Switching-based Adaptive Feedforward Control of Uncertain Linear Systems with Unknown Multi-sinusoidal Disturbances ⋆

Yang Wang [a], Gilberto Pin [b], Andrea Serrani*, [c], Thomas Parisini [d]

[a] *School of Information Science and Technology at Shanghai Technology University, Shanghai, China.*

[b] *Electrolux Italia S.p.A, Italy.*

[c] *Department of Electrical and Computer Engineering, The Ohio State University, Columbus OH, USA.*

[d] *Department of Electrical and Electronic Engineering, Imperial College London, UK and Department of Engineering and Architecture, University of Trieste, Italy*

Abstract

The regulator problem for stable uncertain SISO linear systems perturbed by multi-sinusoidal disturbances with *unknown* frequencies is addressed in this paper. The objective is to recover the regulation of the plant output using a pre-designed family of candidate Adaptive Feedforward Controllers (AFC). A novel State-Norm-Estimator (SNE)-based supervisory unit is proposed to select the controller with the potential to provide satisfactory performance. In this way, we remove the so-called SPR-like conditions, which require certain prior information on the frequency response of the plant. A non-asymptotic frequency estimator is integrated into the proposed scheme to obtain the frequency information. Other notable features of the presented solution include: i) low-order of the overall controller, which is due to the use of notch filters and a decoupled design of the switching logic; ii) guaranteed finite switching, whereas persistence of excitation is not needed.

Key words: output regulation, disturbance rejection, noise control, feedforward control

1 Introduction and Problem Formulation

One of the most relevant problems in control theory is that of reducing the effects of undesirable input signals on the output of the plant to be controlled. In applications, the significance of this problem can be seen in contexts as diverse as active noise control [13], vibration compensation in helicopters [1] and disk drives [9], marine systems [3], attitude tracking [26], active vibration control [14] and control of three-phase inverters [22], to name but a few. From a theoretical standpoint, rejection of periodic disturbance falls within the scope of output regulation [17], which becomes especially challenging when the disturbance and the plant models are both subject to uncertainties.

Systems considered in this paper are assumed to be realizations of the *internally stable* interconnection of an uncertain plant model and a robust stabilizer. This is a classic setup of the AFC problem (see [7]). Specifically, we consider uncertain SISO linear time-invariant (LTI)

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*:Corresponding author.

Email addresses: wangyang4@shanghaitech.edu.cn (Yang Wang), gilbertopin@alice.it (Gilberto Pin), serrani.1@osu.edu (Andrea Serrani*,), t.parisini@imperial.ac.uk (Thomas Parisini).

systems whose dynamic behavior is described by

$$\begin{aligned}\dot{x} &= A(\mu)x + B(\mu)[\,\hat{d}(t) - d(t)\,]\,, \quad x(0) = x_0 \\ y &= C(\mu)x\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ are the immeasurable plant states and $y \in \mathbb{R}$ is the output to be regulated. The unknown vector $\mu \in \mathbb{R}^p$ in (1) collects the uncertain parameters of the plant model, assumed to range over a given known compact set, $\mathcal{P} \subset \mathbb{R}^p$. For future use, we let $W(s) := C(\mu)(sI - A(\mu))^{-1}B(\mu)$ denote the transfer function of system (1). The input of system (1) is given by the difference between the multi-sinusoidal disturbance

$$d(t) = \sum_{i=1}^{n_f} d_i(t) = \sum_{i=1}^{n_f} a_i \sin(\omega_i t + \varphi_i) \tag{2}$$

and its estimated counterpart $\hat{d}(t)$, generated by a module to be designed. Without loss of generality, we assume the a_i and ω_i are all positive constants. The distinct frequencies $\omega_i \in \mathbb{R}_{>0}$, as well as the amplitude and phase $(a_i, \varphi_i)^\top \in \mathbb{R}^2$ of the disturbances $d(t)$ are all *unknown* but ω_i and a_i are assumed to satisfy the following assumption:

Assumption 1 *There exist positive constants $\bar{a}$, $\underline{\omega}$, $\bar{\omega}$ and $\Delta\omega > 0$ such that $0 \le a_i \le \bar{a}$ and $\underline{\omega} \le \omega_i \le \bar{\omega}$ for all $i \in \mathcal{N}_f$. Furthermore, the distinct frequencies satisfy $|\omega_i - \omega_j| \ge \Delta\omega$ for all $i, j \in \mathcal{N}_f$, $i \ne j$.* ◁

Notation Throughout the paper, $\|\cdot\|$ denotes both the Euclidean vector norm and the corresponding induced matrix norm, whereas $\|\cdot\|_{\mathcal{L}_2}$ denotes norms in $\mathcal{L}_2$. A generic identity matrix is denoted by $\boldsymbol{I}$. Eigenvalues of a matrix M are denoted by $\lambda_i(M)$, and $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ are the maximum and minimum eigenvalues of M, respectively. The spectrum of a matrix A is denoted by spec $A(\mu)$.The interior of a set Θ is denoted by int Θ. The operation $\mathrm{mod}(a, m)$ returns the remainder after division of a by m. The notation $\mathrm{col}(x_1, x_2, \cdots, x_n)$ is used for column vectors concatenating each argument x_i.

The control objective is loosely posed as finding $\hat{d}(\cdot)$ such that the boundedness of all internal trajectories is maintained and $y(\cdot)$ is asymptotically regulated. According to the celebrated *Internal Model Principle*, robust regulation can be achieved only if the controller embeds a suitable copy of the system generating the external signals. When the model of the disturbances and the process to be controlled are uncertain, several *adaptive* solutions to the problem have been proposed, including adaptive Youla–Kucera parametrization [14], observer-based methods [19] and *Active Disturbance Rejection Control* [10]. On the other hand, in *Adaptive Feedforward Compensation* (AFC) [7,25,29], which is the framework adopted in this work, a feedforward module is designed to provide complete cancellation of the external disturbance. In a typical AFC setup [7,17], the multi-sinusoidal disturbance $d(t)$ is usually regarded as the sum of the outputs of LTI exosystems given by

$$\begin{aligned}\dot{\nu}_i &= \omega_i T \nu_i\,, \quad \nu_i(0) = \nu_{i,0} \in \mathbb{R}^2, \\ d &= \sum_{i=1}^{n_f} \Gamma \nu_i, \quad i \in \mathcal{N}_f,\end{aligned} \tag{3}$$

where $\mathcal{N}_f := \{1, 2, \cdots, n_f\}$ and

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{4}$$

For the exosystem (3), the corresponding AFC module is given by

$$\begin{aligned}\dot{\hat{\nu}}_i &= \omega_i T \hat{\nu}_i - k b_i y, \quad \hat{\nu}_i(0) = \hat{\nu}_{i,0} \in \mathbb{R}^2, \\ \hat{d} &= \sum_{i=1}^{n_f} \Gamma \hat{\nu}_i, \quad i \in \mathcal{N}_f\end{aligned} \tag{5}$$

where $k > 0$ is a gain parameter to be selected, and $b_i = (\mathrm{sign}\,\mathrm{Re}\{W(j\omega_i)\}, 0)^\top$ or $b_i = (0, -\mathrm{sign}\,\mathrm{Im}\{W(j\omega_i)\})^\top$ are vectors that exploit a priori information on either the real or the imaginary part of $W(j\omega_i)$. Exponential stability of the interconnection between (1), (3) and (5) can be proved to hold for sufficiently small values of k by way of averaging arguments [11,24] and root locus analysis [18]. Knowledge of b_i over the range of frequencies of interest is critical for the implementation of the AFC algorithm. The majority of AFC techniques proposed in the literature assume that b_i is known and time-invariant. This assumption is known as an SPR-like condition [7,18], as strictly positive real systems are internally stable and satisfy $\mathrm{Re}\{W(j\omega)\} > 0$ for all $\omega \in \mathbb{R}$. An adaptive solution that removes the SPR-like condition was pursued in [23], which estimates b_i alongside a_i, φ_i, but its analysis is restricted to an averaged system. Suitable enhancements of the AFC algorithm with frequency adaptation schemes were considered for single and multiple-tone disturbances in [5] and [6], respectively. However, the update law still required sufficient knowledge of the frequency response of the plant. Under the assumption that the frequencies of the disturbance are known, recent works [28,29] have proposed a multiple-model adaptive control scheme that successfully removes the necessity of the SPR-like condition. However, both approaches are characterized by a high dimensionality of the overall controller, especially when the number of distinct harmonics is large. Furthermore, in the presence of sensor noise or frequency misalignment [1], the robustness of the scheme

[1] The difference between the available frequency information (provided by a frequency estimator or known a priori) and the true value is referred to as a “frequency misalignment”.

depends on the choice of the switching strategy and the switching sequence may not terminate. Uncertainties on the frequency of the disturbance are considered in [4], within a nominal plant model. The works [15] and [16] address similar problems under the assumption that the system is minimum-phase with a known relative degree, which is further relaxed in [19], but an SPR-like condition is still required.

In this paper, we consider the significantly challenging case involving uncertainties on both plant and disturbance models, and seek a solution that avoids resorting to SPR-like conditions and prior knowledge of the frequencies of the multi-sinusoidal disturbance. To dispose of SPR-like conditions, we define the unknown parameter vector

$$\vartheta^\top(\mu,\omega) := \begin{pmatrix}\mathrm{Re}\{W(j\omega)\} & \mathrm{Im}\{W(j\omega)\}\end{pmatrix} \in \mathbb{R}^2, \qquad (6)$$

for $\omega \in [\underline{\omega},\overline{\omega}]$ and let the compact set $\varTheta \subset \mathbb{R}^2$ be the annular region defined, for given scalars $0 < \delta_1 < \delta_2$, as

$$\varTheta := \left\{\vartheta \in \mathbb{R}^2 | \delta_1 \le \|\vartheta\| \le \delta_2\right\}. \qquad (7)$$

The assumption of knowledge of the vectors b_i is replaced by the following, much weaker condition:

Assumption 2 *The unknown vector $\vartheta(\mu,\omega)$ in (6) satisfies $\vartheta(\mu,\omega) \in \mathrm{int}\,\varTheta$ for all $(\mu,\omega) \in \mathcal{P} \times [\underline{\omega},\bar{\omega}]$.* ◁

Remark 1.1 *Assumption 2 is weaker than the SPR-like conditions of [7, 18], as it is independent on the sign of the entries of $\vartheta(\mu)$. Hence, we allow the sign of $\mathrm{Re}\{W(j\omega)\}$ and sign of $\mathrm{Im}\{W(j\omega)\}$ to change over the frequency of interest. In practice, one can always find a sufficiently small δ_1 and a sufficiently large δ_2 to verify (7).*

Finally, the plant is assumed to be internally stable:

Assumption 3 *There exist constants $\alpha_0, \alpha_1, \alpha_2 > 0$ such that the parameterized family $P_x : \mathbb{R}^p \to \mathbb{R}^{n_x \times n_x}$ of solutions of the Lyapunov equation*

$$P_x(\mu)A(\mu) + A^\top(\mu)P_x(\mu) = -\boldsymbol{I}$$

satisfies $\alpha_1 I \le P_x(\mu) \le \alpha_2 I$ for all $\mu \in \mathcal{P}$. Moreover, $-\mathrm{Re}\{\lambda\} \ge \alpha_0$ for all $\lambda \in \mathrm{spec}\,A(\mu)$ and all $\mu \in \mathcal{P}$. ◁

The (semi-global) disturbance rejection problem considered in this paper is stated formally as follows:

Problem 1 *Under Assumptions 1–3, given constant estimates $\hat{\omega}_i$, $i \in \mathcal{N}_f$, design a dynamic output-feedback controller of the form*

$$\begin{aligned}\dot{\xi} &= f_a(\xi, y)\,, \quad \xi(0) = \xi_0 \in \mathbb{R}^{n_c}\\ \hat{d} &= h_a(\xi, y)\end{aligned} \qquad (8)$$

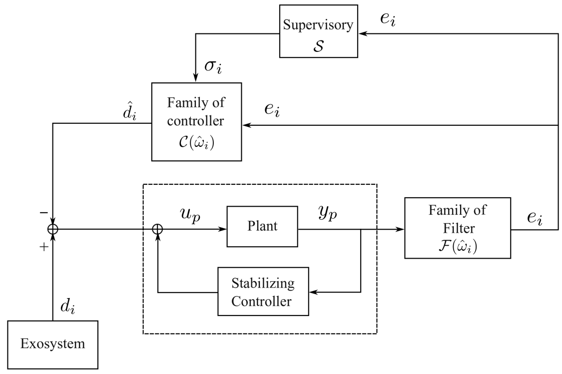

Fig. 1. Switching-based multi-controller architecture.

such that, for all $\mu \in \mathcal{P}$, trajectories of the closed-loop system (1), (2) and (8) originating from all initial conditions $x_0 \in \mathcal{X}_0 := \{x_0 \in \mathbb{R}^{n_x} : \|x_0\| \le r_0\}$ and $\xi_0 \in \mathcal{X}$, where $r_0 > 0$ is arbitrarily fixed and $\mathcal{X} \subset \mathbb{R}^{n_c}$ is a set to be determined, are bounded and satisfy

$$\lim_{t\to\infty} \sup\, y(t) \le r\left(\sum_{i=1}^{n_f} |\tilde{\omega}_i|\right),$$

where $\tilde{\omega}_i := \hat{\omega}_i - \omega_i$ are the frequency estimation errors and $r(\cdot)$ is a class-$\mathcal{K}$ function that depends on the structure of the controller and its tuning parameters. ◁

Remark 1.2 *We account for inaccurate estimates of the frequency of excitation, due to the possible impact of sensor noise. Due to the modularity of the proposed architecture, the deadbeat estimator employed in the paper can be replaced by other existing methods.*

Building on previous work [27, 30], a novel Switching Adaptive Control (SAC) based on State Norm Estimation is proposed to solve Problem 1. As depicted in Fig. 1, the overall control architecture follows the general paradigm of Morse's supervisory control [20]: i) the synthesis of a finite family of controllers $\mathcal{C}_i$, each suitable for a specific operating condition (in our case, corresponding to a different sign of the frequency response of the plant); ii) the design of a supervisor $\mathcal{S}_i$, responsible for selecting the 'best' controller via a switching signal σ_i. However, instead of selecting the potentially optimal candidate according to the current performance index (as done in most SAC approaches, for instance [2]), the proposed supervisor removes unsatisfactory candidates from the family of admissible controllers. In this way, reliable disturbance rejection is achieved within a finite number of switchings. After the switching terminates, the resulting controller stabilizes the closed-loop system, and provides disturbance compensation if the frequency estimate is accurate. If not, the estimation error will be corrected by employing the aforementioned deadbeat frequency estimator. It is worth pointing out that the frequency estimator will be activated only when the plant has reached its steady state and there are frequency misalignments, so the estimates $\hat{\omega}_i$ remain constant during the switching phase. An advantage of the proposed approach is the low-order of the overall controller, which is achieved by a decoupled design of the switching signal for each distinct frequency of excitation.

The organization of the paper is as follows: In Section 2, we first consider a disturbance containing only two distinct harmonics, and design the family of candidate controllers. Then, we prove that there exists at least one candidate in each family that is capable of stabilizing the closed-loop system, even under frequency alignment. Our main contribution, the SNE-based switching mechanism, is presented in detail in Section 3, together with the stability analysis of the overall system. Steady-state errors due to frequency misalginments are handled by updating the estimates provided by the deadbeat estimator introduced in Section 4. Finally, in Section 5, the design is extended to the general case where the disturbance contains more than two harmonic tones. An extensive case study and a brief conclusion are presented in Section 6 and Section 7, respectively.

2 Controller Design

For better illustration, we focus in this section on a disturbance comprised of two distinct excitation frequencies, i.e., $n_f = 2$. Given constant frequency estimates $\hat{\omega}_i$, $i = 1, 2$, are available, the family of candidate controllers $\{C_i^j(\hat{\omega}_i)\}$ is selected as

$$C_i^j(\hat{\omega}_i) = \begin{cases} \dot{\hat{\nu}}_i^j = \hat{\omega}_i T \hat{\nu}_i^j - k b^j e_i\,, \quad \hat{\nu}_i^j(0) = \hat{\nu}_{i,0} \\ \dot{\iota}_i = \hat{\omega}_i F(\gamma)\iota_i + \gamma\hat{\omega}_i G y, \quad \iota_i(0) = 0 \\ e_i = \Gamma \iota_i \\ \hat{d}_i^j = \Gamma \hat{\nu}_i^j\,, \quad j \in \mathcal{J} := \{1,2,3,4\} \end{cases} \tag{9}$$

where $\hat{\nu}_i^j, \iota_i \in \mathbb{R}^2$ are the states, $\hat{d}_i^j \in \mathbb{R}$ provides the estimate of i-th component of the disturbances $d(t)$,

$$F(\gamma) := \begin{pmatrix} -\gamma & 1 \\ -1 & 0 \end{pmatrix}$$

is Hurwitz for all $\gamma \in (0, 2)$, $G = \Gamma^\top$, and $b^j \in \mathbb{R}^2$ are *constant* vectors that span all sign configurations of the plant response at the estimated frequency $\hat{\omega}_i$, i.e.,

$$b^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; b^2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \; b^3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \; b^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{10}$$

In (9), e_i can be regarded as the output of a notch filter whose input-output behaviour can be described by

$$\boldsymbol{e_i}(s) = \gamma\hat{\omega}_i \Gamma \left(sI - \hat{\omega}_i F(\gamma)\right)^{-1} G\boldsymbol{y}(s). \tag{11}$$

The output y of the plant is processed by two notch filters, each driving a family of candidate controllers, where each member generates an estimate of the component d_i. Accordingly, the control signal applied to the plant is

$$\hat{d}(t) := \hat{d}_1^{\sigma_1(t)}(t) + \hat{d}_2^{\sigma_2(t)}(t) \tag{12}$$

where $\sigma_i : [0, \infty) \mapsto \mathcal{J}$, $i = 1, 2$ are piecewise-constant switching signals, generated by a *supervisor*, taking values in the index set of the family of candidate controllers.

Remark 2.1 *The dimension of the overall controller can be reduced by employing a state-shared realization of (9) (see [20]). This allows one to reduce the controller dimension from $10\,n_f$ to $4\,n_f$. While the state-shared realization is most convenient for implementation, we will make use of the form (9) for ease of analysis.*

Next, we show that by a suitable selection of gain parameters there exists at least one member of each group of candidate controllers (9) that solves Problem 1.

Suppose first that the switching terminates, and fixed controllers, say $s_i \in \mathcal{J}$ for $i = 1, 2$, are active for any t greater than a certain time t_0, which represents the last switching time, i.e. $\sigma_i(t) \equiv s_i$, for all $t \geq t_0$. Letting $\Pi_i(\mu)$ be the unique solution of the Sylvester equation

$$\hat{\omega}_i \Pi_i(\mu) T = A(\mu)\Pi_i(\mu) + B(\mu)\Gamma$$

and changing coordinates as $\zeta_i := \hat{\nu}_i^{s_i} - \nu_i$ and $z' := x - \sum_{i=1}^{n_f} \Pi_i(\mu)\zeta_i$, the interconnection of plant (1), exosystem (3) and the active controller in (9) reads as

$$\begin{aligned} \dot{\zeta}_i &= \hat{\omega}_i T\zeta_i - kb^{s_i}e_i + \tilde{\omega}_i T\nu_i\,, \quad \zeta_i(t_0) = \zeta_{i,t_0} \\ \dot{z}' &= A(\mu)z' + k\Pi_1(\mu)b^{s_1}e_1 + k\Pi_2(\mu)b^{s_2}e_2 \\ &\quad -\tilde{\omega}_1\Pi_1(\mu)T\nu_1 - \tilde{\omega}_2\Pi_2(\mu)T\nu_2, \quad z'(t_0) = z'_{t_0} \\ y &= C(\mu)z' + \vartheta_1^\top(\mu,\hat{\omega}_1)\zeta_1 + \vartheta_2^\top(\mu,\hat{\omega}_2)\zeta_2 \end{aligned} \tag{13}$$

where ϑ_i in (6) satisfies $C(\mu)\Pi_i(\mu) = \vartheta_i^\top(\mu, \hat{\omega}_i)$ [29]. To eliminate the frequency misalignment from the dynamics of z', change variables as $z := z' - \sum_{i=1}^{n_f} \tilde{\omega}_i \Xi_i(\mu)\nu_i$, where $\Xi_i(\mu)$ is the unique solution of the Sylvester equation

$$\omega_i \Xi_i(\mu) T = A(\mu)\Xi_i(\mu) - \Pi_i(\mu)T$$

to obtain the *error system*

$$\begin{aligned} \dot{\zeta}_i &= \hat{\omega}_i T\zeta_i - kb^{s_i}e_i + \tilde{\omega}_i T\nu_i\,, \quad \zeta_i(t_0) = \zeta_{i,t_0} \\ \dot{z} &= A(\mu)z + k\Pi_1(\mu)b^{s_1}e_1 + k\Pi_2(\mu)b^{s_2}e_2, \quad z(t_0) = z_{t_0} \\ y &= C(\mu)z + \vartheta_1^\top(\mu,\hat{\omega}_1)\zeta_1 + \vartheta_2^\top(\mu,\hat{\omega}_2)\zeta_2 \\ &\quad + \tilde{\omega}_1\Lambda_1^\top(\mu)\nu_1 + \tilde{\omega}_2\Lambda_2^\top(\mu)\nu_2 \end{aligned} \tag{14}$$

where $\Lambda_i^\top(\mu) := C(\mu)\Xi_i(\mu)$.

Let $\Upsilon_i \in \mathbb{R}^{2\times 2}$ and $\Psi_i \in \mathbb{R}^{2\times 2}$ be the unique solutions of the Sylvester equations

$$\hat{\omega}_i \Upsilon_i T = \hat{\omega}_i F(\gamma)\Upsilon_i + \gamma\hat{\omega}_i G\vartheta_i^\top(\mu,\hat{\omega}_i) \tag{15}$$
$$\hat{\omega}_j \Psi_i T = \hat{\omega}_i F(\gamma)\Psi_i + \gamma\hat{\omega}_i G\vartheta_j^\top(\mu,\hat{\omega}_j) \tag{16}$$

for all $i \neq j$. It can be shown that Ψ_i satisfies

$$\|\Psi_i\| \le \gamma c_0 \tag{17}$$

with $c_0 = (\bar{\omega}\delta_2/\Delta\omega)\sqrt{1+\bar{\omega}^2/\underline{\omega}^2}$ being a known constant (details are simple and omitted for reasons of space). Then, change coordinates for the state of notch filters by replacing ι_i with $\xi_i := \iota_i - \Upsilon_i\zeta_i - \Psi_i\zeta_j$, to obtain an equivalent realization of the filter (11) of the form

$$\begin{aligned}
\dot{\xi}_i =& \hat{\omega}_i F(\gamma)\xi_i + \gamma\hat{\omega}_i G[C(\mu)z + \tilde{\omega}_1\Lambda_1^\top(\mu)\nu_1 + \tilde{\omega}_2\Lambda_2^\top(\mu)\nu_2] \\
&+ k\Upsilon_i b^{s_i} e_i - \tilde{\omega}_i\Upsilon_i T\nu_i + k\Psi_i b^{s_j} e_j - \tilde{\omega}_j\Psi_i T\nu_j \\
e_i =& \Gamma\xi_i + \vartheta_i^\top(\mu,\hat{\omega}_i)\zeta_i + \Gamma\Psi_i\zeta_j,
\end{aligned} \tag{18}$$

for all $i,j \in \{1,2\}$, $j \neq i$, where we have used the fact that $\Gamma\Upsilon_i = \vartheta_i^\top$.

Finally, defining $\eta_i := \mathrm{col}(\zeta_i,\xi_i) \in \mathbb{R}^4$, the cascade connection of the filter and the active controller becomes [2]

$$\begin{aligned}
\dot{\eta}_i =& E_i\eta_i + H_{i,j}\eta_j + \gamma M_i(Cz + \tilde{\omega}_i\Lambda_i^\top\nu_i + \tilde{\omega}_j\Lambda_j^\top\nu_j) \\
&+ N'\tilde{\omega}_i T\nu_i - N(\tilde{\omega}_i\Upsilon_i T\nu_i + \tilde{\omega}_j\Psi_i T\nu_j) \\
e_i =& L_i\eta_i + L_i'\eta_j, \qquad i,j \in \{1,2\}, \quad j \neq i
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
E_i(\gamma) &:= \begin{pmatrix} \hat{\omega}_i T - kb^{s_i}\vartheta_i^\top & -kb^{s_i}\Gamma \\ k\Upsilon_i b^{s_i}\vartheta_i^\top + k\Psi_i b^{s_j}\Gamma\Psi_j & \hat{\omega}_i F(\gamma) + k\Upsilon_i b^{s_i}\Gamma \end{pmatrix}, \\
H_{i,j} &:= \begin{pmatrix} -kb^{s_i}\Gamma\Psi_i & \mathbf{0}_{2\times 2} \\ k\Upsilon_i b^{s_i}\Gamma\Psi_i + k\Psi_i b^{s_j}\vartheta_j^\top & k\Psi_i b^{s_j}\Gamma \end{pmatrix}, \\
N'^\top &:= \begin{pmatrix} \boldsymbol{I}_{2\times 2} & \mathbf{0}_{2\times 2} \end{pmatrix}, N^\top := \begin{pmatrix} \mathbf{0}_{2\times 2} & \boldsymbol{I}_{2\times 2} \end{pmatrix}, \\
L_i &:= \begin{pmatrix} \vartheta_i^\top & \Gamma \end{pmatrix}, L_i' := \begin{pmatrix} \Gamma\Psi_i & \mathbf{0}_{1\times 2} \end{pmatrix}, M_i^\top := \begin{pmatrix} \mathbf{0}_{1\times 2} & \hat{\omega}_i G^\top \end{pmatrix}.
\end{aligned}$$

Note that E_i depends on the gain γ by way of the matrix F. The closed-loop system is given by

$$\begin{aligned}
\dot{\eta}_i =& E_i\eta_i + H_{i,j}\eta_j + \gamma M_i(Cz + \tilde{\omega}_1\Lambda_1^\top\nu_1 + \tilde{\omega}_2\Lambda_2^\top\nu_2) \\
&+ N'\tilde{\omega}_i T\nu_i - N(\tilde{\omega}_i\Upsilon_i T\nu_i + \tilde{\omega}_j\Psi_i T\nu_j), \quad i \in \{1,2\} \\
\dot{z} =& Az + k\Pi_1 b^{\sigma_1}(L_1\eta_1 + L_1'\eta_2) + k\Pi_2 b^{\sigma_2}(L_2\eta_2 + L_2'\eta_1) \\
y =& Cz + L_1''\eta_1 + L_2''\eta_2 + \tilde{\omega}_1\Lambda_1^\top\nu_1 + \tilde{\omega}_2\Lambda_2^\top\nu_2
\end{aligned} \tag{20}$$

[2] In the ensuing analysis, we will often drop the dependence of the matrices on μ for the ease of notation

with $L_i'' := \begin{pmatrix} \vartheta_i^\top & \mathbf{0}_{1\times 2} \end{pmatrix}$. Owing to (17), there exist constants δ_4, δ_5 that depend only on $\hat{\omega}_i$ and ϑ_i such that

$$\|H_{i,j}\| \le k\gamma\delta_4, \quad \|L_i'\| \le \gamma\delta_5 \tag{21}$$

for all $i \in \{1,2\}$. For future use, for a parameterized matrix $X(\mu)$ let

$$\varrho_X := \max_{\mu\in\mathcal{P}} ||X(\mu)||$$

and define $2n_f$ subsets $\mathcal{I}_i$ and $\mathcal{I}_i^*$, $i \in \mathcal{N}_f$ as follows:

$$\mathcal{I}_i := \left\{\sigma_i \in \mathcal{J} : \mathrm{Re}\{\lambda_{\max}(E_i(\gamma))\} < 0\right\}, \tag{22}$$

$$\mathcal{I}_i^* := \left\{\sigma_i \in \mathcal{J} : \mathrm{Re}\{\lambda_{\max}(E_i(\gamma))\} \le -k\alpha\right\} \tag{23}$$

where α is a positive constant to be determined. Clearly, $\mathcal{I}_i^* \subseteq \mathcal{I}_i \subseteq \mathcal{J}$ for all $k > 0$ and $\gamma \in (0,2)$.

In the remaining part of this section, we carry out the stability analysis for the closed-loop system, which shows that, with proper choices of tuning parameters k and γ, the sets $\mathcal{I}_i^*$ and $\mathcal{I}_i$ are non-empty. In other words, for each $\hat{\omega}_i$, there is at least one candidate controller defined in (9) capable of solving Problem 1 by an appropriate selection of the gains. To this end, we first list two properties that are instrumental for the analysis. Arguments are similar to those found in [29], thus omitted.

Property 1 *There exist constants $k_0 > 0$ and $c_3 \ge c_2 > c_1 > 0$ such that the solution $P_o^i : (\vartheta_i, \hat{\omega}_i, \varepsilon) \mapsto \mathbb{R}^{2\times 2}$ of the parametrized family of Lyapunov equations*

$$P_o^i\left[\hat{\omega}_i T - \varepsilon\,\vartheta_i\vartheta_i^\top\right] + \left[\hat{\omega}_i T - \varepsilon\,\vartheta_i\vartheta_i^\top\right]^\top P_o^i = -\varepsilon\,\vartheta_i^\top\vartheta_i\boldsymbol{I}, \tag{24}$$

satisfies $c_1\boldsymbol{I} \le P_o^i \le c_2\boldsymbol{I}$ and $\|P_o^i\| \le c_3$ for all $(\vartheta_i, \hat{\omega}_i, \varepsilon) \in \Theta \times [\underline{\omega}, \bar{\omega}] \times (0, k_0]$. $\triangleleft$

Property 2 *There exist constants $c_6 \ge c_5 > c_4 > 0$ such that the solution $P_f : \gamma \mapsto \mathbb{R}^{2\times 2}$ of the parametrized family of Lyapunov equations*

$$P_f F(\gamma) + F^\top(\gamma) P_f = -\gamma\boldsymbol{I} \tag{25}$$

satisfies $c_4\boldsymbol{I} \le P_f \le c_5\boldsymbol{I}$, $\|P_f\| \le c_6$ for all $\gamma \in (0,2)$. $\triangleleft$

The next result asserts that, for each $\hat{\omega}_i$, at least one candidate controller (9) yields exponential stability.

Property 3 *If Assumptions 1 and 2 hold, there exists a class-$\mathcal{K}$ function $k_1(\cdot)$ such that, the sets $\mathcal{I}_i$ and $\mathcal{I}_i^*$ defined in (22) and (23) are non-empty for all $k \in (0, k_1(\gamma)]$ and all $\gamma \in (0,2)$. Moreover, there exist positive constants $p_3 \ge p_2 > p_1 > 0$ such that the solution $P_e^i : (k,\gamma) \mapsto \mathbb{R}^{4\times 4}$ of the parametrized family of Lyapunov inequalities*

$$P_e^i E_i(\gamma) + E_i^\top(\gamma) P_e^i \le -k\boldsymbol{I}, \quad i = 1,2 \tag{26}$$

satisfies $p_1\boldsymbol{I} \le P_e^i \le p_2\boldsymbol{I}$ and $\|P_e^i\| \le p_3$ for all $k \in (0, k_1(\gamma)]$ and all $\gamma \in (0,2)$. ◁

Proof: See Appendix A. Finally, we show that if, for each $\hat{\omega}_i$, the selected candidate controller belongs to $\mathcal{I}_i^*$, then the closed-loop system (20) is guaranteed to be exponentially stable.

Theorem 4 *Let Assumptions 1-3 hold and let the multi-controller be implemented in the form of (9). Then, there exist positive constants k^* and γ^* such that, Problem 1 is solved with any $\sigma_i \in \mathcal{I}_i^*, k \in (0, k^*)$ and $\gamma \in [k\gamma^*, 2)$.* ◁

Proof: The proof consists of showing that system (20) is ISS with respect to $\tilde{\omega}_i$. The derivative of the Lyapunov function candidate $V(\eta_1, \eta_2, z) := \sum_{i=1}^2 \eta_i^\top P_e^i \eta_i + z^\top P_x z$ along trajectories of (20), yields, for all $(\vartheta_i, k, \gamma) \in \Theta \times (0, k_1(\gamma)] \times (0,2)$,

$$
\begin{aligned}
\dot{V} \le & \sum_{i=1, j\neq i}^{2} [-k\|\eta_i\|^2 - 2\eta_i^\top P_e^i N(\tilde{\omega}_i \Upsilon_i T\nu_i + \tilde{\omega}_j \Psi_i T\nu_j) \\
& + 2\gamma\eta_i^\top P_e^i M_i(Cz + \tilde{\omega}_i \Lambda_i^\top \nu_i + \tilde{\omega}_j \Lambda_j^\top \nu_j) \\
& + 2\eta_i^\top P_e^i H_{i,j}\eta_j + 2\tilde{\omega}_i \eta_i^\top P_e^i N' T\nu_i] \\
& - \|z\|^2 + 2kz^\top P_x \Pi_1 b^{\sigma_1}(L_1\eta_1 + L_1'\eta_2) \\
& + 2kz^\top P_x \Pi_2 b^{\sigma_2}(L_2\eta_2 + L_2'\eta_1) \\
\le & \sum_{i=1, j\neq i}^{2} [-k\|\eta_i\|^2 \\
& + 2|\tilde{\omega}_i|(p_3\varrho_{N'} + p_3\varrho_N\delta_2 + \gamma p_3\varrho_{M_i}\varrho_{\Lambda_i})\|\eta_i\|\|\nu_i\| \\
& + 2|\tilde{\omega}_j|(p_3\varrho_N\gamma c_0 + \gamma p_3\varrho_{M_i}\varrho_{\Lambda_j})\|\eta_i\|\|\nu_j\| \\
& + 2\gamma p_3\varrho_{M_i}\varrho_C\|\eta_i\|\|z\| + 2k\gamma\delta_4 p_3\|\eta_i\|\|\eta_j\|] \\
& - \|z\|^2 + 2k\alpha_2(\delta_2+1)\|z\|(\varrho_{\Pi_1}\|\eta_1\| + \varrho_{\Pi_2}\|\eta_2\|) \\
& + 2k\gamma\delta_5\alpha_2\|z\|(\varrho_{\Pi_2}\|\eta_1\| + \varrho_{\Pi_1}\|\eta_2\|),
\end{aligned}
\qquad (27)
$$

where we made use of the bound given in Property 3, Assumptions 1–3 and the inequalities (21). Applying Young's inequality to the sign-indefinite terms yields

$$
\begin{aligned}
\dot{V} \le & -(1 - \frac{8\gamma^2 p_3^2(\varrho_{M_1}^2 + \varrho_{M_2}^2)\varrho_C^2}{k} - kl_0)\|z\|^2 \\
& - k(\frac{1}{4} - 8\gamma^2\delta_4^2 p_3^2)(\|\eta_1\|^2 + \|\eta_2\|^2) + \frac{1}{k}(l_1\tilde{\omega}_1^2 + l_2\tilde{\omega}_2^2)
\end{aligned}
$$

where $l_0 := 8\alpha_2^2[2(\delta_2+1)^2 + \gamma^2\delta_5^2](\varrho_{\Pi_1}^2 + \varrho_{\Pi_2}^2)$ and $l_i := 8p_3^2(3\varrho_{N'}^2 + 3\varrho_N^2\delta_2^2 + 3\varrho_N^2\gamma^2 c_0^2 + 2\gamma^2(\varrho_{M_1}^2 + \varrho_{M_2}^2)\varrho_{\Lambda_i}^2)\bar{a}^2$, $i = 1, 2$, are positive constants that depend only on known parameters. Defining $l_4 := \sum_{i=1}^{n_f} 8p_3^2\varrho_{M_i}^2\varrho_C^2$ and $\gamma = k\gamma'$, we finally obtain

$$
\begin{aligned}
\dot{V} \le & -\left(1 - k\gamma'^2 l_4 - kl_0\right)\|z\|^2 \\
& - k\left(\frac{1}{4} - 8k^2\gamma'^2\delta_4^2 p_3^2\right)(\|\eta_1\|^2 + \|\eta_2\|^2) + \frac{1}{k}(l_1\tilde{\omega}_1^2 + l_2\tilde{\omega}_2^2).
\end{aligned}
$$

Recall that $k \le k_1(\gamma) := \min\{\gamma\underline{\omega}/4\sqrt{2}c_7, k_0\}$, which requires $\gamma' \ge 4\sqrt{2}c_7/\underline{\omega} := \gamma^*$. Consider the worst case scenario $\gamma' = \gamma^*$ and set

$$
k^* := \min\left\{\frac{1}{2(\gamma^{*2}l_4 + l_0)}, \frac{1}{8\gamma^* p_3\delta_4}, k_0, \frac{2}{\gamma^*}\right\}.
$$

Then, for any $k \in (0, k^*)$ and $\gamma \in [k\gamma^*, 2)$, it follows that

$$
\begin{aligned}
\dot{V} &\le -\frac{1}{2}\|z\|^2 - \frac{1}{8}k(\|\eta_1\|^2 + \|\eta_2\|^2) + \frac{1}{k}(l_1\tilde{\omega}_1^2 + l_2\tilde{\omega}_2^2) \\
&\le -\beta V + \frac{1}{k}(l_1\tilde{\omega}_1^2 + l_2\tilde{\omega}_2^2)
\end{aligned}
\qquad (28)
$$

where $\beta := \min\{1/2\alpha_2, k/8p_2\}$. ◁

Remark 2.2 *The results stated above only imply the existence of controllers (i.e., $C_i^j(\hat{\omega}_i)$ with $j \in \mathcal{I}_i^*$ and $i \in \mathcal{N}_f$) solving Problem 1. Naturally, the next question is how to select those controllers purely on the basis of output measurements. This will be answered by the switching mechanism described in Section 3. It is worth mentioning that, at this stage, even with a viable controller $C_i^j(\hat{\omega}_i)$ in place, the disturbance still may not be completely compensated due to possible frequency estimation errors. These errors will be checked and corrected after the closed-loop system has reached its steady state.*

3 State Norm Estimator-based Supervisor

As the name suggests, the proposed supervisory mechanism is inspired by the state norm estimator of [21]. The core strategy is to define a threshold based on an upper bound of $e_i^2(t)$ under the assumption that $\sigma_i \in \mathcal{I}_i^*$. Then, if the performance index $e_i^2(t)$ violates the threshold $\bar{e}_i$ then $\sigma_i(t) \notin \mathcal{I}_i^*$. Conversely, if the threshold $\bar{e}_i$ holds until the system reaches (approximately) a steady state, this suggests that the active controllers are stabilizing.

3.1 Switching Logic

The proposed supervisor is comprised of the cascade connection of two subsystems: a scheduling logic $\mathcal{O}_i$ and a routing function $h(\cdot) : \{1, 2, 3, \cdots\} \mapsto \mathcal{J}$. The output of the $\mathcal{O}_i$ subsystem $m_i(\cdot) : [0, +\infty) \mapsto \mathbb{Z}^+$, termed *the switching sequence*, is a piecewise-constant signal to be determined. The routing function $h(\cdot)$ employed here is constructed to satisfy the *revisitation property*[3] [20]

$$
h(m_i) := \mathrm{mod}(m_i + \sigma_i(0) - 1, 4) + 1, \qquad (29)
$$

where $\sigma_i(0) \in \mathcal{J}, i \in \mathcal{N}_f$ is the initial selection for the activating controller for each frequency estimate $\hat{\omega}_i$.

[3] The revisitation property claims that, for any $q \in \mathcal{J}$ and any $n \in \mathbb{N}$, there exists an integer $m \ge n$ at which $h(m) = q$.

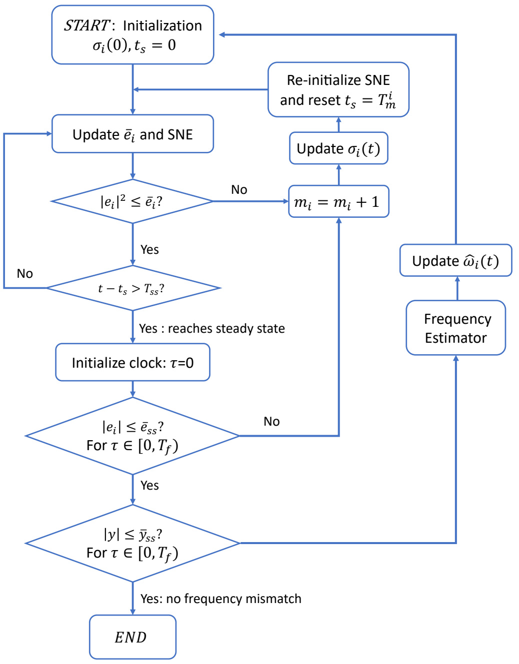

Fig. 2. Flowchart of the switching logic.

The supervisory system keeps adjusting σ_i within the set $\mathcal{J}$ along a pre-specified path $h(m_i)$ until the output e_i is small in a suitable sense. The scheduling logic $\mathcal{O}_i$ exploits two auxiliary signals $\bar{e}_i$ and $\bar{e}_{ss}$ that are generated by the State Norm Estimators (SNEs) to make decisions. The flow chart of the complete switching logic is given in Fig. 2, where the thresholds $\bar{e}_i, \bar{e}_{ss}, \bar{y}_{ss}$, and times T_{ss} and T_f will be determined and explained in the sequel.

For each $i \in \mathcal{N}_f$, let the sequence $\{T_m^i\}_{m=1}^{\infty}$ denote the set of time instants at which the switching signal $\sigma_i(t)$ changes. The same controller is kept active in the time interval $[T_m^i, T_{m+1}^i)$.

3.2 Generation of Switching Criteria

Referring to system (19), to develop the most critical threshold $\bar{e}_i$, we first need to establish the norm bound for the state $\eta_i(t)$, denoted by $\bar{\eta}_i$ in (31). For $t \in [T_m^i, T_{m+1}^i)$, the SNE for $\eta_i(t)$ is designed as follows:

$$\begin{aligned}\dot{\boldsymbol{\eta}}_i(t) = &-\frac{k}{2p_2}\boldsymbol{\eta}_i(t) + 8k\gamma^2 p_2^2\delta_4^2\bar{\eta}_j(t)\\ &+\frac{8}{k}p_2^2\gamma^2\hat{\omega}_i^2\varrho_C^2\bar{z}(t) + \frac{1}{k}\bar{\nu}_i,\end{aligned}\tag{30}$$

with $\boldsymbol{\eta}_i(T_m^i) = 0$. The signal $\bar{z}(t)$ is the norm-bound of the state $z(t)$ (see equation (33) below) and $\bar{\nu}_i$ is a constant taking care of the impact of all the terms involving the frequency misalignment:

$$\bar{\nu}_i = 24\bar{\omega}^2p_2^2(1+\delta_2^2+\gamma^2c_0^2)\bar{a}^2+16\gamma^2p_2^2\varrho_{M_i}^2\bar{\omega}^2(\varrho_{\Lambda_1}^2+\varrho_{\Lambda_2}^2)\bar{a}^2.$$

Then the norm bound $\bar{\eta}_i(t)$ can be calculated as

$$\bar{\eta}_i(t) = \frac{\boldsymbol{\eta}_i(t)}{p_1} + \frac{p_2\rho_i^2(T_m^i)}{p_1}\mathrm{e}^{-\frac{k}{2p_2}(t-T_m^i)},\tag{31}$$

where $\rho_i(T_m^i)$ is the upper bound of $||\eta_i(T_m^i)||$, which needs to be computed separately as follows:

$$\begin{aligned}&\|\eta_i(T_m^i)\| \leq \|\zeta_i(T_m^i)\| + \|\xi_i(T_m^i)\|\\ &\leq \|\zeta_i(T_m^i)\| + \|\iota_i(T_m^i)\| + \|\Upsilon_i\zeta_i(T_m^i)\| + \|\Psi_i\zeta_j(T_m^i)\|\\ &\leq (1+\delta_2)(\|\hat{\nu}_i^{\sigma_i}(T_m^i)\| + \bar{a}) + \|\iota_i(T_m^i)\|\\ &\quad + \gamma c_0(\|\hat{\nu}_j^{\sigma_j}(T_m^i)\| + \bar{a}) =: \rho_i(T_m^i).\end{aligned}\tag{32}$$

The SNE and norm-bound for $z(t)$ are given by:

$$\begin{aligned}\dot{\boldsymbol{z}}(t) &= -\frac{1}{2\alpha_2}\boldsymbol{z}(t) + 2n_f\alpha_2^2k^2(\varrho_{\Pi_1}^2e_1^2(t) + \varrho_{\Pi_2}^2e_2^2(t)),\\ \bar{z}(t) &= \frac{\boldsymbol{z}(t)}{\alpha_1} + \frac{\alpha_2\rho_z^2}{\alpha_1}\mathrm{e}^{-\frac{1}{2\alpha_2}t},\end{aligned}\tag{33}$$

with $\boldsymbol{z}(0) = 0$ and ρ_z represents the upper norm-bound of $||z(0)||$ obtained as:

$$\begin{aligned}&\|z(0)\| \leq \|x_0\| + \varrho_{\Pi_1}\|\zeta_1(0)\| + \varrho_{\Pi_2}\|\zeta_2(0)\|\\ &\leq r_0 + \varrho_{\Pi_1}(\bar{a} + \|\hat{\nu}_{1,0}\|) + \varrho_{\Pi_2}(\bar{a} + \|\hat{\nu}_{2,0}\|) =: \rho_z.\end{aligned}\tag{34}$$

The initial conditions $\boldsymbol{\eta}_i(T_m^i)$, $\boldsymbol{z}(0)$ are set to zero for simplicity (see [12]). Clearly, the SNE $\boldsymbol{\eta}_i(t)$ and the norm bound $\bar{\eta}_i(t)$ need be re-initialized after each switching, since $\|\hat{\nu}_i^{\sigma_i}(T_m^i)\|$ is different for each candidate controller.

Next, we establish the switching criterion for $\sigma_i(t)$:

Lemma 5 *Given Assumptions 2–3, for any $k \in (0, k^*)$, $\gamma \in [k\gamma^*, 2)$ and $\sigma_i(t) \in \mathcal{I}_i^*$, the filtered output $e_i(t)$ is norm-bounded by:*

$$|e_i(t)|^2 \leq \bar{e}_i := (1+\delta_2^2)\bar{\eta}_i(t) + \gamma^2\delta_5^2\bar{\eta}_j(t)\tag{35}$$

for all $i \in \mathcal{N}_f$, $i \neq j$ and $t \geq T_m^i$. ◁

The constants k^* and γ^* are defined in Theorem 4. The proof of Lemma 5 is given in Appendix B.

Remark 3.1 *Property 3 and Theorem 4 establish the existence of a controller that enforces boundedness of the filtered output by the threshold given in Lemma 5. Therefore, there exists a positive integer $\bar{m}_i \leq 4$ such that the selection of the candidate controller in the family $\{C_i\}$ stops after $T_{\bar{m}_i}^i$ units of time, and the thresholds $\bar{e}_i$ are fulfilled for all $t > T_{\bar{m}_i}^i$. The proof of this result, omitted for reasons of space, follows from arguments similar to those presented in [30].*

We are ready to show that, with a suitable choice of the gain parameters, the condition $|e_i(t)|^2 \leq \bar{e}_i, \forall i \in \mathcal{N}_f$, holds for all $t \geq T_s := \max\{T^1_{\bar{m}_1}, T^2_{\bar{m}_2}\}$, thus proving that $e_i \in \mathcal{L}_\infty$, and all closed-loop trajectories are bounded.

Lemma 6 *If Assumptions 2–3 hold, there exists a positive constant k^{**} such that, for all $k \in (0, k^{**})$ and $\gamma \in [k\gamma^*, 2)$, the active controllers selected by the conditions (35) guarantee that the trajectories of the closed-loop system (20) are bounded.* ◁

Proof: Assuming that the switching stops after $t \geq T_s$, in the light of (30) and (31), one obtains

$$\begin{aligned}\|\bar{\eta}_i\|_\infty &\leq \frac{\|\boldsymbol{\eta}_i\|_\infty}{p_1} + \frac{p_2\rho_i^2(T_s)}{p_1}\mathrm{e}^{-\frac{k}{2p_2}(t-T_s)} \\ &\leq \frac{1}{p_1}\left(8k\gamma^2p_2^2\delta_4^2\|\bar{\eta}_j\|_\infty + \frac{8}{k}p_2^2\gamma^2\hat{\omega}_i^2\varrho_C^2\|\bar{z}\|_\infty + \frac{1}{k}\bar{\nu}_i\right) \\ &\quad + \frac{p_2[\rho_i^2(T_s)+\boldsymbol{\eta}_i(T_s)]}{p_1}\mathrm{e}^{-\frac{k}{2p_2}(t-T_s)}, \qquad (36)\end{aligned}$$

where we denote the upper bound of a signal $\boldsymbol{x}(t)$, $t \in [T_s, \infty)$ with $\|\boldsymbol{x}\|_\infty$, and the dependence on t of variables is dropped for simplicity. Rewrite (36) as

$$\|\bar{\eta}_i\|_\infty \leq k\gamma^2\beta_1\|\bar{\eta}_j\|_\infty + \frac{\gamma^2}{k}\beta_2\|\bar{z}\|_\infty + \frac{\beta_3}{k}\bar{\nu}_i + \varepsilon_i(t), \quad (37)$$

for positive constants $\beta_1, \beta_2, \beta_3$ and exponentially decaying signal $\varepsilon_i(t)$. Owing to (33), it follows that

$$\|\bar{z}\|_\infty \leq k^2\beta_4(\|e_1\|^2_\infty + \|e_2\|^2_\infty) + \varepsilon_z(t) \qquad (38)$$

where $\beta_4 > 0$ and $\varepsilon_z(t)$ decays to zero exponentially fast. Using inequalities (35), one obtains

$$\|e_1\|^2_\infty + \|e_2\|^2_\infty \leq (1+\delta_2^2+\gamma^2\delta_5^2)(\|\bar{\eta}_1\|_\infty + \|\bar{\eta}_2\|_\infty) \quad (39)$$

Substituting (38) and (39) into (37) and multiplying both sides of the resulting inequality by k, it follows that

$$\begin{aligned}&k(\|\bar{\eta}_1\|_\infty + \|\bar{\eta}_2\|_\infty) \\ &\leq 2k^2\beta_1(\|\bar{\eta}_1\|_\infty + \|\bar{\eta}_2\|_\infty) + \beta_3(\bar{\nu}_1 + \bar{\nu}_2) \\ &\quad + 8k^2\beta_2\beta_4(1+\delta_2^2+4\delta_5^2)(\|\bar{\eta}_1\|_\infty + \|\bar{\eta}_2\|_\infty) \\ &\quad + k\varepsilon_1(t) + k\varepsilon_2(t) + 8\beta_2\varepsilon_z(t), \qquad (40)\end{aligned}$$

where we have exploited the fact that $\gamma < 2$. From (40), one concludes that

$$\begin{aligned}\|\bar{\eta}_1\|_\infty + \|\bar{\eta}_2\|_\infty \leq{}& \frac{2\beta_3}{k}(\bar{\nu}_1+\bar{\nu}_2) + 2\varepsilon_1(t) + 2\varepsilon_2(t) \\ &+ \frac{8\beta_2}{k}\varepsilon_z(t) \leq \infty \qquad (41)\end{aligned}$$

for all $\gamma \in [k\gamma^*, 2)$ and all $k \in (0, k^{**})$, where

$$k^{**} := \min\left\{k^*, \frac{1}{4\beta_1 + 16\beta_2\beta_4(1+\delta_2^2+\gamma^2\delta_5^2)}\right\}.$$

Since $\bar{\eta}_i$ are norm-bounds for the state variables η_i, it follows that $\eta_i \in \mathcal{L}_\infty$, $i = 1, 2$. By virtue of (38) and (39), it follows that $e_i \in \mathcal{L}_\infty$ and $z \in \mathcal{L}_\infty$. ◁

Lemma 6 guarantees the boundedness of all internal variables; however, this does not yet indicate that Problem 1 is solved, as two cases are possible:

(1) The closed-loop system is internally stable.
(2) The closed-loop system is "neutrally stable", which is a pathological case that shall be avoided.

Hence, the penultimate step in the switching mechanism is to check the steady state of e_i to exclude case (2) above. Then, the steady state of $y(t)$ and $\hat{d}_i(t)$ will be examined to see whether there is a frequency misalignment and which frequency estimate is inaccurate.

3.3 Generation of Thresholds

To generate the thresholds for the steady state of the system, we need to first estimate the time at which the steady state is approximately reached. Referring to system (20) and defining $\varsigma := (\eta_1, \eta_2, z) \in \mathbb{R}^{n_\varsigma}$ with $n_\varsigma = 4n_f + n_x$, we rewrite the closed-loop system as

$$\begin{aligned}\dot{\varsigma} &= E_\varsigma\varsigma + \tilde{\omega}_1Q_1\nu_1 + \tilde{\omega}_2Q_2\nu_2 \\ y &= C_\varsigma\varsigma + \tilde{\omega}_1\Lambda_1^\top\nu_1 + \tilde{\omega}_2\Lambda_2^\top\nu_2 \qquad (42)\end{aligned}$$

for $t \in [t_s, \infty)$, where $t_s := \max\{T^1_m, T^2_m\}$. In the above system, Q_1 and Q_2 are constant matrices, and

$$E_\varsigma = \begin{pmatrix} E_1(\gamma) & H_{1,2} & \gamma M_1 C \\ H_{2,1} & E_2(\gamma) & \gamma M_2 C \\ V_1 & V_2 & A \end{pmatrix}, \quad C_\varsigma^\top = \begin{pmatrix} L_1''^\top \\ L_2''^\top \\ C^\top \end{pmatrix}$$

where $V_1 := k\Pi_1 b^{\sigma_1}L_1 + k\Pi_2 b^{\sigma_2}L_2'$ and $V_2 := k\Pi_1 b^{\sigma_1}L_1' + k\Pi_2 b^{\sigma_2}L_2$.

According to Theorem 4 and inequality (28), the matrix $E_\varsigma \in \mathbb{R}^{n_\varsigma \times n_\varsigma}$ is Hurwitz if $k \in (0, k^{**})$, $\gamma \in [k\gamma^*, 2)$ and $\sigma_i \in \mathcal{I}_i^*$ for all $i \in \mathcal{N}_f$. Moreover, it is easy to show that there exist positive constants $p_6 \geq p_5 > p_4 > 0$ and $\beta > 0$ such that the solution $P_\varsigma : (k, \gamma) \mapsto \mathbb{R}^{n_\varsigma \times n_\varsigma}$ of

$$P_\varsigma E_\varsigma + E_\varsigma^\top P_\varsigma \leq -\beta\boldsymbol{I}, \qquad (43)$$

satisfies $p_4\boldsymbol{I} \leq P_\varsigma \leq p_5\boldsymbol{I}$ and $\|P_\varsigma\| \leq p_6$ for all $k \in (0, k^{**})$, $\gamma \in [k\gamma^*, 2)$ and $\sigma_i \in \mathcal{I}_i^*$. Owing to equalities (30) and (33), the initial state of (42) is bounded by

$$\|\varsigma(t_s)\|^2 \leq \sum_{i=1}^{n_f}\bar{\eta}_i(t_s) + \bar{z}(t_s) := \rho_\varsigma(t_s).$$

System (42) is deemed to have reached steady state if

$$\mathrm{e}^{-\beta(t-t_s)}\frac{p_5}{p_4}\rho_\varsigma(t_s) \le \varepsilon_0$$

holds for all $t \ge t_s$, where $\varepsilon_0 > 0$ is a small positive constant chosen by the designer. After each switching, the time at which steady state is attained is estimated as

$$T_{ss}(t_s) := \log\left[\frac{\bar\rho(t_s)}{\varepsilon_0}\right]^{\frac{1}{\beta}} \tag{44}$$

with $\bar\rho(t_s) = \rho_\varsigma(t_s)p_5/p_4$.

Next, we develop the threshold for the steady state of the filtered output $e_i(t)$. Letting Σ_i be the unique solution of

$$\omega_i\Sigma_i T = E_\varsigma\Sigma_i + Q_i, \quad i \in \mathcal{N}_f$$

and changing coordinates as $\varsigma' := \varsigma - \tilde\omega_1\Sigma_1\nu_1 - \tilde\omega_2\Sigma_2\nu_2$, the closed-loop system (42) reads as

$$\begin{aligned}\dot\varsigma' &= E_\varsigma\varsigma'\\ y &= C_\varsigma\varsigma' + \tilde\omega_1\Phi_1^\top\nu_1 + \tilde\omega_2\Phi_2^\top\nu_2,\end{aligned} \tag{45}$$

where $\Phi_i^\top = C_\varsigma\Sigma_i + \Lambda_i^\top \in \mathbb{R}^{1\times 2}$. Substituting y from (45) in the filter defined in (9) yields

$$\begin{aligned}\dot\iota_i &= \hat\omega_i F(\gamma)\iota_i + \gamma\hat\omega_i G[C_\varsigma\varsigma' + \tilde\omega_i\Phi_i^\top\nu_i + \tilde\omega_j\Phi_j^\top\nu_j]\\ e_i &= \Gamma\iota_i, \qquad i \in \mathcal{N}_f\end{aligned} \tag{46}$$

To obtain a suitable upper bound of the steady state of e_i, we again consider the coordinate change $\iota_i' := \iota_i - \tilde\omega_i\Upsilon_i'\nu_i - \tilde\omega_j\Psi_i'\nu_j$ where the matrix Υ_i' and Ψ_i' are the unique solutions of the Sylvester equations

$$\omega_i\Upsilon_i'T = \hat\omega_i F(\gamma)\Upsilon_i' + \gamma\hat\omega_i G\Phi_i^\top, \tag{47}$$

$$\omega_j\Psi_i'T = \hat\omega_i F(\gamma)\Psi_i' + \gamma\hat\omega_i G\Phi_j^\top, \quad i \in \{1,2\} \tag{48}$$

to obtain

$$\begin{aligned}\dot\iota_i' &= \hat\omega_i F(\gamma)\iota_i' + \gamma\hat\omega_i G C_\varsigma\varsigma'\\ e_i &= \Gamma\iota_i' + \tilde\omega_i\Gamma\Upsilon_i'\nu_i + \tilde\omega_j\Gamma\Psi_i'\nu_j.\end{aligned} \tag{49}$$

Using (44), it is easy to show that the steady-state trajectory of (49) shall be bounded as follows:

$$e_{i,ss}(t_s) := \sup_{t\ge t_s+T_{ss}}|e_i(t)| \le \varepsilon_0 + |\tilde\omega_i|\|\Upsilon_i'\|\bar a + |\tilde\omega_j|\|\Psi_i'\|\bar a \tag{50}$$

with gains $k \in (0, k^{**})$, $\gamma \in [k\gamma^*, 2)$, and index of the active controller $\sigma_i \in \mathcal{I}_i^*$. Owing to (47) and (48)

$$|\tilde\omega_i|\|\Upsilon_i'\| \le \gamma r_1, \quad |\tilde\omega_j|\|\Psi_i'\| \le \gamma r_2 \tag{51}$$

for positive constants r_1 and r_2. The derivation of the bounds can be found in Appendix C.

Without loss of generality, set the threshold $\bar e_{ss} := 2\varepsilon_0$. Accordingly, the steady state of $e_i(t)$ can be bounded by

$$e_{i,ss}(t_s) \le \bar e_{ss}, \qquad i \in \{1,2\} \tag{52}$$

for all $\sigma_i \in \mathcal{I}_i^*$ with any $k \in (0, \min\{k^{**}, \gamma^{**}/\gamma^*\})$, $\gamma \in [k\gamma^*, \gamma^{**}]$, where $\gamma^{**} := \frac{\varepsilon_0}{(r_1+r_2)\bar a}$. It is worth pointing out that condition (52) can be satisfied for arbitrarily small $\bar e_{ss}$ as long as $\sigma_i \in \mathcal{I}_i^*$ and the tuning parameters k and γ are sufficiently small.

The following lemma indicates that, if (52) is satisfied for $T_f := 2\pi/\underline{\omega}$, the pathological case (2) has been excluded. Otherwise, the active controller does not belong to $\mathcal{I}_i^*$ and the switching shall continue.

Lemma 7 *If Assumptions 2–3 hold, then for arbitrary $\varepsilon_0 > 0$ there exist positive constants $\bar k$ and $\bar\gamma$ such that, for any $k \in (0, \bar k)$ and $\gamma \in (k\gamma^*, \bar\gamma]$, if the active controllers verify (52) for all $t \in [t_s + T_{ss}, t_s + T_{ss} + T_f]$, then the closed-loop system (45) is internally stable.* ◁

The proof, along with the definitions of $\bar k$ and $\bar\gamma$, is given in Appendix D.

Remark 3.2 *The proposed pre-routed switching logic differs from the one presented in [20] in that the stabilizing controller will be selected within at most $4n_f$ switchings, which grows linearly with the number of frequencies. According to [20], if the cardinality of $\mathcal{J}$ is large, the performance of the pre-routed switching will be significantly degraded. This is avoided in the proposed switching mechanism, due to the low cardinality of $\mathcal{J}$ and the guaranteed termination of the switching sequence.*

The last step is to check whether the frequency estimate is accurate. In view of the closed-loop system (45), if $\tilde\omega_i = 0$ for all $i \in \mathcal{N}_f$, then the steady state trajectories of the system output should be bounded by

$$y_{ss}(t_s) := \sup_{t\ge t_s+T_{ss}}|y(t)| \le \|C_\varsigma\|\varepsilon_0 \le \varrho_{C_\varsigma}\varepsilon_0 \tag{53}$$

for any $k \in (0, \bar k)$ and $\gamma \in [k\gamma^*, \bar\gamma)$. For practical implementation, one can choose a sufficiently small ε_0 that verifies the condition

$$\varepsilon_0 \le \frac{1}{\varrho_{C_\varsigma}}\left(\bar y_{ss} - n_f\rho_{\tilde\omega}\varrho_{\Phi_i}\bar a\right) \tag{54}$$

where $\bar y_{ss}$ is the desired norm-bound of the regulated output, and $\rho_{\tilde\omega}$ is the tolerance of estimation error which depends on the frequency estimation algorithm.

If $y_{ss}(t_s) \le \bar y_{ss}$ holds for another T_f seconds, then one can claim the disturbance is completely rejected by the active AFC modules. Otherwise, there must be a frequency misalignment, and the output will be fed to the frequency estimator to obtain the correct estimates.

4 Frequency Estimator

As stated in Section 1, several well-developed frequency estimation schemes can be employed to correct the potential frequency misalignment, due to the modularity of the proposed scheme. However, the frequency estimator must ensure that the estimation error $\tilde{\omega}_i := \hat{\omega}_i - \omega_i$ is ISS with respect to additive bounded noise. Here, we employ the deadbeat estimator of [8], since it permits exact identification of the unknown parameters within a finite time. Note that the finite-time convergence property is not essential, and an asymptotic estimator is also feasible. However, in this latter case, additional care will be needed to ensure that the frequency estimate is sufficiently accurate to be fed to the AFC module.

As shown in Fig. 2, the deadbeat estimator is activated and driven by $y(t)$ after the system (approximately) reaches a steady state. By virtue of (45), for all $t \in [T_0, \infty)$, $T_0 := t_s + T_{ss} + 2T_f$, the output $y(t)$ can be rewritten as

$$y(t) = y_{ss}(t) + \varepsilon_y(t) \tag{55}$$

with steady-state response

$$y_{ss}(t) = \sum_{l \in \mathcal{I}'} a_l |W(j\omega_l)| \sin(\omega_l t + \phi_l + \angle W(j\omega_l)),$$

where $\mathcal{I}' := \{i \in \mathcal{N}_f | \tilde{\omega}_i \neq 0\}$ denotes the index of the sinusoidal components that have not been rejected due to the frequency misalignments. The cardinality of $\mathcal{I}'$ is denoted by $n'_f \in \mathbb{N}$. The exponentially decaying signal $\varepsilon_y(\cdot)$ obeys the condition $\varepsilon_y(t) \leq \varrho_{C_\varsigma} \varepsilon_0 := \bar{\varepsilon}_0$ for all $t \geq T_0$. 0 For the sake of simplicity, the time variable will be redefined as $t := t - T_0$ in the remaining part of this section. The sinusoidal signal considered in [8] admits the more general form

$$d_g = \sum_{i=1}^{n'_f} A_i e^{\rho_i t} \sin(\omega_i t + \phi_i) + A_0, \tag{56}$$

and the unknown parameters considered include frequency, amplitude, phase and damping factor ρ_i. Here, as the signal $y(t)$ is purely sinusoidal, only information on the frequencies $\omega_i, \dots, \omega_{n'_f}$ is needed, which can be obtained directly from the imaginary part of the roots of the characteristic equation

$$\begin{aligned} P(s) &:= s \prod_{i=1}^{n'_f} [(s - \rho_i)^2 + \omega_i^2]^2 \\ &= s^{2n'_f+1} + \alpha'_{2n'_f} s^{2n'_f} + \dots + \alpha'_2 s^2 + \alpha'_1 s = 0. \end{aligned} \tag{57}$$

The coefficient vector $\boldsymbol{\Lambda}^\top = [\alpha'_1, \alpha'_2, \dots, \alpha'_{2n'_f}]^\top \in \mathbb{R}^{2n'_f}$ together with the unknown initial state of the observer canonical form of the exosystem $\boldsymbol{v}_f^\top := [v_0(0), \dots, v_{2n'_f}(0)]^\top \in \mathbb{R}^{2n'_f+1}$ comprise the vector $\boldsymbol{\theta} := [\boldsymbol{\Lambda}, \boldsymbol{v}_f]^\top \in \mathbb{R}^{4n'_f+1}$, which contains all the parameters of the disturbance $d_g(t)$. The vector $\boldsymbol{\theta}$ is estimated by the estimation law

$$\hat{\boldsymbol{\theta}}(t) = \begin{cases} \hat{\boldsymbol{\theta}}(t^-), & \min \operatorname{eig}(\boldsymbol{R}_f(t)) < \sigma \\ \boldsymbol{R}_f^{-1}(t) \boldsymbol{S}_f(t), & \min \operatorname{eig}(\boldsymbol{R}_f(t)) \geq \sigma \end{cases} \tag{58}$$

where $\hat{\boldsymbol{\theta}}(0) = \boldsymbol{\theta}_0 \in \mathbb{R}^{4n'_f+1}$ is an initial guess and σ is an invertibility threshold set arbitrarily in the interval $0 < \sigma \leq \epsilon e^{-g t_\epsilon}$, where ϵ, t_ϵ and g are positive constants related to the persistence of excitation of the sinusoidal signals [24]. To obtain the matrix-valued signals $\boldsymbol{R}_f(t)$ and $\boldsymbol{S}_f(t)$, define auxiliary signals $\boldsymbol{z}_a(t) \in \mathbb{R}^{2n'_f}$, $\boldsymbol{z}_f(t) \in \mathbb{R}$ as

$$\begin{aligned} \boldsymbol{z}_a^\top(t) &:= [-[V_{K^{(1)}} y](t), [V_{K^{(2)}} y](t), \dots, [V_{K^{(2n'_f)}} y](t)]^\top \\ \boldsymbol{z}_f(t) &:= [V_{K^{(2n'_f+1)}} y](t) \end{aligned} \tag{59}$$

where $[V_{K^{(i)}} y](t)$ results by processing the output $y(t)$ through Volterra operators (see [8] for more details). Then, the filtered state $\boldsymbol{R}_f(t) \in \mathbb{R}^{(4n'_f+1)\times(4n'_f+1)}$ and $\boldsymbol{S}_f(t) \in \mathbb{R}^{(4n'_f+1)}$ in (58) are obtained from

$$\begin{cases} \dot{\boldsymbol{S}}_f(t) = -g\boldsymbol{S}_f(t) + \boldsymbol{S}(t), & \boldsymbol{S}_f(0) = \boldsymbol{0} \\ \dot{\boldsymbol{R}}_f(t) = -g\boldsymbol{R}_f(t) + \boldsymbol{R}(t) & \boldsymbol{R}_f(0) = \boldsymbol{0}. \end{cases} \tag{60}$$

where $\boldsymbol{S}(t) := \boldsymbol{v}(t)\boldsymbol{z}_f(t)$, $\boldsymbol{R}(t) := \boldsymbol{v}(t)\boldsymbol{v}^\top(t)$ are time-varying matrices depending on the regressor vector $\boldsymbol{v}(t) := [\boldsymbol{z}_a(t), \boldsymbol{\Gamma}_f(t)]^\top \in \mathbb{R}^{4n'_f+1}$, $\boldsymbol{\Gamma}_f^\top(t) := [\gamma_0, \gamma_1, \cdots, \gamma_{2n'_f}]^\top \in \mathbb{R}^{2n'_f+1}$, and each element in $\boldsymbol{\Gamma}_f$ is a known function given by

$$\gamma_l(t) = \sum_{j=1}^{2n'_f+3} e^{-j\beta' t} \mathcal{F}_{l,j}, \quad l = 0, 1, \dots, 2n'_f \tag{61}$$

with $\mathcal{F}_{l,j} = \frac{(-1)^{2n'_f-l+j}(2n'_f+2)!}{(2n'_f+3-j)!(j-1)!}(j\beta')^{2n'_f-l}$. The signals $\boldsymbol{z}_a(t)$ and $\boldsymbol{z}_f(t)$ can be regarded as the output of the stable LTI system

$$\begin{cases} \dot{\boldsymbol{\xi}}(t) = \boldsymbol{G}_\xi \boldsymbol{\xi}(t) + \boldsymbol{E} y(t), \ \boldsymbol{\xi}(0) = \boldsymbol{0} \in \mathbb{R}^{n_\xi} \\ \boldsymbol{z}'(t) = \boldsymbol{H}\boldsymbol{\xi}(t) \end{cases} \tag{62}$$

where $n_\xi := 4n'^2_f + 8n'_f + 3$, $\boldsymbol{z}'(t) := [\boldsymbol{z}_a(t), \boldsymbol{z}_f(t)]^\top$, the state matrix $\boldsymbol{G}_\xi \in \mathbb{R}^{n_\xi \times n_\xi}$, given by

$$\boldsymbol{G}_\xi := \operatorname{diag}[\boldsymbol{G}, \dots, \boldsymbol{G}], \ \boldsymbol{G} = \operatorname{diag}[-\beta', \dots, -(2n'_f+3)\beta']$$

is Hurwitz with $\beta' > 0$, and $\boldsymbol{E} \in \mathbb{R}^{n_\xi}$ is given by

$$\boldsymbol{E} = \left[\boldsymbol{E}_1^\top, \ldots, \boldsymbol{E}_{2n_f'+1}^\top\right]^\top, \; \boldsymbol{E}_i = \left[\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,2n_f'+3}\right]^\top$$

with $\lambda_{i,j} := \frac{(-1)^{i+j}(2n_f'+2)!}{(2n_f'+3-j)!(j-1)!}(j\beta')^i$. Finally, $\boldsymbol{H} :=$ diag $\left[\mathbf{1}^\top, \ldots, \mathbf{1}^\top\right]^\top \in \mathbb{R}^{(2n_f'+1)\times n_\xi}$ where $\mathbf{1}^\top$ denotes a row vector of ones with $2n_f'+3$ elements.

Lemma 8 [8, Theorem 4.1, Lemma 3.1] *For a persistently exciting sinusoidal signal $y(\cdot)$ perturbed by additive noise $n(\cdot)$, the estimation error $\tilde{\omega}(\cdot)$ is ISS with respect to the additive disturbance, with bound*

$$\sum_{i=1}^{n_f'} \|\tilde{\omega}_i(\cdot)\|_\infty \le \rho_{\tilde{\omega}}(\bar{\varepsilon}_0) \tag{63}$$

where $\rho_{\tilde{\omega}}(\cdot)$ is a class-$\mathcal{K}$ function, and $\|n(\cdot)\|_\infty \le \bar{\varepsilon}_0$.

5 Extension to More than Two Frequencies

This section is devoted to extending the mechanism described in the previous sections to the more general case where the disturbance contains more than two frequencies, i.e., $n_f \ge 2$. For each distinct frequency, the family of candidate controllers takes the same form as the one presented in the previous section. However, the SNEs and the switching criteria are slightly different. Specifically, for $t \in [T_m^i, T_{m+1}^i)$, the general form of SNEs and norm-bounds for $\eta_i(t)$ are as follows:

$$\begin{aligned}
\dot{\boldsymbol{\eta}}_i =& -\frac{k}{2p_2}\boldsymbol{\eta}_i + \gamma^2 \sum_{j\in\mathcal{N}_f'} 8k(n_f-1)p_3^2\delta_4^2\bar{\eta}_j \\
&+ \frac{8}{k}\gamma^2 p_3^2 \hat{\omega}_i^2 \varrho_C^2 \bar{z} + \frac{1}{k}\bar{\nu}_i, \quad \boldsymbol{\eta}_i(T_m^i) = 0 \\
\bar{\eta}_i =& \frac{\boldsymbol{\eta}_i}{p_1} + \frac{p_2\rho_i^2(T_m^i)}{p_1} \mathrm{e}^{-\frac{k}{2p_2}(t-T_m^i)}, \quad i \in \mathcal{N}_f.
\end{aligned} \tag{64}$$

where $\mathcal{N}_f' := \mathcal{N}_f \setminus \{i\}$, and

$$\begin{aligned}
\bar{\nu}_i =& 24\bar{\omega}^2 p_2^2[1+\delta_2^2+(n_f-1)^2\gamma^2 c_0^2]\bar{a}^2 \\
&+ 8n_f\gamma^2 p_2^2 \varrho_{M_i}^2 \bar{\omega}^2 \bar{a}^2 \sum_{l=1}^{n_f} \varrho_{\Lambda_l}^2, \\
\rho_i(T_m^i) =& (1+\delta_2)(\|\hat{\nu}_i^{\sigma_i}(T_m^i)\| + \bar{a}) + \|\iota_i(T_m^i)\| \\
&+ \sum_{j\in\mathcal{N}_f'} \gamma c_0(\|\hat{\nu}_j^{\sigma_j}(T_m^i)\| + \bar{a}).
\end{aligned} \tag{65}$$

Taking all sinusoidal components into consideration, ρ_z is recomputed as

$$\rho_z := r_0 + \sum_{i=1}^{n_f} \varrho_{\Pi_i}(\bar{a} + \|\hat{\nu}_{i,0}\|). \tag{66}$$

Finally, owing to (33), (64), (65) and (66), one obtains the norm bound of the filtered output $e_i^2(t)$ as

$$\bar{e}_i := 2(1+\delta_2)^2\bar{\eta}_i + 2\gamma^2\delta_5^2 \sum_{j\in\mathcal{N}_f'} \bar{\eta}_j. \tag{67}$$

For the thresholds employed in the switching, we first set $\bar{y}_{ss}$, then choose a sufficiently small ε_0 verifying (54), and finally calculate $\bar{e}_{ss} := 2\varepsilon_0$ and T_{ss} from (44).

The last step is to determine the values of the gains. The feasible ranges $k \in (0, \bar{k})$ and $\gamma \in (k\gamma^*, \bar{\gamma})$ for the gains detailed in Lemma 5 through 7, are modified as follows

$$\begin{aligned}
\bar{k} &:= \min\{k^{**}, \bar{\gamma}/\gamma^*\}, \quad \bar{\gamma} := \min\{2, \gamma^{**}\}, \\
\gamma^* &:= \frac{4\sqrt{2}c_7}{\underline{\omega}}, \quad \gamma^{**} := \frac{\varepsilon_0}{\bar{a}\sum_{i\in\mathcal{N}_f} r_i} \\
k^* &:= \min\left\{\frac{1}{l_{k1}^*}, \frac{1}{l_{k2}^*}, k_0, \frac{2}{\gamma^*}, \frac{1}{\beta^*}\right\}
\end{aligned} \tag{68}$$

where

$$\begin{aligned}
l_{k1}^* &:= 2(\gamma^{*2}l_4 + l_0), \quad l_{k2}^* := 8(n_f-1)\gamma^* p_3\delta_4 \\
l_4 &:= \sum_{i=1}^{n_f} 8p_3^2\varrho_{M_i}^2\varrho_C^2 \\
l_0 &:= 4n_f\alpha_2^2 \sum_{l=1}^{n_f} \varrho_{\Pi_l}^2[(\delta_2+1)^2 + (n_f-1)\gamma^2\delta_5^2] \\
\beta^* &:= 8\beta_1 + 8\beta_2\beta_4(1+\delta_2^2+\gamma^2\delta_5^2).
\end{aligned}$$

The derivation of the above parameters follows the same line of the case $n_f = 2$, found in the proof of Theorem 4.

Remark 5.1 *As seen from* (68)*, the suitable range of the gains k and γ becomes narrower as the number of frequencies increases, which may lead to poor transient behavior. Increasing the number of candidate controllers could in principle provide a larger feasible range for the gains. Nevertheless, this may in turn require more switchings until the 'best' candidates are selected, therefore establishing a tradeoff. It is worth pointing out that, in practice, the admissible range of gain parameters is larger than what is presented in* (68)*, as the bounds derived in this section are likely conservative.*

6 Illustrative Example

Consider the stable non-minimum phase plant model

$$W(s) = \frac{2s-2}{s^2+2s+5} \tag{69}$$

perturbed by the sinusoidal signal with two harmonics

$$d(t) = \sin(t+\pi) + \cos(3t + \tfrac{\pi}{4}).$$

Table 1
Frequency response and the sets of stabilizing controllers

Frequency	$\vartheta_i^\top$	$\mathcal{I}_i^*$	$\mathcal{I}_i$
$\omega_1 = 1$[rad/s]	(-0.2,0.6)	{4}	{3,4}
$\omega_2 = 3$[rad/s]	(0.85,-0.23)	{1}	{1,2}
$\omega_3 = 2$[rad/s]	(0.82,0.71)	{1}	{1,4}

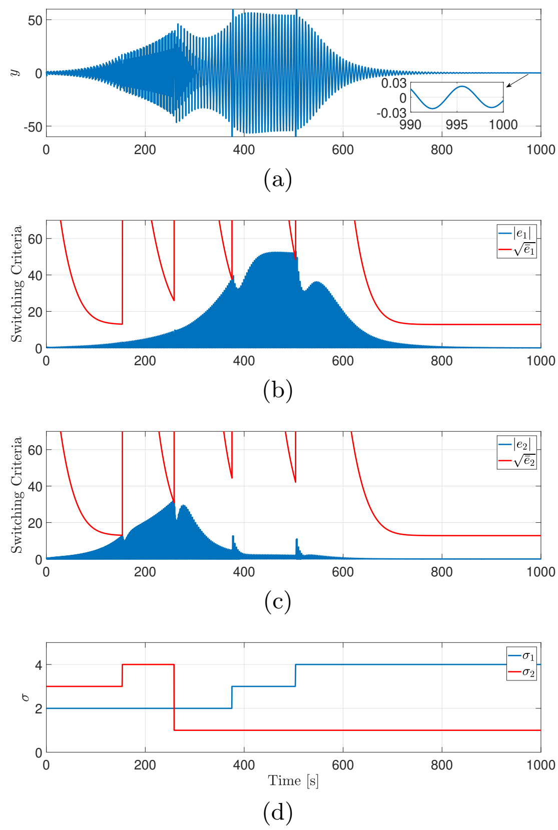

Fig. 3. Case 1): (a) Time history of output trajectory $y(t)$; (b,c) Time history of switching criteria; (d) Time history of switching signals.

The information on the frequency response together with the set of stabilizing controllers is listed in Table 1. The gains of the controllers are selected as $k = 0.05, \gamma = 0.1, \varepsilon_0 = 0.1$, and the initial conditions are $x(0) = [1, 1]^\top$. The Runge-Kutta method has been employed for all simulations with fixed sampling interval $T_s = 10^{-3}$s.

Four cases have been considered:

Case 1) In the first case, the frequencies are known, but the worst-case initial scenario is considered where all the initial controllers are mismatched. From Fig. 3, it is seen that the switching for each frequency occurs independently from each other. Although the transient behavior is somewhat less than satisfactory due to the conservative choice of the tuning parameters, the 'best' controllers are selected after 250 [s] and 500 [s], respectively, and the disturbance is rejected at around 900 [s].

Case 2) Next, the influence of frequency misalignment is considered, as the initial guesses of the frequencies are 2 [rad/s] and 3 [rad/s]. The results in Fig. 4 show that, with frequency estimation error, the output can not be regulated to zero (see Fig. 4(a) between $400 - 500$ [s]), but the stability of the overall system is retained. The estimation error is then detected and corrected by the deadbeat estimator at around 550 [s]. To determine the incorrect frequency estimate, we check the output $\hat{d}_i$ of the active controller, which will be close to zero if the frequency does not match the true value. Finally, the new 'best' candidate controller is selected after three switchings, and the disturbance is completely rejected after 1500 [s]. It is worth noting that the proposed algorithm is able to correct not only an initial frequency estimation error but also a sudden change in the frequency of the disturbance after the system has reached its steady state.

Case 3) The robustness of the proposed scheme is tested by adding a random noise $v(t)$ with uniform distribution within the interval $[-0.02, 0.02]$ at the output of the plant, that is, the available output $y(t)$ becomes

$$y_d(t) = y(t) + v(t).$$

Fig. 5 shows that the sinusoidal disturbance is still attenuated in spite of the measurement noise.

Case 4) Finally, we consider a case where the disturbance model is over-parameterized. Specifically, the family of candidate controllers accounts for more distinct frequencies than the ones carried by the disturbance. Here, we construct a copy of the exosystem model with order 6, containing three frequencies $\hat{\omega}_1 = 1, \hat{\omega}_2 = 2$ and $\hat{\omega}_3 = 3$. The external disturbance considered is the same as the one in Case 1). Fig. 6 shows that the disturbance is still successfully rejected. This result suggests that the absence of the persistence of excitation of the disturbance (with respect to the order of the disturbance model, see [24]) does not compromise the effectiveness of the proposed methodology. In Fig. 6, the estimated disturbance $\hat{d}_1, \hat{d}_3$ track the disturbance d_1, d_3 while $\hat{d}_2$ eventually decays to zero. Remarkably, even if the sinusoidal component with frequency $\omega = 2$ does not exist, the corresponding supervisory unit still selects a stabilizing controller without affecting the switching in other loops.

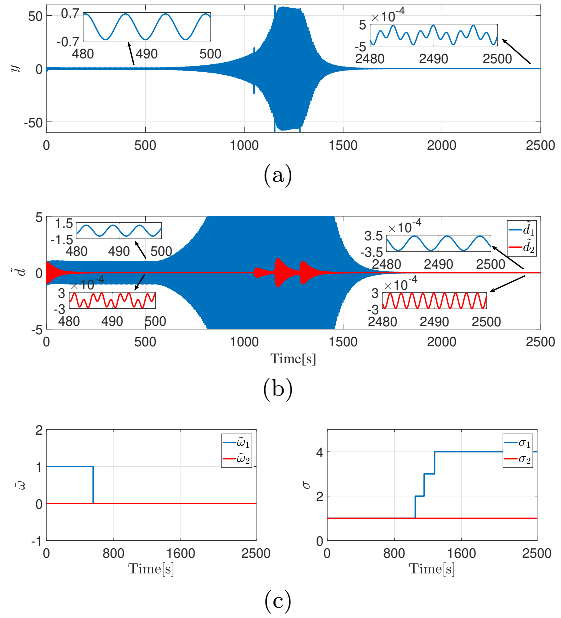

Fig. 4. Case 2): (a) Output trajectory $y(t)$; (b) Error between the disturbance $\hat{d}_i$ and its estimate $\tilde{d}_i$, $i = 1, 2$; (c) Frequency estimation errors $\tilde{\omega}_i$ and switching signals $\sigma_i, i = 1, 2$

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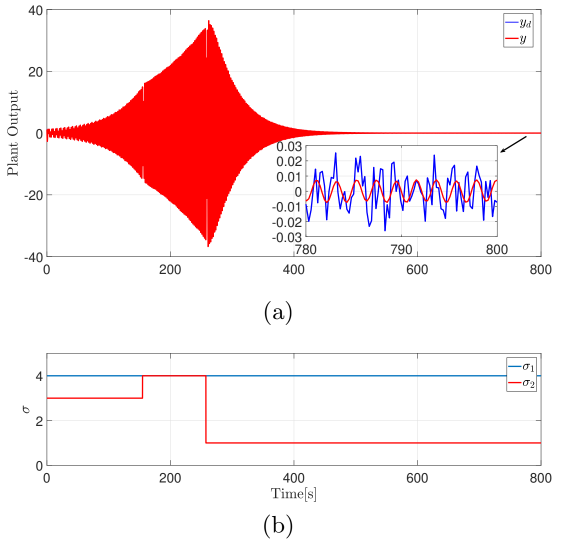

Fig. 5. Case 3): (a) Time history of the output of the plant y and the output with noise y_d; (b) Time history of the switching signals.

7 Concluding Remarks

A switching-based approach has been proposed in this paper to remove a longstanding requirement in the output regulation problem for an uncertain linear system, namely the necessity to impose SPR-like conditions on the transfer function at frequencies of interest. The rationale behind the method is to design an SNE-based switching mechanism to remove destabilizing controllers from a finite set of candidate controllers. The use of notch filters within a decoupled design of the switching mechanism allows the proposed scheme to reject unknown multi-sinusoidal disturbances, whereas the incorporation of a deadbeat frequency estimator allows for uncertainty in the disturbance model to be taken into account as well. The proposed method is also shown to exhibit good robustness features with respect to additive sensor noise. Drawbacks of the approach, which are worthy of future investigation, relating to the complexity of the design of both the switching algorithm and the frequency estimator, especially in the presence of multiple mismatched frequencies in the controllers. The switching logic may also suffer from a slow convergence speed, due to the conservative choice of the tuning parameters.

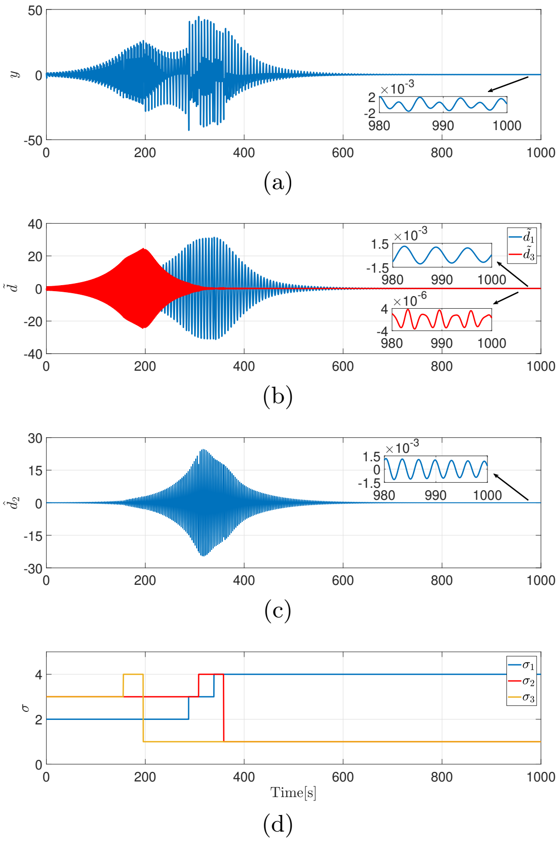

Fig. 6. Case 4): (a) Time history of the output $y(t)$; (b) Time history of the estimation errors $\tilde{d}_1$ and $\tilde{d}_3$; (c) Time history of the estimation for the estimate $\hat{d}_2$ of the non-existing harmonic; (d) Time history of the switching signals.

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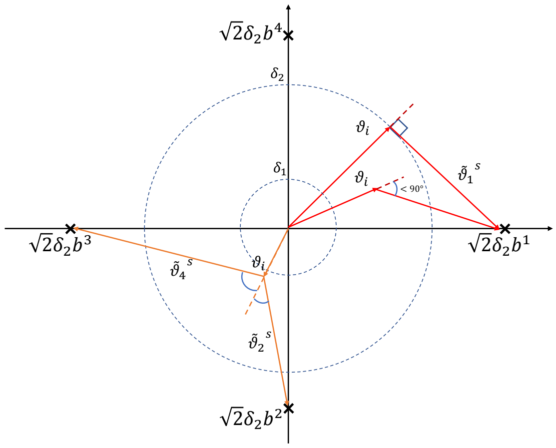

Fig. A.1. Dot product of vectors ϑ and $\tilde{\vartheta}^\sigma, \sigma \in \mathcal{I}$.

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A Proof of Property 3

For the autonomous system

$$
\begin{aligned}
\dot{\xi}_i &= \hat{\omega}_i F(\gamma)\xi_i + k\Upsilon_i b^{s_i}(\Gamma\xi_i + \vartheta_i^\top \zeta_i) + k\Psi_i b^{s_j}\Gamma\Psi_j\zeta_i \\
\dot{\zeta}_i &= \hat{\omega}_i T\zeta_i - kb^{s_i}(\Gamma\xi_i + \vartheta_i^\top \zeta_i), \quad i = 1,2
\end{aligned} \tag{A.1}
$$

rewrite the dynamics of ζ_i as

$$\dot{\zeta}_i = (\hat{\omega}_i T - \frac{k}{\sqrt{2}\delta_2}\vartheta_i\vartheta_i^\top)\zeta_i - \frac{k}{\sqrt{2}\delta_2}\tilde{\vartheta}_i\vartheta_i^\top\zeta_i - kb^{s_i}\Gamma\xi_i$$

with $\tilde{\vartheta}_i := \sqrt{2}\delta_2 b^{s_i} - \vartheta_i$. Owning to Property 1 and Assumption 2, evaluating the time derivative of the Lyapunov function candidate $V_1(\zeta_i) := \zeta_i^\top P_o^i \zeta_i$ along trajectories of system (A.1) yields

$$\dot{V}_1 \leq -\frac{k\delta_1^2}{\sqrt{2}\delta_2}\|\zeta_i\|^2 - \frac{2k}{\sqrt{2}\delta_2}\zeta_i^\top P_o^i\tilde{\vartheta}_i\vartheta_i^\top\zeta_i + 2kc_3\|\zeta_i\|\|\xi_i\|$$

for all $(\vartheta_i, k) \in \Theta \times (0, \sqrt{2}k_0\delta_2]$. Referring to Fig. A.1, it is easy to see that for any $\vartheta_i \in \Theta$, there exists at least one vector b^{σ_i}, among the choices in (10), with index σ_i belonging to the set

$$\mathcal{A}_i^* := \{\sigma_i \in \mathcal{J} : \vartheta_i^\top\tilde{\vartheta}_i^{\sigma_i} \geq 0\}, \tag{A.2}$$

where $\tilde{\vartheta}_i^\sigma := \sqrt{2}\delta_2 b^{\sigma_i} - \vartheta_i$. For $s_i \in \mathcal{A}_i^*$, it follows that

$$\lambda_{\min}(\tilde{\vartheta}_i\vartheta_i^\top) = 0, \quad \lambda_{\max}(\tilde{\vartheta}_i\vartheta_i^\top) = \vartheta_i^\top\tilde{\vartheta}_i \geq 0.$$

Consequently, we have $-\frac{2k}{\sqrt{2}\delta_2}\zeta_i^\top P_o^i\tilde{\vartheta}_i\vartheta_i^\top\zeta_i \leq 0$, so that the application of Young's inequality yields

$$\dot{V}_1 \leq -\frac{k\delta_1^2}{2\sqrt{2}\delta_2}\|\zeta_i\|^2 + \frac{2\sqrt{2}kc_3^2\delta_2}{\delta_1^2}\|\xi_i\|^2. \tag{A.3}$$

For the ξ_i-dynamics, consider the Lyapunov function candidate $V_2(\xi_i) := \xi_i^\top P_f\xi_i$, whose derivative along trajectories of system (A.1) yields

$$\begin{aligned}\dot{V}_2 \leq &-\gamma\hat{\omega}_i\|\xi_i\|^2 + 2k\xi_i^\top P_f\Upsilon_i b^{s_i}\Gamma\xi_i + 2k\xi_i^\top P_f\Upsilon_i b^{s_i}\vartheta_i^\top\zeta_i \\ &+ 2k\xi_i^\top P_f\Psi_i b^{s_j}\Gamma\Psi_j\zeta_i \\ \leq &-(\gamma\hat{\omega}_i - 2\sqrt{2}kc_6\delta_2)\|\xi_i\|^2 + 2kc_6(\sqrt{2}\delta_2^2 + 4c_0^2)\|\xi_i\|\|\zeta_i\|\end{aligned}$$

where we have applied the bound on P_f defined in Property 2 and made use of the fact that

$$\|\Upsilon_i\| \leq \sqrt{2}\delta_2, \quad \|\Psi_i\| \leq \gamma c_0 \leq 2c_0, \quad \forall i \in \mathcal{N}_f,\ \forall\gamma \in (0,2).$$

Finally, we consider the storage function $V(\eta_i) = V_1(\zeta_i) + V_2(\xi_i)$. Using the previously obtained bounds, the derivative of $V(\eta_i)$ along the solution of system (A.1) satisfies

$$\begin{aligned}\dot{V} \leq &-\frac{k\delta_1^2}{2\sqrt{2}\delta_2}\|\zeta_i\|^2 - \left[\gamma\hat{\omega}_i - 2\sqrt{2}k\left(c_6\delta_2 + \frac{c_3^2\delta_2}{\delta_1^2}\right)\right]\|\xi_i\|^2 \\ &+ 2\sqrt{2}k(c_6\delta_2^2 + 2\sqrt{2}c_0)\|\xi_i\|\|\zeta_i\| \\ \leq &-\frac{k\delta_1^2}{4\sqrt{2}\delta_2}\|\zeta_1\|^2 - (\gamma\hat{\omega}_i - 2\sqrt{2}kc_7)\|\xi_i\|^2\end{aligned}$$

for all $(\vartheta_i, k) \in \Theta \times (0, \sqrt{2}\delta_2 k_0]$, where $c_7 := c_6\delta_2 + c_3^2\delta_2/\delta_1^2 + 32\sqrt{2}(c_6\delta_2^2 + 2\sqrt{2}c_0)^2\delta_2/\delta_1^2$. Setting $k_1(\gamma) := \min\{\gamma\underline{\omega}/4\sqrt{2}c_7, k_0\}$, one can easily verify that

$$\gamma\hat{\omega}_i - 2\sqrt{2}kc_7 \geq 2\sqrt{2}kc_7 \geq \frac{k\delta_1^2}{4\sqrt{2}\delta_2}$$

for all $k \in (0, k_1(\gamma)]$.Then, it follows that

$$\dot{V} \leq -\frac{k\delta_1^2}{4\sqrt{2}\delta_2}(\|\zeta_i\|^2 + \|\xi_i\|^2) \leq -k\beta_0\|\eta_i\|^2 \tag{A.4}$$

for all $\gamma \in (0,2)$ and $k \in (0, k_1(\gamma)]$, where $\beta_0 := \frac{\delta_1^2}{4\sqrt{2}\delta_2}$. Let $P_e^i := \frac{1}{\beta_0}\mathrm{diag}(P_o^i, P_f)$, and write the Lyapunov function as $V(\eta_i) = \beta_0\eta_i^\top P_e^i\eta_i$ and its derivative as

$$\dot{V} = \beta_0\eta_i^\top(P_e E_i + E_i^\top P_e)\eta_i$$

Owning to (A.4), one obtains

$$P_e E_i + E_i^\top P_e \leq -k\boldsymbol{I}$$

which yields (26). Due to boundedness of P_o^i, P_f and β_0, there exist positive constants p_1, p_2, p_3 such that

$$p_1\boldsymbol{I} \leq P_e^i \leq p_2\boldsymbol{I}, \quad \|P_e^i\| \leq p_3$$

for all $k \in (0, k_1(\gamma)]$ and all $\gamma \in (0,2)$.

Next, we derive the upper bound on the eigenvalues of E_i. Rewrite (A.4) as

$$\dot{V} \leq -k\alpha_0 V \tag{A.5}$$

with $\alpha_0 := \beta_0/p_3$. Integrating both sides of (A.5) from t_0 to t yields

$$V(t) \leq \mathrm{e}^{-k\alpha_0(t-t_0)}V(t_0) \tag{A.6}$$

for all $t \in [t_0, \infty)$ and $\vartheta_i \in \Theta$. Consequently, system (A.1) is exponentially stable, hence E_i is Hurwitz, if $\sigma_i \in \mathcal{A}_i^*$ and $(k,\gamma) \in (0, k_1(\gamma)] \times (0,2)$.

Finally, we show that the absolute value of the eigenvalues of $E_i, i = 1,2$ is lower bounded by $\frac{1}{2}\alpha_0 k$. Let λ_j and v_j denote the eigenvalues and corresponding eigenvectors of matrix E_i. Setting the initial value $\eta_i(t_0) := v_j$, it follows that

$$\begin{aligned}\eta_i(t) = \mathrm{e}^{E_i(t-t_0)}\eta_i(t_0) &= \sum_{l=1}^{\infty}\frac{(t-t_0)^l}{l!}E_i^l v_j \\ &= \sum_{l=1}^{\infty}\frac{(t-t_0)^l}{l!}\lambda_j^l v_j = \mathrm{e}^{\lambda_j(t-t_0)}\eta_i(t_0)\end{aligned}$$

Using (A.6), one obtains

$$\|\eta_i(t)\| = \mathrm{e}^{\lambda_j(t-t_0)}\|\eta_i(t_0)\| \leq \sqrt{\rho_0}\mathrm{e}^{-0.5\alpha_0 k(t-t_0)}\|\eta_i(t_0)\|$$

with $\rho_0 := p_2/p_1$, which yields

$$\mathrm{e}^{[\lambda_j+0.5\alpha_0 k](t-t_0)} \leq \sqrt{\rho_0}, \quad t \geq t_0 \tag{A.7}$$

Since (A.7) holds for all eigenvalues of E_i, the inequality

$$-\mathrm{Re}\{\max_j\{\lambda_j\}\} \geq 0.5\alpha_0 k \tag{A.8}$$

must hold as well. Therefore, letting $\mathcal{I}_i^* := \mathcal{A}_i^*$ and $\alpha = 0.5\alpha_0$, the set $\mathcal{I}_i^*$ defined in (23) is non-empty for all $k \in (0, k_1(\gamma)]$.

B Proof of Lemma 5

Consider the Lyapunov function candidate $V_i(\eta_i) = \eta_i^\top P_e^i \eta_i$, whose derivative along solutions of (20) reads as

$$\begin{aligned}\dot{V}_i \leq &- k\eta_i^\top\eta_i + 2\eta_i^\top P_e^i H_{i,j}\eta_j + 2\gamma\eta_i^\top P_e^i M_i(Cz \\ &+ \tilde{\omega}_1 \Lambda_1^\top \nu_1 + \tilde{\omega}_2\Lambda_2^\top\nu_2) + 2\tilde{\omega}_i\eta_i^\top P_e^i N' T\nu_i \\ &+ 2\eta_i^\top P_e^i N(\tilde{\omega}_i \Upsilon_i T\nu_i + \tilde{\omega}_j\Psi_i T\nu_j) \\ \leq &- \frac{k}{2}\|\eta_i\|^2 + \nu_i \leq -\frac{k}{2p_2}V_i + \nu_i\end{aligned} \tag{B.1}$$

where

$$\begin{aligned}\nu_i :=& 8k\gamma^2 p_2^2\delta_4^2\bar{\eta}_j + \frac{8}{k}\gamma^2 p_2^2\hat{\omega}_i^2\varrho_C^2\bar{z} + \frac{24}{k}\bar{\omega}^2 p_2^2(1+\delta_2^2 \\ &+ \gamma^2 c_0^2)\bar{a}^2 + \frac{16}{k}\gamma^2 p_2^2\varrho_{M_i}^2\bar{\omega}^2(\varrho_{\Lambda_1}^2 + \varrho_{\Lambda_2}^2)\bar{a}^2\end{aligned}$$

Substituting the identity $\nu_i = \dot{\boldsymbol{\eta}}_i + \frac{k}{2p_2}\boldsymbol{\eta}_i$ from (30) into (B.1), one obtains

$$\dot{V}_i - \dot{\boldsymbol{\eta}}_i \leq -\frac{k}{2p_2}(V - \boldsymbol{\eta}_i)$$

which implies, for all $t \geq T_m^i$,

$$\begin{aligned}V_i(t) &\leq \boldsymbol{\eta}_i(t) + \mathrm{e}^{-\frac{k}{2p_2}(t-T_m^i)}(V(T_m^i) - \boldsymbol{\eta}_i(T_m^i)) \\ &\leq \boldsymbol{\eta}_i(t) + \mathrm{e}^{-\frac{k}{2p_2}(t-T_m^1)}p_2\rho_i^2(T_m^i)\end{aligned}$$

where $\rho_i(T_m^i)$ is the norm-bound on η_i defined in (32). Finally, one obtains

$$\|\eta_i(t)\|^2 \leq \frac{\boldsymbol{\eta}_i(t)}{p_1} + \frac{p_2\rho_i^2(T_m^i)}{p_1}\mathrm{e}^{-\frac{k}{2p_2}(t-T_m^i)}, \quad t \geq T_m^i$$

which verifies the inequality

$$\|\eta_i(t)\|^2 \leq \bar{\eta}_i, \quad t \geq T_m^i \tag{B.2}$$

The condition $\|z(t)\|^2 \leq \bar{z}$, $t \geq T_m^i$, can also be easily verified by considering the Lyapunov function candidate

$$V(z) = z^\top P_x z$$

and analyzing its derivative along the same lines of the proof of (B.2), which is omitted to avoid redundancy.

In summary, by virtue of (19), (21) and (B.2), if $\sigma_i \in \mathcal{I}_i^*$, then the filtered output e_i is norm-bounded by

$$\|e_i(t)\|^2 \leq \|L_i\|^2\bar{\eta}_i + \|L_i'\|^2\bar{\eta}_j \leq (1+\delta_2^2)\bar{\eta}_i + \gamma^2\delta_5^2\bar{\eta}_j := \bar{e}_i$$

for all $t \geq T_m^i$.

C Proof of inequalities (51)

The entries r_{ij} of the matrix $\Upsilon_i' \in \mathbb{R}^{2\times 2}$, solution of the Sylvester equations (47), are given by

$$\begin{aligned}r_{11} &= \gamma\frac{\gamma\varphi_{i,1}' + (1-\Delta_i)\varphi_{i,2}'}{\gamma^2 + (1-\Delta_i^2)}, \quad r_{21} = -\Delta_i r_{12} \\ r_{12} &= \gamma\frac{\gamma\varphi_{j,2}' + (\Delta_i - 1)\varphi_{j,1}'}{\gamma^2 + (1-\Delta_i^2)}, \quad r_{22} = \Delta_i r_{11}\end{aligned}$$

with $\Delta_i = \hat{\omega}_i/\omega_i$ and $\varphi_{i,l}', l = 1, 2$ represents the l-th elements of parameter vector Φ_i defined in (45). The Frobenius norm of Υ_i reads as

$$\begin{aligned}\|\Upsilon_i'\|_F^2 &= \frac{(1+\Delta_i^2)\gamma^2}{(\gamma^2 + (1-\Delta_i)^2)^2}\left[(\gamma\varphi_{j,1}' + (1-\Delta_i)\varphi_{l,2}')^2\right. \\ &\qquad \left. + (\gamma\vartheta_{j,2} + (\Delta_i - 1)\vartheta_{l,1})^2\right] \\ &= \frac{(1+\Delta_i^2)\gamma^2}{(\gamma^2 + (1-\Delta_i)^2)^2}\|\Phi_i\|^2\end{aligned} \tag{C.1}$$

By virtue of Assumption 1, it follows that

$$(1-\Delta_i)^2 = \frac{\tilde{\omega}_i^2}{\omega_i^2} \geq \frac{\tilde{\omega}_i^2}{\bar{\omega}^2}, \quad 1 + \Delta_i^2 \leq 1 + \frac{\hat{\omega}_i^2}{\underline{\omega}^2}$$

Consequently,

$$\begin{aligned}|\tilde{\omega}_i|^2\|\Upsilon_i'\|^2 &\leq (1 + \frac{\hat{\omega}_i^2}{\underline{\omega}^2})\frac{\gamma^2\tilde{\omega}_i^2}{\gamma^2 + \frac{\tilde{\omega}_i^2}{\bar{\omega}^2}}\|\Phi_i\|^2 \\ &\leq \gamma^2(1 + \frac{\hat{\omega}_i^2}{\underline{\omega}^2})\bar{\omega}^2\|\Phi_i\|^2,\end{aligned}$$

which verifies the first inequality in (51) with $r_1 = \sqrt{(1 + \frac{\hat{\omega}_i^2}{\underline{\omega}^2})}\bar{\omega}\|\Phi_i\|$. Similarly, we obtain the norm bound of the solution $\Psi_i' \in \mathbb{R}^{2\times 2}$ of the Sylvester equations (48) as

$$\|\Psi_i\| \leq \frac{(1+\Delta_j^2)\gamma^2}{(\gamma^2 + (1-\Delta_j)^2)}\|\Phi_j\|^2$$

where $\Delta_j = \hat{\omega}_i/\omega_j$. From $|\omega_i - \omega_j| \geq \Delta\omega$, it follows that

$$(1-\Delta_j)^2 = \frac{(\hat{\omega}_i - \omega_j)^2}{\omega_i^2} \geq \frac{\Delta\omega^2}{4\bar{\omega}^2}$$

Therefore, the norm of $\tilde{\omega}_j \Psi_i'$ is bounded by

$$|\tilde{\omega}_j| \|\Psi_i\| \leq |\bar{\omega}| \|\Psi_i\|_F \leq \gamma r_2$$

with $r_2 = \bar{\omega}\sqrt{1 + \frac{\hat{\omega}_i^2}{\underline{\omega}^2} \frac{4\bar{\omega}}{\Delta\omega}} \|\Phi_j\|$.

D Proof of Lemma 7

By virtue of Lemma 6, each active controller either stabilizes the closed-loop system or introduces a pair of eigenvalues on the imaginary axis. Let $\bar{k} := \min\{k^{**}, \bar{\gamma}/\gamma^*\}$ and $\bar{\gamma} := \min\{2, \gamma^{**}\}$ and recall the dynamic equation of the active controller and corresponding filter given in (9), where $i \in \mathcal{N}_f$, $j = \sigma_i$. As the system approaches steady state, the input of each active controller becomes bounded by (52). The sinusoidal signal generated by the active controller admits the form

$$\hat{d}_i(t) = \sin(\hat{\omega}_i t + \phi) + \epsilon_i(t)$$

where ϵ_i is an exponentially decay term, ultimately bounded by ε_0. Consequently, it is impossible for the active controller C_i^j to introduce a pair of eigenvalues on imaginary axis other than $\pm j\hat{\omega}_i$. On the other hand, if C_i^j does introduce a pair of eigenvalues on imaginary axis equal to $j\hat{\omega}_i$, then the output $y(t)$ will have a sinusoidal component with frequency equals to $\hat{\omega}_i$ which will filtered out by the notch filter, thus the condition (52) will not be verified. This contradiction proves that, if the condition (52) is verified for the possible longest period T_f, the active controllers are guaranteed to be stabilizing.