Example 12

The Sturm-Liouville problem of example 9 has the set of orthogonal characteristic functions $\{\phi_n\}$, where $\phi_n(x) = c_n \sin nx$ (n = 1, 2, 3,...... ; $0 \le x \le \pi$) and c_n (n = 1, 2, 3....) are nonzero constants. We now form the sequence of orthonormal characteristic functions $\{k_n\phi_n\}$, where k_n is defined as above. We have

$$K_n = \int_0^{\pi} (c_n \ \sin nx)^2 \ (1)\,dx = \frac{c_n^2\pi}{2},$$

$$k_n = \frac{1}{\sqrt{K_n}} = \frac{1}{c_n}\sqrt{\frac{2}{\pi}},$$

$$k_n\phi_n(x) = \left(\frac{1}{c_n}\sqrt{\frac{2}{\pi}}\right) (c_n \ \sin nx) = \sqrt{\frac{2}{\pi}} \ \sin nx \qquad (n = 1, 2, 3.....)$$

Thus the Sturm-Liouville problem under consideration has the set of orthonormal characteristic functions $\{\psi_n\}$, where $\psi_n(x) = \sqrt{2/\pi}\ \sin nx$ (n = 1, 2, 3, ...; $0 \le x \le \pi$). We see that this is the set of orthonormal functions considered in