{zm}(z_{\text{top of m}} - z)}$$

where

$$E_{xm\uparrow} = H_{m\uparrow} k_{zm}/(\omega \epsilon_{xm} \epsilon_0) \quad , \quad E_{xm\downarrow} = -H_{m\downarrow} k_{zm}/(\omega \epsilon_{xm} \epsilon_0)$$
$$E_{zm\uparrow} = -H_{m\uparrow} k_x/(\omega \epsilon_{zm} \epsilon_0) \quad , \quad E_{zm\downarrow} = -H_{m\downarrow} k_x/(\omega \epsilon_{zm} \epsilon_0)$$

(except that $X_{m\uparrow}=0$ in layer 0 (which has no bottom) and $X_{m\downarrow}=0$ in layer $(N-1)$ (which has no top)). $X_{m\uparrow}$ describes the component that decays to 0 as z becomes more positive, while $X_{m\downarrow}$ describes the component that decays to 0 as z becomes more negative.

Check Faraday's law:

$$-i\omega(\mu_y\mu_0 H) = \partial_t B \stackrel{?}{=} -\nabla\times E \quad\rightarrow\quad H \stackrel{?}{=} (-i/\mu_0\mu_y\omega)\nabla\times E$$

$$H_y \stackrel{?}{=} \frac{-i}{\mu_0\mu_{ym}\omega}\left(\partial_z E_x - \partial_x E_z\right) = \frac{H_{m\uparrow}e^{ik_{zm}(z-z_{\text{bottom of m}})}}{\mu_0\mu_{ym}\omega^2\epsilon_0}\left(\frac{k_{zm}^2}{\epsilon_{xm}}+\frac{k_x^2}{\epsilon_{zm}}\right)+\frac{H_{m\downarrow}e^{ik_{zm}(z_{\text{top of m}}-z)}}{\mu_0\mu_{ym}\omega^2\epsilon_0}\left(\frac{k_{zm}^2}{\epsilon_{xm}}+\frac{k_x^2}{\epsilon_{zm}}\right)$$

It works!!

Check Ampere's law:

$$\nabla\times H \stackrel{?}{=} -i\omega(\epsilon\epsilon_0)E \quad\rightarrow\quad E \stackrel{?}{=} (i/(\omega\epsilon\epsilon_0))\nabla\times H$$

$$E_z \stackrel{?}{=} \frac{i}{\omega\epsilon_{zm}\epsilon_0}\partial_x H_y = \frac{i}{\omega\epsilon_{zm}\epsilon_0}\left(ik_x H_{m\uparrow}e^{ik_{zm}(z-z_{\text{bottom of m}})} + ik_x H_{m\downarrow}e^{ik_{zm}(z_{\text{top of m}}-z)}\right)$$

$$E_x \stackrel{?}{=} \frac{-i}{\omega\epsilon_{xm}\epsilon_0}\partial_z H_y = \frac{-i}{\omega\epsilon_{xm}\epsilon_0}\left(ik_{zm} H_{m\uparrow}e^{ik_{zm}(z-z_{\text{bottom of m}})} - ik_{zm} H_{m\downarrow}e^{ik_{zm}(z_{\text{top of m}}-z)}\right)$$

It works!!

Check Gauss's law:

$$\nabla\cdot\vec{D} \stackrel{?}{=} 0 \quad\rightarrow\quad \epsilon_x\partial_x E_x + \epsilon_z\partial_z E_z \stackrel{?}{=} 0$$

$$0 \stackrel{?}{=} \left(i\epsilon_{xm}k_x E_{xm\uparrow}e^{ik_{zm}(z-z_{\text{bottom of m}})} + i\epsilon_{xm}k_x E_{xm\downarrow}e^{ik_{zm}(z_{\text{top of m}}-z)}\right)$$
$$+\left(i\epsilon_{zm}k_{zm} E_{zm\uparrow}e^{ik_{zm}(z-z_{\text{bottom of m}})} - i\epsilon_{zm}k_{zm} E_{zm\downarrow}e^{ik_{zm}(z_{\text{top of m}}-z)}\right)$$

It works!!

3 Solving strategy

I *guess* k_x. Then calculate all the k_{zm}'s. I have $2N-2$ unknowns (all the $H_{m\uparrow}, H_{m\downarrow}$, except $H_{0\downarrow}$ and $H_{N-1,\uparrow}$) and I have $(N-1)$ boundaries, each of which give me two continuity equations (E_x is continuous and $\epsilon_z E_z$ is continuous. I think H_y is continuous too, but I think that's redundant with the other two.) Therefore this is a system of linear equations with a nontrivial solution. There is an associated matrix whose determinant must be zero. I can calculate that determinant for every possible k_x, and use that as a figure of merit to home in on the actual solution.

E_x is continuous:

$$E_{x0\downarrow} = E_{x1\uparrow} + E_{x1\downarrow}e^{ik_{z1}d_1}$$
$$E_{x1\uparrow}e^{ik_{z1}d_1} + E_{x1\downarrow} = E_{x2\uparrow} + E_{x2\downarrow}e^{ik_{z2}d_2}$$
$$\cdots$$
$$E_{x(N-2)\uparrow}e^{ik_{z(N-2)}d_{(N-2)}} + E_{x(N-2)\downarrow} = E_{x(N-1)\uparrow}$$

$\epsilon_z E_z$ is continuous:

$$\epsilon_{z0} E_{z0\downarrow} = \epsilon_{z1} E_{z1\uparrow} + \epsilon_{z1} E_{z1\downarrow} e^{i k_{z1} d_1}$$

$$\epsilon_{z1} E_{z1\uparrow} e^{i k_{z1} d_1} + \epsilon_{z1} E_{z1\downarrow} = \epsilon_{z2} E_{z2\uparrow} + \epsilon_{z2} E_{z2\downarrow} e^{i k_{z2} d_2}$$

$$\cdots$$

$$\epsilon_{z(N-1)} E_{z(N-2)\uparrow} e^{i k_{z(N-2)} d_{(N-2)}} + \epsilon_{z(N-2)} E_{z(N-2)\downarrow} = \epsilon_{z(N-1)} E_{z(N-1)\uparrow}$$

For brevity, define $\delta_m = e^{i k_{zm} d_m}$.

$$\left(\begin{array}{cccccc}
\frac{E_{x0\downarrow}}{H_{0\downarrow}} & -\frac{E_{x1\uparrow}}{H_{1\uparrow}} & -\frac{E_{x1\downarrow}}{H_{1\downarrow}}\delta_1 & 0 & 0 & 0 \\
0 & \frac{E_{x1\uparrow}}{H_{1\uparrow}}\delta_1 & \frac{E_{x1\downarrow}}{H_{1\downarrow}} & -\frac{E_{x2\uparrow}}{H_{2\uparrow}} & -\frac{E_{x2\downarrow}}{H_{2\downarrow}}\delta_2 & 0 \\
0 & 0 & 0 & \frac{E_{x2\uparrow}}{H_{2\uparrow}}\delta_2 & \frac{E_{x2\downarrow}}{H_{2\downarrow}} & -\frac{E_{x3\uparrow}}{H_{3\uparrow}} \\
\hline
\epsilon_{z0}\frac{E_{z0\downarrow}}{H_{0\downarrow}} & -\epsilon_{z1}\frac{E_{z1\uparrow}}{H_{1\uparrow}} & -\epsilon_{z1}\frac{E_{z1\downarrow}}{H_{1\downarrow}}\delta_1 & 0 & 0 & 0 \\
0 & \epsilon_{z1}\frac{E_{z1\uparrow}}{H_{1\uparrow}}\delta_1 & \epsilon_{z1}\frac{E_{z1\downarrow}}{H_{1\downarrow}} & -\epsilon_{z2}\frac{E_{z2\uparrow}}{H_{2\uparrow}} & -\epsilon_{z2}\frac{E_{z2\downarrow}}{H_{2\downarrow}}\delta_2 & 0 \\
0 & 0 & 0 & \epsilon_{z2}\frac{E_{z2\uparrow}}{H_{2\uparrow}}\delta_2 & \epsilon_{z2}\frac{E_{z2\downarrow}}{H_{2\downarrow}} & -\epsilon_{z3}\frac{E_{z3\uparrow}}{H_{3\uparrow}}
\end{array}\right)
\begin{pmatrix} H_{0\downarrow} \\ H_{1\uparrow} \\ H_{1\downarrow} \\ H_{2\uparrow} \\ H_{2\downarrow} \\ H_{3\uparrow} \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(The horizontal line separates the E_x equations from the E_z equations.)

4 Poynting vector

Time-averaged Poynting vector is (Jackson (6.132)): $S = \frac{1}{2} E \times H^*$. The real part of S indicates time-averaged power flow. I am only interested in the x-component of S, $S_x = -(1/2) E_z H_y^*$.

$$S_x = -(1/2) E_z H_y^* = -(1/2)\left(\begin{matrix} E_{zm\uparrow} e^{i k_{zm}(z - z_{\text{bottom of m}})} + \\ + E_{zm\downarrow} e^{i k_{zm}(z_{\text{top of m}} - z)} \end{matrix}\right)\left(\begin{matrix} H^*_{m\uparrow} e^{-i k^*_{zm}(z - z_{\text{bottom of m}})} + \\ + H^*_{m\downarrow} e^{-i k^*_{zm}(z_{\text{top of m}} - z)} \end{matrix}\right)$$

$$= \frac{-E_{zm\uparrow} H^*_{m\uparrow}}{2} e^{-2\,\mathrm{Im}(k_{zm})(z - z_{\text{bottom of m}})} + \frac{-E_{zm\downarrow} H^*_{m\downarrow}}{2} e^{-2\,\mathrm{Im}(k_{zm})(z_{\text{top of m}} - z)}$$

$$+ \frac{-E_{zm\downarrow} H^*_{m\uparrow}}{2} e^{i k_{zm} d_m} e^{-2i\,\mathrm{Re}(k_{zm})(z - z_{\text{bottom of m}})} + \frac{-E_{zm\uparrow} H^*_{m\downarrow}}{2} e^{i k_{zm} d_m} e^{-2i\,\mathrm{Re}(k_{zm})(z_{\text{top of m}} - z)}$$

Integrate this to get...

$$\int_{z_{\text{bottom of m}}}^{z_{\text{top of m}}} S_x = \frac{-E_{zm\uparrow} H^*_{m\uparrow}}{4\,\mathrm{Im}(k_{zm})}\left(1 - e^{-2\,\mathrm{Im}(k_{zm}) d_m}\right) + \frac{-E_{zm\downarrow} H^*_{m\downarrow}}{4\,\mathrm{Im}(k_{zm})}\left(1 - e^{-2\,\mathrm{Im}(k_{zm}) d_m}\right)$$

$$+ \frac{-E_{zm\downarrow} H^*_{m\uparrow}}{4i\,\mathrm{Re}(k_{zm})} e^{i k_{zm} d_m}\left(1 - e^{-2i\,\mathrm{Re}(k_{zm}) d_m}\right) + \frac{-E_{zm\uparrow} H^*_{m\downarrow}}{4i\,\mathrm{Re}(k_{zm})} e^{i k_{zm} d_m}\left(1 - e^{-2i\,\mathrm{Re}(k_{zm}) d_m}\right)$$