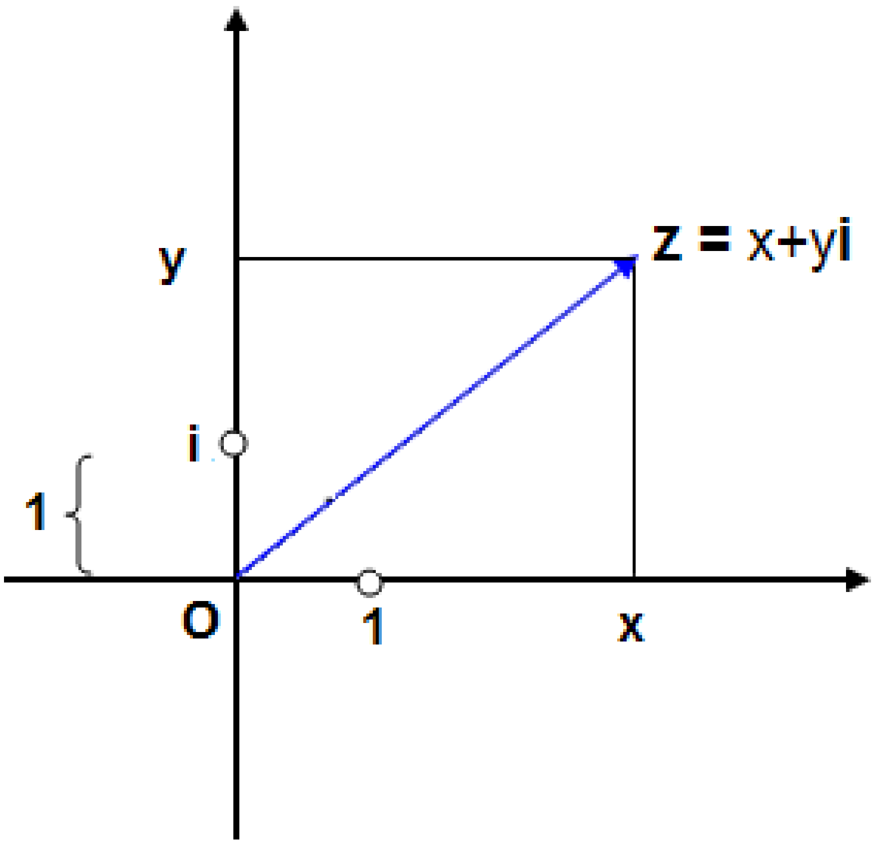
Kompleksiluvut

eli miten tason pisteitä lasketaan yhteen, vähennetään, **kerrotaan** ja **jaetaan**.

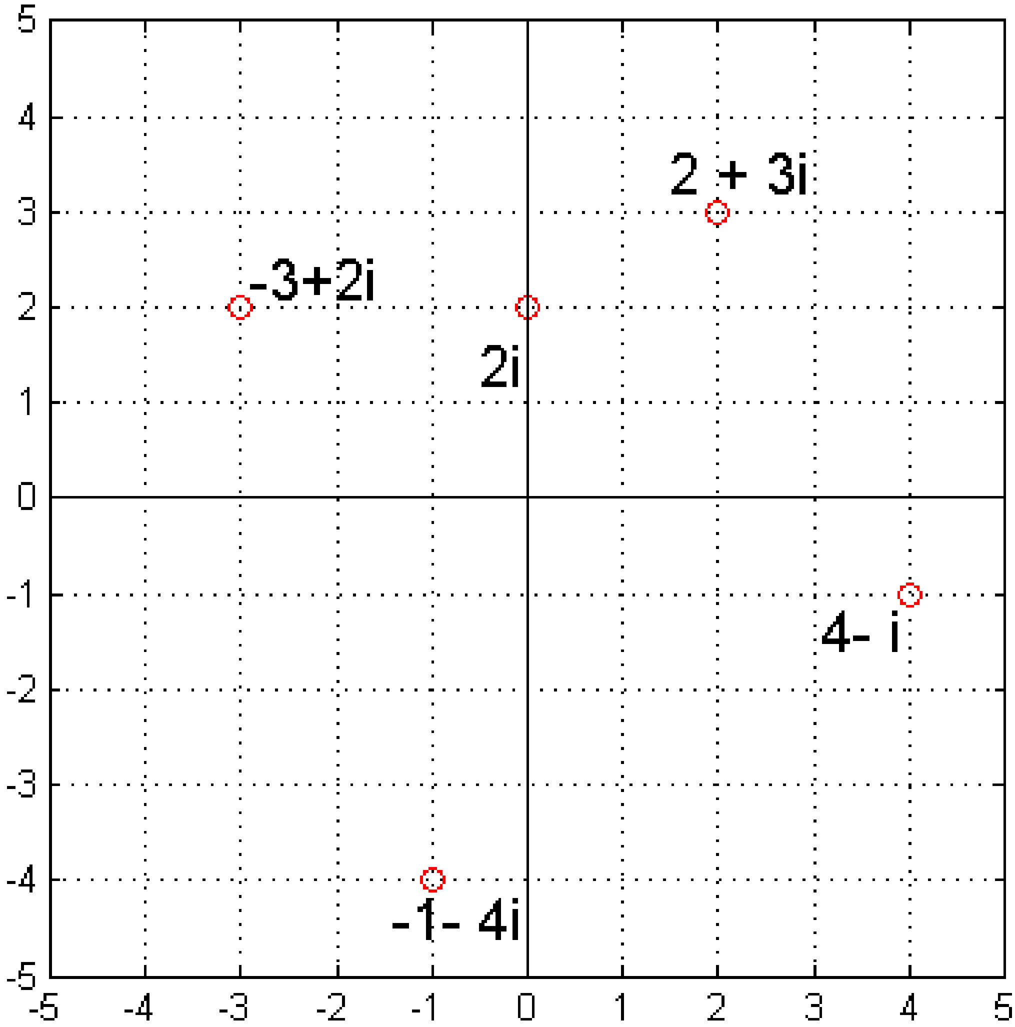


Suorakulmainen muoto: $z = x + y\mathbf{i}$

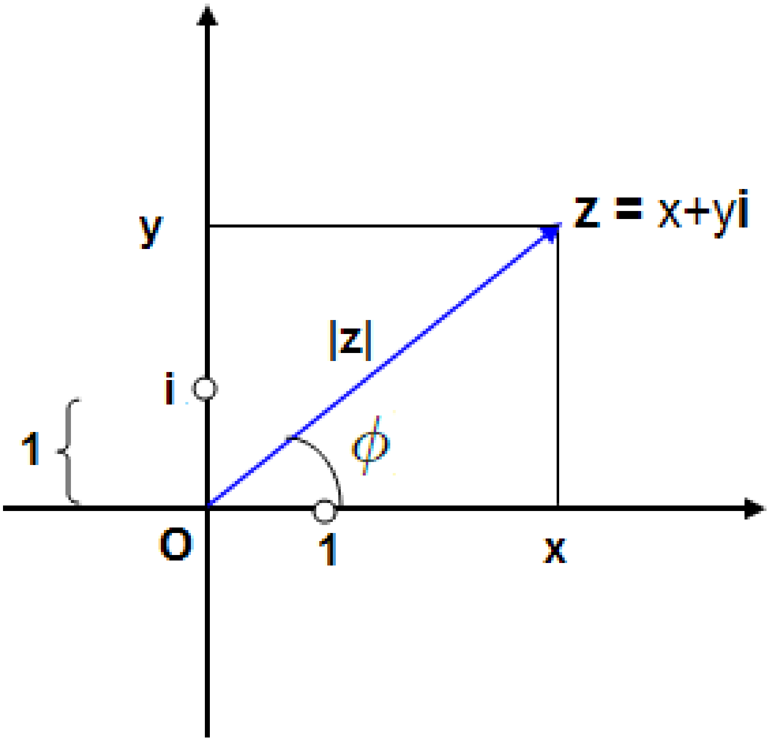
x on z:n reaaliosa

y on z:n imaginääriosa

$\mathbf{i} = 0 + 1\mathbf{i}$ on imaginääriyksikkö



Kulmamuoto: $z = |z|\angle\phi$



$|z|$ on z:n pituus eli itseisarvo

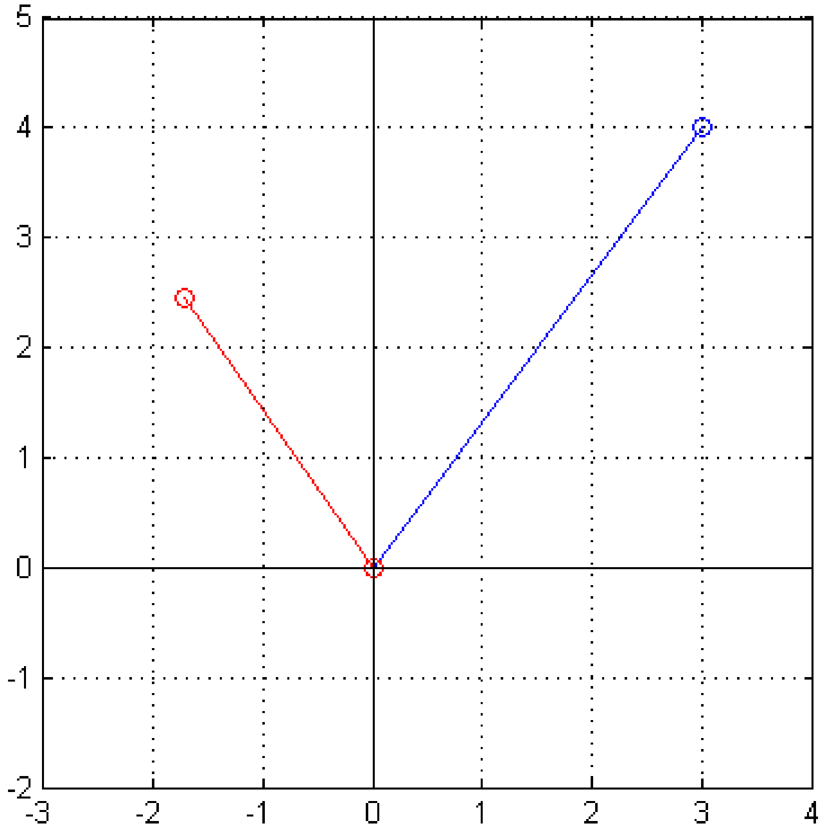
ϕ on z:n suuntakulma

$|z| = \sqrt{x^2 + y^2}$, $\phi = \text{atan2}(y, x)$

$x = |z|\cos(\phi)$, $y = |z|\sin(\phi)$

$3 + 4\mathbf{i} = 5\angle 53^\circ$

$3\angle 125^\circ = -1.7 + 2.5\mathbf{i}$



Laskutoimitukset:

$z = x + y\mathbf{i}$, $w = a + b\mathbf{i}$

Yhteen- ja vähennyslasku:

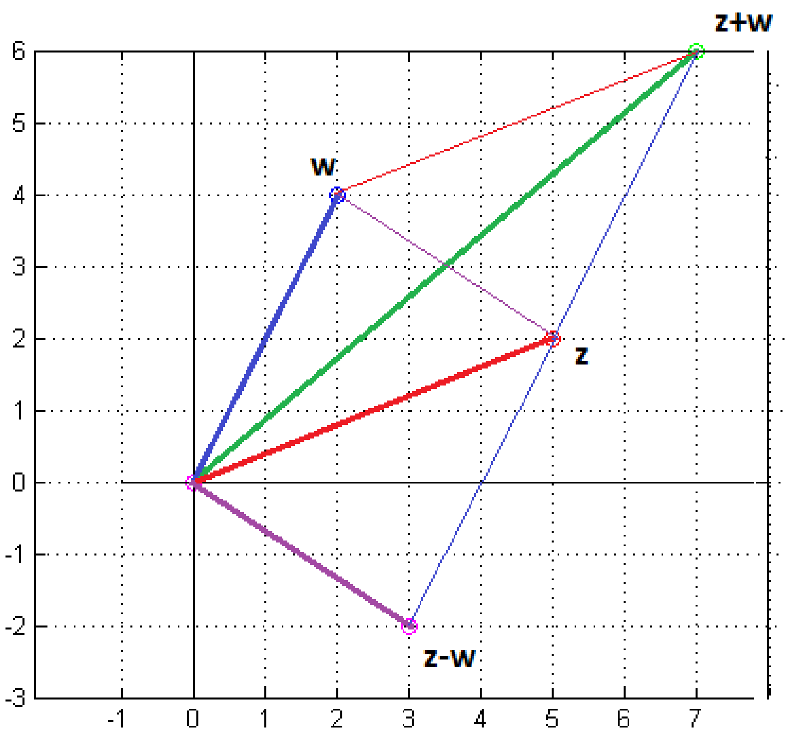
$z + w = (x + a) + (y + b)\mathbf{i}$

$z - w = (x - a) + (y - b)\mathbf{i}$

Esim. $z = 5 + 2\mathbf{i}$ ja $w = 2 + 4\mathbf{i}$

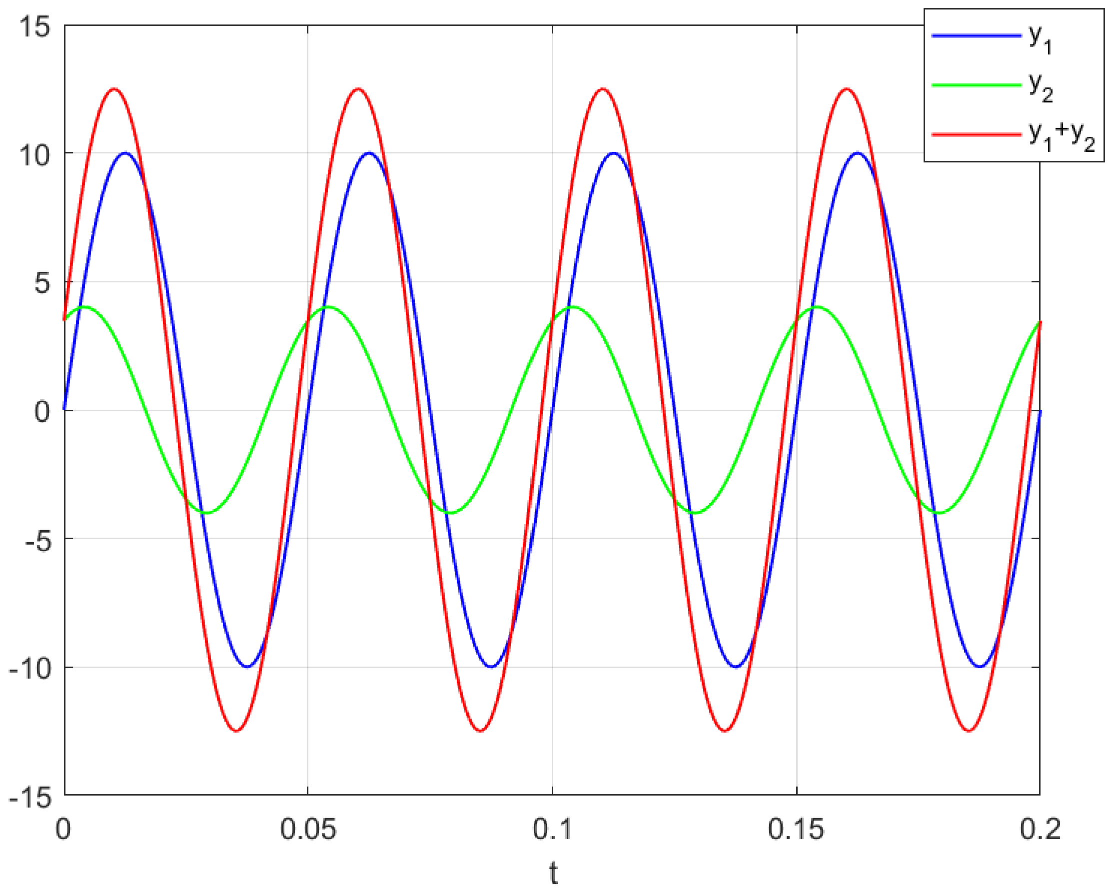
$z + w = 7 + 6\mathbf{i}$

$z - w = 3 - 2\mathbf{i}$



Esim. sinikäyrien yhteenlasku

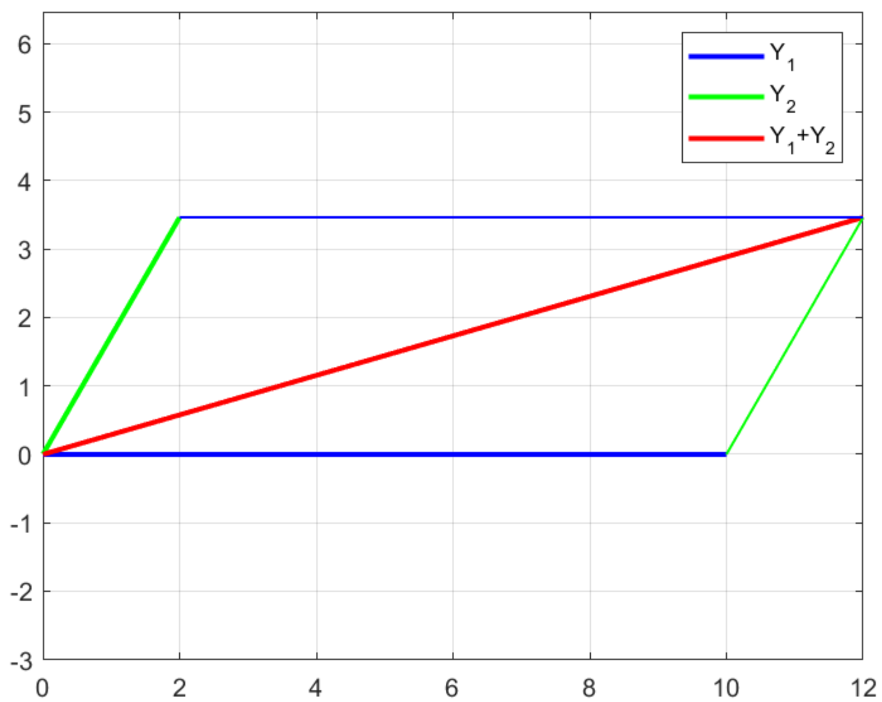
$$y_1 = 10\sin(\omega t)$$
$$y_2 = 4\sin(\omega t + \pi/3)$$
$$y_1 + y_2 = 12.5\sin(\omega t + 0.28)$$



sinikäyrä $y = A \sin(\omega t + \phi) \leftrightarrow$

kompleksiluku $Y = A\angle\phi$

$Y_1 = 10\angle 0 = 10 + 0\mathbf{i}$
$Y_2 = 4\angle \pi/3 = 2 + 3.46\mathbf{i}$
$Y_1 + Y_2 = 12 + 3.46\mathbf{i} = 12.5\angle 0.28$



Kertolasku:

$$
\begin{aligned}
zw &= (x + y\mathbf{i})(a + b\mathbf{i}) \\
&= (xa - yb) + (xb + ya)\mathbf{i}
\end{aligned}
$$

Jakolasku:

$$
\begin{aligned}
\frac{z}{w} &= \frac{x + y\mathbf{i}}{a + b\mathbf{i}} \\
&= \frac{xa + yb}{a^2 + b^2} + \frac{ya - xb}{a^2 + b^2}\,\mathbf{i}
\end{aligned}
$$

Esim. $z = 5 + 2\mathbf{i}$ ja $w = 2 + 4\mathbf{i}$

$zw = 2 + 24\mathbf{i}$

$$\frac{z}{w} = \frac{18}{20} - \frac{16}{20}\mathbf{i} = 0.9 - 0.8\mathbf{i}$$

idea: kerrotaan sulut auki ja käytetään laskusääntöä $\underline{\mathbf{i}^2 = -1}$!!!

$$zw = (x + y\mathbf{i})(a + b\mathbf{i})$$

$$= xa + xb\mathbf{i} + ya\mathbf{i} + yb\mathbf{i}^2$$

$$= xa + xb\mathbf{i} + ya\mathbf{i} + yb(-1)$$

$$= (xa - yb) + (xb + ya)\mathbf{i}$$

idea: lavennetaan konjugaatilla $\overline{w} = a - b\mathbf{i}$

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}}$$

$$= \frac{(x + y\mathbf{i})(a - b\mathbf{i})}{(a + b\mathbf{i})(a - b\mathbf{i})}$$

$$= \frac{(xa + yb) + (ya - xb)\mathbf{i}}{a^2 + b^2}$$

$$= \frac{xa + yb}{a^2 + b^2} + \frac{ya - xb}{a^2 + b^2}\,\mathbf{i}$$

Kerto- ja jakolasku ovat havainnollisempia kulmamuodossa:

Jos $z = |z|\angle\varphi$ ja $w = |w|\angle\phi$, niin

$$zw = |z||w|\angle\varphi + \phi$$

$$\frac{z}{w} = \frac{|z|}{|w|}\angle\varphi - \phi$$

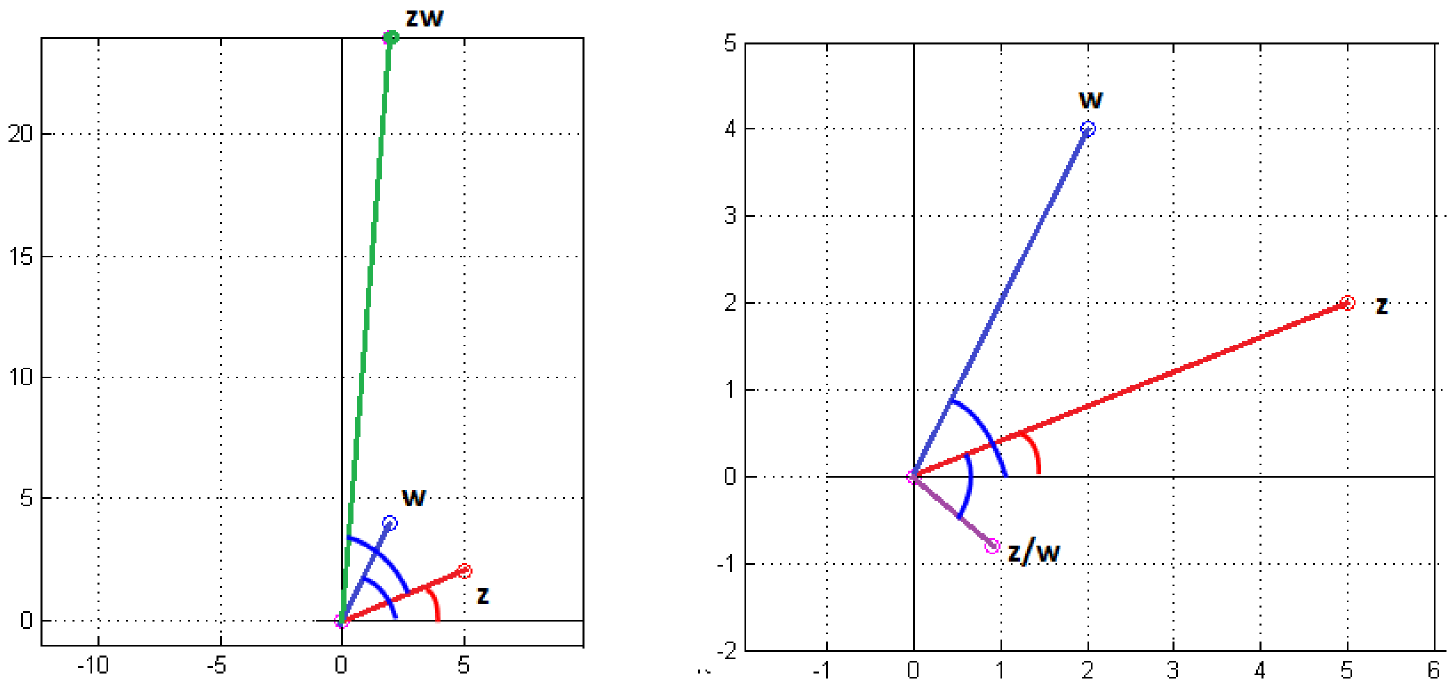
eli **kertolaskussa kerrotaan pituudet ja lasketaan kulmat yhteen**, ja **jakolaskussa jaetaan pituudet ja vähennetään kulmat** .

Esim.

$$z = 5 + 2\mathbf{i} = 5.39\angle 21.8^\circ$$
$$w = 2 + 4\mathbf{i} = 4.47\angle 63.4^\circ$$

$$zw = 2 + 24\mathbf{i} = 24.1\angle 85.2^\circ$$

$$\frac{z}{w} = 0.9 - 0.8\mathbf{i} = 1.2\angle - 41.6^\circ$$



Syy: jos $z = x + y\mathbf{i} = |z|\angle\varphi$

eli $z = \underbrace{|z|\cos\varphi}_{x} + \underbrace{|z|\sin\varphi}_{y}\,\mathbf{i}$

ja $w = a + b\mathbf{i} = |w|\angle\phi$

eli $w = \underbrace{|w|\cos\phi}_{a} + \underbrace{|w|\sin\phi}_{b}\,\mathbf{i}$

niin

$$zw = (xa - yb) + (xb + ya)\mathbf{i}$$

$$= |z||w|(\underbrace{\cos\varphi\cos\phi - \sin\varphi\sin\phi}_{=\cos(\varphi+\phi)})$$

$$+|z||w|(\underbrace{\cos\varphi\sin\phi + \sin\varphi\cos\phi}_{=\sin(\varphi+\phi)})\mathbf{i}$$

eli $zw = |z||w|\angle\varphi + \phi$

$$\frac{z}{w} = \frac{xa + yb}{a^2 + b^2} + \frac{ya - xb}{a^2 + b^2}\,\mathbf{i}$$

$$= \frac{|z||w|(\cos\varphi\cos\phi + \sin\varphi\sin\phi)}{|w|^2}$$

$$+ \frac{|z||w|(\sin\varphi\cos\phi - \cos\varphi\sin\phi)}{|w|^2}\,\mathbf{i}$$

$$= \frac{|z|}{|w|}(\underbrace{\cos\varphi\cos\phi + \sin\varphi\sin\phi}_{=\cos(\varphi-\phi)})$$

$$+\frac{|z|}{|w|}(\underbrace{\sin\varphi\cos\phi - \cos\varphi\sin\phi}_{=\sin(\varphi-\phi)})\ \mathbf{i}$$

eli $\dfrac{z}{w} = \dfrac{|z|}{|w|}\angle\varphi - \phi$

Ohmin laki: $I = \dfrac{U}{Z}$ eli jos

jännite $U = |U|\angle\varphi$ ja

impedanssi $Z = |Z|\angle\phi$, niin

virta $I = \dfrac{|U|}{|Z|}\angle\varphi - \phi$

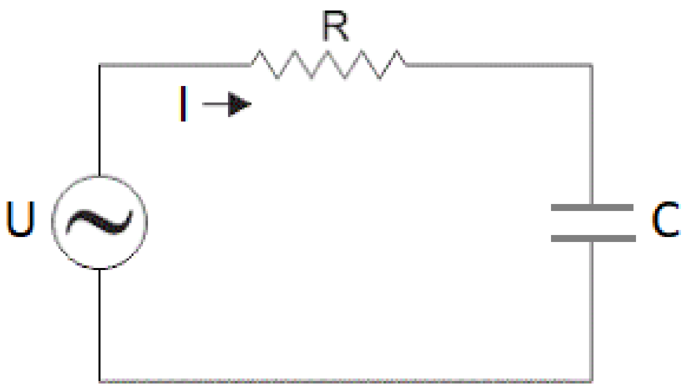
Eli, jos jännite on sinikäyrä

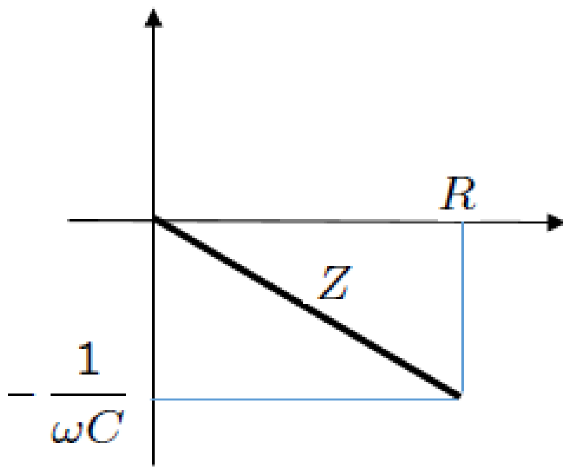
$$u = |U| \sin(\omega t + \varphi)$$

niin virta on sinikäyrä

$$i = \frac{|U|}{|Z|} \sin(\omega t + \varphi - \phi)$$

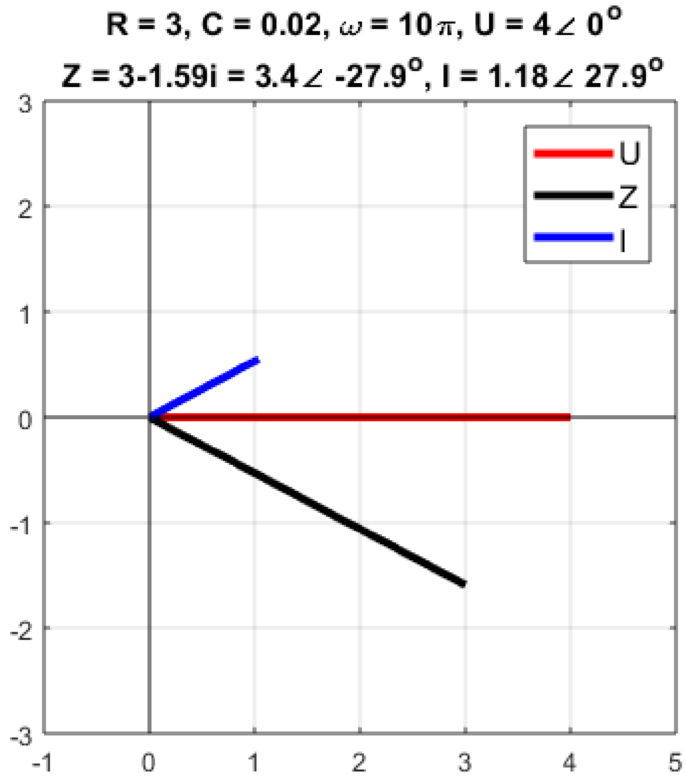
RC-piiri: $Z = R - \frac{1}{\omega C}\mathbf{i}$

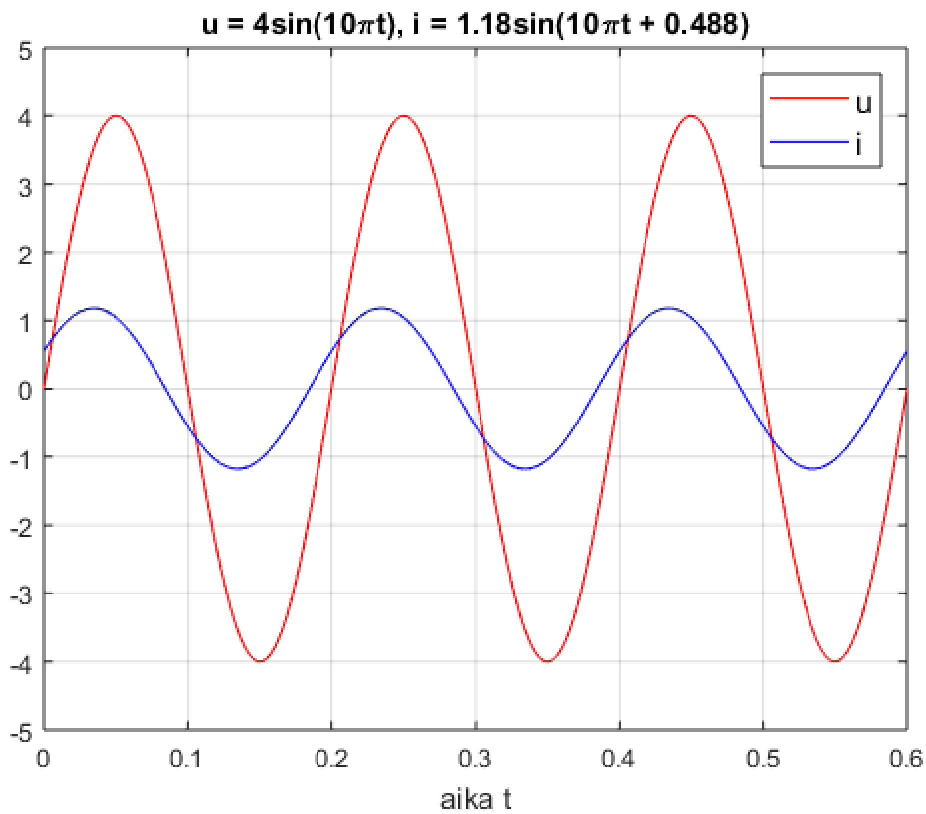




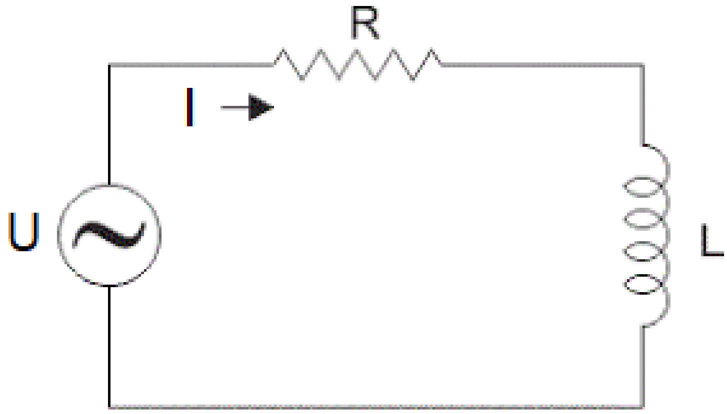
$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

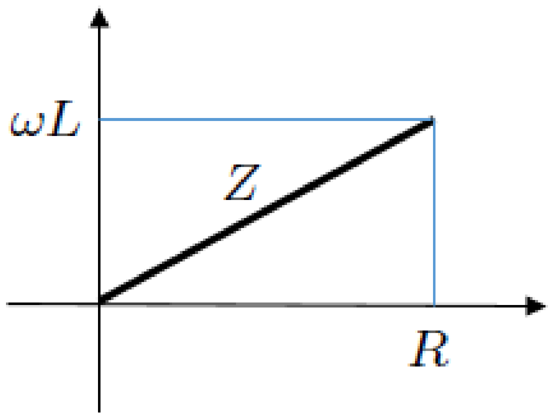
$$\angle Z = \mathrm{atan2}\left(-\frac{1}{\omega C}, R\right)$$


R = 3, C = 0.02, ω = 10π, U = 4∠ 0°
Z = 3-1.59i = 3.4∠ -27.9°, I = 1.18∠ 27.9°
U
Z
I
3
2
1
0
-1
-2
-3
-1
0
1
2
3
4
5


u = 4sin(10πt), i = 1.18sin(10πt + 0.488)
u
i
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
0.1
0.2
0.3
0.4
0.5
0.6
aika t

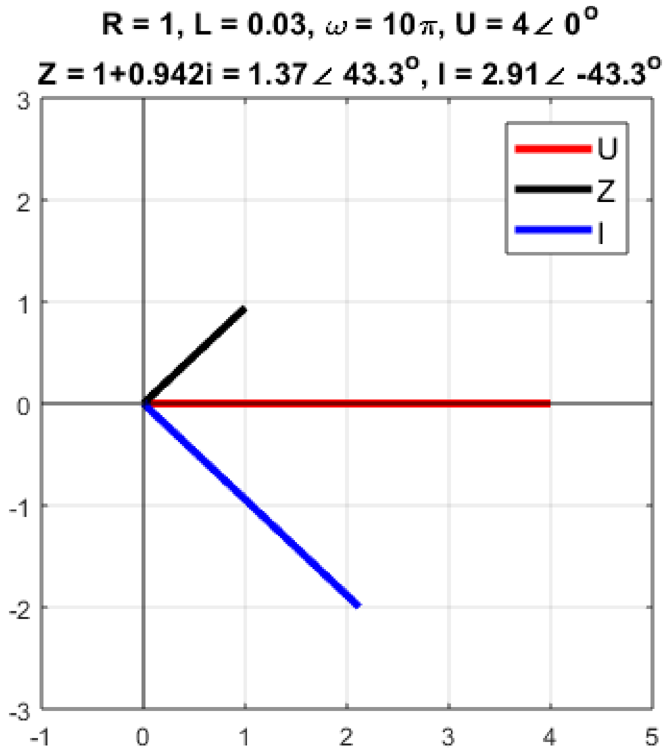
RL-piiri: $Z = R + \omega L\,\mathbf{i}$

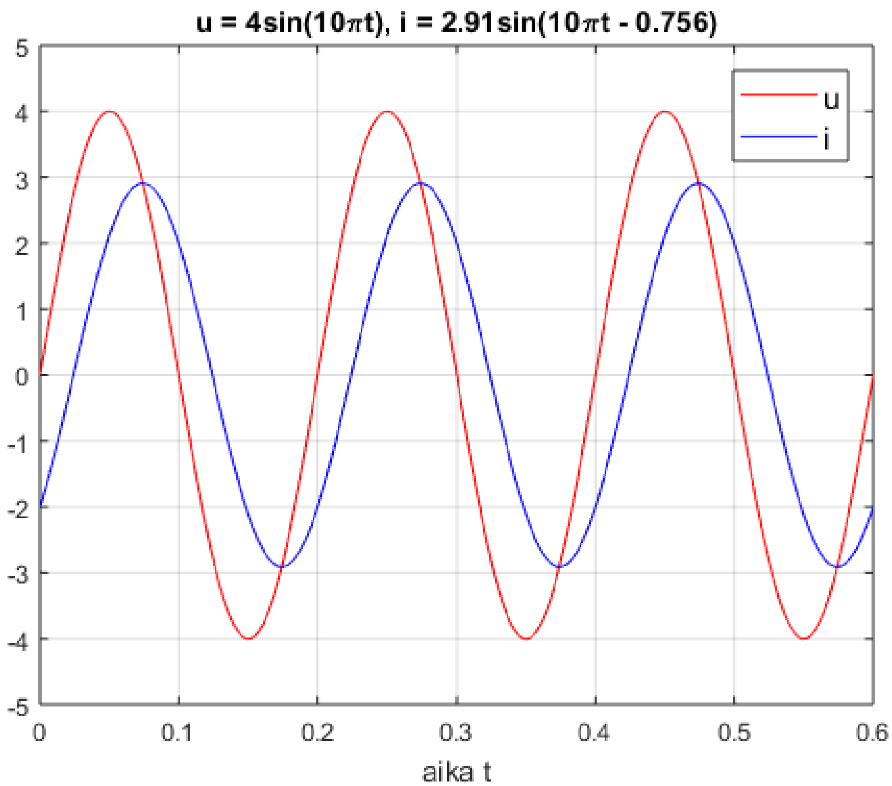




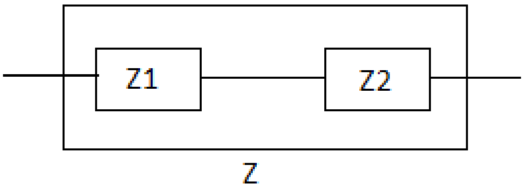
$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

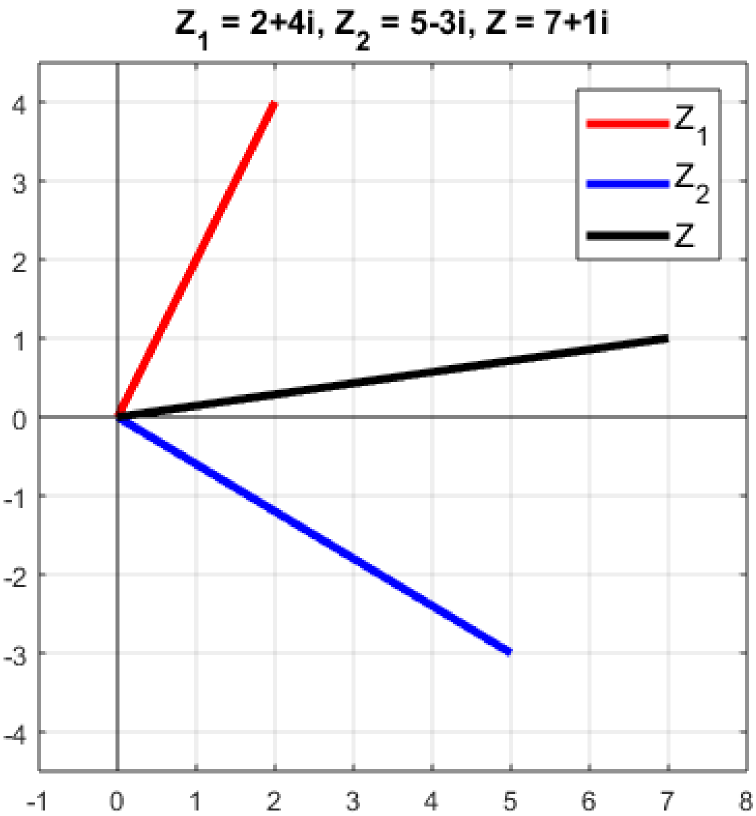
$$\angle Z = \mathrm{atan2}\,(\omega L, R)$$


R = 1, L = 0.03, ω = 10π, U = 4∠ 0°
Z = 1+0.942i = 1.37∠ 43.3°, I = 2.91∠ -43.3°
U
Z
I

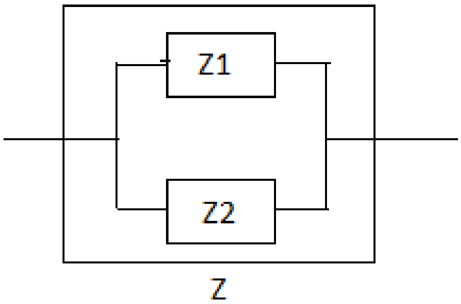

u = 4sin(10πt), i = 2.91sin(10πt - 0.756)
u
i
aika t

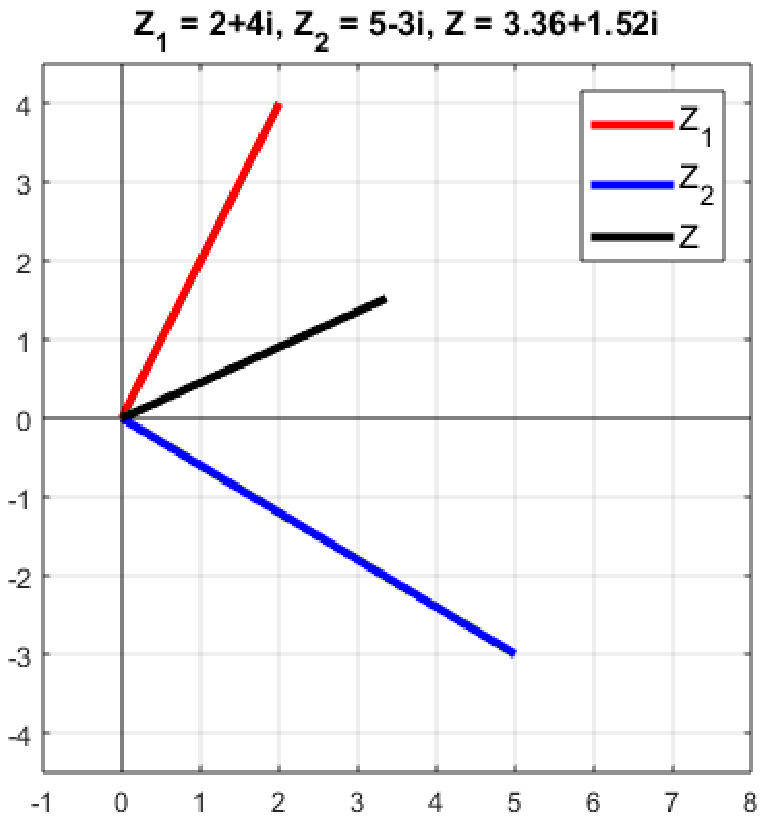
Sarjaankytkentä: $Z = Z_1 + Z_2$





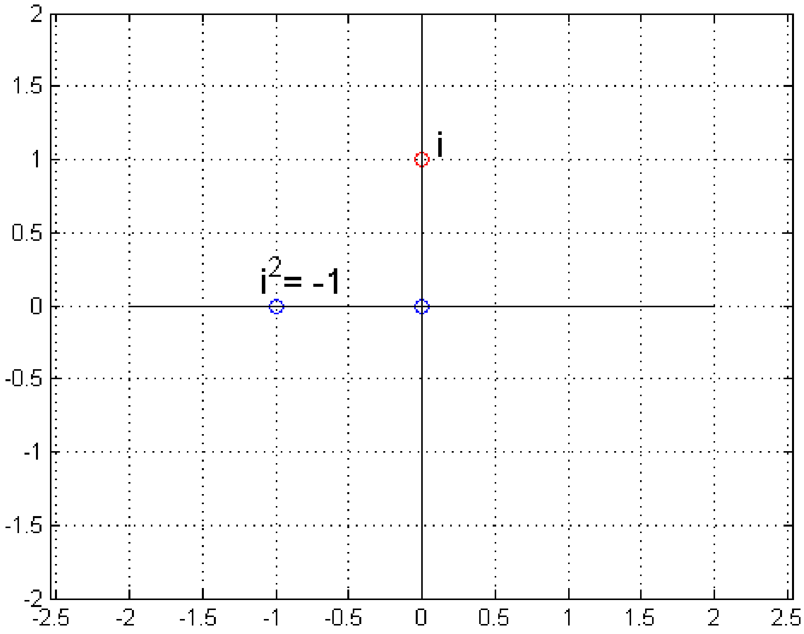
Rinnakytkentä: $Z = \dfrac{Z_1 Z_2}{Z_1 + Z_2}$





Huom: Laskusääntö

$\mathbf{i}^2 = -1$ eli $\mathbf{i} = \sqrt{-1}$!!!



Selitys: $\mathbf{i} = 1\angle 90^\circ$, joten

$\mathbf{i}^2 = \mathbf{i} \cdot \mathbf{i} = 1\angle 180^\circ = -1$

Näin saadaan kaikille negatiivisille luvuille neliöjuuret: esimerkiksi

$$\begin{aligned}
\sqrt{-4} &= \sqrt{4 \cdot (-1)} \\
&= \sqrt{4}\,\sqrt{-1} \\
&= 2\mathbf{i}
\end{aligned}$$

koska

$$(2\mathbf{i})^2 = 4\mathbf{i}^2 = 4 \cdot (-1) = -4$$

Huom: Toisen asteen yhtälön ratkaisukaava:

$$ax^2 + bx + c = 0 \leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Jos $b^2 - 4ac < 0$, niin ratkaisut kompleksilukuja

$$\begin{aligned} x &= \frac{-b \pm \sqrt{4ac - b^2}\,\mathbf{i}}{2a} \\ &= -\frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a}\,\mathbf{i} \end{aligned}$$

Esimerkiksi, yhtälön

$$x^2 + 2x + 5 = 0$$

ratkaisut ovat

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm 4\mathbf{i}}{2} \\ &= -1 \pm 2\mathbf{i} \end{aligned}$$