3.4 Homework III

(Only submit solutions to Questions 1,4,6,9,14. Unless otherwise specified, a Brownian motion is assume to be 1-dimensional with zero initial value.)

1. a) Let $X \sim N(0,t)$ which means its pdf is $f_X = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$. Define its moment generating function $\varphi(u) = \mathbb{E}[e^{uX}] = \int_{\mathbb{R}} e^{ux} f_X(x)dx$. Show that $\varphi(u) = e^{\frac{1}{2}u^2 t}$.

   b) If $X \sim N(\mu, \sigma^2)$, show that the so called log-normally distributed random variable $S = e^X$ has mean $e^{\mu + \frac{1}{2}\sigma^2}$. [Hint: To simplify the calculation, consider $\mathbb{E}[e^X] = e^{\mu}\mathbb{E}[e^{X-\mu}]$.]

   c) What is the variance of e^X if $X \sim N(\mu, \sigma^2)$?

   Solution: (1)

$$\varphi(u) = \int_{\mathbb{R}} e^{ux} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = e^{\frac{1}{2}u^2 t} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-ut)^2}{2t}} dx = e^{\frac{u^2}{2}t}.$$

   (2) $X - \mu \sim N(0, \sigma^2)$. $\mathbb{E}[e^X] = e^{\mu}\mathbb{E}[e^{X-\mu}] = e^{\mu} e^{\frac{1}{2}\sigma^2}$ by part (1) with $u = 1$ and $t = \sigma^2$.

   (3) $2X \sim N(2\mu, 4\sigma^2)$. $\mathrm{Var}[e^X] = \mathbb{E}(e^X)^2 - \left(\mathbb{E}e^X\right)^2 = \mathbb{E}e^{2X} - \left(e^{\mu+\frac{1}{2}\sigma^2}\right)^2 = e^{2\mu+\frac{1}{2}4\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$.

2. The kurtosis of a random variable is defined to be the ratio of its fourth central moment to the square of its variance. Prove that for a normal random variable $X \sim N(\mu, \sigma^2)$, the kurtosis $\frac{\mathbb{E}[(X-\mu)^4]}{(\mathbb{E}[(X-\mu)^2])^2} = 3$. (Hint: If $\mu = 0$, we can use the moment generating function $\varphi(u) = \mathbb{E}[e^{uX}]$ defined in Question 1. Taking 2 derivatives with respect to u, we get $\varphi''(u) = \mathbb{E}[X^2 e^{\mu X}]$, which is $(t + tu)e^{\frac{1}{2}u^2 t}$ since $\varphi(u) = e^{\frac{1}{2}u^2 t}$. By setting $u = 0$, we obtain $\mathbb{E}[X^2] = t$. By taking 4 derivatives, one can calculate $\mathbb{E}[X^4]$.) Then show that

$$E[|W_t - W_s|^4] = 3|t-s|^2, \qquad 0 \le t, s \le T.$$

   Hence (3.39) is true with $D = 3$, $\alpha = 4$, $\beta = 1$ and "$\le$" is indeed "$=$" for the Brownian motion case.

   Solution: Define $\varphi(u) = \mathbb{E}[e^{u(X-\mu)}]$. Then $\varphi(u) = e^{\frac{1}{2}u^2\sigma^2}$ by Question 1.

$$\varphi^{(4)}(u) = \mathbb{E}[(X-\mu)^4 e^{u(X-\mu)}] = \left(3\sigma^4 + 6u^2\sigma^6 + u^4\sigma^8\right) e^{\frac{1}{2}u^2\sigma^2}.$$

   Letting $u = 0$, we get

$$\mathbb{E}[(X-\mu)^4] = 3\sigma^4.$$

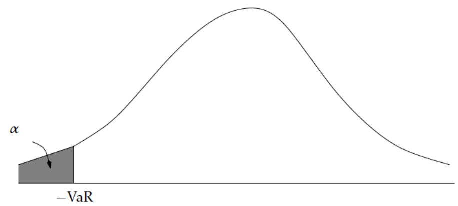
   Since $W_t - W_s \sim N(0, t-s)$, $E[|W_t - W_s|^4] = 3|t-s|^2$.

3. Value-at-risk (VaR) denotes, within a confidence level, the maximum loss a portfolio could suffer. To be more precise, denote by X the change in the market value of a portfolio

during a given time period. Then the VaR with confidence level $1-\alpha$ is defined to be the value VaR in

$$\mathbb{P}(X \le -\text{VaR}) = \alpha. \tag{3.43}$$

In other words, with probability $1-\alpha$, the maximum loss will not exceed VaR.



Show that if X is assumed to be normally distributed with mean μ and variance σ^2, the VaR with confidence level $1-\alpha$ is given by

$$\text{VaR} = z_\alpha \sigma - \mu \tag{3.44}$$

where z_α satisfies $N(-z_\alpha) = \alpha$ with function N defined in (3.15). For example, $z_{0.1} = 1.2816$, $z_{0.05} = 1.6449$, $z_{0.01} = 2.3263$, $z_{0.005} = 2.5758$.

Proof: $X \sim N(\mu, \sigma^2)$ implies that $\frac{X-\mu}{\sigma} \sim N(0,1)$. Hence

$$\begin{aligned} N(-z_\alpha) = \alpha = \mathbb{P}(X \le -\text{VaR}) &= \mathbb{P}\left(\frac{X-\mu}{\sigma} \le \frac{-\text{VaR}-\mu}{\sigma}\right) \\ &= \int_{-\infty}^{-\frac{\text{VaR}+\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = N\left(-\frac{\text{VaR}+\mu}{\sigma}\right). \end{aligned}$$

So $z_\alpha = \frac{\text{VaR}+\mu}{\sigma}$ which proved (3.44). $\square$

4. A binary call option with maturity T pays one dollar when the stock price at time T is at or above a certain level K and pays nothing otherwise. The payoff can be written in the form of an indicator function

$$X = 1_{\{S_T \ge K\}}.$$

Suppose in the risk neutral world $\log S_T$ is normally distributed with mean $\log S_0 + (r - \frac{\sigma^2}{2})T$ and variance $\sigma^2 T$. Show that the discounted expected payoff $e^{-rT}\mathbb{E}X$ is

$$e^{-rT}\mathbb{E}X = e^{-rT}N(d_-), \tag{3.45}$$

where the function N is defined in (3.15) and the d_- is defined in (3.14). [Hint: Note that $\mathbb{E}X = \mathbb{P}(\log S_T \ge \log K)$.]

Solution: $\mathbb{E}X = \mathbb{P}(S_T \geq K) \times 1 + \mathbb{P}(S_T < K) \times 0 = \mathbb{P}(S_T \geq K) = \mathbb{P}(\log S_T \geq \log K)$. Since $\log S_T$ is normally distributed, $Z \stackrel{\text{def}}{=} \frac{\log S_T - \log S_0 - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \sim N(0,1)$,

$$\begin{aligned}
\mathbb{P}(\log S_T \geq \log K) &= \mathbb{P}\left(\frac{\log S_T - \log S_0 - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \geq \frac{\log \frac{K}{S_0} - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \\
&= \mathbb{P}\left(Z \geq -\left(\frac{\log \frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\right) \\
&= \mathbb{P}\left(Z \geq -d_-\right) = \int_{-d_-}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = N(d_-). \qquad \square
\end{aligned}$$

5. Let $f(t)$ be a function defined for $0 \leq t \leq T$. The quadratic variation of f up to time T is defined to be

$$[f,f](T) = \lim_{\max_j |t_{j+1} - t_j| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|^2 \tag{3.46}$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of the interval $[0,T]$. Recall that the mean value theorem says that if f has continuous derivative, for any $[t_j, t_{j+1}]$, there is a $t_j^* \in (t_j, t_{j+1})$ so that $f(t_{j+1}) - f(t_j) = f'(t_j^*)(t_{j+1} - t_j)$.

Prove that if f has continuous derivative on $[0,T]$ with $T < \infty$, then

$$[f,f](T) = 0.$$

Here you can use the fact that f has continuous derivative on $[0,T]$ implies that $M \stackrel{\text{def}}{=} \sup_{s \in [0,T]} |f'(s)| < \infty$.

Proof:

$$\begin{aligned}
0 \leq [f,f](T) &= \lim_{\max_j |t_{j+1} - t_j| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|^2 \\
&= \lim_{\max_j |t_{j+1} - t_j| \to 0} \sum_{j=0}^{n-1} |f'(t_j^*)|^2 |t_{j+1} - t_j|^2 \\
&\leq \lim_{\max_j |t_{j+1} - t_j| \to 0} \sum_{j=0}^{n-1} \left(M^2 \max_j |t_{j+1} - t_j|\right) |t_{j+1} - t_j| \\
&= \lim_{\max_j |t_{j+1} - t_j| \to 0} \left(M^2 \max_j |t_{j+1} - t_j|\right) \sum_{j=0}^{n-1} |t_{j+1} - t_j| \\
&= \lim_{\max_j |t_{j+1} - t_j| \to 0} \left(M^2 \max_j |t_{j+1} - t_j|\right) T = 0.
\end{aligned}$$

6. By definition, we say that $X_n(\omega) \to X(\omega)$ in mean square sense if $\mathbb{E}[(X_n - X)^2] \to 0$ as $n \to \infty$. Given a Brownian motion W_t, define a random variable $[W,W]_{\Pi,T}$ by

$$[W,W]_{\Pi,T}(\omega) = \sum_{j=0}^{n-1} |W_{t_{j+1}}(\omega) - W_{t_j}(\omega)|^2$$

where $\Pi = \{0 = t_0, t_1, \cdots, t_n = T\}$ is the partition of $[0,T]$. This question asks you to show that the random variable $[W,W]_{\Pi,T} \to T$ in the mean square sense when $\|\Pi\| \stackrel{\text{def}}{=} \max_j |t_{j+1} - t_j| \to 0$ [31] . The proof is split into two steps:

a) Show that
$$\mathbb{E}[[W,W]_{\Pi,T}] = T. \tag{3.48}$$

b) By (3.48) and $\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$, the variance of $[W,W]_{\Pi,T}$ is $\mathrm{Var}[[W,W]_{\Pi,T}] = \mathbb{E}[([W,W]_{\Pi,T} - T)^2]$. Show that
$$\mathrm{Var}[[W,W]_{\Pi,T}] = \sum_{j=0}^{n-1} \mathrm{Var}[|W_{t_{j+1}} - W_{t_j}|^2] \to 0$$
as $\|\Pi\| \to 0$.

Proof: (a) Since $W_{t_{j+1}} - W_j$ and $W_{t_{i+1}} - W_i$ are independent when $i \neq j$,

$$\mathbb{E}[[W,W]_{\Pi,T}] = \sum_{j=0}^{n-1} \mathbb{E}[|W_{t_{j+1}} - W_{t_j}|^2] = \sum_{j=0}^{n-1} t_{j+1} - t_j.$$

In the last step, we have used that fact that $W_t - W_s \sim N(0, t-s)$ or $\sqrt{t-s}N(0,1)$ when $s < t$. Hence $\mathbb{E}[[W,W]_{\Pi,T}] = t_n - t_0 = T$.

(b) Since $W_{t_{j+1}} - W_j$ and $W_{t_{i+1}} - W_i$ are independent when $i \neq j$,

$$\mathrm{Var}[[W,W]_{\Pi,T}] = \sum_{j=0}^{n-1} \mathrm{Var}[|W_{t_{j+1}} - W_{t_j}|^2].$$

We know $W_t - W_s \sim \sqrt{t-s}N(0,1)$. Let $Y \sim N(0,1)$. Then $\mathrm{Var}[|W_{t_{j+1}} - W_{t_j}|^2] = \mathrm{Var}[(t_{j+1} - t_j)Y^2] = (t_{j+1} - t_j)^2 \mathrm{Var}[Y^2] =$ some constant C times $(t_{j+1} - t_j)^2$. So

$$\mathrm{Var}[[W,W]_{\Pi,T}] = \sum_{j=0}^{n-1} C(t_{j+1} - t_j)^2 \leq CT\|\Pi\| \to 0 \qquad \text{as } \|\Pi\| \to 0.$$

[31] Theorem 3.4.3 of Shreve II says that

$$[W,W]_{\Pi,T} \to T \text{ almost surely when } \max_j |t_{j+1} - t_j| \to 0. \tag{3.47}$$

Note that we say that a statement A that depends on ω is true almost surely if $\mathbb{P}(\{\omega : A(\omega) \text{ is true}\}) = 1$. Hence what we will prove in this question is that the convergence in (3.47) happens in mean square sense.

By the way, we can show that this constant $C = 2$: $C = \text{Var}[Y^2] = \mathbb{E}[(Y^2 - \mathbb{E}[Y^2])^2] = \mathbb{E}[(Y^2-1)^2]$. Question 2 says that $\mathbb{E}[(Y - \mathbb{E}[Y])^4] = 3(\text{Var}[Y])^2$ when Y has normal distribution. Hence when $Y \sim N(0,1)$, $\mathbb{E}[Y^4] = \mathbb{E}[(Y-\mathbb{E}[Y])^4] = 3(\text{Var}[Y])^2 = 3$ and $C = \mathbb{E}[(Y^2-1)^2] = \mathbb{E}[Y^4] - 2\mathbb{E}[Y^2] + 1 = 3 - 2 + 1 = 2$.

7. We now continue with Example 2.11 of Chapter 2 to introduce Markov property:

   Definition 3.5 *(Definition 2.5.1 of Shreve I) Consider the binomial asset-pricing model. Let X_0, X_1, ... , X_N be a sequence of random variables, with each X_n depending only on the first n coin tosses (and X_0 constant). If, for every n between 0 and $N-1$ and for every function $f(x)$, there is another function $g(x)$ (depending on f and n) such that*

$$\tilde{\mathbb{E}}_n[f(X_{n+1})] = g(X_n), \tag{3.49}$$

   we say that $X_0, X_1, \cdots, X_n$ is a Markov process.

   The Markov property says that the dependence of $\tilde{\mathbb{E}}_n[f(X_{n+1})]$ on the first n coin tosses occurs through X_n (i.e., the information about the coin tosses one needs in order to evaluate $\tilde{\mathbb{E}}[f(X_{n+1})]$ is summarized by X_n).

   It also implies the "two-step ahead" property:

$$\tilde{\mathbb{E}}_n[h(X_{n+2})] \overset{(2.31)}{=} \tilde{\mathbb{E}}_n[\tilde{\mathbb{E}}_{n+1}[h(X_{n+2})]] = \tilde{\mathbb{E}}_n[f(X_{n+1})] = g(X_n), \tag{3.50}$$

   or "multi-step ahead" property: For any function f and any $0 \le n \le m \le N$, there is a function g so that

$$\tilde{\mathbb{E}}_n[h(X_m)] = g(X_n). \tag{3.51}$$

   Now comes the question: Consider the stock price model

$$S_{n+1}(\omega_1 \cdots \omega_n \omega_{n+1}) = \begin{cases} uS_n(\omega_1 \cdots \omega_n) & \text{if } \omega_{n+1} = H, \\ dS_n(\omega_1 \cdots \omega_n) & \text{if } \omega_{n+1} = T. \end{cases}$$

   Find the function g so that

$$\tilde{\mathbb{E}}_n[f(S_{n+1})](\omega_1 \cdots \omega_n) = g(S_n).$$

   Solution: $g(x) = q_u f(ux) + q_d f(dx)$.

8. (Conditional distribution) The conditional probability of event A happens under the condition that event B already happens is denoted as $\mathbb{P}(A|B)$ and we have the relation

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(AB)}{\mathbb{P}(B)} \tag{3.52}$$

if $\mathbb{P}(B) > 0$. If we have joint density function $\rho_{X,Y}$ for two continuous random variables X and Y, we define marginal density function $\rho_X(x)$ and $\rho_Y(y)$ as in (3.19). Then we can calculate conditional distribution

$$
\begin{aligned}
\mathbb{P}(X \in A|Y = y) &\stackrel{\text{def}}{=} \lim_{a\downarrow 0} \mathbb{P}(X \in A|Y \in [y-a, y+a]) \stackrel{(3.52)}{=} \lim_{a\downarrow 0} \frac{\mathbb{P}(X \in A, Y \in [y-a, y+a])}{\mathbb{P}(Y \in [y-\Delta, y+\Delta])} \\
&= \lim_{a\downarrow 0} \frac{\int_{[y-a,y+a]} \left(\int_A \rho_{X,Y}(x,v)dx\right) dv}{\int_{[y-a,y+a]} \rho_Y(v)dv} \\
&= \lim_{a\downarrow 0} \frac{\left(\int_A \rho_{X,Y}(x,y)dx\right) \times (2a) + \text{ higher order term of } a}{\rho_Y(y) \times (2a) + \text{ higher order term of } a}.
\end{aligned}
$$

In the last step, we used again the fact that $\int_a^b f(v)dv = f(\frac{a+b}{2})(b-a)+$ higher order term of $(b-a)$. Now, dividing $2a$ from both the numerator and denomonator, we get

$$
\begin{aligned}
\mathbb{P}(X \in A|Y = y) &= \lim_{a\downarrow 0} \frac{\int_A \rho_{X,Y}(x,y)dx + \frac{\text{higher order term of } a}{2a}}{\rho_Y(y) + \frac{\text{higher order term of } a}{2a}} \\
&= \frac{\int_A \rho_{X,Y}(x,y)dx + \lim_{a\downarrow 0} \frac{\text{higher order term of } a}{2a}}{\rho_Y(y) + \lim_{a\downarrow 0} \frac{\text{higher order term of } a}{2a}}.
\end{aligned}
$$

In the last step, we have used $\lim_{a\to b} \frac{f(a)}{g(a)} = \frac{\lim_{a\to b} f(a)}{\lim_{a\to b} g(a)}$. Continue, we get

$$
\mathbb{P}(X \in A|Y = y) = \frac{\int_A \rho_{X,Y}(x,y)dx + 0}{\rho_Y(y) + 0} = \int_A \frac{\rho_{X,Y}(x,y)}{\rho_Y(y)} dx.
$$

Hence by (3.1), the conditional density of X given $Y = y$ is

$$
\frac{\rho_{X,Y}(x,y)}{\rho_Y(y)} \stackrel{\text{def}}{=} \rho_{X|Y}(x|y), \tag{3.53}
$$

which, by the way, also depends on y. Since we now have the density function, by (3.10), we know

$$
\mathbb{E}[h(X)|Y = y] = \int_{\mathbb{R}} h(x)\rho_{X|Y}(x|y)dx \tag{3.54}
$$

which is a function of y. By the way, if X and Y are independent, by (3.21), $\rho_{X|Y}(x|y) = \frac{\rho_{X,Y}(x,y)}{\rho_Y(y)} = \rho_X(x)$, then

$$
\mathbb{E}[h(X)|Y = y] = \int_{\mathbb{R}} h(x)\rho_{X|Y}(x|y)dx = \int_{\mathbb{R}} h(x)\rho_X(x)dx = \mathbb{E}[h(X)]. \tag{3.55}
$$

Go back to (3.54). Note that $\mathbb{E}[h(X)|Y] \stackrel{\text{def}}{=} \mathbb{E}[h(X)|Y = Y]$ is a function of Y which is hence a random variable by itself. As we already know the density function of Y, by (3.10), we can compute

$$
\mathbb{E}[\mathbb{E}[h(X)|Y]] = \int_{\mathbb{R}} \mathbb{E}[h(X)|Y = y]\, \rho_Y(y)dy.
$$

Here comes the question: Prove the iterated conditioning property

$$\mathbb{E}[\mathbb{E}[h(X)|Y]] = \mathbb{E}[h(X)]. \tag{3.56}$$

Remark: You should compare $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$ with

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}_m[X]] = \mathbb{E}[\mathbb{E}[X|\text{given } \omega_1 \cdots \omega_m]] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_m]], \tag{3.57}$$

which we have learned before in Theorem 2.2 (we used $\tilde{\mathbb{E}}$ there to stress that we were using the risk-neutral probability. But apparently, it is true for any probability). The precise meaning of the last term, $\mathbb{E}[\mathbb{E}[X|\mathcal{F}_m]]$, will be clarified in Theorem 4.3 of the next chapter.

Proof:

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[h(X)|Y]] &\overset{(3.10)}{=} \int_{\mathbb{R}} \mathbb{E}[h(X)|Y = y]\rho_Y(y)dy \\
&\overset{(3.54)}{=} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x)\rho_{X|Y}(x|y)dx \right) \rho_Y(y)dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x)\frac{\rho_{X,Y}(x,y)}{\cancel{\rho_Y(y)}}dx \right) \cancel{\rho_Y(y)}dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x)\rho_{X,Y}(x,y)dx \right) dy \\
&= \mathbb{E}[h(X)].
\end{aligned}$$

9. (Continue with Question 8.) Let S_1 and S_2 be the prices of two assets. Assume that $X = \log S_1$ and $Y = \log S_2$ have a joint density function

$$\rho_{X,Y} = \frac{\sqrt{3}}{4\pi}e^{-\frac{1}{2}(x^2-xy+y^2)}, \qquad \text{for } x, y \in \mathbb{R}.$$

Determine $\mathbb{E}[X|Y]$.

Solution:

$$\begin{aligned}
\rho_Y(y) &= \int_{\mathbb{R}} \rho_{X,Y}(x,y)dx = \int_{\mathbb{R}} \frac{\sqrt{3}}{4\pi}e^{-\frac{1}{2}(x^2-xy+y^2)}dx \\
&= \sqrt{\frac{3}{8\pi}}e^{-\frac{3}{8}y^2} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\frac{1}{2}y)^2}dx \\
&\overset{u=x-\frac{y}{2}}{=} \sqrt{\frac{3}{8\pi}}e^{-\frac{3}{8}y^2} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du \\
&= \sqrt{\frac{3}{8\pi}}e^{-\frac{3}{8}y^2}.
\end{aligned}$$

Hence

$$\rho_{X|Y}(x|y) = \frac{\rho_{X,Y}(x,y)}{\rho_Y(y)} = \frac{\frac{\sqrt{3}}{4\pi}e^{-\frac{1}{2}(x^2-xy+y^2)}}{\sqrt{\frac{3}{8\pi}}e^{-\frac{3}{8}y^2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\frac{1}{2}y)^2}.$$

(By the way, we can check that $\int_{\mathbb{R}} \rho_Y(y)dy = 1$ and $\int_{\mathbb{R}} \rho_{X|Y}(x|y)dx = 1$.)

$$\begin{aligned}\mathbb{E}[X|Y=y] &= \int_{\mathbb{R}} x\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\frac{1}{2}y)^2}dx \overset{u=x-\frac{1}{2}y}{=} \int_{\mathbb{R}}(u+\frac{1}{2}y)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du \\ &= \frac{1}{2}y\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du = \frac{1}{2}y,\end{aligned}$$

where we have used the fact that $\int_{\mathbb{R}} u\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2}du = \int_{\mathbb{R}}$ odd function of u $du = 0$. Hence $\mathbb{E}[X|Y] = \frac{1}{2}Y$.

10. (This problem is rather difficult and won't be tested. But I hope you can read it so that you have a better understanding of the density function of Brownian motion. This example is taken from Question 2.12 of Hui Wang, "Monte Carlo Simulations with Applications to Finance". Please bear with me for the complicated computations. If your probability/finance intuition already tells you that the conclusion is obvious, you do not need to read the proof.) Given an arbitrary constant θ, let $B = \{B_t : t \geq 0\}$ be a Brownian motion with drift θ, i.e.,

$$B_t = W_t + \theta t, \qquad t \geq 0, \tag{3.58}$$

where $W = \{W_t : t \geq 0\}$ is a Brownian motion. Given $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, show that the conditional distribution of the $(n-1)$-dimensional random variable $X = (B_{t_1}, \cdots, B_{t_{n-1}})$ given $Y = B_T = y$ does not depend on θ. In particular, letting $\theta = 0$, we conclude that the conditional distribution of $\{B_t : 0 \leq t \leq T\}$ given $B_T = y$ is the same as the conditional distribution of $\{W_t : 0 \leq t \leq T\}$ given $W_T = y$.

Proof: Recall (3.33)

$$\rho_{W_{t_1},\cdots,W_{t_n}}(x_1,\cdots x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\prod_{i=1}^n(t_i-t_{i-1})}}\exp\left(-\sum_{i=1}^n\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)$$

and $W_{t_n} \sim N(x_0, t_n)$ which means

$$\rho_{W_{t_n}}(x_n) = \frac{1}{\sqrt{2\pi(t_n-t_0)}}\exp(-\frac{(x_n-x_0)^2}{2t_n}).$$

Since $B_{t_i} = W_{t_i} + \theta t_i \sim N(x_0+\theta t_i, t_i)$, by the change of variable formula in multivariable calculus or probability, which is (3.35), we get

$$\rho_{B_{t_1},\cdots,B_{t_n}}(x_1,\cdots x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\prod_{i=1}^n(t_i-t_{i-1})}}\exp\left(-\sum_{i=1}^n\frac{[(x_i-\theta t_i)-(x_{i-1}-\theta t_{i-1})]^2}{2(t_i-t_{i-1})}\right)$$

and

$$\rho_{B_{t_n}}(x_n) = \frac{1}{\sqrt{2\pi(t_n - t_0)}} \exp\left(-\frac{(x_n - \theta t_n - x_0)^2}{2t_n}\right).$$

To calculate $\frac{\rho_{B_{t_1},\cdots,B_{t_n}}(x_1,\cdots x_n)}{\rho_{B_{t_n}}(x_n)}$, we need to compute

$$\begin{aligned}
&-\sum_{i=1}^{n} \frac{[(x_i - x_{i-1}) - \theta(t_i - t_{i-1})]^2}{2(t_i - t_{i-1})} + \frac{[(x_n - x_0) - \theta t_n]^2}{2t_n} \\
&= -\sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} + \cancel{\sum_{i=1}^{n} \theta(x_i - x_{i-1})} - \cancel{\frac{\theta^2}{2}\sum_{i=1}^{n}(t_i - t_{i-1})} - \frac{(x_n - x_0)^2}{2t_n} + \cancel{\theta(x_n - x_0)} + \cancel{\frac{\theta^2}{2}t^n} \\
&= -\sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} + \frac{(x_n - x_0)^2}{2t_n}.
\end{aligned}$$

Hence the conditional density function of $(B_{t_1}, \cdots, B_{t_{n-1}})$ given $B_{t_n} = x_n$ is

$$\frac{\rho_{B_{t_1},\cdots,B_{t_n}}(x_1,\cdots x_n)}{\rho_{B_{t_n}}(x_n)} = \frac{\sqrt{t_n - t_0}}{(2\pi)^{(n-1)/2}\sqrt{\prod_{i=1}^{n}(t_i - t_{i-1})}} \exp\left(-\sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} + \frac{(x_n - x_0)^2}{2t_n}\right). \tag{3.59}$$

The parameter θ has disappeared from the above formula. This proves the result. □

11. (This problem is rather difficult and won't be tested. This example is taken from Page 40 of Hui Wang, "Monte Carlo Simulations with Applications to Finance". It shows an application of conditional expectation. The result is useful for pricing, say, a look back call option whose payoff is $(\max_{0\le t\le T} S_t - K)^+$ and is path-dependent. See Example 2.6 of "Monte Carlo Simulations with Applications to Finance" by Hui Wang for more details.) We say a function is path-dependent if it depends on the sample paths of the relevant process. For example, we can define

$$h(W_{[0,T]}) = \max_{0\le t\le T} W_t - \min_{0\le t\le T} W_t - W_T \tag{3.60}$$

whose value depends on the entire sample path $W_{[0,T]} = \{W_t : 0 \le t \le T\}$.

Introduce B_t and W_t as in Question 10 with $W_0 = 0$. Prove that for any path dependent function h,

$$\mathbb{E}\left[h(B_{[0,T]})\right] = \mathbb{E}\left[e^{\theta W_T - \frac{1}{2}\theta^2 T} h(W_{[0,T]})\right]. \tag{3.61}$$

Proof: Since $W_T \sim N(0,T)$, by (3.56),

$$\begin{aligned}RHS &= \int_{\mathbb{R}} \mathbb{E}\left[e^{\theta W_T - \frac{1}{2}\theta^2 T} h(W_{[0,T]}) \middle| W_T = x \right] \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx \\ &= \int_{\mathbb{R}} \mathbb{E}\left[e^{\theta x - \frac{1}{2}\theta^2 T} h(W_{[0,T]}) \middle| W_T = x \right] \frac{1}{\sqrt{2\pi T}} e^{-\frac{x^2}{2T}} dx \\ &= \int_{\mathbb{R}} \mathbb{E}\left[h(W_{[0,T]}) \middle| W_T = x \right] \frac{1}{\sqrt{2\pi T}} e^{\theta x - \frac{1}{2}\theta^2 T - \frac{x^2}{2T}} dx \\ &= \int_{\mathbb{R}} \mathbb{E}\left[h(W_{[0,T]}) \middle| W_T = x \right] \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(x-\theta T)^2} dx.\end{aligned}$$

By Question 10, $\mathbb{E}\left[h(W_{[0,T]}) \middle| W_T = x \right] = \mathbb{E}\left[h(B_{[0,T]}) \middle| B_T = x \right]$. Hence

$$RHS = \int_{\mathbb{R}} \mathbb{E}\left[h(B_{[0,T]}) \middle| B_T = x \right] \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(x-\theta T)^2} dx$$

and $\frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(x-\theta T)^2}$ is pdf of B_T. Hence

$$RHS = \int_{\mathbb{R}} \mathbb{E}\left[h(B_{[0,T]}) \middle| B_T = x \right] \rho_{B_T}(x) dx = \mathbb{E}\left[\mathbb{E}\left[h(B_{[0,T]}) \middle| B_T \right] \right] = \mathbb{E}[h(B_{[0,T]})]$$

by (3.56). This finishes the proof.

12. Let $\{W_t, t \geq 0\}$ be a Brownian motion. Find $\mathbb{P}(W_3 \leq 1 | W_2 = \frac{1}{2})$.

Solution: Note that W_2 and $W_3 - W_2$ are independent.

$$\mathbb{P}(W_3 \leq 1 | W_2 = \frac{1}{2}) = \mathbb{P}(W_3 - W_2 \leq \frac{1}{2} | W_2 = \frac{1}{2}) = \mathbb{P}(W_3 - W_2 \leq \frac{1}{2})$$

Since $W_3 - W_2 \sim N(0,1)$, $\mathbb{P}(W_3 - W_2 \leq \frac{1}{2}) = \int_{-\infty}^{1/2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \approx 0.6915$.

13. (To compare with Question 12.) Let $\{W_t, t \geq 0\}$ be a Brownian motion with $W_0 = 0$. Find $\mathbb{P}(W_2 \leq 1 | W_3 = \frac{1}{2})$.

Solution: By (3.53), if $t_1 < t_2$,

$$\rho_{W_{t_1}|W_{t_2}}(x_1|x_2) = \frac{\rho_{W_{t_1},W_{t_2}}(x_1,x_2)}{\rho_{W_{t_2}}(x_2)} = \frac{\sqrt{t_2}}{\sqrt{2\pi}\sqrt{t_1(t_2-t_1)}} e^{-\frac{x_1^2}{2t_1} - \frac{(x_2-x_1)^2}{2(t_2-t_1)} + \frac{x_2^2}{2t_2}}$$

$$= \frac{1}{\sqrt{2\pi \frac{t_1(t_2-t_1)}{t_2}}} e^{-\frac{(t_2 x_1 - t_1 x_2)^2}{2t_1 t_2 (t_2-t_1)}} = \frac{1}{\sqrt{2\pi \frac{t_1(t_2-t_1)}{t_2}}} e^{-\frac{(x_1 - \frac{t_1}{t_2}x_2)^2}{2\frac{t_1(t_2-t_1)}{t_2}}} \sim N(\frac{t_1}{t_2}x_2, \frac{t_1(t_2-t_1)}{t_2}). \quad (3.62)$$

Hence $W_2 | W_3 = \frac{1}{2} \sim N(\frac{2}{3}\frac{1}{2}, \frac{2}{3}) = N(\frac{1}{3}, \frac{2}{3})$ with pdf $\frac{1}{\sqrt{2\pi\frac{2}{3}}} e^{-\frac{(x_2-\frac{1}{3})^2}{2\times\frac{2}{3}}}$,

$$\mathbb{P}(W_2 \leq 1 | W_3 = \frac{1}{2}) = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi\frac{2}{3}}} e^{-\frac{(x_2-\frac{1}{3})^2}{2\times\frac{2}{3}}} dx_2 \overset{u=\frac{x_2-\frac{1}{3}}{\sqrt{\frac{2}{3}}}}{=} \int_{-\infty}^{\sqrt{2/3}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \approx 0.7929.$$

14. Let $X_t = e^{W_t - \frac{1}{2}t}$ be the price of a stock where W_t is a Brownian motion with $W_0 = 0$. Find $\mathbb{P}(X_3 \leq 3)$. [Hint: Event $X_3 \leq 3$ and event $\log X_3 \leq \log 3$ are equivalent.]

    Solution:

$$\begin{aligned}\mathbb{P}(X_3 \leq 3) &= \mathbb{P}(\log X_3 \leq \log 3) = \mathbb{P}(W_3 - \frac{3}{2} \leq \log 3) = \mathbb{P}(W_3 \leq \frac{3}{2} + \log 3) \\ &= \mathbb{P}(\frac{W_3}{\sqrt{3}} \leq \frac{\frac{3}{2} + \log 3}{\sqrt{3}}) = \int_{-\infty}^{\frac{\frac{3}{2}+\log 3}{\sqrt{3}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx 0.9332.\end{aligned}$$

15. Stock prices are sometimes modeled by distributions other than lognormal in order to fit the empirical data more accurately. For instance, soon after his Black-Scholes-Merton work, Merton introduces a jump diffusion model for stock prices. A special case of Merton's model assumes that the underlying stock price S satisfies

$$S = e^Y, \qquad Y = X_1 + \sum_{i=1}^{X_2} Z_i \tag{3.63}$$

    where X_1 is $N(\mu, \sigma^2)$, X_2 is Poisson with parameter λ [32], Z_i is $N(0, \nu^2)$, and X_1, X_2, $\{Z_i\}$ are all independent. The evaluation of call options involves expected values such as

$$\mathbb{E}[(S - K)^+], \tag{3.64}$$

    where K is some positive constant. Compute this expected value.

    Solution: For every $n \geq 0$, we can compute the conditional expected value

$$v_n = \mathbb{E}[(S - K)^+ | X_2 = n].$$

    Recall that the sum of jointly normally distributed random variables are still normally distributed. So, conditional on $X_2 = n$, $Y = X_1 + Z_1 + \cdots + Z_n$ is normally distributed as $N(\mu, \sigma^2 + n\nu^2)$ since $\mathbb{E}[Y] = \mu$ and $\text{Var}[Y] = \sigma^2 + n\nu^2$ by the independence assumption. By (3.16), we get

$$v_n = \mathbb{E}[(S - K)^+ | X_2 = n] = e^{\mu + \frac{1}{2}(\sigma^2 + n\nu^2)} N(d_{n,+}) - KN(d_{n,-}) \tag{3.65}$$

    with $d_{n,-} = \frac{\log \frac{1}{K} + \mu}{\sqrt{\sigma^2 + n\nu^2}}$ and $d_{n,+} = d_{n,-} + \sqrt{\sigma^2 + n\nu^2}$. By (3.56) of Question 8,

$$\begin{aligned}\mathbb{E}[(S - K)^+] &= \mathbb{E}[\mathbb{E}[(S - K)^+ | X_2]] = \sum_{n=0}^{\infty} \mathbb{E}[(S - K)^+ | X_2 = n] \mathbb{P}(X_2 = n) \\ &= \sum_{n=0}^{\infty} v_n \mathbb{P}(X_2 = n) = e^{-\lambda} \sum_{n=0}^{\infty} v_n \frac{\lambda^n}{n!}.\end{aligned}$$

    The last expression can be evaluated numerically.

[32] which means $\mathbb{P}(X_2 = n) = e^{-\lambda} \frac{\lambda^n}{n!}$ for $n = 0, 1, 2, 3, \ldots$.

16. (Cholesky factorization and independent increment) Show that the right hand side of (3.38), called Σ, has a decomposition

$$\Sigma = \mathbb{E}[(Z-M)(Z-M)^\top] = \begin{pmatrix} t_1 & t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & t_2 & \cdots & t_2 \\ t_1 & t_2 & t_3 & \cdots & t_3 \\ \vdots & \vdots & \vdots & & \vdots \\ t_1 & t_2 & t_3 & \cdots & t_n \end{pmatrix} = AA^\top \tag{3.66}$$

where

$$A = \begin{pmatrix} \sqrt{t_1} & 0 & 0 & \cdots & 0 \\ \sqrt{t_1} & \sqrt{t_2-t_1} & 0 & \cdots & 0 \\ \sqrt{t_1} & \sqrt{t_2-t_1} & \sqrt{t_3-t_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \sqrt{t_1} & \sqrt{t_2-t_1} & \sqrt{t_3-t_2} & \cdots & \sqrt{t_n-t_{n-1}} \end{pmatrix}. \tag{3.67}$$

Let $X = (X_1, \cdots, X_n)^\top \sim N(\mathbf{0}, I_n)$, which means $X_i \in N(0,1)$ and X_i's are independent. **Show** that $AX \sim N(\mathbf{0}, \Sigma)$. By the way, as an application of the result, one can generate $Z = (W_{t_1}, \cdots, W_{t_n})^\top$ as AX. This is precisely the telescoping sum (3.36) and is also the method used in the Matlab code at the beginning of this Chapter:

```
for j = 2:N                    % start the iteration
    dW(j) = sqrt(dt)*randn;  % general increment
    W(j) = W(j-1) + dW(j);
end
```

Proof: By direct calculation, one can easily check that $AA^\top = \Sigma$. By (3.5), we know $AX \sim N(\mathbf{0}, AI_nA^\top) = N(\mathbf{0}, AA^\top) = N(\mathbf{0}, \Sigma)$.

4.2 Homework IV

(Only submit solutions to Questions 6,9,10.)

1. (The weak law of large number) Suppose $\{Y_i\}$ are i.i.d. (independent identically distributed) random variables with $\mathbb{E}Y_i = a$ and $\text{Var}Y_i = b < \infty$. Show that $\frac{1}{N}\sum_{i=1}^N Y_i \to a$ (which is equivalent to $\frac{1}{N}\sum_{i=1}^N (Y_i - a) \to 0$) in the mean square sense as $N \to \infty$.

   Proof: $\mathbb{E}[(\frac{1}{N}\sum_{i=1}^N Y_i - a)^2] = \frac{1}{N^2}\mathbb{E}[(\sum_{i=1}^N Y_i - Na)^2] = \frac{1}{N^2}\sum_{i=1}^N \mathbb{E}[(Y_i - a)^2] = \frac{b}{N} \to 0$ as $N \to \infty$.

2. (Another proof of Question 6 of Homework III) Let $\{W_s : 0 \le s \le T\}$ be a 1-dimensional Brownian motion with $W_{t_0} = W_0 = 0$. Given $t > 0$ and $N \in \mathbb{Z}_+$, let $\delta t = t/N$ and $t_j = j\delta t$. Use the weak law of large number to prove that

$$\lim_{\delta t \to 0} \sum_{j=0}^{N-1} (W_{t_{j+1}} - W_{t_j})^2 = t \tag{4.38}$$

   in the mean square sense. [44]

   Proof: $\mathbb{E}\left(\frac{(W_{t_{j+1}} - W_{t_j})^2}{\delta t}\right) = 1$.

$$\lim_{\delta t \to 0} \sum_j (W_{t_{j+1}} - W_{t_j})^2 = t \lim_{N \to \infty} \frac{\sum_{j=0}^{N-1} \frac{(W_{t_{j+1}} - W_{t_j})^2}{\delta t}}{N} = t \times 1 = 1. \tag{4.39}$$

3. Use the same notation as in Question 2. Show that W_t has unbounded first variation, which means

$$\lim_{N \to \infty} \sum_{j=0}^{N-1} |W_{t_{j+1}} - W_{t_j}| = \infty. \tag{4.40}$$

   Proof: Since

$$\sum_{j=1}^{N-1} (W_{t_{j+1}} - W_{t_j})^2 \le \left(\max_{0 \le k \le N-1} |W_{t_{k+1}} - W_{t_k}|\right)\left(\sum_{j=1}^{N-1} |W_{t_{j+1}} - W_{t_j}|\right),$$

   we know

$$\infty > \sum_{j=1}^{N-1} |W_{t_{j+1}} - W_{t_j}| \ge \frac{\sum_{j=1}^{N-1} (W_{t_{j+1}} - W_{t_j})^2}{\max_{0 \le k \le N-1} |W_{t_{k+1}} - W_{t_k}|}. \tag{4.41}$$

[44] FYI: Let Y_i, $i = 1, 2, \cdots$, be a sequence of independent and identically distributed random variables with mean $E[Y_i] = \mu$. The strong law of large number says that $\frac{\sum_{i=1}^N (Y_i - \mu)}{N} \to 0$ with probability 1. It means that for almost all ω, $\frac{\sum_{i=1}^N (Y_i(\omega) - \mu)}{N} \to 0$. So, if we use the strong law of large number, we can conclude that (4.38) happens with probability 1.

As W_t is continuous, $\lim_{N\to\infty}\max_{0\le k\le N-1}|W_{t_{k+1}} - W_{t_k}| = 0$. Using (4.38) and letting $N\to\infty$ in (4.41), we obtain

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}|W_{t_{j+1}} - W_{t_j}| = \infty.$$

4. Let Y_i, $i = 1, 2, \cdots$, be a sequence of independent and identically distributed random variables with mean $E[Y_i] = \mu$ and $\text{Var}[Y_i] = \sigma^2$. The central limit theorem says that

$$\lim_{n\to\infty}\frac{\sum_{i=1}^n (Y_i - \mu)}{\sqrt{n}} \to Z \sim N(0, \sigma^2) \tag{4.42}$$

in the sense of distribution. It means that as n increases, the distribution of the random variable $\frac{\sum_{i=1}^n (Y_i-\mu)}{\sqrt{n}}$ becomes closer and closer to that of $N(0,\sigma^2)$ random variable Z.

Now, consider the symmetric random walk (defined in Example 2.11)

$$M_k = \sum_{i=1}^k Z_i$$

where $M_0 = 0$, $\mathbb{P}(Z_i = 1) = \mathbb{P}(Z_i = -1) = \frac{1}{2}$. Fix t, define $W_t^{(n)} = \sqrt{\frac{t}{n}}M_n$. Show that

$$\lim_{n\to\infty} W_t^{(n)} \to Z \sim N(0, t) \tag{4.43}$$

in the sense of distribution.

Solution: $\mathbb{E}[Z_i] = 1 \times \frac{1}{2} + (-1)\times\frac{1}{2} = 0$. $\text{Var}[Z_i] = \mathbb{E}[Z_0^2] - (\mathbb{E}[Z_i])^2 = 1$. By the central limit theorem

$$\frac{\sum_{i=1}^k (Z_i - \mathbb{E}[Z_i])}{\sqrt{n}} \to Z \sim N(0,1)$$

in the sense of distribution. Hence

$$W_t^{(n)} = \sqrt{t}\,\frac{\sum_{i=1}^k (Z_i - \mathbb{E}[Z_i])}{\sqrt{n}} \to \sqrt{t}Z \sim N(0,t)$$

in the sense of distribution.

5. Recall (2.28)

$$\tilde{\mathbb{E}}_n[X](\omega_1\cdots\omega_n) = \sum_{\omega_{n+1}\cdots\omega_N} q_u^{\#H(\omega_{n+1}\cdots\omega_N)} q_d^{\#T(\omega_{n+1}\cdots\omega_N)} X(\omega_1\cdots\omega_n\omega_{n+1}\cdots\omega_N)$$

and the S_3 defined in Example 2.4. **Show** that

$$\tilde{\mathbb{E}}_2[S_3](HH)\mathbb{P}(A_{HH}) = \sum_{\omega\in A_{HH}} S_3(\omega)\mathbb{P}(\omega), \tag{4.44}$$

where $A_{HH} = \{HHH, HHT\}$ is defined in (2.40). Since $\tilde{\mathbb{E}}_2[S_3](\omega)$ does not change value on A_{HH}, (4.44) can be rewritten as

$$\int_{A_{HH}} \tilde{\mathbb{E}}_2[S_3](\omega)d\mathbb{P}(\omega) = \int_{A_{HH}} S_3(\omega)d\mathbb{P}(\omega). \tag{4.45}$$

Similarly, **prove**

$$\int_{A_{HT}} \tilde{\mathbb{E}}_2[S_3](\omega)d\mathbb{P}(\omega) = \int_{A_{HT}} S_3(\omega)d\mathbb{P}(\omega).\text{[45]} \tag{4.47}$$

6. Let W_t be 1-dimensional Brownian motion with $W_0 = 0$ and let $X_t = e^{W_t^2}$. Show that

$$\mathbb{E}[X_t^2] = \frac{1}{\sqrt{1-4t}}, \qquad t \in [0, 1/4).$$

Solution: Note that one can write $2 - \frac{1}{2t}$ as $-\frac{1}{2a}$ with $a = \frac{1}{\frac{1}{t}-4}$.

$$\begin{aligned}\mathbb{E}[X_t^2] =& \mathbb{E}[e^{2W_t^2}] = \int_{-\infty}^{\infty} e^{2x^2}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}e^{\left(2-\frac{1}{2t}\right)x^2}dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2a}}dx \\ =& \int_{-\infty}^{\infty}\sqrt{\frac{a}{t}}\frac{1}{\sqrt{2\pi a}}e^{-\frac{x^2}{2a}}dx = \sqrt{\frac{a}{t}} = \frac{1}{\sqrt{1-4t}}.\end{aligned}$$

7. Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \geq 0$, be a filtration for this Brownian motion. Show that $W^2(t) - t$ is a martingale, i.e., $\mathbb{E}[W^2(t) - t|\mathcal{F}(s)] = W^2(s) - s$ or equivalently $\mathbb{E}[W^2(t) - W^2(s)|\mathcal{F}(s)] = t - s$ for $0 \leq s \leq t$.

   Proof:

$$\begin{aligned}&\mathbb{E}[W^2(t) - W^2(s)|\mathcal{F}(s)] \\ =&\mathbb{E}[(W(t) - W(s))^2 + 2W(t)W(s) - 2W^2(s)|\mathcal{F}(s)] \\ \overset{(4.22)}{=}&\mathbb{E}[(W(t) - W(s))^2|\mathcal{F}(s)] + 2W(s)\mathbb{E}[W(t) - W(s)|\mathcal{F}(s)] = t - s.\end{aligned}$$

8. Use the Itô isometry which is Corollary 4.1 and property (iii) of Theorem 4.1 to answer the following questions. Let W_t be 1-dimensional Brownian motion with $W_0 = 0$. Define

$$X = \int_0^T t dW_t, \qquad \text{and} \qquad Y = \int_0^T (T-t)dW_t.$$

[45] By the same method, one can prove that

$$\int_A \tilde{\mathbb{E}}_2[S_3](\omega)d\mathbb{P}(\omega) = \int_A S_3(\omega)d\mathbb{P}(\omega) \tag{4.47}$$

for $A \in \{A_{HH}, A_{HT}, A_{TH}, A_{TT}\}$ or more generally for any $A \in \mathcal{F}_2$ defined by (2.41). Hence if one recalls the standard definition of conditional expectation of $\tilde{\mathbb{E}}[S_3|\mathcal{F}_2]$ in (4.20), we have $\tilde{\mathbb{E}}_2[S_3] = \tilde{\mathbb{E}}[S_3|\mathcal{F}_2]$.

Determine $\mathbb{E}[X]$, $\text{Var}[X]$, $\mathbb{E}[Y]$, $\text{Var}[Y]$. Note that

$$X + Y = \int_0^T T dW_t = TW_T.$$

Determine $\text{Var}[X+Y]$ and then determine $\text{Cov}(X,Y) \stackrel{\text{def}}{=} \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$.

Solution: $\mathbb{E}X = \mathbb{E}Y = 0$. $\text{Var}X = \mathbb{E}[X^2] = \mathbb{E}[\left(\int_0^T t dW_t\right)^2] = \mathbb{E}\int_0^T t^2 dt = \frac{T^3}{3}$.

$\text{Var}Y = \mathbb{E}[Y^2] = \mathbb{E}[\left(\int_0^T (T-t) dW_t\right)^2] = \mathbb{E}\int_0^T (T-t)^2 dt = \frac{T^3}{3}$.

$\text{Var}[X+Y] = \mathbb{E}[(X+Y)^2] = T^2 T = T^3$.

Since $\mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2 + Y^2 + 2XY]$,

$$\mathbb{E}[XY] = \frac{1}{2}\left(\mathbb{E}[(X+Y)^2] - \mathbb{E}[X^2] - \mathbb{E}[Y^2]\right) = \frac{T^3}{6}.$$

$$\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{T^3}{6}.$$

9. For the M_t defined in (4.28), prove that

$$\mathbb{E}[M_t] = 0.$$

(Hint: Use (4.24) with $\mathcal{G} = \mathcal{F}_0$.) Then find the variance of

$$M_t = \int_0^t e^{-\alpha s} dW_s$$

for $\alpha > 0$.

Proof: $\mathbb{E}[M_t] = \mathbb{E}[\mathbb{E}[M_t|\mathcal{F}_0]] = \mathbb{E}[M_0] = \mathbb{E}[0] = 0$.

By Itô isometry (4.12), $\text{Var}[M_t] = \mathbb{E}[(M_t - \mathbb{E}[M_t])^2] = \mathbb{E}\left[\left(\int_0^t e^{-\alpha s} dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t e^{-2\alpha s} ds\right] = \frac{1-e^{-2\alpha t}}{2\alpha}$.

10. Let

$$Y_t = \int_0^t \sqrt{|W_s|} dW_s,$$

where $|W_s|$ denotes the absolute value of W_s. Determine $\text{Var}[Y]$.

Solution:

$$\begin{aligned}\mathbb{E}[|W_t|] &= \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \stackrel{y=x/\sqrt{t}}{=} \sqrt{\frac{t}{2\pi}} \int_{-\infty}^{\infty} |y| e^{-\frac{y^2}{2}} dy \\ &= \sqrt{\frac{2t}{\pi}} \int_0^{\infty} y e^{-\frac{y^2}{2}} dy \stackrel{u=y^2/2}{=} \sqrt{\frac{2t}{\pi}} \int_0^{\infty} e^{-u} du = \sqrt{\frac{2t}{\pi}}\end{aligned}$$

By Itô isometry,

$$\mathrm{Var}[Y_t] = \mathbb{E}[Y_t^2] = \mathbb{E}\left(\int_0^t |W_s| ds\right) = \int_0^t \mathbb{E}|W_s| ds = \int_0^t \sqrt{\frac{2}{\pi}}\sqrt{s} ds = \sqrt{\frac{2}{\pi}}\frac{2}{3}t^{3/2} = \frac{1}{3\sqrt{\pi}}(2t)^{3/2}.$$

11. Let W_t be one-dimensional Brownian motion, $\sigma \in \mathbb{R}$ be constant and $s \geq t \geq 0$. (1) Use Question 1 of Homework III to prove that

$$\mathbb{E}[e^{\sigma(W_s - W_t)}] = e^{\frac{1}{2}\sigma^2(s-t)}. \tag{4.48}$$

(2) Prove directly from the definition that

$$M_t = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}, \qquad t \geq 0 \tag{4.49}$$

is a martingale with respect to $\mathcal{F}_t$ ((4.27)). Then use this result to prove that $\mathbb{E}[M_t] = 1$ for all $t \geq 0$ if $W_0 = 0$. (Hint: If $s \geq t$, then $\mathbb{E}[M_s|\mathcal{F}_t] =$
$\mathbb{E}[M_t e^{\sigma(W_s - W_t) - \frac{1}{2}\sigma^2(s-t)}|\mathcal{F}_t] \overset{(4.22)}{=} M_t e^{-\frac{1}{2}\sigma^2(s-t)}\mathbb{E}[e^{\sigma(W_s-W_t)}|\mathcal{F}_t]$.)

Proof: (1) $W_s - W_t \sim N(0, s-t)$. $\mathbb{E}[e^{\sigma(W_s-W_t)}] = e^{\frac{1}{2}\sigma^2(s-t)}$.

(2)

$$\begin{aligned}
\mathbb{E}[M_s|\mathcal{F}_t] =& M_t e^{-\frac{1}{2}\sigma^2(s-t)}\mathbb{E}[e^{\sigma(W_s-W_t)}|\mathcal{F}_t] \\
\overset{(4.25),(3.12)}{=}& M_t e^{-\frac{1}{2}\sigma^2(s-t)}\mathbb{E}[e^{\sigma(W_s-W_t)}] \\
=& M_t e^{-\frac{1}{2}\sigma^2(s-t)} e^{\frac{1}{2}\sigma^2(s-t)} = M_t.
\end{aligned}$$

$\mathbb{E}[M_t] \overset{(4.24)}{=} \mathbb{E}[\mathbb{E}[M_t|\mathcal{F}_0]] = \mathbb{E}[M_0] = \mathbb{E}[1] = 1.$

12. (Page 324 of "Dynamic Asset Pricing Theory", 3rd edition, by Duffie. An equivalent definition of conditional expectation. This is for your information only, in case you will read Duffie or other books later in your career. It won't be tested.) For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Ω finite, if $\mathcal{G}$ is a sub-σ-algebra[46], then $\mathcal{G}$ represents in some sense "less information". The conditional expectation of X given a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ is defined as any $\mathcal{G}$-measurable random variable denoted by $\mathbb{E}[X|\mathcal{G}]$, satisfying the property that

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]Z] = \int_\Omega \mathbb{E}[X|\mathcal{G}](\omega)Z(\omega)d\mathbb{P}(\omega) = \int_\Omega X(\omega)Z(\omega)d\mathbb{P}(\omega) = \mathbb{E}[XZ] \tag{4.50}$$

for any $\mathcal{G}$-measurable random variable Z. Please compare it with (4.20) and show that (4.50) implies (4.20). Then use (4.22) to show that (4.20) implies (4.50).

[46] meaning $\mathcal{G}$ is a σ-algebra and is also a subset of $\mathcal{F}$.

Proof: "(4.50)$\Rightarrow$(4.20)": For any set $A \in \mathcal{G}$, define $I_A = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{otherwise.} \end{cases}$ Then 1_A is $\mathcal{G}$-measurable since $A \in \mathcal{G}$. So, we can let $Z = 1_A$ in (4.50). This leads to

$$\int_\Omega \mathbb{E}[X|\mathcal{G}](\omega) 1_A(\omega) d\mathbb{P}(\omega) = \int_\Omega X(\omega) 1_A(\omega) d\mathbb{P}(\omega)$$

or

$$\int_A \mathbb{E}[X|\mathcal{G}](\omega) d\mathbb{P}(\omega) = \int_A X(\omega) d\mathbb{P}(\omega)$$

which is precisely (4.20).

"(4.20)$\Rightarrow$(4.50)": (4.20) implies (4.22) $\mathbb{E}[ZX|\mathcal{G}] = Z\mathbb{E}[X|\mathcal{G}]$. Taking $\mathbb{E}$ on both sides, we obtain

$$\mathbb{E}[\mathbb{E}[ZX|\mathcal{G}]] = \mathbb{E}[Z\mathbb{E}[X|\mathcal{G}]].$$

Since the left hand side is $\mathbb{E}[ZX]$ by (4.24), we get (4.50).

13. (Continue with Question 12. Page 324 of "Dynamic Asset Pricing Theory", 3rd edition, by Duffie. This is for your information only. It won't be tested.) If Y is a nonnegative random variable with $\mathbb{E}Y = 1$, then we can create a new probability measure $\tilde{\mathbb{P}}$ from the old probability measure $\mathbb{P}$ by defining

$$\tilde{\mathbb{P}}(A) = \mathbb{E}(1_A Y) \tag{4.51}$$

for any event A, where $1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{otherwise.} \end{cases}$ Note that (4.51) can also be written as $\int_A d\tilde{\mathbb{P}}(\omega) = \int_\Omega Y(\omega) 1_A(\omega) d\mathbb{P}(\omega) = \int_A Y(\omega) d\mathbb{P}(\omega)$. So, we write $Y = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ and call Y the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$. (4.51) can also be written as $\tilde{\mathbb{E}}(1_A) = \mathbb{E}(Y 1_A)$ for any set $A \in \Omega$, where $\tilde{\mathbb{E}}$ denotes the expectation of under $\tilde{\mathbb{P}}$ and $\mathbb{E}$ denotes the expectation of under $\mathbb{P}$. With some standard mathematics/probability technics which you do not need to know the details, the last equation implies that for any random variable X,

$$\tilde{\mathbb{E}}(X) = \mathbb{E}(YX). \tag{4.52}$$

Definition 4.7 *If $\tilde{\mathbb{P}}(A) > 0$ whenever $\mathbb{P}(A) > 0$, and vice verse, then $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are said to be equivalent measures; they have the same events of probability zero.*

Prove that if $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$ and $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$, then

$$\tilde{\mathbb{E}}(Z|\mathcal{G}) = \frac{1}{\mathbb{E}(\xi|\mathcal{G})} \mathbb{E}(\xi Z|\mathcal{G}), \tag{4.53}$$

where $\xi = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$.

Proof: By definition (4.50), we need to prove that for any random variable Y being $\mathcal{G}$-measurable,

$$\tilde{\mathbb{E}}[ZY] = \tilde{\mathbb{E}}[\frac{1}{\mathbb{E}(\xi|\mathcal{G})}\mathbb{E}(\xi Z|\mathcal{G})Y], \tag{4.54}$$

which, by the definition of $\tilde{\mathbb{E}}$, is equivalent to

$$\mathbb{E}[\xi ZY] = \mathbb{E}[\xi\frac{1}{\mathbb{E}(\xi|\mathcal{G})}\mathbb{E}(\xi Z|\mathcal{G})Y]. \tag{4.55}$$

But by (4.24) $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]]$, the right hand side of (4.55) is

$$\mathbb{E}\left[\mathbb{E}\left[\xi\frac{1}{\mathbb{E}(\xi|\mathcal{G})}\mathbb{E}(\xi Z|\mathcal{G})Y\middle|\mathcal{G}\right]\right] \stackrel{(4.22)}{=} \mathbb{E}\left[\mathbb{E}[\xi|\mathcal{G}]\frac{1}{\mathbb{E}(\xi|\mathcal{G})}\mathbb{E}(\xi Z|\mathcal{G})Y\right]$$

since Y and $\frac{1}{\mathbb{E}(\xi|\mathcal{G})}\mathbb{E}(\xi Z|\mathcal{G})$ are already $\mathcal{G}$-measurable. Hence the right hand side of (4.55) becomes $\mathbb{E}[\mathbb{E}(\xi Z|\mathcal{G})Y] \stackrel{(4.22)}{=} \mathbb{E}[\mathbb{E}[\xi ZY|\mathcal{G}]] \stackrel{(4.24)}{=} \mathbb{E}[\xi ZY]$. This proves (4.55).

14. (Generalization of Question 9 of Homework III) Let X and Y be a pair of jointly normal random variables with joint density

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right) \\ &= \frac{1}{2\pi\det(\Sigma)^{1/2}}\exp\left(-\frac{1}{2}(x-\mu_1, y-\mu_2)\Sigma^{-1}\begin{pmatrix} x-\mu_1 \\ y-\mu_2 \end{pmatrix}\right) \end{aligned} \tag{4.56}$$

where σ_1, $\sigma_2 > 0$, $|\rho| < 1$, μ_1, μ_2 are real numbers, and $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. Define $W = Y - \frac{\rho\sigma_2}{\sigma_1}X$. Prove W and X are independent and show that

$$\mathbb{E}[Y|X] = \frac{\rho\sigma_2}{\sigma_1}(X-\mu_1) + \mu_2. \tag{4.57}$$

Proof: Since we know Σ is the covariance matrix, we know $\text{Var}(X) = \sigma_1^2$, $\text{Cov}(X,Y) = \mathbb{E}[(X-\mu_1)(Y-\mu_2)] = \rho\sigma_1\sigma_2$. Then

$$\text{Cov}(X,W) = \text{Cov}(X,Y) - \frac{\rho\sigma_2}{\sigma_1}\text{Cov}(X,X) = \rho\sigma_1\sigma_2 - \rho\sigma_1\sigma_2 = 0.$$

Since X and W are jointly normal distributed and are un-correlated, X and W are independent.

Because $Y = \frac{\rho\sigma_2}{\sigma_1}X + W$ and X and W are indenpendent,

$$\mathbb{E}[Y|X] = \mathbb{E}[\frac{\rho\sigma_2}{\sigma_1}X + W|X] \stackrel{(4.21)}{=} \frac{\rho\sigma_2}{\sigma_1}X + \mathbb{E}[W|X] = \frac{\rho\sigma_2}{\sigma_1}(X-\mu_1) + \mu_2.$$

In the last step, we have used $\mathbb{E}[W|X] \stackrel{(4.25)}{=} \mathbb{E}[W] = \mu_2 - \frac{\rho\sigma_2}{\sigma_1}\mu_1$.

15. Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \geq 0$, be a filtration for this Brownian motion. Show that $W^2(t) - t$ is a martingale, i.e., $\mathbb{E}[W^2(t) - t|\mathcal{F}(s)] = W^2(s) - s$ or equivalently $\mathbb{E}[W^2(t) - W^2(s)|\mathcal{F}(s)] = t - s$ for $0 \leq s \leq t$. (Hint: Write $W^2(t)$ as $(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s)$.)

Proof:

$$\begin{aligned}\mathbb{E}[W^2(t) - W^2(s)|\mathcal{F}(s)] =&\mathbb{E}[(W(t) - W(s))^2 + 2W(t)W(s) - 2W^2(s)|\mathcal{F}(s)]\\ =&\mathbb{E}[(W(t) - W(s))^2|\mathcal{F}(s)] + 2W(s)\mathbb{E}[W(t) - W(s)|\mathcal{F}(s)] = t - s.\end{aligned}$$

16. Let $\{W_s : s \geq 0\}$ be a 1-dimensional Brownian motion with $W_0 = 0$. Show that the covariance of $\int_0^s W_u du$ and $\int_0^t W_v dv$ is

$$\text{Cov}\left(\int_0^s W_u du, \int_0^t W_v dv\right) = \frac{1}{3}\min\{s^3, t^3\} + \frac{1}{2}|t - s| \min\{s^2, t^2\}. \tag{4.58}$$

Proof: By definition, when $t \geq s$,

$$\begin{aligned}&\text{Cov}\left(\int_0^s W_u du, \int_0^t W_v dv\right)\\ =&\mathbb{E}\left[\left(\int_0^s W_u du\right)\left(\int_0^t W_v dv)\right)\right] - \mathbb{E}\left[\left(\int_0^s W_u du\right)\right]\mathbb{E}\left[\left(\int_0^t W_v dv)\right)\right]\\ =&\mathbb{E}\left[\int_0^s W_u\left(\int_0^t W_v dv\right) du\right] = \mathbb{E}\left[\int_0^s\left(\int_0^t W_u W_v dv\right) du\right]\\ =&\left[\int_0^s\left(\int_0^t \mathbb{E}[W_u W_v] dv\right) du\right] = \left[\int_0^s\left(\int_0^t \min(u, v) dv\right) du\right]\\ =&\left[\int_0^s\left(\int_u^t \min(u, v) dv\right) du\right] + \left[\int_0^s\left(\int_0^u \min(u, v) dv\right) du\right] \qquad \text{use } t \geq s\\ =&\left[\int_0^s\left(\int_u^t u dv\right) du\right] + \left[\int_0^s\left(\int_0^u v dv\right) du\right]\\ =&\left[\int_0^s u(t - u) du\right] + \left[\int_0^s u^2/2 du\right]\\ =&ts^2/2 - s^3/6 = \frac{1}{3}s^3 + \frac{1}{2}(t - s)s^2.\end{aligned}$$

When $t \leq s$, we switch s and t in the above computation and get

$$\text{Cov}\left(\int_0^s W_u du, \int_0^t W_v dv\right) = \text{Cov}\left(\int_0^t W_v dv, \int_0^s W_u du\right) = st^2/2 - t^3/6 = \frac{1}{3}t^3 + \frac{1}{2}(s - t)t^2.$$

Combining them together, we have proved the desired result.

17. Let $\{W_s : s \geq 0\}$ be a 1-dimensional Brownian motion with $W_0 = 0$. Show that $X_t = W_t^3 - 3tW_t$ is a martingale, i.e., show that for $s \leq t$

$$\mathbb{E}[W_t^3 - 3tW_t|\mathcal{F}_s] = W_s^3 - 3sW_s.$$

[Hint: Rewrite $W_t^3 - 3tW_t$ in terms of the increment $W_t - W_s$ and derive $W_t^3 - 3tW_t = (W_t - W_s)^3 + 3(W_t - W_s)^2 W_s + 3(W_t - W_s)W_s^2 + W_s^3 - 3t(W_t - W_s) - 3tW_s$.]