Functions

Quadratic functions

Objectives

At the end of this section you should be able to:

- Solve a quadratic equation using 'the formula'.
- Solve a quadratic equation given its factorization.
- Sketch the graph of a quadratic function using a table of function values.
- Sketch the graph of a quadratic function by finding the coordinates of the intercepts.
- Determine equilibrium price and quantity given a pair of quadratic demand and supply functions.

The simplest non-linear function is known as a ***quadratic*** and takes the form

$$f(x) = ax^2 + bx + c$$

for some parameters a, b and c. (In fact, even if the demand function is linear, functions derived from it, such as total revenue and profit, turn out to be quadratic. We investigate these functions in the next section.) For the moment we concentrate on the mathematics of quadratics and show how to sketch graphs of quadratic functions and how to solve quadratic equations.

Consider the elementary equation

$$x^2 - 9 = 0$$

It is easy to see that the expression on the left-hand side is a special case of the above with $a = 1$, $b = 0$ and $c = -9$. To solve this equation we add 9 to both sides to get

$$x^2 = 9$$

x^2 is an abbreviation for $x \times x$

so we need to find a number, x, which when multiplied by itself produces the value 9. A moment's thought should convince you that there are exactly two numbers that work, namely 3 and −3 because

$$3 \times 3 = 9 \quad \text{and} \quad (-3) \times (-3) = 9$$

These two solutions are called the ***square roots*** of 9. The symbol √ is reserved for the positive square root, so in this notation the solutions are $\sqrt{9}$ and $-\sqrt{9}$. These are usually combined and written $\pm\sqrt{9}$. The equation

$$x^2 - 9 = 0$$

is trivial to solve because the number 9 has obvious square roots. In general, it is necessary to use a calculator to evaluate square roots. For example, the equation

$$x^2 - 2 = 0$$

can be written as

$$x^2 = 2$$

and so has solutions $x = \pm\sqrt{2}$. My calculator gives 1.414 213 56 (correct to 8 decimal places) for the square root of 2, so the above equation has solutions

$$1.414\,213\,56 \quad \text{and} \quad -1.414\,213\,56$$

Example

Solve the following quadratic equations:

(a) $5x^2 - 80 = 0$ **(b)** $x^2 + 64 = 0$ **(c)** $(x + 4)^2 = 81$

Solution

(a) $5x^2 - 80 = 0$

$$5x^2 = 80 \quad \text{(add 80 to both sides)}$$

$$x^2 = 16 \quad \text{(divide both sides by 5)}$$

$$x = \pm 4 \quad \text{(square root both sides)}$$

(b) $x^2 + 64 = 0$

$$x^2 = -64 \quad \text{(subtract 64 from both sides)}$$

This equation does not have a solution because you cannot square a real number and get a negative answer.

(c) $(x + 4)^2 = 81$

$$x + 4 = \pm 9 \quad \text{(square root both sides)}$$

The two solutions are obtained by taking the + and – signs separately. Taking the + sign,

$$x + 4 = 9 \quad \text{so} \quad x = 9 - 4 = 5$$

Taking the – sign,

$$x + 4 = -9 \quad \text{so} \quad x = -9 - 4 = -13$$

The two solutions are 5 and –13.

Problem

1 Solve the following quadratic equations. (Round your solutions to 2 decimal places if necessary.)

(a) $x^2 - 100 = 0$ **(b)** $2x^2 - 8 = 0$ **(c)** $x^2 - 3 = 0$ **(d)** $x^2 - 5.72 = 0$

(e) $x^2 + 1 = 0$ **(f)** $3x^2 + 6.21 = 0$ **(g)** $x^2 = 0$

All of the equations considered in Problem 1 are of the special form

$$ax^2 + c = 0$$

in which the coefficient of x is zero. To solve more general quadratic equations we use a formula that enables the solutions to be calculated in a few lines of working. It can be shown that

$$ax^2 + bx + c = 0$$

has solutions

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

The following example describes how to use this formula. It also illustrates the fact (which you have already discovered in Practice Problem 1) that a quadratic equation can have two solutions, one solution or no solutions.

Example

Solve the quadratic equations

(a) $2x^2 + 9x + 5 = 0$

(b) $x^2 - 4x + 4 = 0$

(c) $3x^2 - 5x + 6 = 0$

Solution

(a) For the equation

$$2x^2 + 9x + 5 = 0$$

we have $a = 2$, $b = 9$ and $c = 5$. Substituting these values into the formula

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

gives

$$x = \frac{-9 \pm \sqrt{(9^2 - 4(2)(5))}}{2(2)}$$

$$= \frac{-9 \pm \sqrt{(81 - 40)}}{4}$$

$$= \frac{-9 \pm \sqrt{41}}{4}$$

The two solutions are obtained by taking the + and – signs separately: that is,

$$\frac{-9 + \sqrt{41}}{4} = -0.649 \quad \text{(correct to 3 decimal places)}$$

$$\frac{-9 - \sqrt{41}}{4} = -3.851 \quad \text{(correct to 3 decimal places)}$$

It is easy to check that these are solutions by substituting them into the original equation. For example, putting $x = -0.649$ into

$$2x^2 + 9x + 5$$

gives

$$2(-0.649)^2 + 9(-0.649) + 5 = 0.001\ 402$$

which is close to zero, as required. We cannot expect to produce an exact value of zero because we rounded $\sqrt{41}$ to 3 decimal places. You might like to check for yourself that –3.851 is also a solution.

(b) For the equation

$$x^2 - 4x + 4 = 0$$

we have $a = 1$, $b = -4$ and $c = 4$. Substituting these values into the formula

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

gives

$$x = \frac{-(-4) \pm \sqrt{((-4^2) - 4(1)(4))}}{2(1)}$$

$$= \frac{4 \pm \sqrt{(16 - 16)}}{2}$$

$$= \frac{4 \pm \sqrt{0}}{2}$$

$$= \frac{4 \pm 0}{2}$$

Clearly we get the same answer irrespective of whether we take the + or the – sign here. In other words, this equation has only one solution, $x = 2$. As a check, substitution of $x = 2$ into the original equation gives

$$(2)^2 - 4(2) + 4 = 0$$

(c) For the equation

$$3x^2 - 5x + 6 = 0$$

we have $a = 3$, $b = -5$ and $c = 6$. Substituting these values into the formula

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

gives

$$x = \frac{-(-5) \pm \sqrt{((-5^2) - 4(3)(6))}}{2(3)}$$

$$= \frac{5 \pm \sqrt{(25 - 72)}}{6}$$

$$= \frac{5 \pm \sqrt{(-47)}}{6}$$

The number under the square root sign is negative and, as you discovered in Practice Problem 1, it is impossible to find the square root of a negative number. We conclude that the quadratic equation

$$3x^2 - 5x + 6 = 0$$

has no solutions.

This example demonstrates the three cases that can occur when solving quadratic equations. The precise number of solutions that an equation can have depends on whether the number under the square root sign is positive, zero or negative. The number $b^2 - 4ac$ is called the ***discriminant*** because the sign of this number discriminates between the three cases that can occur.

- If $b^2 - 4ac > 0$ then there are two solutions

$$x = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

- If $b^2 - 4ac = 0$ then there is one solution

$$x = \frac{-b \pm \sqrt{0}}{2a} = \frac{-b}{2a}$$

- If $b^2 - 4ac < 0$ then there are no solutions because $\sqrt{(b^2 - 4ac)}$ does not exist.

Practice Problem

2 Solve the following quadratic equations (where possible):

(a) $2x^2 - 19x - 10 = 0$ **(b)** $4x^2 + 12x + 9 = 0$

(c) $x^2 + x + 1 = 0$ **(d)** $x^2 - 3x + 10 = 2x + 4$

You may be familiar with another method for solving quadratic equations. This is based on the factorization of a quadratic into the product of two linear factors. Section 1.4 described how to multiply out two brackets. One of the examples in that section showed that

$$(x + 1)(x + 2) = x^2 + 3x + 2$$

Consequently, the solutions of the equation

$$x^2 + 3x + 2 = 0$$

are the same as those of

$$(x + 1)(x + 2) = 0$$

Now the only way that two numbers can be multiplied together to produce a value of zero is when (at least) one of the numbers is zero.

if $ab = 0$ then either $a = 0$ or $b = 0$ (or both)

It follows that either

$x + 1 = 0$ with solution $x = -1$

or

$x + 2 = 0$ with solution $x = -2$

The quadratic equation

$$x^2 + 3x + 2 = 0$$

therefore has two solutions, $x = -1$ and $x = -2$.

The difficulty with this approach is that it is impossible, except in very simple cases, to work out the factorization from any given quadratic, so the preferred method is to use the formula. However, if you are lucky enough to be given the factorization, or perhaps clever enough to spot the factorization for yourself, then it does provide a viable alternative.

Example

Write down the solutions to the following quadratic equations:

(a) $x(3x - 4) = 0$ **(b)** $(x - 7)^2 = 0$

Solution

(a) If $x(3x - 4) = 0$ then either $x = 0$ or $3x - 4 = 0$

The first gives the solution $x = 0$ and the second gives $x = 4/3$.

(b) If $(x - 7)(x - 7) = 0$ then either $x - 7 = 0$ or $x - 7 = 0$

Both options lead to the same solution, $x = 7$.

Practice Problem

3 Write down the solutions to the following quadratic equations. (There is no need to multiply out the brackets.)

(a) $(x - 4)(x + 3) = 0$

(b) $x(10 - 2x) = 0$

(c) $(2x - 6)^2 = 0$

One important feature of linear functions is that their graphs are always straight lines. Obviously the intercept and slope vary from function to function, but the shape is always the same. It turns out that a similar property holds for quadratic functions. Now, whenever you are asked to produce a graph of an unfamiliar function, it is often a good idea to tabulate the function, to plot these points on graph paper and to join them up with a smooth curve. The precise number of points to be taken depends on the function but, as a general rule, between 5 and 10 points usually produce a good picture.

Example

Sketch a graph of the square function, $f(x) = x^2$.

Solution

A table of values for the simple square function

$$f(x) = x^2$$

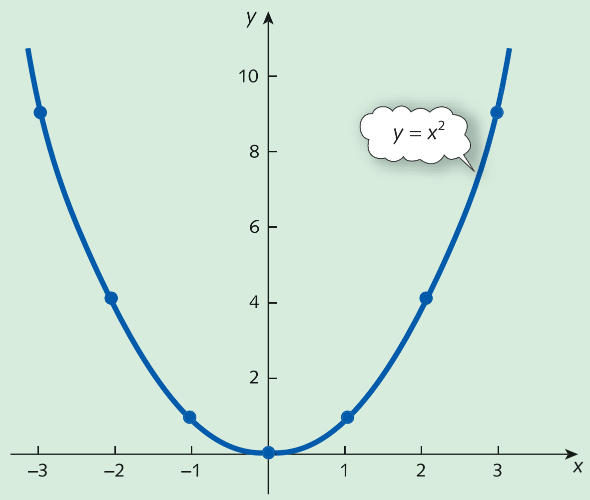
is given by

x	−3	−2	−1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9

The first row of the table gives a selection of 'incoming' numbers, x, while the second row shows the corresponding 'outgoing' numbers, y. Points with coordinates (x, y) are then plotted on graph paper to produce the curve shown in Figure 2.1. For convenience, different scales are used on the x and y axes.

Mathematicians call this curve a ***parabola***, whereas economists refer to it as ***U-shaped***. Notice that the graph is symmetric about the y axis with a minimum point at the origin; if a mirror is placed along the y axis then the left-hand part is the image of the right-hand part.

Figure 2.1



Advice

The following problem is designed to give you an opportunity to tabulate and sketch graphs of more general quadratic functions. Please remember that when you substitute numbers into a formula you must use BIDMAS to decide the order of the operations. For example, in part (a) you need to substitute $x = -1$ into $4x^2 - 12x + 5$. You get

$$4(-1)^2 - 12(-1) + 5$$
$$= 4 + 12 + 5$$
$$= 21$$

Note also that when using a calculator you must use brackets when squaring negative numbers. In this case a possible sequence of key presses might be

$\boxed{4}$ $\boxed{(}$ $\boxed{(-)}$ $\boxed{1}$ $\boxed{)}$ $\boxed{x^2}$ $\boxed{-}$ $\boxed{12}$ $\boxed{\times}$ $\boxed{(-)}$ $\boxed{1}$ $\boxed{+}$ $\boxed{5}$ $\boxed{=}$

Practice Problem

4 Complete the following tables of function values and hence sketch a graph of each quadratic function.

(a) $f(x) = 4x^2 - 12x + 5$

x	-1	0	1	2	3	4
$f(x)$						

(b) $f(x) = -x^2 + 6x - 9$

x	0	1	2	3	4	5	6
$f(x)$							

(c) $f(x) = -2x^2 + 4x - 6$

x	-2	-1	0	1	2	3	4
$f(x)$							

The results of Practice Problem 4 suggest that the graph of a quadratic is always parabolic. Furthermore, whenever the coefficient of x^2 is positive, the graph bends upwards and is a 'happy' parabola (U shape). A selection of U-shaped curves is shown in Figure 2.2. Similarly, when the coefficient of x^2 is negative, the graph bends downwards and is a 'sad' parabola (inverted U shape). A selection of inverted U-shaped curves is shown in Figure 2.3.

The task of sketching graphs from a table of function values is extremely tedious, particularly if only a rough sketch is required. It is usually more convenient just to determine a few key points on the curve. The obvious points to find are the intercepts with the coordinate axes, since these enable us to 'tether' the parabola down in the various positions shown in Figures 2.2 and 2.3. The curve crosses the y axis when $x = 0$. Evaluating the function

$$f(x) = ax^2 + bx + c$$

Figure 2.2

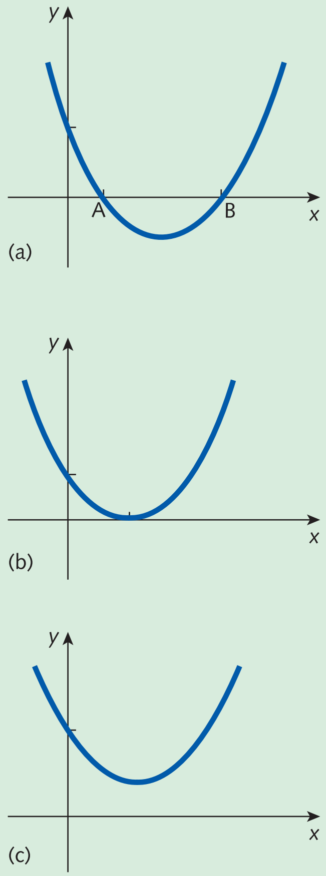
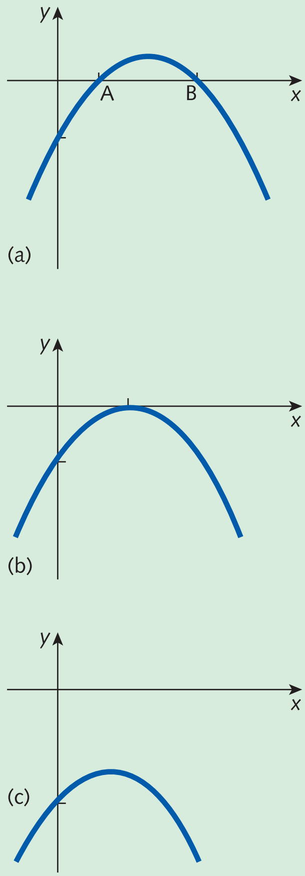


Figure 2.3



at $x = 0$ gives

$$f(0) = a(0)^2 + b(0) + c = c$$

so the constant term determines where the curve cuts the vertical axis (as it did for linear functions). The curve crosses the x axis when $y = 0$ or, equivalently, when $f(x) = 0$, so we need to solve the quadratic equation

$$ax^2 + bx + c = 0$$

This can be done using 'the formula' and the solutions are the points where the graph cuts the horizontal axis. In general, a quadratic equation can have two, one or no solutions and these possibilities are illustrated in cases (a), (b) and (c) in Figures 2.2 and 2.3. In case (a) the curve crosses the x axis at A, turns round and crosses it again at B, so there are two solutions. In case (b) the curve turns round just as it touches the x axis, so there is only one solution. Finally, in case (c) the curve turns round before it has a chance to cross the x axis, so there are no solutions.

The strategy for sketching the graph of a quadratic function

$$f(x) = ax^2 + bx + c$$

may now be stated.

Step 1

Determine the basic shape. The graph has a U shape if $a > 0$, and an inverted U shape if $a < 0$.

Step 2

Determine the y intercept. This is obtained by substituting $x = 0$ into the function, which gives $y = c$.

Step 3

Determine the x intercepts (if any). These are obtained by solving the quadratic equation

$$ax^2 + bx + c = 0$$

This three-step strategy is illustrated in the following example.

Example

Give a rough sketch of the graph of the following quadratic function:

$$f(x) = -x^2 + 8x - 12$$

Solution

For the function

$$f(x) = -x^2 + 8x - 12$$

the strategy is as follows.

Step 1

The coefficient of x^2 is -1, which is negative, so the graph is a 'sad' parabola with an inverted U shape.

Step 2

The constant term is -12, so the graph crosses the vertical axis at $y = -12$.

Step 3

For the quadratic equation

$$-x^2 + 8x - 12 = 0$$

the formula gives

$$x = \frac{-8 \pm \sqrt{(8^2 - 4(-1)(-12))}}{2(-1)} = \frac{-8 \pm \sqrt{(64 - 48)}}{-2}$$

$$= \frac{-8 \pm \sqrt{16}}{-2} = \frac{-8 \pm 4}{-2}$$

so the graph crosses the horizontal axis at

$$x = \frac{-8 + 4}{-2} = 2$$

and

$$x = \frac{-8 - 4}{-2} = 6$$

The information obtained in steps 1–3 is sufficient to produce the sketch shown in Figure 2.4.

In fact, we can go even further in this case and locate the coordinates of the turning point – that is, the maximum point – on the curve. By symmetry, the x coordinate of this point occurs exactly halfway between $x = 2$ and $x = 6$: that is, at

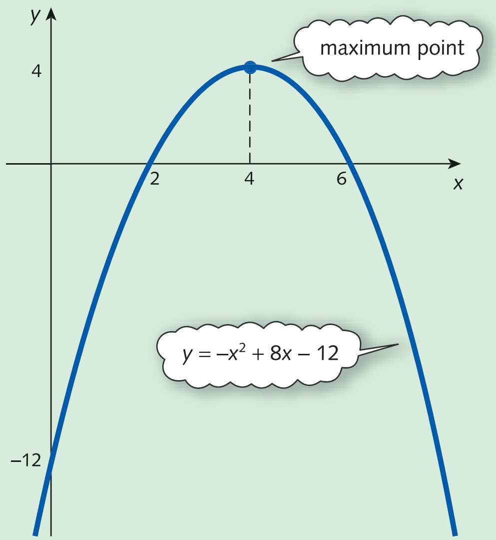
$$x = \tfrac{1}{2}(2 + 6) = 4$$

The corresponding y coordinate is found by substituting $x = 4$ into the function to get

$$f(4) = -(4)^2 + 8(4) - 12 = 4$$

The maximum point on the curve therefore has coordinates (4, 4).

Figure 5.2



Practice Problem

5 Use the three-step strategy to produce rough graphs of the following quadratic functions:

(a) $f(x) = 2x^2 - 11x - 6$ **(b)** $f(x) = x^2 - 6x + 9$

We conclude this section by seeing how to solve a particular problem in microeconomics. In Section 1.3 the concept of market equilibrium was introduced and in each of the problems the supply and demand functions were always given to be linear. The following example shows this to be an unnecessary restriction and indicates that it is almost as easy to manipulate quadratic supply and demand functions.

Example

Given the supply and demand functions

$$P = Q_S^2 + 14Q_S + 22$$
$$P = -Q_D^2 - 10Q_D + 150$$

calculate the equilibrium price and quantity.

Solution

In equilibrium, $Q_S = Q_D$, so if we denote this equilibrium quantity by Q, the supply and demand functions become

$$P = Q^2 + 14Q + 22$$
$$P = -Q^2 - 10Q + 150$$

Hence

$$Q^2 + 14Q + 22 = -Q^2 - 10Q + 150$$

since both sides are equal to P. Collecting like terms gives

$$2Q^2 + 24Q - 128 = 0$$

which is just a quadratic equation in the variable Q. Before using the formula to solve this it is a good idea to divide both sides by 2 to avoid large numbers. This gives

$$Q^2 + 12Q - 64 = 0$$

and so

$$Q = \frac{-12 \pm \sqrt{((12^2) - 4(1)(-64))}}{2(1)}$$
$$= \frac{-12 \pm \sqrt{(400)}}{2}$$
$$= \frac{-12 \pm 20}{2}$$

The quadratic equation has solutions $Q = -16$ and $Q = 4$. Now the solution $Q = -16$ can obviously be ignored because a negative quantity does not make sense. The equilibrium quantity is therefore 4. The equilibrium price can be calculated by substituting this value into either the original supply or demand equation.

From the supply equation,

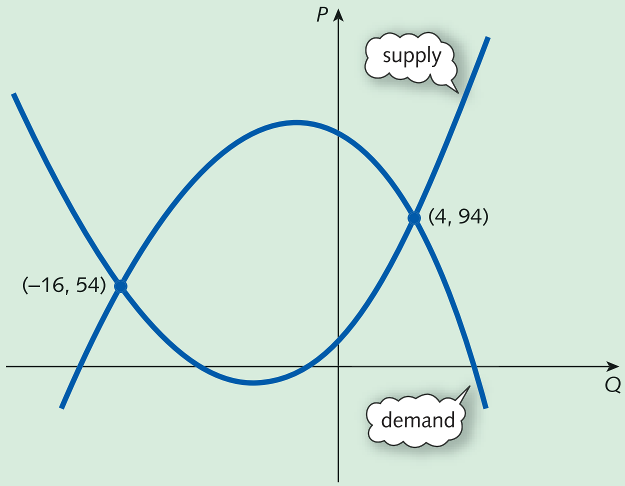
$$P = 4^2 + 14(4) + 22 = 94$$

As a check, the demand equation gives

$$P = -(4)^2 - 10(4) + 150 = 94 \quad ✓$$

You might be puzzled by the fact that we actually obtain two possible solutions, one of which does not make economic sense. The supply and demand curves are sketched in Figure 2.5. This shows that there are indeed two points of intersection confirming the mathematical solution. However, in economics the quantity and price are both positive, so the functions are only defined in the top right-hand (that is, positive) quadrant. In this region there is just one point of intersection, at (4, 94).

Figure 2.5



Practice Problem

6 Given the supply and demand functions

$$P = 2Q_S^2 + 10Q_S + 10$$

$$P = -Q_D^2 - 5Q_D + 52$$

calculate the equilibrium price and quantity.

Key Terms

Discriminant The number $b^2 - 4ac$ which is used to indicate the number of solutions of the quadratic equation $ax^2 + bx + c = 0$.

Parabola The shape of the graph of a quadratic function.

Quadratic function A function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$.

Square root A number that when multiplied by itself equals a given number; the solutions of the equation $x^2 = c$ which are written $\pm\sqrt{c}$.

U-shaped curve A term used by economists to describe a curve, such as a parabola, which bends upwards, like the letter U.