Numerical Approximation of PDEs by Finite Differences and Finite Volumes
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Exercise sheet 1 19. April 2023

Exercise 1

Let $u : \mathbb{R} \supset [x_{\min}, x_{\max}] \to \mathbb{R}$ be a smooth function. For a fixed $\Delta x > 0$ consider the following finite-difference approximations of various derivatives of u

1. Forward difference approximation of the first derivative of u

$$D_1 u(x) = \frac{u(x+\Delta x) - u(x)}{\Delta x}. \tag{1}$$

2. Central difference approximation of the second derivative of u

$$D_2 u(x) = \frac{-\frac{1}{12}u(x-2\Delta x) + \frac{4}{3}u(x-\Delta x) - \frac{5}{2}u(x) + \frac{4}{3}u(x+\Delta x) - \frac{1}{12}u(x+2\Delta x)}{\Delta x^2}.$$

Determine the order of accuracy (in the sense of definition 1.1 from the lecture notes) of the above approximations. Compare the order of accuracy of (1) with that of the center difference scheme (example 1.2 in lecture notes).

Exercise 2

Under the settings of exercise 1 consider the following finite-difference approximation of the third derivative of u

$$D_3 u(x) = \frac{1}{\Delta x^3}\left(a_1 u(x-2\Delta x) + a_2 u(x-\Delta x) + a_3 u(x+\Delta x) + a_4 u(x+2\Delta x)\right)$$

with the coefficients $a_1, a_2, a_3, a_4 \in \mathbb{R}$. Determine the values of these coefficients for the approximation to have a second-order accuracy.

Exercise 3

Write a code to compute the numerical derivatives D_1, D_2, D_3 above of the function $u : [-\pi, \pi] \to \mathbb{R}$ given by

$$u(x) = \sin(10x)\cos(5x)\exp(-x^2)x^5.$$

Analyze the convergence of the approximate derivatives as $\Delta x \to 0$ by plotting, on a logarithmic scale, the root-mean-squared error of the approximations as a function of Δx.

Exercise 4

Write a program to solve one of the two following problems.

1. Consider the linear advection equation with periodic boundary conditions introduced in section 1.1 in the lecture notes

$$\begin{aligned} \partial_t u(t,x) + \partial_x u(t,x) &= 0 \qquad \text{in } (0,T) \times [-1,1] \\ u(0,x) &= \sin(\pi x) \qquad \text{for } x \in [-1,1]. \end{aligned}$$

   Solve this equation using a second-order accurate central difference method to approximate the derivative in space and a third-order accurate Runge-Kuta method with uniform time step size to numerically integrate in time. Plot the root mean squared error of the approximation as a function of Δx for different time steps.

2. Consider the infinite-dimensional eigenvalue problem

$$-\frac{1}{2}u_i''(x) + \frac{1}{2}x^2 u_i(x) = E_i u_i(x)$$

   on the domain of interest $[-5,5]$. Solve this equation, i.e., find approximate eigenpairs (u_i, E_i), using a fourth-order accurate central difference approximation of u_i''. Make a table with the absolute error in the smallest 5 eigenvalues and L^2 errors in the computation of their corresponding eigenvectors for several values of Δx.

   Hint: Use a finite difference formula to approximate the operator $\frac{d}{dx^2}$ and compute the approximated operator on a grid. Use your favorite eigensolver to diagonalize the resulting matrix. For computing the errors, note that the exact eigenvalues are given by $E_i = 1/2 + n, n = 0, 1, 2, \dots$ and the exact eigenfunctions are the so-called Hermite functions. Normalize the approximate eigenfunctions in L^2 before comparison with the exact solutions.