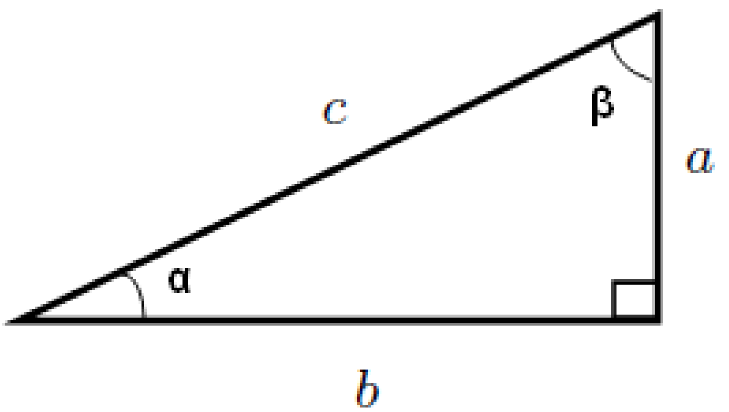
SUORAKULMAINEN KOLMIO



Yksi kulmista suora eli 90°

kateetit a ja b, **hypotenuusa** c (suoran kulman vastainen sivu)

PERUSFAKTAT:

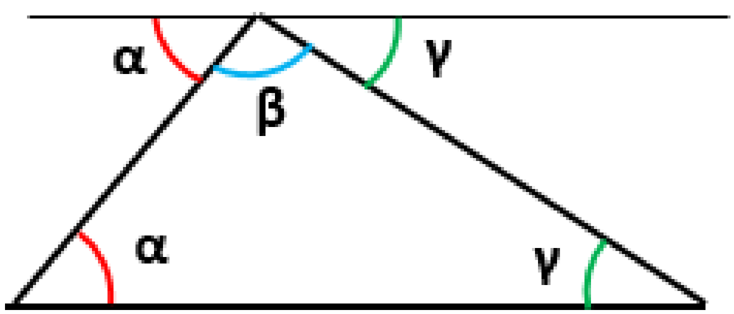
Kolmion **kulmien summa on** 180° eli

$\alpha + \beta = 90^\circ$

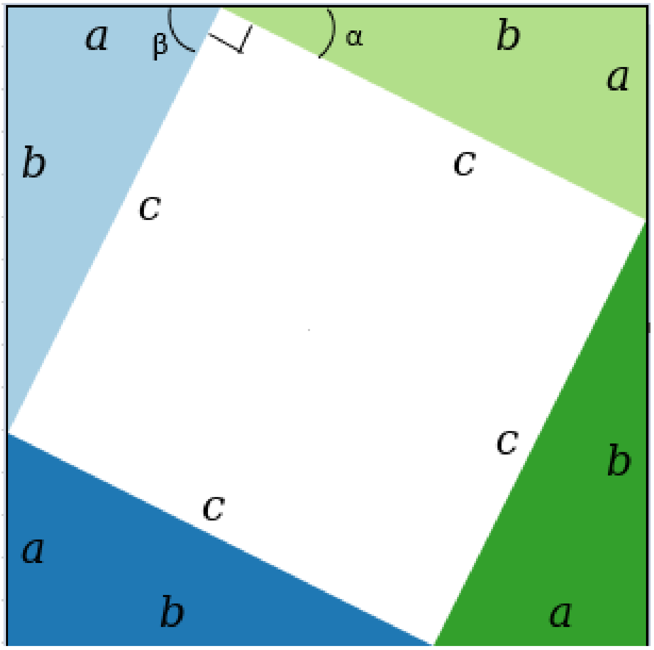
Pythagoraan lause:

$a^2 + b^2 = c^2$

$$\alpha + \beta + \gamma = 180^\circ.$$



$a^2 + b^2 = c^2$



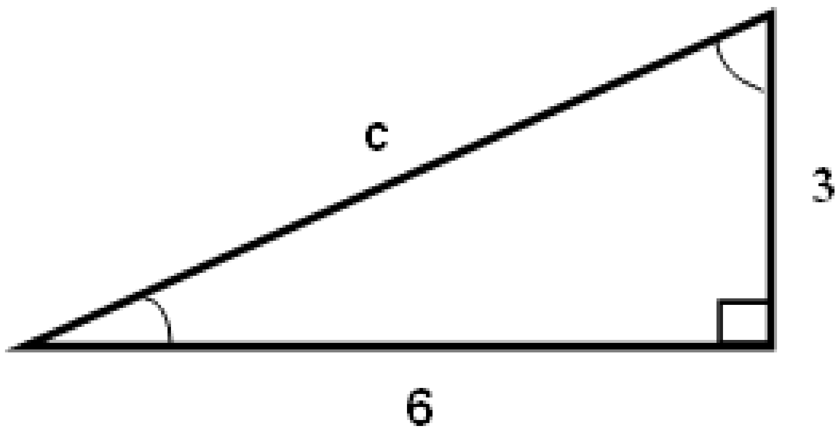
ison neliön ala $=$ 4 $\times$ kolmion ala $+$ pienen neliön ala eli

$$(a+b)^2 = 4 \cdot \frac{1}{2}ab + c^2$$

$$a^2 + \cancel{2ab} + b^2 = \cancel{2ab} + c^2$$

$$a^2 + b^2 = c^2$$

Esim.



$$c^2 = 3^2 + 6^2$$

$$c = \sqrt{3^2 + 6^2} \approx 6.7$$

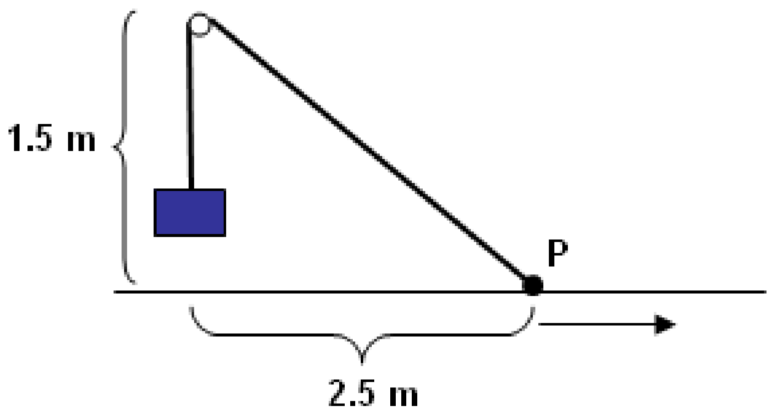
Esim.

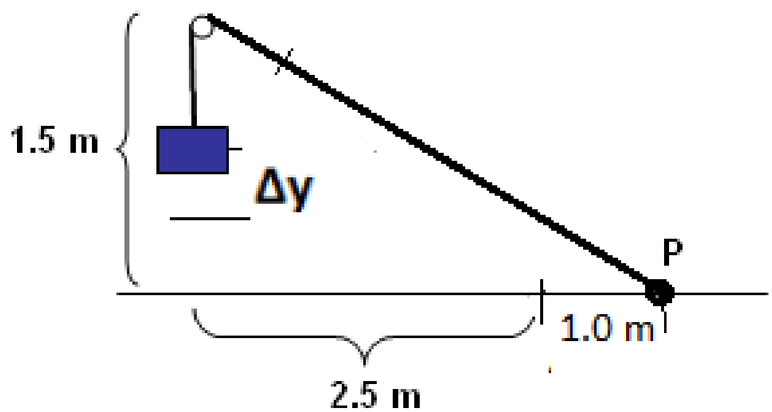
$$11.4^2 = a^2 + 8.7^2$$

$$a^2 = 11.4^2 - 8.7^2$$

$$a = \sqrt{11.4^2 - 8.7^2} \approx 7.4$$

Esim. Paljonko taakka nousee, jos P liikkuu 1.0 m oikealle ?

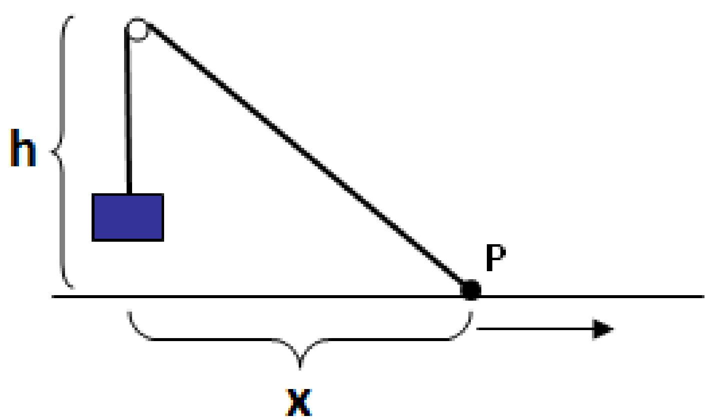


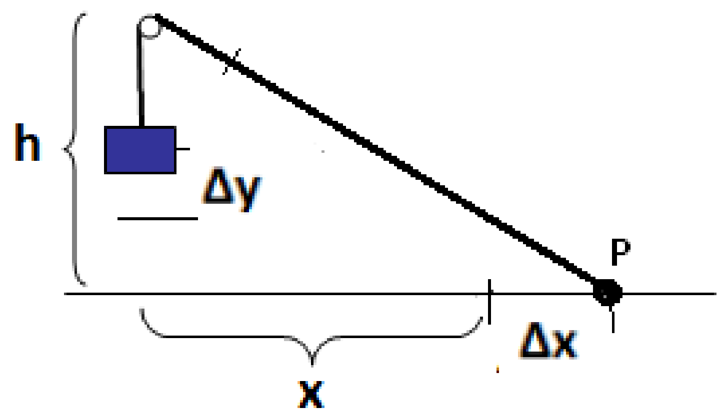


Taakka nousee sen verran kuin köyden vino osa pitenee eli

$$\Delta y = \sqrt{3.5^2 + 1.5^2} - \sqrt{2.5^2 + 1.5^2} \approx 0.9$$

Esim. Paljonko taakka nousee,
jos P liikkuu Δx:n verran oikealle ?

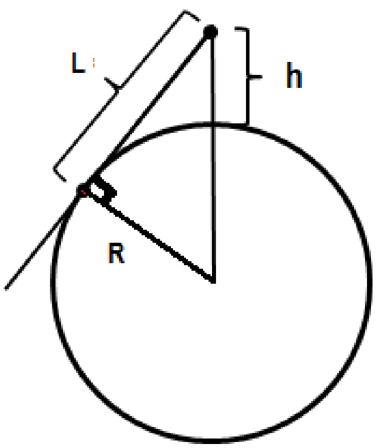


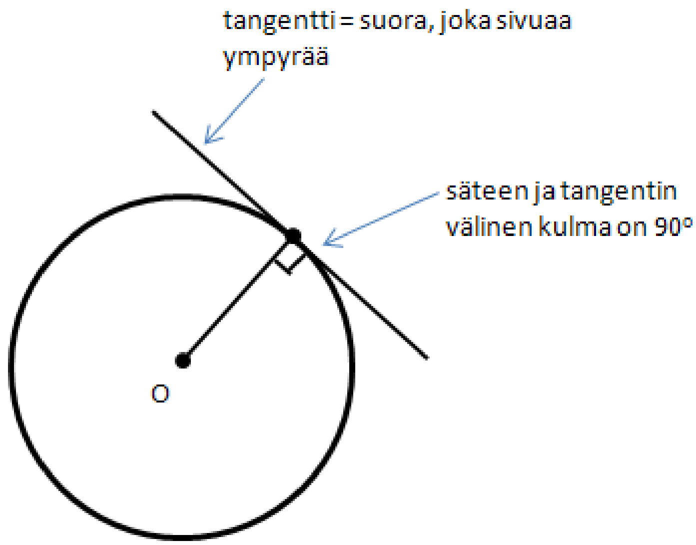


Taakka nousee sen verran kuin köyden vino osa pitenee eli

$$\Delta y = \sqrt{(x + \Delta x)^2 + h^2} - \sqrt{x^2 + h^2}$$

Esim. Laske mitta L mittojen h ja R avulla





Huom: ympyrän säde ja tangentti ovat kohtisuoria !

$$(R+h)^2 = R^2 + L^2$$

$$L^2 = (R+h)^2 - R^2 = 2Rh + h^2$$

$$L = \sqrt{2Rh + h^2}$$

Wolfram alpha

solve (R+h)^2=R^2+L^2,L

Result

$L = \pm\left(\sqrt{h}\,\sqrt{h+2R}\right)$

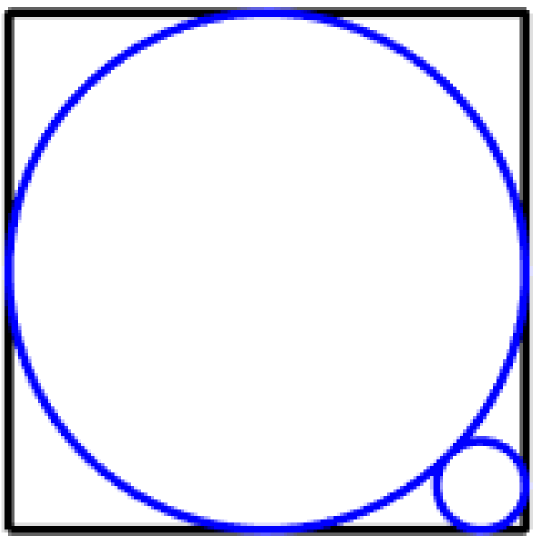
MATLAB + Symbolic math toolbox

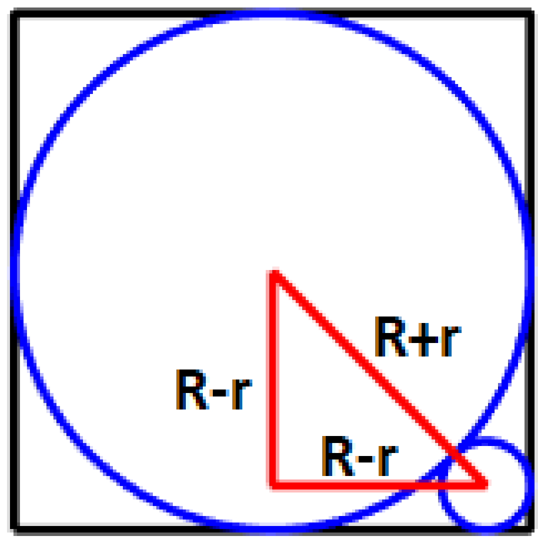
```
syms R h L %kirjainmuuttujat
solve((R+h)^2==R^2+L^2,L)
```

```
ans =

  h^(1/2)*(2*R + h)^(1/2)
 -h^(1/2)*(2*R + h)^(1/2)
```

Esim jos $R = 6400$ (km, maapallon säde) ja $h = 0.1$, niin $L \approx 36$ eli 100 m korkeasta tornista näkyy 36 km:n päähän.

Esim: Laske pienen ympyrän säde r, jos ison ympyrän säde on R





$$(R+r)^2 = (R-r)^2 + (R-r)^2$$

$$r^2 - 6Rr + R^2 = 0$$

$$r = \frac{6R \pm \sqrt{(-6R)^2 - 4R^2}}{2} = (3 \pm \sqrt{8})R$$

$$r = (3-\sqrt{8})R \approx 0.17R$$

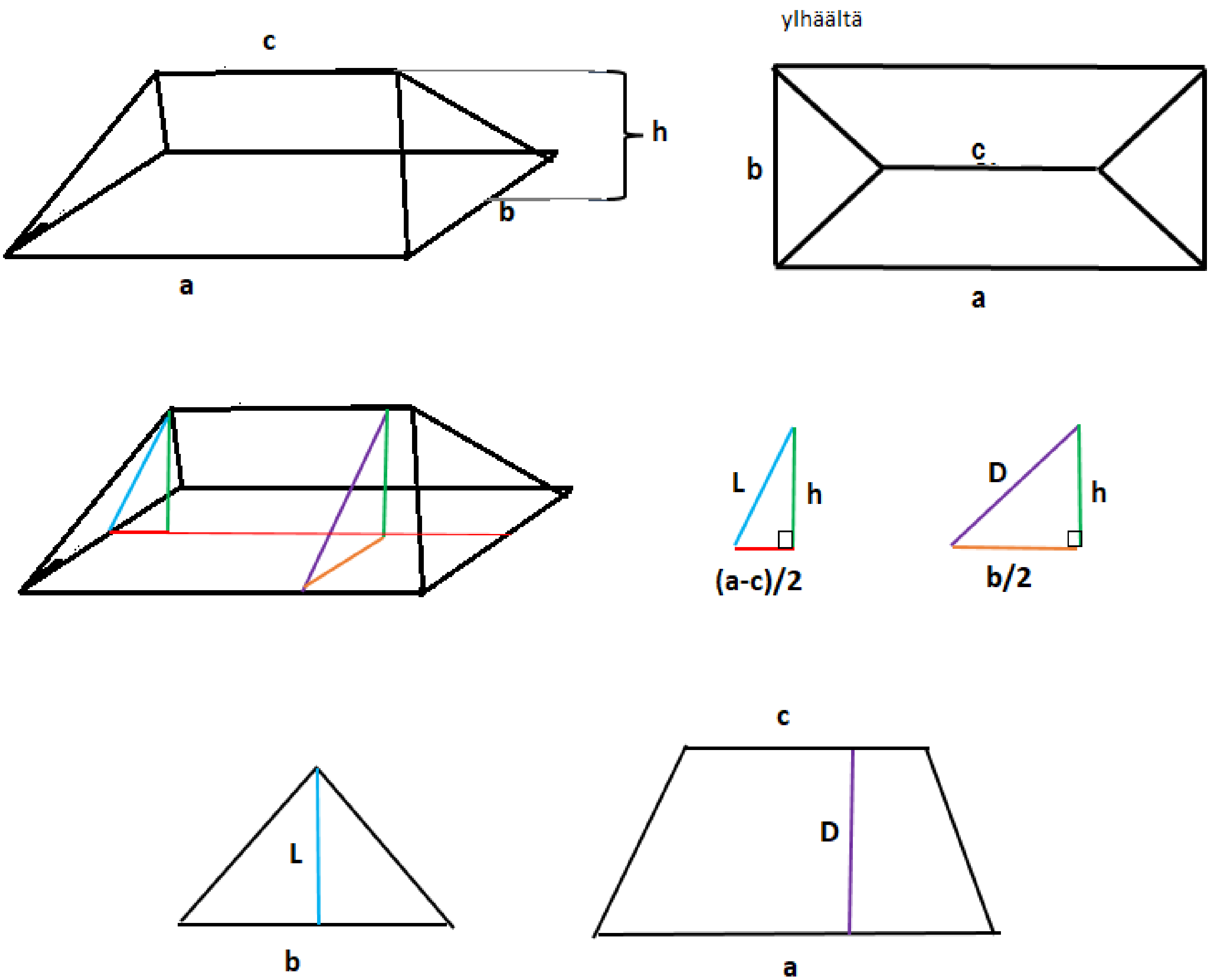
```
solve (R+r)^2=2*(R-r)^2,r
```

$r = (3 - 2\sqrt{2})R$

$r = (3 + 2\sqrt{2})R$

```
syms R r
solve((R+r)^2==2*(R-r)^2,r)

ans =

 3*R - 2*2^(1/2)*R
 3*R + 2*2^(1/2)*R
```

Example: Laske allaolevan "katon" pinta-ala
(= 2 kolmiota ja 2 puolisuunnikasta)



Kolmion korkeus $L = \sqrt{h^2 + \left(\frac{a-c}{2}\right)^2}$

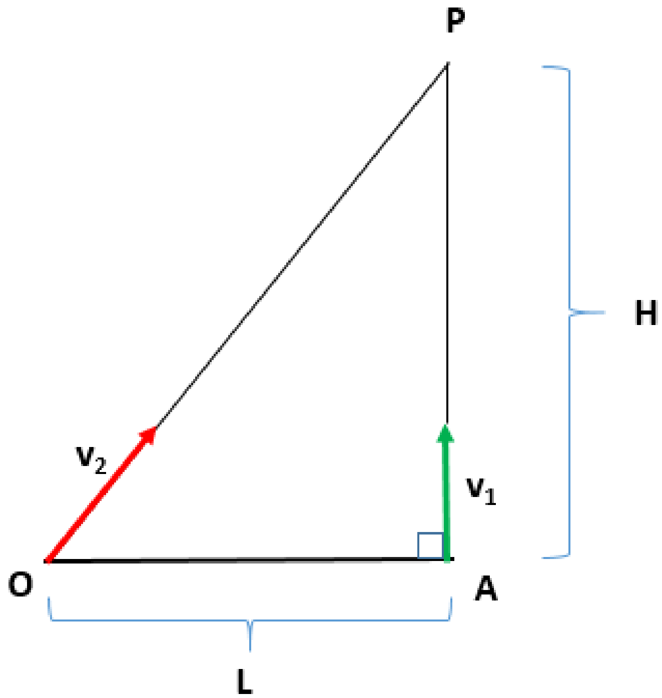
ja pinta-ala $A_1 = \frac{1}{2} bL$

Puolisuunnikkaan korkeus $D = \sqrt{h^2 + \left(\frac{b}{2}\right)^2}$

ja pinta-ala $A_2 = \frac{1}{2}(a+c)D$

Katon pinta-ala $A = 2A_1 + 2A_2$

Esim. Otus 1 lähtee A:sta vauhdilla v_1 ja otus 2 samaan aikaan O:sta vauhdilla $v_2 > v_1$. Määrää H niin, että ne kohtaavat pisteessä P.



Otukset kohtaavat P:ssä, jos matka-ajat $A \to P$ ja $O \to P$ ovat yhtäsuuret eli

$$\frac{AP}{v_1} = \frac{OP}{v_2}$$

$$\frac{H}{v_1} = \frac{\sqrt{L^2 + H^2}}{v_2}$$

$$H = \frac{v_1}{\sqrt{v_2^2 - v_1^2}} \cdot L$$

```
solve H/v1=sqrt(H^2+L^2)/v2,H
```

$$H = -\frac{L\,v1}{\sqrt{v2^2 - v1^2}}$$

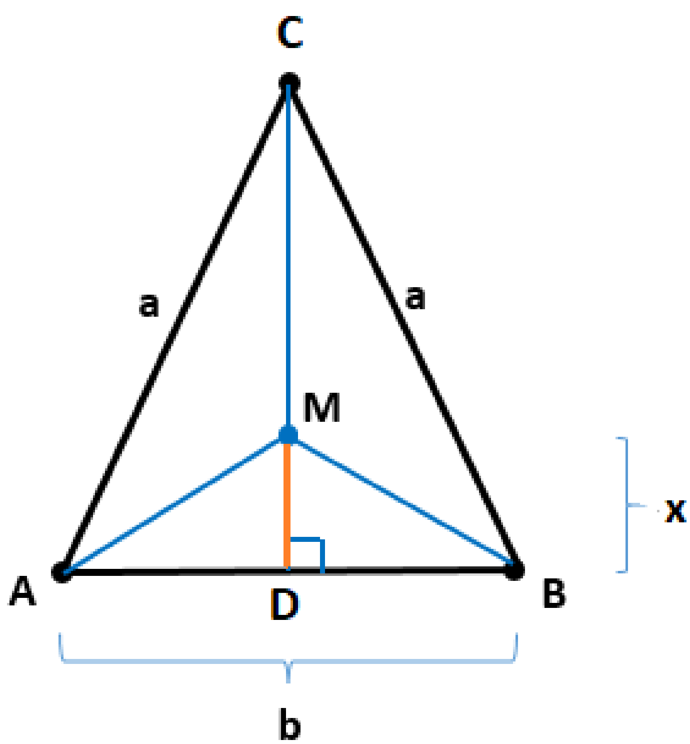
$$H = \frac{L\,v1}{\sqrt{v2^2 - v1^2}}$$

```
syms H v1 v2 L
solve(H/v1==sqrt(H^2+L^2)/v2,H)
simplify(ans)

ans =

 -(L*v1)/(v2^2 - v1^2)^(1/2)
  (L*v1)/(v2^2 - v1^2)^(1/2)
```

Esim. Laske etäisyyksien summa

$s = MA + MB + MC$

mittojen a, b ja x avulla.



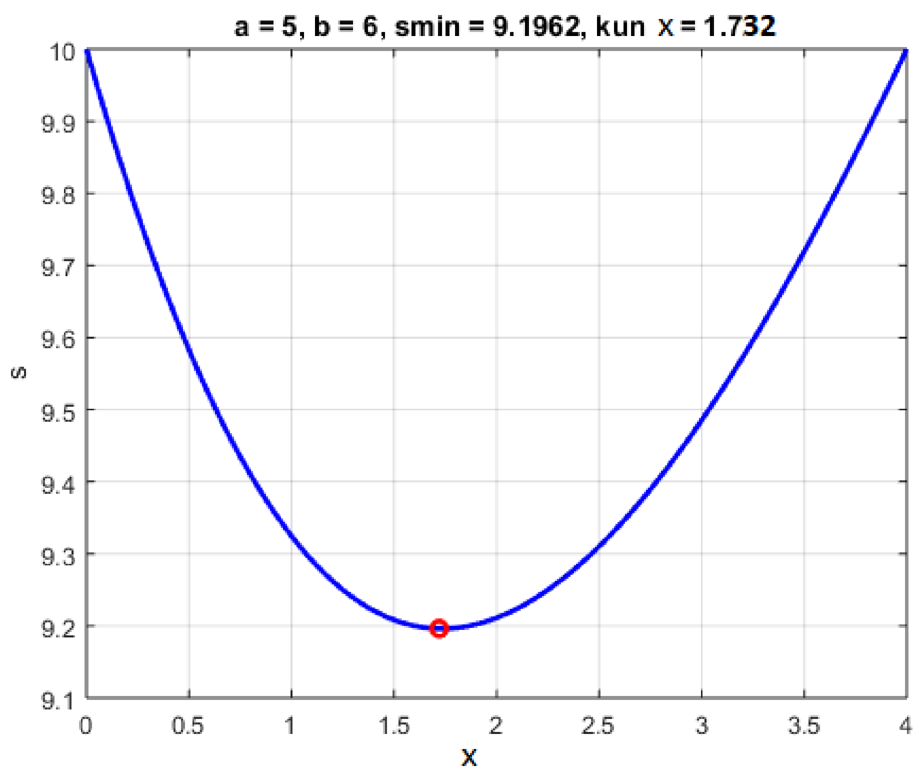
$$DC = \sqrt{a^2 - \left(\frac{b}{2}\right)^2}$$

$$MA = \sqrt{\left(\frac{b}{2}\right)^2 + x^2} = MB$$

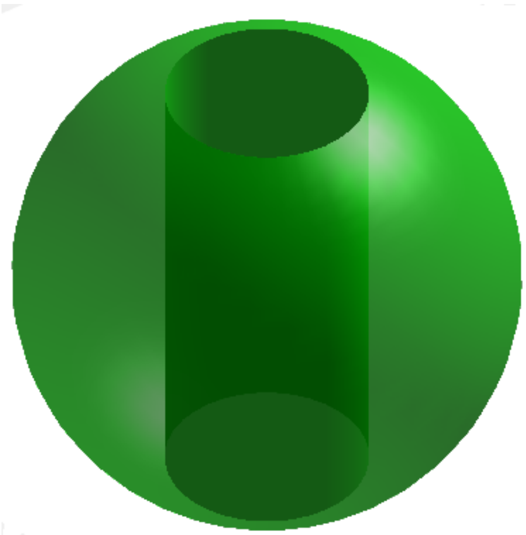
$$MC = DC - x$$

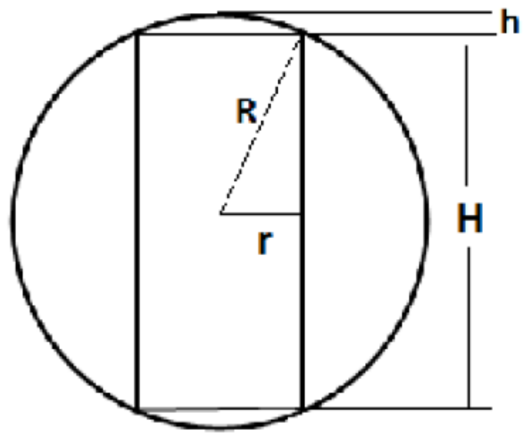
$$s = 2\sqrt{\left(\frac{b}{2}\right)^2 + x^2} + \sqrt{a^2 - \left(\frac{b}{2}\right)^2} - x$$

s:n arvojen kuvaaja, kun $x = 0 \ldots DC$



Esim. Pallon (säde R) läpi porataan reikä (säde r). Kuinka suuri osa pallon tilavuudesta ja pinta-alasta poistuu ?





Poistuva osa on

lieriö, korkeus $H = 2\sqrt{R^2 - r^2}$, tilavuus $\pi r^2 H$,

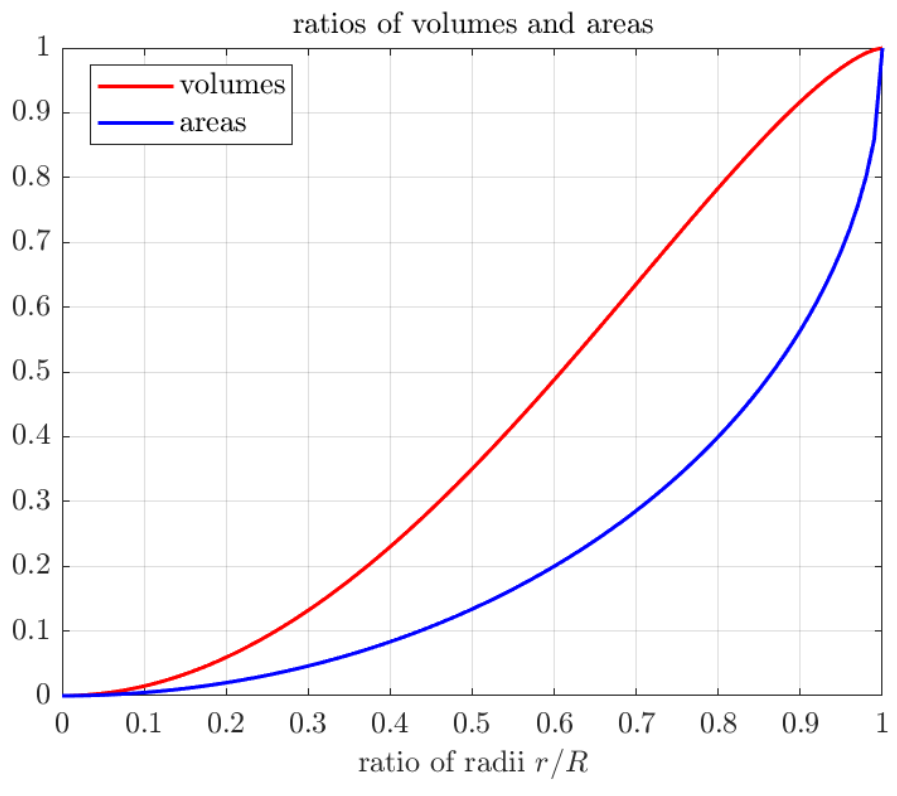
2 pallosegmenttiä, korkeus $h = R - H/2$, tilavuus $\pi h^2(R - h/3)$, pinta-ala $2\pi Rh$

Tilavuuksien suhde

$$\frac{\text{poistunut osa}}{\text{pallo}} = \frac{\pi r^2 H + 2 \cdot \pi h^2 (R - h/3)}{\frac{4}{3}\pi R^3}$$

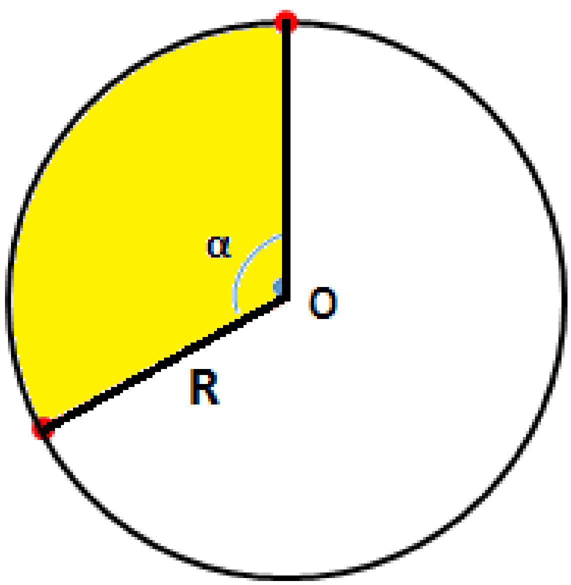
Pinta-alojen suhde

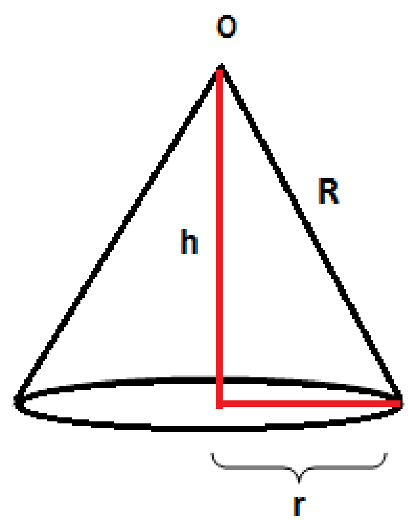
$$\frac{\text{poistunut osa}}{\text{pallo}} = \frac{2 \cdot 2\pi R h}{4\pi R^2} = \frac{h}{R}$$



Ex. Taivutetaan keltainen sektori kartioksi.

Laske r ja h.



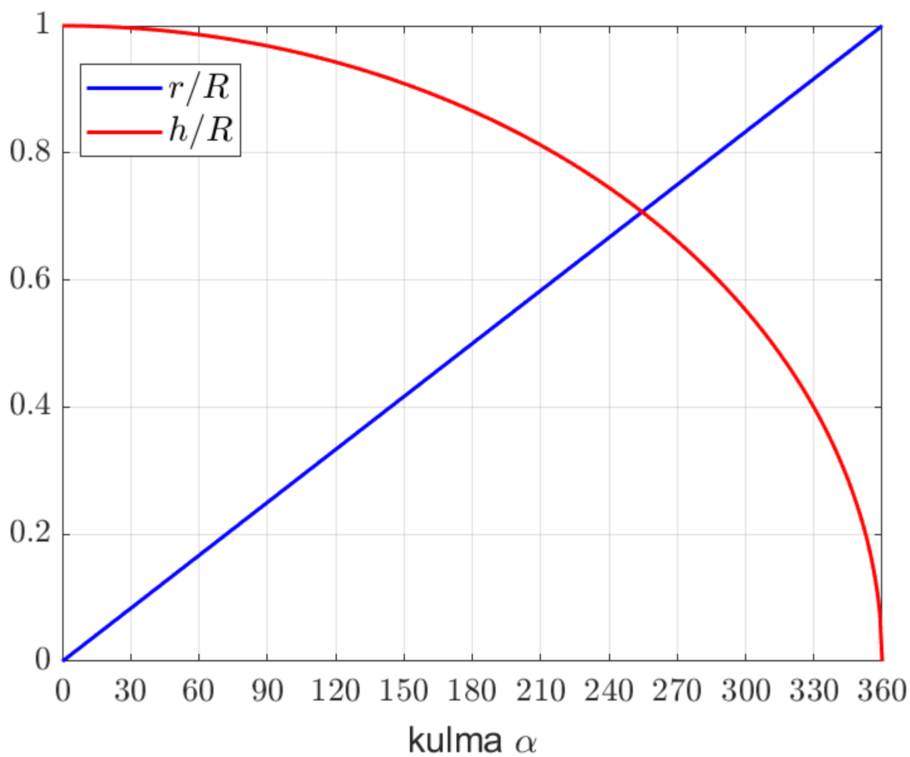


Sektorin kaaren pituus = kartion pohjaympyrän piiri eli

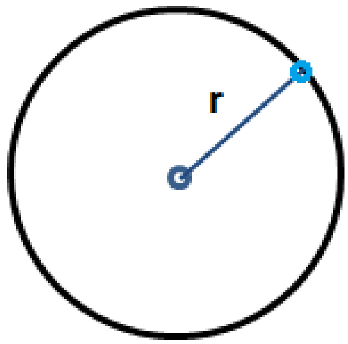
$$\frac{\alpha}{360} \cdot 2\pi R = 2\pi r \to r = \frac{\alpha}{360} \cdot R$$

Kartion sivujanan pituus = R, joten

$$h = \sqrt{R^2 - r^2} = \sqrt{1 - \left(\frac{\alpha}{360}\right)^2} \cdot R$$



RADIAANI :



Ympyrän piiri (ympärysmitta) on $2\pi r$, missä pii $\pi \approx 3.14159...$

Asteina koko ympyrä $= 360^\circ$,

radiaaneina $2\pi \approx 6.28$

MUUNTOKERTOIMET:

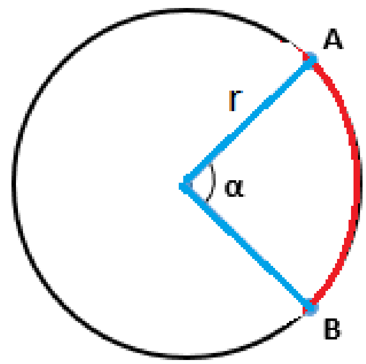
$$360^\circ = 2\pi \text{ radiaania}$$

$$1^\circ = \frac{2\pi}{360} = \frac{\pi}{180} \approx 0.0175 \text{ rad}$$

$$1 \text{ rad } = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi} \approx 57^\circ$$

aste	30	45	60	90	180
rad	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π

POINTTI: kun kulma α on radiaaneina, niin kaaren AB pituus on

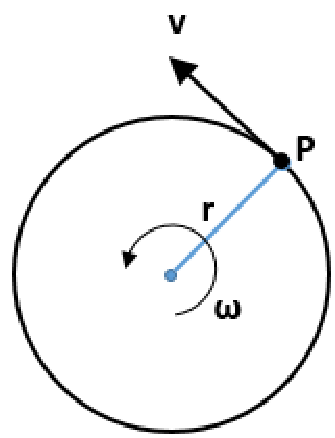


$$\frac{\alpha}{2\pi} \cdot 2\pi r = \alpha \cdot r = \text{kulma}\cdot\text{säde}$$

eli

$$\alpha = \frac{\text{kaaren AB pituus}}{\text{säde}}$$

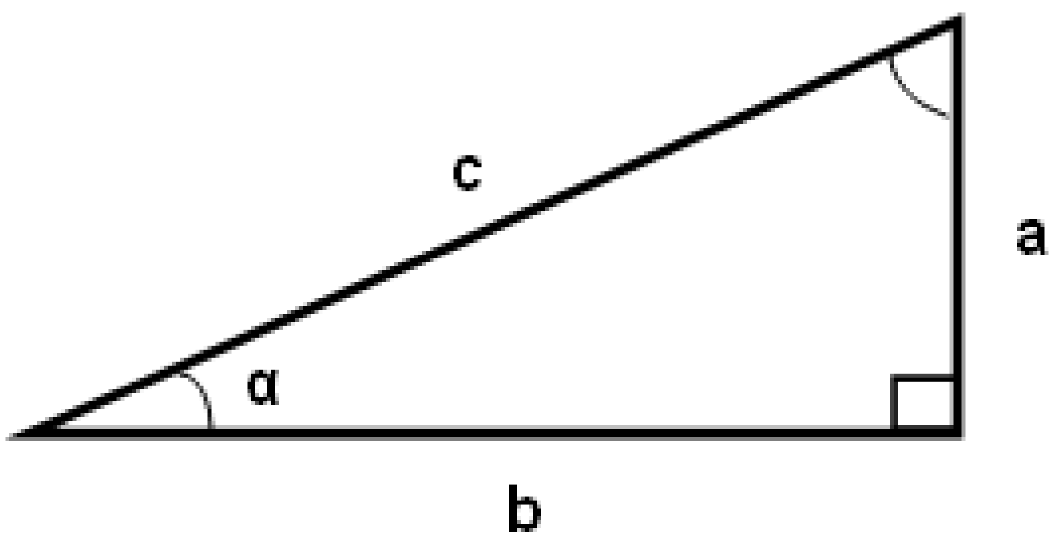
Esim: Jos ympyrän säde r metriä ja pisteen P pyörimisnopeus ω radiaania/sekunti, niin P:n vauhti $v = \omega r$ m/s.



Syy: aikavälillä Δt piste P kiertää ympyrää kulman $\omega \cdot \Delta t$ verran eli kulkee $\omega \cdot \Delta t \cdot r$:n pituisen matkan

SINI, KOSINI, TANGENTTI

kulmat $\rightarrow$ sivun pituudet



$$\sin(\alpha) = \frac{a}{c} \qquad \left(= \frac{\text{vastainen}}{\text{hypotenuusa}}\right)$$

$$\cos(\alpha) = \frac{b}{c} \qquad \left(= \frac{\text{viereinen}}{\text{hypotenuusa}}\right)$$

$$\tan(\alpha) = \frac{a}{b} \qquad \left(= \frac{\text{vastainen}}{\text{viereinen}}\right)$$

MATLAB/Octave:

$\sin(\alpha), \cos(\alpha), \tan(\alpha)$, kun α radiaaneina

sind(α),cosd(α),tand(α), kun α asteina (degree)

Esimerkiksi, laskukone kertoo että

$\sin(55^\circ) \approx 0.82$

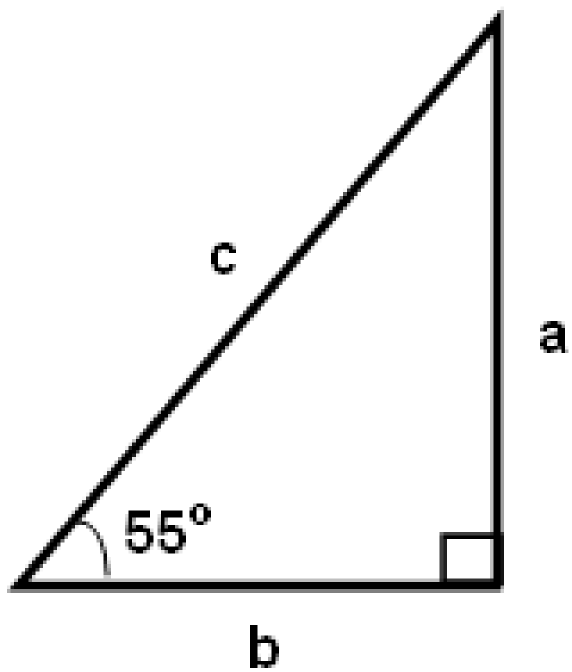
$\cos(55^\circ) \approx 0.57$

$\tan(55^\circ) \approx 1.43$

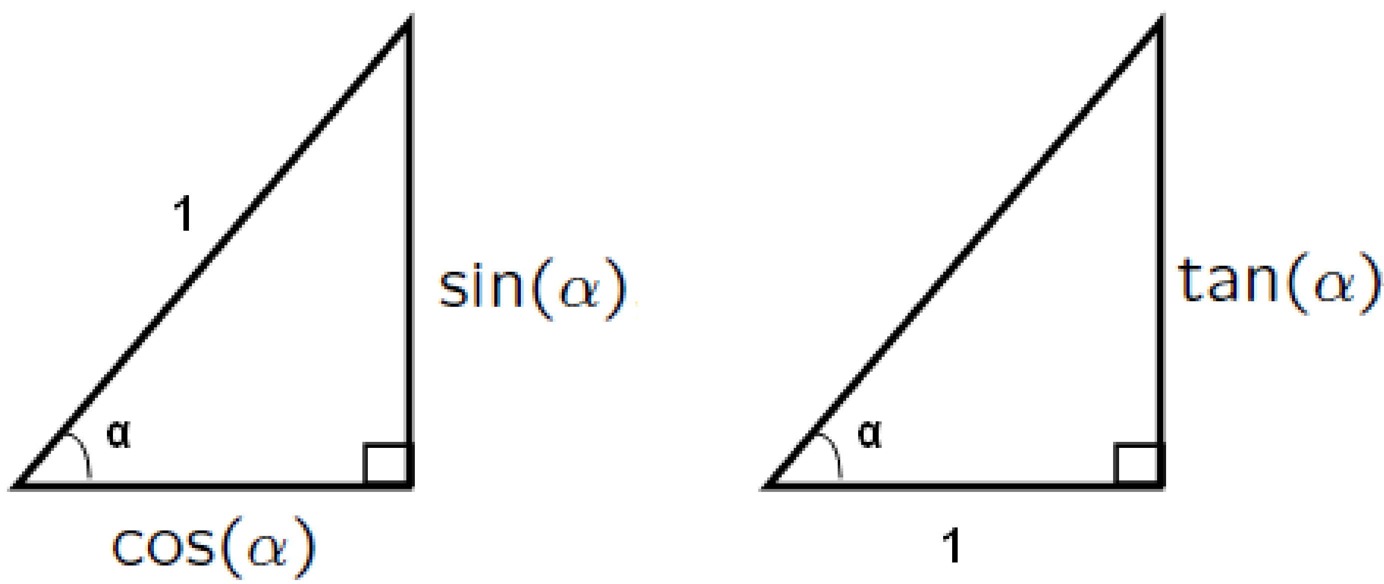
Tämä tarkoittaa yksinkertaisesti sitä, että viereisessä kuvassa sivujen pituuksien suhteet

$$\frac{a}{c} = 0.82,\ \frac{b}{c} = 0.57,\ \frac{a}{b} = 1.43$$

olipa kolmio minkä kokoinen tahansa !!!.

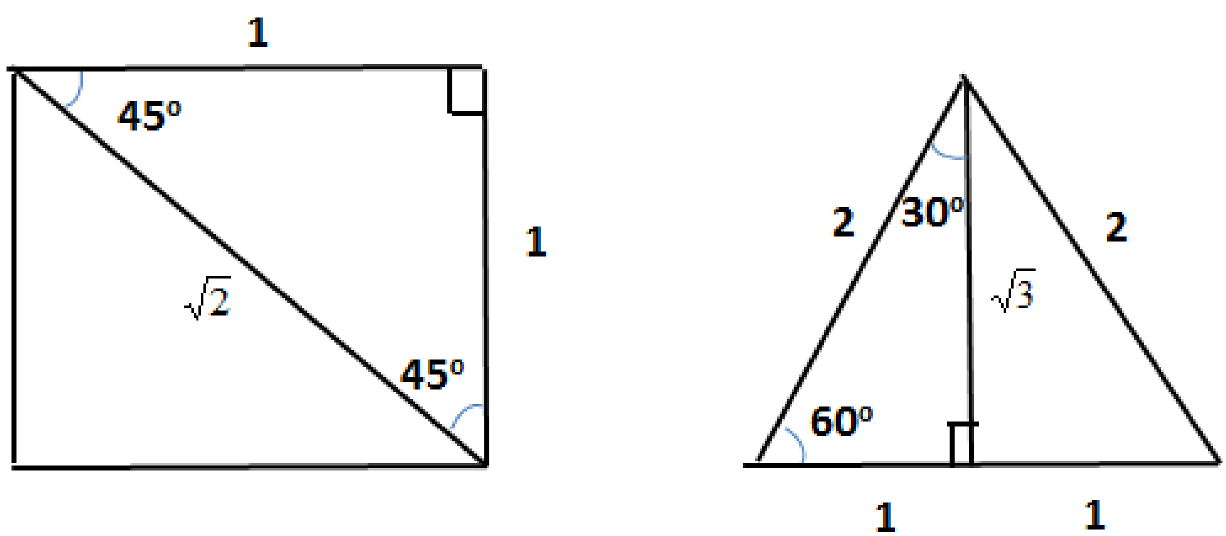


Geometrisesti $\sin(\alpha)$, $\cos(\alpha)$, $\tan(\alpha)$ ovat allaolevan kuvan mukaiset mitat:

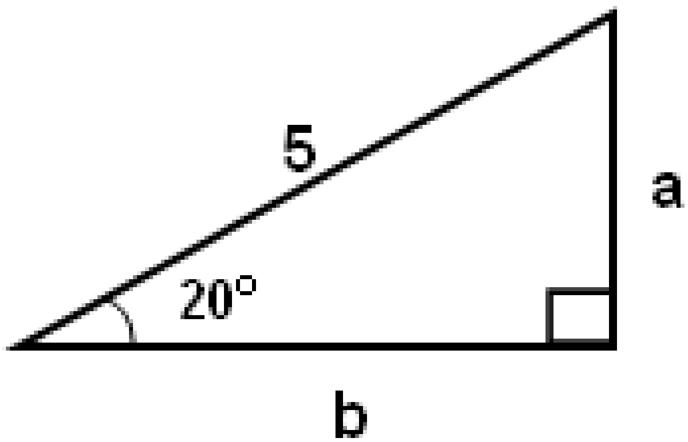


Perusarvoja:

α	$\sin(\alpha)$	$\cos(\alpha)$	$\tan(\alpha)$
30°	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
45°	$1/\sqrt{2}$	$1/\sqrt{2}$	1
60°	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$



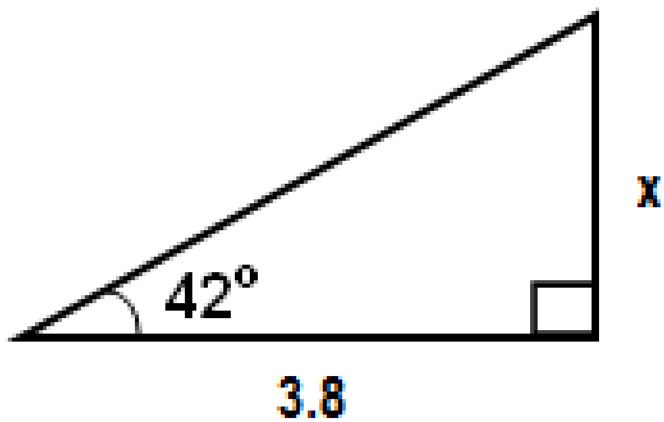
Esim.



$$\frac{a}{5} = \sin(20^\circ) \rightarrow a = 5 \cdot \sin(20^\circ) \approx 1.7$$

$$\frac{b}{5} = \cos(20^\circ) \rightarrow b = 5 \cdot \cos(20^\circ) \approx 4.7$$

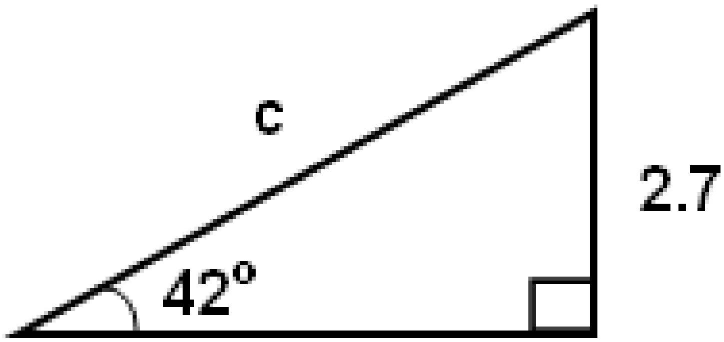
Esim.



$$\frac{x}{3.8} = \tan(42^\circ)$$

$$\rightarrow x = 3.8 \tan(42^\circ) \approx 3.4$$

Esim.



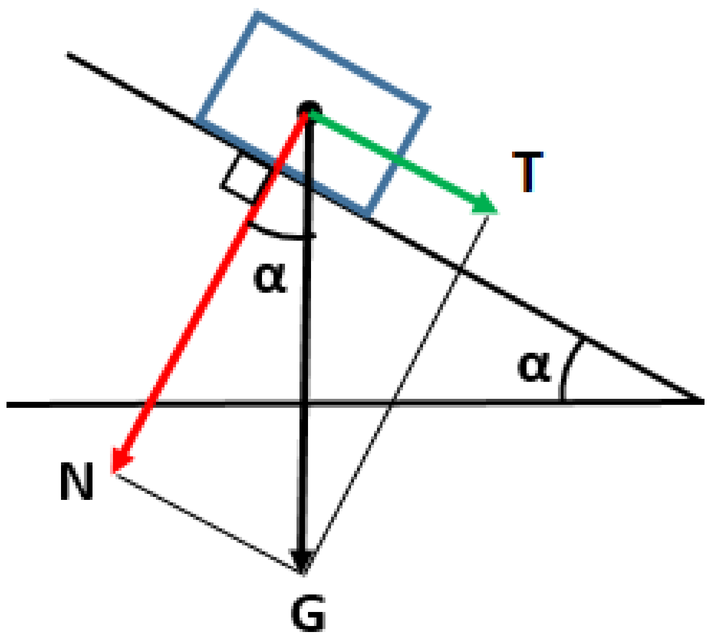
$$\frac{2.7}{c} = \sin(42^\circ)$$

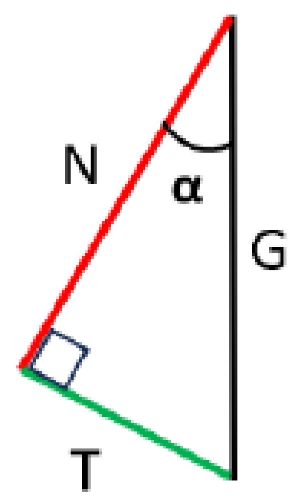
$$\rightarrow c = \frac{2.7}{\sin(42^\circ)} \approx 4.0$$

Ex: Painovoima G, komponentit

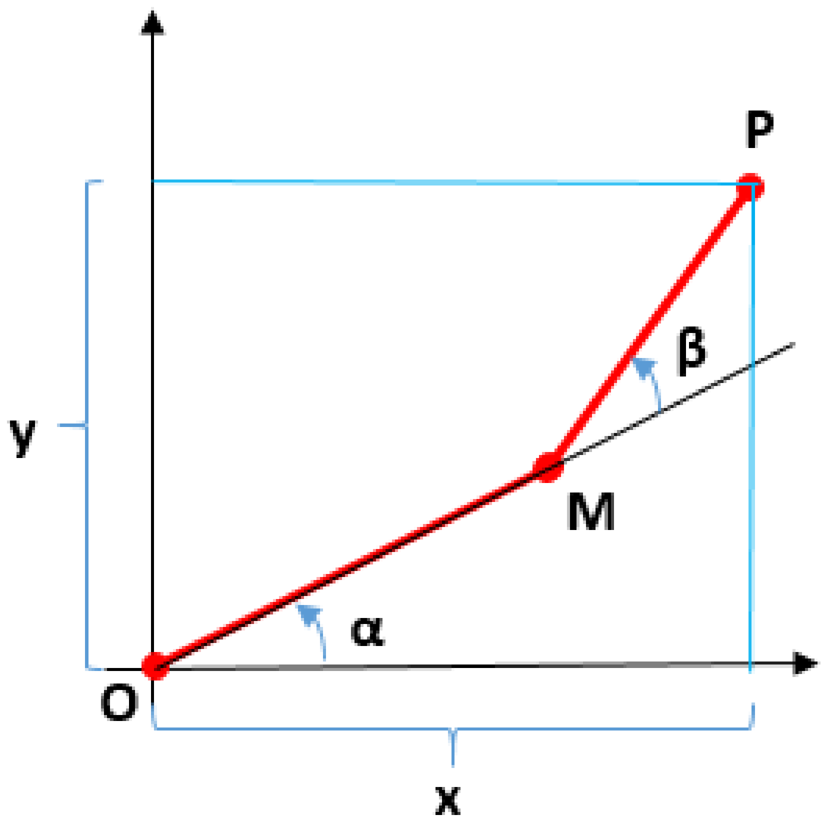
$$\frac{T}{G} = \sin(\alpha) \rightarrow T = G\sin(\alpha)$$

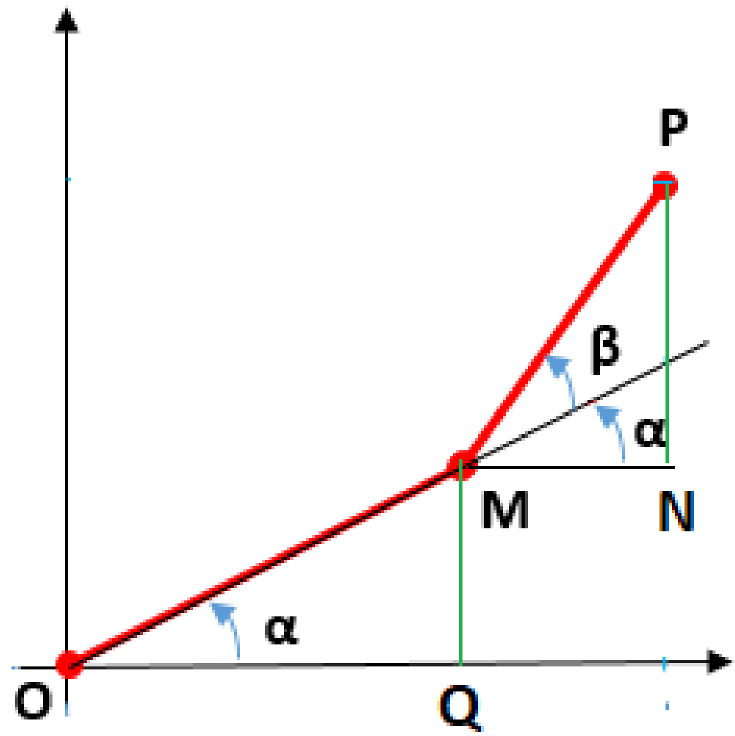
$$\frac{N}{G} = \cos(\alpha) \rightarrow N = G\cos(\alpha)$$





Esim. Laske mitat x ja y mittojen OM ja MP ja kulmien α ja β avulla





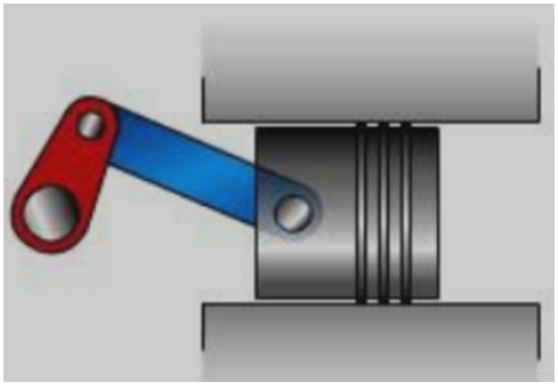
$OQ = OM\cos(\alpha)$, $MN = MP\cos(\alpha+\beta)$

$x = OQ + MN$

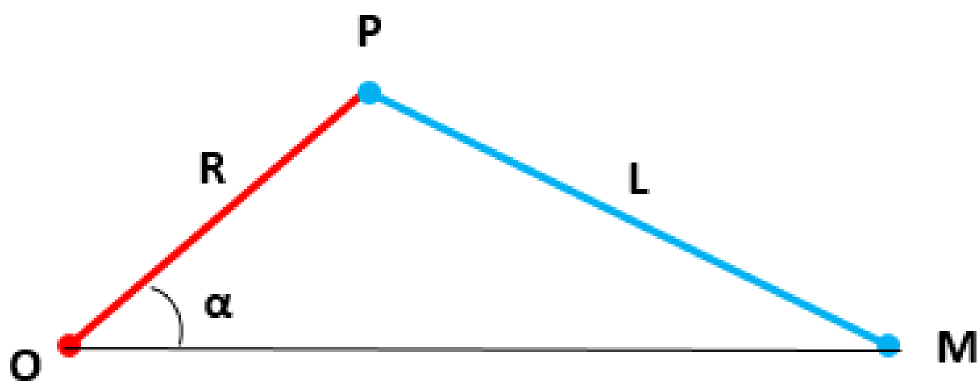
$QM = OM\sin(\alpha)$, $NP = MP\sin(\alpha+\beta)$

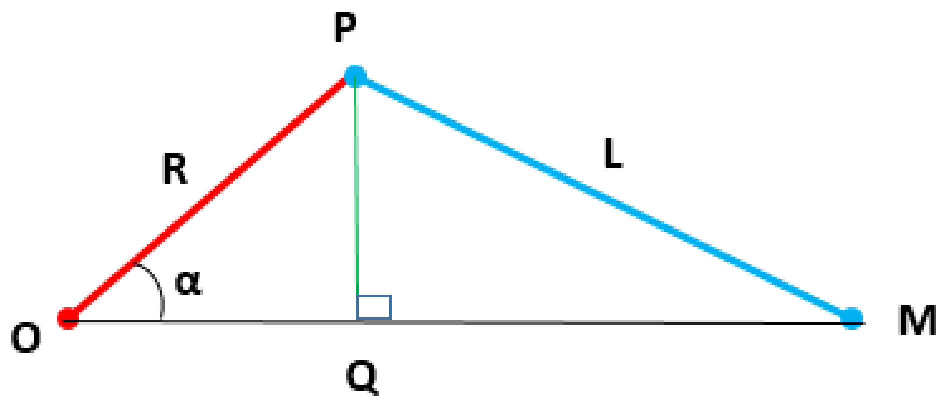
$y = QM + NP$

Esim: Slider crank mechanism



Laske mitta OM mittojen R ja L ja kulman α avulla





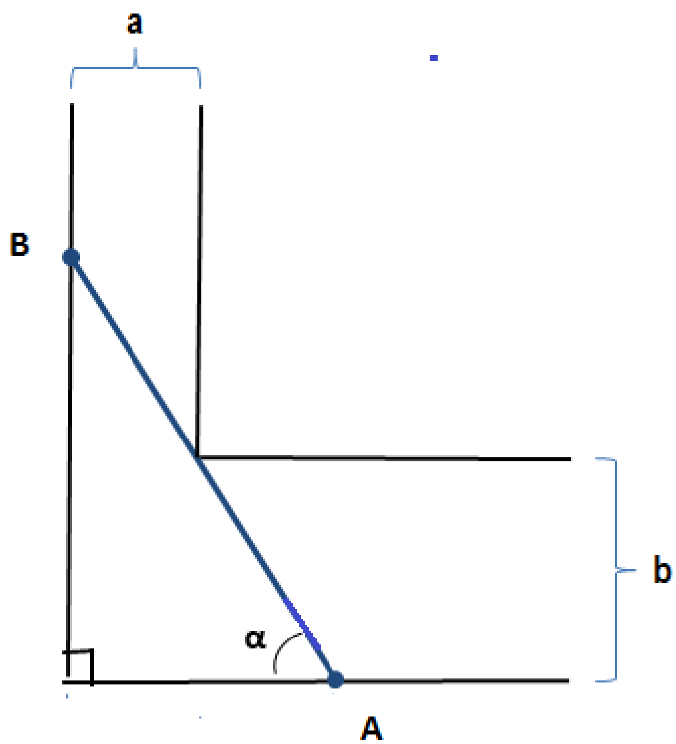
$$OQ = R\cos(\alpha)$$

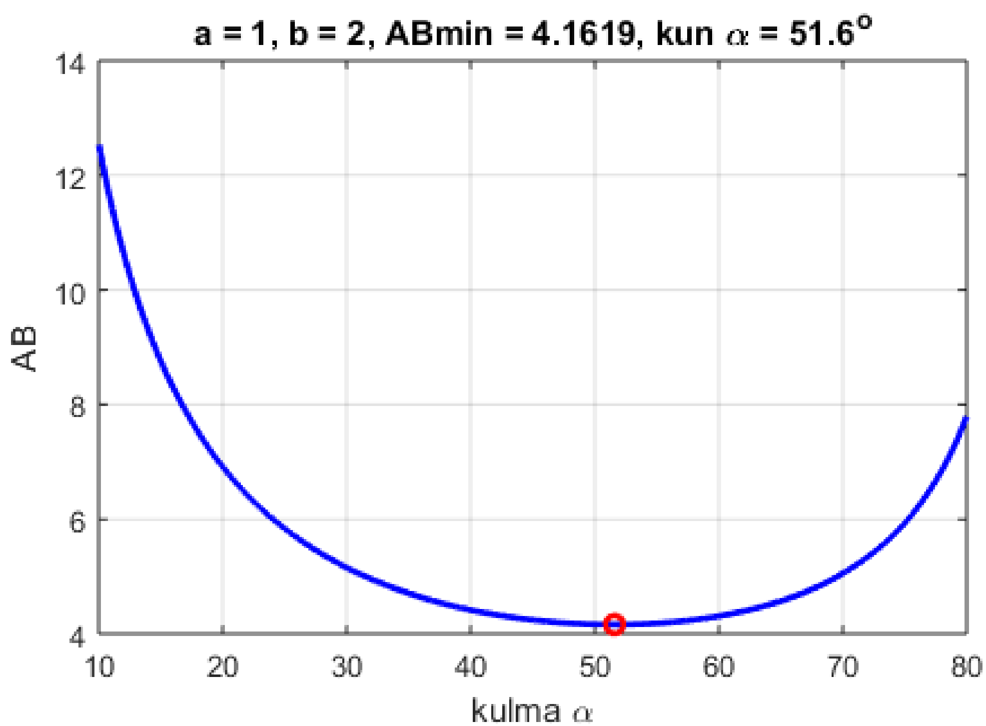
$$QP = R\sin(\alpha)$$

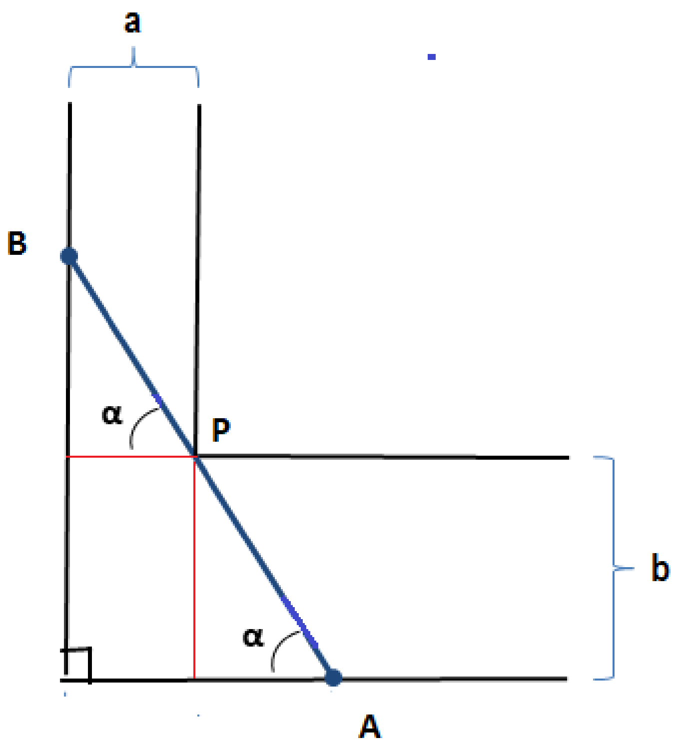
$$QM = \sqrt{L^2 - QP^2}$$

$$OM = OQ + QM$$

Esim. Kuinka pitkä keppi sopii kulmasta eli laske pituus AB mittojen a ja b ja kulman α avulla.





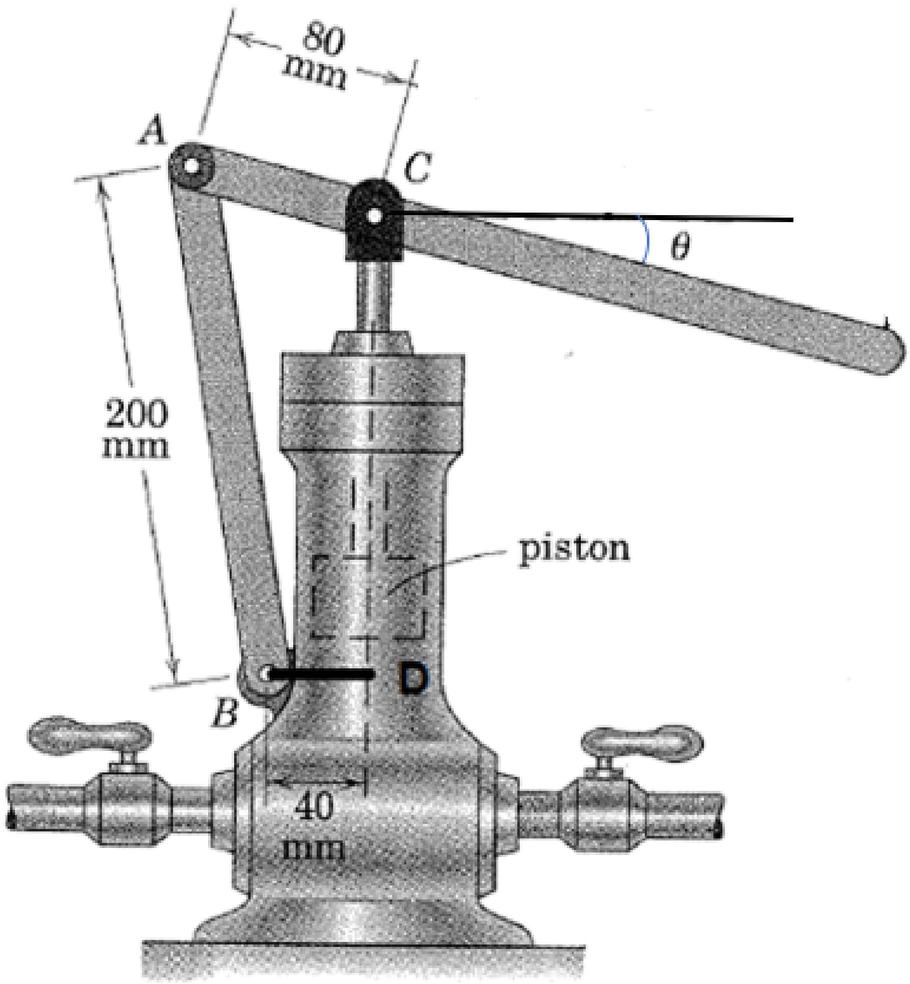


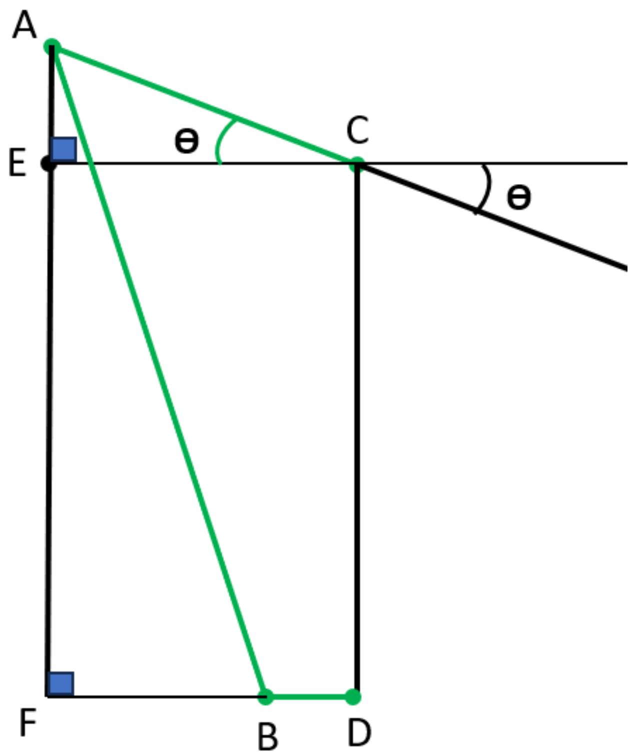
$$\frac{b}{AP} = \sin(\alpha) \rightarrow AP = \frac{b}{\sin(\alpha)}$$

$$\frac{a}{PB} = \cos(\alpha) \rightarrow PB = \frac{a}{\cos(\alpha)}$$

$$AB = AP + PB$$

Esim: Laske mitta CD mittojen AB, AC ja BD ja kulman θ avulla.





$$CE = AC\cos(\theta)$$

$$AE = AC\sin(\theta)$$

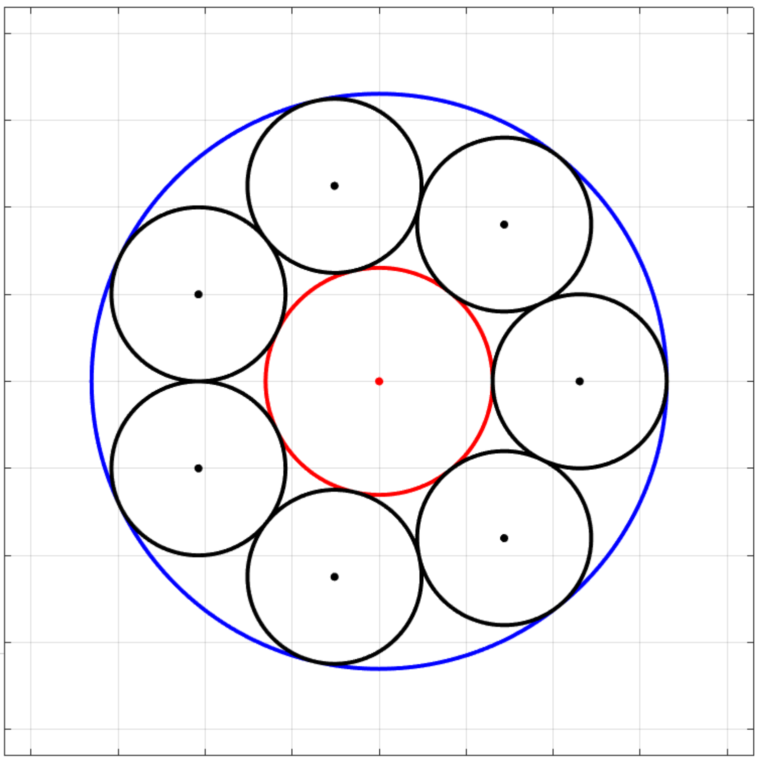
$$FB = CE - BD$$

$$AF = \sqrt{AB^2 - FB^2}$$

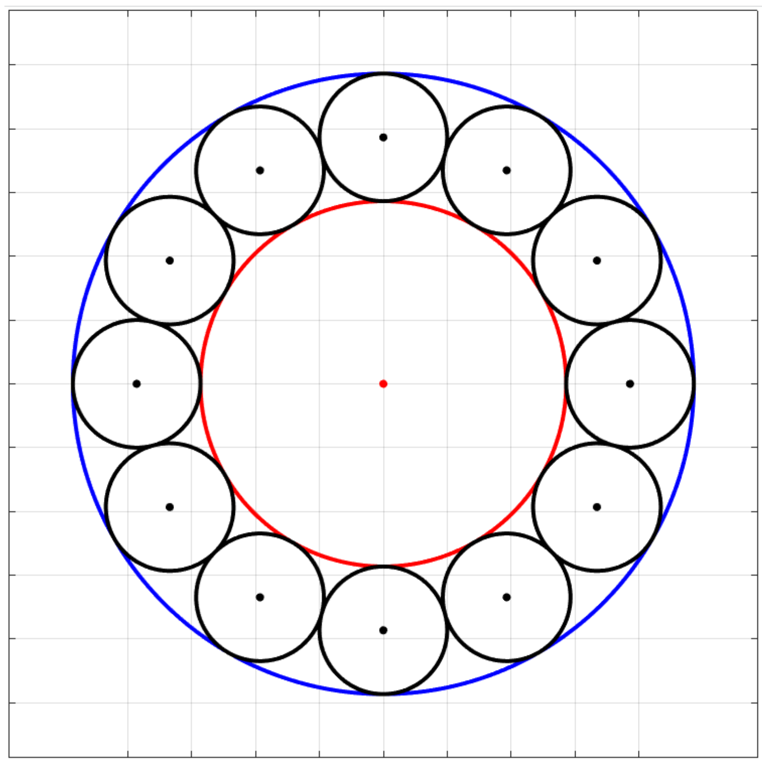
$$CD = AF - AE$$

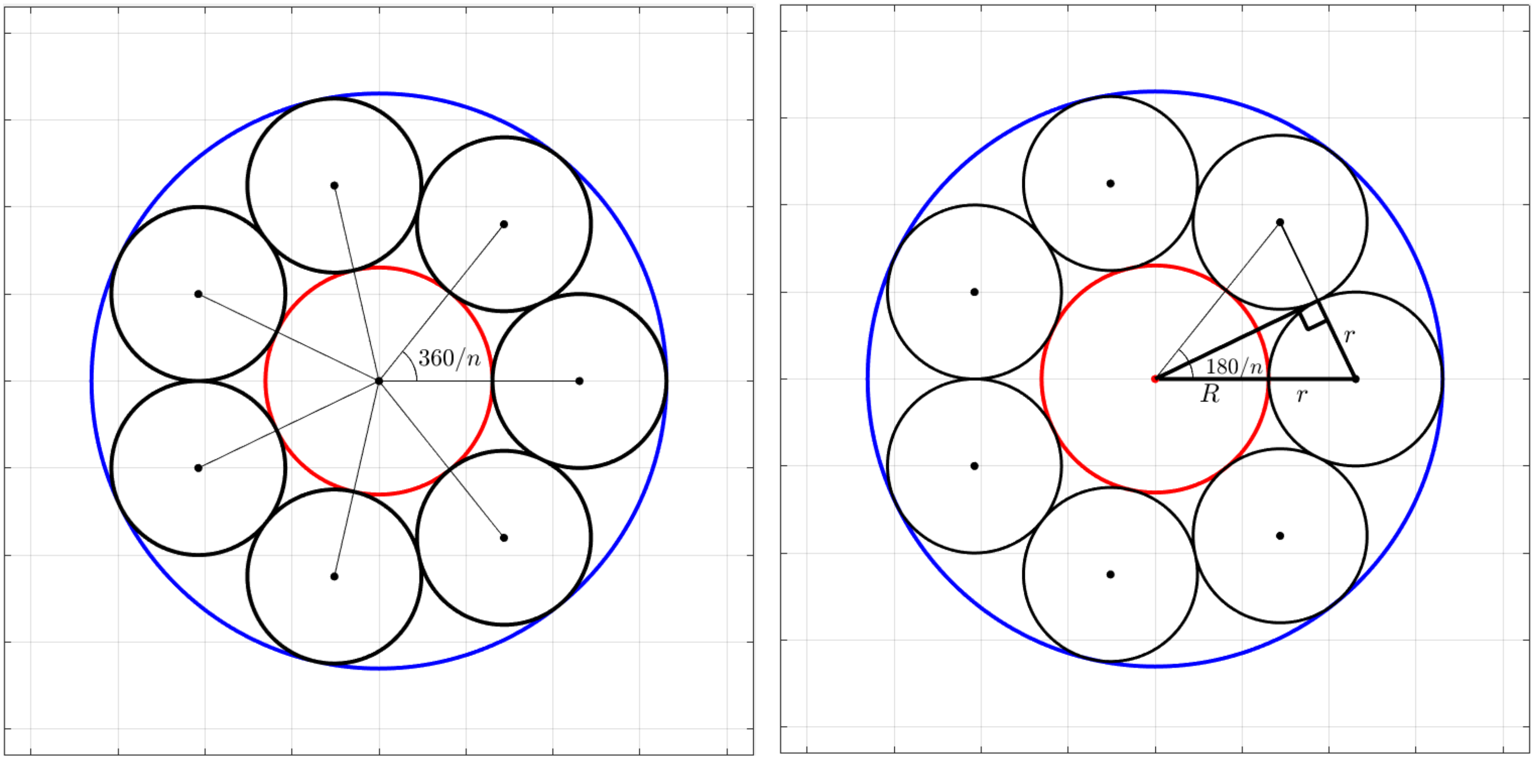
Esim. Mustia ympyröitä on n kpl ja niiden säde on r. Laske punaisen ympyrän säde R.

$n = 7$, $r = 1$, $R = 1.3048$



$n = 12$, $r = 1$, $R = 2.8637$



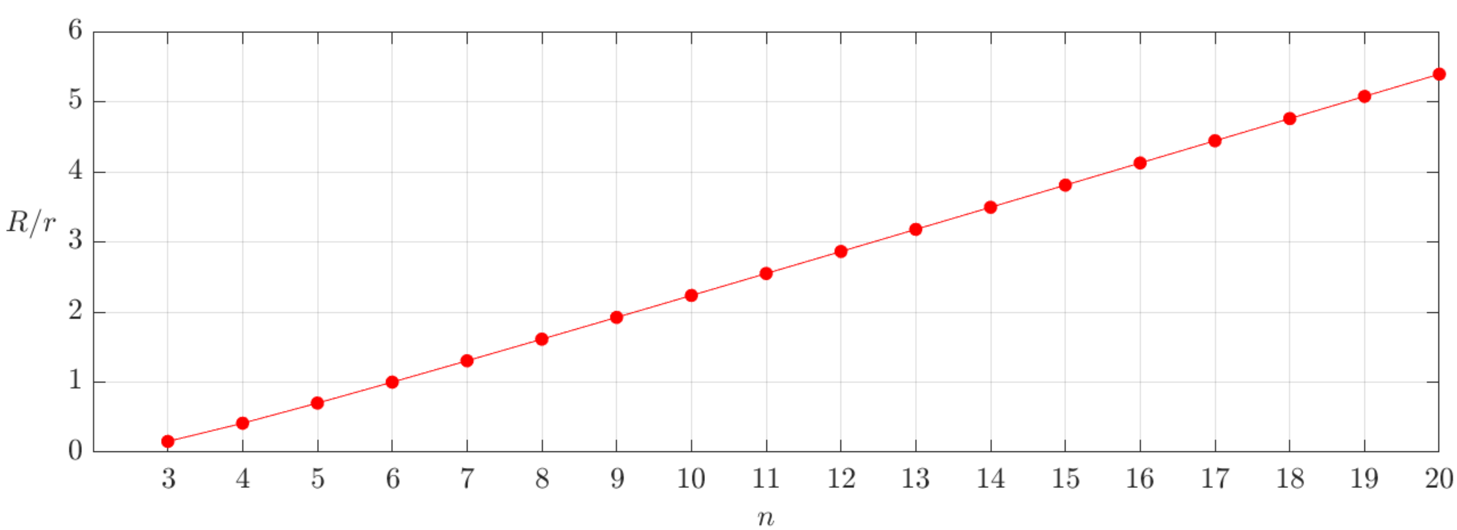


$$\frac{r}{R+r} = \sin(180/n) \rightarrow R = \frac{1-\sin(180/n)}{\sin(180/n)} \cdot r$$

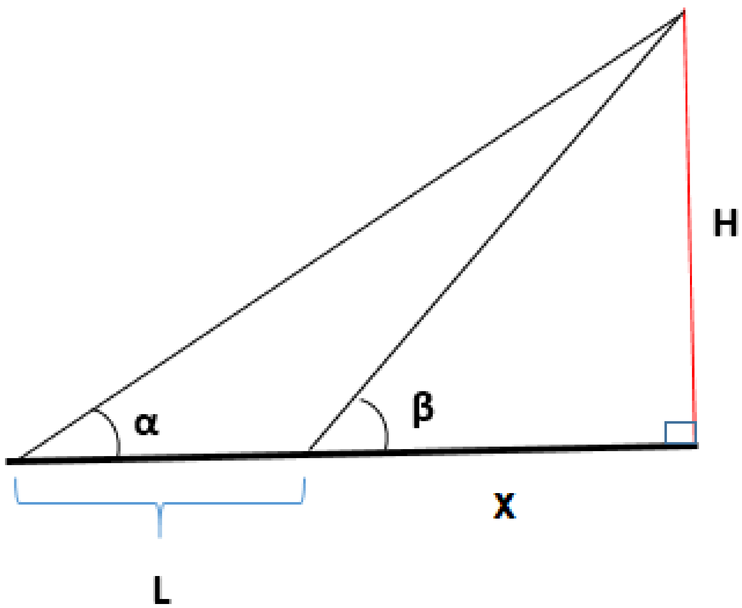
solve r/(R+r)=s,R

$$R = r\left(\frac{1}{s} - 1\right)$$

```
syms R r s
solve(r/(R+r)==s,R)
ans =
r/s - r
```



Esim. $L, \alpha, \beta \to H, x$



$$\frac{H}{x} = \tan(\beta) \ , \quad \frac{H}{L+x} = \tan(\alpha) \quad \to$$

$$H = -\frac{\tan(\alpha)\tan(\beta)L}{\tan(\alpha)-\tan(\beta)}, \quad x = -\frac{\tan(\alpha)L}{\tan(\alpha)-\tan(\beta)}$$

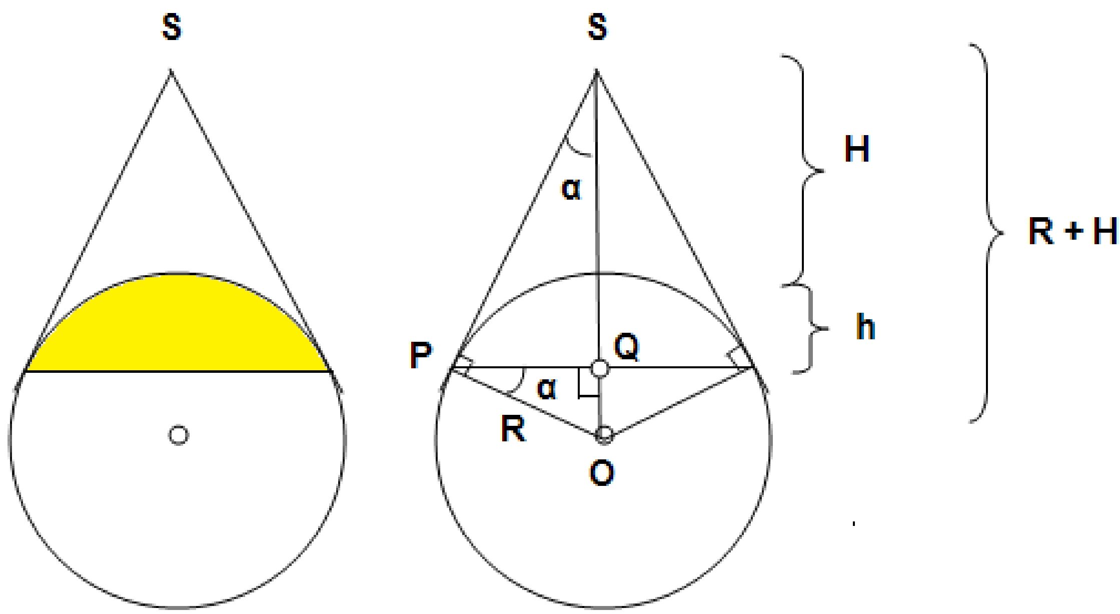
solve H/x=t2, H/(x+L)=t1,H,x

$$H = -\frac{L\,\mathrm{t1}\,\mathrm{t2}}{\mathrm{t1} - \mathrm{t2}} \quad \text{and} \quad x = -\frac{L\,\mathrm{t1}}{\mathrm{t1} - \mathrm{t2}}$$

```
syms H x L a b
solve(H/x==tan(b),H/(L+x)==tan(a),H,x)
H=ans.H
x=ans.x

%H=-(L*tan(a)*tan(b))/(tan(a) - tan(b))
%x=-(L*tan(a))/(tan(a) - tan(b))
```

Esim. $R, H \to h$

eli kuinka korkea segmentti näkyy etäisyydeltä H ympyrän/pallon (säde R) pinnasta

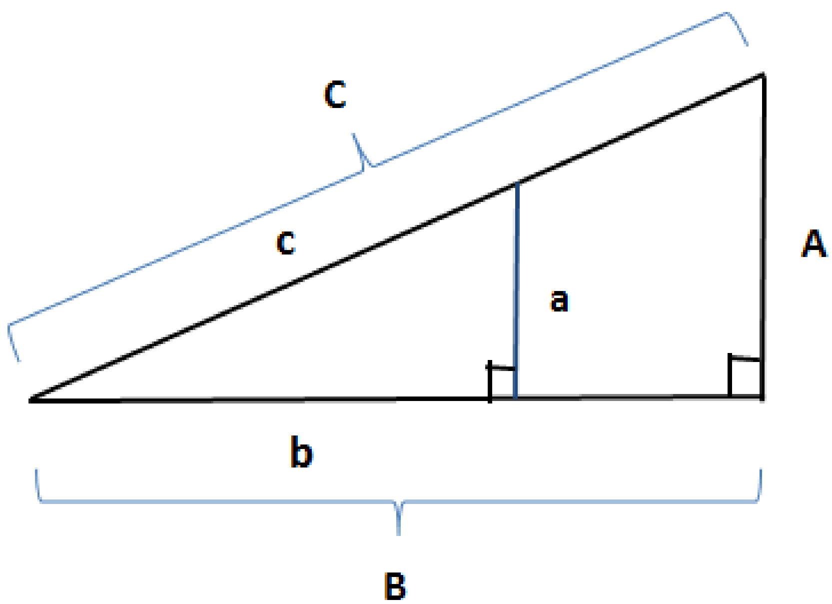


kolmio OPS: $\sin(\alpha) = \dfrac{OP}{OS} = \dfrac{R}{R+H}$

kolmio OPQ: $OQ = R\sin(\alpha) = \dfrac{R^2}{R+H}$

$\rightarrow h = R - OQ = \dfrac{RH}{R+H}$

HUOM: Suorakulmaisessa kolmiossa sivujen pituuksien suhteet eivät riipu kolmion koosta vaan pelkästään sen muodosta eli kulmista, eli esimerkiksi allaolevan kuvan tilanteessa

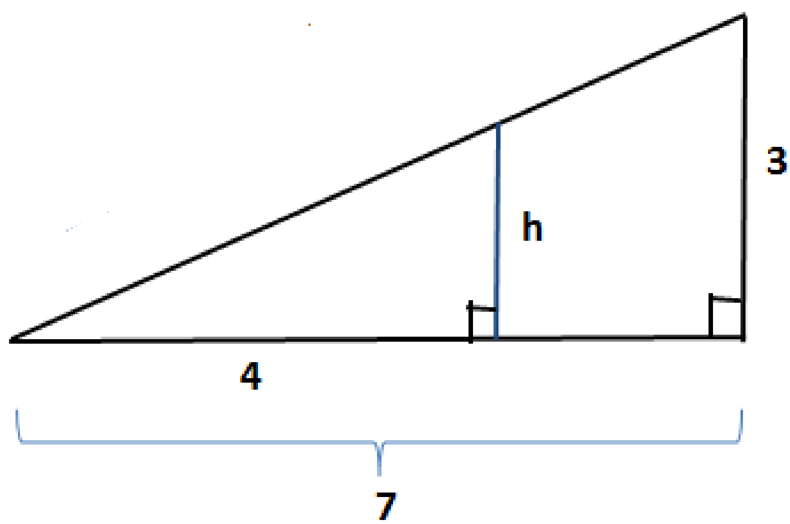


$$\frac{a}{c} = \frac{A}{C}, \quad \frac{b}{c} = \frac{B}{C}, \quad \frac{a}{b} = \frac{A}{B}$$

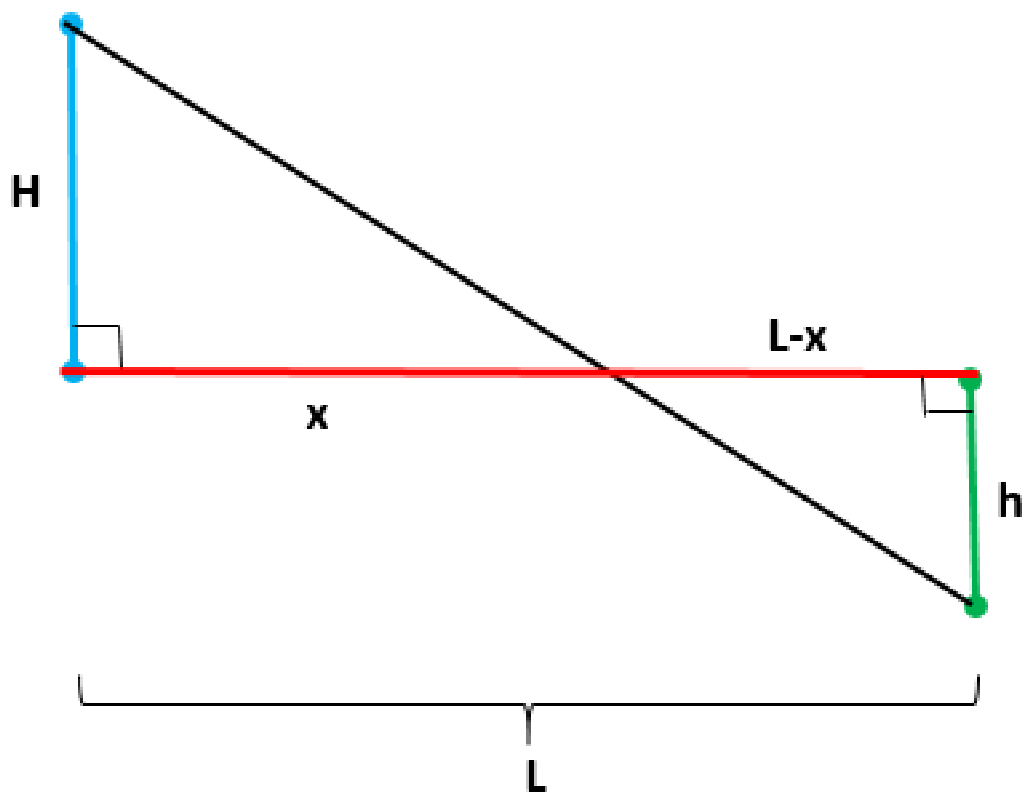
eli vastinsivujen pituuksien suhde on sama :

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$$

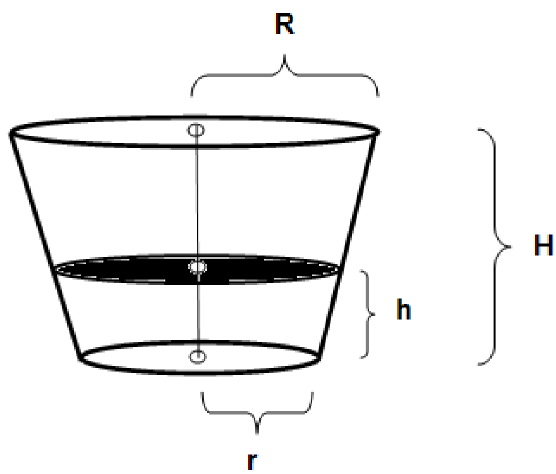
Esim. $\dfrac{h}{4} = \dfrac{3}{7} \to h = \dfrac{12}{7} \approx 1.7$

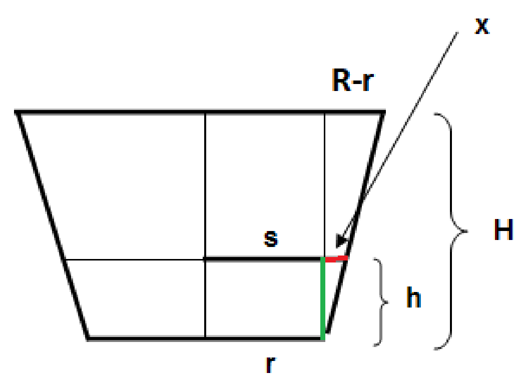


Esim. $\dfrac{x}{H} = \dfrac{L-x}{h}$ eli $x = \dfrac{LH}{L+h}$



Esim. $r, R, H, h \to V_{neste}$





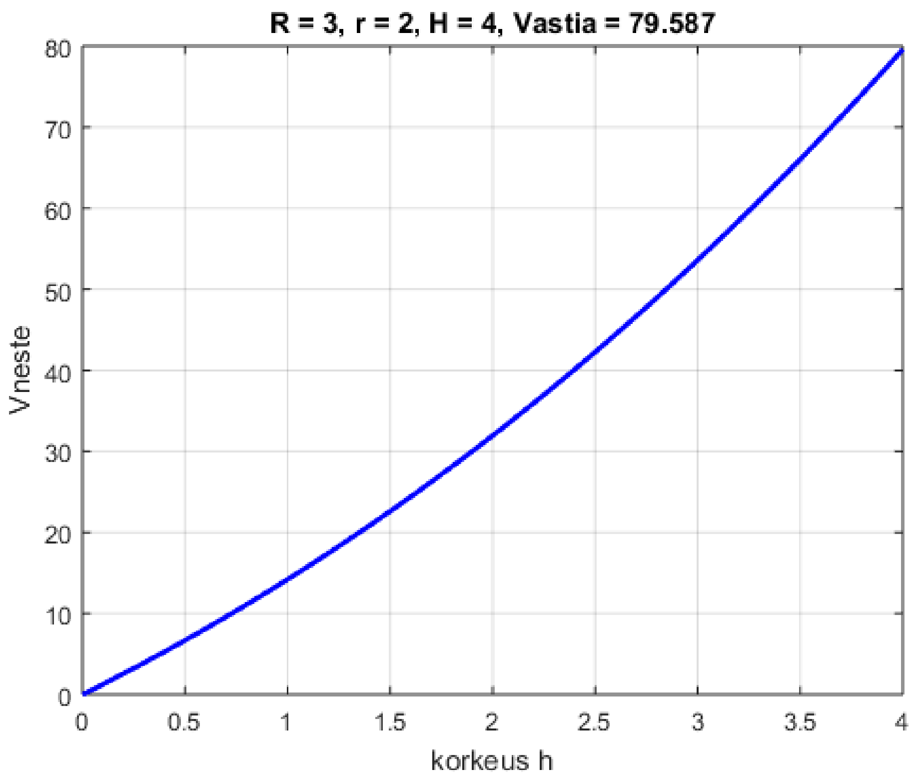
$$\frac{x}{h} = \frac{R-r}{H} \to x = \frac{h}{H} \cdot (R-r) \text{ eli}$$

$$s = r + x = r + \frac{h}{H} \cdot (R-r)$$

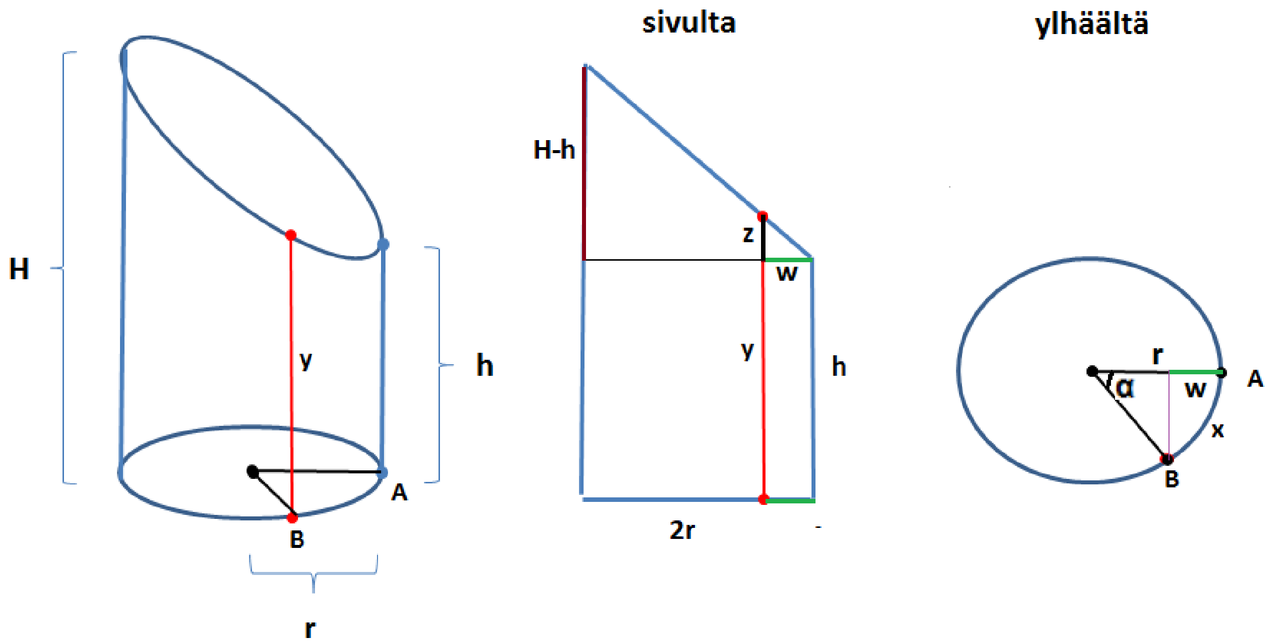
Tilavuudet (katkaistu kartio)

$$V_{astia} = \frac{1}{3}\pi(R^2 + Rr + r^2)H$$

$$V_{neste} = \frac{1}{3}\pi(s^2 + sr + r^2)h$$



Esim. Laske korkeus y, kun kaaren AB pituus on x

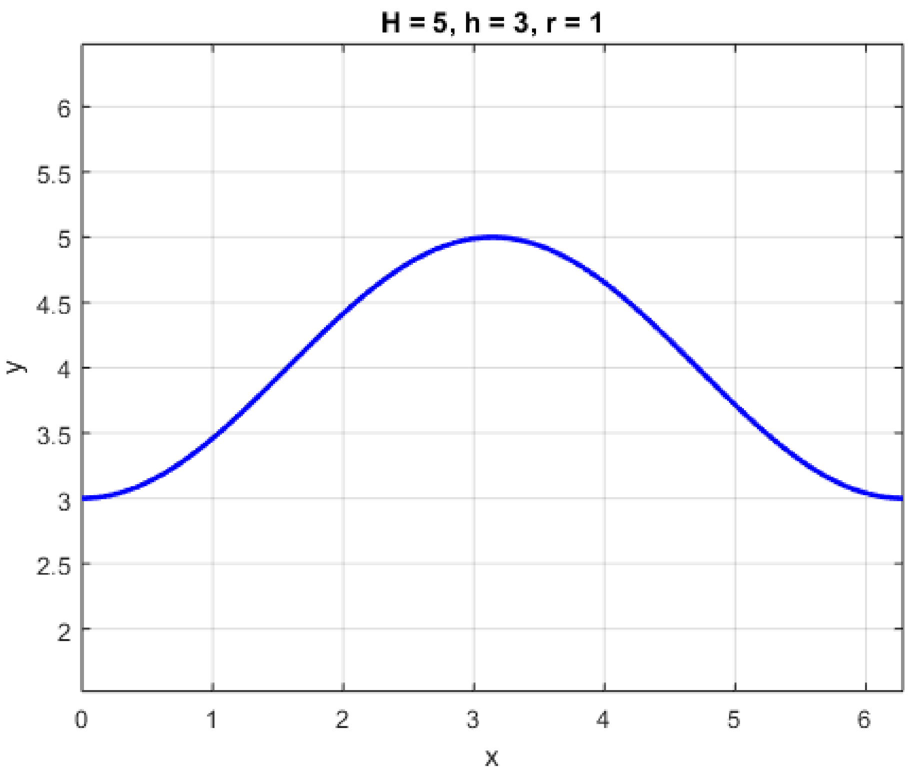


eli minkä muotoisesta levystä saadaan rullalle taivuttamalla vino lieriö

$$\alpha = x/r,\ w = r - r\cos(\alpha)$$

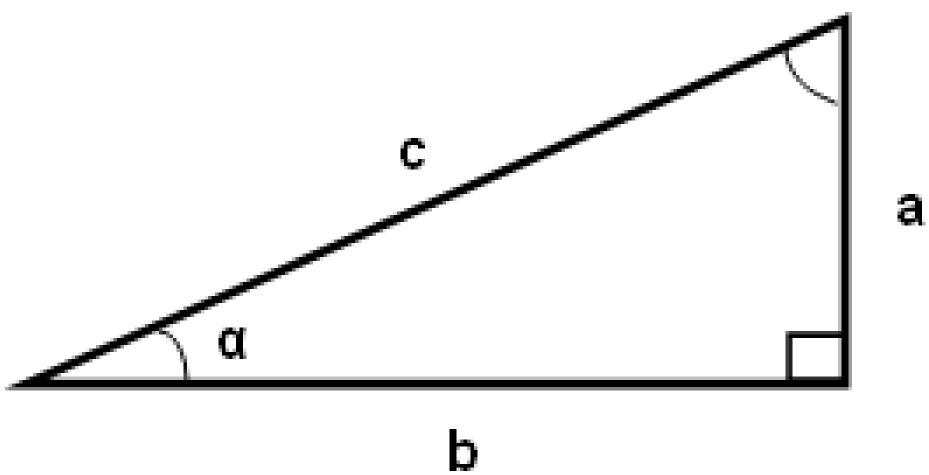
$$z = \frac{(H-h)w}{2r} = \frac{(H-h)(1-\cos(\alpha))}{2}$$

$$y = h + z$$



Arcussini, -kosini ja -tangentti

sivun pituudet $\rightarrow$ kulmat



$$\sin(\alpha) = \frac{a}{c} \quad \leftrightarrow \quad \alpha = \sin^{-1}\left(\frac{a}{c}\right) = \arcsin\left(\frac{a}{c}\right)$$

$$\cos(\alpha) = \frac{b}{c} \quad \leftrightarrow \quad \alpha = \cos^{-1}\left(\frac{b}{c}\right) = \arccos\left(\frac{b}{c}\right)$$

$$\tan(\alpha) = \frac{a}{b} \quad \leftrightarrow \quad \alpha = \tan^{-1}\left(\frac{a}{b}\right) = \arctan\left(\frac{a}{b}\right)$$

MATLAB/Octave:

asin(a/c),acos(b/c),atan(a/b) radiaaneina

asind(a/c),acosd(b/c),atand(a/b) asteina

Esimerkiksi

$\sin^{-1}(0.35)$

on sellaisen kulman suuruus, jonka sini $= 0.35$

Laskukoneella saadaan

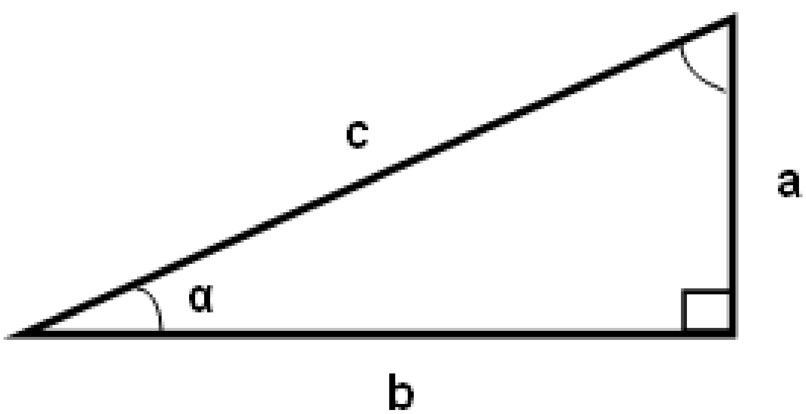
$\sin^{-1}(0.35) \approx 20.5^\circ$

eli jos

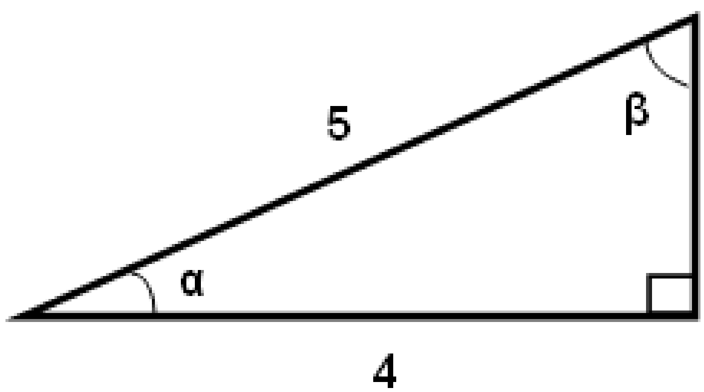
$$\sin(\alpha) = \frac{a}{c} = 0.35$$

niin

$$\alpha = \sin^{-1}(0.35) \approx 20.5^\circ$$



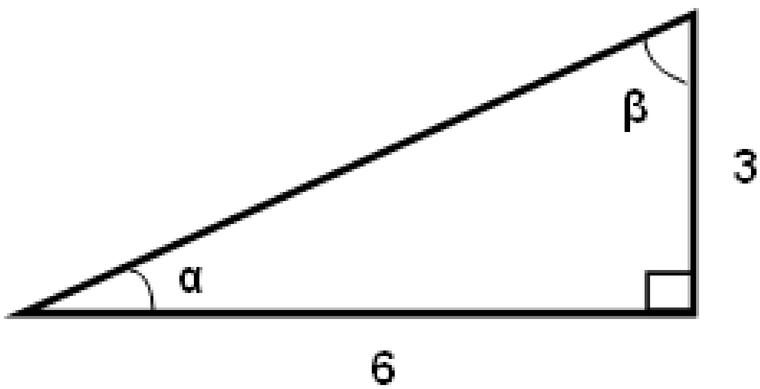
Esim.



$$\alpha = \cos^{-1}\left(\frac{4}{5}\right) \approx 36.9^\circ$$

$$\beta = \sin^{-1}\left(\frac{4}{5}\right) \approx 53.1^\circ$$

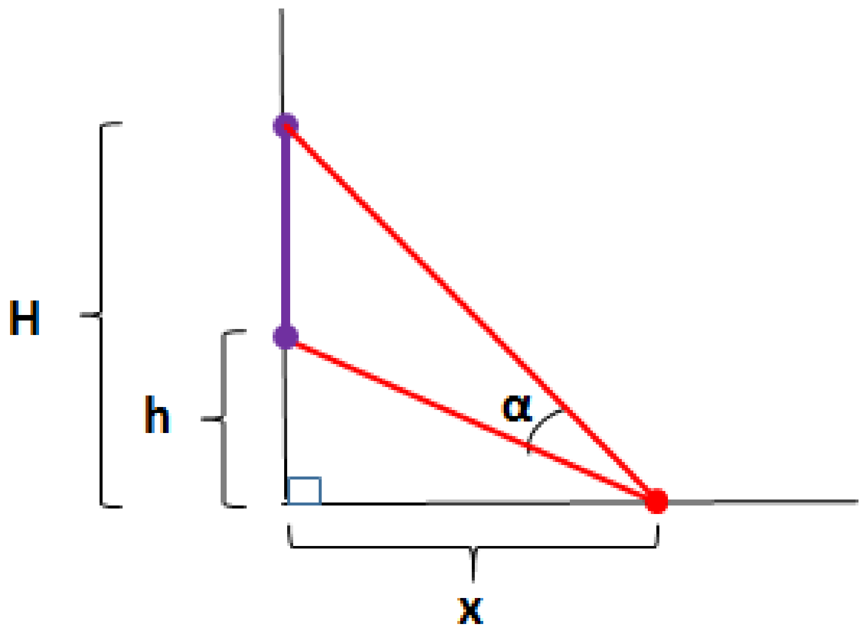
Esim.

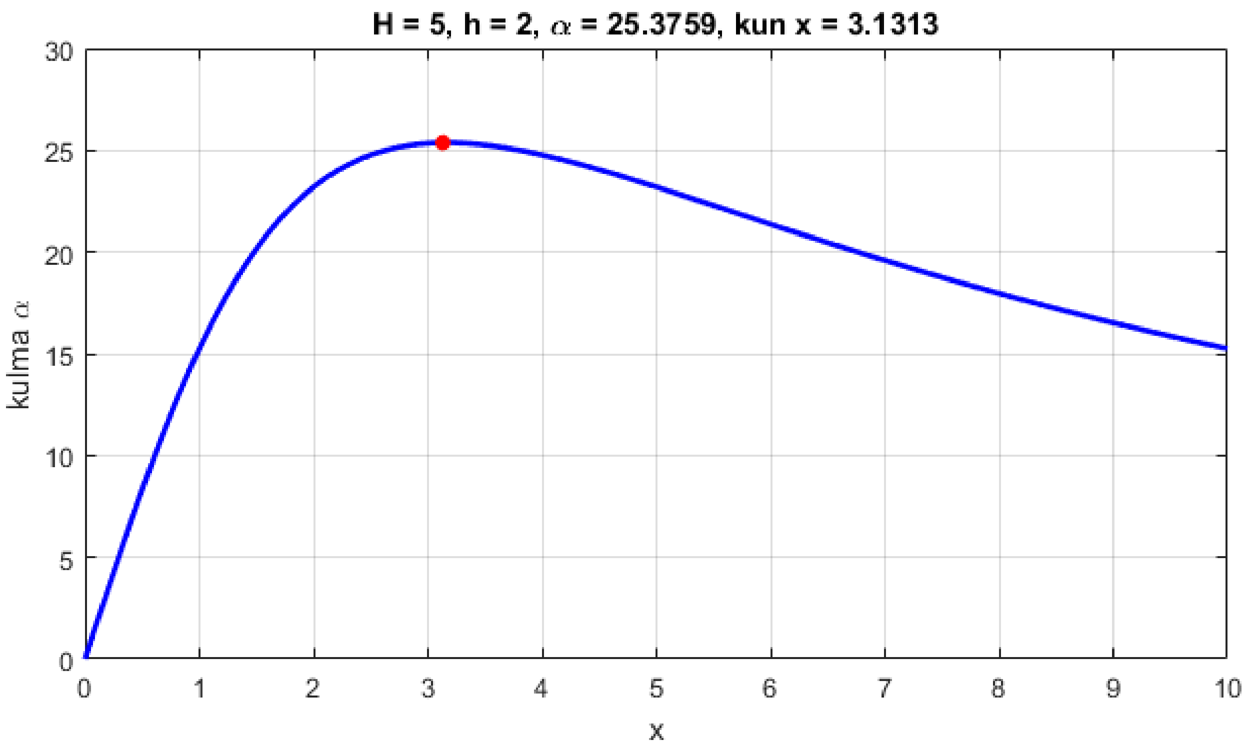


$$\alpha = \tan^{-1}\left(\frac{3}{6}\right) \approx 26.6^\circ$$

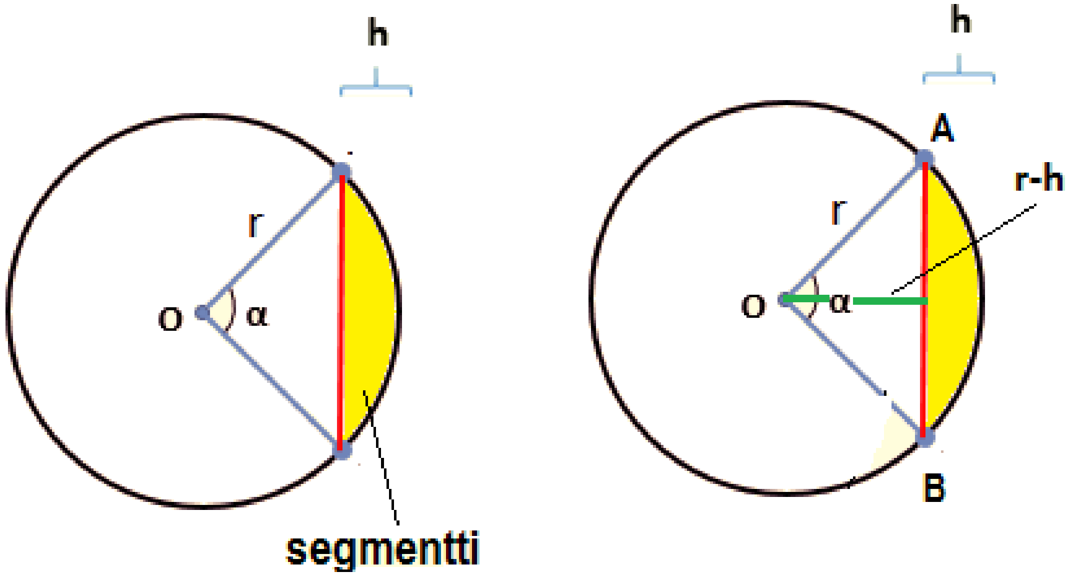
$$\beta = \tan^{-1}\left(\frac{6}{3}\right) \approx 63.4^\circ$$

Esim. $\alpha = \tan^{-1}\left(\frac{H}{x}\right) - \tan^{-1}\left(\frac{h}{x}\right)$




H = 5, h = 2, α = 25.3759, kun x = 3.1313
kulma α
x
0
5
10
15
20
25
30
0
1
2
3
4
5
6
7
8
9
10

Esim. $\alpha = 2\cos^{-1}\left(\frac{r-h}{r}\right)$



Huom: ympyrän segmentin pinta-ala on

$$\frac{1}{2}(\alpha - \sin(\alpha))r^2$$

kun keskuskulma α on radiaaneina.

Syy: sektorin ala on

$$\frac{\alpha}{2\pi} \cdot \pi r^2 = \frac{1}{2}\alpha r^2$$

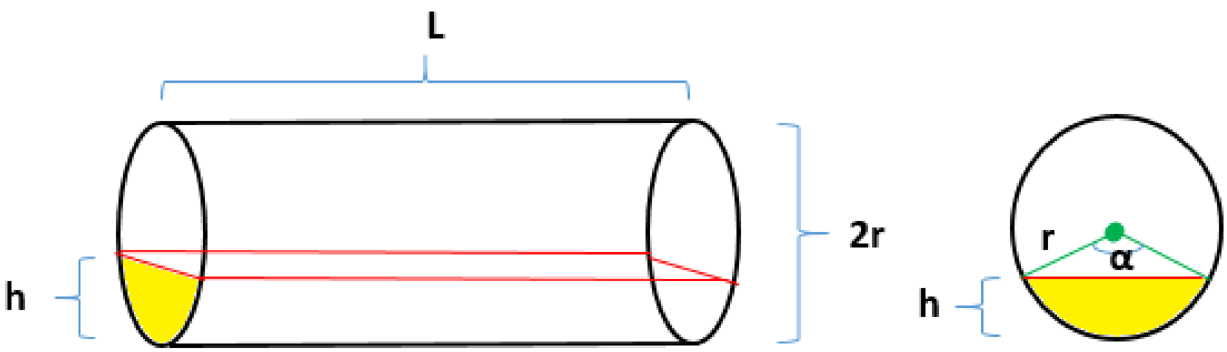
ja kolmion ala

$$\tfrac{1}{2} \cdot \underbrace{r\cos(\alpha/2)}_{leveys} \cdot \underbrace{2r\sin(\alpha/2)}_{korkeus}$$

$$= \tfrac{1}{2} \cdot \underbrace{2\cos(\alpha/2)\sin(\alpha/2)}_{=\sin(\alpha)} \cdot r^2$$

$$= \tfrac{1}{2}\sin(\alpha)r^2$$

Esim. säiliössä olevan nesteen korkeus h



Nesteen tilavuus

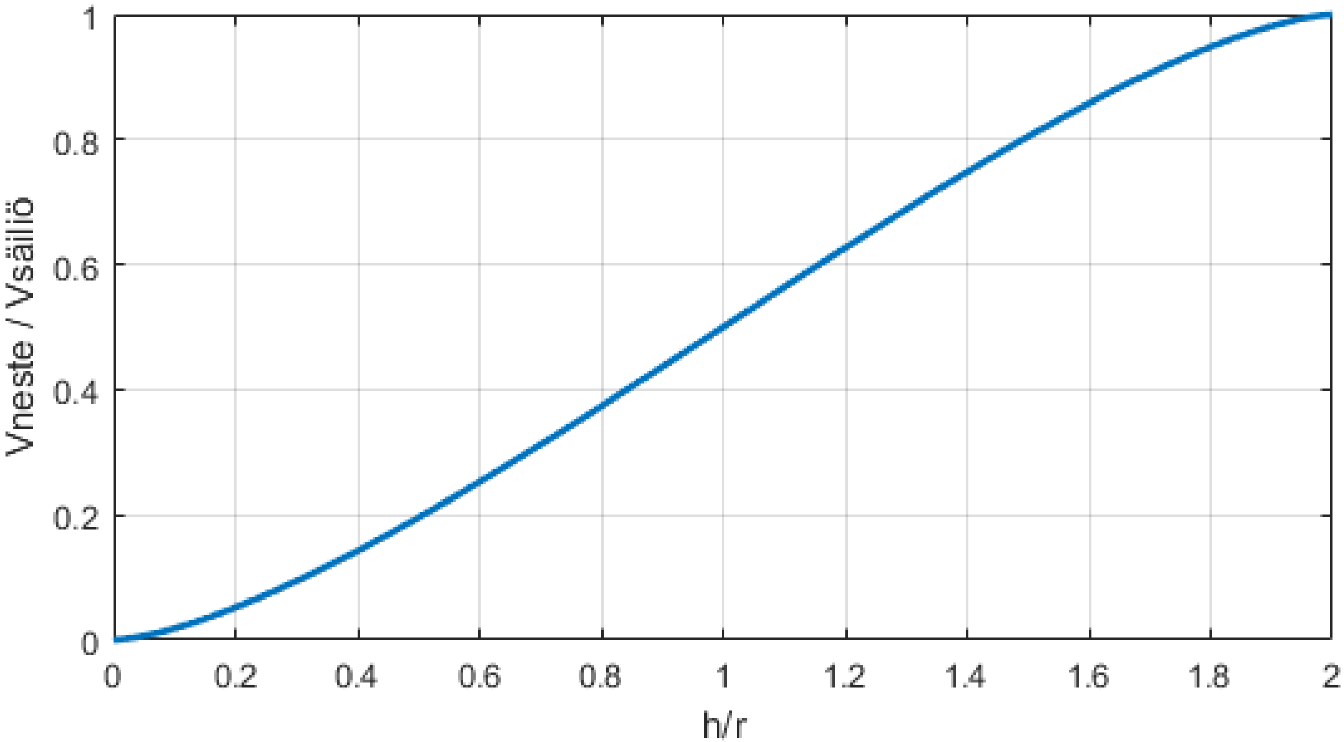
$$V_{neste} = \text{segmentin ala} \times \text{leveys } L$$

$$= \frac{1}{2}(\alpha - \sin(\alpha))r^2 L$$

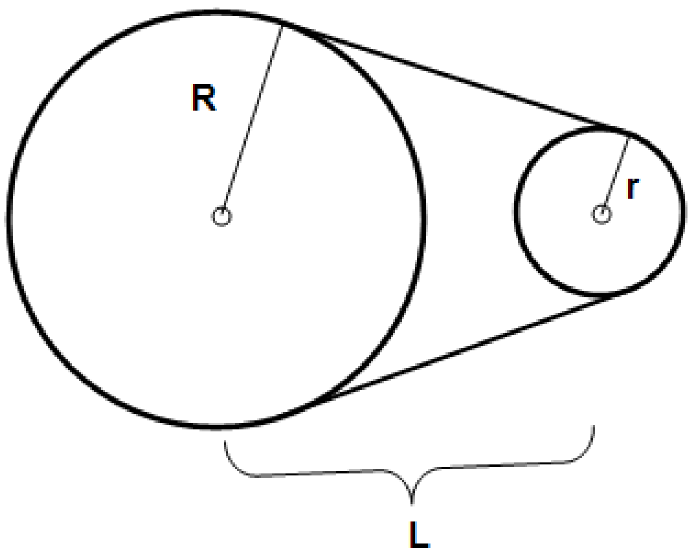
missä

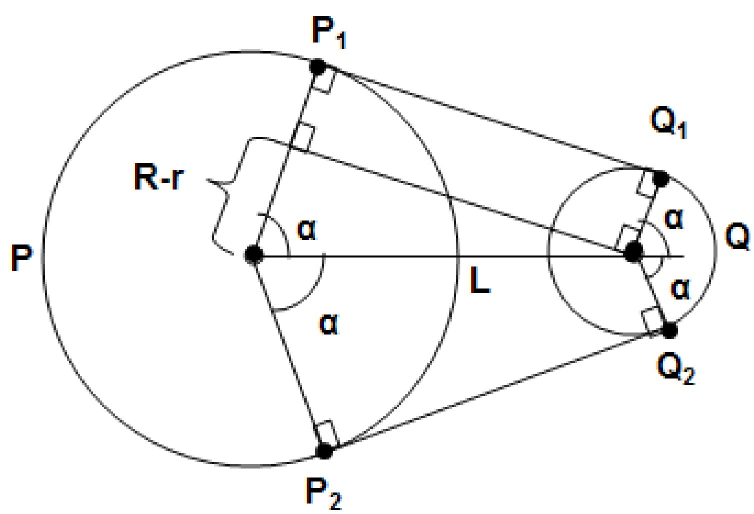
$$\alpha = 2\cos^{-1}\left(\frac{r-h}{r}\right) \text{ (rad)}$$

$$\frac{V_{neste}}{V_{säiliö}} = \frac{\frac{1}{2}(\alpha - \sin(\alpha))r^2 L}{\pi r^2 L} = \frac{1}{2\pi}(\alpha - \sin(\alpha))$$



Esim. $R, r, L \to$ hihnan pituus





$$P_1Q_1 = P_2Q_2 = \sqrt{L^2 - (R-r)^2}$$

$$\alpha = \cos^{-1}\left(\frac{R-r}{L}\right) \text{ (rad)}$$

kaari $Q_1QQ_2 = 2\alpha r$

kaari $P_1PP_2 = (2\pi - 2\alpha)R$

Trigonometrian kaavoja:

Merkintä: $\sin^2(\alpha) = (\sin(\alpha))^2$ jne

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

$$\tan(2\alpha) = \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)}$$

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$$

$$\cot(\alpha) = \frac{1}{\tan(\alpha)} = \frac{\cos(\alpha)}{\sin(\alpha)} \qquad \text{(kotangentti)}$$

$$\sec(\alpha) = \frac{1}{\cos(\alpha)} \qquad \text{(sekantti)}$$

$$\csc(\alpha) = \frac{1}{\sin(\alpha)} \qquad \text{(kosekantti)}$$

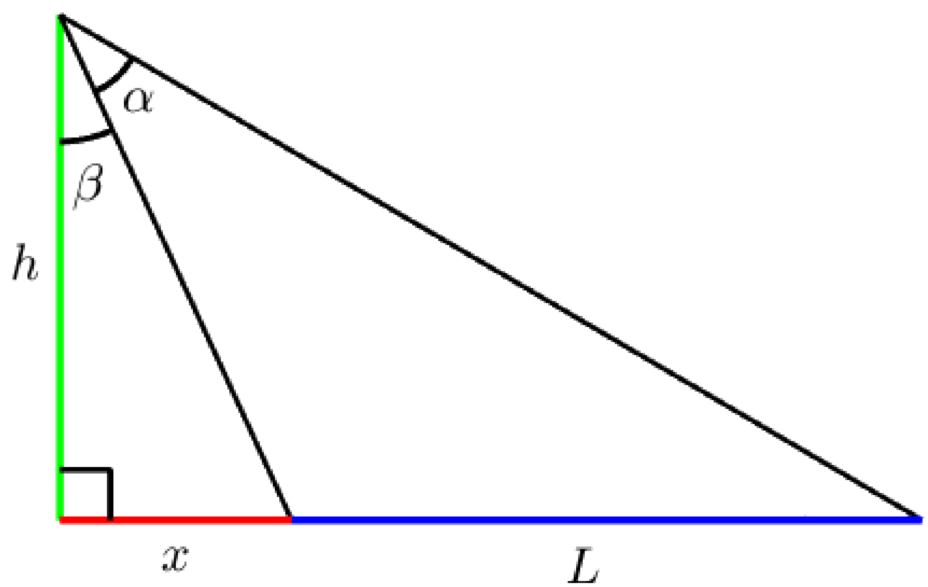
MATLAB/Octave:

$\text{cot}(\alpha), \text{sec}(\alpha), \text{csc}(\alpha)$ (α radiaaneina)

$\text{cotd}(\alpha), \text{secd}(\alpha), \text{cscd}(\alpha)$ (α asteina)

Esim: $h, L, \alpha \to x, \beta$

(eli määrää "kameran" sijainti ja kallistuskulma, jotta se näkisi L:n pituisen alueen)



$$\tan(\beta) = \frac{x}{h}$$

$$\tan(\alpha + \beta) = \frac{x + L}{h}$$

solve tan(b)=x/h,tan(a+b)=(x+L)/h,x,b

$$x = 0.5\left(-h\csc(a)\sqrt{\frac{\sin^2(a)\left(4\,h\,L\cot(a) - 4\,h^2 + L^2\right)}{h^2}} - L\right)$$

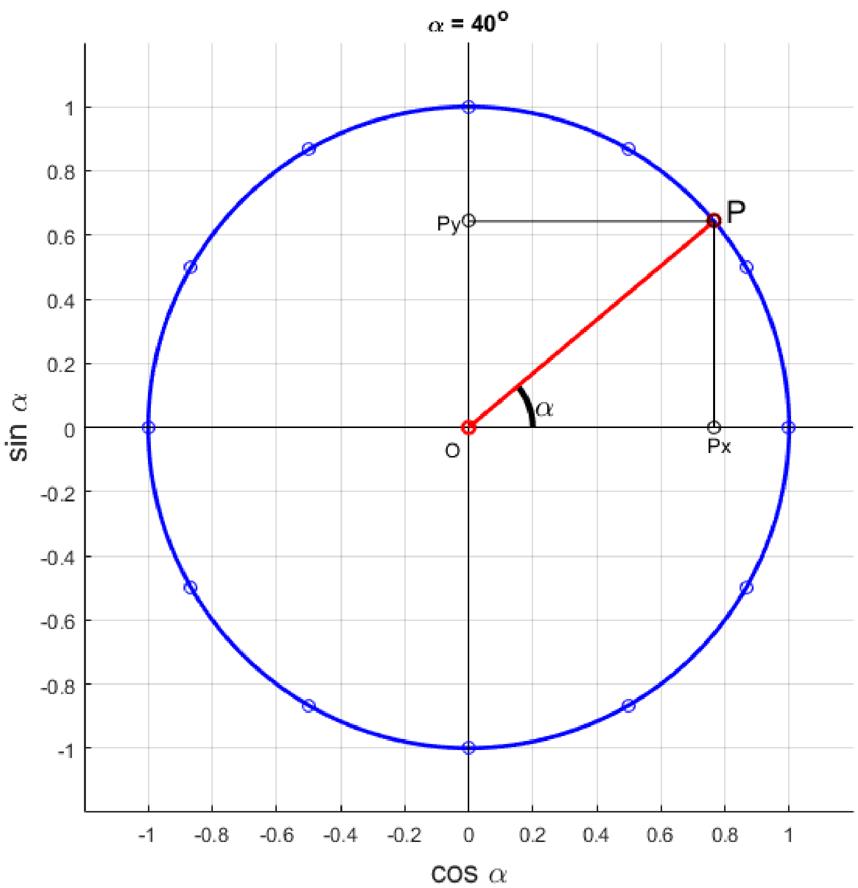
$$x = \frac{1}{2}\left(h\csc(a)\sqrt{\frac{\sin^2(a)\left(4\,h\,L\cot(a) - 4\,h^2 + L^2\right)}{h^2}} - L\right)$$

```
syms x a b h L
solve(tan(b)==x/h,tan(a+b)==(x+L)/h,x,b)
x=ans.x

x =
 -(L*tan(a) - (tan(a)*(tan(a)*L^2 + 4*L*h - 4*tan(a)*h^2))^(1/2))/(2*tan(a))
 -(L*tan(a) + (tan(a)*(tan(a)*L^2 + 4*L*h - 4*tan(a)*h^2))^(1/2))/(2*tan(a))
```

$$\beta = \tan^{-1}\left(\frac{x}{h}\right)$$

YKSIKKÖYMPYRÄ

eli mitä $\sin(\alpha), \cos(\alpha)$ ja $\tan(\alpha)$ oikeasti tarkoittavat:

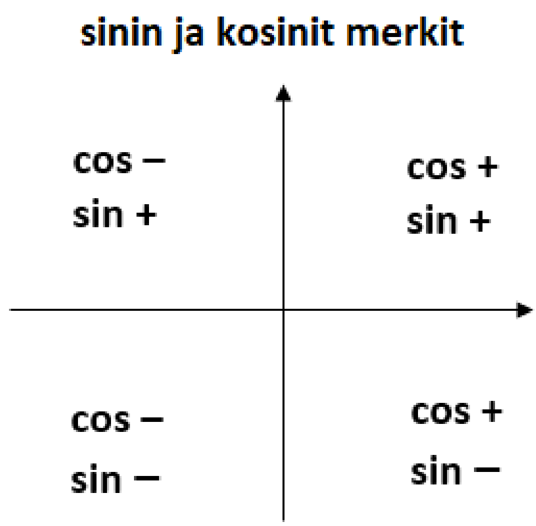


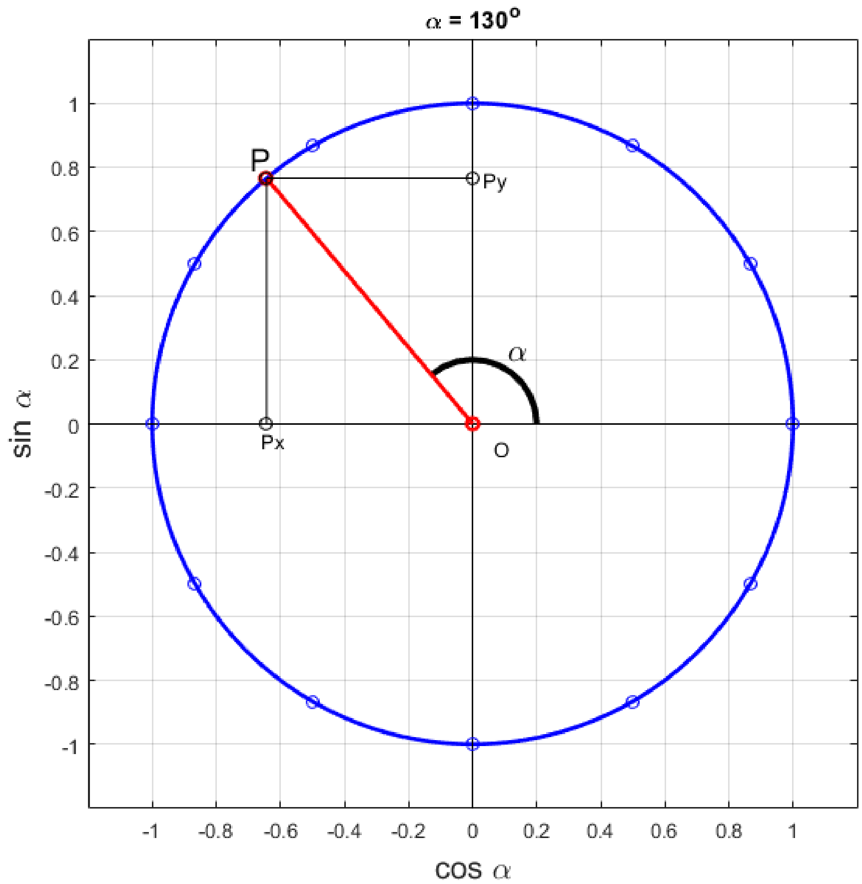
$Px = \cos(40^\circ) \approx 0.77, \quad Py = \sin(40^\circ) \approx 0.64$

Yksikköympyrän (säde =1) pisteen P koordinaatit eli sen vaaka- ja pystysuuntaiset etäisyydet O:sta (oikealle plus, vasemmalle miinus, ylös plus, alas miinus) ovat

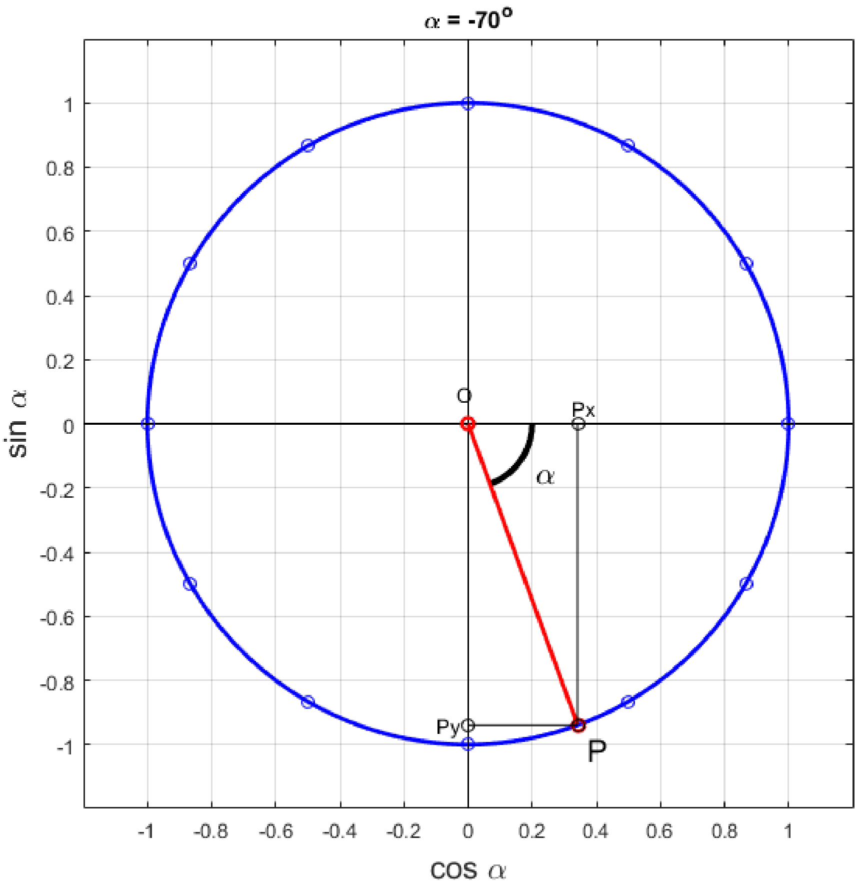
$Px = \cos(\alpha)$, $Py = \sin(\alpha)$

Näin $\cos(\alpha)$ ja $\sin(\alpha)$ ovat järkeviä lukuja olipa kulma α kuinka suuri tahansa



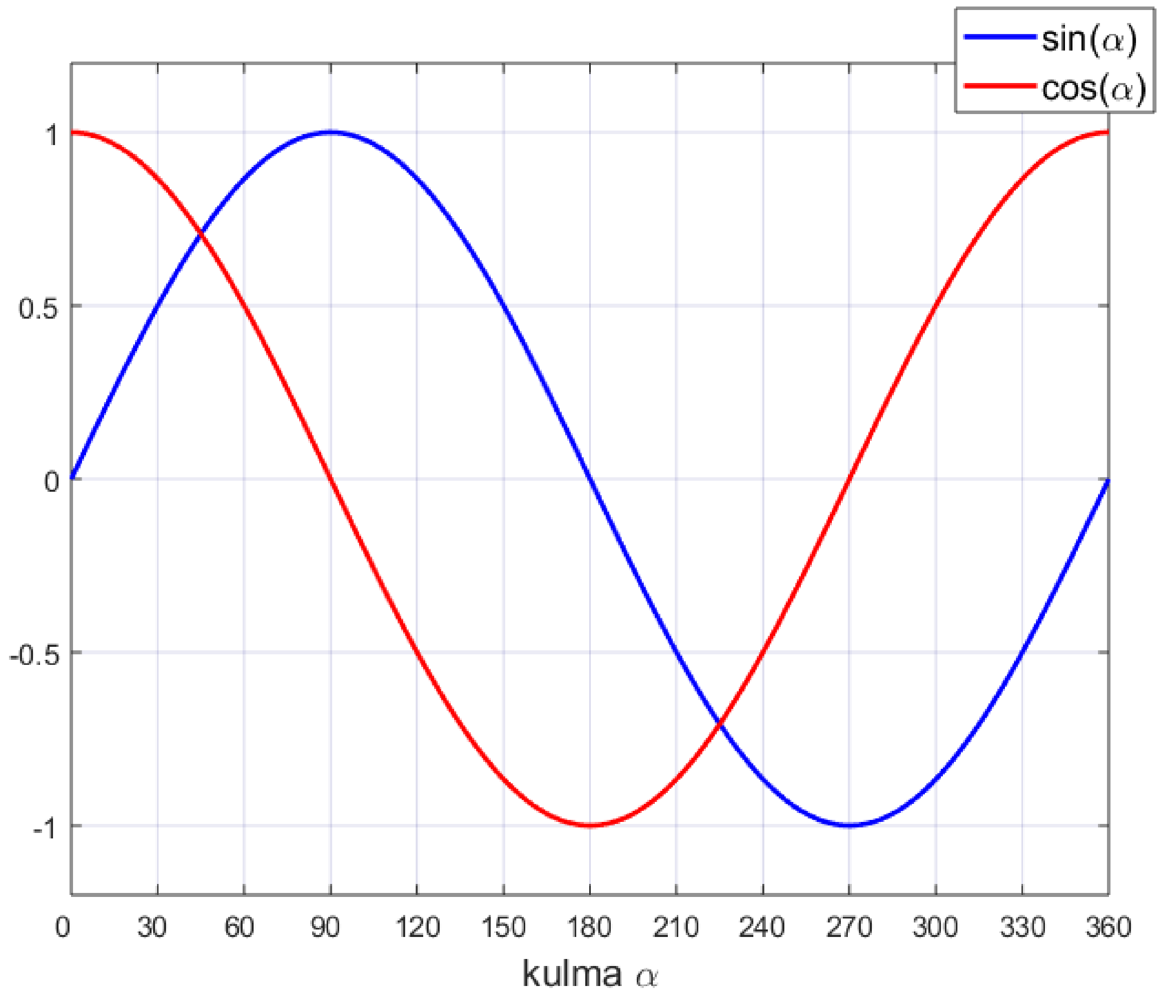


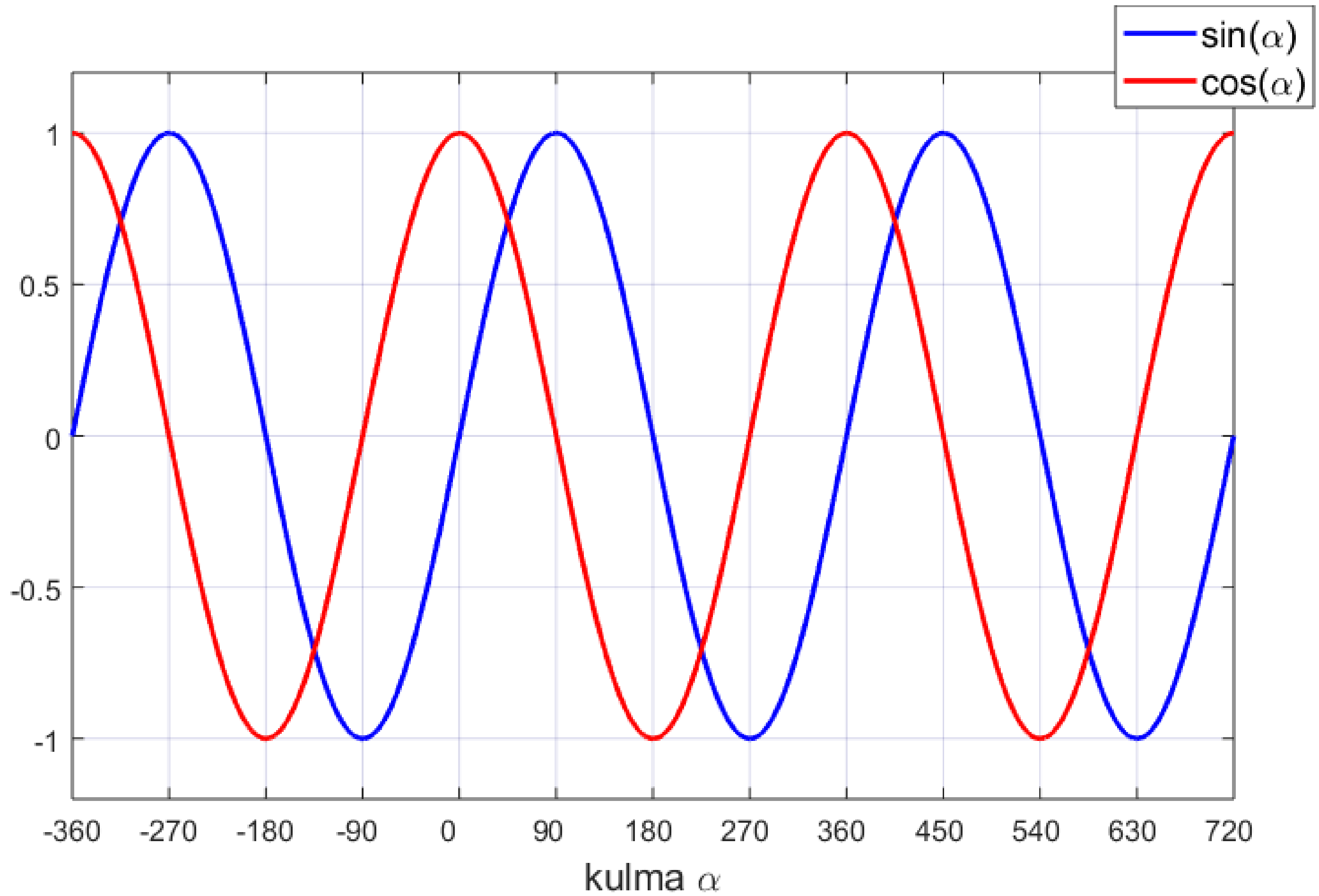
$$\cos(130^\circ) \approx -0.64, \sin(130^\circ) \approx 0.77$$



$$\cos(-70^\circ) = \cos(290^\circ) \approx 0.34$$
$$\sin(-70^\circ) = \sin(290^\circ) \approx -0.94$$

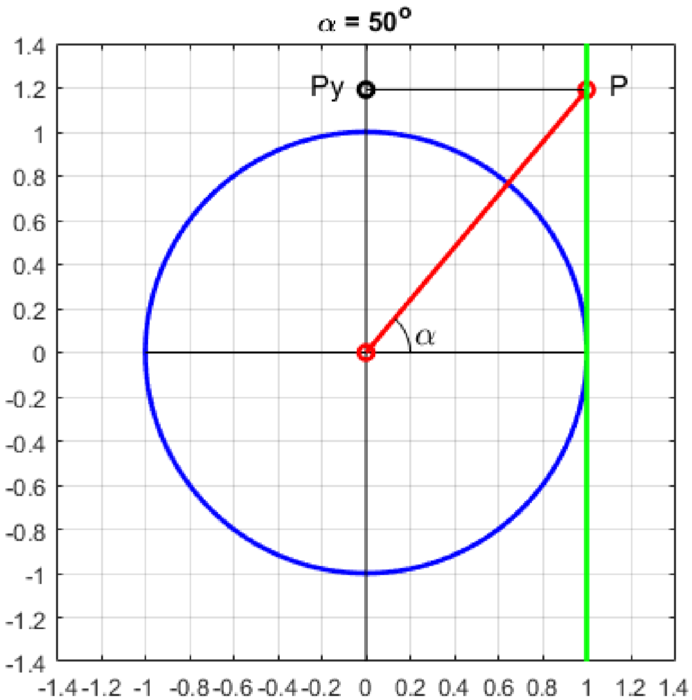
Sinin ja kosinin arvot vaihtelevat välillä $-1 \dots 1$ ja ne toistuvat kierroksen eli 360°:een välein.

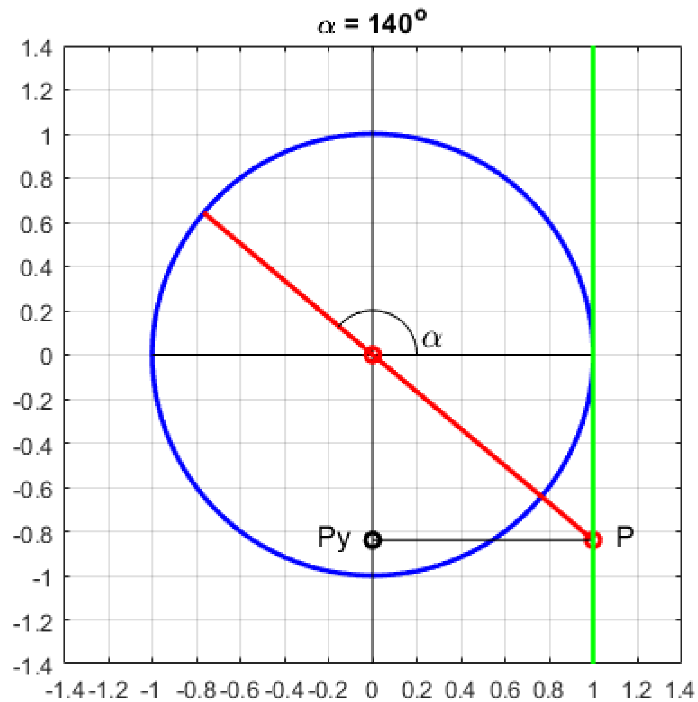




sin(α)
cos(α)
1
0.5
0
-0.5
-1
-360
-270
-180
-90
0
90
180
270
360
450
540
630
720
kulma α

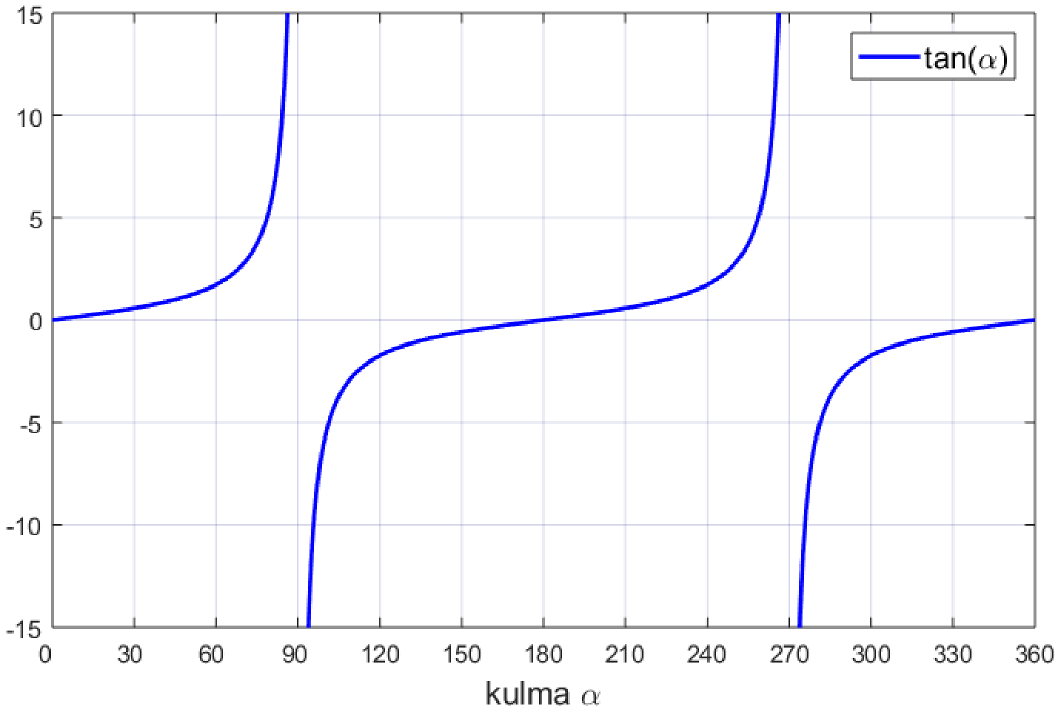
$$Py = \tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$$





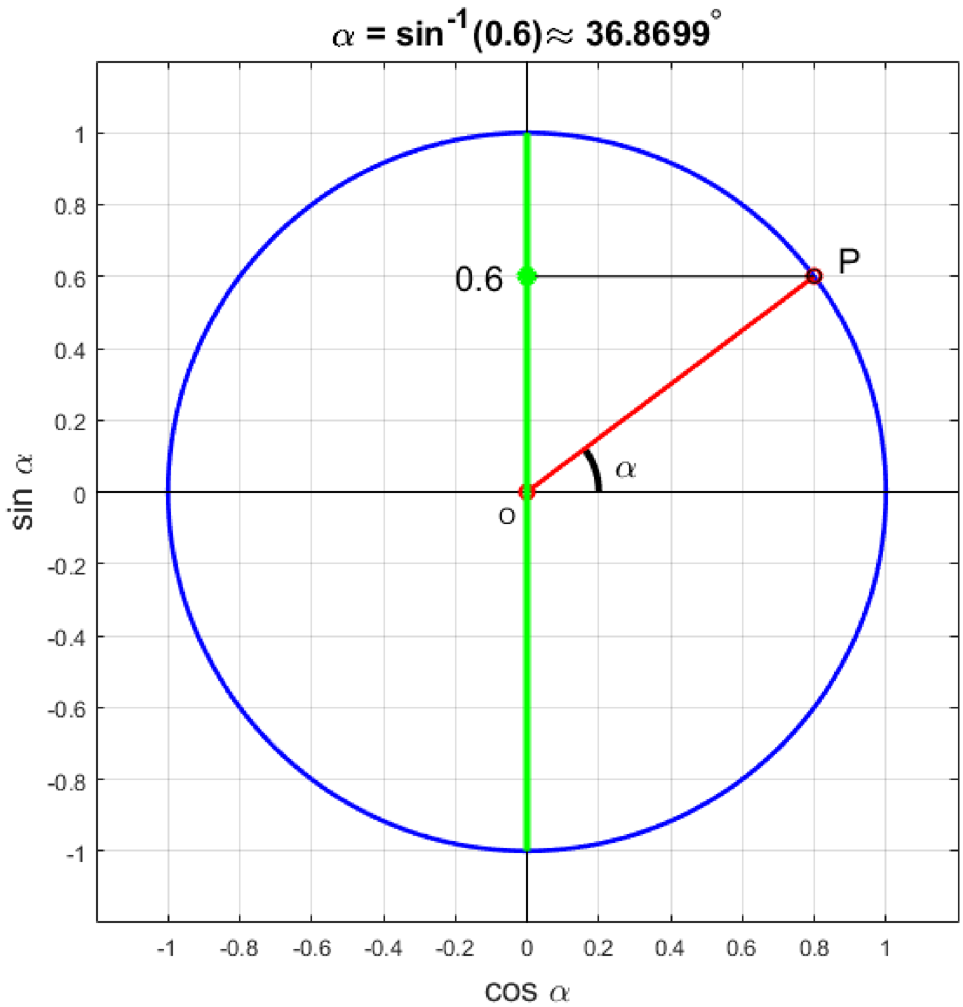
$$\tan(50^\circ) \approx 1.2$$

$$\tan(140^\circ) = \tan(-40^\circ) \approx -0.84$$

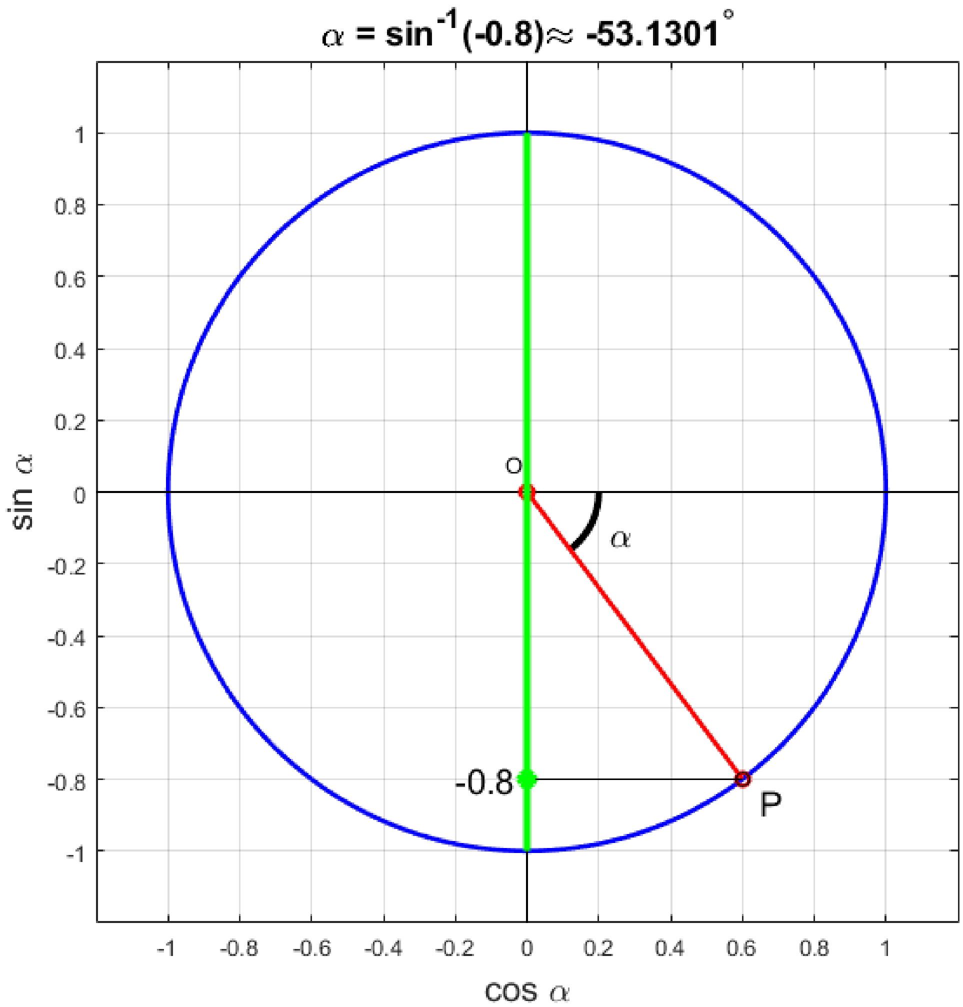


tan(α):aa ei ole olemassa, kun $\alpha = 90^\circ$ tai 270°

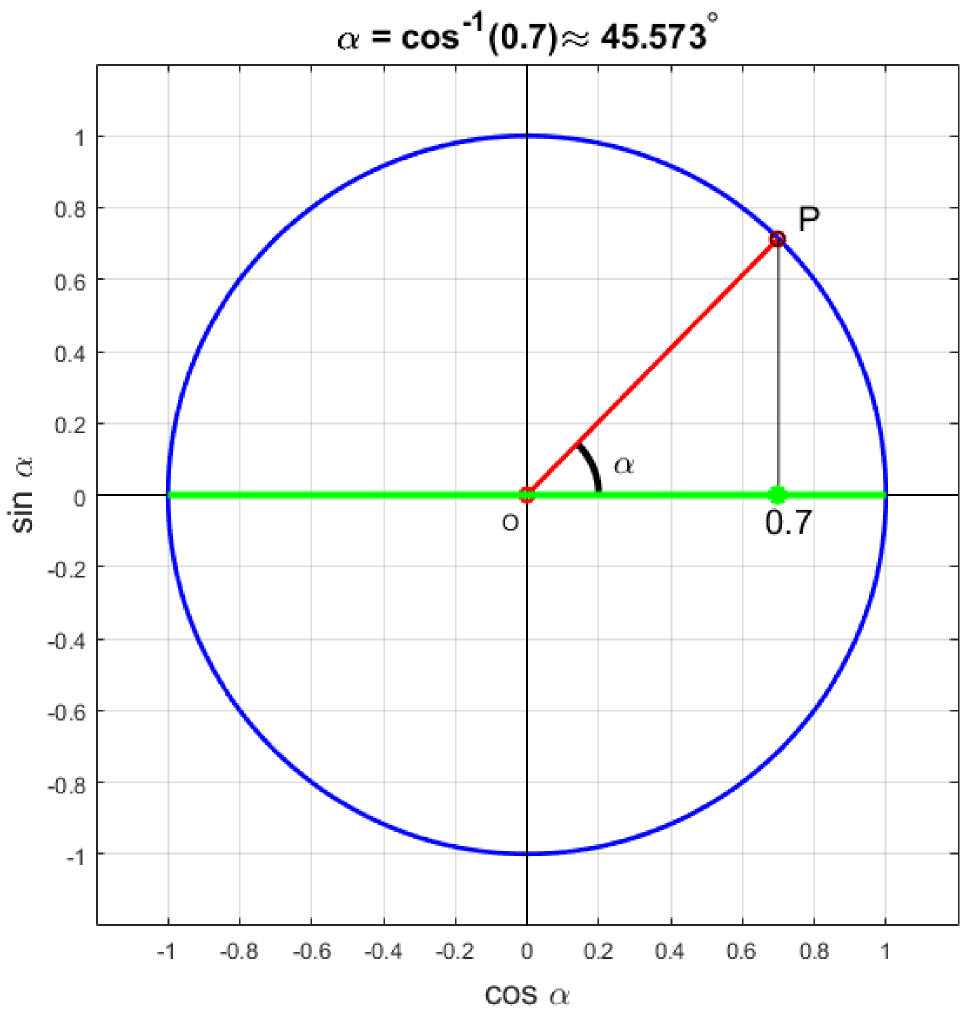
$\alpha = \sin^{-1}(y) = $ se kulma $\underline{\text{väliltä } -90^\circ \ldots 90^\circ}$, jonka sini on y



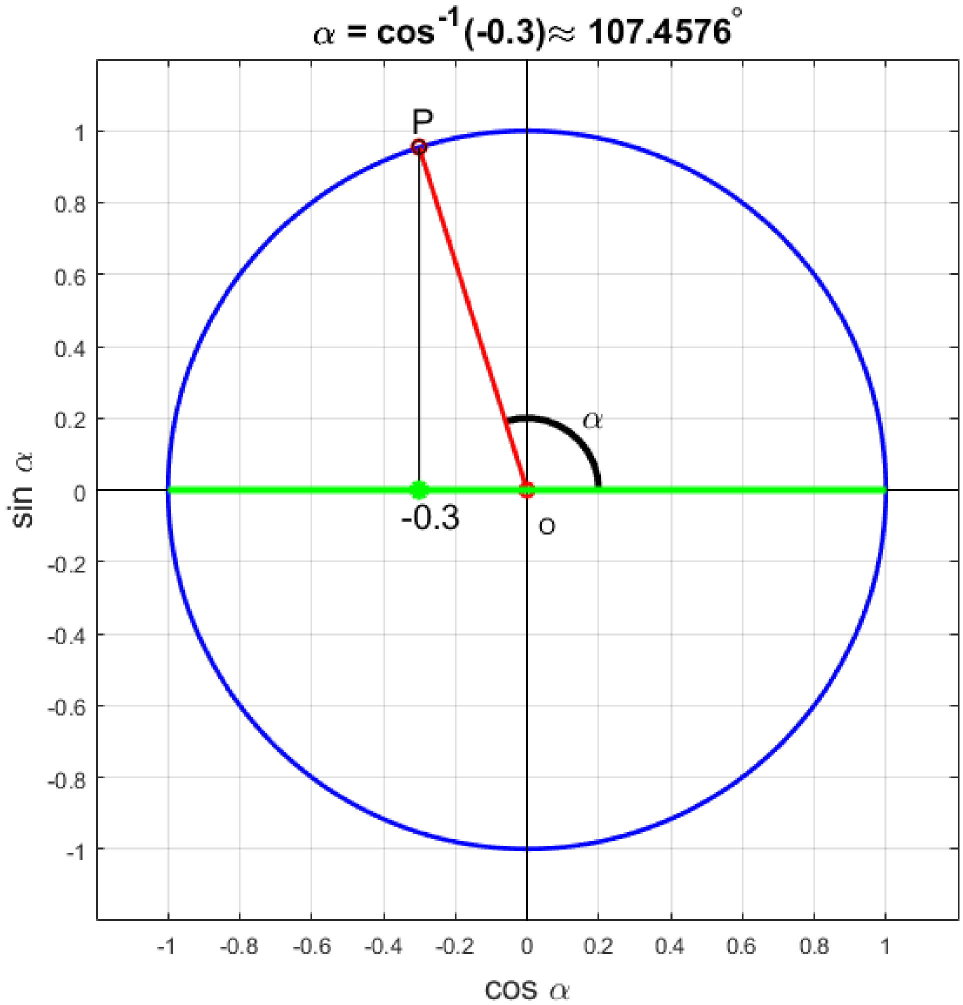
Huom: jotta kulma $\sin^{-1}(y)$ olisi olemassa, niin y:n pitää olla välillä $-1 \ldots 1$



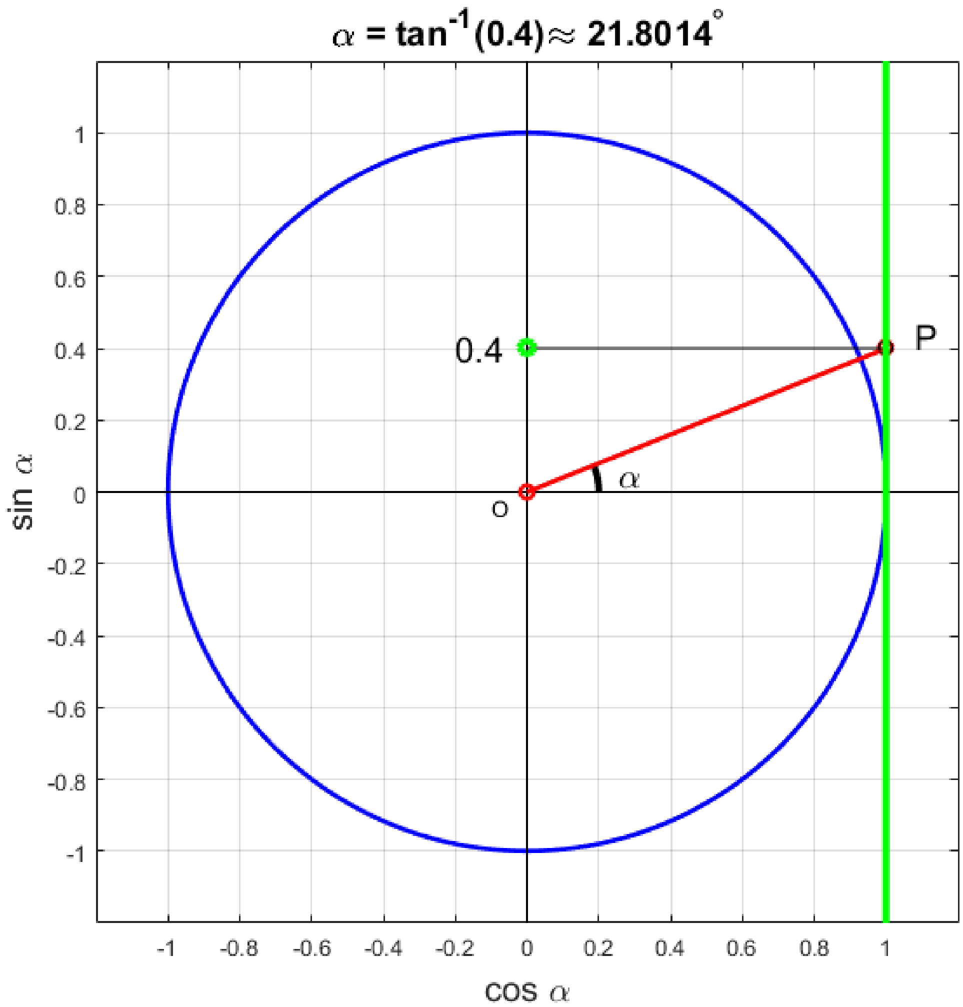
$\alpha = \cos^{-1}(x) =$ se kulma <u>väliltä $0 \ldots 180°$</u>, jonka kosini on x



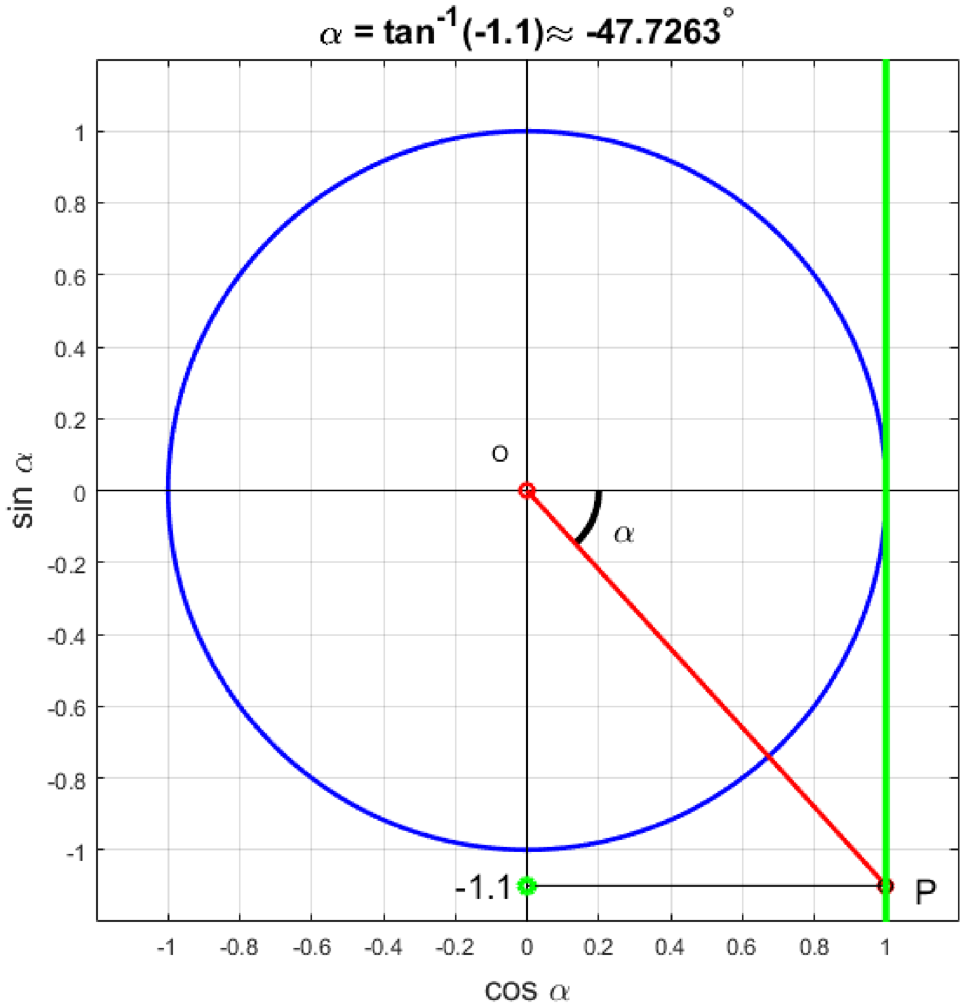
Huom: jotta kulma $\cos^{-1}(x)$ olisi olemassa, niin x:n pitää olla välillä $-1 \ldots 1$



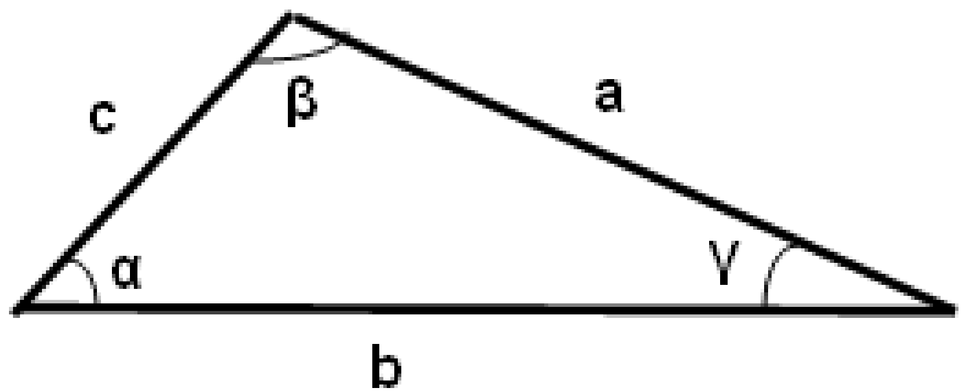
$\alpha = \tan^{-1}(y) =$ se kulma $\underline{\text{väliltä } -90^\circ \ldots 90^\circ}$, jonka tangentti on y



Huom: kulma $\tan^{-1}(y)$ on olemassa, olipa y mitä tahansa



YLEINEN KOLMIO



Sivu-kulma-parit
$\alpha \leftrightarrow a$, $\beta \leftrightarrow b$ ja $\gamma \leftrightarrow c$

Kulmien summa $\alpha + \beta + \gamma = 180°$

Kosinilause (law of cosines):

(c on kulman γ vastainen sivu)

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma) \qquad \text{eli}$$

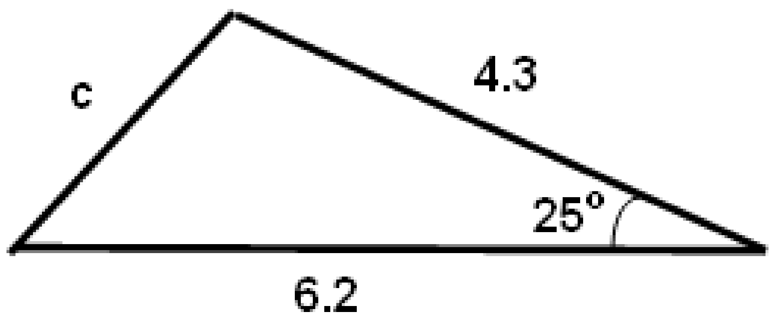
$$c = \sqrt{a^2 + b^2 - 2ab\cos(\gamma)}$$

tai

$$\cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab} \qquad \text{eli}$$

$$\gamma = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

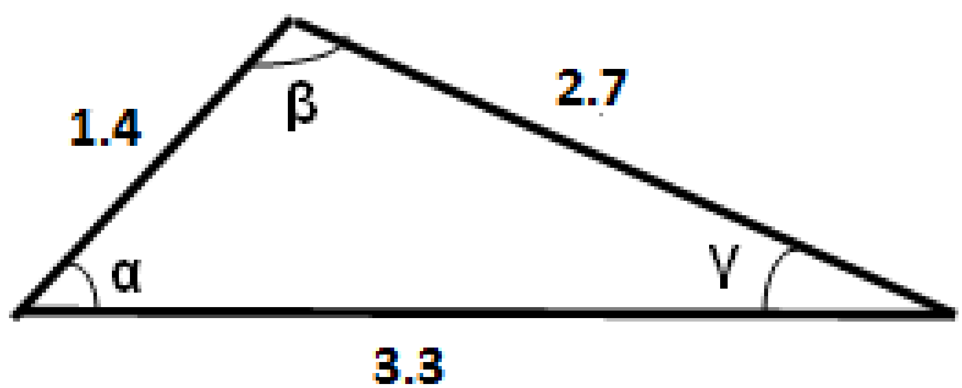
Esim. lasketaan kolmion kolmas sivu, kun tunnetaan kaksi muuta sivua ja niiden välinen kulma



$$a = 4.3, b = 6.2, \gamma = 25^\circ$$

$$c = \sqrt{a^2 + b^2 - 2ab\cos(\gamma)} \approx 2.9$$

Esim. lasketaan kolmion kulmat, kun tunnetaan kaikki sivut

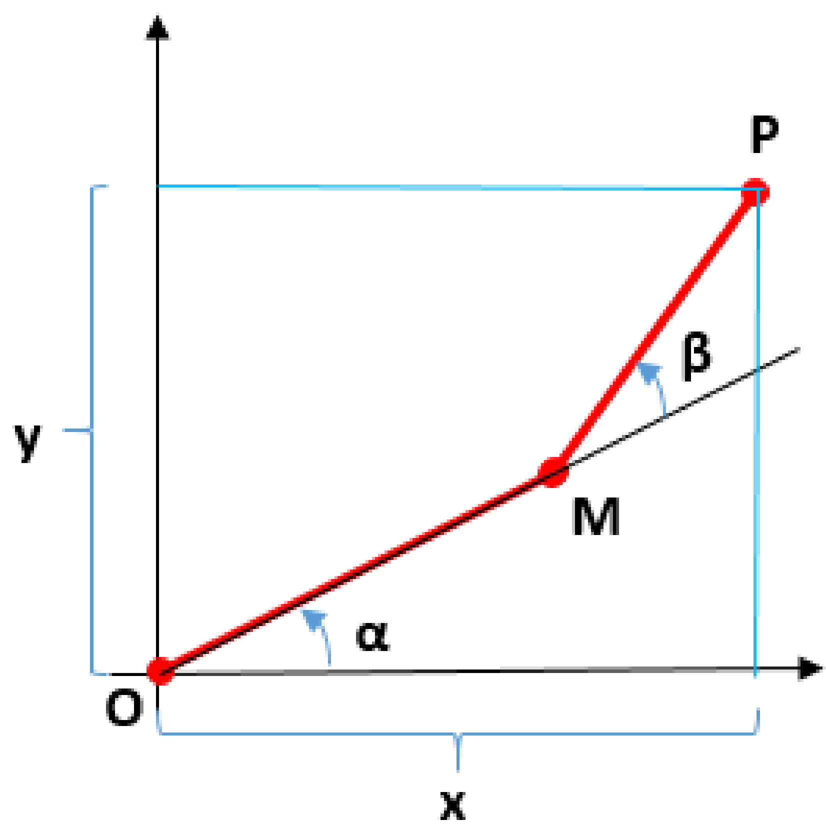


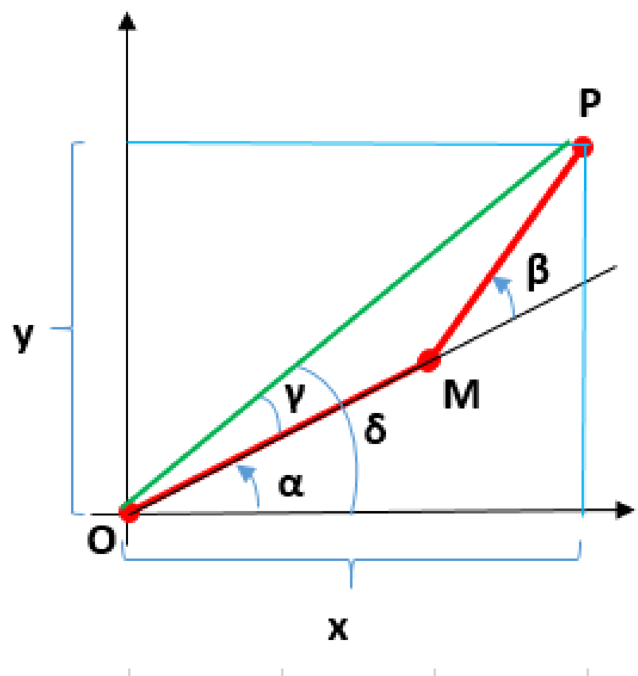
$$\gamma = \cos^{-1}\left(\frac{2.7^2 + 3.3^2 - 1.4^2}{2 \cdot 2.7 \cdot 3.3}\right) \approx 24.5^\circ$$

$$\alpha = \cos^{-1}\left(\frac{1.4^2 + 3.3^2 - 2.7^2}{2 \cdot 1.4 \cdot 3.3}\right) \approx 53.0^\circ$$

$$\beta = \cos^{-1}\left(\frac{1.4^2 + 2.7^2 - 3.3^2}{2 \cdot 1.4 \cdot 2.7}\right) \approx 102.5^\circ$$

Esim. Laske kulmat α ja β mittojen OM, MP, x ja y avulla





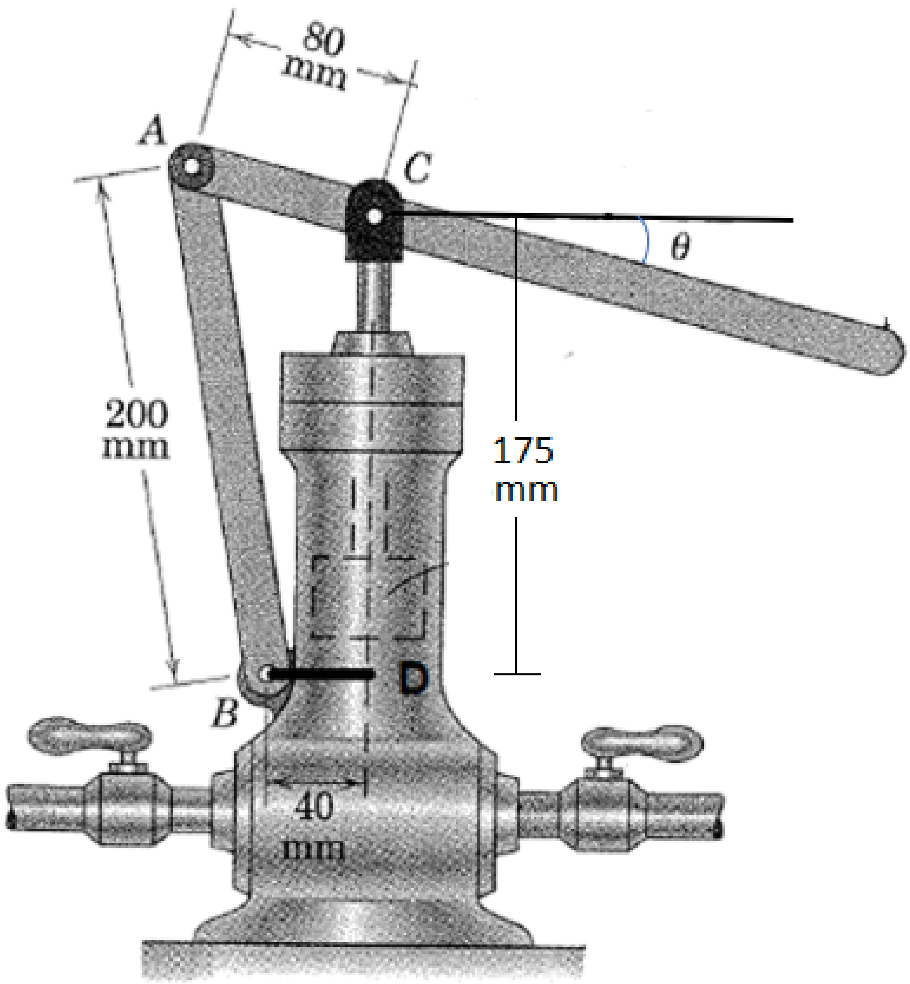
$$OP = \sqrt{x^2 + y^2}, \delta = \tan^{-1}(y/x)$$

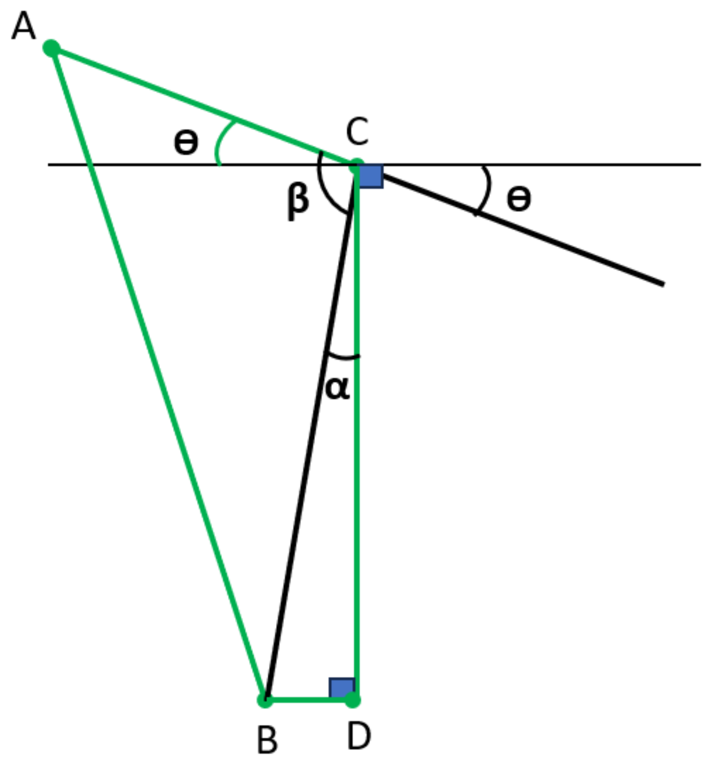
$$\gamma = \cos^{-1}\left(\frac{OP^2 + OM^2 - MP^2}{2 \cdot OP \cdot OM}\right)$$

$$\alpha = \delta - \gamma$$

$$\beta = 180^\circ - \cos^{-1}\left(\frac{MP^2 + OM^2 - OP^2}{2 \cdot MP \cdot OM}\right)$$

Esim: Laske kulma θ mittojen AB, AC, BD ja CD avulla avulla.





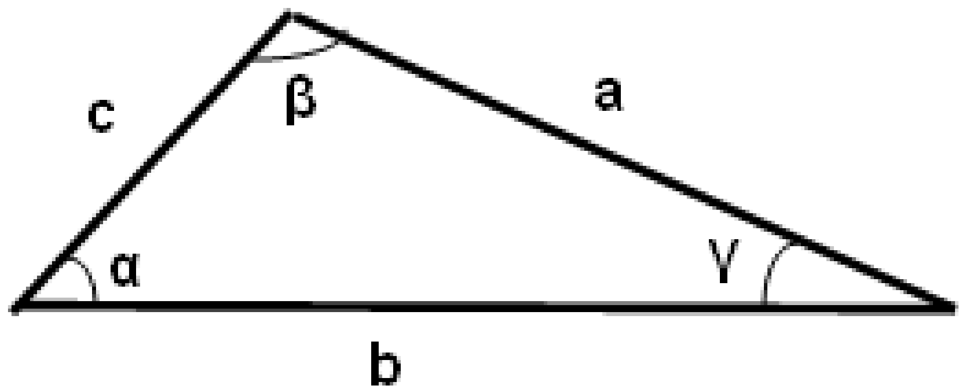
$$BC = \sqrt{BD^2 + CD^2}$$

$$\alpha = \tan^{-1}\left(\frac{BD}{CD}\right)$$

$$\beta = \cos^{-1}\left(\frac{BC^2 + AC^2 - AB^2}{2 \cdot BC \cdot AC}\right)$$

$$\theta = \alpha + \beta - 90^\circ$$

Sinilause (law of sines):



$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$

tai toisinpäin

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

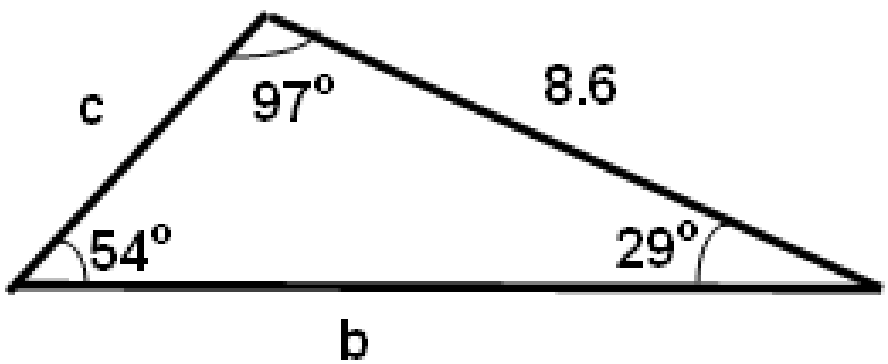
Yleensä sinilauseella lasketaan kolmion puuttuvat sivut, kun tunnetaan sen kulmat ja **yksi** sivuista.

Esimerkiksi, jos sivu a tunnetaan, niin

$$\frac{b}{\sin(\beta)} = \frac{a}{\sin(\alpha)} \to b = \frac{a}{\sin(\alpha)} \cdot \sin(\beta)$$

$$\frac{c}{\sin(\gamma)} = \frac{a}{\sin(\alpha)} \to c = \frac{a}{\sin(\alpha)} \cdot \sin(\gamma)$$

Esim.



Nyt tunnetaan kulma-sivu-pari $54^\circ \leftrightarrow 8.6$ ja

$$\frac{8.6}{\sin(54^\circ)} = 10.63$$

joten

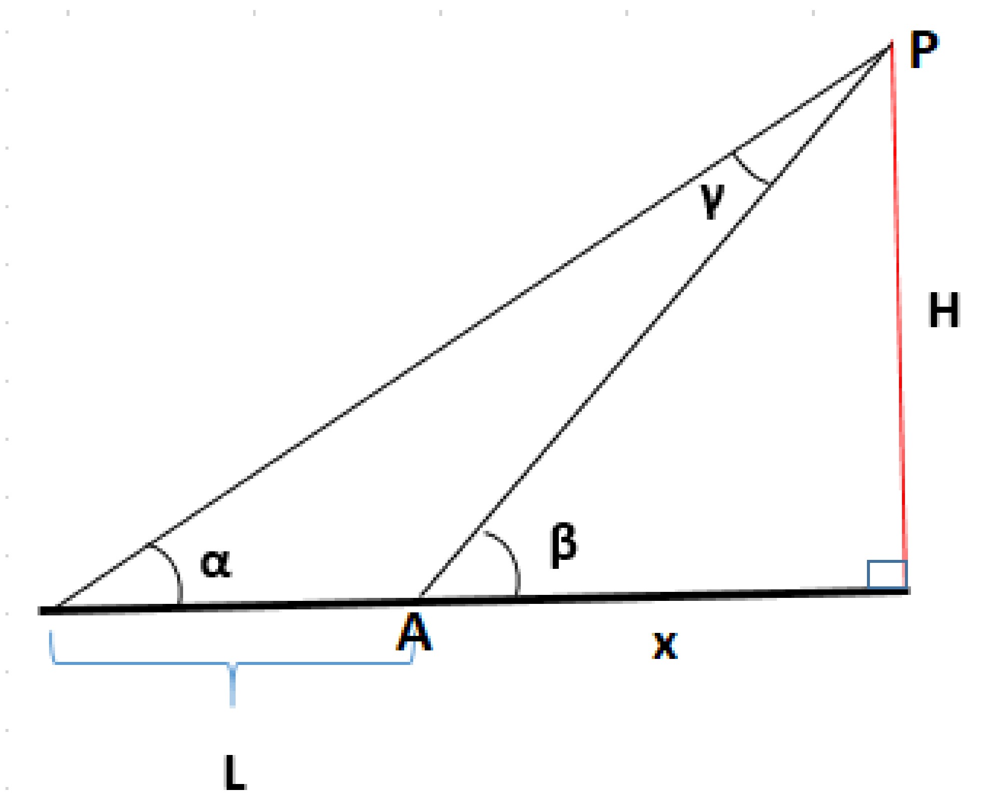
$$\frac{b}{\sin(97^\circ)} = 10.63 \text{ eli}$$

$$b = 10.63 \cdot \sin(97^\circ) = 10.6$$

$$\frac{c}{\sin(29^\circ)} = 10.63 \text{ eli}$$

$$c = 10.63 \cdot \sin(29^\circ) = 5.2$$

Esim. $L, \alpha, \beta \to H, x$.



$$\gamma = 180 - (\alpha + (180 - \beta)) = \beta - \alpha$$

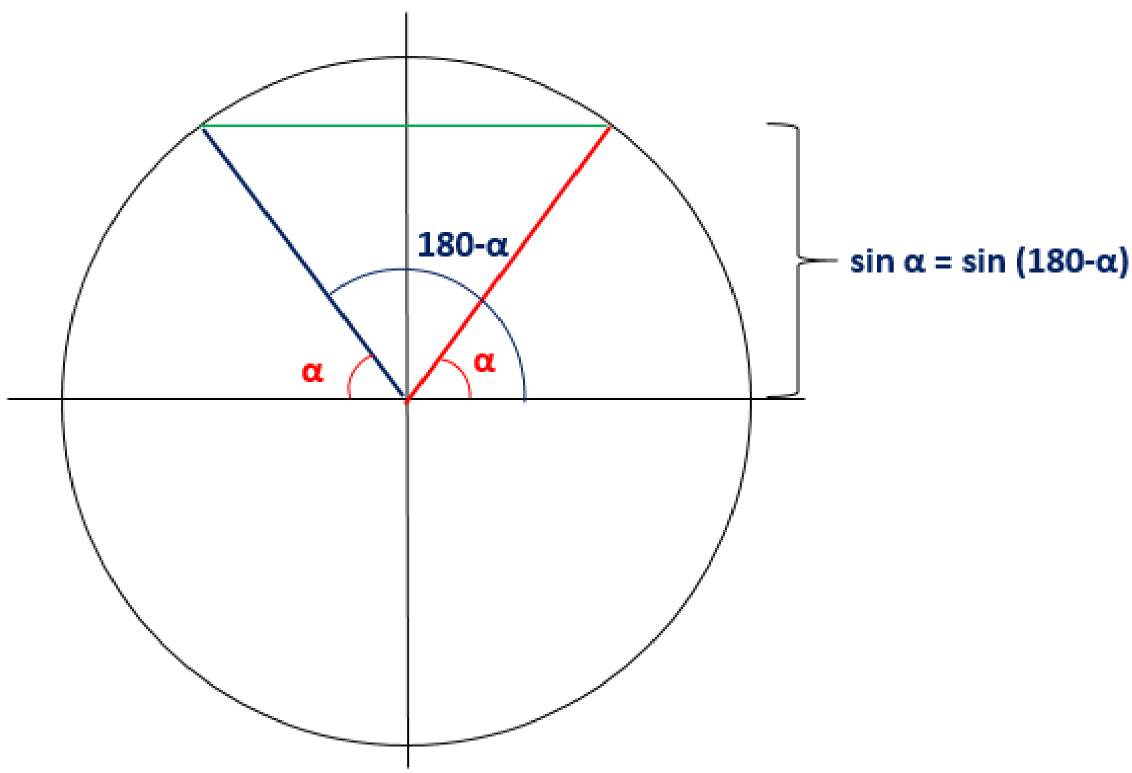
$$\frac{AP}{\sin(\alpha)} = \frac{L}{\sin(\gamma)} \text{ eli } AP = \frac{\sin(\alpha)L}{\sin(\gamma)}$$

$$H = AP\sin(\beta) = \frac{\sin(\alpha)\sin(\beta)L}{\sin(\gamma)}$$

$$x = AP\cos(\beta) = \frac{\sin(\alpha)\cos(\beta)L}{\sin(\gamma)}$$

Sinilauseella voidaan laskea myös kulmia ,mutta silloin pitää ottaa huomioon sinin ominaisuus

$$\sin(\alpha) = \sin(180^\circ - \alpha)$$



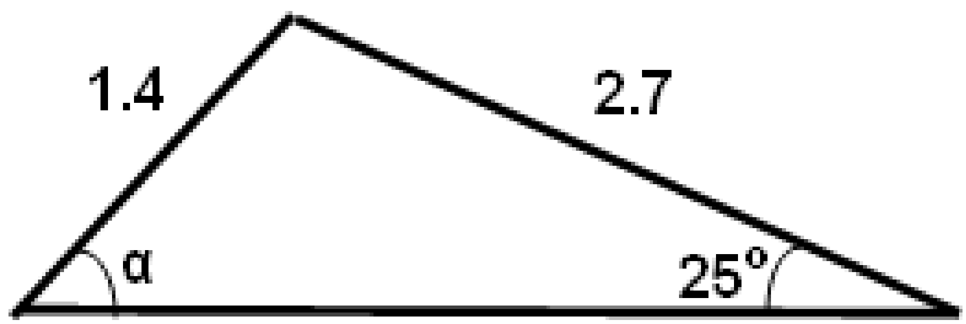
eli esimerkiksi

$\sin(42^\circ) = \sin(138^\circ){=}0.67.$

Laskukoneet antavat näistä pienemmän eli

$\sin^{-1}(0.67) = 42^\circ$

Esim.



Nyt tunnetaan kulma-sivu-pari $25^\circ \leftrightarrow 1.4$ ja

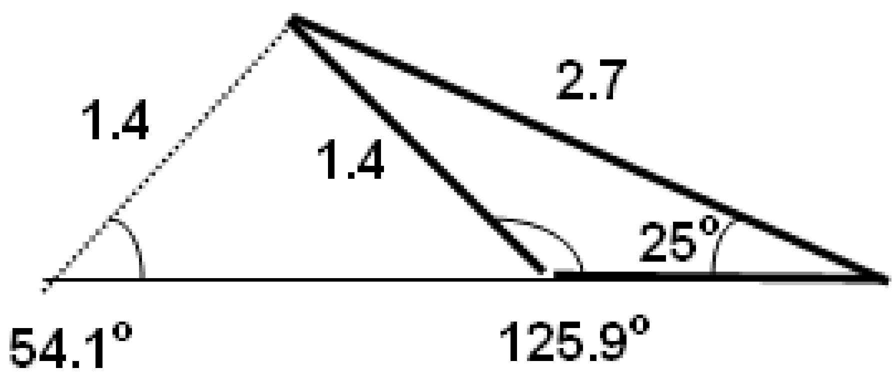
$$\frac{\sin(25^\circ)}{1.4} = 0.30$$

joten $\quad \dfrac{\sin(\alpha)}{2.7} = 0.30$ eli

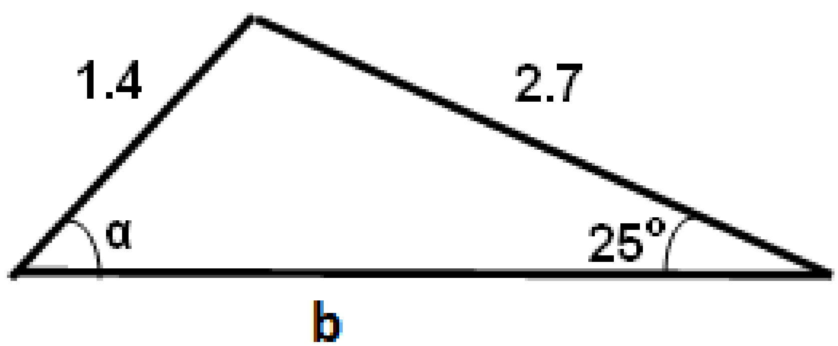
$\sin(\alpha) = 2.7 \cdot 0.30 = 0.81$ eli

$\alpha = \sin^{-1}(0.81) = 54.1^\circ$ **TAI**

$\alpha = 180^\circ - 54.1^\circ = 125.9^\circ$.



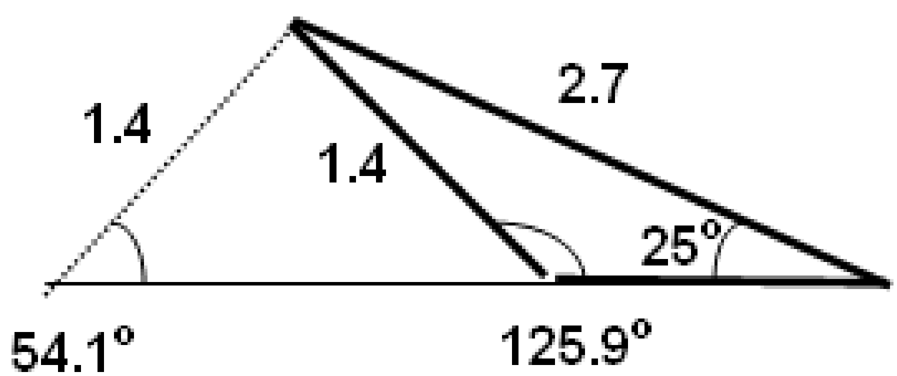
Sama kosinilauseella:



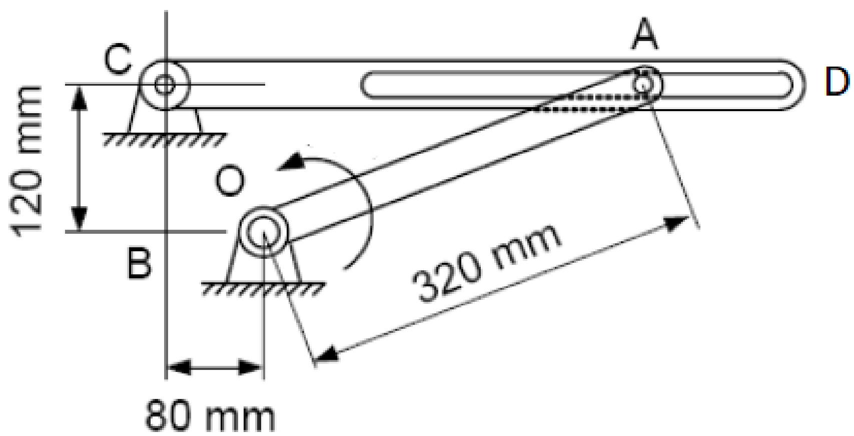
$$1.4^2 = b^2 + 2.7^2 - 2 \cdot b \cdot 2.7 \cdot \cos(25^\circ)$$

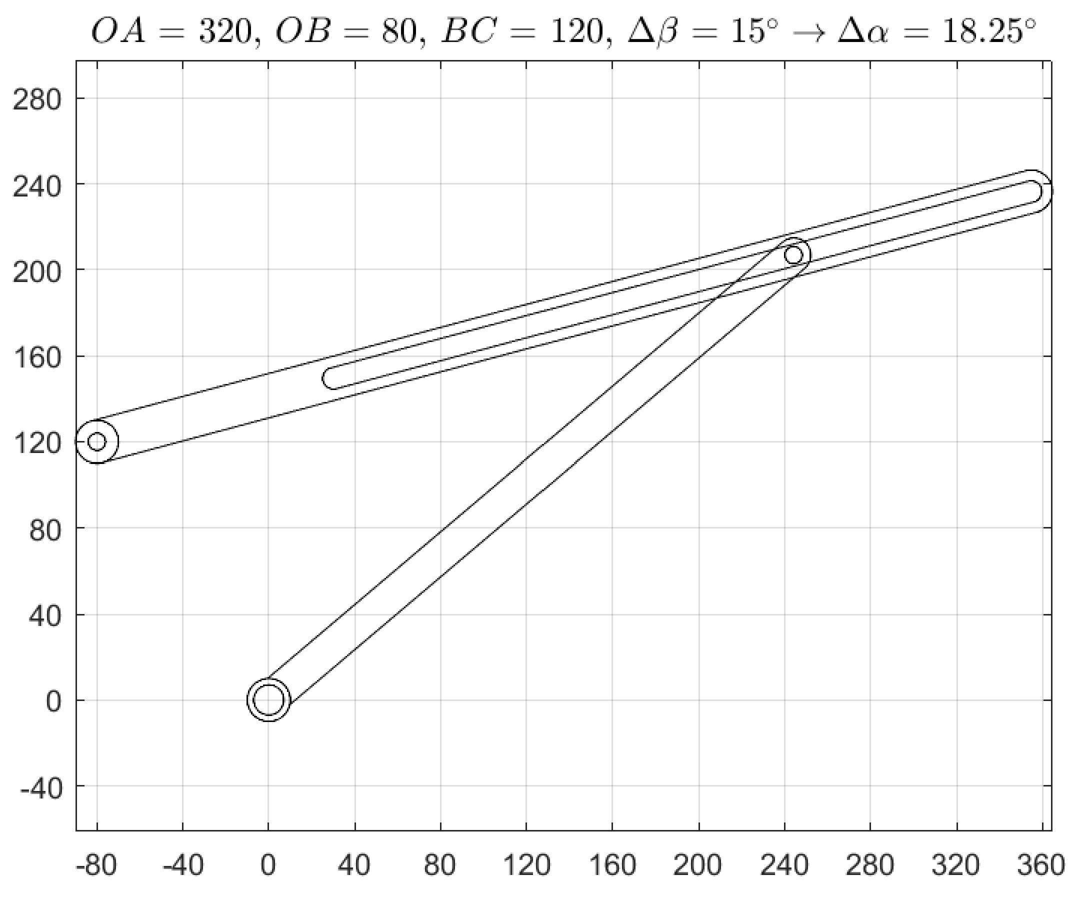
toisen asteen yhtälö, ratkaisut

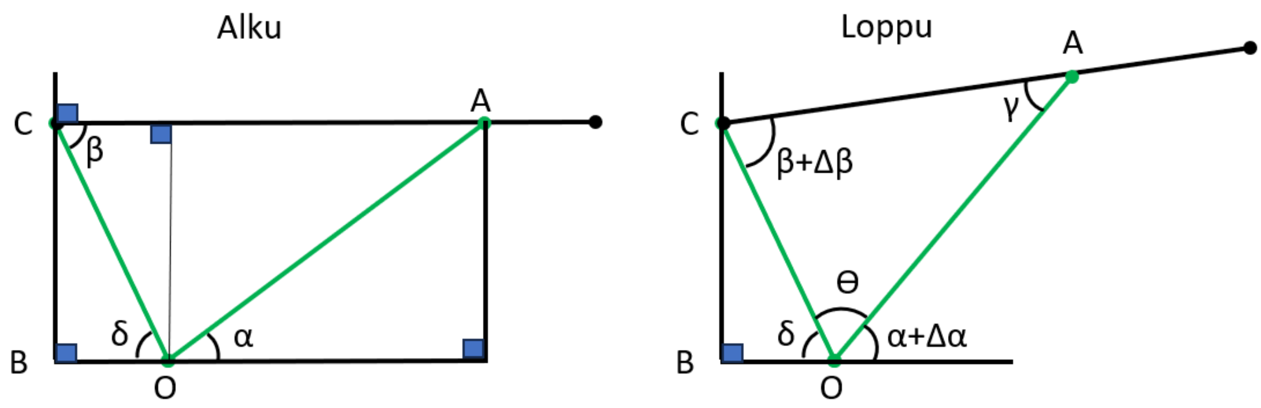
$b = 1.64$ tai $b = 3.26$



Esim: Jos varsi OA kääntyy kuvan asennosta kulman $\Delta\alpha$ verran, niin varsi CD kääntyy kulman $\Delta\beta$:n verran. Laske $\Delta\alpha$ mittojen OA, OB ja BC ja $\Delta\beta$:n avulla







Alku:

$$OC = \sqrt{OB^2 + BC^2},\ \alpha = \sin^{-1}\left(\frac{BC}{OA}\right)$$

$$\beta = \delta = \tan^{-1}\left(\frac{BC}{OB}\right)$$

Loppu:

$$\frac{\sin(\gamma)}{OC} = \frac{\sin(\beta + \Delta\beta)}{OA} \rightarrow \sin(\gamma) = \frac{\sin(\beta + \Delta\beta)}{OA} \cdot OC$$

$$\gamma = \sin^{-1}\left(\frac{\sin(\beta + \Delta\beta)}{OA} \cdot OC\right) \quad (\gamma < 90^\circ)$$

$$\theta = 180^\circ - (\beta + \Delta\beta) - \gamma$$

$$\Delta\alpha = \underbrace{(180^\circ - \theta - \delta)}_{\alpha + \Delta\alpha} - \alpha$$