Mini Paper: The Penney Ante Problem

Saleh Hindi

December 4, 2016

1 Introduction

Imagine a two player game where each player is assigned a sequence, for example THTH and HTHH, and a coin is flipped until either player sees their sequence. The first player to see their sequence appear wins. Given two sequences, which sequence is expected to come first in the sequence of coin flips? What is the probability of a certain player winning? Although these two questions sound similar, the result is that in our example, the expected number of turns for sequence A to appear is 20 and for B it is 18. Meanwhile, the probability of A winning is 9/14 while the probability of B winning is 5/14. Additionally the game has the property that for any sequence A chooses, B can always find a sequence that has a higher probability of winning. These counterintuitive results are the core of the Penney Ante problem, discovered by Walter Penney [3]. My thesis will study this problem through three approaches based in Markov chains, martingales, and a combinatoric approach.

2 Definitions and Notation

Let X_t be a random variable for $t \geq 1$, and let (X_t) be a stochastic sequence of letters chosen uniformly and randomly from a q-letter alphabet. In our case, $q = 2$ and (X_t) represents the sequence of coin flips. Let $A = a_1a_2a_3...$ and $B = b_1b_2b_3...$ be sequences of n letters chosen from the q-letter alphabet. We say sequence A or B wins if it is the first sequence to appear within (X_t). Let τ_A and τ_B be random variables denoting the number of turns for A or B to appear. We denote the probability of A winning as $P(\tau_A < \tau_B)$ and we denote the expected time for sequence A and B to appear as $E(\tau_A)$ and $E(\tau_B)$.

As stated earlier, $E(\tau_A) < E(\tau_B)$ does not imply $P(\tau_A < \tau_B) > \frac{1}{2}$. To prove and understand this result, we will be using a mathematical object called Markov chains which has the probability that each state occuring, ie each X_k, is based solely on X_{k-1} for all $k > 1$. Markov chains imagine the game as a graph with vertices representing (Be precise about this. Each state represents how close you are to winning right? I should define this thing here) some subset of (X_t) and edges denoting probabilities between edges. More formally,

Definition 2.1 ([7]) *XXXNote that Xk is not an event, is a single random variableXXX A Markov chain is a stochastic sequence such that*

$$P(X_{k+1} = x|X_1X_2...X_k) = P(X_{k+1} = x|X_k)$$

That is, the probability that an event occurs given the entire history of previous events is only dependent on the most recent event. Denote the probability that event y occurs given x as $P(x,y)$.

The Penney ante game is a Markov chain because each coin flip X_k is chosen uniformly and randomly from the set of possible coin flips so it does not depend on.... It's because each vertex is an event like HHT and this is only dependent on the previous event. We can use Markov chains to find $E(\tau_A)$ and $P(\tau_A < \tau_B)$ by modelling each X if for 1 <= i <= n as a systems of equations which will be done next section.

We can also study the Penney ante problem using another mathematical object called a martingale. Let τ be any betting strategy on (X_t) and let W_t be the earnings until time t (define a betting strategy). A martingale is a game such that regardless of the betting strategy, the expected earnings is the same at any time. A martingale with an expected earnings of 0 is called a fair game. More precisely,

Definition 2.2 ([5]) *(Is this correct?) A martingale is a stochastic sequence (X_t) such that for any integer k and for any finite expected value $E(|X_k|)$,*

$$E(X_{k+1}|X_1,...,X_k) = X_k$$

(How do I know this game is a martingale?) Let's define a betting strategy on the game. At each time, a new player enters the game and bets 1 on X_t being a_t. The player bets double or nothing on each subsequent coin flip. If any of the players lose at any point, they stop betting. Given two sequences A and B, define the operation AB as the total earnings of the players if they bet on B given A occurs (check this). Conway devised a "magic algorithm" which let us computer AB which will be used to find the expected winning time and the probability of A winning.

And finally, we will be using a combinatoric method for finding $E(\tau_A)$ based around finding the generative function for XXX. A generative function is a function that generates... In our game, the generative function for XXX is XXX. Blah Blah Blah.

Finally, we define nontransitivity. In this context, a game is nontransitive if for any sequence A, player B can choose a sequence B such that $P(\tau_A < \tau_B)$. In later sections we will see that the Penney ante game is nontransitive for $n > 4$. In general nontransitivity is....

3 Results

Using Markov chains, we can construct a system of equations for the expected value and probability. Let p_x denote the probability of A winning given that x has occured. To find $p_\emptyset$, the probability of A winning given nothing has happened, we can construct a system of $|\Omega|$ equations for each $x \in \Omega$ composed of the sum of the possible ways to reach x times their probabilities times the probability of reaching x from y.

$$p_x = \sum_{y-x} P(x,y) p_y$$

In this game, for a $q = 2$ alphabet each sum will have two terms since there will only be at most two states x that are reachable from each state y. Similarly, we can find the expected value this method but with a small change. Let E_x denote the time to win given x has occurred. Then we can write a system of $|\Omega|$ equations for each $x \in \Omega$,

$$E_x = \sum_{y-x} P(x,y)(1 + E_y)$$

Note the we add 1 to E_y because....

For when thinking about the Penney ante problem in terms of martingales, we can use Conway's algorithm to find $P(\tau_A < \tau_B)$ and $E(\tau_A)$. Conway devised an algorithm for computing these two values which has been described as an algorithm that "cranks out the answer as if by magic" [3]. The algorithm is as follows,

Theorem 3.1 *(Conway's Algorithm [3]) Given two n-tuples A and B, we find the binary representation of the operation $A \oplus B$ by the following algorithm:*

1. *loop through integers 1, 2, ..., n,*
2. *At every ith iteration we look at the ith through nth digits of A and compare it to the 1st through the (n-i)th digits of B.*

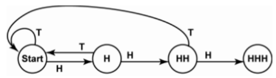

Figure 1: The graph of Ω_A

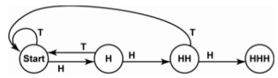

Figure 2: The graph of Ω_B

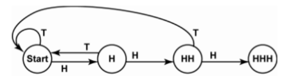

Figure 3: Another figure

3. *If these subsequences are equal, the ith digit in the binary representation is 1 but 0 otherwise.*

The binary representation of $A \oplus B$ is then converted to a decimal number. Once we find $A \oplus A$, $A \oplus B$, $B \oplus A$, and $B \oplus B$, the probability that A precedes B is

$$P(\tau_A < \tau_B) = \frac{A \oplus A - A \oplus B}{(B \oplus B - B \oplus A) + (A \oplus A - A \oplus B)}$$

Furthermore,

$$E(\tau_A) = 2(A \oplus A)$$

Consider $THTH \oplus THTH$ as an example.

1010
<u>THTH</u>
THTH
  THTH
    THTH
      THTH

The binary number 1010 in decimal is 9 so $E(\tau_{THTH}) = 18$.

Using the generative approach, we find that ....

4 Examples

As an example, let $A = THTH$ and $B = HTHH$. Although not all possible sequences have the property that, $E(\tau_A) < E(\tau_B)$ while $P(\tau_A < \tau_B) > \frac{1}{2}$, these two do so it is a worthwhile example to consider. (Could there be any other good examples? Maybe for a dice?) As stated in the introduction, $E(\tau_A)$ is 20, $E(\tau_B)$ is 18, and $P(\tau_A > \tau_B) = \frac{9}{14}$. As a sanity check, we should check that the three methods described in the previous section give the same answer for $P(\tau_A > \tau_B)$ and $E(\tau_B)$. For Ω, $\Omega = n^q$ which is 16 in this case (this is wrong because we only count states that get us closer to A). Here are a couple of equations as an example with the Mathematica code to solve the equations

(How should I work through this long example? Should I show all equations). For the probability we have,

$$\begin{aligned}
p_{\emptyset} &= \frac{1}{2}XX + \frac{1}{2}XX \\
p_{T} &= \frac{1}{2}XX + \frac{1}{2}XX \\
p_{TH} &= \frac{1}{2}XX + \frac{1}{2}XX \\
p_{THT} &= \frac{1}{2}XX + \frac{1}{2}XX \\
p_{THTH} &= 1
\end{aligned} \tag{1}$$

Using Mathematica we find that $p_{THTH} = \frac{9}{14}$. For the expected value we have,

$$\begin{aligned}
E_{\emptyset} &= \frac{1}{2}(1+XX) + \frac{1}{2}(1+XX) \\
E_{T} &= \frac{1}{2}(1+XX) + \frac{1}{2}(1+XX) \\
E_{TH} &= \frac{1}{2}(1+XX) + \frac{1}{2}(1+XX) \\
E_{THT} &= \frac{1}{2}(1+XX) + \frac{1}{2}(1+XX) \\
E_{THTH} &= \frac{1}{2}(1+XX) + \frac{1}{2}(1+XX)
\end{aligned} \tag{2}$$

The calculation for B is omitted because it is essentially the same. Using the same Mathematica code we find that $E_{THTHT} = 20$.

5 Connection Between Techniques

Blah

6 Future Work

In the future I will prove the results stated in the results section. It will also be worthwhile to give a proof of the nontransitivity property which was not stated in the results.

References

[1] Boyer, Robert S., and J. Strother Moore. "A Fast String Searching Algorithm." Communications of the ACM 20.10 (1977): 762-72. Web.

[2] Breen, Stephen, Waterman Michael S., and Zhang Ning. "Renewal Theory for Several Patterns." Journal of Applied Probability 22.1 (1985): 228-34. Web.

[3] Gardner, Martin. "Mathematical Games: On the Paradoxical Situations That Arise from Nontransitive Relations." Scientific American 10 (1974): 120-25. Print.

[4] Guibas, L.j, and A.m Odlyzko. "String Overlaps, Pattern Matching, and Nontransitive Games." Journal of Combinatorial Theory, Series A 30.2 (1981): 183-208. Web.

[5] Li, Shuo-Yen Robert. "A Martingale Approach to the Study of Occurrence of Sequence Patterns in Repeated Experiments." The Annals of Probability 8.6 (1980): 1171-176. Web.

[6] Nickerson, R. S. "Penney Ante: Counterintuitive Probabilities in Coin Tossing." The UMAP Journal 28.4 (2007): 503-32. JSTOR. Web. 8 Sept. 2016.

[7] Levin, David Asher, Y. Peres, and Elizabeth L. Wilmer. Markov Chains and Mixing times. Providence, RI: American Mathematical Society, 2009. Print.