Question 1 :names: sami serhan=327876298 , najah kamal=325829133, salam qais=327876116

$$f_{a,4}(x) = \begin{Bmatrix} 1 & if\ ax \geq 4 \\ 0 & otherwise \end{Bmatrix}$$

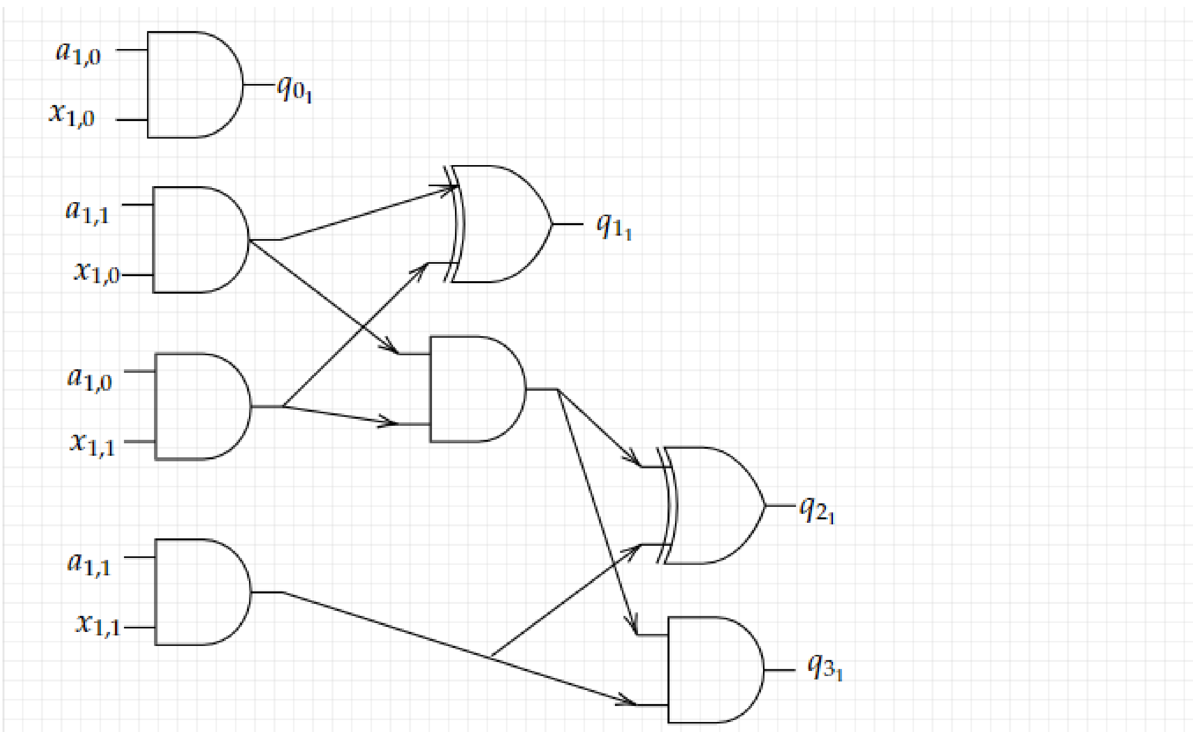
$input$	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$a = 0$	0	0	0	0
$a = 1$	0	0	0	0
$a = 2$	0	0	1	1
$a = 3$	0	0	1	1

Question 2 :

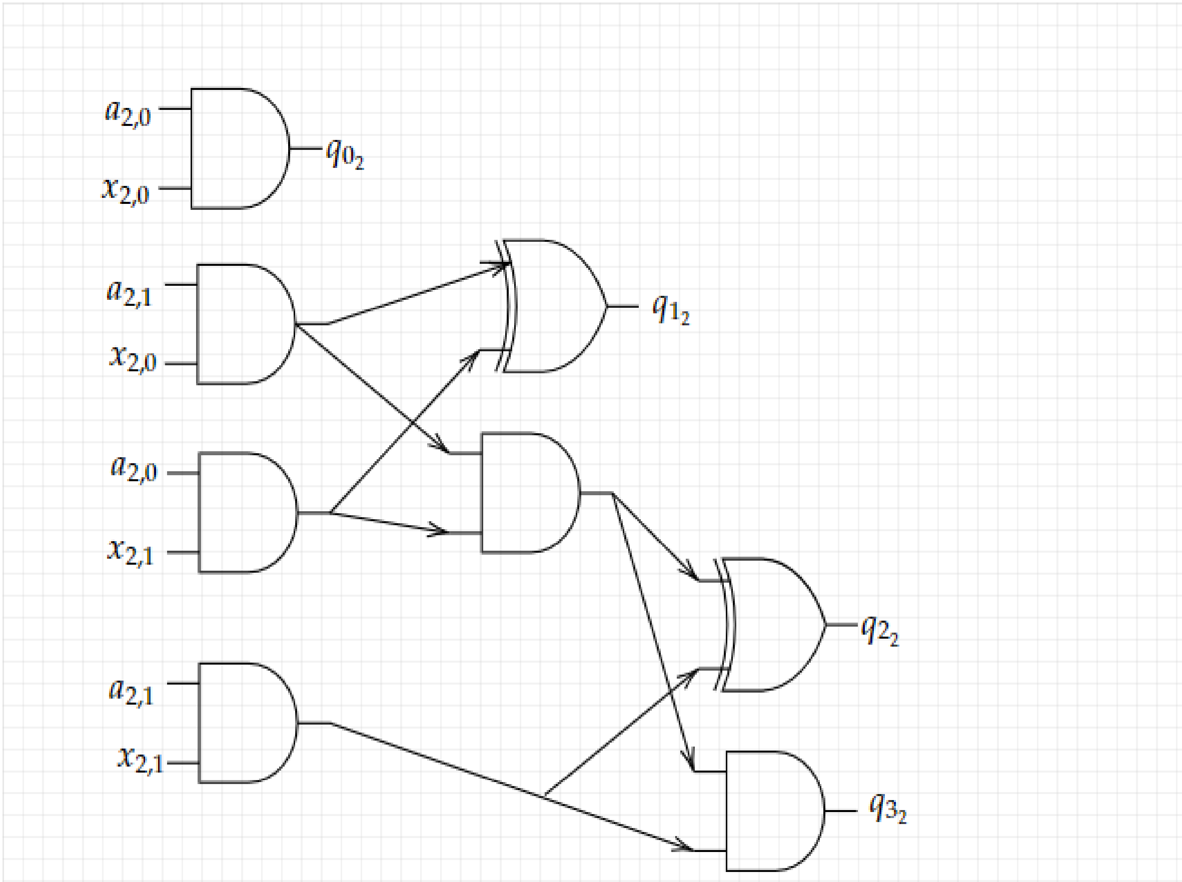
$$f_{\vec{a},A}(x_1,\ x_2) = \begin{cases} 1 & f\ a_1x_1\ +\ a_2x_2\ \geq\ 4 \\ 0 & otherwise \end{cases}$$

$a_1 = (a_{1,1}, a_{1,0})$
$a_2 = (a_{2,0}, a_{2,1})$
$x_1 = (x_{1,1}, a_{1,0})$
$x_2 = (x_{2,0}, a_{2,1})$

This is how we can mull two binary numbers A∗B , just by using XOR, AND gates
$MULL_1$:



$MULL_2$

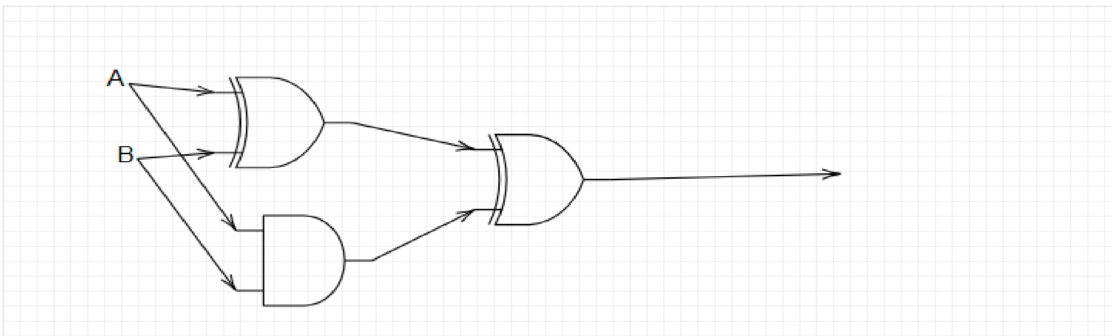


$so\,,after\;will\;made\;a_1{*}x_1 \quad ,\;a_2x_2$

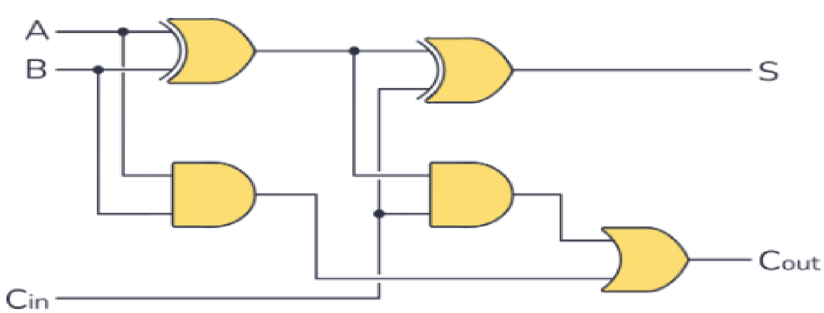
$and\;we\;got\;\; q_{0_1},q_{1_1},q_{2_1},q_{3_1} \qquad q_{0_2},q_{1_2},q_{2_2},q_{3_2}$
$a_1x_1 = q_{3_1}q_{2_1}q_{1_1}q_{0_1}$
$a_2x_2 = q_{3_2}q_{2_2}q_{1_2}q_{0_2}$

$A\;OR\;B = (A \oplus B) \oplus (A \wedge B)$

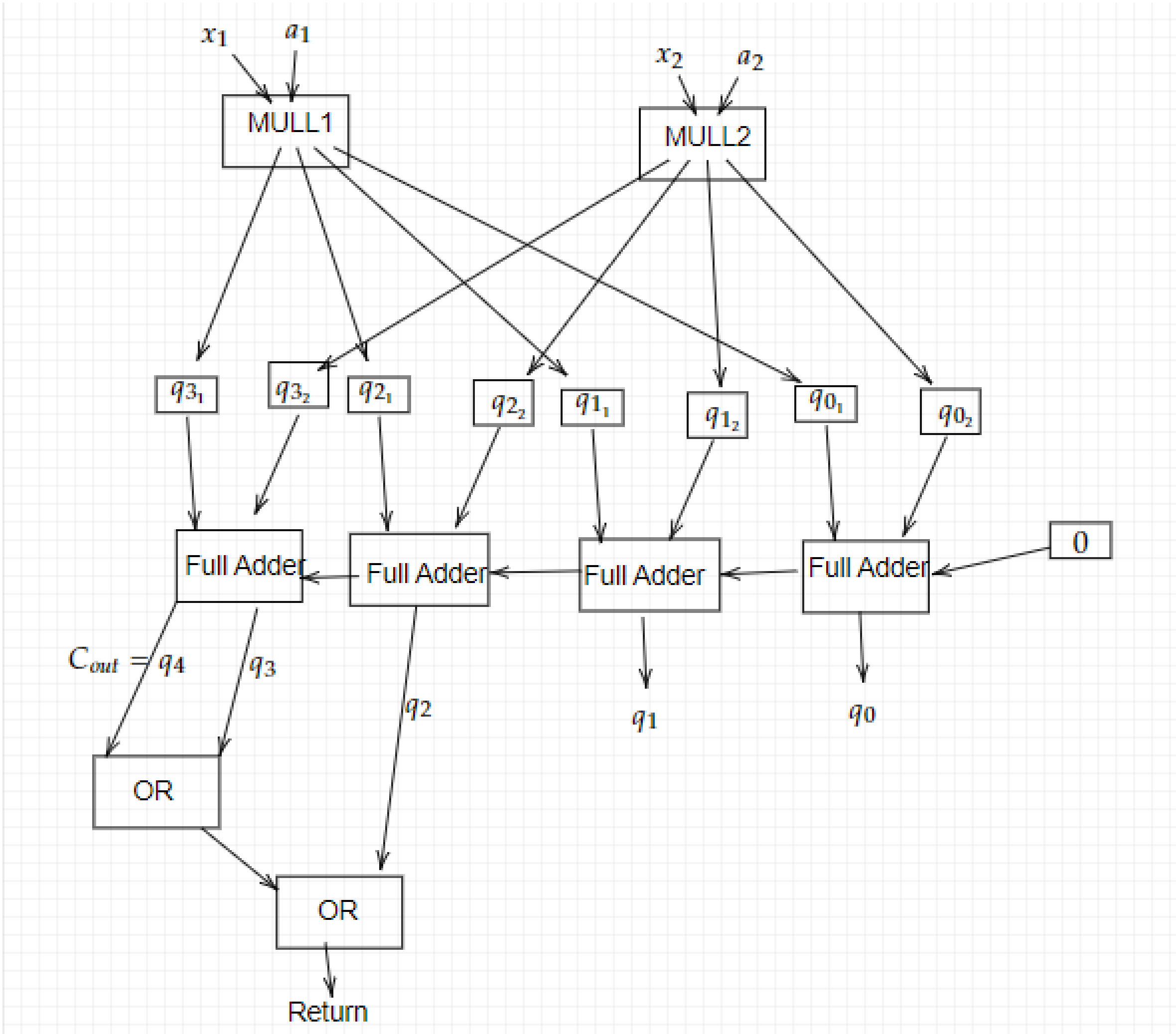
This is how we can Build OR gate by XOR, AND gates



$FULL\;ADDER:$
We learned this in Introduction to Hardware
$C_{in} = carry\;in$
$C_{out} = carry\;out$
$S = result$
$the\;last\;result\;of\;sum\;two\;number\; = (C_{out}S)$



ok now we use the Full adder to sum the Numbers :
lets build a big adder



x_1
a_1
MULL1
x_2
a_2
MULL2
$q3_1$
$q3_2$
$q2_1$
$q2_2$
$q1_1$
$q1_2$
$q0_1$
$q0_2$
Full Adder
Full Adder
Full Adder
Full Adder
0
$C_{out} = q_4$
q_3
q_2
q_1
q_0
OR
OR
Return

Question 3:

$f_{a_1,a_2,4}(x_1,x_2) = \left\{\begin{matrix} 1 & if\ a_1x_1 + a_2x_2 \geq 4 \\ 0 & otherwise \end{matrix}\right\}$

$x_1, x_2\ a_1, a_2$

$as\ we\ know\ from\ the\ law:$

$z(x) = x^{p-1} mod(p) = \left\{\begin{matrix} 0 & if\ x = 0 \\ 1 & else \end{matrix}\right\}\ in\ our\ case\ p = 11$

$let's\ X' = a_1x_1 + a_2x_2\ wanna\ check\ if\ X \geq 4$

$will\ check\ if\ (X' = 0\ or\ X' = 1\ or\ X' = 2\ or\ X' = 3)\ doing\ that\ by:$

$Z(X' - i)\ i = 1 \ldots 3$

$how\ to\ subtract\ in\ GP(11)\ to\ do\ that\ we\ find\ the\ inverse\ number\ for\ each\ one\ of\ (0,1,2,3)$

$in\ GP(11)\ the\ inverse = \{11 - r\ |r = \{0, \ldots 10\}\}\ and\ then\ adding\ it\ to\ the\ X'\ for\ example:$

$X' = 7\ and\ we\ want\ to\ calculate\ X' - 3$

$1.\ get\ the\ inverse\ number\ which\ is\ 11 - 3 = 8$

$2\ add\ the\ inverse\ to\ X' \rightarrow 7 + 8 = 15\ mod 11 = 4 = 7 - 3$

$so\ the\ inverse\ in\ GP(11)\ for\ 0\ is\ 11, 1\ is\ 10, 2\ is\ 9, 3\ is\ 8$

$$f_{a_1,a_2,4}(x_1,x_2) = \left(\prod_{i=1}^{10}((x_1 \times a_1) + (x_2 \times a_2))\right) \times \left(\prod_{i=1}^{10}(((x_1 \times a_1) + (x_2 \times a_2)) + 10)\right) \times (\prod_{i=1}^{10}\left(\left((x_1 \times a_1) + (x_2 \times a_2)\right) + 9\right) \times \left(\prod_{i=1}^{10}(((x_1 \times a_1) + (x_2 \times a_2)) + 8)\right)$$

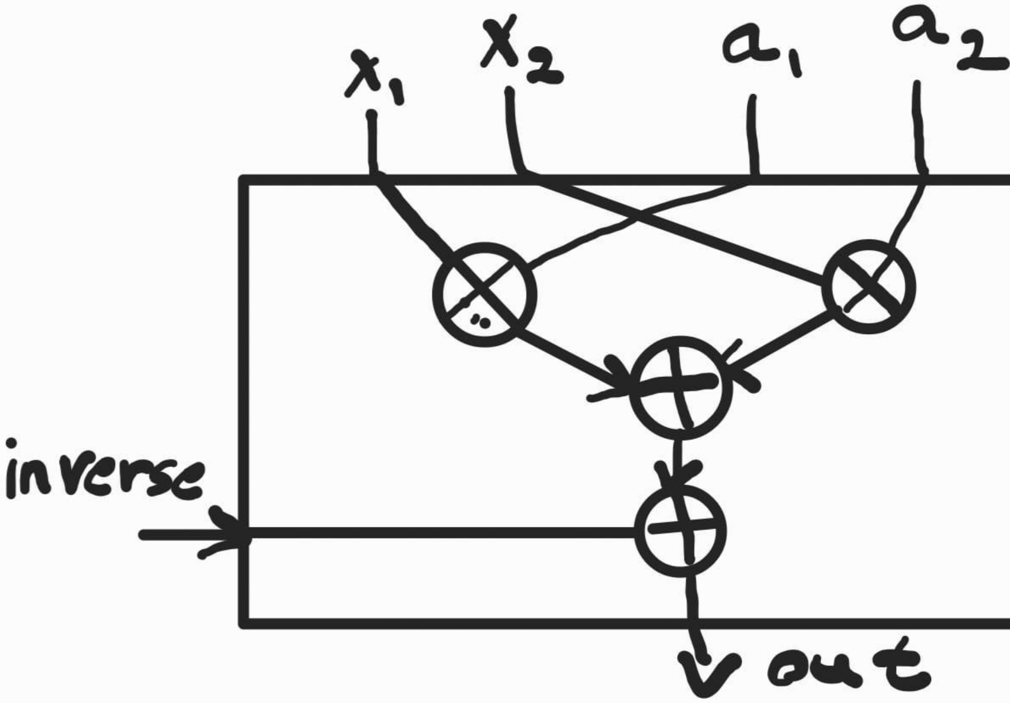
$$= \prod_{j=8}^{11}\prod_{i=1}^{10}\left(\left(\left((x_1 \times a_1) + (x_2 \times a_2)\right)\right) + j\right)$$

and this will give exactly $\left\{\begin{matrix} 1 & if\ a_1x_1 + a_2x_2 \geq 4 \\ 0 & otherwise \end{matrix}\right\}$ because if X' equal to 0 or 1 or 2 or 3 one of

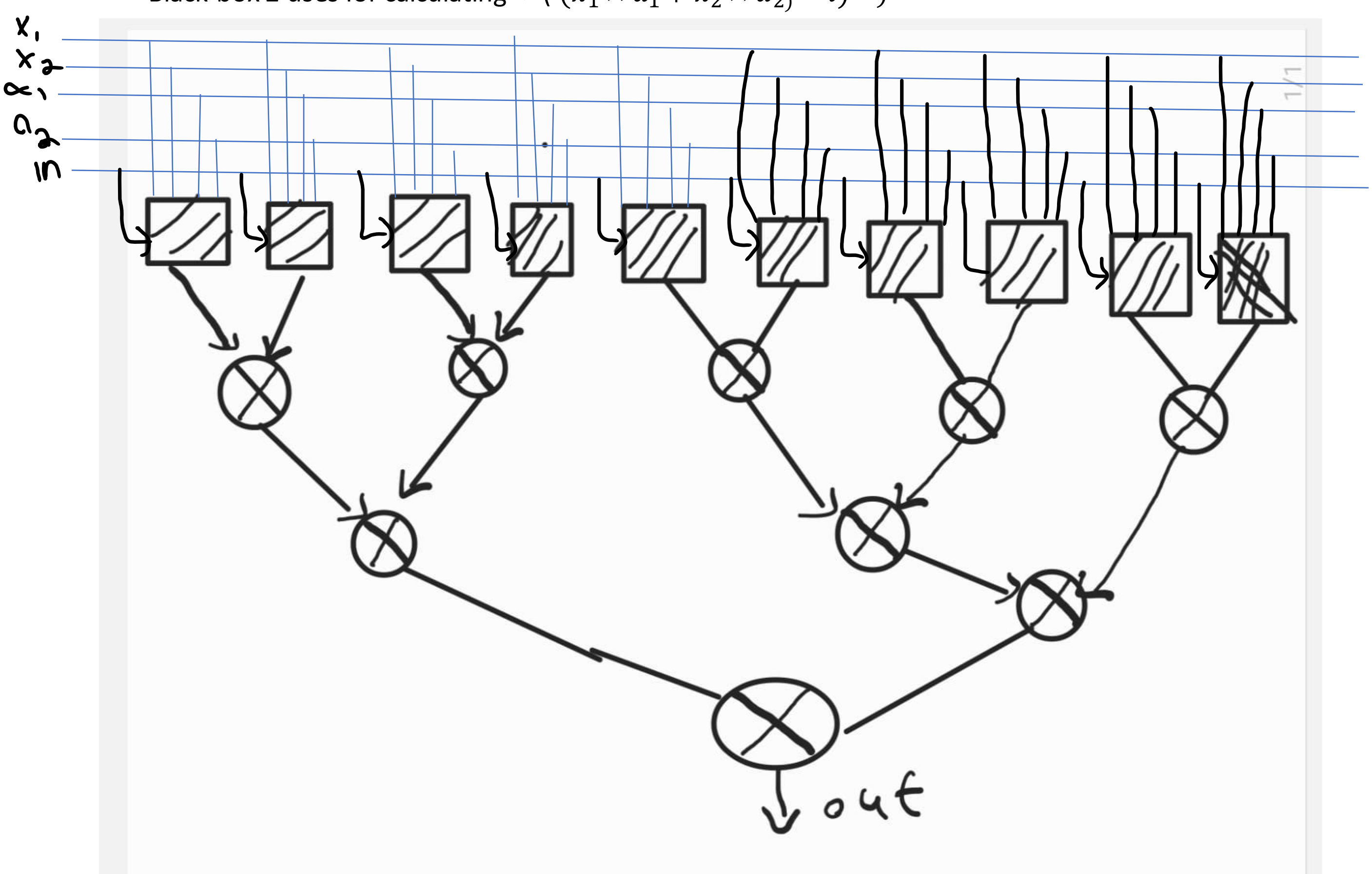
$Z(X' - i) = \prod_{i=1}^{10}\left(\left(\left((x_1 \times a_1) + (x_2 \times a_2)\right)\right) + j\right)$

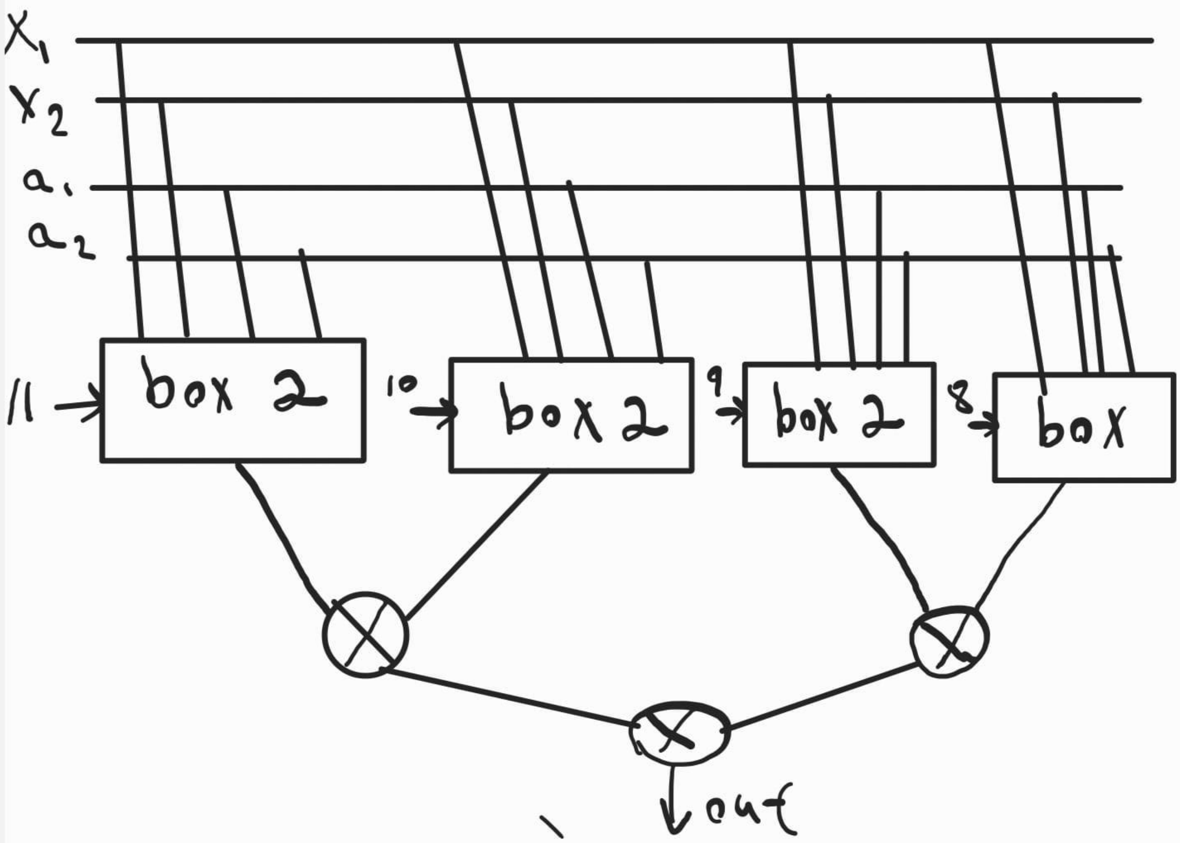
$will\ return\ 0\ so\ doing\ multiplication\ between\ them\ will\ ensure\ that\ else\ X'\ do\ not\ equal\ any\ of\ them\ so\ we\ will\ get\ 1.$

Black box 1 uses for($x_1 \times a_1 + x_2 \times a_2) - i$:



Black box 2 uses for calculating -> ($\left(x_1 \times a_1 + x_2 \times a_{2)} - i\right)^{10}$)





X1
X2
a1
a2
11
box 2
10
box 2
9
box 2
8
box
out

Question 4:

Analyzing the complexity of question 2:

- Circuit Size:
  - In each MULL, we have 8 operations, and we have 2 MULLs: 8*2=16 operations.
  - In each Full Adder, we have 5 operations, and we have 4 Full Adders: 5*4=20 operations.
  - After that, we have 2 ORs.

  → We have 16+20+2=38 operations.

  → Circuit size = 38

- Circuit Depth:
  - The maximal depth in the MULL is 2.
  - The maximal depth in the Full Adder is 3.

  → The maximal depth can be calculate by going to the MULL, and then going to the Full Adder and then 2 ORs.

  That's mean the maximal depth is 2+3+2=7.

  → Circuit Depth = 7.

- X-Depth:
  - The maximal X-Depth in the MULL is 1.
  - The maximal X-Depth in the Full-Adder is 1.

  And in the big picture we don't have any multiplication operation, so the maximal X-Depth will not change and it will be 2.

  → X-Depth = 2.

- #MULT:
  - In each MULL, we have 6 ANDs, and we have 2 MULLs: 6*2=12 MULT operations.
  - In each Full Adder, we have 2 ANDs, and we have 4 Full Adders: 2*4=8 MULT operations.

  And in the big picture we don't have any multiplication operation, so the maximal X-Depth will not change and it will be 12+8=20.

  → #MULT = 20.

Analyzing the complexity of question 3:

- Circuit Size:
  - In each Black Box 1, we have 4 operations (2 multiply and 2 sum).

- In Black Box 2, we have 10 Black Boxes 2, after that we have 7 operations: $\rightarrow 10 * 4 + 7 = 47$ operations in Black Box 2.
- And in the last picture, we have 4 Black Boxes 2, and after that we have 3 operations: $\rightarrow 47 * 4 + 3 = 191$ operations.

→ Circuit size = 167.

- Circuit Depth:
  - The maximal depth in the Black Box 1 is 4.
  - The maximal depth in Black Box 2 is: $4 + 4 = 8$.
  - In the big picture, we have one edge and the maximal depth of Black Box 2, and after that 2 operations.
  - $\rightarrow 8 + 1 + 2 = 11$

  That means the maximal depth is 11.

  → Circuit Depth = 11.

- X-Depth:
  - The maximal X-Depth in the Black Box 1 is 2.
  - The maximal X-Depth in the Black Box 2 is: 2+4=6 .
  - In the big picture, we have one edge and the maximal depth of Black Box 2, and after that 2 operations.
  - $\rightarrow 6 + 1 + 2 = 9$

  That means the maximal depth is 11.

  → X-Depth = 9.

- #MULT:
  - In each Black Box 1, we have 2 multiply operations.
  - In each Black Box 2, we have 10 Black Boxes 1, and after that we have 8 multiply operations: $\rightarrow 2 * 10 + 8 = 28$.
  - In the big picture, we have 4 Black Boxes 2, and after that 3 multiply operations: $\rightarrow 28 * 4 + 3 = 115$

  → #MULT = 115.