1.6 Homework I

(Only submit solutions to Questions 1,4,5,9.)

1. A straddle is a portfolio with long positions in a European call and a European put with the same strike price, maturity, and underlying. The straddle is seen to benefit from a movement in either direction away from the strike price. Show that the payoff of a straddle is $|S_T - K|$ if we ignore the premium by constructing the payoff table as in Example 1.2.

   Solution: The following is the payoff table for this strategy.

strategy	$S_T \leq K$	$K < S_T$
A long call at K	0	$S_T - K$
A long put at K	$K - S_T$	0
Total	$K - S_T$	$S_T - K$

2. Suppose that an amount A is invested for n years at an interest rate of R per annum. If the rate is compounded m times per annum, the terminal value of the investment is

$$A\left(1+\frac{R}{m}\right)^{mn}.$$

   If the interest rate is 10% and is measured with semiannual compounding, What is the value of \$100 at end of 1 year?

   Determine the limit

$$\lim_{m\to\infty} A\left(1+\frac{R}{m}\right)^{mn}.$$

   Solution:

$$\$100 \times (1+0.05)^2 = \$110.25.$$

   Recall $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$ and $\lim_{x\to\infty}(f(x))^k = (\lim_{x\to\infty} f(x))^k$. Hence

$$\lim_{m\to\infty} A\left(1+\frac{R}{m}\right)^{mn} = A\lim_{\frac{m}{R}\to\infty}\left(1+\frac{R}{m}\right)^{\frac{m}{R}Rn} = A\left(\lim_{\frac{m}{R}\to\infty}\left(1+\frac{R}{m}\right)^{\frac{m}{R}}\right)^{Rn} = Ae^{Rn}.$$

3. Prove Theorem 1.2. [Hint: Prove by contradiction. If the conclusion is false, one can construct a portfolio $\Phi_c = \Phi_1 - \Phi_2 + B$ which has arbitrage opportunity. You need to specify the value of B_t at some $t = t^*$.]

   Proof: We prove by contradiction. If the conclusion is false, then there exists a $t^* \in [0, T)$ such that

$$V_{t^*}(\Phi_1) < V_{t^*}(\Phi_2).$$

   Now, we define

$$E = V_{t^*}(\Phi_2) - V_{t^*}(\Phi_1) > 0 \tag{1.29}$$

and construct a portfolio Φ_c at $t = t^*$

$$\Phi_c = \Phi_1 - \Phi_2 + B$$

where B is the risk-free asset (bond) of the market that satisfies $B_{t^*} = E$. We now claim that Φ_c has arbitrage opportunity in $[t^*, T]$. This proves (1.12).

To prove the claim, note that

$$\begin{aligned} V_{t^*}(\Phi_c) &= V_{t^*}(\Phi_1) - V_{t^*}(\Phi_2) + B_{t^*} = 0 \text{ by (1.29) and } B_{t^*} = E \\ V_T(\Phi_c) &= V_T(\Phi_1) - V_T(\Phi_2) + B_T > 0 \text{ by (1.11).} \qquad \square \end{aligned}$$

4. Consider a European put option on a non-dividend-paying stock when the stock price is \$38, the strike price is \$40, the time to maturity is 3 months, and the risk-free rate of interest is 10% per annum. Find a lower bound for the option price.

   Solution $S_0 = 38$, $K = 40$, $T = 0.25$, and $r = 0.10$. From (1.16), a lower bound for p_0 is $Ke^{-rT} - S_0$ which is $40e^{-0.1\times 0.25} - 38 = 1.01$.

5. The price of a non-dividend paying stock is \$19 and the price of a three-month European call option on the stock with a strike price of \$20 is \$1. The risk-free rate is 4% per annum. What is the price of a three-month European put option with a strike price \$20?

   Solution $c_0 = 1$, $T = 0.25$, $S_0 = 10$, $K = 20$, and $r = 0.04$. From put-call parity

   $$p_0 = c_0 + Ke^{-rT} - S_0$$

   or $p_0 = 1 + 20e^{-0.04\times 0.25} - 19 = 1.80$. So the European put price is \$1.80.

6. We want to prove (1.25): For any $K_0, K_1 \geq 0$, if $\theta \in [0, 1]$, $c_t(\theta K_1 + (1-\theta)K_0) \leq \theta c_t(K_1) + (1-\theta)c_t(K_0)$. To do that, we construct two portfolios at $t = 0$:

   $$\Phi_1 = \theta c(K_1) + (1-\theta)c(K_0), \qquad \Phi_2 = c(\theta K_1 + (1-\theta)K_0).$$

   Prove that $V_T(\Phi_1) \geq V_T(\Phi_2)$. Then (1.25) follows from Theorem 1.2.

   Proof: On the expiration date $t = T$,

   $$V_T(\Phi_1) = \theta(S_T - K_1)^+ + (1-\theta)(S_T - K_0)^+, \tag{1.30}$$
   $$V_T(\Phi_2) = (S_T - \theta K_1 - (1-\theta)K_0)^+. \tag{1.31}$$

   Without loss of generality, we can assume $K_1 \geq K_0$. Let $K_\theta = \theta K_1 + (1-\theta)K_0$. $K_1 \geq K_\theta \geq K_0$. There are 4 cases:

   1. $S_T \geq K_1$: $V_T(\Phi_1) = S_T - K_\theta = V_2(\Phi_2)$.
   2. $K_\theta \leq S_T < K_1$: $V_T(\Phi_1) = (1-\theta)(S_T - K_2)$. $V_T(\Phi_2) = (S_T - K_\theta) = \theta(S_T - K_1) + (1-\theta)(S_T - K_2) \leq V_T(\Phi_1)$.

3. $K_0 \le S_T < K_\theta$: $V_T(\Phi_1) = (1-\theta)(S_T - K_2)$. $V_T(\Phi_2) = 0 \le V_T(\Phi_1)$.
4. $S_T < K_0$: $V_T(\Phi_1) = 0 = V_2(\Phi_2)$.

Hence $V_T(\Phi_1) \ge V_T(\Phi_2)$. $\square$

7. A 1-month European put option on a non-dividend-paying stock is currently selling for \$2.50. The stock price is \$47, the strike price is \$50, and the risk-free interest rate is 6% per annum. What opportunities are there for an arbitrageur?

Solution: For European put option, if the market is arbitrage-free, we should have (1.16)

$$(Ke^{-rT} - S_0)^+ < p_0 < Ke^{-rT}$$

which implies

$$(50e^{-0.06\times(1/12)} - 47)^+ < p_0 < 50e^{-0.06\times(1/12)},$$

$$49.75 - 47 < p_0 < 49.75 \quad \Rightarrow \quad 2.75 < p_0 < 49.75.$$

Hence $(Ke^{-rT} - S_0)^+ < p_0$ is violated. Since the European put option is undervalued, the arbitrageur should buy the put option. Then he also need to buy the stock so that he can sell the stock to the option seller to close out. Hence an arbitrageur can build a portfolio

$$\Phi = p + S - B$$

by buying the put option, buying the stock, and borrowing B_0 dollars from the money market at $t = 0$. (This portfolio can also be seen from the condition $p_0 + S_0 - Ke^{-rT} < 0$.) The value of B_0 is determined by the requirement that

$$V_0(\Phi) = 0 = p_0 + S_0 - B_0.$$

Then

$$V_T(\Phi) = (K - S_T)^+ + S_T - B_0e^{rT} = (K - S_T)^+ + S_T - (p_0 + S_0)e^{rT}$$

Right now, $p_0 < (Ke^{-rT} - S_0)$. $-p_0e^{rT} > -(Ke^{-rT} - S_0)e^{rT} = -(K - S_0e^{rT})$. Hence

$$V_T(\Phi) > (K - S_T)^+ + S_T - (K - \cancel{S_0e^{rT}}) - \cancel{S_0e^{rT}} \ge 0.$$

In the last step, we have used the fact that $a^+ - a = \max(a, 0) - a \ge 0$.

If one uses $p_0 = (Ke^{-rT} - S_0) - \varepsilon$ with $\varepsilon = \$0.25$ for this problem, then $V_T(\Phi) \ge \varepsilon e^{rT}$.

8. A European call option and put option on a stock both have strike price of \$20 and an expiration date in three months. Both sell for \$3. The risk-free interest rate is 10% per annum, the current stock price is \$18. Identity the arbitrage opportunity open to a trader.

Solution: By (1.17),

$$c_0 + Ke^{-rT} = p_0 + S_0. \tag{1.32}$$

$$3 + 20e^{-0.1\times(3/12)} = p_0 + 18 \quad \Rightarrow \quad p_0 = 4.51$$

The put is then undervalued relative to the call. So, the arbitrageur should buy the put and sell the call. He also need to buy a stock so that he can sell it to the the buyer of the call if the stock price is high at the end of three months. This portfolio can also be seen from the violation of (1.32):

$$p_0 + S_0 - c_0 - Ke^{-rT} < 0. \tag{1.33}$$

So, the arbitrageur should build a portfolio

$$\Phi = p + S - c - B$$

by borrowing money B from the money market to buy the put, buy the stock, and sell the call. B_0 is determined by

$$0 = V_0(\Phi) = p_0 + S_0 - c_0 - B_0.$$

Since $p_0 + S_0 - c_0 < Ke^{-rT}$ by (1.33), we able to choose an $\varepsilon > 0$ so that $p_0 + S_0 - c_0 = Ke^{-rT} - \varepsilon$. For this problem, $\varepsilon = 20e^{-0.1\times 0.25} - 18 = 1.51$.

$$\begin{aligned}
V_T(\Phi) =& (K - S_T)^+ + S_T - (S_T - K)^+ - (p_0 + S_0 - c_0)e^{rT} \\
=& (K - S_T)^+ + S_T - (S_T - K)^+ - (Ke^{-rT} - \varepsilon)e^{rT} \\
=& \left\{ \begin{array}{ll} K & \text{if } S_T \le K \\ K & \text{if } S_T > K \end{array} \right\} - K + \varepsilon e^{rT} \\
=& \varepsilon e^{rT} > 0.
\end{aligned}$$

We know exactly the gain of the arbitrageur.

Remark: The above solution indeed proves why the put-call parity $p_0 + S_0 - c_0 - Ke^{-rT} = 0$ should be valid in an arbitrage-free market.

9. (Dividend Put-Call Parity Formula) Note that by the arbitrage-free principle, it is easy to show that when a stock pays a dividend D_1 at t_1, the stock's value is immediately reduced by the amount of the dividend. In other words, $\lim_{t\uparrow t_1} S_t - D_1 = \lim_{t\downarrow t_1} S_t$.

   Assume that a dividend D_j is paid at time t_j, where $0 < t_1 < t_2 < \cdots < t_n \le T$. Let D denote the present value of the dividend stream:

$$D = e^{-rt_1}D_1 + e^{-rt_2}D_2 + \cdots e^{-rt_n}D_n. \tag{1.34}$$

   Consider an European call option and an European put option, each with strike price K, maturity T, and underlying one share of S, assumed to be dividend-paying. We want to prove

$$c_0 + Ke^{-rT} + D = p_0 + S_0. \tag{1.35}$$

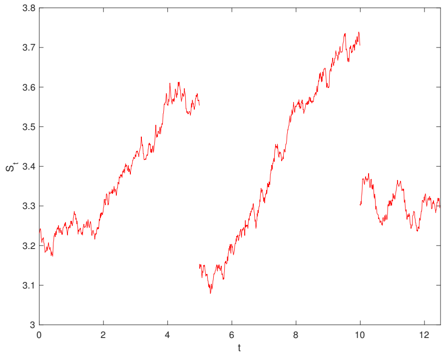

Figure 1.2: $t_1 = 5,\ t_2 = 10,\ D_1 = 0.4,\ D_2 = 0.4.$

The idea is to consider two portfolios at $t = 0$:

$$\Phi_1 = c + Ke^{-rT} + D,$$
$$\Phi_2 = p + S.$$

It means that Φ_1 consists of a call option, a zero-coupon bond with face value (par value) K and maturity T, and n zero-coupon bonds with face values D_k and maturity times t_k, $k = 1, 2, \cdots, n$. Φ_2 consists of a put option and the stock which will pay dividend D_j at t_j.

Now, you are asked to **prove** $V_T(\Phi_1) = V_T(\Phi_2)$. Then we can conclude $V_0(\Phi_1) = V_0(\Phi_2)$ which is precisely (1.35) by Corollary 1.1.

Solution: Please note that D contains n bonds. The ith bond will pay D_i at t_i. Then, at T, D_i becomes $D_i e^{r(T-t_i)}$. Hence

$$V_T(\Phi_1) = c_T + Ke^{-rT}e^{rT} + V_T(D) = (S_T - K)^+ + K + \sum_{i=1}^{n} D_i e^{r(T-t_i)} = \max(S_T, K) + \sum_{i=1}^{n} D_i e^{r(T-t_i)}.$$

For Φ_2, holding S means receiving n dividend payments by time T. The ith dividend D_i is paid at t_i. Then, by T, D_i becomes $D_i e^{r(T-t_i)}$. Note that we requires $t_n \le T$ which means that the nth dividend payment could happen right at T. If this is the case, at time T, stock price drops by D_n and becomes S_T. This means $S_T = \lim_{t \uparrow T} S_t - D_n$. Hence, no matter $t_n = T$ or $t_n < T$, we always have

$$V_T(\Phi_2) = p_T + S_T + \sum_{i=1}^{n} D_i e^{r(T-t_i)} = (K - S_T)^+ + S_T + \sum_{i=1}^{n} D_i e^{r(T-t_i)} = \max(S_T, K) + \sum_{i=1}^{n} D_i e^{r(T-t_i)}.$$

So, we have proved $V_T(\Phi_1) = V_T(\Phi_2)$.

10. (No arbitrage delivery price of a forward) A forward contract is an agreement to buy or sell an asset at a certain future time (expiration date) for a certain price (delivery price). One of the parties to a forward contract assumes a long position and agrees to buy the

underlying asset on expiration date for certain delivery price F. The other party assumes a short position and agrees to sell the asset at the expiration date for the same price. Using arbitrage-free principle to show that the delivery price F on a non-dividend-paying asset with spot price S_0 is given by

$$K = S_0 e^{rT}$$

where r is the risk-free interest rate and T is the time to expiry of the forward contract.

Solution: Let F denote a forward contract and construct a portfolio

$$\Phi = -B + S - F,$$

with $V_0(B) = S_0$. It means that at $t = 0$, one borrows S_0 dollars from the bank to buy the asset S from the market, and one also sell a forward contract. This means he/she agrees to sell S at the expiration date for the delivery price K. So, $V_0(F) = 0$ and $V_T(-F) = K - S_T$. The latter equation says that since he/she sold a forward contract, by T, he/she has to sell S (whose price is S_T) for price K to the buyer of the forward contract. Hence

$$V_0(\Phi) = -S_0 + S_0 + 0 = 0,$$
$$V_T(\Phi) = -S_0 e^{rT} + S_T + K - S_T = K - S_0 e^{rT}.$$

So, if $K > S_0 e^{rT}$, $V_0(\Phi) = 0$ and $V_T(\Phi) > 0$, one has an arbitrage opportunity. On the other hand, if $K < S_0 e^{rT}$, $V_0(-\Phi) = 0$ and $V_T(-\Phi) > 0$, one has an arbitrage opportunity by building a portfolio $-\Phi$ which is $B - S + F$. Since we assume the market is arbitrage-free, we must have $K = S_0 e^{rT}$.

11. Consider a European call c_1 with a strike price of K_1 and a second European call c_2 on the same stock with a strike price of $K_2 > K_1$. Both call options have the same expiration date. Let $c(t, K)$ denote the price of the European call option at time t with strike price K. So, $c(t, K_i) = V_t(c_i)$ for $i = 1, 2$. Prove that

$$-e^{-r(T-t)}(K_2 - K_1) < c(t, K_2) - c(t, K_1) < 0.$$

Furthermore, deduce that

$$-e^{-r(T-t)} \le \frac{\partial c}{\partial K}(t, K) \le 0. \tag{1.36}$$

In other words, suppose the call price can be expressed as a differentiable function of the strike price, then the derivative must be non-positive and no greater in absolute value than the price of a zero-coupon bond with face value of unity and the same maturity.

Solution: Consider the following portfolio at $t = 0$:

$$\Phi = c_2 - c_1 + e^{-rT}(K_2 - K_1).$$

Then

$$V_T(\Phi) = (S_T - K_2)^+ - (S_T - K_1)^+ + (K_2 - K_1) = \begin{cases} (K_2 - K_1) & \text{if } S_T \le K_1 \\ -S_T + K_2 & \text{if } K_1 < S_T \le K_2 \\ 0 & \text{if } K_2 < S_T \end{cases}$$

Hence $V_T(\Phi) \ge 0$ and $\mathbb{P}(V_T(\Phi) > 0) > 0$. By Definition 1.1, we must have

$$V_t(\Phi) > 0, \qquad \forall t \in [0, T). \tag{1.37}$$

(If not, then $V_{t^*}(\Phi) \le 0$ for some $t^* \in [0, T)$, which means Φ has arbitrage opportunity by Definition 1.1.) Equation (1.37) means

$$V_t(\Phi_1) = c(t, K_2) - c(t, K_1) + e^{-r(T-t)}(K_2 - K_1) > 0.$$

Next, we consider $\Phi = c_1 - c_2$, Then

$$V_T(\Phi) = (S_T - K_1)^+ - (S_T - K_2)^+ = \begin{cases} 0 & \text{if } S_T \le K_1 \\ S_T - K_1 & \text{if } K_1 < S_T \le K_2 \\ K_2 - K_1 & \text{if } K_2 < S_T \end{cases}$$

Hence $V_T(\Phi) \ge 0$ and $\mathbb{P}(V_T(\Phi) > 0) > 0$. By Definition 1.1, we must have

$$V_t(\Phi) > 0, \qquad \forall t \in [0, T).$$

This means

$$V_t(\Phi_1) = c(t, K_1) - c(t, K_2) > 0.$$

Finally, by the definition of derivative,

$$\frac{\partial c}{\partial K}(t, K) = \lim_{h \downarrow 0} \frac{c(t, K + h) - c(t, K)}{h}.$$

Since $h > 0$, $-e^{-r(T-t)} < \frac{c(t,K+h)-c(t,K)}{h} < 0$, we get (1.36) by letting $h \downarrow 0$.

2.10 Homework II

(Only submit solutions to Questions 1,2,3,6,10,15.)

1. Using the following three-step binomial tree to compute the price of an European call option with strike price \$90 and $T = 6$ months. The initial price for the underlying stock is \$80. $r = 0.05$ per annum. $\delta t = 2$ months. $u = \frac{5}{4}$. $d = \frac{4}{5}$.

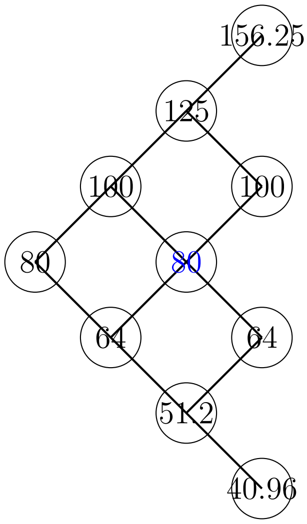

Figure 2.10: A binomial tree

Solution: $\rho = e^{r\delta t} = e^{0.05\times 2/12}$. $q_u = \frac{\rho - d}{u-d} = \frac{e^{0.05/6}-0.8}{1.25-0.8} \approx 0.4630$. $q_d \approx 0.5370$. The solution is shown in Figure 2.11.

One can also modify the Matlab code in Section 2.5 to get the same result.

```
S0 = 80;
K = 90;
T = 0.5;
r = 0.05;
M = 3; % number of time steps
dt = T/M;
u = 1.25;
d = 0.8;
rho = exp(r*dt);
qu = (rho-d)/(u-d);
qd = 1-qu;
Call = max(S0*u.^([M:-1:0]).*d.^([0:1:M])- K,0);
% We proceed backward to compute option value at time 0.
for i = M:-1:1
```

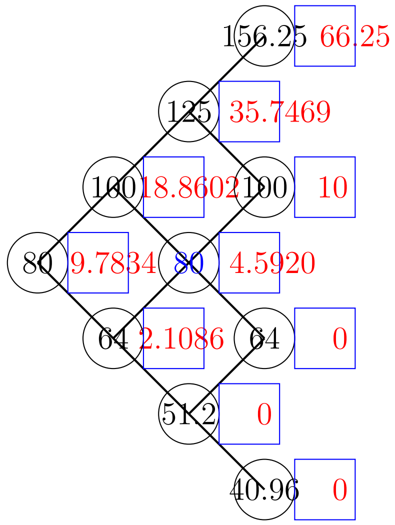

Figure 2.11: A binomial tree

```
    Call = (qu*Call(1:i) + qd*Call(2:i+1))/rho;
end
```

2. (stochastic volatility, random interest rate) Consider a binomial pricing model, but at each time $n \geq 1$, the "up factor" $u_n(\omega_1 \cdots \omega_n)$, the "down factor" $d_n(\omega_1 \cdots \omega_n)$, and the interest rate $r_n(\omega_1 \cdots \omega_n)$ are allowed to depend on n and on the first n coin tosses $\omega_1 \cdots \omega_n$. The initial up factor u_0, the initial down factor d_0, and the initial interest rate r_0 are not random. More specifically, the stock price at time one is given by

$$S_1(\omega_1) = \begin{cases} u_0 S_0 & \text{if } \omega_1 = H, \\ d_0 S_0 & \text{if } \omega_1 = T, \end{cases}$$

and, for $n \geq 1$, the stock price at time $n+1$ is given by

$$S_{n+1}(\omega_1 \cdots \omega_{n+1}) = \begin{cases} u_n(\omega_1 \cdots \omega_n) S_n(\omega_1 \cdots \omega_n) & \text{if } \omega_{n+1} = H, \\ d_n(\omega_1 \cdots \omega_n) S_n(\omega_1 \cdots \omega_n) & \text{if } \omega_{n+1} = T. \end{cases}$$

One dollar invested in or borrowed from the money market at time $t = 0$ grows to an invest or debt of $\rho = e^{r_0 \delta t}$ at time $t_1 = \delta t$, and, for $n \geq 1$, one dollar invested in or borrowed from the money market at time t_n grows to an investment or debt of $e^{r_n(\omega_1 \cdots \omega_n)\delta t}$ at time t_{n+1}. We assume that for each n and for all $\omega_1 \cdots \omega_n$, the no-arbitrage condition

$$0 < d_n(\omega_1 \cdots \omega_n) < \rho_n = e^{r_n(\omega_1 \cdots \omega_n)\delta t} < u_n(\omega_1 \cdots \omega_n)$$

holds. We also assume that $0 < d_0 < \rho_0 = e^{r_0 \delta t} < u_0$.

i) Let N be a positive integer and let U_n be the price at time $t_n = n\delta t$ of a derivative security. Derive the formula that relates $U_n(\omega_1 \cdots \omega_n)$ to random variables

$U_{n+1}(\omega_1 \cdots \omega_n H)$, $U_{n+1}(\omega_1 \cdots \omega_n T)$, $q_{u,n}(\omega_1 \cdots \omega_n) = \frac{\rho_n(\omega_1\cdots\omega_n) - d_n(\omega_1\cdots\omega_n)}{u_n(\omega_1\cdots\omega_n) - d_n(\omega_1\cdots\omega_n)}$, and $q_{d,n}(\omega_1 \cdots \omega_n) = \frac{u_n(\omega_1\cdots\omega_n) - \rho_n(\omega_1\cdots\omega_n)}{u_n(\omega_1\cdots\omega_n) - d_n(\omega_1\cdots\omega_n)}$.

ii) Suppose the initial stock price is $S_0 = 80$, with each head the stock price increase by 10, and with each tail the stock price decrease by 10. In another words, $S_1(H) = 90$, $S_1(T) = 70$, $S_2(HH) = 100$. Assume that the interest rate is always zero. Consider a European call with strike price 80, expiring at $t_3 = 3\delta t$. What is the price of this call at time $t = 0$?

Solution: i) Let $\omega = \omega_1 \cdots \omega_n$. Let us build a profile

$$\Phi(\omega) = U_n(\omega) - \Delta_n(\omega)S_n(\omega), \qquad \Delta_n(\omega) = \frac{U_{n+1}(\omega H) - U_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)}.$$

Then

$$\begin{aligned} V_{t_{n+1}}(\Phi)(\omega H) =& U_{n+1}(\omega H) - \frac{U_{n+1}(\omega H) - U_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)} S_{n+1}(\omega H) \\ =& U_{n+1}(\omega T) - \frac{U_{n+1}(\omega H) - U_{n+1}(\omega T)}{S_{n+1}(\omega H) - S_{n+1}(\omega T)} S_{n+1}(\omega T) = V_{t_{n+1}}(\Phi)(\omega T). \end{aligned}$$

Suppose there is bank deposit $B_n(\omega)$ made at t_n. We choose $B_n(\omega)$ so that $V_{t_{n+1}}(\Phi)(\omega H) = V_{t_{n+1}}(\Phi)(\omega T) = \rho_n(\omega)B_n(\omega)$.

As the market is arbitrage-free during $[t_n, t_{n+1}]$, there must be

$$B_n(\omega) = V_{t_n}(\Phi)(\omega) = U_n(\omega) - \Delta_n(\omega)S_n(\omega).$$

Hence

$$\begin{aligned} U_n(\omega) =& B_n(\omega) + \Delta_n(\omega)S_n(\omega) \\ =& \frac{1}{\rho_n(\omega)} \left(\frac{u_n(\omega)}{u_n(\omega) - d_n(\omega)} U_{n+1}(\omega T) - \frac{d_n(\omega)}{u_n(\omega) - d_n(\omega)} U_{n+1}(\omega H) \right) + \frac{U_{n+1}(\omega H) - U_{n+1}(\omega T)}{u_n(\omega) - d_n(\omega)} \\ =& \frac{1}{\rho_n(\omega)} \left(q_{u,n}(\omega)U_{n+1}(\omega H) + q_{d,n}(\omega)U_{n+1}(\omega T) \right). \end{aligned}$$

ii) Note that if the price n goes to $n+10$ and $n-10$, $q_u = \frac{1 - \frac{n-10}{n}}{\frac{n+10}{n} - \frac{n-10}{n}} = \frac{1}{2}$, $q_d = \frac{1}{2}$.

The solution is shown in Figure 2.12. The price is 7.5.

3. Consider a European call option on an underlying stock with its present price $S_0 =$\$50 per share. Suppose that at the expiry date T the stock has only two possible values $S^u = \$80$ and $S^d = \$40$. Assume the strike price $K = \$60$ and the risk-free interest rate is $r = 0$.

Here is how to determine the option price by replication: If we construct a portfolio at $t = 0$ which consists of a debt of \$20 and $\frac{1}{2}$ shares of the stock

$$\Phi = -20 + \frac{1}{2}S.$$

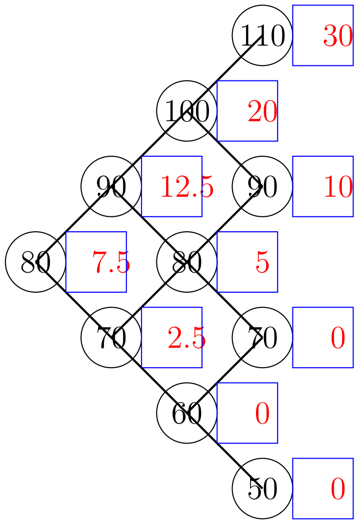

Figure 2.12: A binomial tree

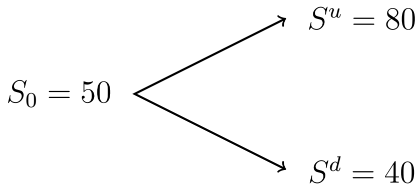

Figure 2.13: A single period binomial tree

Show that $V_T(\Phi) = V_T(\textcircled{p})$ no matter whether the stock price will go up or go down at $t = T$. We say that such a portfolio replicates the given option. Then use Corollary 1.1 to determine $\textcircled{p}_0 = V_0(\textcircled{p})$. Is this value the same as the one determined by (2.5)?

Solution:

$$V_0(\Phi) = -20 + \frac{1}{2} \times S_0 = 5.$$

At the expiry date the value of the portfolio will be

$$V_T(\Phi) = \left\{ \begin{array}{ll} -20 + \frac{1}{2} \times S^u = -20 + \frac{1}{2} \times 80 = 20, & \text{stock price goes up} \\ -20 + \frac{1}{2} \times S^d = 0, & \text{stock price goes down} \end{array} \right\} = V_T(\textcircled{p}).$$

Hence $\textcircled{p}_0 = V_0(\Phi) = \$5$.

If use (2.5), $u = \frac{8}{5}$, $d = \frac{4}{5}$, $q_u = \frac{1-\frac{4}{5}}{\frac{8}{5}-\frac{4}{5}} = \frac{1}{4}$. $\textcircled{p}_0 = \frac{1}{4}(80-60)^+ + \frac{3}{4}(40-60)^+ = \$5$.

4. (A generalization of the last problem: Replication in the one period binomial model) Given an option, consider a portfolio Φ consisting of a risk-free asset with interest rate r

and an underlying stock:

$$\Phi = B + \Delta S \tag{2.43}$$

where Δ is the number of shares of the underlying stock. We would like to choose $B_0 = V_0(B)$ and Δ so that $V_T(\Phi) = V_T(\textcircled{p}) = \begin{cases} \textcircled{p}^u & \text{if } V_T(S) = S^u, \\ \textcircled{p}^d & \text{if } V_T(S) = S^d, \end{cases}$. Find the formula for B_0 and Δ using r, T, $\textcircled{p}^u$, $\textcircled{p}^d$, S_0, S^u, and S^d. Can you determine $\textcircled{p}_0$ using Corollary 1.1?

Solution:

$$\begin{cases} e^{rT}B_0 + \Delta S^u = \textcircled{p}^u \\ e^{rT}B_0 + \Delta S^d = \textcircled{p}^d \end{cases}$$

or equivalently

$$\begin{bmatrix} e^{rT} & S^u \\ e^{rT} & S^d \end{bmatrix} \begin{bmatrix} B_0 \\ \Delta \end{bmatrix} = \begin{bmatrix} \textcircled{p}^u \\ \textcircled{p}^d \end{bmatrix}.$$

Hence

$$\begin{bmatrix} B_0 \\ \Delta \end{bmatrix} = \frac{1}{e^{rT}(S^d - S^u)} \begin{bmatrix} S^d & -S^u \\ -e^{rT} & e^{rT} \end{bmatrix} \begin{bmatrix} \textcircled{p}^u \\ \textcircled{p}^d \end{bmatrix}.$$

Thus

$$B_0 = \frac{1}{e^{rT}} \left(\frac{-S^d}{S^u - S^d} \textcircled{p}^u + \frac{S^u}{S^u - S^d} \textcircled{p}^d \right),$$
$$\Delta = \frac{\textcircled{p}^u - \textcircled{p}^d}{S^u - S^d}.$$

We mention in passing that the Δ used to build the portfolio (2.43) that replicate a option equals to the Δ used to build a risk-free portfolio in (2.2).

$$\textcircled{p}_0 = V_0(\Phi) = B_0 + \Delta S_0 = \frac{1}{\rho} \left(\frac{\rho S_0 - S^d}{S^u - S^d} \textcircled{p}^u + \frac{S^u - \rho S_0}{S^u - S^d} \textcircled{p}^d \right)$$

with $\rho = e^{rT}$. The above formula for $\textcircled{p}_0$ is the same as the formula we derived in (2.5).

5. (Replication in the multiperiod binomial model) Consider the multiperiod binomial model introduced in Section 2.3. Suppose we define

$$\textcircled{p}_n(\omega_1 \cdots \omega_n) = e^{-r\delta t} \left(q_u \textcircled{p}_{n+1}(\omega_1 \cdots \omega_n H) + q_d \textcircled{p}_{n+1}(\omega_1 \cdots \omega_n T) \right) \tag{2.44}$$

and

$$\Delta_n(\omega_1 \cdots \omega_n) = \frac{\textcircled{p}_{n+1}(\omega_1 \cdots \omega_n H) - \textcircled{p}_{n+1}(\omega_1 \cdots \omega_n T)}{S_{n+1}(\omega_1 \cdots \omega_n H) - S_{n+1}(\omega_1 \cdots \omega_n T)}, \tag{2.45}$$

with $n = N-1, \cdots, 0$. Prove that if we set $\Phi_0 = V_0$ and define recursively forward in time the portfolio values $\Phi_1, \Phi_2, \cdots \Phi_N$ by

$$\Phi_{n+1} = e^{r\delta t} \left(\Phi_n - \Delta_n S_n \right) + \Delta_n S_{n+1}, \tag{2.46}$$

then

$$\Phi_N(\omega_1\cdots\omega_N) = ⓟ_N(\omega_1\cdots\omega_N) \qquad \text{for all } \omega_1\cdots\omega_N. \tag{2.47}$$

Proof: We prove by induction in n that

$$\Phi_n(\omega_1\cdots\omega_n) = ⓟ_n(\omega_1\cdots\omega_n) \qquad \text{for all } \omega_1\cdots\omega_n \tag{2.48}$$

for $n = 0, ..., N$.

We know (2.48) is true when $n = 0$. We assume (2.48) is true for n and show that it is true for $n+1$.

By (2.46),

$$\Phi_{n+1}(\omega_1\cdots\omega_n H) = e^{r\delta t}\left(\Phi_n(\omega_1\cdots\omega_n) - \Delta_n(\omega_1\cdots\omega_n)S_n(\omega_1\cdots\omega_n)\right) + \Delta_n(\omega_1\cdots\omega_n)uS_n(\omega_1\cdots\omega_n).$$

To simplify the notation, we suppress $\omega_1\cdots\omega_n$ and write the equation simply as

$$\Phi_{n+1}(H) = e^{r\delta t}\left(\Phi_n - \Delta_n S_n\right) + \Delta_n u S_n. \tag{2.49}$$

With $\omega_1\cdots\omega_n$ suppressed, (2.45) can be written as

$$\Delta_n = \frac{ⓟ_{n+1}(H) - ⓟ_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} = \frac{ⓟ_{n+1}(H) - ⓟ_{n+1}(T)}{(u-d)S_n}.$$

Plugging the above equation into (2.49), we get

$$\begin{aligned}\Phi_{n+1}(H) &= e^{r\delta t}\Phi_n + \left(u - e^{r\delta t}\right)\Delta_n S_n = e^{r\delta t}\Phi_n + \left(u - e^{r\delta t}\right)\frac{ⓟ_{n+1}(H) - ⓟ_{n+1}(T)}{u-d} \\ &\overset{\text{induction assumption and def of } q_d}{=} e^{r\delta t}ⓟ_n + q_d\left(ⓟ_{n+1}(H) - ⓟ_{n+1}(T)\right) \\ &\overset{(2.44)}{=} q_u ⓟ_{n+1}(H) + q_d ⓟ_{n+1}(T) + q_d\left(ⓟ_{n+1}(H) - ⓟ_{n+1}(T)\right) \\ &= ⓟ_{n+1}(H).\end{aligned}$$

A similar argument shows that $\Phi_{n+1}(T) = ⓟ_{n+1}(T)$.

6. (The call can never be less risky than the underlying stock) With the same setup as in Example 2.8, consider a one period binomial tree model. Show that for put option, $\Omega \le 0$, and for call option $\Omega \ge 1$.

   [Hint: For call option, you need to prove $ⓟ^u - ⓟ^d \ge (u-d)ⓟ_0 > 0$. Use (2.5) to show that you only need to prove $dⓟ^u - uⓟ^d \ge 0$. Now, use $ⓟ^u = \max(uS_0 - K, 0)$ and $ⓟ^d = \max(dS_0 - K, 0)$ for the one period model.] As a matter of fact, this result is also true for the multi-period model. See Page 187 of "Options Markets" by Cox and Rubinstein.

   Solution: Recall

$$\Omega = \frac{(ⓟ^u - ⓟ^d)/ⓟ_0}{(uS_0 - dS_0)/S_0}.$$

For put option, $u > d$ implies $uS_0 > dS_0$. Hence $K - uS_0 < K - dS_0$ and $\textcircled{p}^u = \max(K - S^u, 0) = \max(K - uS_0, 0) \leq \max(K - dS_0, 0) = \textcircled{p}^d$. So, $\Omega \leq 0$.

For call option, We need to prove $\textcircled{p}^u - \textcircled{p}^d \geq (u-d)\textcircled{p}_0 > 0$.

$$(u-d)\textcircled{p}_0 \overset{(2.5)}{=} \frac{\rho - d}{\rho}\textcircled{p}^u + \frac{u - \rho}{\rho}\textcircled{p}^d.$$

Hence we only need to prove

$$d\textcircled{p}^u - u\textcircled{p}^d \geq 0.$$

This is true because

$$\begin{aligned} d\textcircled{p}^u - u\textcircled{p}^d &= d\max(uS_0 - K, 0) - u\max(dS_0 - K, 0) \\ &= \begin{cases} d(uS_0 - K) - u(dS_0 - K) = (u-d)K & \text{if } uS_0 > dS_0 \geq K \\ d(uS_0 - K) & \text{if } uS_0 \geq K \geq dS_0 \\ 0 & \text{if } K \geq uS_0 > dS_0 \end{cases} \\ &\geq 0. \end{aligned}$$

7. (Bernoulli Random Variable) An experiment, whose outcome can be classified as either a success or a failure is performed. Let $X = 1$ when the outcome is a success, and $X = 0$ if the outcome is a failure. Then the probability mass function of X is given by

$$\begin{aligned} \mathbb{P}(X = 0) &= 1 - p \\ \mathbb{P}(X = 1) &= p \end{aligned}$$

where $p \in [0, 1]$ is the probability that the trial is a success. A random variable X is said to be a **Bernoulli random variable** if its probability mass function is given as above. Prove that $E[X] = p$ and $\mathrm{Var}[X] = p(1-p)$.

Solution:

$$E[X] = 0 \times \mathbb{P}(X = 0) + 1 \times \mathbb{P}(X = 1) = p.$$
$$E[X^2] = 0^2 \times \mathbb{P}(X = 0) + 1^2 \times \mathbb{P}(X = 1) = p.$$
$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = p - p^2 = p(1-p).$$

8. If X is a discrete random variable with

$$\begin{aligned} \mathbb{P}(X = a) &= p \\ \mathbb{P}(X = b) &= 1 - p \end{aligned}$$

where $p \in [0, 1]$. Prove that $E[X] = ap + b(1-p)$ and $\mathrm{Var}[X] = (a-b)^2 p(1-p)$.

Solution:

$$E[X] = a \times \mathbb{P}(X = a) + b \times \mathbb{P}(X = b) = ap + b(1-p).$$
$$E[X^2] = a^2 p + b^2(1-p).$$

$$\begin{aligned}\mathrm{Var}(X) &= E[X^2] - (E[X])^2 = a^2p + b^2(1-p) - (ap + b(1-p))^2 \\ &= (a-b)^2 p(1-p).\end{aligned}$$

9. (Binomial Random Variable) Suppose n independent trials, each results in a success with probability p or in a failure with probability $1-p$, are to be performed. If Y represents the number of successes occur in the n trials, then Y is said to be **binomial random variable** with parameters (n, p), and denoted as $Y \sim B(n, p)$. Prove that $E[Y] = np$ and $\mathrm{Var}[Y] = np(1-p)$.

   Solution: Note that

$$P(Y = i) = \binom{n}{k} p^i (1-p)^{n-i} \quad \text{with } i = 0, 1, 2, ..., n.$$

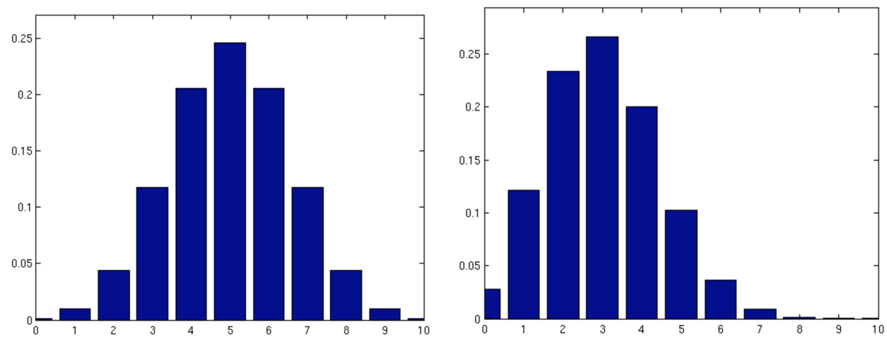

Figure 2.14: $\binom{n}{i} p^i (1-p)^{n-i}$ with $n = 10$, $i = 0, 1, 2, \ldots, n$. Left: $p = 0.5$. Right: $p = 0.3$

Then, we can prove the statements using definition. We start with computing the kth order moment

$$\begin{aligned} E[Y^k] &= \sum_{i=0}^{n} i^k \mathbb{P}(Y = i) = \sum_{i=0}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i} \\ &= \sum_{i=0}^{n} i^{k-1} np \binom{n-1}{i-1} p^{i-1} (1-p)^{(n-1)-(i-1)} \\ &= \sum_{i=1}^{n} i^{k-1} np \binom{n-1}{i-1} p^{i-1} (1-p)^{(n-1)-(i-1)} \\ &\overset{j=i-1}{=} np \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^j (1-p)^{(n-1)-j} \\ &= npE[(Z+1)^{k-1}] \end{aligned} \tag{2.50}$$

with $Z \sim B(n-1, p)$. Hence

$$E[Y] = npE[(Z+1)^0] = npE[1] = np. \tag{2.51}$$

$$E[Y^2] = npE[(Z+1)] = np\left(E[Z]+1\right) \overset{(2.51)}{=} np\left((n-1)p+1\right). \tag{2.52}$$

$$\mathrm{Var}[Y] = E[Y^2] - (E[Y])^2 = np\left((n-1)p+1\right) - (np)^2 = np(1-p).$$

For later reference, I would like to present another (easier) proof using a decomposition:

$$Y = X_1 + X_2 + \cdots + X_n$$

where $X_i = \begin{cases} 1 & \text{if the } i\text{th trial is a success} \\ 0 & \text{if the } i\text{th trial is a failure} \end{cases}$ and X_i's are independent. The definition of independence will be introduced in the next Chapter where one can show that if U and V are independent, then $\mathrm{Var}[U+V] = \mathrm{Var}[U] + \mathrm{Var}[V]$. Note that we already have $E[U+V] = E[U] + E[V]$ no matter U, V are independent or not. Since each X_i is a Bernoulli random variable, $E[X] = p$ and $\mathrm{Var}[X] = p(1-p)$. Hence

$$E[Y] = E[X_1] + E[X_2] + \cdots + E[X_n] = np,$$

$$\mathrm{Var}[Y] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \cdots + \mathrm{Var}[X_n] = np(1-p).$$

10. (Skewness) The skewness of a random variable X is defined to be

$$Sk = E\left[\left(\frac{X - E[X]}{\sigma}\right)^3\right]$$

where $\sigma = \sqrt{\mathrm{Var}[X]}$. Prove that the skewness of $B(n,p)$ distribution is

$$Sk(n,p) = \frac{1-2p}{\sqrt{np(1-p)}}.$$

[Hint: Use (2.50) and $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.] [20]

[20]For your information, the kurtosis of a random variable X is defined to be

$$K = E\left[\left(\frac{X - E[X]}{\sigma}\right)^4\right]$$

where $\sigma = \sqrt{\mathrm{Var}[X]}$. Using the same idea as in the computation of $Sk(n,p)$, one can prove that the kurtosis of $B(n,p)$ distribution is

$$K(n,p) = 3 + \frac{1 - 6p(1-p)}{np(1-p)}.$$

You are not asked to prove the above formula of $K(n,p)$ in the homework.

Solution: By (2.50),

$$E[X^3] = npE[(Z+1)^2] = npE[Z^2+2Z+1] = np[(n-1)p(1-p)+(n-1)^2p^2+2(n-1)p+1]$$

Since $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ (see the Pascal's triangle),

$$
\begin{aligned}
& Sk(n,p) \\
=& E\left[\left(\frac{X-E[X]}{\sigma}\right)^3\right] \\
=& \frac{1}{\sigma^3}E[(X-np)^3] \\
=& \frac{1}{[np(1-p)]^{3/2}}\left(E[X^3] - 3npE[X^2] + 3(np)^2E[X] - (np)^3\right) \\
=& \frac{np}{[np(1-p)]^{3/2}}\Big\{(n-1)p(1-p) + (n-1)^2p^2 + 2(n-1)p + 1 - 3np(1-p) - \cancel{3n^2p^2} + \cancel{3(np)^2} - (np)^2 \\
=& \frac{np}{[np(1-p)]^{3/2}}\left\{(n-1)p[3-2p+np] + 1 - 3np + 3np^2 - n^2p^2\right\} \\
=& \frac{np}{[np(1-p)]^{3/2}}\left\{1 - 3p + 2p^2\right\} \\
=& \frac{np(1-p)}{[np(1-p)]^{3/2}}\{1-2p\} \\
=& \frac{1-2p}{[np(1-p)]^{1/2}}.
\end{aligned}
$$

11. Suppose X is a random variable. Define $M_k = \tilde{\mathbb{E}}_k[X]$. Prove that $\{M_k\}$ is a martingale.

    Proof: $\tilde{\mathbb{E}}_n[M_{n+1}] = \tilde{\mathbb{E}}_n[\tilde{\mathbb{E}}_{n+1}[X]] \overset{(2.31)}{=} \tilde{\mathbb{E}}_n[X] = M_n$.

12. (random walk) Toss a coin repeatedly. Assume the probability of head on each toss is $\frac{1}{2}$, so is the probability of tail. Let $X_j = 1$ if the jth toss results in a head and $X_j = -1$ if the jth toss results in a tail. Consider $M_1, M_1, M_2, \cdots$ (which is an example of stochastic process) defined by $M_0 = 0$ and

$$M_n = \sum_{j=1}^{n} X_j, \qquad n \geq 1.$$

    This is called a symmetric random walk; with each head, it steps up one, and with each tails, it steps down one. Using Theorem 2.2 to show that $M_1, M_1, M_2, \cdots, M_n, \cdots$ is a martingale.

    Proof: M_n only depends on $\omega_1 \cdots \omega_n$.

$$\tilde{\mathbb{E}}_n[M_{n+1}] = \tilde{\mathbb{E}}_n[M_n + X_{n+1}] = \tilde{\mathbb{E}}_n[M_n] + \tilde{\mathbb{E}}_n[X_{n+1}] = M_n + \mathbb{E}[X_{n+1}] = M_n.$$

13. (discrete-time stochastic integral) Suppose $M_0, M_1, \cdots, M_N$ is a martingale, and let $\Delta_0, \Delta_1, \cdots, \Delta_{N-1}$ be an adapted process (see definition 2.4). Define the discrete-time stochastic integral $I_0, I_1, \cdots, I_N$ by setting $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} \Delta_j (M_{j+1} - M_j), \qquad n = 1, ..., N. \tag{2.53}$$

Show that $I_0, I_1, \cdots, I_N$ is a martingale.

Proof: I_n only depends on $\omega_1 \cdots \omega_n$ and $I_0, I_1, \cdots, I_N$ is therefore an adapted stochastic process.

$$\begin{aligned}\tilde{\mathbb{E}}_n[I_{n+1}] =& \tilde{\mathbb{E}}_n[I_n + \Delta_n(M_{n+1} - M_n)] \overset{(2.29)}{=} \tilde{\mathbb{E}}_n[I_n] + \tilde{\mathbb{E}}_n[\Delta_n(M_{n+1} - M_n)] \\ \overset{(2.30)}{=}& I_n + \Delta_n \tilde{\mathbb{E}}_n[M_{n+1} - M_n] \\ =& I_n.\end{aligned}$$

In the last step, we have used $\tilde{\mathbb{E}}_n[M_{n+1} - M_n] = \tilde{\mathbb{E}}_n[M_{n+1}] - M_n \overset{M_n \text{ is martingale}}{=} 0$.

14. (i) Consider the dice-toss space similar to the coin-toss space. A typical element ω in this space is an infinite sequence $\omega = \omega_1\omega_2\omega_3 \cdots$ with $\omega_i \in \{1, 2, \cdots, 6\}$. Define a random variable

$$X(\omega) = \omega_1$$

and a function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 4 \\ 0 & \text{if } x < 4 \end{cases}.$$

Recall the definition of $\sigma(X)$ (Defintion 2.7). Let $\Omega = \{\omega : \omega = \omega_1\omega_2\omega_3 \cdots\}$ and $A_i = \{\omega : \omega_1 = i\}$. It is not hard to see that $\sigma(X) = \{\emptyset, \Omega, A_i,\ A_i \cup A_j,\ A_i \cup A_j \cup A_k,\ A_i \cup A_j \cup A_k \cup A_l,\ A_i \cup A_j \cup A_k \cup A_l \cup A_m,\ i, j, k, l, m \text{ are not equal pairwisely},\ i, j, k, l, m = 1, \cdots, 6\}$.

Since $f(X)$ is also a random variable defined on Ω, we can also define $\sigma(f(X))$. What is $\sigma(f(X))$?

(ii) In general, if X is a random variable, can the σ-algebra generated by $f(X)$ ever be strictly larger than the σ-algebra generated by X?

Solution: (i) $\sigma(f(X)) = \{\emptyset, \Omega, \{\omega : \omega_1 = 1, \text{or } 2, \text{or } 3\}, \{\omega : \omega_1 = 4, \text{or } 5, \text{or } 6\}\}$.

(ii) No. $\sigma(f(X))$ is always a subset of $\sigma(X)$.

15. Consider the symmetric random walk $M_1, M_1, M_2, \cdots$ defined in Example 2.11. Let $\sigma > 0$ be a constant.

   a) Define $J_0 = 0$ and

$$J_n = \sum_{j=0}^{n-1} e^{\sigma M_j}(M_{j+1} - M_j), \quad n = 1, 2, \cdots. \tag{2.54}$$

Show that $J_0, J_1, \cdots, J_N$ is a martingale, which means, you need to prove that $\mathbb{E}_n[J_{n+1}] = J_n$.

b) Define

$$K_n = \sum_{j=0}^{n-1} M_{j+1}(M_{j+1} - M_j), \quad n = 1, 2, \cdots. \tag{2.55}$$

Show that

$$K_n = \frac{1}{2}M_n^2 + \frac{n}{2}.$$

Solution:

(a)

$$\begin{aligned}\mathbb{E}_n[J_{n+1}] =& \mathbb{E}_n[J_n + e^{\sigma M_n}(M_{n+1} - M_n)] = \mathbb{E}_n[J_n] + \mathbb{E}_n[e^{\sigma M_n} X_{n+1}] \\ =& J_n + e^{\sigma M_n}\mathbb{E}_n[X_{n+1}] \\ =& J_n + 0 = J_n.\end{aligned}$$

(b)

$$\begin{aligned}K_n &= \sum_{j=0}^{n-1} M_{j+1}(M_{j+1} - M_j) = \frac{1}{2}\sum_{j=0}^{n-1}(M_{j+1}^2 - M_j^2) + \frac{1}{2}\sum_{j=0}^{n-1}(M_{j+1} - M_j)^2 \\ &= \frac{1}{2}(M_n^2 - M_0^2) + \frac{1}{2}\sum_{j=0}^{n-1} X_{j+1}^2 = \frac{M_n^2}{2} + \frac{n}{2}.\end{aligned}$$