# The Penney Ante Problem

## *Analysis using Markov Chains, Martingales, and Combinatoric Techniques*

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## Abstract

Imagine a two player game where each player is assigned a sequence, for example THTH and HTHH, and a coin is flipped until either player sees their sequence. The first player to see their sequence appear wins. Given two sequences, which sequence is expected to come first in the sequence of coin flips? What is the probability of a certain player winning? Although these two questions sound similar, the result is that in our example, the expected number of turns for sequence A to appear is 20 and for B it is 18. Meanwhile, the probability of A winning is 9/14 while the probability of B winning is 5/14. Additionally the game has the property that for any sequence A chooses, B can always find a sequence that has a higher probability of winning. These counterintuitive results are the core of the Penney Ante problem, discovered by Walter Penney [3]. My thesis will study this problem through three approaches based in Markov chains, martingales, and a combinatoric approach.

## 1 The Game

Let $(S_t)$ for $t \geq 1$ be a stochastic sequence of letters chosen uniformly and randomly from a $q$-letter alphabet. Let $A = a_1a_2a_3...$ and $B = b_1b_2b_3...$ be sequences of $n$ letters chosen from the $q$-letter alphabet. We say sequence $A$ or $B$ wins if it is the first sequence to appear within $(S_t)$. Let $\tau_A$ and $\tau_B$ be the number of turns for $A$ or $B$ to appear. Note that $\tau$ is not a random variable. We denote the probability of $A$ winning as $P(\tau_A < \tau_B)$ and we denote the expected time for sequence $A$ and $B$ to appear as $E(\tau_A)$ and $E(\tau_B)$.

## 2 Markov Chains

We can use Markov chains, the assignment of probabilities connecting states of the game, as one approach to study this problem.

**Definition 2.1** ([7]). *A Markov chain is a stochastic sequence such that*

$$P(x = X_{k+1}|X_1X_2...X_k) = P(x = X_{k+1}|X_k)$$

*That is, the probability that an event occurs given the entire history of previous events is only dependent on the most recent event. Denote the probability that event $y$ occurs given $x$ as $P(x, y)$.*

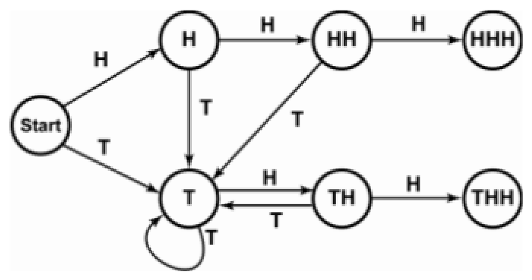


**Figure 1:** Markov chain for the probability of HHH or THH appearing first [6]

To visualize the game, consider the above graph where each vertex represents the current running for who is winning and each edge represents the probability of getting H or T. Since the probability of each state only depends on the previous state, we can calculate the probability $P(\tau_A < \tau_B)$ by writing a system of $n + 1$ equations.

## 3 Non Transitivity

This game has the property that no matter what sequence A chooses, player B can choose a $B$ such that $P(\tau_B < \tau_A) > \frac{1}{2}$. This is called non transitivity.

Guibas and Odlyzko point out that for large $n$, $a_1$ can be chosen so that player B has odds of beating player A of $\frac{q}{q-1}$. [4]

| B \ A | HHH | HHT | HTH | HTT | THH | THT | TTH | TTT |
|---|---|---|---|---|---|---|---|---|
| HHH | | 1/2 | 2/5 | 2/5 | 1/8 | 5/12 | 3/10 | 1/2 |
| HHT | 1/2 | | 2/3 | 2/3 | 1/4 | 5/8 | 1/2 | 7/10 |
| HTH | 3/5 | 1/3 | | 1/2 | 1/2 | 1/2 | 3/8 | 7/12 |
| HTT | 3/5 | 1/3 | 1/2 | | 1/2 | 1/2 | 3/4 | 7/8 |
| THH | 7/8 | 3/4 | 1/2 | 1/2 | | 1/2 | 1/3 | 3/5 |
| THT | 7/12 | 3/8 | 1/2 | 1/2 | 1/2 | | 1/3 | 3/5 |
| TTH | 7/10 | 1/2 | 5/8 | 1/4 | 2/3 | 2/3 | | 1/2 |
| TTT | 1/2 | 3/10 | 5/12 | 1/8 | 2/5 | 2/5 | 1/2 | |

**Figure 2:** A table showing $P(\tau_B < \tau_A)$. As an exercise, pick an A (a column), and try to find a row in that column with a value over $\frac{1}{2}$. [3]

## 4 Martingales

**Definition 4.1** ([5]). *A martingale is a stochastic sequence $S = s_1, s_2, ...$ such that for any integer $k$ and for any finite expected value $E(|s_k|)$,*

$$E(s_{k+1}|s_1, ..., s_k) = s_k$$

Note that although Markov chains and martingales are both stochastic sequences, they differ in that Markov chains is a stochastic sequence such that the next state in the process depends only on the current state. A martingale is a stochastic sequence such that the expected payoff in a martingale is constant.

In the coin flipping game, imagine at each time $t$, a gambler is allowed to pay a wager of $w$ on the next coin flip. If they are correct, they will win $2w$ otherwise they receive 0. The gambler is allowed to play for however long they want and can employ a stopping strategy for example, stop if they lose twice in a row or bet on THH and stop. We denote the earnings of the gambler for $s$ bets starting at time $t$ as $N_t^s$. It turns out that $N_t^s$ is a martingale and has the property that $E(N_t^s) = 0$. [7]

## 5 Combinatoric Method

The notion of counting the period of subsequence overlaps is the basis for the combinatorial approach for solving the penney ante problem for a generalized $q$-alphabet. We can count subsequences using Conway's algorithm presented in the next section.

## 6 Conway's Algorithm

**Theorem 6.1.** *(Conway's Algorithm [3]) Given two n-tuples A and B, we find the binary representation of the operation $A \oplus B$ by the following algorithm:*

1. *loop through integers 1, 2, ..., n,*
2. *At every ith iteration we look at the ith through nth digits of A and compare it to the 1st through the (n-i)th digits of B.*
3. *If these subsequences are equal, the ith digit in the binary representation is 1 but 0 otherwise.*

*The binary representation of $A \oplus B$ is then converted to a decimal number. Once we find $A \oplus A$, $A \oplus B$, $B \oplus A$, and $B \oplus B$, the probability that A precedes B is*

$$\frac{A \oplus A - A \oplus B}{(B \oplus B - B \oplus A) + (A \oplus A - A \oplus B)}$$

*Furthermore,*

$$E(\tau_A) = 2(A \oplus A)$$

*Consider $THTH \oplus THTH$ as an example.*

*1010*
*<u>THTH</u>*
*THTH*
  *THTH*
    *THTH*
      *THTH*

*The binary number 1010 in decimal is 9 so $E(\tau_{THTH}) = 18$.*

## 7 Further Research

This paper examines string searching for $q = 2$ and general $q$ alphabet strings of length $n$. However, what happens when searching for a $q$ alphabet string that is biased towards certain letters in the alphabet? Another potential area of research is examining properties of nontransitive games in general, outside of this game.

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