Combination

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content

1 Number of k-combinations

The number of k-combinations from a given set S of n elements is often denoted in elementary combinatorics texts by C(n,k), or by a variation such as C_k^n,${}_nC_k$,${}^nC_k$, $C_{n,k}$ or evenC_n^k(the latter form was standard in French, Romanian, Russian, Chinese and Polish texts[citation needed]). The same number however occurs in many other mathematical contexts, where it is denoted by $\binom{n}{k}$ (often read as "n choose k"); notably it occurs as a coefficient in the binomial formula, hence its name binomial coefficient. One can define $\binom{n}{k}$for all natural numbers k at once by the relation:

$$(1+X)^n = \sum_{k\geq 0} \binom{n}{k} X^k$$

from which it is clear that

$$\binom{n}{0} = \binom{n}{n} = 1$$

To see that these coefficients count k-combinations from S, one can first

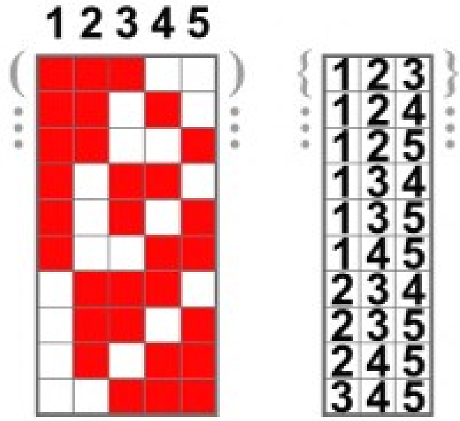

Figure 1: 3-element subsets of a 5-element set

consider a collection of n distinct variablesX_s labeled by the elements s of S, and expand the product over all elements of S:

$$\prod_{s \in S} (1 + X_s);$$

it has2^n distinct terms corresponding to all the subsets of S, each subset giving the product of the corresponding variablesX_s. Now setting all of theX_s equal to the unlabeled variable X, so that the product becomes $(1 + X)^n$, the term for each k-combination from S becomes X^k, so that the coefficient of that power in the result equals the number of such k-combinations.
Binomial coefficients can be computed explicitly in various ways. To get all of them for the expansions up to $(1 + X)^n$, one can use (in addition to the basic cases already given) the recursion relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

for $0 < k < n$, which follows from $(1+X)^n = (1+X)^{n-1}(1+X)$; this leads to the construction of Pascal's triangle. For determining an individual binomial coefficient, it is more practical to use the formula

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k)}{k!}$$

The numerator gives the number of k-permutations of n, i.e., of sequences of k distinct elements of S, while the denominator gives the number of such k-permutations that give the same k-combination when the order is ignored. When k exceeds n/2, the above formula contains factors common to the numerator and the denominator, and canceling them out gives the relation

$$\binom{n}{k} = \binom{n}{n-k}$$

for $0 \leq k \leq n$. This expresses a symmetry that is evident from the binomial formula, and can also be understood in terms of k-combinations by taking the complement of such a combination, which is an $(n-k)$-combination. Finally there is a formula which exhibits this symmetry directly, and has the merit of being easy to remember:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

where n! denotes the factorial of n. It is obtained from the previous formula by multiplying denominator and numerator by (n - k)!, so it is certainly inferior as a method of computation to that formula.

The last formula can be understood directly, by considering the n! permutations of all the elements of S. Each such permutation gives a k-combination by selecting its first k elements. There are many duplicate selections: any combined permutation of the first k elements among each other, and of the final (n - k) elements among each other produces the same combination; this explains the division in the formula.

From the above formulas follow relations between adjacent numbers in Pascal's triangle in all three directions:

$$\binom{n}{k} = \begin{cases} \binom{n}{k-1}\frac{n-k+1}{k} & \text{if } k > 0 \\ \binom{n-1}{k}\frac{n}{n-k} & \text{if } k < n \\ \binom{n-1}{k-1}\frac{n}{k} & \text{if } n, k > 0 \end{cases}$$

Together with the basic cases $\binom{n}{0} = 1 = \binom{n}{n}$, these allow successive computation of respectively all numbers of combinations from the same set (a row in Pascal's triangle), of k-combinations of sets of growing sizes, and of combinations with a complement of fixed size n - k.

1.1 Example of counting combinations

As a specific example, one can compute the number of five-card hands possible from a standard fifty-two card deck as:

$$\binom{52}{5} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2,598,960$$

1.2 Enumerating k-combinations

One can enumerate all k-combinations of a given set S of n elements in some fixed order, which establishes a bijection from an interval of $\binom{n}{k}$ integers with the set of those k-combinations. Assuming S is itself ordered, for instance $S = \{1, 2, \cdots, n\}$, there are two natural possibilities for ordering its k-combinations: by comparing their smallest elements first (as in the illustrations above) or by comparing their largest elements first. The latter option has the advantage that adding a new largest element to S will not change the initial part of the enumeration, but just add the new k-combinations of the larger set after the previous ones. Repeating this process, the enumeration can be extended indefinitely with k-combinations of ever larger sets. If moreover the intervals of the integers are taken to start at 0, then the k-combination at a given place i in the enumeration can be computed easily from i, and the bijection so obtained is known as the combinatorial number system. It is also known as "rank"/"ranking" and "unranking" in computational mathematics.

There are many ways to enumerate k combinations. One way is to visit all the binary numbers less than 2^n. Choose those numbers having k nonzero bits, although this is very inefficient even for small n (e.g. $n = 20$ would require visiting about one million numbers while the maximum number of allowed k combinations is about 186 thousand for k = 10). The positions of these 1 bits in such a number is a specific k-combination of the set $\{1, \ldots, n\}$. Another simple, faster way is to track k index numbers of the elements selected, starting with $\{0, \cdots, k-1\}$ (zero-based) or $\{1, \cdots, k\}$ (one-based) as the first allowed k-combination and then repeatedly moving to the next allowed k-combination by incrementing the last index number if it is lower than $n-1$ (zero-based) or n (one-based) or the last index number x that is less than the index number following it minus one if such an index exists and resetting the index numbers after x to $\{x+1, x+2, \dots\}$.

2 Number of combinations with repetition

2.1 Example of counting multisubsets

No.	3-Multiset	Eq. Solution	Stars and Bars
1	1,1,1	[3,0,0,0]	$\star\star\star\vert\vert\vert$
2	1,1,2	[2,1,0,0]	$\star\star\vert\star\vert\vert$
3	1,1,3	[2,0,1,0]	$\star\star\vert\vert\star\vert$
4	1,1,4	[2,0,0,1]	$\star\star\vert\vert\vert\star$
5	1,2,2	[1,2,0,0]	$\star\vert\star\star\vert\vert$
6	1,2,3	[1,1,1,0]	$\star\vert\star\vert\star\vert$
7	1,2,4	[1,1,0,1]	$\star\vert\star\vert\vert\star$
8	1,3,3	[1,0,2,0]	$\star\vert\vert\star\star\vert$
9	1,3,4	[1,0,1,1]	$\star\vert\vert\star\vert\star$
10	1,4,4	[1,0,0,2]	$\star\vert\vert\vert\star\star$
11	2,2,2	[0,3,0,0]	$\vert\star\star\star\vert\vert$
12	2,2,3	[0,2,1,0]	$\vert\star\star\vert\star\vert$
13	2,2,4	[0,2,0,1]	$\vert\star\star\vert\vert\star$
14	2,3,3	[0,1,2,0]	$\vert\star\vert\star\star\vert$
15	2,3,4	[0,1,1,1]	$\vert\star\vert\star\vert\star$
16	2,4,4	[0,1,0,2]	$\vert\star\vert\vert\star\star$
17	3,3,3	[0,0,3,0]	$\vert\vert\star\star\star\vert$
18	3,3,4	[0,0,2,1]	$\vert\vert\star\star\vert\star$
19	3,4,4	[0,0,1,2]	$\vert\vert\star\vert\star\star$
20	4,4,4	[0,0,0,3]	$\vert\vert\vert\star\star\star$

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