Numerical Approximation of PDEs by Finite Differences and Finite Volumes
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Exercise sheet 2 05. May 2023

Exercise 1

Consider the ODE

$$\frac{d}{dt}x(t) = \lambda x(t)$$
$$x(t_0) = x_0,$$

for some $\lambda \in \mathbb{C}$. Set $t_n = t_0 + n\Delta t$ for some $\Delta t > 0, n = 1, 2, \dots$. Consider the following methods to propagate from t_n to t_{n+1}:

- Euler method:
$$x_{n+1} = x_n + \Delta t \lambda x_n$$

- Implicit Euler method:
$$x_{n+1} = x_n + \Delta t \lambda x_{n+1}$$

Determine the regions of stability for both methods. Assume that λ lies in the left half-plane, i.e., it has a negative real part. Under what conditions on Δt is the implicit Euler method stable? Assume that λ is purely imaginary, i.e., it has a zero real part. Is Euler's method a good choice for solving the Cauchy problem defined above?

Exercise 2

Consider a p-th order accurate summation by part operator given by a grid $\underline{x}$, a first derivative operator D and a mass matrix M.

- Assume that M is diagonal and show that $\underline{1}^T M \underline{x}^{2q} = \frac{1}{2q+1}(x_{\max}^{2q+1} - x_{\min}^{2q+1})$ for $q \in \{1, \dots, p-1\}$.

  Remark: This shows that M induces a quadrature rule that integrates monomials of even degrees $\leq 2p-1$ exactly. It, hence, completes the proof of theorem 1.15 in the lecture notes.

- Assuming that M is not necessary diagonal, show that $\underline{1}^T M$ still yields a quadrature rule that is at least $(p-1)$-th order accurate.

Exercise 3

Consider the problem

$$\begin{aligned}
\partial_t u(t,x) + a\partial_x u(t,x) &= 0 \qquad \text{in } (0,T)\times[0,2]\\
u(0,x) &= \sin(\pi x) \qquad \text{for } x\in[-1,1]\\
u(t,0) &= -\sin(\pi t) \qquad \text{for } t\in[0,T]
\end{aligned}$$

for some $a = 1/2$. Solve this equation using the SBP operator from example 1.10 and a third or fourth order Runge-Kuta method. Conduct convergence studies where you consider as a metric the L^2 error, i.e.,

$$\|u - u_{\text{approximate}}\|_{L^2} = \left(\int_0^2 \left(u(x) - u_{\text{approximate}}(x)\right)^2\, dx\right)^{1/2}.$$

In the convergence studies differentiate between the transient region $\mathrm{TR} = \{x \in [0,2] \mid x > at\}$ and the asymptotic region $\mathrm{AR} = [0,2]\setminus \mathrm{TR}$. Repeat the convergence study for different values of $a > 0$ and a higher-order SBP operator.

Hint: To compute the error remember that the mass matrix M induces a quadrature rule. You can impose the boundary condition either weakly or strongly. An example of a weak imposition is given in the lecture notes.

Exercise 4

Consider the system of equations

$$\begin{aligned}
\partial_t u(t,x) + \partial_x v(t,x) &= 0 \qquad \text{in } (0,T)\times(0,1)\\
\partial_t v(t,x) - \partial_x u(t,x) &= 0 \qquad \text{in } (0,T)\times(0,1)\\
u(0,x) &= \sin(2\pi x) \qquad \text{for } x\in[0,1]\\
v(0,x) &= -\sin(2\pi x) \qquad \text{for } x\in[0,1]\\
u(t,1) &= v(t,0) \qquad \text{for } t\in[0,T]\\
v(t,1) &= u(t,1) \qquad \text{for } t\in[0,T]
\end{aligned}$$

Obtain an energy estimate for this problem. Solve the problem using an SBP discretization in space and a third- or fourth-order Runge-Kuta method. Impose the boundary conditions weakly.