FE5222 Solutions to Homework 1

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1. (Q1) Let $W(t)$ be a Brownian motion, $t_1 < t_2 < \ldots < t_n$. Prove that the random vector

$$[W(t_1), W(t_2), \ldots, W(t_n)]$$

is a joint normal distribution. Compute its mean and covariance matrix.

Solution: Let $X(\omega) = [X_1(\omega), \ldots, X_n(\omega)]^T$ be a random vector. It is a joint normal distribution if one of the following equivalent conditions hold

(a) There exists a vector $\mu = [\mu_1, \ldots, \mu_n]^T$ and a positive definite matrix Σ such that for any $t = [t_1, \ldots, t_n]$, the moment generating function of X is

$$M_X(t) = \mathbb{E}e^{t^T X} = e^{t^T\mu + \frac{1}{2}t^T\Sigma t}$$

(b) The p.d.f. of X is

$$p(x_1, \ldots, x_n) = \frac{1}{(2\pi)^n 2\sqrt{|\Sigma|}} e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$$

where $x = [x_1, \ldots, x_n]$.

(c) For any vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha^T X = \alpha_1 X_1 + \ldots \alpha_n X_n$ is a (univariate) normal distribution.

A common mistake about joint normal distribution is if each of $X_1, \ldots, X_n$ is normal distribution, then X is a joint normal distribution. This is not the case in general. A simple counter-example is $X = [X_1, -X_1]$ where X_1 is a standard normal distribution.

A few useful facts about joint normal distribution

(a) If $X_1, \ldots, X_n$ are independent and normally distributed (not necessarily identically distributed), then X is a joint normal.

(b) If X is a joint normal distribution, A is an non-singular matrix and b is a vector, then $AX + b$ is a joint normal distribution.

To prove $W(t)$ is jointly normal, we notice that

$$W(t) = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 1 & 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} W_1(t) \\ W_2(t) - W_1(t) \\ \vdots \\ W_n(t) - W_{n-1}(t) \end{bmatrix}$$

Since

$$X = \begin{bmatrix} W_1(t) \\ W_2(t) - W_1(t) \\ \vdots \\ W_n(t) - W_{n-1}(t) \end{bmatrix}$$

is a vector of independent normal distributions, using the first fact above, X is a joint normal distribution. W is a linear transformation of X which is again a joint normal distribution by the second fact.

The mean and covariance matrix are easy.

2. (Q3) Let $\Delta(t)$ be a simple process and $I(t) = \int_0^t \Delta(t)dW(t)$. Prove that

   (a) $I(t)$ is $\mathcal{F}_t$-measurable.

   (b) $I(t)$ is a martingale

   (c) **Ito Isometry** $\mathbb{E}[I^2(t)] = \mathbb{E}\left[\int_0^t \Delta^2(s)ds\right]$

   (d) **Quadratic Variation** $[I, I](t) = \int_0^t \Delta^2(s)ds$

   Solution: See Section 4.2, Shreve's book

3. (Q4) Let $X(t)$ be an Ito process as

$$X(t) = X(0) + \int_0^t \Delta(s)dW(s) + \int_0^t \Theta(s)ds$$

Prove that

$$[X, X](t) = \int_0^t \Delta^2(s)ds$$

Solution: The proof can be found in Section 4.4, Shreve's book.

A common mistake is to use

$$dX(t) = \Theta(t)dt + \Delta t dW(t)$$

and argue that

$$dX(t)dX(t) = (\Theta(t)dt + \Delta t dW(t))^2 = \Delta^2(t)dt$$

using the rules $dtdt = 0, dW(t)dt = 0$ and $dW(t)dW(t) = dt$. It then follows

$$[X, X](t) = \int_0^t \Delta^2(s)ds$$

The issue with this argument is that you have applied the fact that

$$[X, X](t) = \int_0^t dX(s)dX(s)ds$$

but this is essentially what you are asked to prove. For this exercise you need to prove it from first principles (i.e., using definition). Going forward, you can use this result though.

4. (Q5) Let $W_i(t)$ and $W_j(t)$ be two independent Brownian motions. Prove that

$$\lim_{||\Pi||\to 0} \sum_{k=1}^{n} \left(W_i(t_k) - W_i(t_{k-1})\right)\left(W_j(t_k) - W_j(t_{k-1})\right) = 0$$

where $\Pi : t_0 < t_1 < \ldots < t_n$ is a partition of $[0, T]$ and the limit converges in probability. This limit is called covariation of two processes and denoted by $[W_i, W_j](t)$. This exercise justifies the notation

$$dW_i(t)dW_j(t) = 0$$

Hint: Prove the limit converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ which in turn implies convergence in probability.

Solution: Let X_n be a sequence of random variables and X be a random variable. X_n is said to converge to X in probability as $n \to \infty$ if for any $\delta > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \delta\right) = 0$$

It is usually more difficult to prove the convergence in probability from definition. A sufficient condition is based on the fact that: if $\lim_{n\to\infty} \mathbb{E}\left[(X_n - X)^2\right] = 0$, then X_n converges to X in probability. This can be proved with the aid of Chebyshev inequality

$$\mathbb{P}\left(|X_n - X| > \delta\right) \le \frac{\mathbb{E}\left[(X_n - X)^2\right]}{\delta^2}$$

From Chebyshev inequality, it is easy to see that $\lim_{n\to\infty} \mathbb{E}\left[(X_n - X)^2\right] = 0$ implies

$$\lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \delta\right) = 0$$

Hence to prove $[W_i, W_j](t) = 0$, it is sufficient to show that

$$\lim_{||\Pi||\to 0} \mathbb{E}\left(\sum_{k=1}^{n} \left(W_i(t_k) - W_i(t_{k-1})\right)\left(W_j(t_k) - W_j(t_{k-1})\right)\right)^2 = 0$$

Note that

$$\begin{aligned}
& \mathbb{E}\left(\textstyle\sum_{k=1}^{n} \left(W_i(t_k) - W_i(t_{k-1})\right)\left(W_j(t_k) - W_j(t_{k-1})\right)\right)^2 \\
= \;& \mathbb{E}\Big[\textstyle\sum_{k=1}^{n} \left(W_i(t_k) - W_i(t_{k-1})\right)^2\left(W_j(t_k) - W_j(t_{k-1})\right)^2 \\
+ \;& 2\textstyle\sum_{k<l} \left(W_i(t_k) - W_i(t_{k-1})\right)\left(W_j(t_k) - W_j(t_{k-1})\right)\left(W_i(t_l) - W_i(t_{l-1})\right)\left(W_j(t_l) - W_j(t_{l-1})\right)\Big] \\
= \;& \mathbb{E}\left[\textstyle\sum_{k=1}^{n} \left(W_i(t_k) - W_i(t_{k-1})\right)^2\left(W_j(t_k) - W_j(t_{k-1})\right)^2\right] \\
+ \;& 2\mathbb{E}\left[\textstyle\sum_{k<l} \left(W_i(t_k) - W_i(t_{k-1})\right)\left(W_j(t_k) - W_j(t_{k-1})\right)\left(W_i(t_l) - W_i(t_{l-1})\right)\left(W_j(t_l) - W_j(t_{l-1})\right)\right]
\end{aligned}$$

Since $W_i(t_k) - W_i(t_{k-1})$, $(W_j(t_k) - W_j(t_{k-1})$, $W_i(t_l) - W_i(t_{l-1})$ and $W_j(t_l) - W_j(t_{l-1})$ are independent for $k \neq l$ and each has mean zero, the second term in the last equation is zero. Hence

$$\begin{aligned}
& \mathbb{E}\left(\textstyle\sum_{k=1}^{n} \left(W_i(t_k) - W_i(t_{k-1})\right)\left(W_j(t_k) - W_j(t_{k-1})\right)\right)^2 \\
= \;& \mathbb{E}\left[\textstyle\sum_{k=1}^{n} \left(W_i(t_k) - W_i(t_{k-1})\right)^2\left(W_j(t_k) - W_j(t_{k-1})\right)^2\right] \\
= \;& \textstyle\sum_{k=1}^{n} \mathbb{E}\left[\left(W_i(t_k) - W_i(t_{k-1})\right)^2\left(W_j(t_k) - W_j(t_{k-1})\right)^2\right] \\
= \;& \textstyle\sum_{k=1}^{n} \mathbb{E}\left[\left(W_i(t_k) - W_i(t_{k-1})\right)^2\right] \mathbb{E}\left[\left(W_j(t_k) - W_j(t_{k-1})\right)^2\right] \\
= \;& \textstyle\sum_{k=1}^{n} (t_k - t_{k-1})^2
\end{aligned}$$

where the second to last equality comes from the independence between $W_i(t_k) - W_i(t_{k-1})$ and $W_i(t_k) - W_i(t_{k-1})$. Since

$$\sum_{k=1}^{n}(t_k - t_{k-1})^2 \leq t||\Pi||$$

it converges to zero as $||\Pi||$ converges to zero. Hence

$$\lim_{||\Pi||\to 0} \mathbb{E}\left(\sum_{k=1}^{n}\left(W_i(t_k) - W_i(t_{k-1})\right)\left(W_j(t_k) - W_j(t_{k-1})\right)\right)^2 = \lim_{||\Pi||\to 0}\sum_{k=1}^{n}(t_k - t_{k-1})^2 = 0$$

which implies the convergence in probability.

5. (Q7) Let $B_1(t)$ and $B_2(t)$ be two Brownian motions such that

$$dB_1(t)dB_2(t) = \rho(t)dt$$

where $-1 < \rho(t) < 1$ is a stochastic process. Define

$$B_1(t) = W_1(t)$$
$$B_2(t) = \int_0^t \rho(t)dW_1(t) + \int_0^t \sqrt{1-\rho^2(t)}dW_2(t)$$

Prove that W_1 and W_2 are two independent Brownian motions.

Solution: We need to prove two things:

(a) W_1 and W_2 are Brownian motions

(b) W_1 and W_2 are independent.

It is trivial that W_1 is a Brownian motion. To prove $W_2(t)$ is a Brownian motion, we use Levy's Theorem. That is we need to prove

(a) $W_2(t)$ is continuous

(b) $W_2(t)$ is a martingale

(c) $W_2(t)$ has unit quadratic variation, i.e., $dW_2(t)dW_2(t) = dt$.

Note that we can solve $W_2(t)$ as

$$W_2(t) = \int_0^t \frac{1}{\sqrt{1-\rho^2(t)}}dB_2(t) - \int_0^t \frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t)$$

It is the difference of two Ito's integrals. Since an Ito's integral is continuous and a martingale, $W_2(t)$ is continuous and a martingale. Since

$$dW_2(t) = \frac{1}{\sqrt{1-\rho^2(t)}}dB_2(t) - \frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t)$$

$$\begin{aligned}
dW_2(t)dW_2(t) &= \left(\tfrac{1}{\sqrt{1-\rho^2(t)}}dB_2(t) - \tfrac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t)\right)^2 \\
&= \left(\tfrac{1}{\sqrt{1-\rho^2(t)}}dB_2(t)\right)^2 + \left(\tfrac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t)\right)^2 - 2\tfrac{1}{\sqrt{1-\rho^2(t)}}\tfrac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t)dB_2(t) \\
&= \tfrac{1}{1-\rho^2(t)}dt + \tfrac{\rho^2(t)}{1-\rho^2(t)}dt - 2\tfrac{\rho^2(t)}{1-\rho^2(t)}dt \\
&= dt
\end{aligned}$$

where in deriving the second to last equality we use the fact that $dB_1(t)dB_2(t) = \rho(t)dt$. Hence by Levy's Theorem $W_2(t)$ is a Brownian motion.

By two dimensional Levy's Theorem, to prove $W_1(t)$ and $W_2(t)$ are independent, it is sufficient to prove $[W_1, W_2](t) = 0$. This follows from the derivation below.

$$\begin{aligned}
dW_1(t)dW_2(t) &= dB_1(t)\left(\frac{1}{\sqrt{1-\rho^2(t)}}dB_2(t) - \frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t)\right) \\
&= \frac{1}{\sqrt{1-\rho^2(t)}}dB_1(t)dB_2(t) - \frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t)dB_1(t) \\
&= \frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dt - \frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dt \\
&= 0
\end{aligned}$$