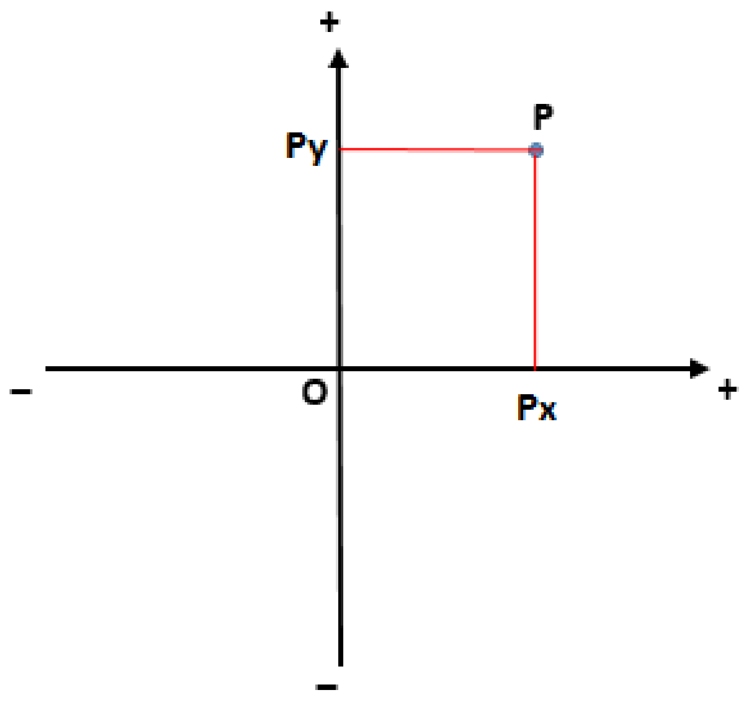
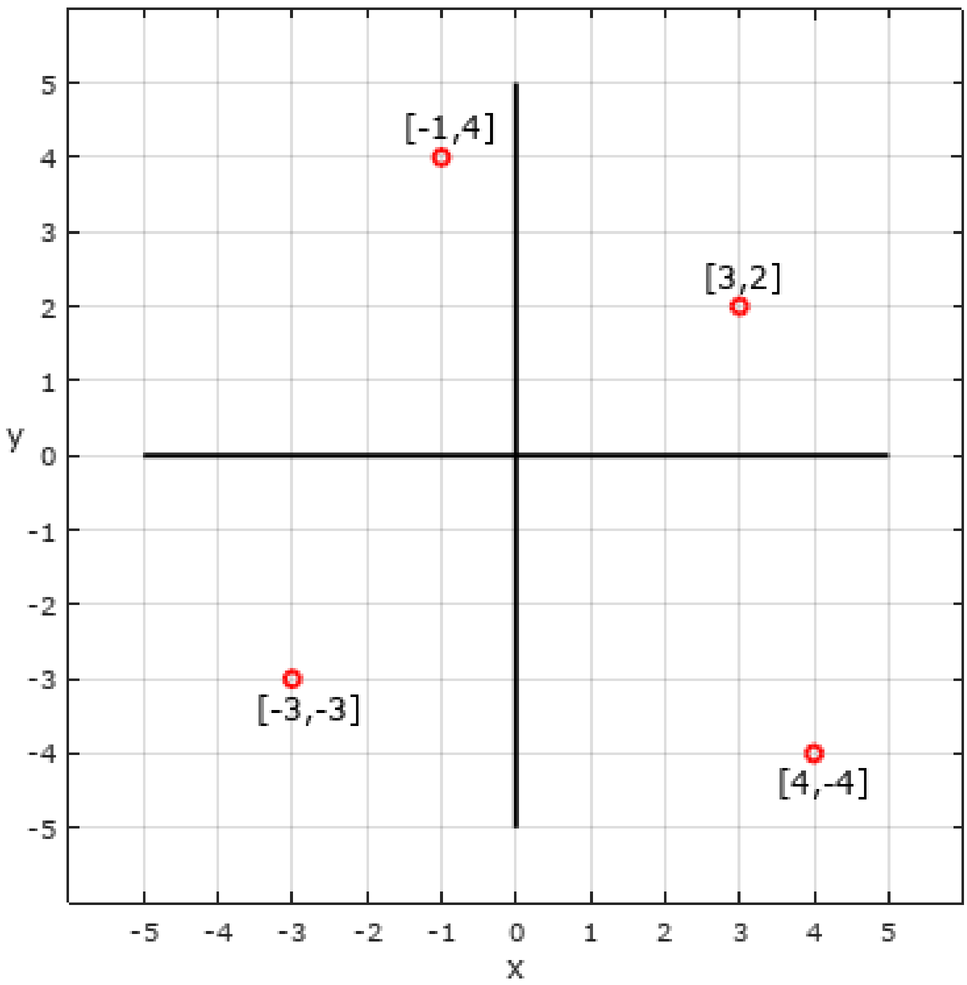
Geometriaa koordinaattien avulla

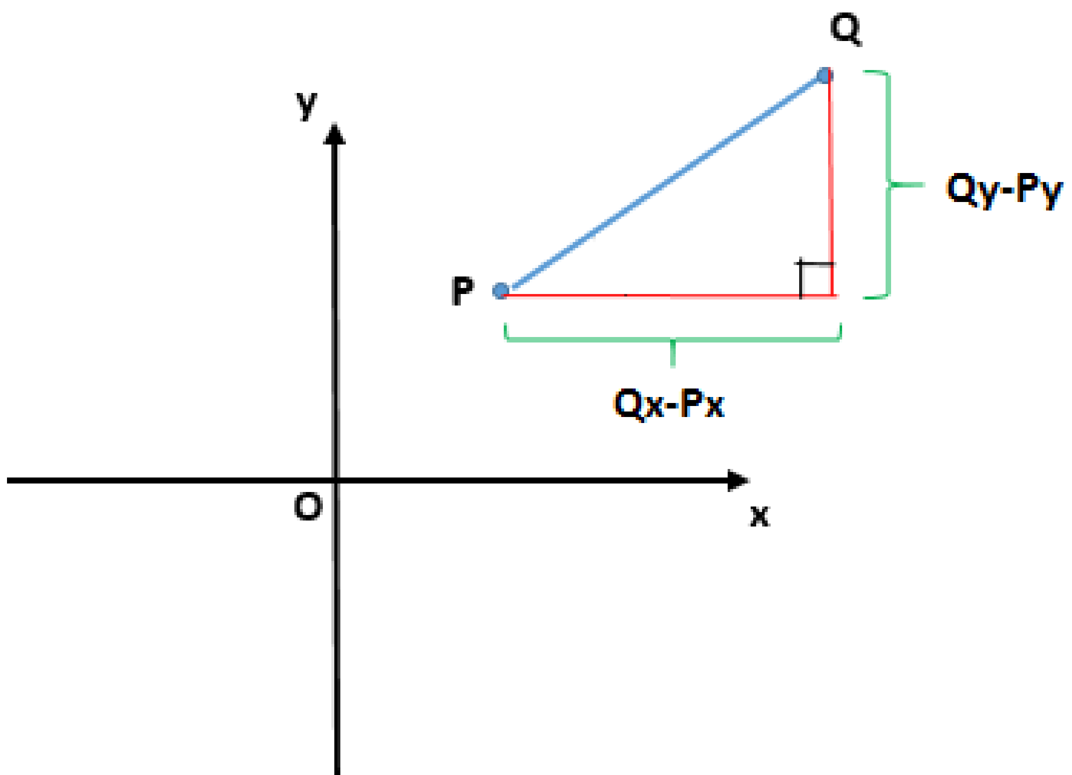
Pisteen $P = [Px, Py]$ koordinaatit Px ja Py kertovat sen vaaka- ja pystysuoran etäisyyden origosta $O = [0, 0]$




[-1,4]
[3,2]
[-3,-3]
[4,-4]
y
x
5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5

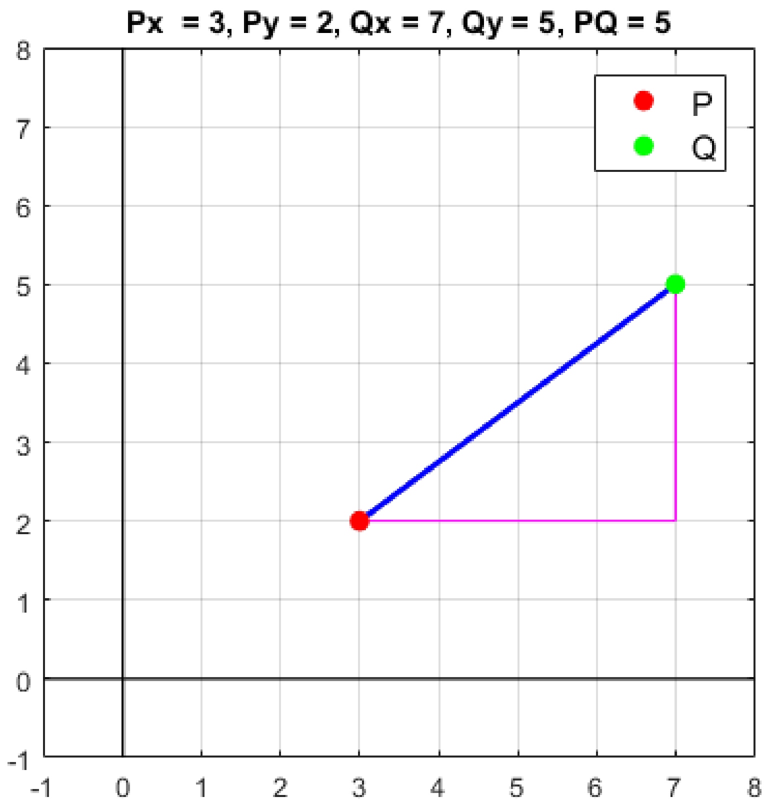
Pisteiden $P = [Px, Py]$ ja $Q = [Qx, Qy]$ välinen etäisyys

$$PQ = \sqrt{(Qx - Px)^2 + (Qy - Py)^2}$$

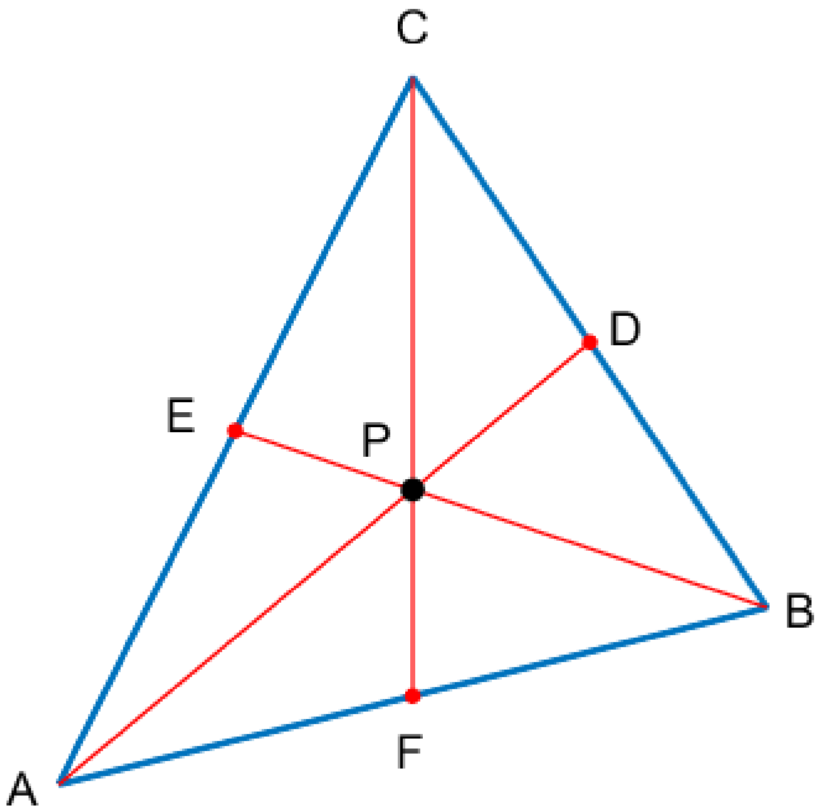


Esim. jos $P = [3, 2]$ ja $Q = [7, 5]$, niin

$$PQ = \sqrt{(7-3)^2 + (5-2)^2} = \sqrt{25} = 5$$



Esim: kolmion ABC mediaanien leikkauspiste



D, E ja F ovat sivujen keskipisteet

Koordinaatit:

$Dx = \frac{1}{2}(Bx + Cx)$, $Dy = \frac{1}{2}(By + Cy)$

$Ex = \frac{1}{2}(Ax + Cx)$, $Ey = \frac{1}{2}(Ay + Cy)$

$Fx = \frac{1}{2}(Ax + Bx)$, $Fy = \frac{1}{2}(Ay + By)$

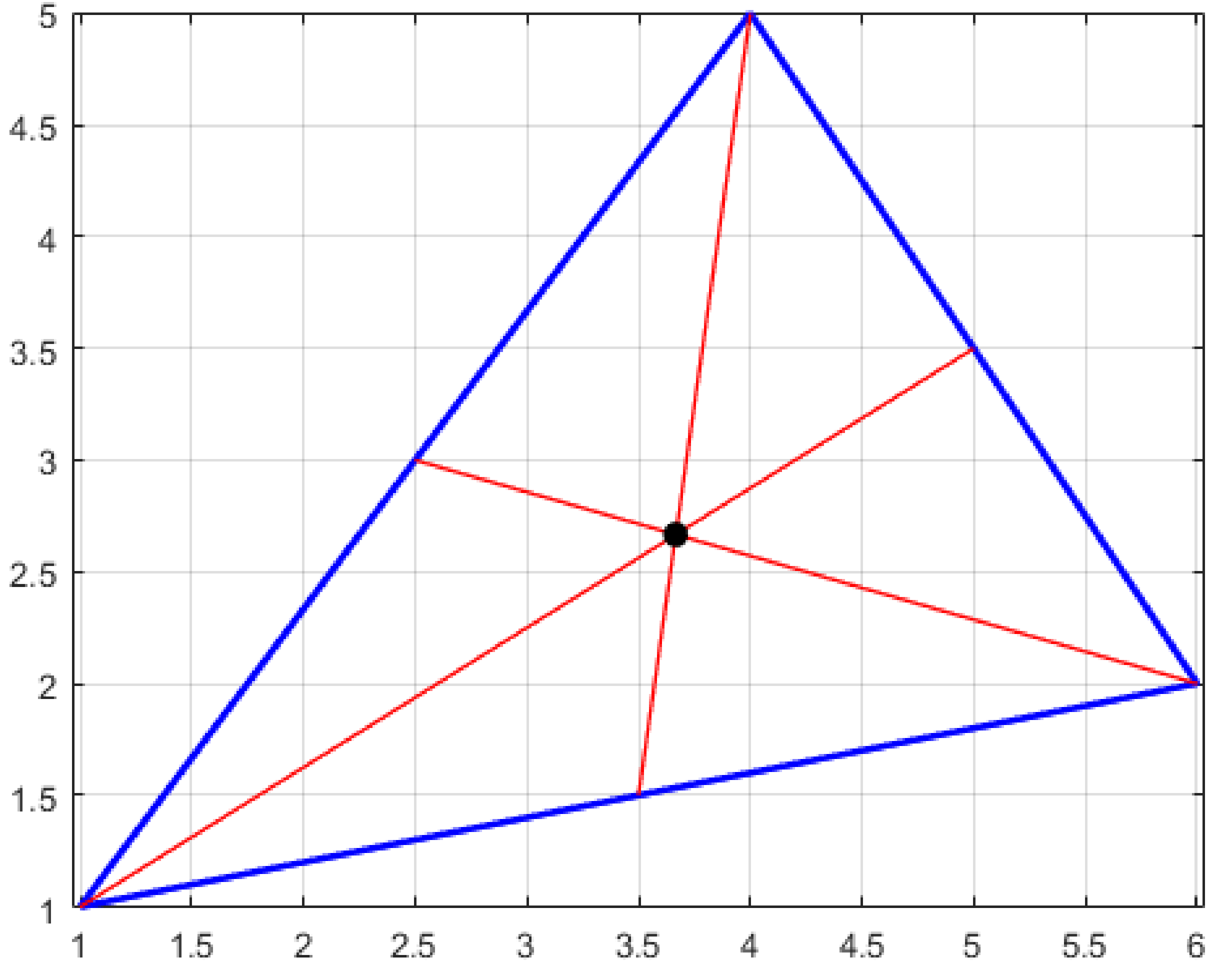
Mediaanit AD, BE ja CF leikkaavat kolmion painopisteessä P, koordinaatit

$Px = \frac{1}{3}(Ax + Bx + Cx)$

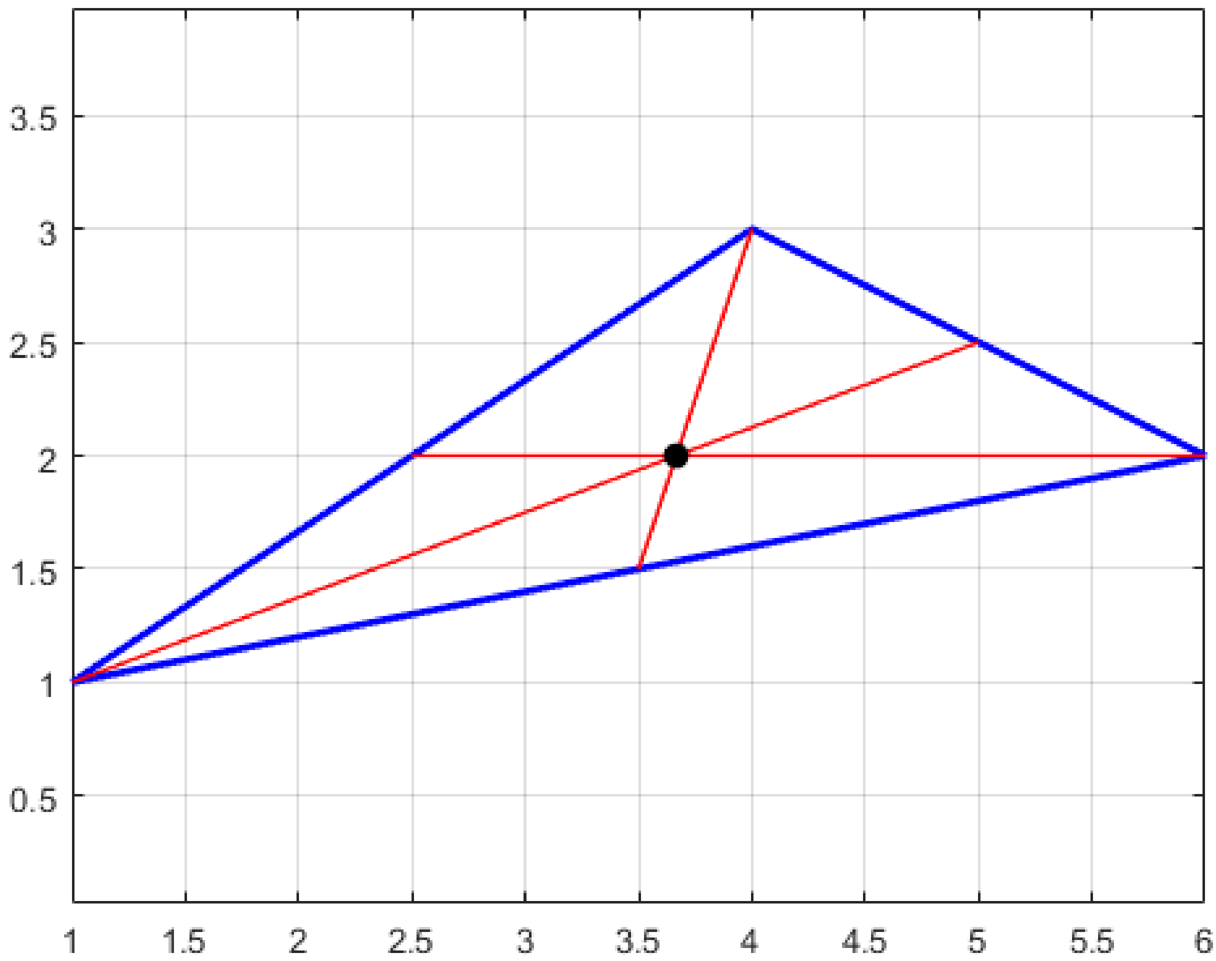
$Py = \frac{1}{3}(Ay + By + Cy)$

joka jakaa ne 2:1 eli

$AP = \frac{2}{3}AD$, $BP = \frac{2}{3}BE$, $CP = \frac{2}{3}CF$

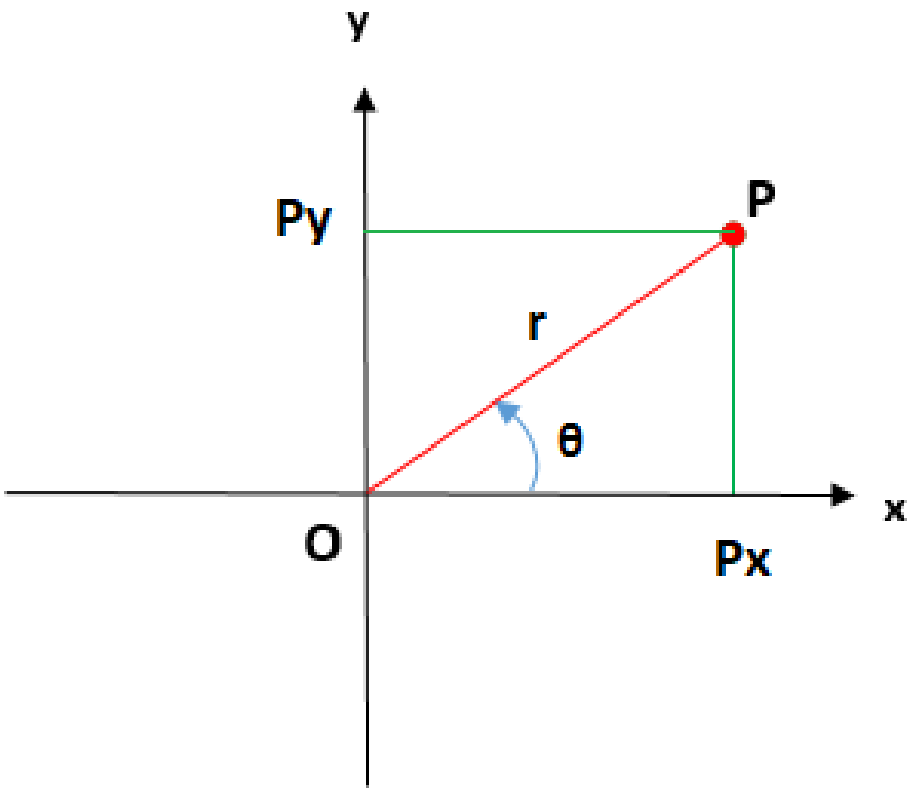


5
4.5
4
3.5
3
2.5
2
1.5
1
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6



3.5
3
2.5
2
1.5
1
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6

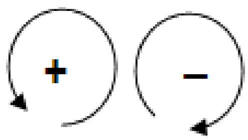
Napakoordinaatit (polar coordinates) r, θ:



r on OP:n pituus eli O:n ja P:n välinen etäisyys

θ on OP:n **suuntakulma**, joka mitataan x-akselilta kuvan mukaisesti

Kulman θ merkki:



$\underline{Px, Py \to r, \theta}$:

$r = \sqrt{Px^2 + Py^2}$

MATLAB/Octave:

$\theta =$ atan2d(Py, Px) on OP:n suuntakulma asteina väliltä $-180^\circ \ldots 180^\circ$

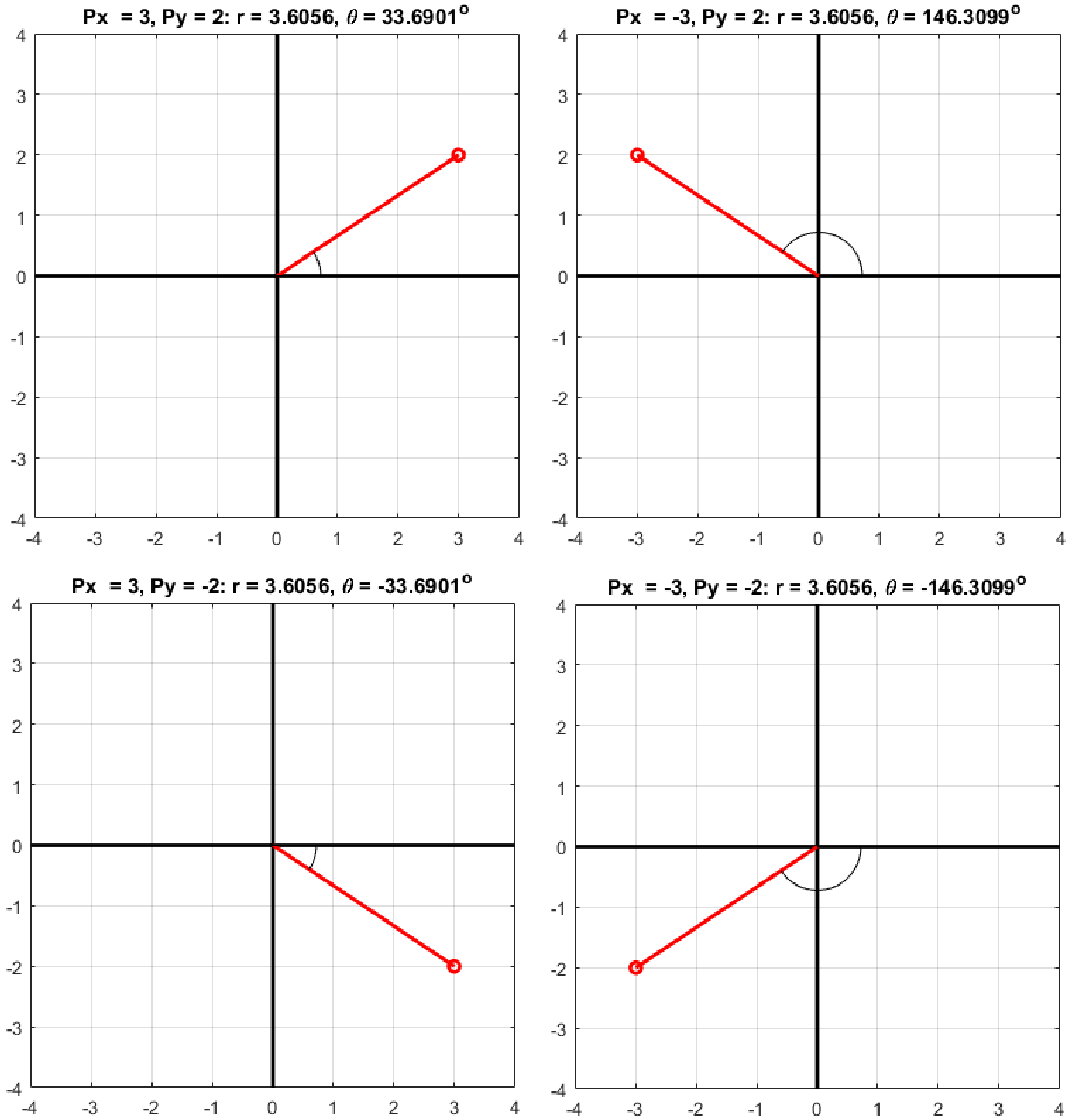
$\theta =$ atan2(Py, Px) on OP:n suuntakulma radiaaneina väliltä $-\pi \ldots \pi$

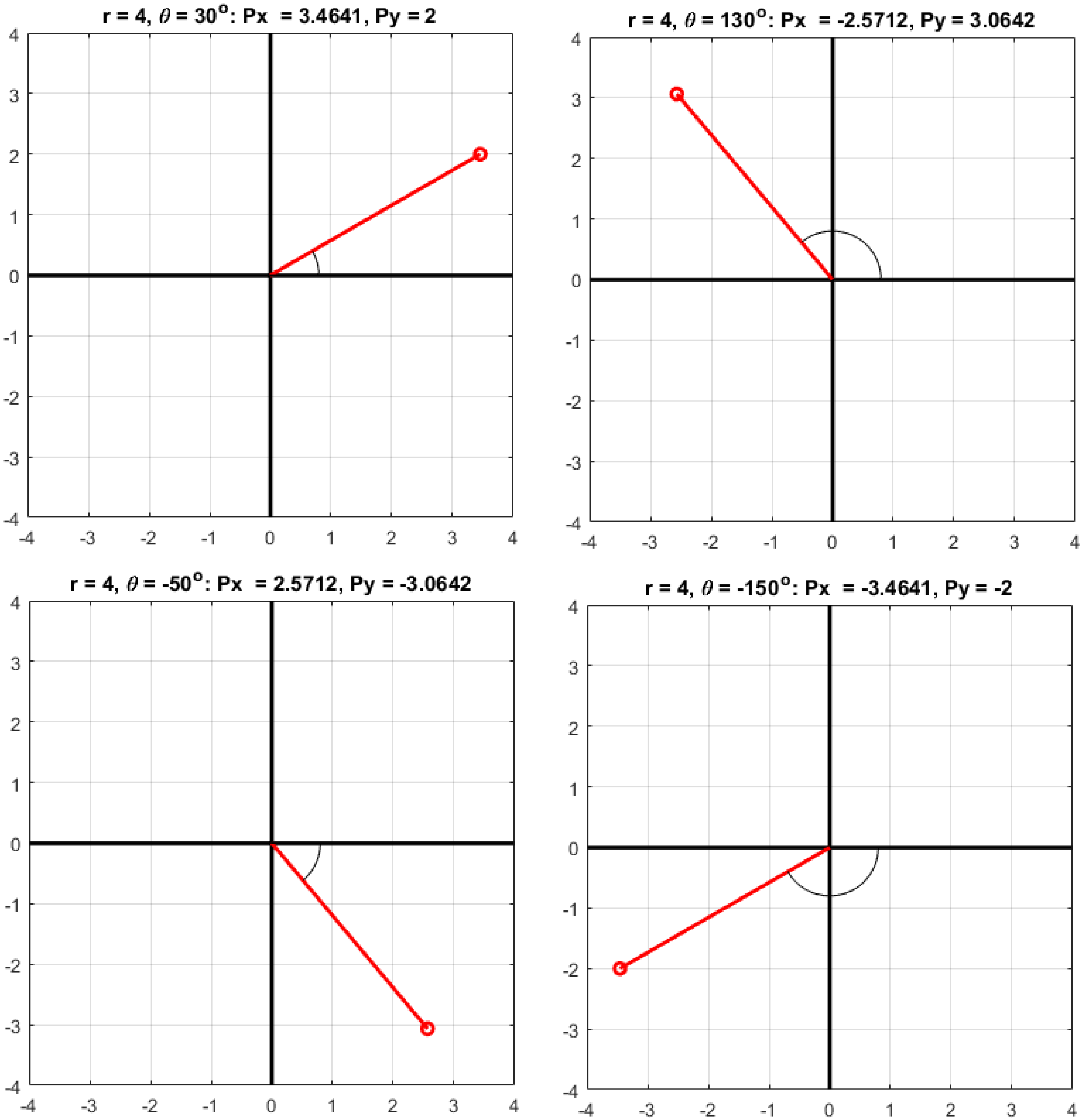
(huomaa järjestys Py, Px !!)

$\underline{r,\ \theta \to Px, Py}$:

$Px = r \cdot \cos(\theta)$

$Py = r \cdot \sin(\theta)$


Px = 3, Py = 2: r = 3.6056, θ = 33.6901°
Px = -3, Py = 2: r = 3.6056, θ = 146.3099°
Px = 3, Py = -2: r = 3.6056, θ = -33.6901°
Px = -3, Py = -2: r = 3.6056, θ = -146.3099°
4
3
2
1
0
-1
-2
-3
-4



r = 4, θ = 30°: Px = 3.4641, Py = 2
r = 4, θ = 130°: Px = -2.5712, Py = 3.0642
r = 4, θ = -50°: Px = 2.5712, Py = -3.0642
r = 4, θ = -150°: Px = -3.4641, Py = -2

Viiva pisteе**stä** P pistee**seen** Q:

vaakasuora osa $Qx - Px$, pystysuora $Qy - Py$

$\underline{Px, Py, Qx, Qy \rightarrow PQ, \theta}$:

pituus $PQ = \sqrt{(Qx - Px)^2 + (Qy - Py)^2}$

suuntakulma $\theta = \theta_{PQ}$

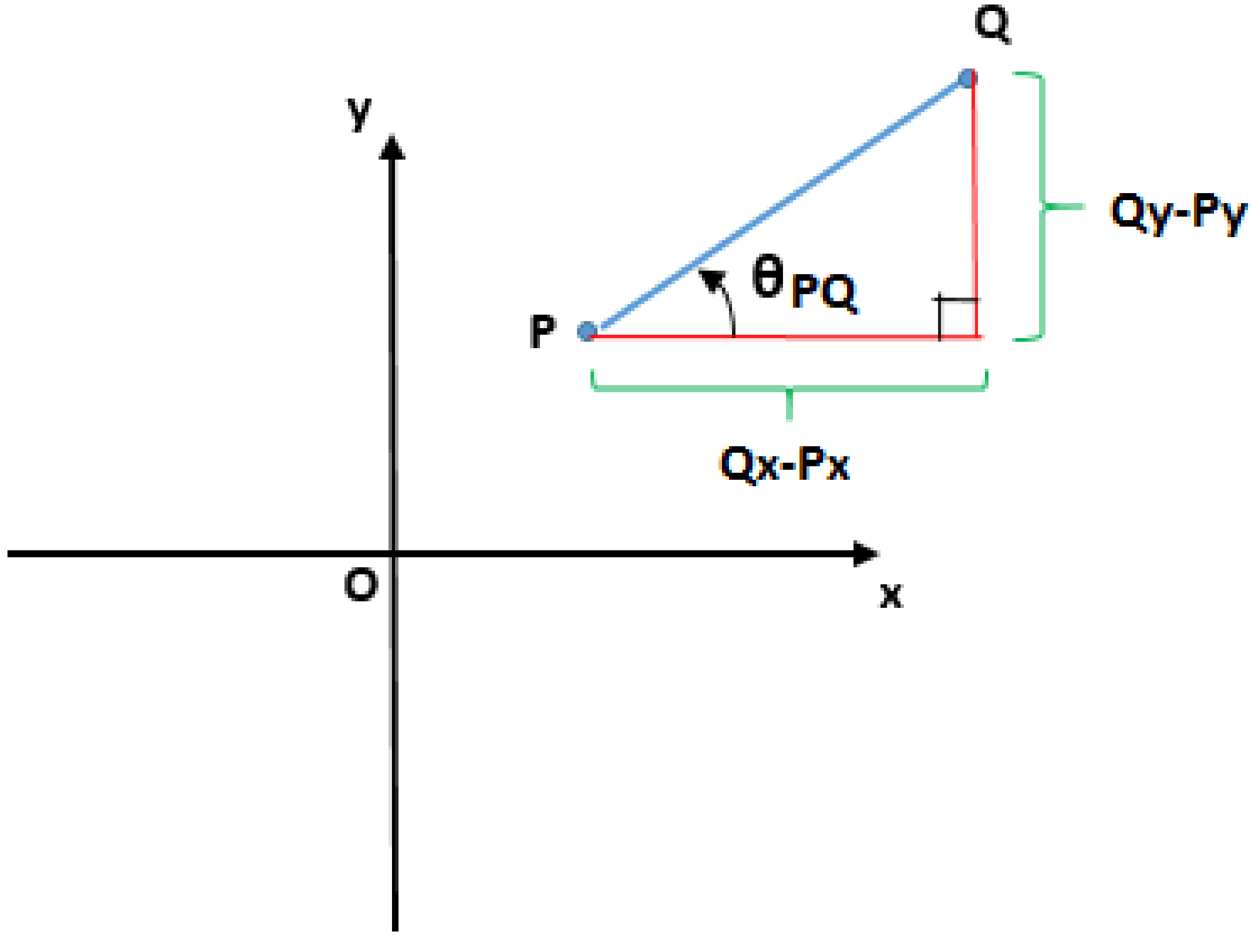
$\theta = \text{atan2}(Qy - Py, Qx - Px)$ (rad)

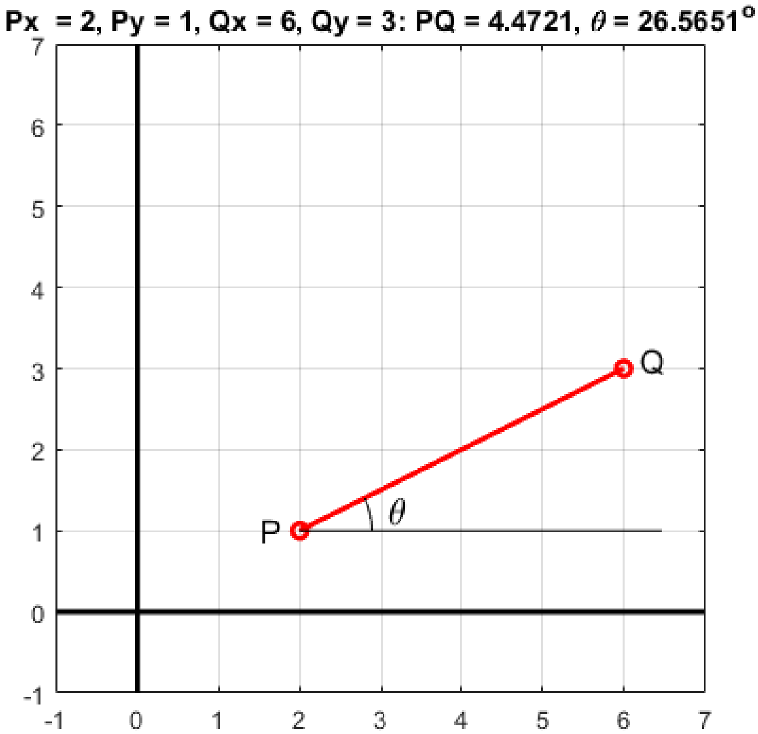
$\theta = \text{atan2d}(Qy - Py, Qx - Px)$ (aste)

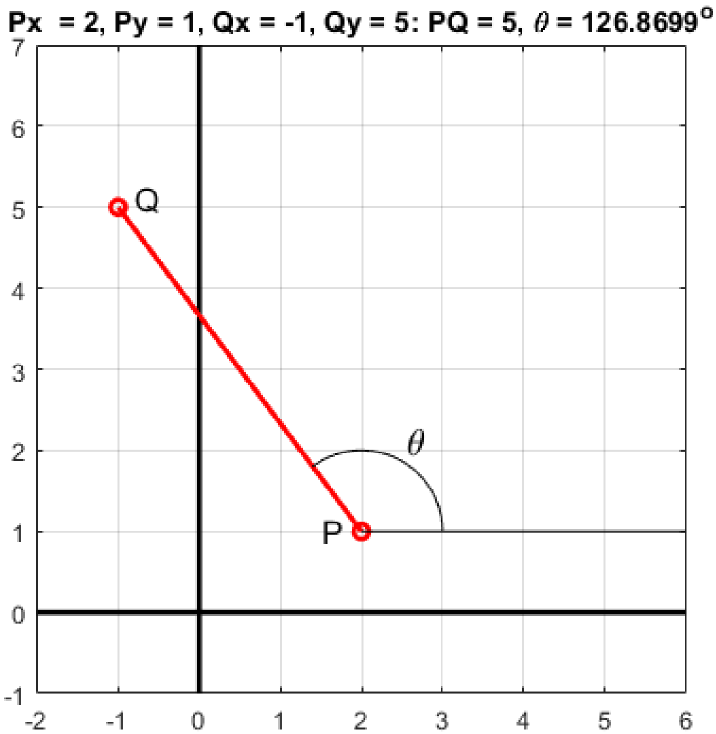
$\underline{Px, Py, PQ, \theta \rightarrow Qx, Qy}$:

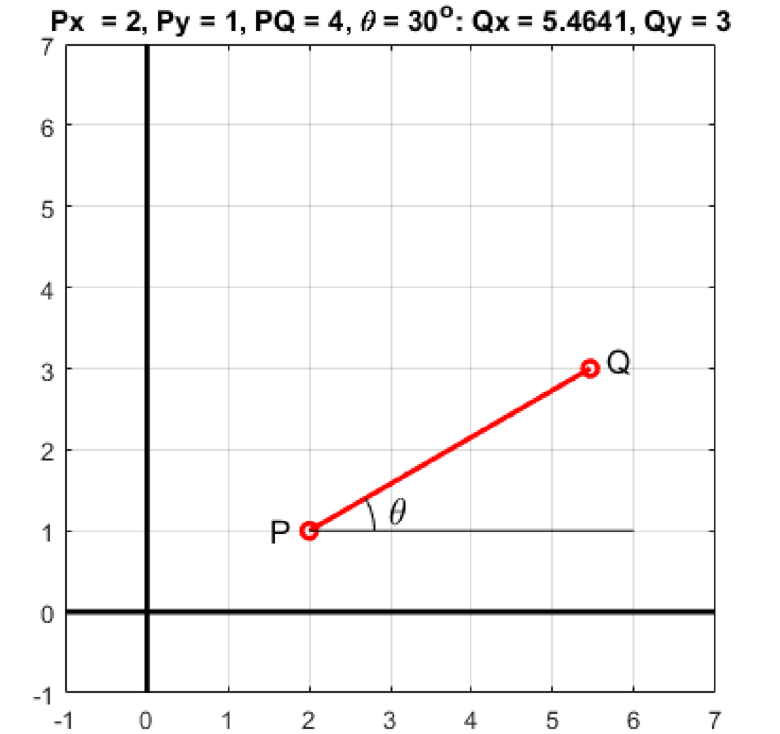
$Qx = Px + PQ\cos(\theta)$

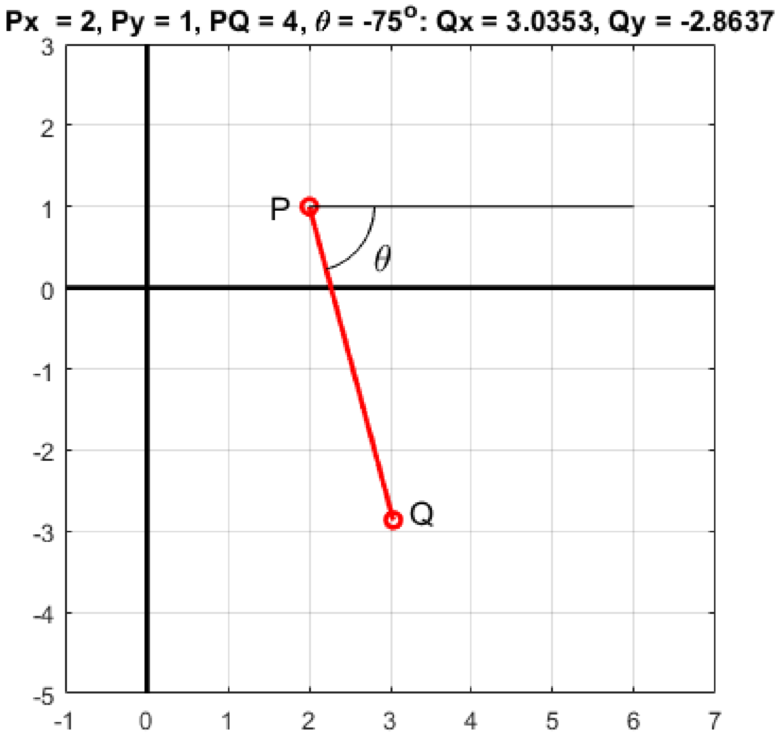
$Qy = Py + PQ\sin(\theta)$


Q
y
Qy-Py
θPQ
P
Qx-Px
O
x

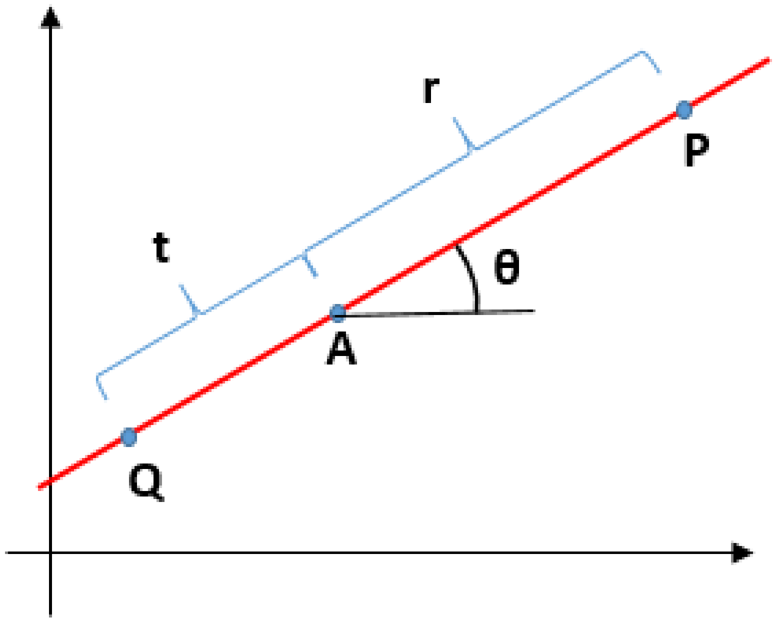

Px = 2, Py = 1, Qx = 6, Qy = 3: PQ = 4.4721, θ = 26.5651°
Q
P
θ


Px = 2, Py = 1, Qx = -1, Qy = 5: PQ = 5, θ = 126.8699°
Q
P
θ


Px = 2, Py = 1, PQ = 4, θ = 30°: Qx = 5.4641, Qy = 3
Q
P
θ


Px = 2, Py = 1, PQ = 4, θ = -75°: Qx = 3.0353, Qy = -2.8637
P
θ
Q

Esim: Suora A, θ, joka kulkee pisteen A kautta ja jonka suuntakulma on θ:



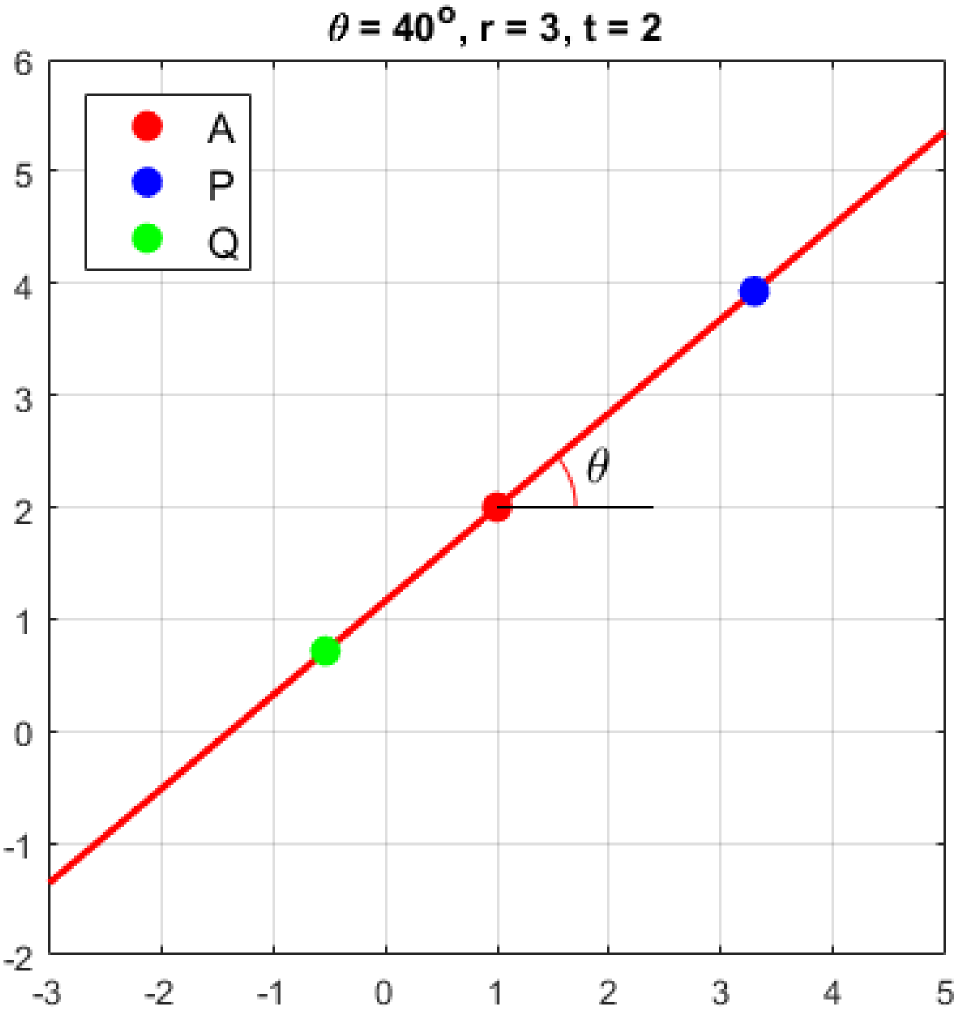
Kuvan mukaisten suoran pisteiden koordinaatit:

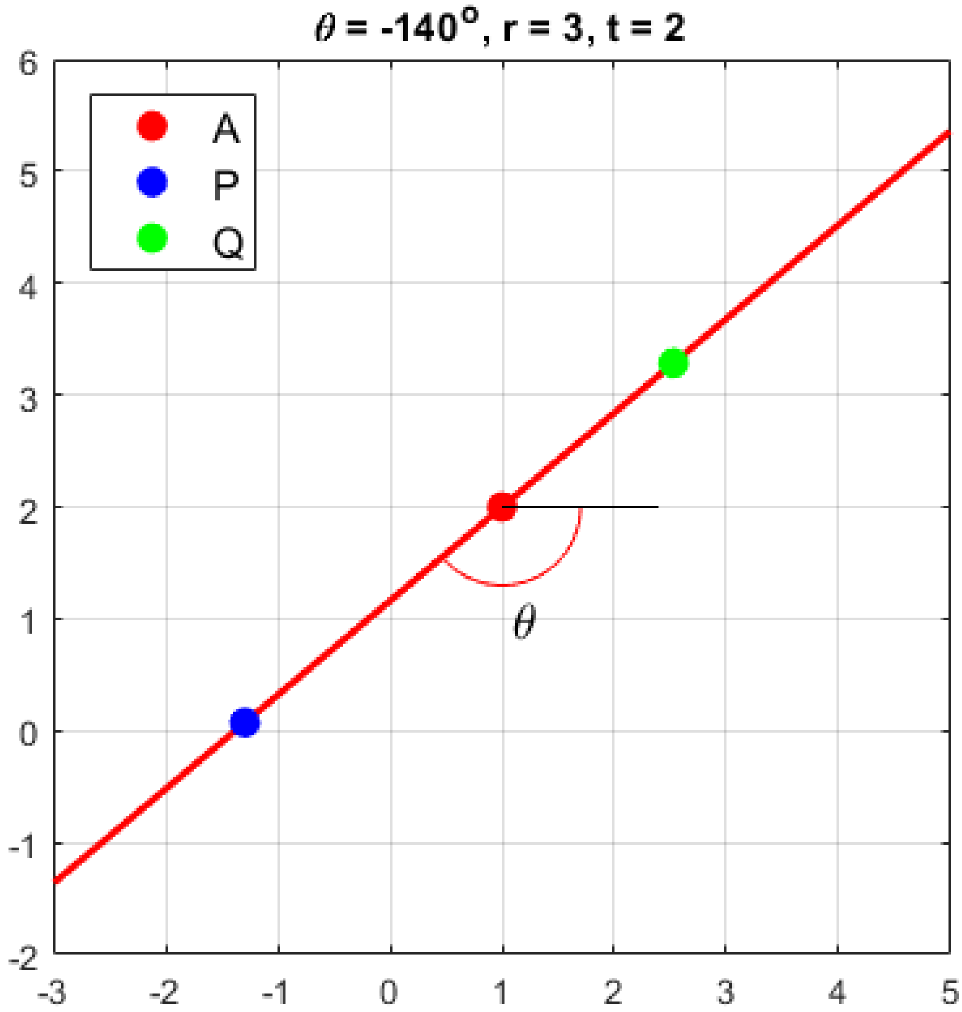
$$Px = Ax + r\cos(\theta)$$

$$Py = Ay + r\sin(\theta)$$

$$Qx = Ax - t\cos(\theta)$$

$$Qy = Ay - t\sin(\theta)$$

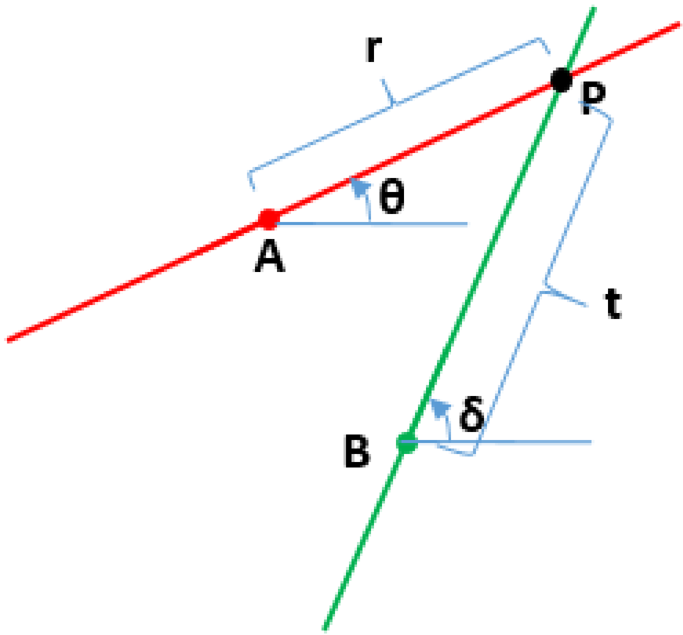

θ = 40°, r = 3, t = 2
A
P
Q
θ
6
5
4
3
2
1
0
-1
-2
-3
-2
-1
0
1
2
3
4
5


θ = -140°, r = 3, t = 2
A
P
Q
θ
6
5
4
3
2
1
0
-1
-2
-3
-2
-1
0
1
2
3
4
5

Esim: Suorien A, θ ja B, δ leikkauspiste P:

$$Px = Ax + r\cos(\theta) = Bx + t\cos(\delta)$$

$$Py = Ay + r\sin(\theta) = By + t\sin(\delta)$$



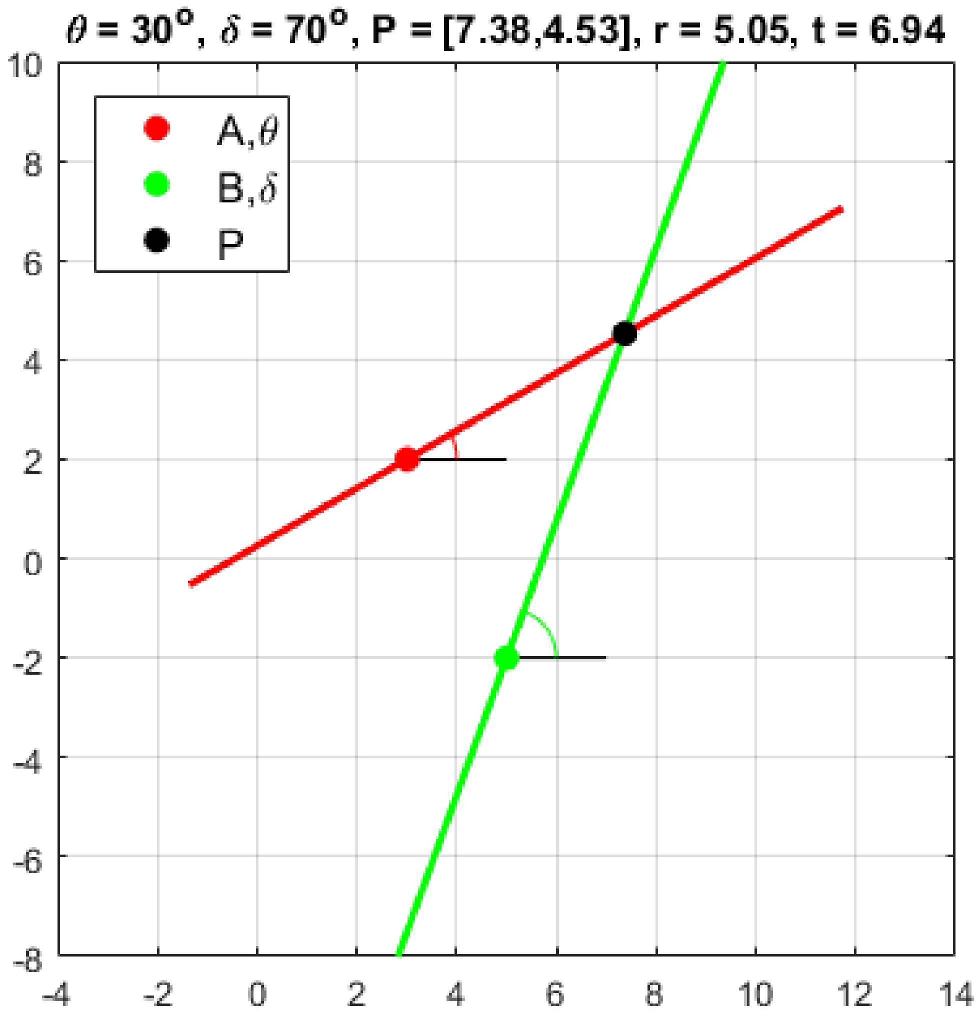
eli saadaan yhtälöpari

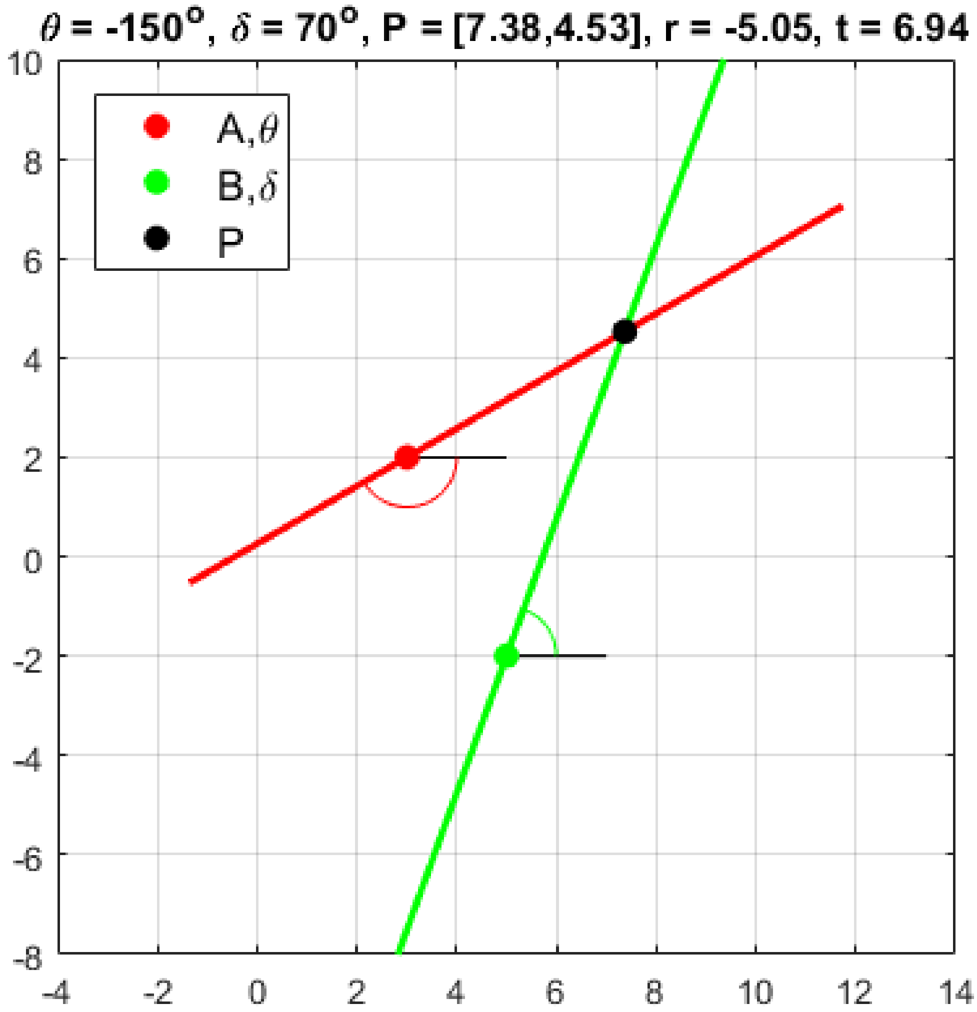
$$\overbrace{\cos(\theta)}^{a} \cdot r \overbrace{- \cos(\delta)}^{b} \cdot t = \overbrace{Bx - Ax}^{e}$$

$$\underbrace{\sin(\theta)}_{c} \cdot r \underbrace{- \sin(\delta)}_{d} \cdot t = \underbrace{By - Ay}_{f}$$

jonka ratkaisu

$$r = \frac{de - bf}{ad - bc}, \quad t = \frac{af - ce}{ad - bc}$$


θ = 30°, δ = 70°, P = [7.38,4.53], r = 5.05, t = 6.94
A,θ
B,δ
P
10
8
6
4
2
0
-2
-4
-6
-8
-4
-2
0
2
4
6
8
10
12
14

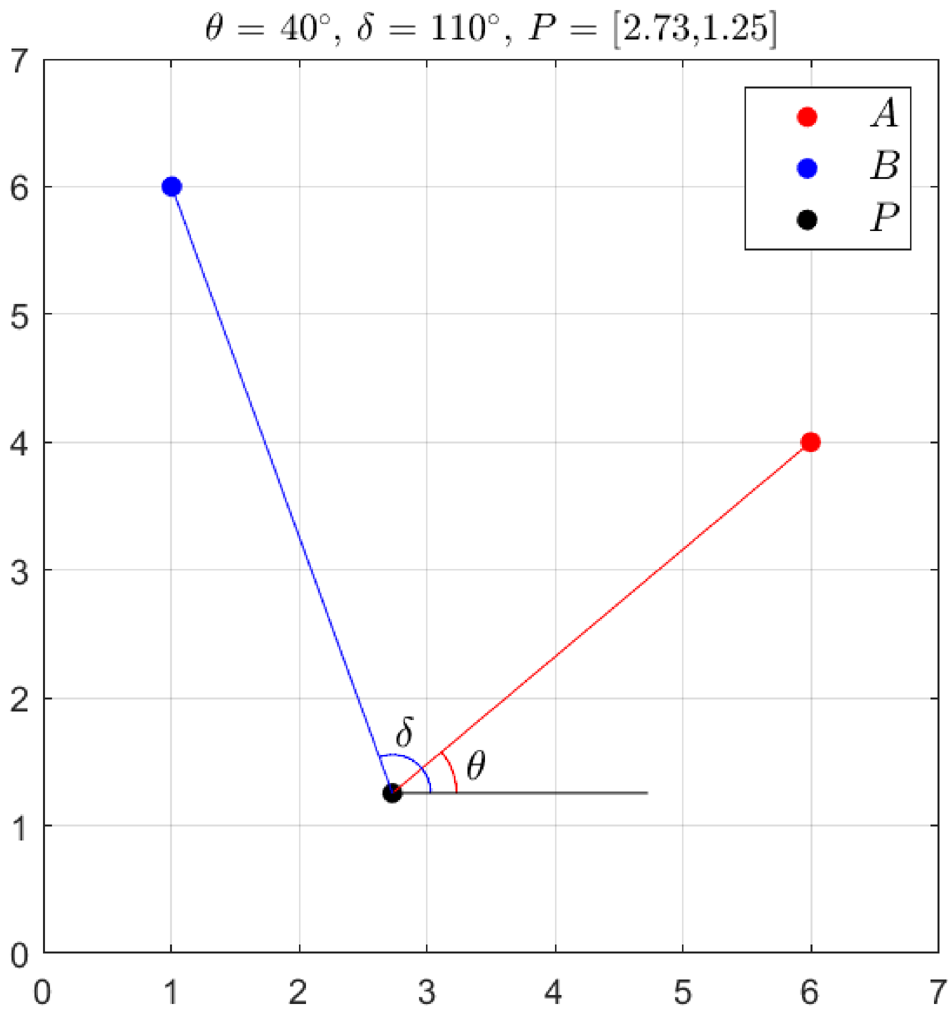

θ = -150°, δ = 70°, P = [7.38,4.53], r = -5.05, t = 6.94
A,θ
B,δ
P
10
8
6
4
2
0
-2
-4
-6
-8
-4
-2
0
2
4
6
8
10
12
14

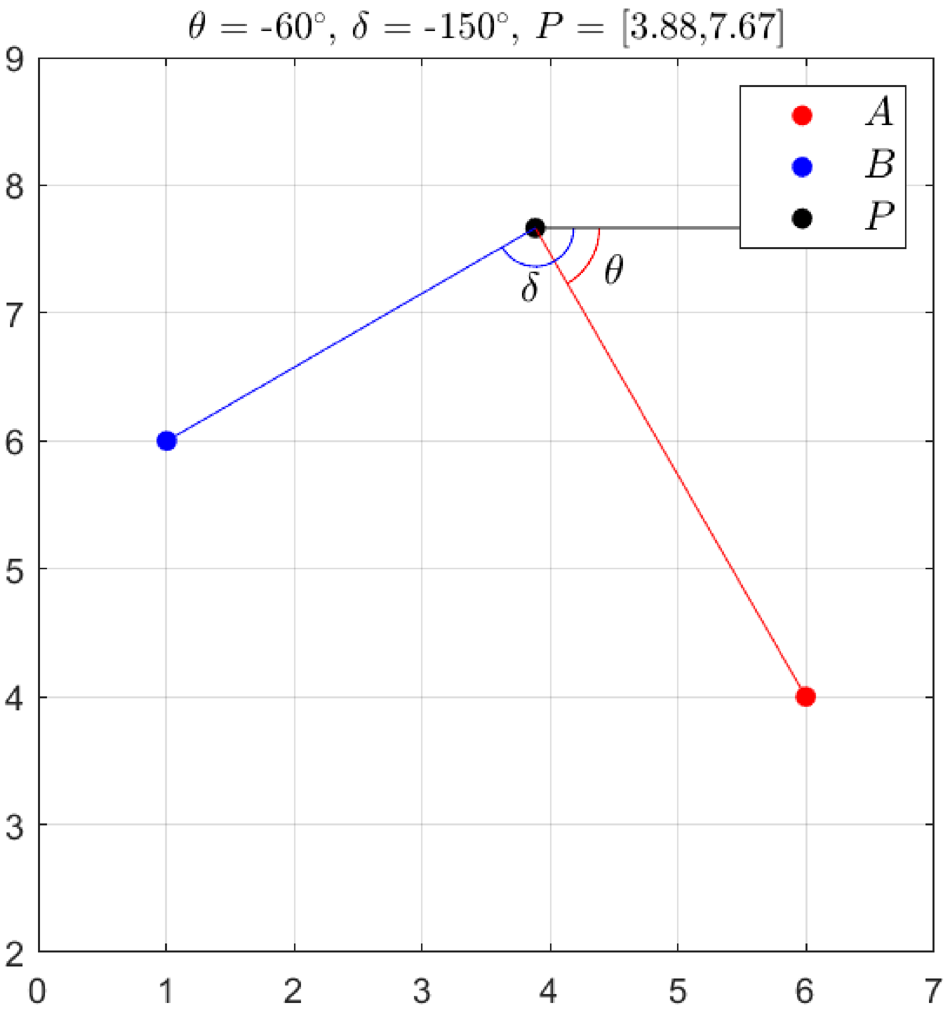
Esim: Paikannus suuntamittauksista

Tunnetut tukiasemat A ja B, tuntematon sijainti P

Mitataan PA:n ja PB:n suuntakulmat θ ja δ

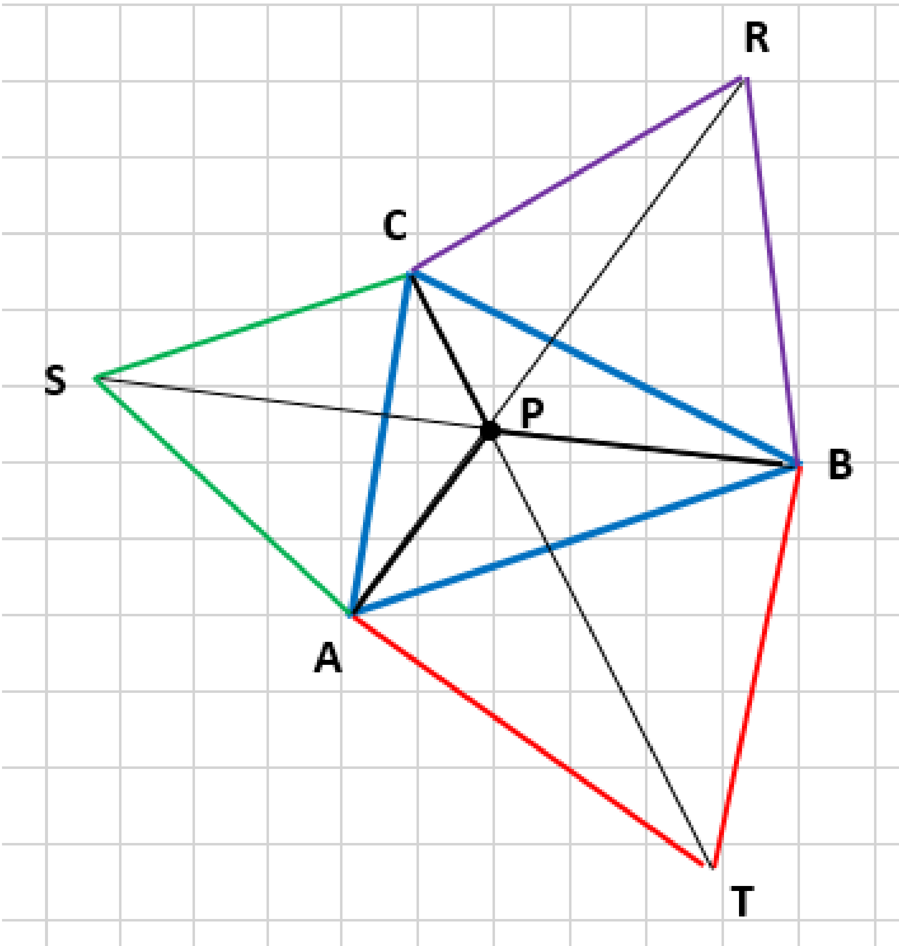
P on suorien A, θ ja B, δ leikkauspiste





θ = -60°, δ = -150°, P = [3.88,7.67]
A
B
P
δ
θ
0
1
2
3
4
5
6
7
2
3
4
5
6
7
8
9

Esim: kolmion ABC Fermatin piste P, jolle etäisyyksien summa $AP+BP+CP$ on pienin



Kolmiot ABT, BCR ja ACS ovat tasasivuisia (eli $AT = BT = AC$ jne), eli niiden kulmat ovat 60°

Suuntakulmat

$$\theta_{AT} = \theta_{AB} - 60^\circ$$

$$\theta_{BR} = \theta_{BC} - 60^\circ$$

$$\theta_{CS} = \theta_{CA} - 60^\circ$$

Koordinaatit

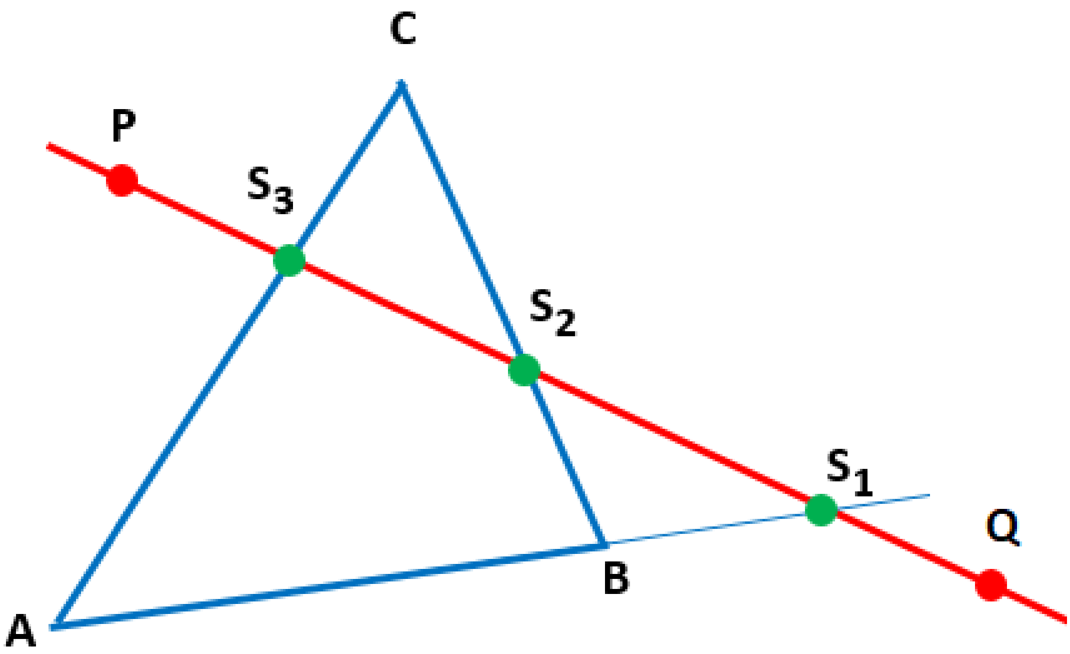
$$Tx = Ax + AB\cos(\theta_{AT})$$
$$Ty = Ay + AB\sin(\theta_{AT})$$

$$Rx = Bx + BC\cos(\theta_{BC})$$
$$Ry = By + BC\sin(\theta_{BC})$$

$$Sx = Cx + CA\cos(\theta_{CA})$$
$$Sy = Cy + CA\sin(\theta_{CA})$$

P on suorien A, θ_{AR}, B, θ_{BS} ja C, θ_{CT} leikkauspiste

Esim: Suoran PQ ja kolmion ABC leikkaus-pisteet



Lasketaan suoran P, θ_{PQ} ja suorien A, θ_{AB}, B, θ_{BC}, C, θ_{CA} leikkauspisteet S_1, S_2, S_3

$$S_{1x} = Ax + r_1 \cos(\theta_{AB})$$
$$S_{1y} = Ay + r_1 \sin(\theta_{AB})$$

$$S_{2x} = Bx + r_2 \cos(\theta_{BC})$$
$$S_{2y} = By + r_2 \sin(\theta_{BC})$$

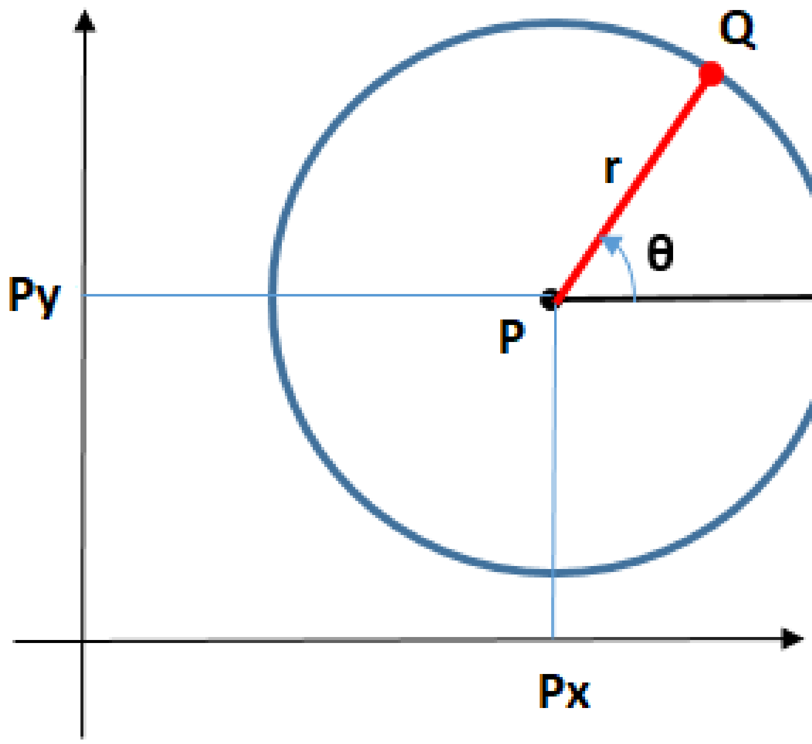
$$S_{3x} = Cx + r_3 \cos(\theta_{CA})$$
$$S_{3y} = Cy + r_3 \sin(\theta_{CA})$$

S_1 on sivulla AB, jos r_1 on välillä $0 \ldots AB$

S_2 on sivulla BC, jos r_2 on välillä $0 \ldots BC$

S_3 on sivulla CA, jos r_3 on välillä $0 \ldots CA$

Esim: Ympyrä, keskipiste P ja säde r.



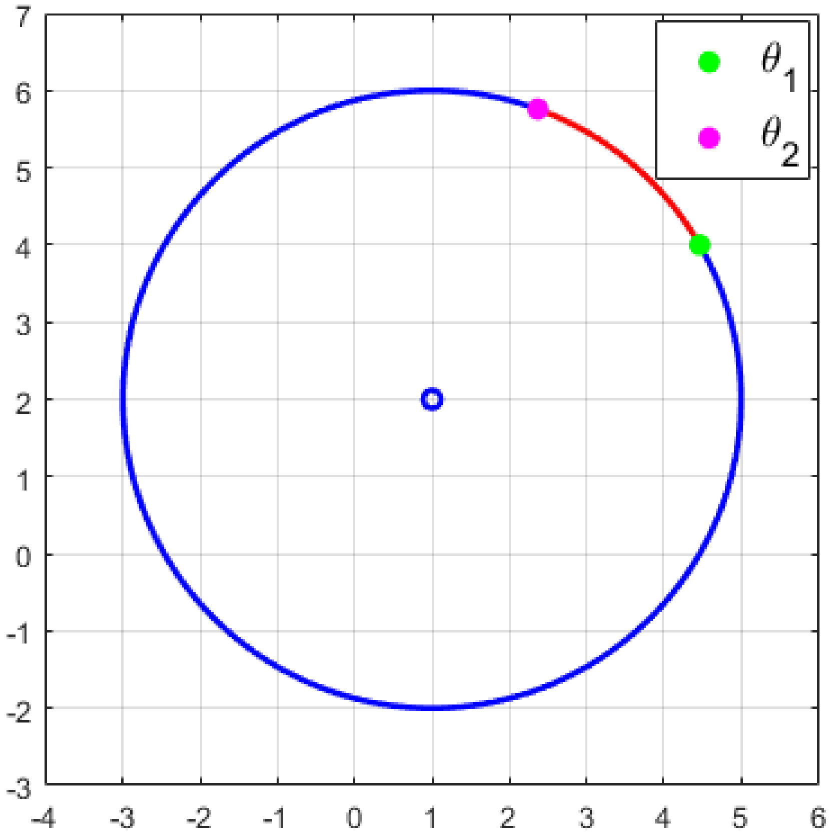
Kiertokulmaa θ vastaava ympyrän piste Q:

$$Qx = Px + r\,\mathsf{cos}(\theta)$$

$$Qy = Py + r\,\mathsf{sin}(\theta)$$

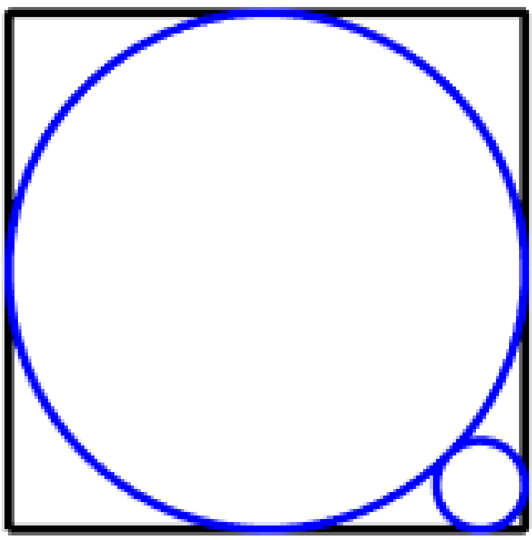
Koko ympyrä: $\theta = 0 \ldots 360^\circ$

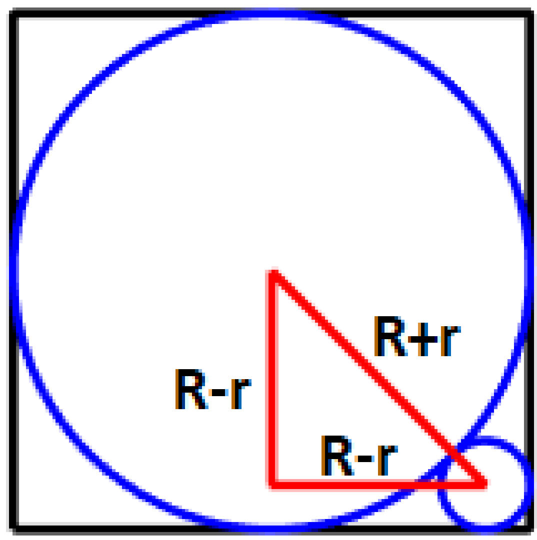
Kaari: $\theta = \theta_1 \ldots \theta_2$



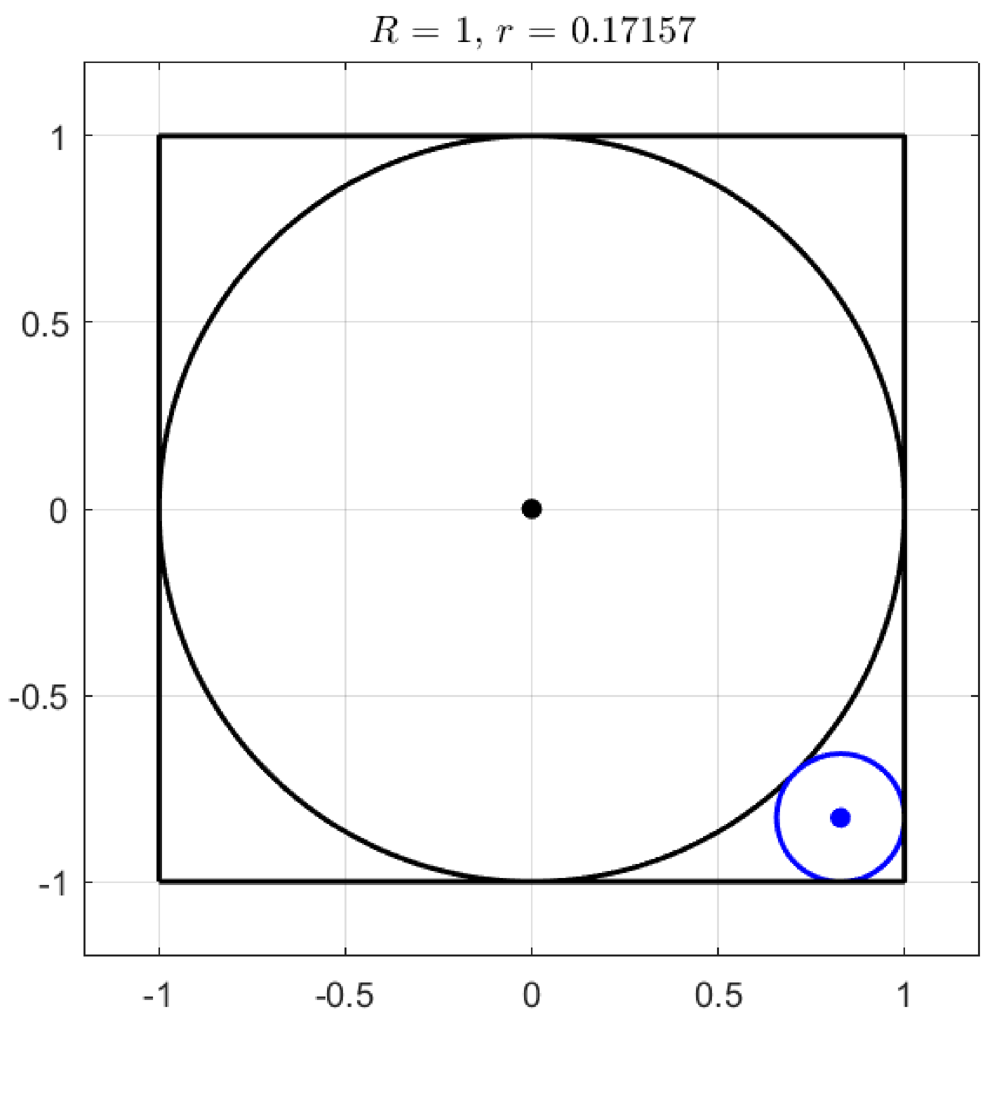
Esim. Jos ison ympyrän säde on R ja pienen r, niin

$$(R+r)^2 = 2(R-r)^2 \to r = (3-\sqrt{8})R \approx 0.17R$$

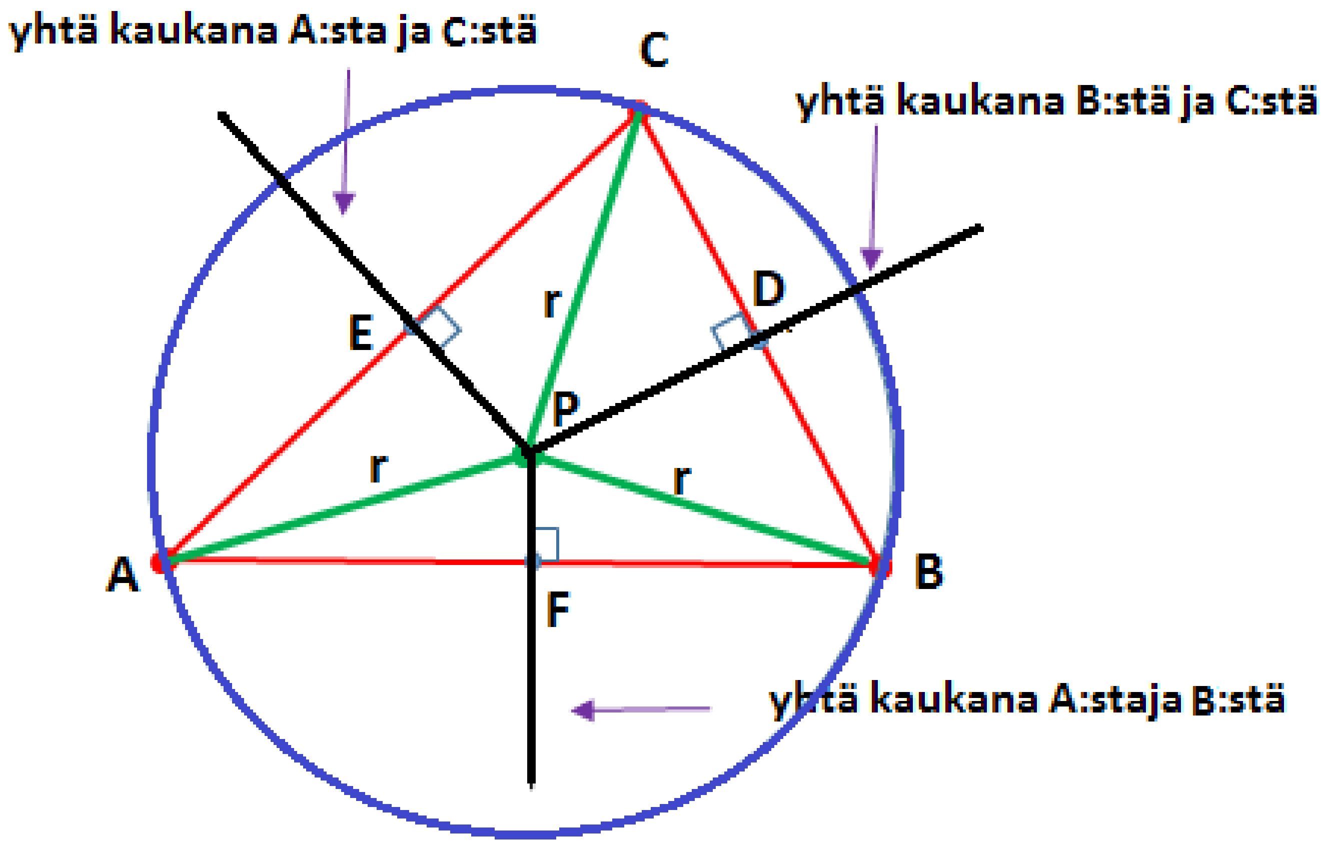




Jos ison ympyrän keskipiste on $[0, 0]$, niin pienen ympyrän keskipiste on $[R - r, -(R - r)]$



Esim: Pisteiden A, B ja C kautta kulkevan ympyrän keskipiste P on kolmion ABC sivujen keskinormaalien leikkauspiste.

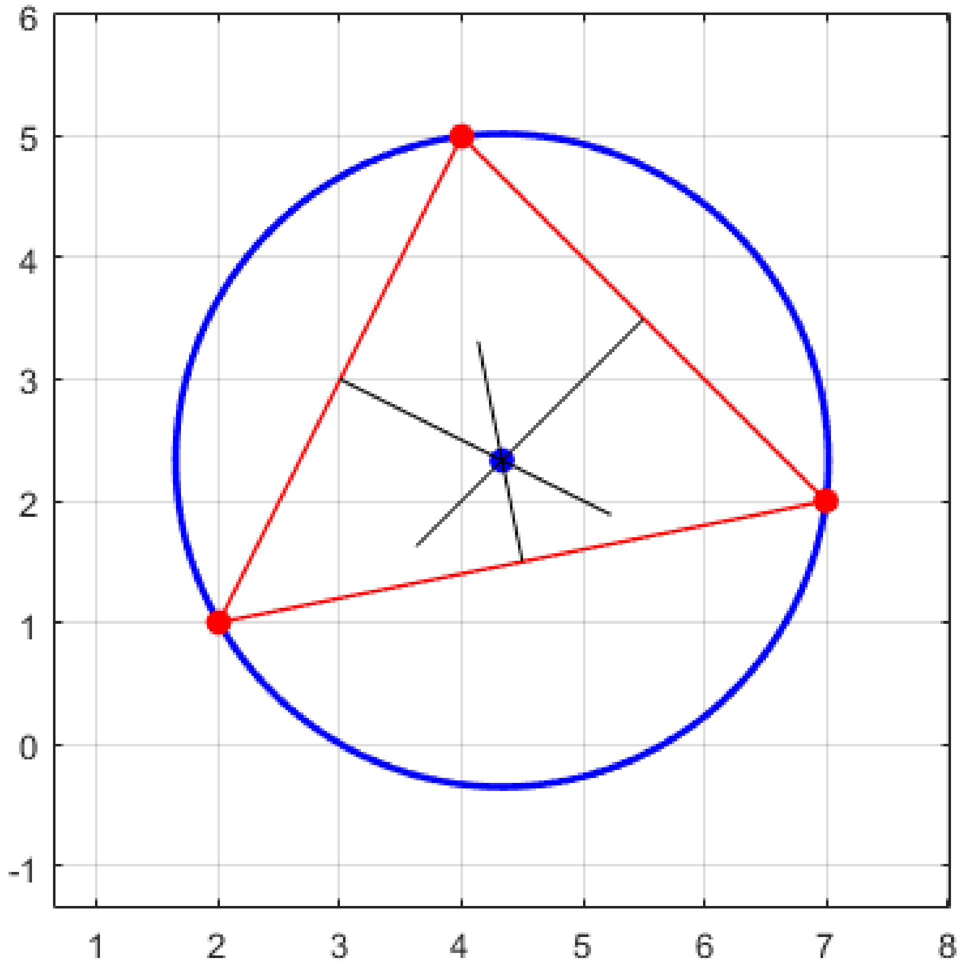


Sivun keskinormaali = sivun keskipisteen kautta kulkeva, sivua vastaan kohtisuora suora = ne pisteet jotka ovat yhtä kaukana sivun päätepisteistä

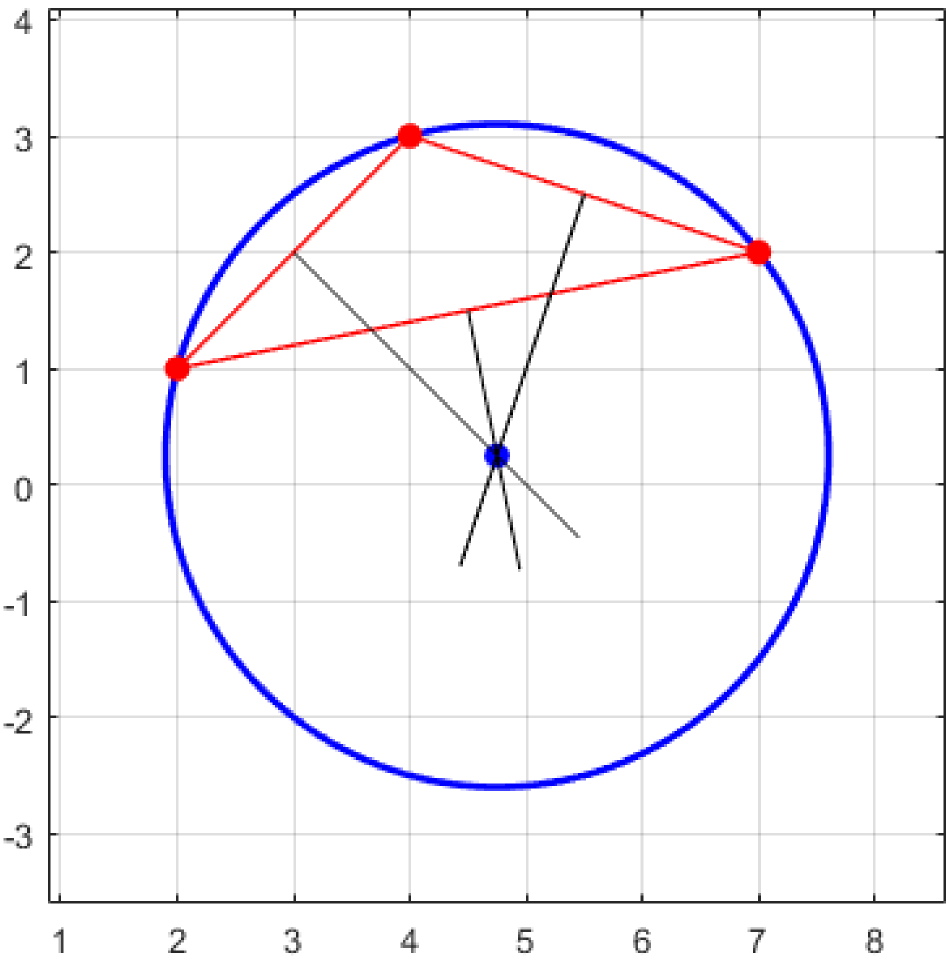
Eli, jos D, E ja F ovat kolmion sivujen keskipisteet, niin P on suorien

$D, \theta_{BC} + 90°$, $E, \theta_{CA} + 90°$ ja $F, \theta_{AB} + 90°$

leikkauspiste



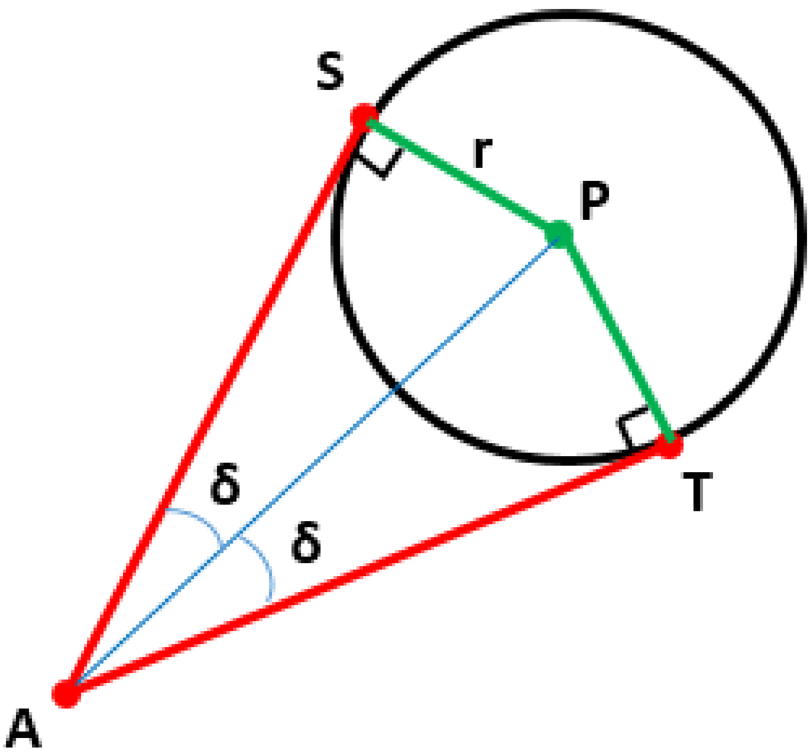
6
5
4
3
2
1
0
-1
1
2
3
4
5
6
7
8



4
3
2
1
0
-1
-2
-3
1
2
3
4
5
6
7
8

Esim: Tangentit ympyrälle P, r pisteestä A

$Ax, Ay, Px, Py, r \rightarrow Sx, Sy, Tx, Ty$



$AS = AT = \sqrt{AP^2 - r^2}$

$\delta = \sin^{-1}(r/AP)$

Suuntakulmat:

$$\theta_{AS} = \theta_{AP} + \delta$$

$$\theta_{AT} = \theta_{AP} - \delta$$

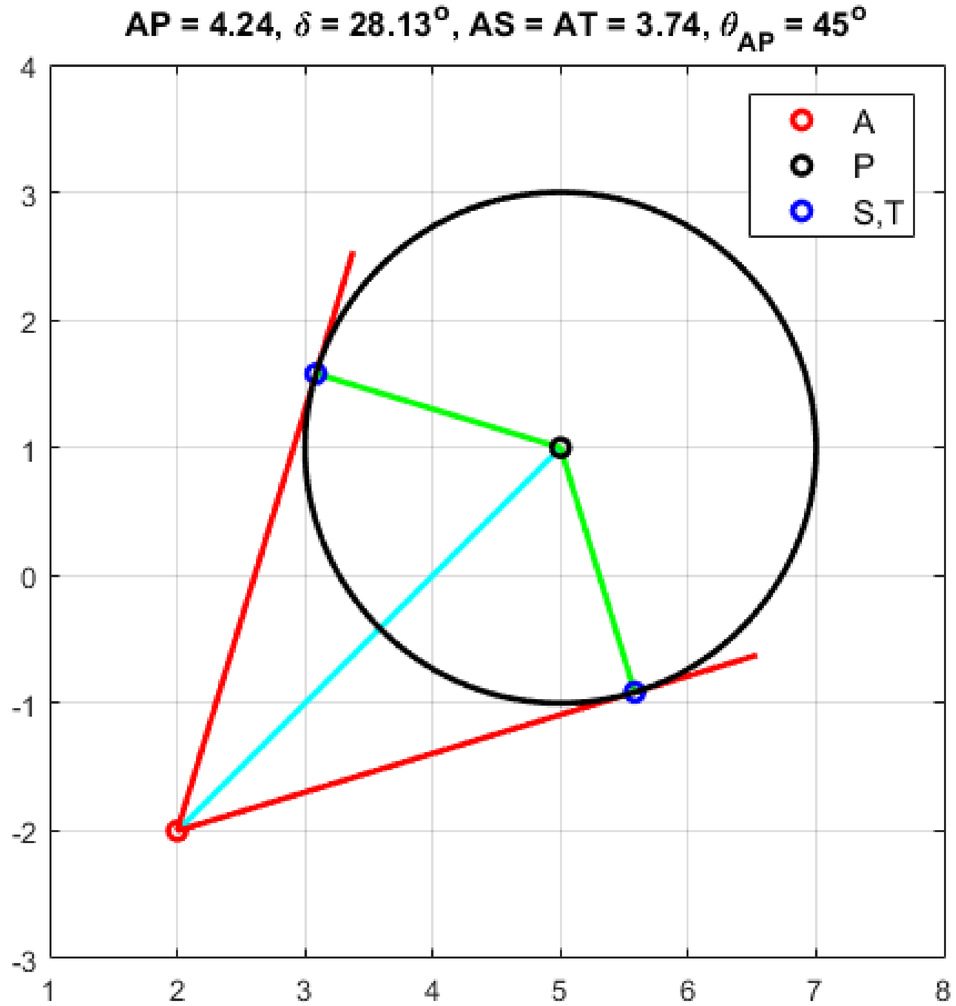
S:n ja T:n koordinaatit:

$$Sx = Ax + AS \cos(\theta_{AS})$$

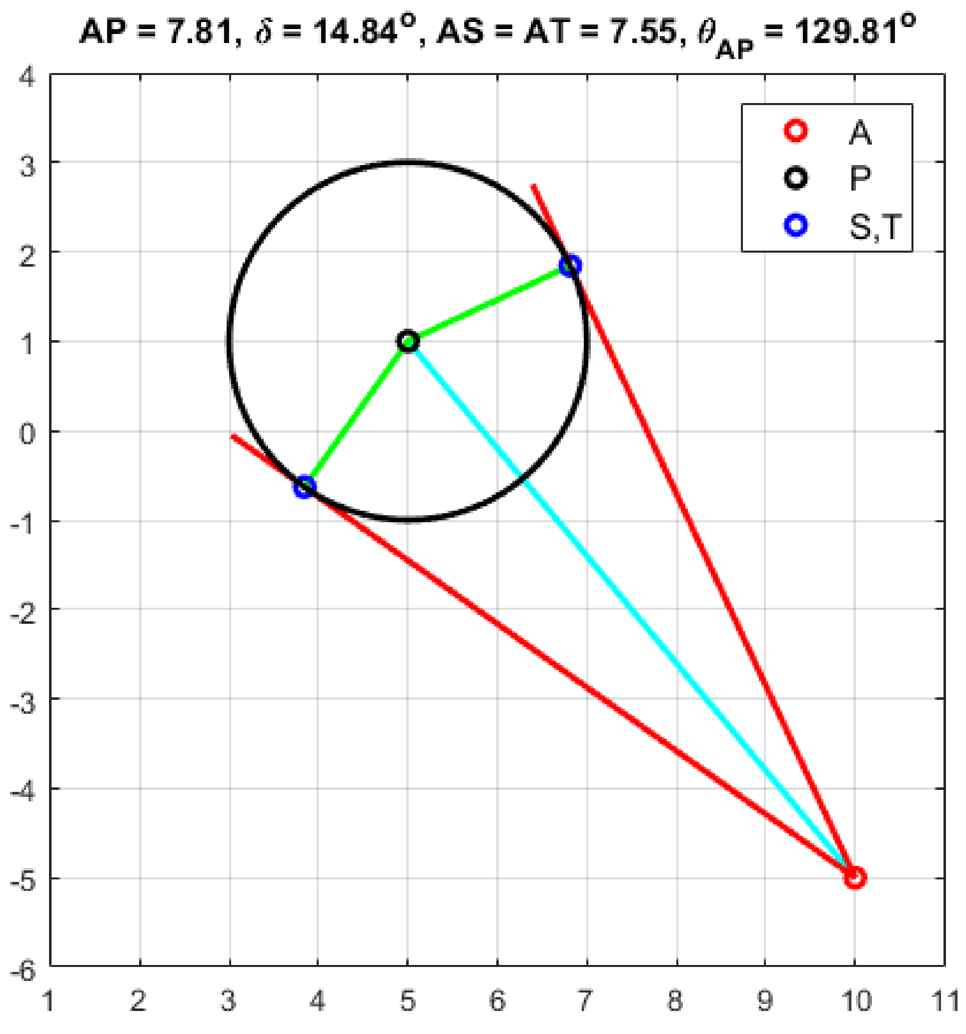
$$Sy = Ay + AS \sin(\theta_{AS})$$

$$Tx = Ax + AT \cos(\theta_{AT})$$

$$Ty = Ay + AT \sin(\theta_{AT})$$

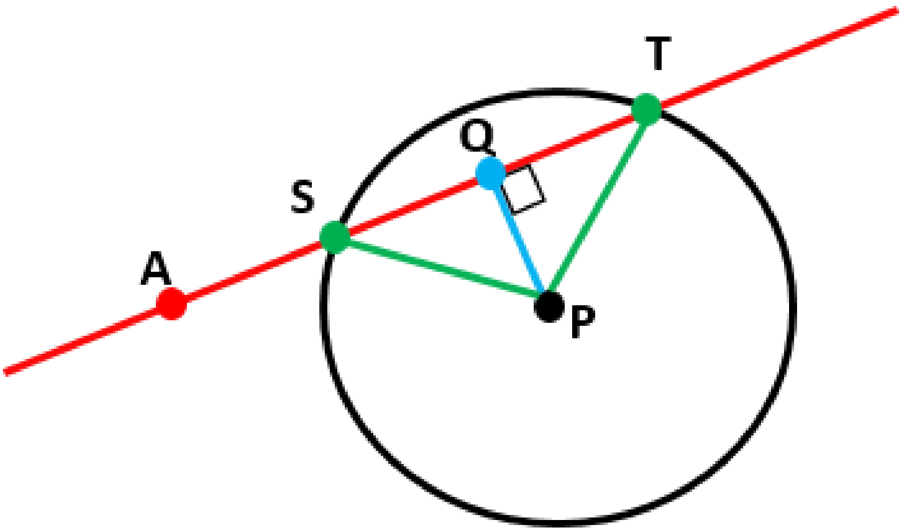


AP = 4.24, δ = 28.13^o, AS = AT = 3.74, θ_{AP} = 45^o
A
P
S,T
4
3
2
1
0
-1
-2
-3
1
2
3
4
5
6
7
8



AP = 7.81, δ = 14.84°, AS = AT = 7.55, θ_{AP} = 129.81°
A
P
S,T
4
3
2
1
0
-1
-2
-3
-4
-5
-6
1
2
3
4
5
6
7
8
9
10
11

Esim: Suoran A, θ ja ympyrän P, r leikkauspisteet



Q on suorien A, θ ja $P, \theta + 90^\circ$ leikkauspiste

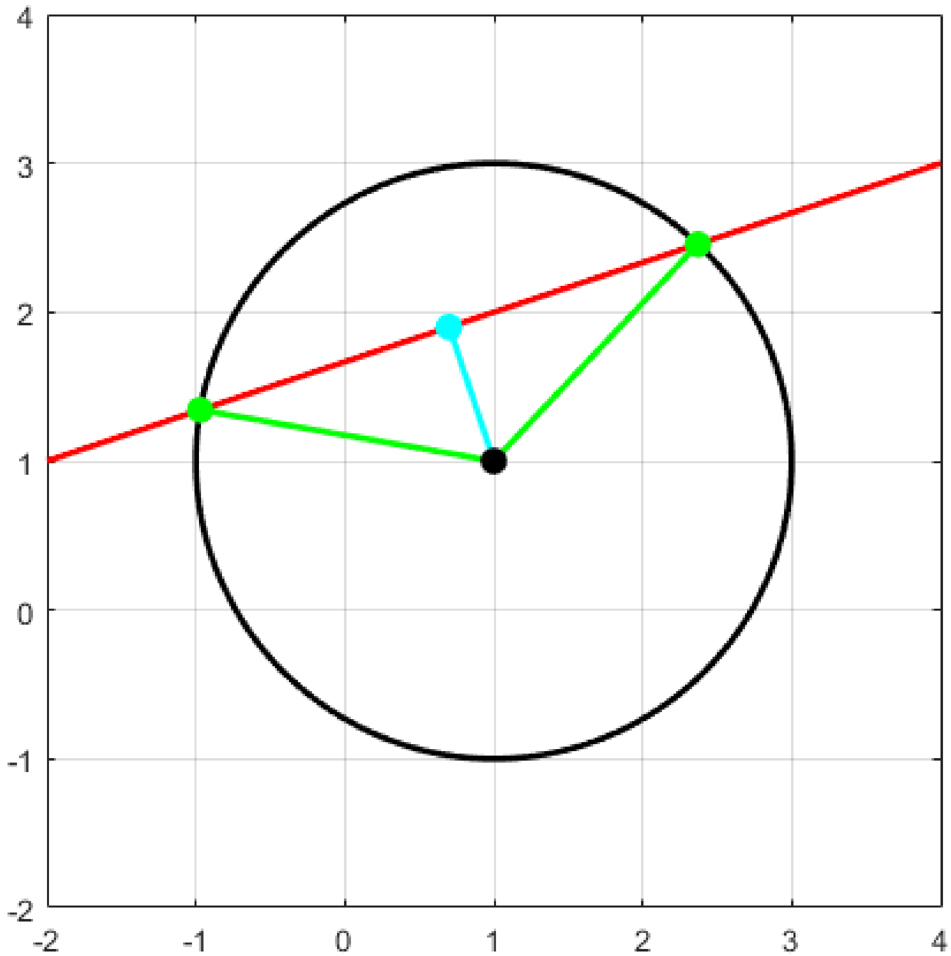
Suora ja ympyrä leikkaavat, jos $PQ \leq r$. Tällöin

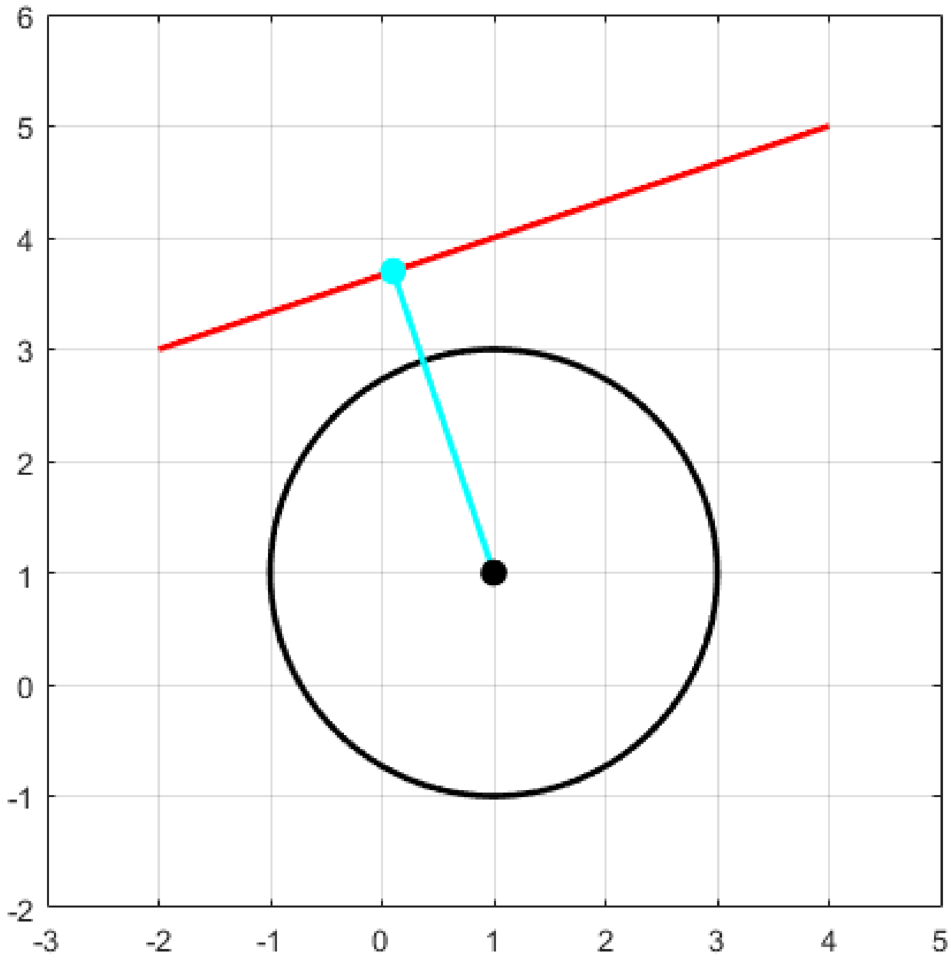
$$QS = QT = \sqrt{r^2 - PQ^2}$$

ja leikkauspisteiden koordinaatit ovat:

$$Sx = Qx - QS\cos(\theta)$$
$$Sy = Qy - QS\sin(\theta)$$

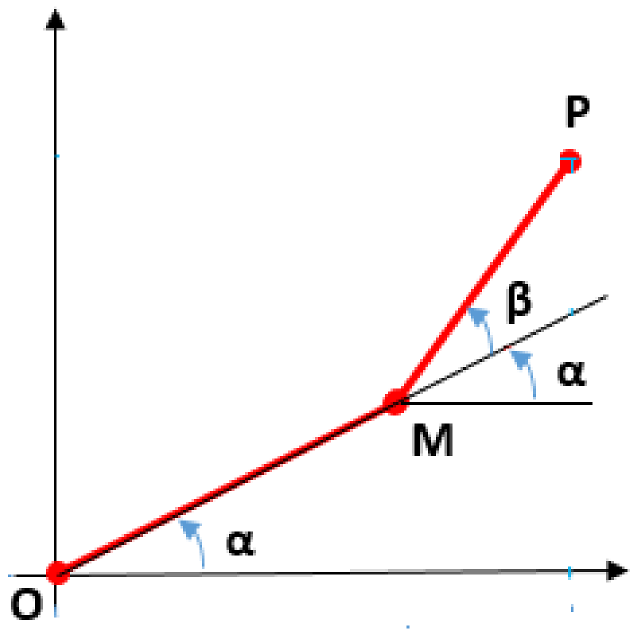
$$Tx = Qx + QT\cos(\theta)$$
$$Ty = Qy + QT\sin(\theta)$$

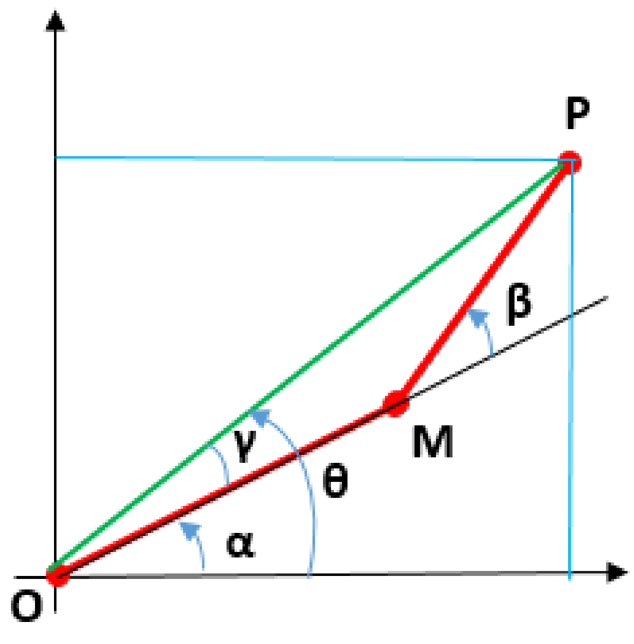

4
3
2
1
0
-1
-2
-2
-1
0
1
2
3
4



6
5
4
3
2
1
0
-1
-2
-3
-2
-1
0
1
2
3
4
5

Esim: Käsivarsi (two-arm robot), varsien pituudet OM ja MP





suora kinematiikka: $\alpha, \beta \rightarrow Px, Py$

$$Mx = OM \cos(\alpha)$$
$$My = OM \sin(\alpha)$$

$$Px = Mx + MP \cos(\alpha + \beta)$$
$$Py = My + MP \sin(\alpha + \beta)$$

käänteinen kinematiikka: $Px, Py \rightarrow \alpha, \beta$

$$OP = \sqrt{Px^2 + Py^2}$$

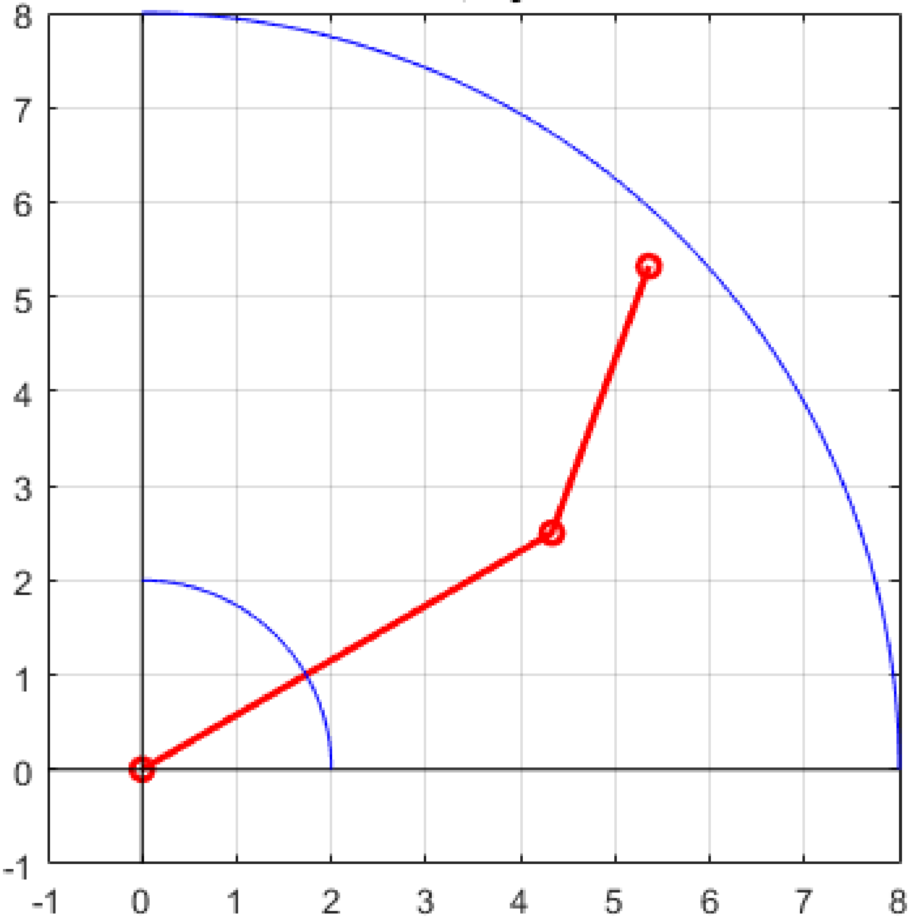
$$\theta = \text{atan2d}(Py, Px)$$

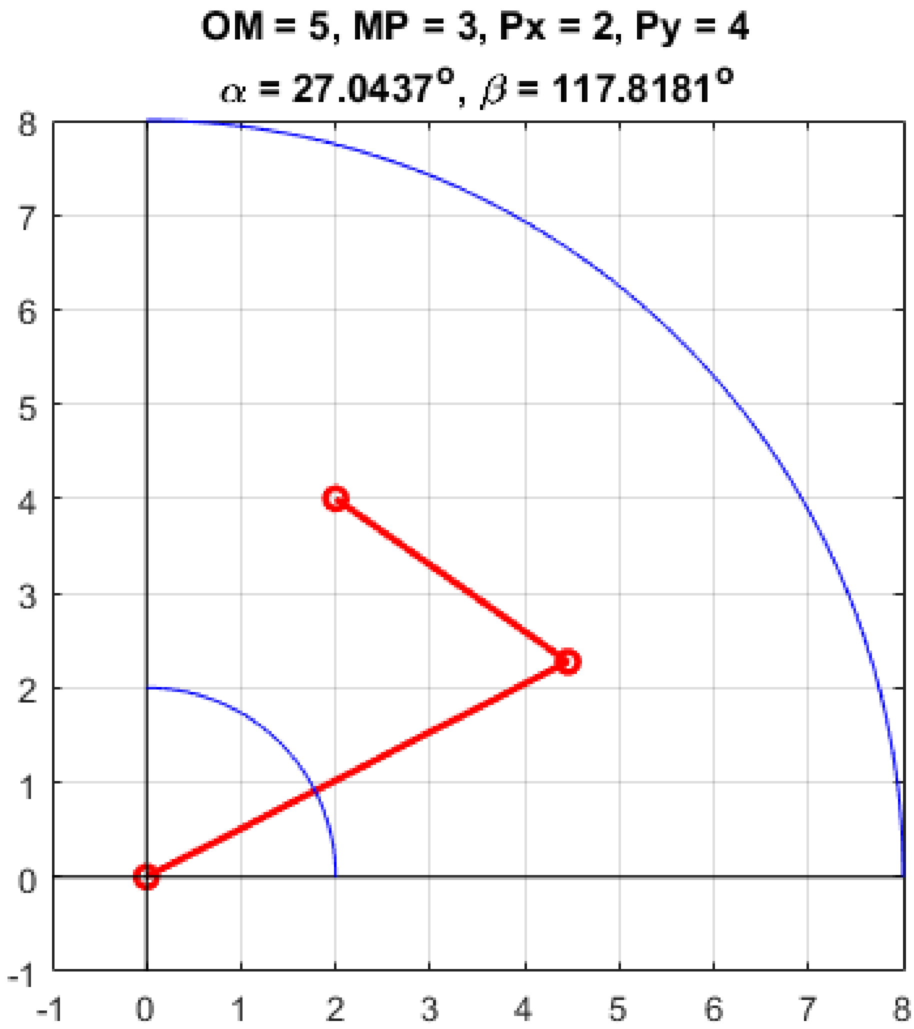
$$\gamma = \cos^{-1}\left(\frac{OP^2 + OM^2 - MP^2}{2 \cdot OP \cdot OM}\right)$$

$$\alpha = \theta - \gamma$$

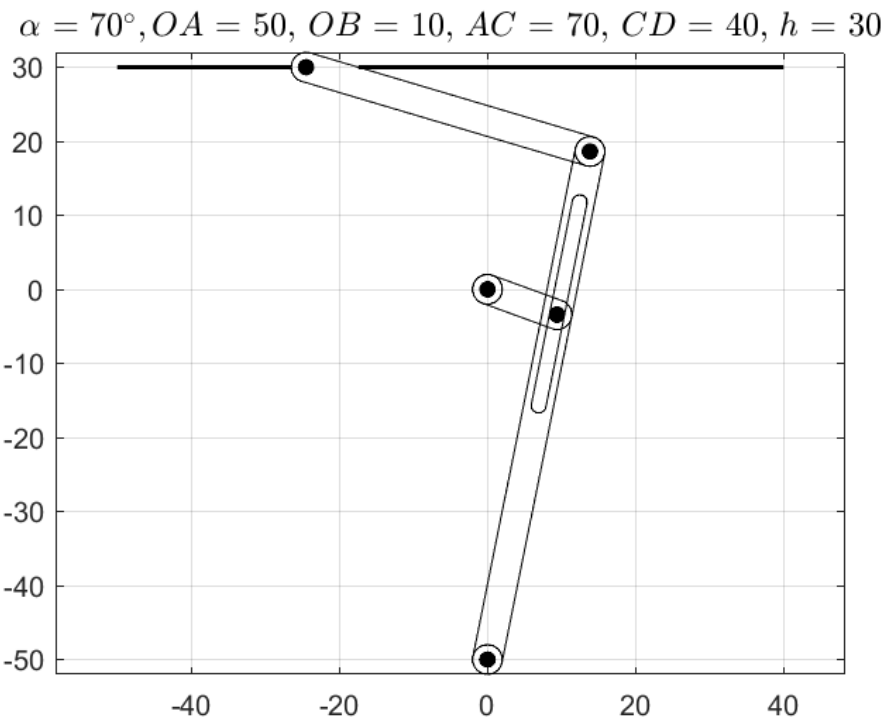
$$\beta = 180^\circ - \cos^{-1}\left(\frac{MP^2 + OM^2 - OP^2}{2 \cdot MP \cdot OM}\right)$$

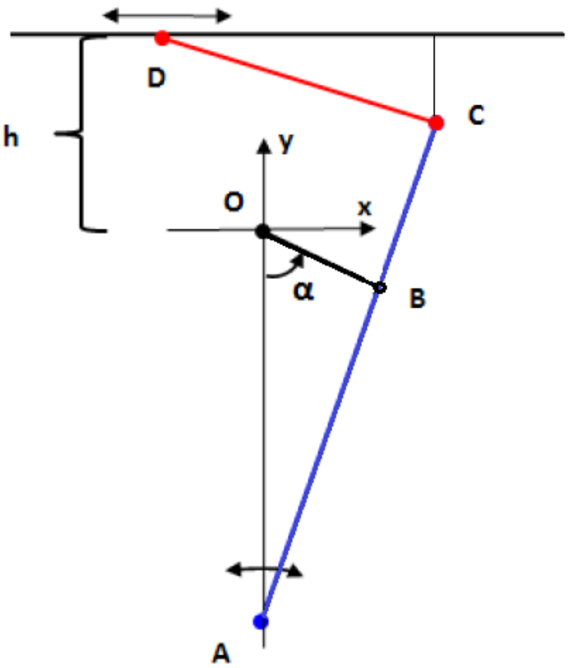
OM = 5, MP = 3, α = 30°, β = 40°
Px = 5.3562, Py = 5.3191




OM = 5, MP = 3, Px = 2, Py = 4
α = 27.0437°, β = 117.8181°
8
7
6
5
4
3
2
1
0
-1
-1 0 1 2 3 4 5 6 7 8

Esim: "Siirtäjä": O ja A kiinteitä, OB pyörii ja B liukuu pitkin AC:tä, D liikkuu edestakaisin korkeudella h





Annetut mitat: OA, OB, AC, CD, h

Koordinaatit:

$Ax = 0,\ Ay = -OA$

$\theta_{OB} = -90^\circ + \alpha$

$Bx = OB\cos(\theta_{OB})$

$By = OB\sin(\theta_{OB})$

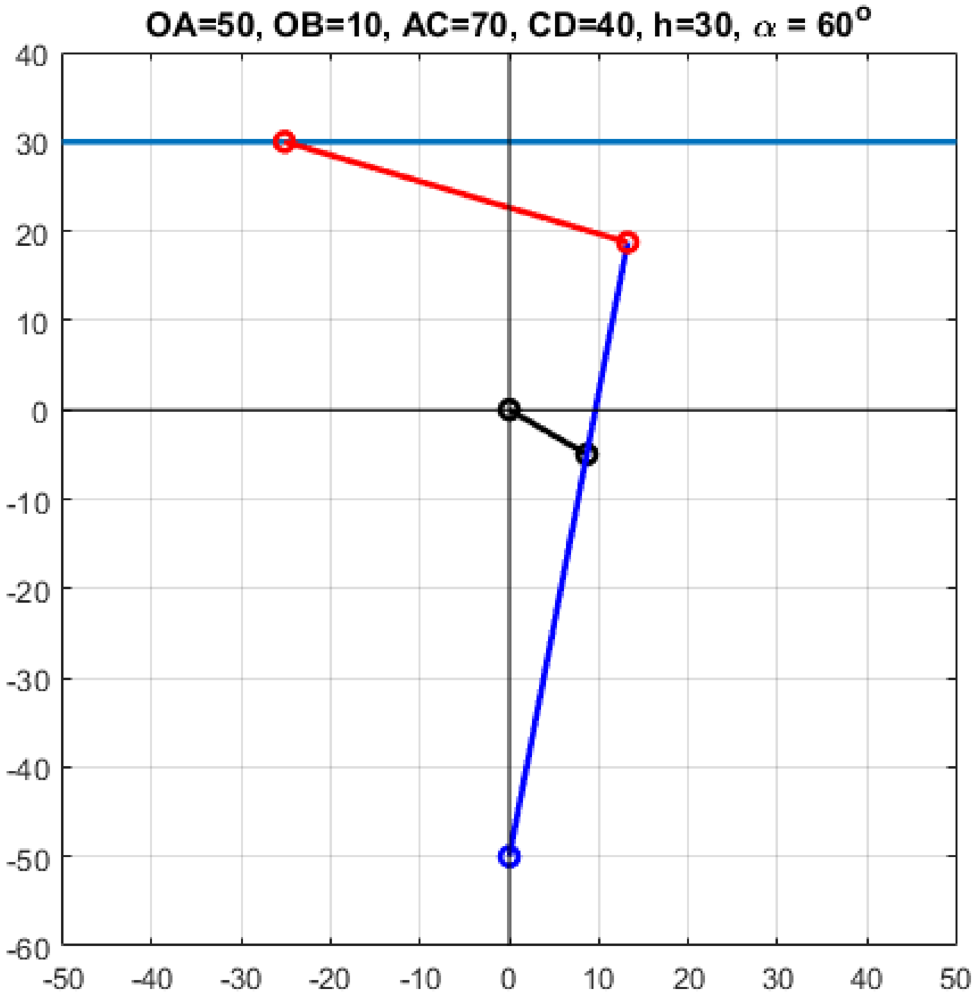
$\theta_{AC} = \theta_{AB} = \mathrm{atan2d}(By - Ay, Bx - Ax)$

$Cx = Ax + AC\cos(\theta_{AC})$

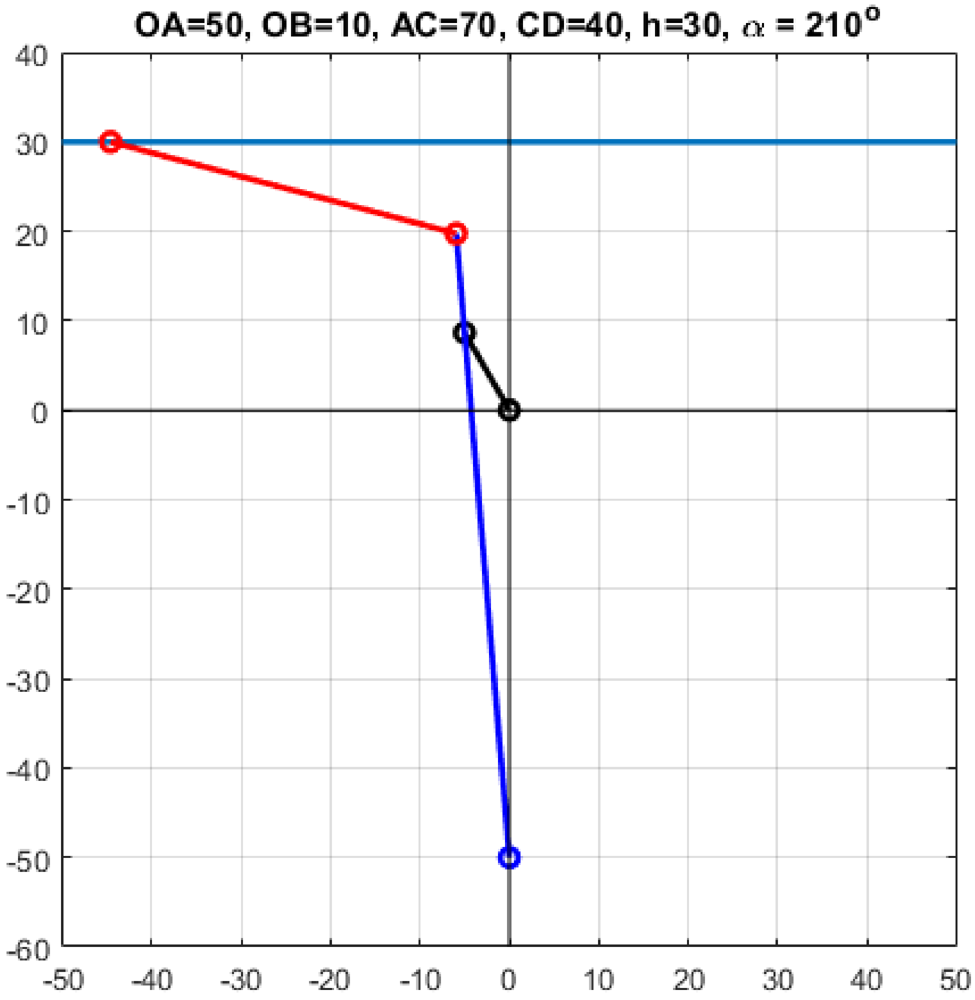
$Cy = Ay + AC\sin(\theta_{AC})$

$Dx = Cx - \sqrt{CD^2 - (Dy - Cy)^2}$

$Dy = h$



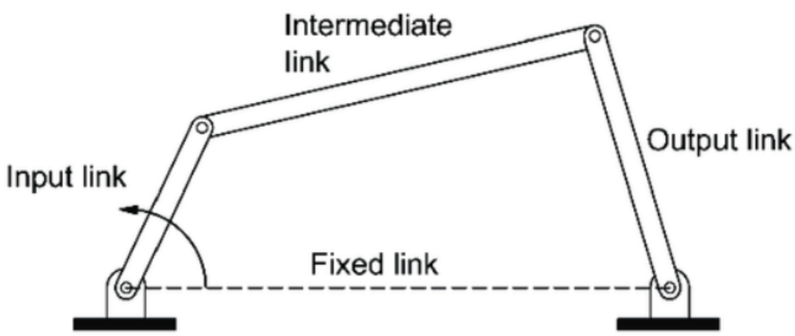
OA=50, OB=10, AC=70, CD=40, h=30, α = 60°
40
30
20
10
0
-10
-20
-30
-40
-50
-60
-50
-40
-30
-20
-10
0
10
20
30
40
50

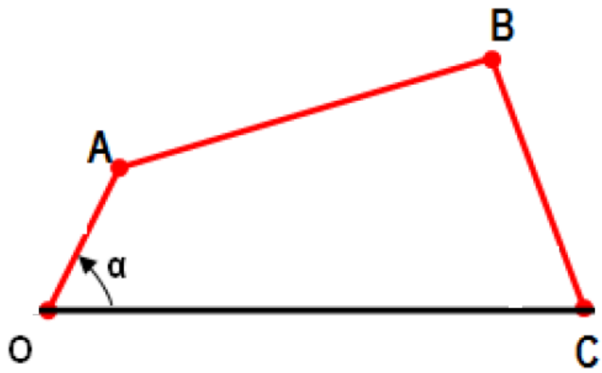

OA=50, OB=10, AC=70, CD=40, h=30, α = 210^o
40
30
20
10
0
-10
-20
-30
-40
-50
-60
-50
-40
-30
-20
-10
0
10
20
30
40
50

Esim. Nelikulmio-mekanismi, four-bar mechanism

O ja C kiinteitä, mitat OA, AB, BC ja OC

OA pyörii



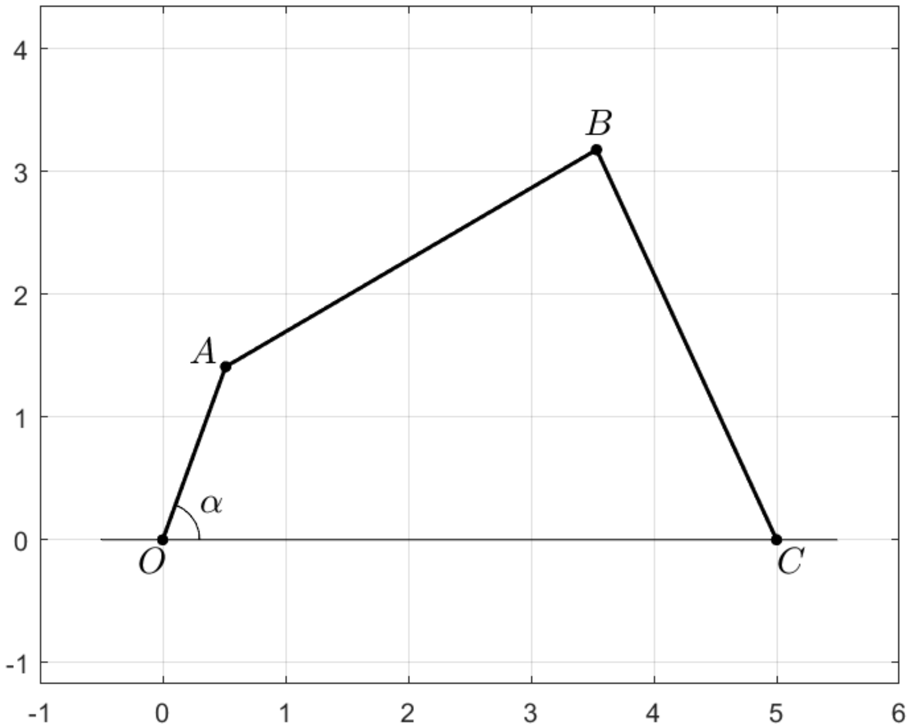


Jos OA on lyhin ja OC pisin ja

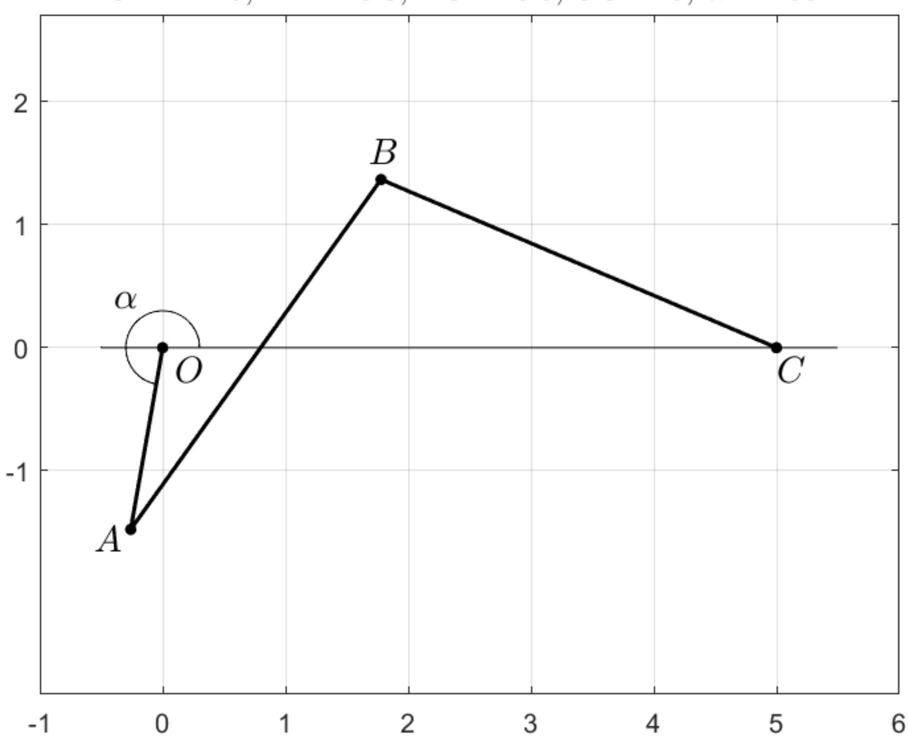
$$OA + OC \leq AB + BC$$

niin OA pääsee pyörimään koko kierroksen (Grashof condition)

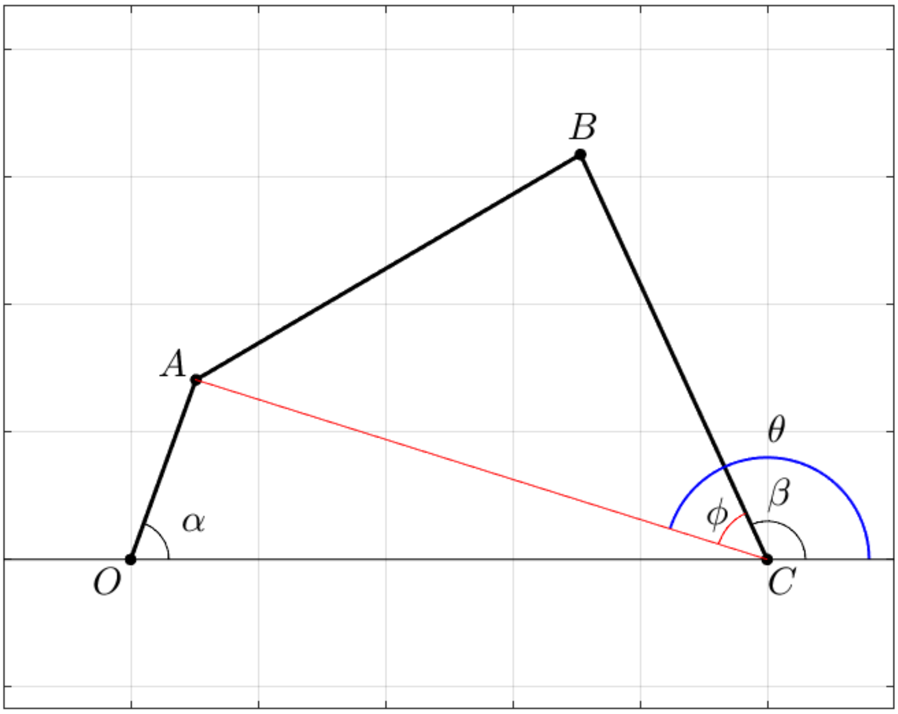
$OA = 1.5,\ AB = 3.5,\ BC = 3.5,\ OC = 5,\ \alpha = 70^\circ$



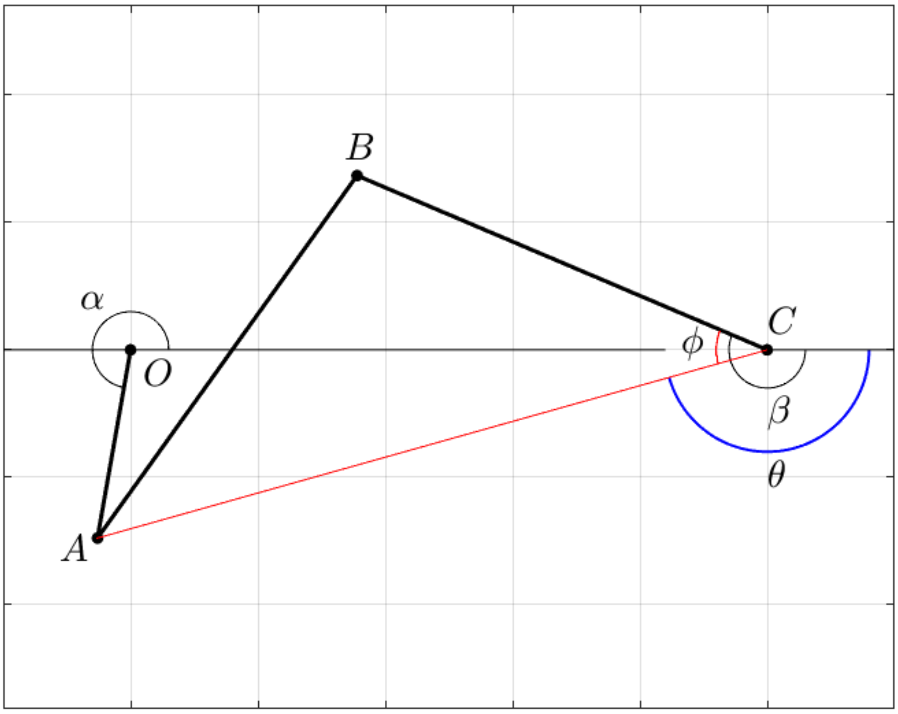
$OA = 1.5,\ AB = 3.5,\ BC = 3.5,\ OC = 5,\ \alpha = 260^\circ$



α = 70°, θ = 162.6°, ϕ = 47.79°, β = 114.77°



α = 260°, θ = -164.3°, ϕ = 38.69°, β = -203°



Koordinaatit:

$$Ox = 0, Oy = 0$$

$$Cx = OC, Cy = 0$$

$$Ax = OA\cos(\alpha),\ Ay = OA\sin(\alpha)$$

$$AC = \sqrt{(Ax - Cx)^2 + (Ay - Cy)^2}$$

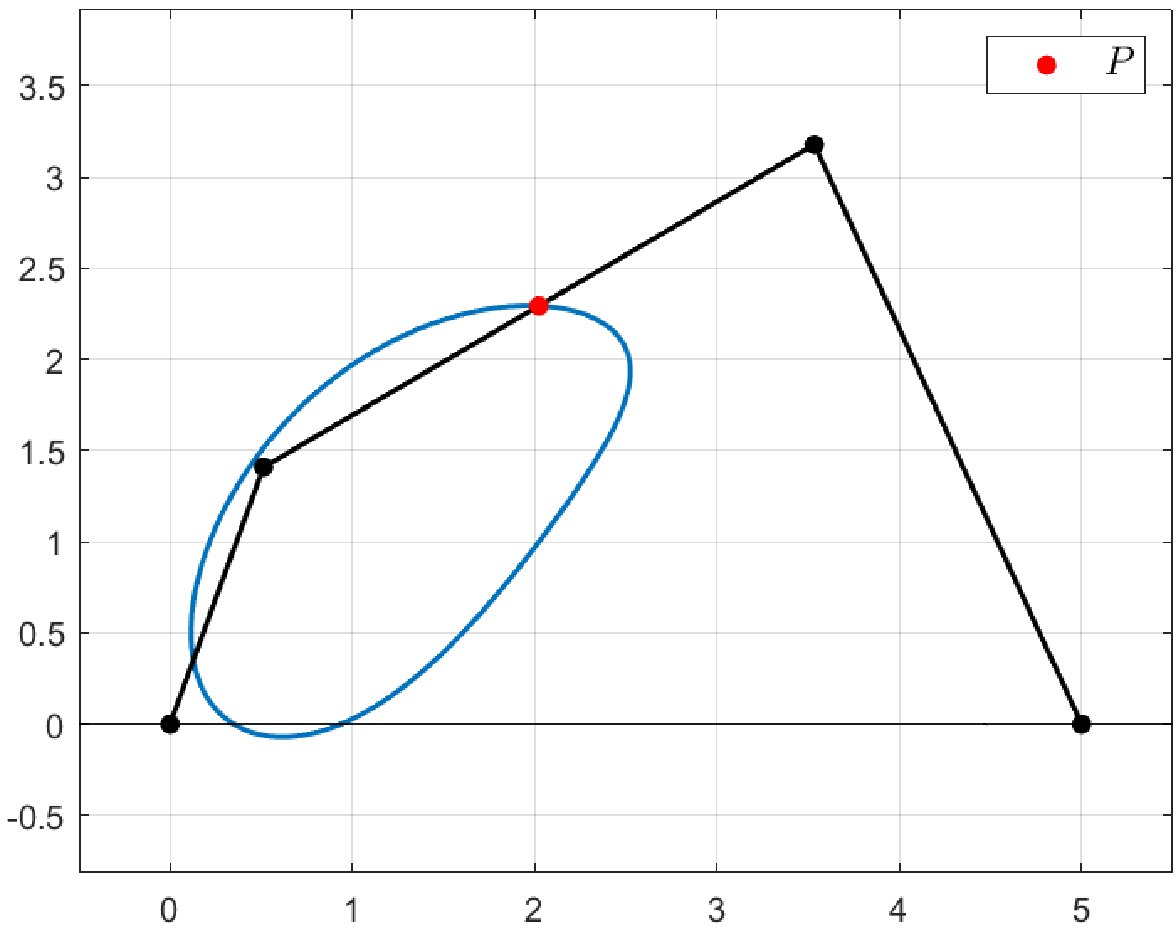
$$\theta = \theta_{CA} = \mathrm{atan2}\,(Ay - Cy, Ax - Cx)$$

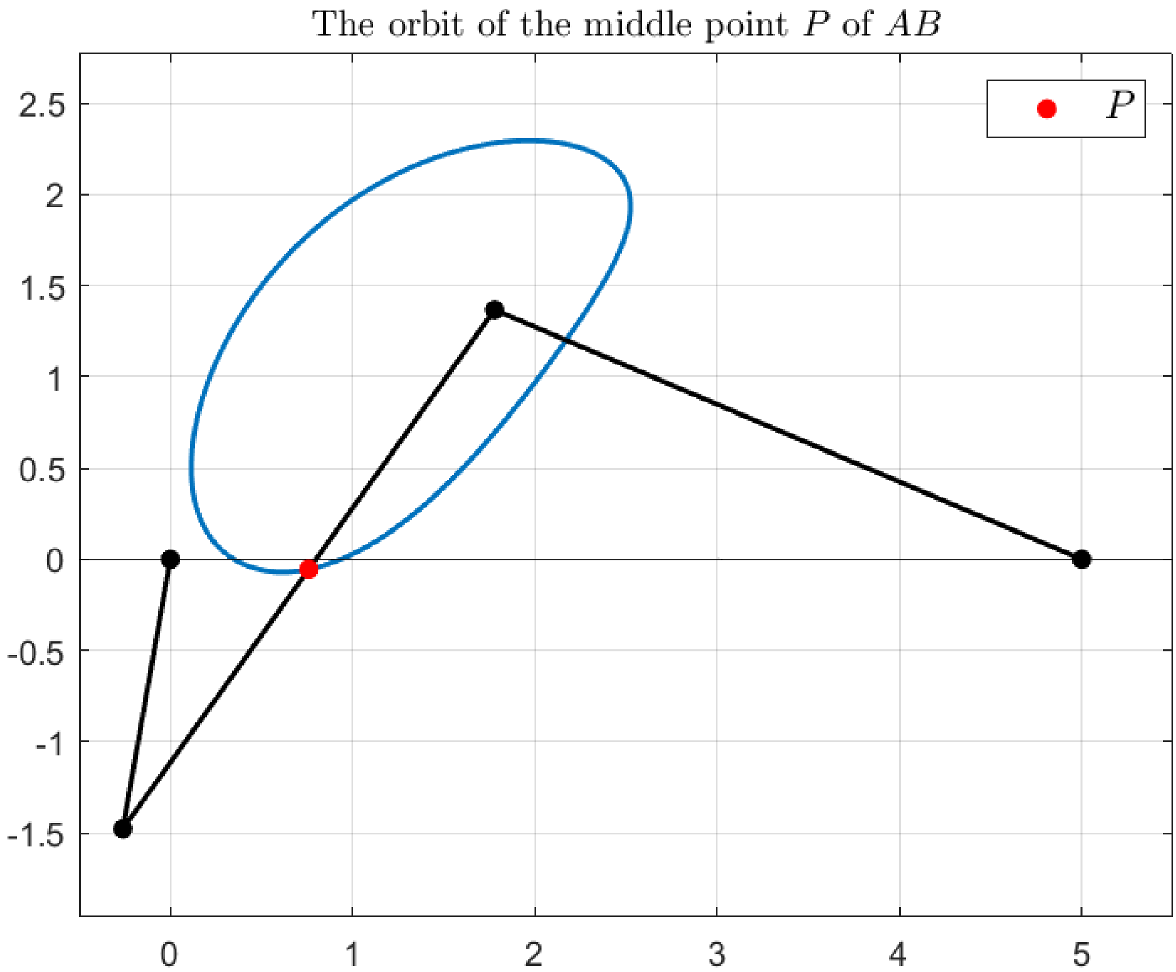
$$\phi = \cos^{-1}\left(\frac{AC^2 + BC^2 - AB^2}{2 \cdot AC \cdot BC}\right)$$

$$\beta = \theta - \phi$$

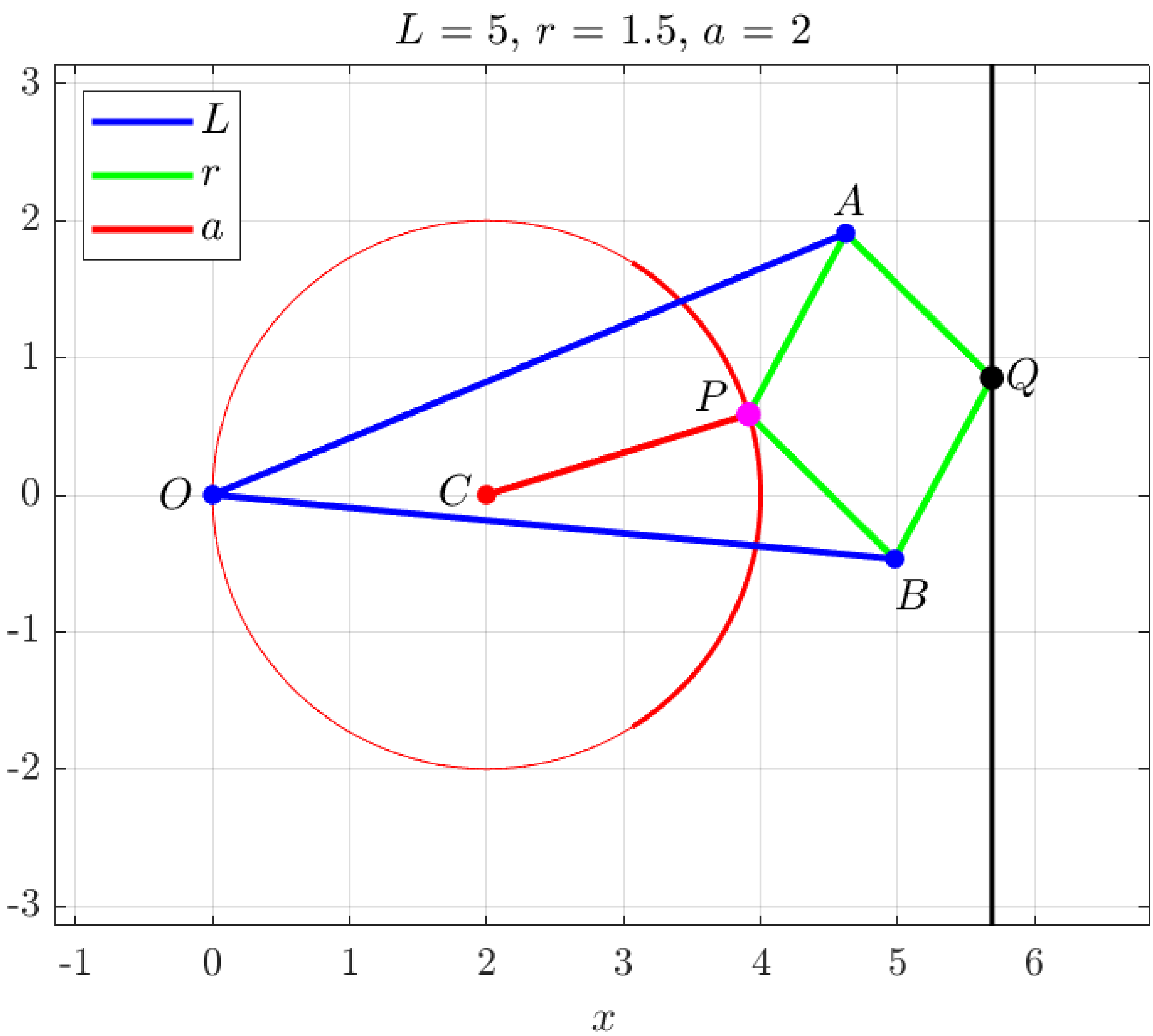
$$Bx = Cx + BC\cos(\beta),\ By = Cy + BC\sin(\beta)$$

The orbit of the middle point P of AB




The orbit of the middle point P of AB
P
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
1
2
3
4
5

Esim: Peaucellierin mekanismi (1864) muuntaa pyörimisliikkeen suoraviivaiseksi liikkeeksi



Varsien pituudet

$$OA = OB = L$$

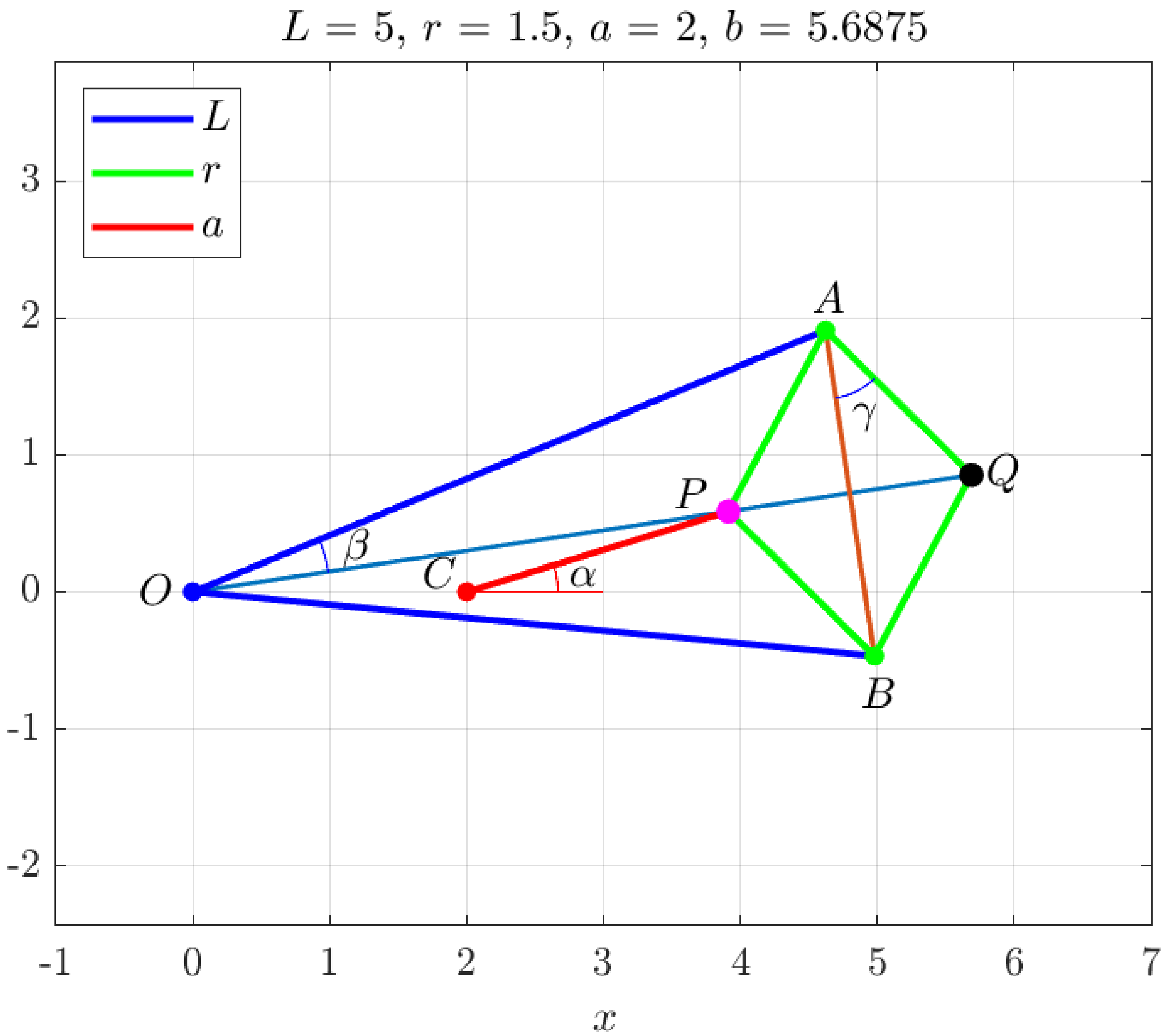
$$AP = AQ = BP = BQ = r$$

$$CP = a$$

Jos $O = [0, 0]$ ja P liikkuu pitkin a-säteistä ympyrää (keskipiste $C = [a, 0]$), niin Q liikkuu pitkin suoraa

$$x = \frac{L^2 - r^2}{2a}$$

Koordinaatit kulman $\alpha = \theta_{CP}$ avulla:



$O = [0, 0]$, $C = [a, 0]$

Symmetria $\rightarrow O, P$ ja Q samalla suoralla

$$Px = Cx + a\cos(\alpha),\ Py = Cy + a\sin(\alpha)$$

$$OP = \sqrt{Px^2 + Py^2},\ \theta_{OP} = \mathrm{atan2}(Py, Px)$$

$$\beta = \cos^{-1}\left(\frac{L^2 + OP^2 - r^2}{2 \cdot L \cdot OP}\right)$$

$$\theta_{OA} = \theta_{OP} + \beta,\ \theta_{OB} = \theta_{OP} - \beta$$

$$Ax = L\cos(\theta_{OA}),\ Ay = L\sin(\theta_{OA})$$

$$Bx = L\cos(\theta_{OB}),\ By = L\sin(\theta_{OB})$$

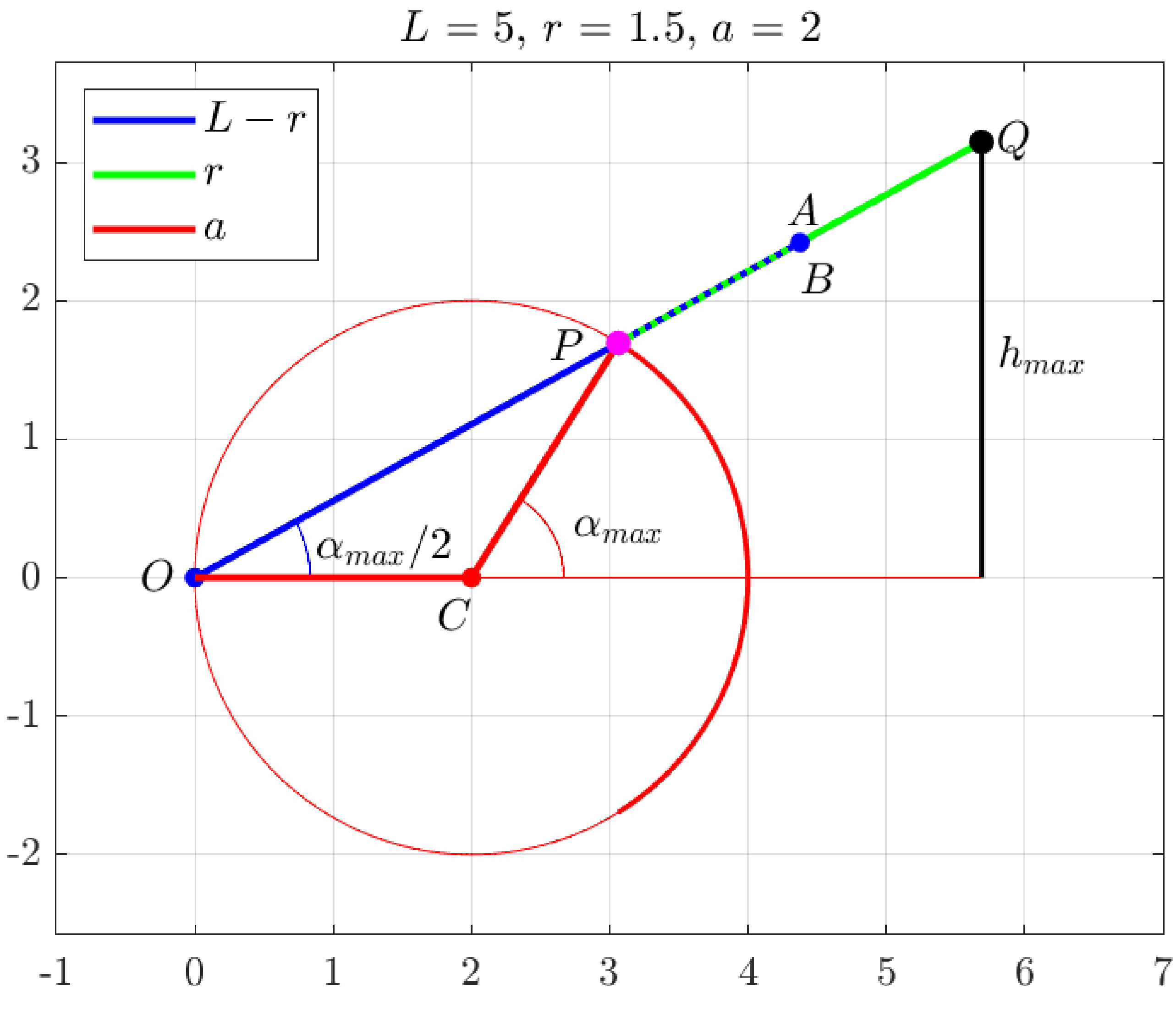
$$AB = \sqrt{(Bx - Ax)^2 + (By - Ay)^2}$$

$$\theta_{AB} = \mathrm{atan2}(By - Ay, Bx - Ax)$$

$$\gamma = \cos^{-1}\left(\frac{AB^2 + r^2 - r^2}{2 \cdot AB \cdot r}\right)$$

$$\theta_{AQ} = \theta_{AB} + \gamma$$

$$Qx = Ax + r\cos(\theta_{AQ}), Qy = Ay + r\sin(\theta_{AQ})$$



$L = 5,\ r = 1.5,\ a = 2$
$L - r$
r
a
Q
A
B
P
h_{max}
α_{max}
$\alpha_{max}/2$
O
C
3
2
1
0
-1
-2
-1
0
1
2
3
4
5
6
7
x

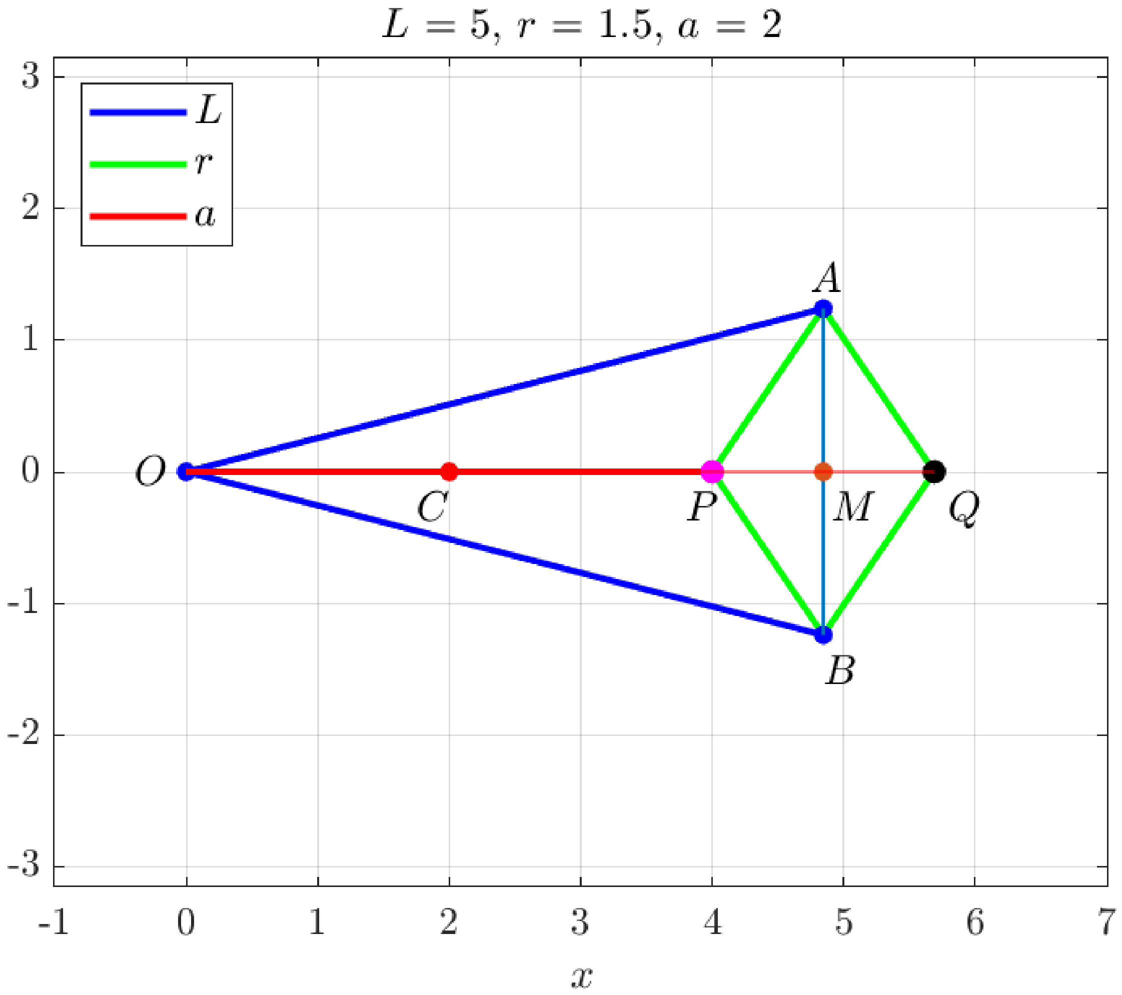
$$\alpha = -\alpha_{max} \ldots \alpha_{max}, \text{ missä}$$

$$\alpha_{max} = 180 - \cos^{-1}\left(\frac{a^2 + a^2 - (L - r)^2}{2 \cdot a \cdot a}\right)$$

$$Qy = -h_{max} \ldots h_{max}, \text{ missä}$$

$$h_{max} = (L + r)\sin(\alpha_{max}/2)$$

Selitys sille, että $Qx = \dfrac{L^2 - r^2}{2a}$

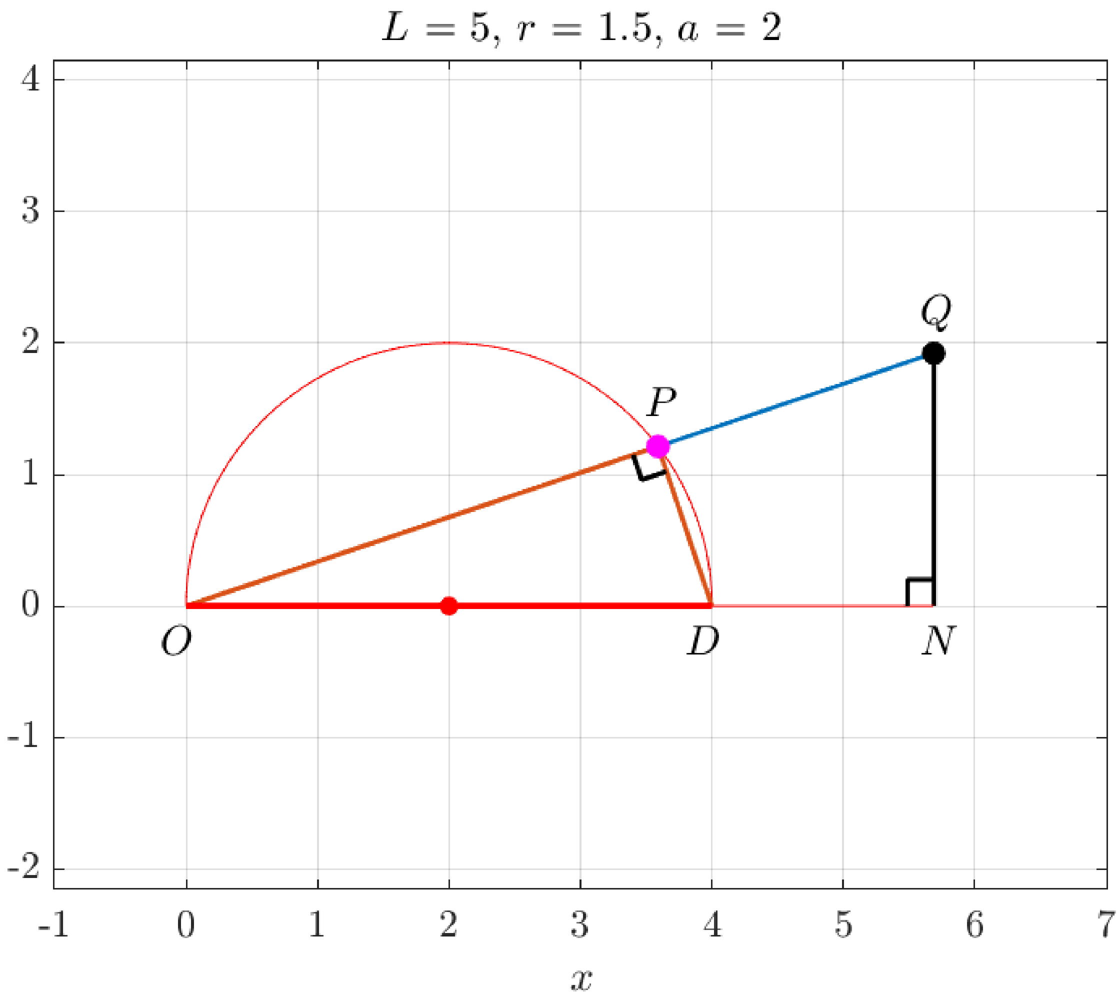


$$OP = OM - MP, OQ = OM + MP$$

$$\rightarrow OP \cdot OQ = OM^2 - MP^2$$

$$MP^2 = r^2 - MA^2, OM^2 + MA^2 = L^2$$

$$\rightarrow OP \cdot OQ = OM^2 - r^2 + MA^2 = L^2 - r^2$$



$$\frac{OP}{OD} = \frac{ON}{OQ}$$

$$\rightarrow ON = \frac{OP \cdot QQ}{OD} = \frac{L^2 - r^2}{2a}$$