Yhtälö

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = F$$

esittää (yleensä) kierrettyä ja/tai siirrettyä ellipsiä, hyperbeliä tai paraabelia
(poikkeustapauksissa yhtä tai kahta suoraa, yhtä pistettä tai tyhjää joukkoa).

Matriisimuodossa

$$[x, y] \underbrace{\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}}_{M} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{[D, E]}_{N} \begin{bmatrix} x \\ y \end{bmatrix} = F$$

Kulman θ verran kierretyssä uv-koordinaatistossa:

$$K = K_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}, \quad c = \cos(\theta),\ s = \sin(\theta)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = K \begin{bmatrix} u \\ v \end{bmatrix}, \quad [x, y] = [u, v]K^T$$

$$[u, v]K^T MK \begin{bmatrix} u \\ v \end{bmatrix} + NK \begin{bmatrix} u \\ v \end{bmatrix} = F$$

eli yhtälö uv-koordinaatistossa on

$$\alpha u^2 + \beta uv + \gamma v^2 + \delta u + \varepsilon v = F$$

missä

$$K^T MK = \begin{bmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{bmatrix}, \quad NK = [\delta, \varepsilon]$$

ja

$$\begin{cases} \alpha = Ac^2 + Bcs + Cs^2 \\ \\ \beta = -2(A - C)cs + B(c^2 - s^2) \\ \\ \gamma = As^2 - Bcs + Cc^2 \\ \\ \delta = Dc + Es \\ \\ \varepsilon = -Ds + Ec \end{cases}$$

Etsitään kiertokulma θ niin, että

$$\beta = -2(A - C)cs + B(c^2 - s^2) = 0$$

Trigonometrian kaavat:

$$c^2 - s^2 = \cos(2\theta), \quad 2cs = \sin(2\theta)$$

eli

$$\beta = -(A - C)\sin(2\theta) + B\cos(2\theta)$$

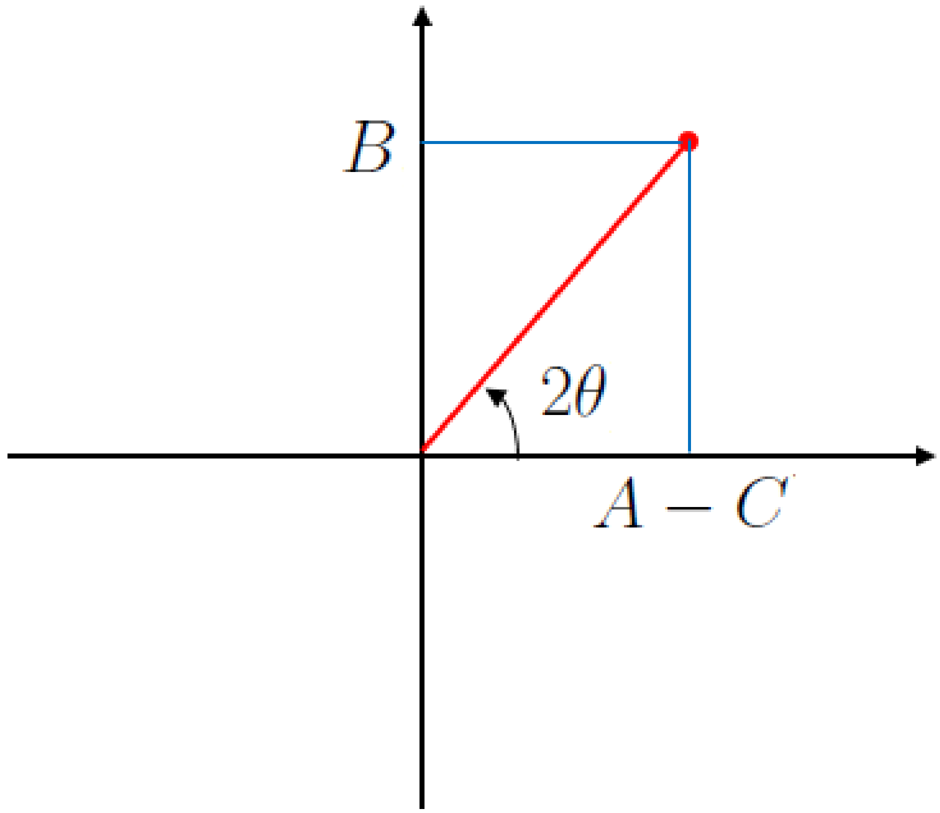
Jos

$$\theta = \frac{1}{2}\,\mathrm{atan2}(B, A - C)$$

niin

$$\cos(2\theta) = \frac{A - C}{\sqrt{B^2 + (A - C)^2}}$$

$$\sin(2\theta) = \frac{B}{\sqrt{B^2 + (A - C)^2}}$$



ja

$$\beta = -(A - C)\sin(2\theta) + B\cos(2\theta) = 0$$

eli käyrän yhtälö on

$$\alpha u^2 + \gamma v^2 + \delta u + \varepsilon v = F$$

Tapaus 1: $\alpha \neq 0$ ja $\gamma \neq 0$: täydennetään neliöksi

$$\alpha u^2 + \gamma v^2 + \delta u + \varepsilon v = F$$

$$\alpha \left(u^2 + 2\,\frac{\delta}{2\alpha}\,u + \left(\frac{\delta}{2\alpha}\right)^2 \right) + \gamma \left(v^2 + 2\,\frac{\varepsilon}{2\gamma}\,v + \left(\frac{\varepsilon}{2\gamma}\right)^2 \right)$$

$$= F + \alpha \left(\frac{\delta}{2\alpha}\right)^2 + \gamma \left(\frac{\varepsilon}{2\gamma}\right)^2$$

$$\alpha \left(u + \frac{\delta}{2\alpha} \right)^2 + \gamma \left(v + \frac{\varepsilon}{2\gamma} \right)^2 = F + \alpha \left(\frac{\delta}{2\alpha}\right)^2 + \gamma \left(\frac{\varepsilon}{2\gamma}\right)^2$$

$$\alpha (u - u_0)^2 + \gamma (v - v_0)^2 = k$$

missä

$$u_0 = -\frac{\delta}{2\alpha},\ v_0 = -\frac{\varepsilon}{2\gamma},\ k = F + \alpha \left(\frac{\delta}{2\alpha}\right)^2 + \gamma \left(\frac{\varepsilon}{2\gamma}\right)^2$$

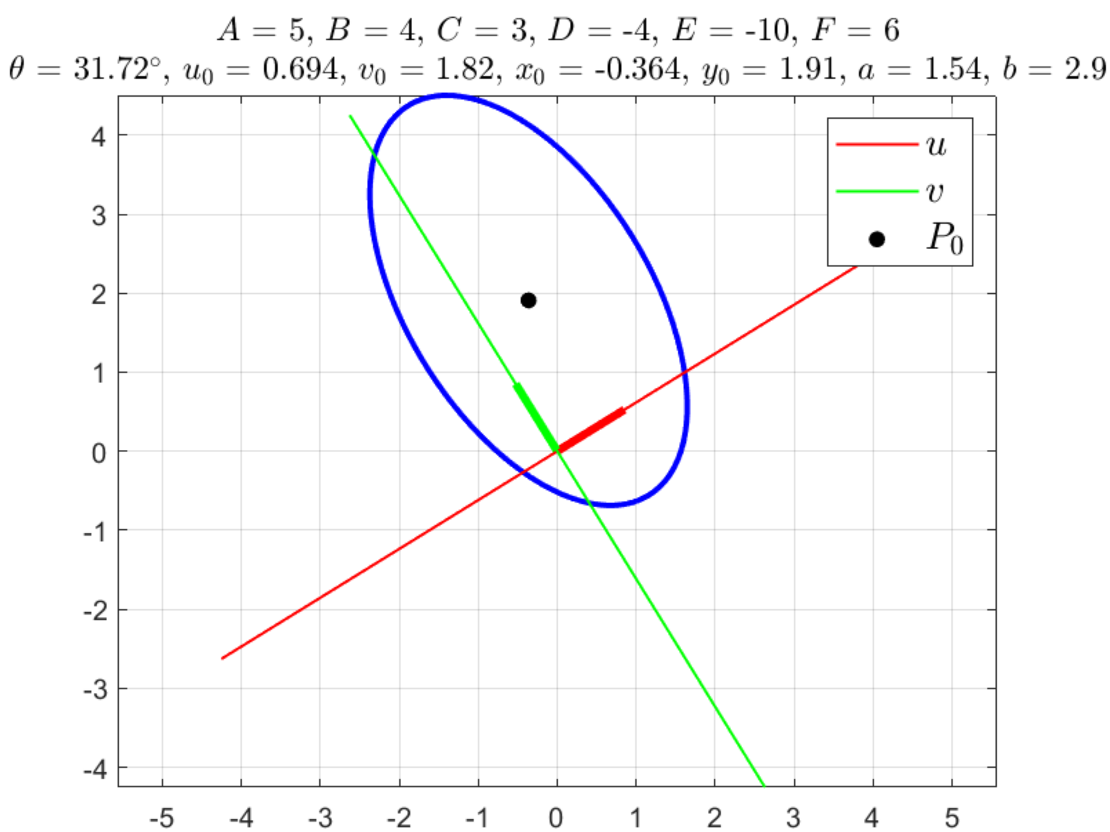
i) $\alpha, \gamma, k > 0$ tai $\alpha, \gamma, k < 0$:

$$\alpha(u - u_0)^2 + \gamma(v - v_0)^2 = k$$

$$\frac{(u - u_0)^2}{a^2} + \frac{(v - v_0)^2}{b^2} = 1$$

$$a = \sqrt{\frac{k}{\alpha}},\ b = \sqrt{\frac{k}{\gamma}}$$

ellipsi, keskipiste u_0, v_0 ja puoliakselit a, b.



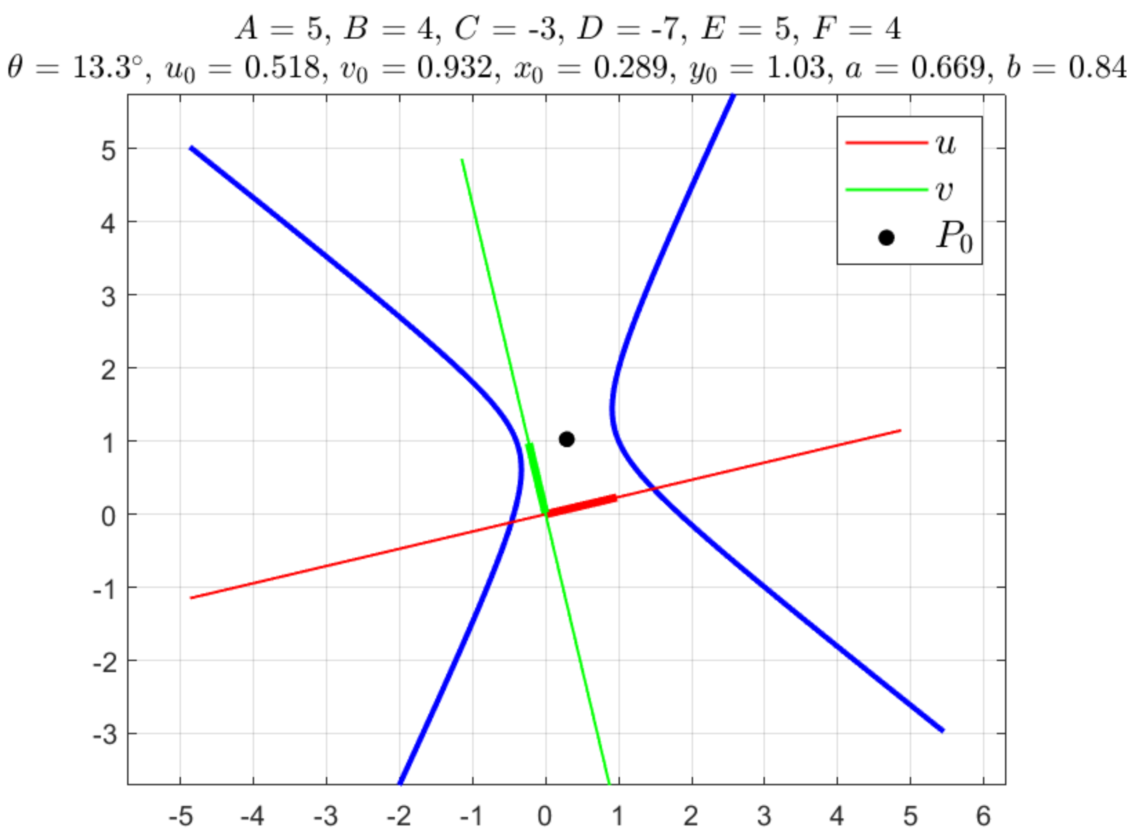
ii) $\alpha, k > 0, \gamma < 0$ tai $\alpha, k < 0, \gamma > 0$:

$$\alpha(u-u_0)^2 + \gamma(v-v_0)^2 = k$$

$$\frac{(u-u_0)^2}{a^2} - \frac{(v-v_0)^2}{b^2} = 1$$

$$a = \sqrt{\frac{k}{\alpha}},\ b = \sqrt{-\frac{k}{\gamma}}$$

hyperbeli, keskipiste u_0, v_0, puoliakselit a, b, "aukeaa u-akselin suuntaan":



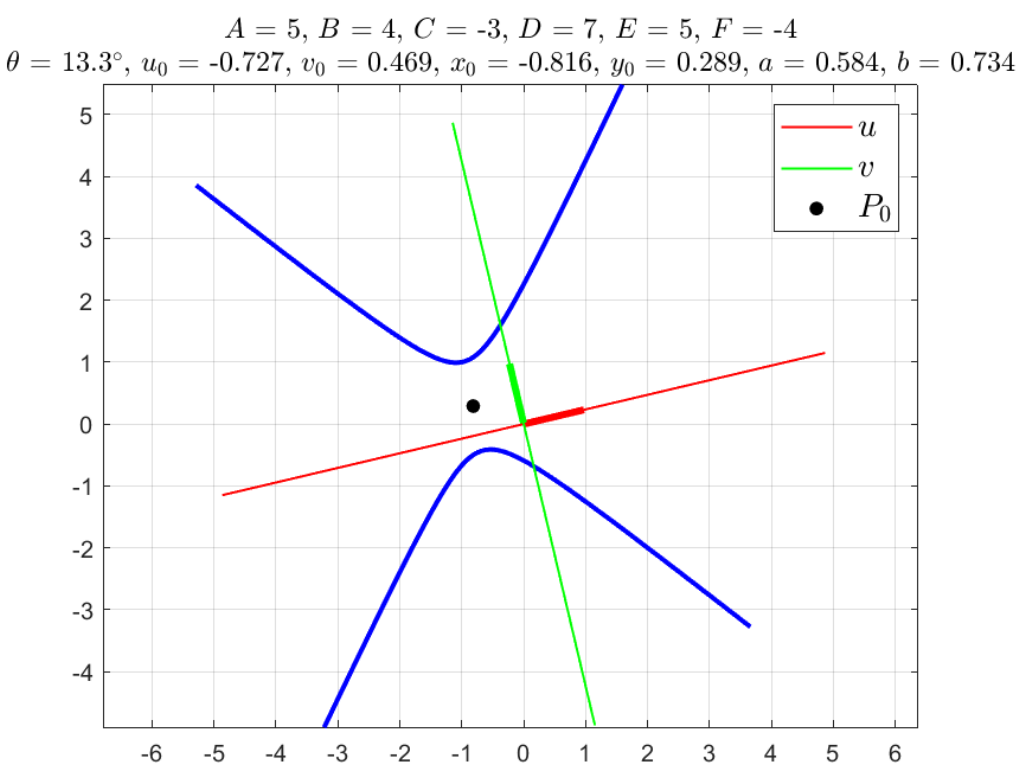
iii) $\alpha < 0, \gamma, k > 0$ tai $\alpha < 0, \gamma, k > 0$:

$$\alpha(u - u_0)^2 + \gamma(v - v_0)^2 = k$$

$$-\frac{(u - u_0)^2}{a^2} + \frac{(v - v_0)^2}{b^2} = 1$$

$$a = \sqrt{-\frac{k}{\alpha}},\ b = \sqrt{\frac{k}{\gamma}}$$

hyperbeli, keskipiste u_0, v_0, puoliakselit a, b
"aukeaa v-akselin suuntaan":



Tapaus 2: $\alpha = 0$ tai $\gamma = 0$

$$\alpha u^2 + \gamma v^2 + \delta u + \varepsilon v = F$$

i) $\alpha \neq 0, \gamma = 0, \varepsilon \neq 0$:

$$\alpha u^2 + \delta u + \varepsilon v = F$$

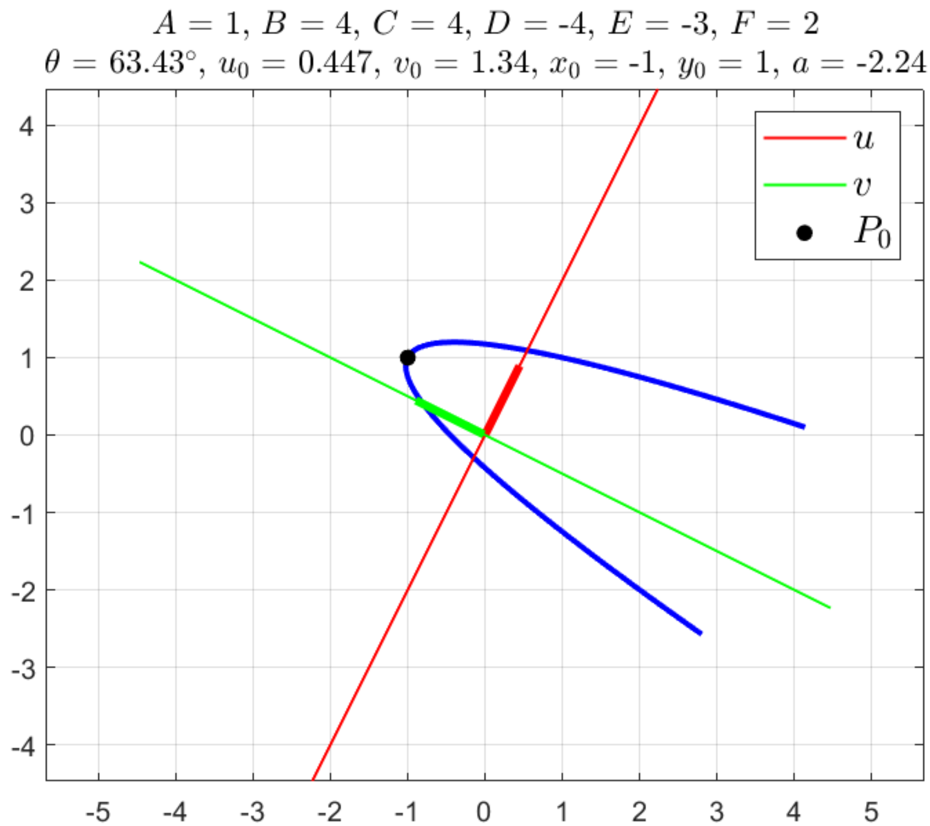
$$v = -\frac{\alpha}{\varepsilon} u^2 - \frac{\delta}{\varepsilon} u + \frac{F}{\varepsilon}$$

$$= -\frac{\alpha}{\varepsilon}\left(u^2 + 2\,\frac{\delta}{2\alpha}u \, + \left(\frac{\delta}{2\alpha}\right)^2\right) + \frac{F}{\varepsilon} + \frac{\alpha}{\varepsilon}\left(\frac{\delta}{2\alpha}\right)^2$$

$$= a(u - u_0)^2 + v_0$$

$$a = -\frac{\alpha}{\varepsilon}, \quad u_0 = -\frac{\delta}{2\alpha}, \quad v_0 = \frac{F}{\varepsilon} + \frac{\alpha}{\varepsilon}\left(\frac{\delta}{2\alpha}\right)^2$$

paraabeli, huippu u_0, v_0, aukeaa v-akselin suuntaan



ii) $\alpha = 0,\ \gamma \neq 0, \delta \neq 0$:

$$\gamma v^2 + \delta u + \varepsilon v = F$$

$$u = -\frac{\gamma}{\delta} v^2 - \frac{\varepsilon}{\delta}\, v + \frac{F}{\delta}$$

$$= -\frac{\gamma}{\delta}\left(v^2 + 2\,\frac{\varepsilon}{2\gamma}\, v\ + \left(\frac{\varepsilon}{2\gamma}\right)^2\right) + \frac{F}{\delta} + \frac{\gamma}{\delta}\left(\frac{\varepsilon}{2\gamma}\right)^2$$

$$= a(v - v_0)^2 + u_0$$

$$a = -\frac{\gamma}{\delta},\ v_0 = -\frac{\varepsilon}{2\gamma},\ u_0 = \frac{F}{\delta} + \frac{\gamma}{\delta}\left(\frac{\varepsilon}{2\gamma}\right)^2$$

paraabeli, huippu u_0, v_0, "aukeaa u-akselin suuntaan"