Section 3. HJM model: No-Arbitrage of Entire Yield Curve

NUS National University of Singapore | RMI Advancing Risk Management for Singapore and Beyond

Liu Xiaoqing

The HJM model is about the evolution of forward instantaneous rates.

– The HJM model is different from short rate models as it captures all dynamics of the entire forward rate curve, while the short-rate models only capture the dynamics of a single point.

– This is like calculus for functions versus functional analysis in mathematics.

Robert Jarrow:

"I guess it all started, in some sense, way back when in my graduate program at MIT. I did my thesis on the term structure of interest rates and the expectation hypothesis. My thesis was partly an empirical investigation showing that the expectations hypothesis does not hold. This is a topic directly related to the HJM."

3.1. Expectation Hypothesis (EH)

A forward rate is equal to the expectations of the respective future spot rate.

Ask your own intuition if this is right!

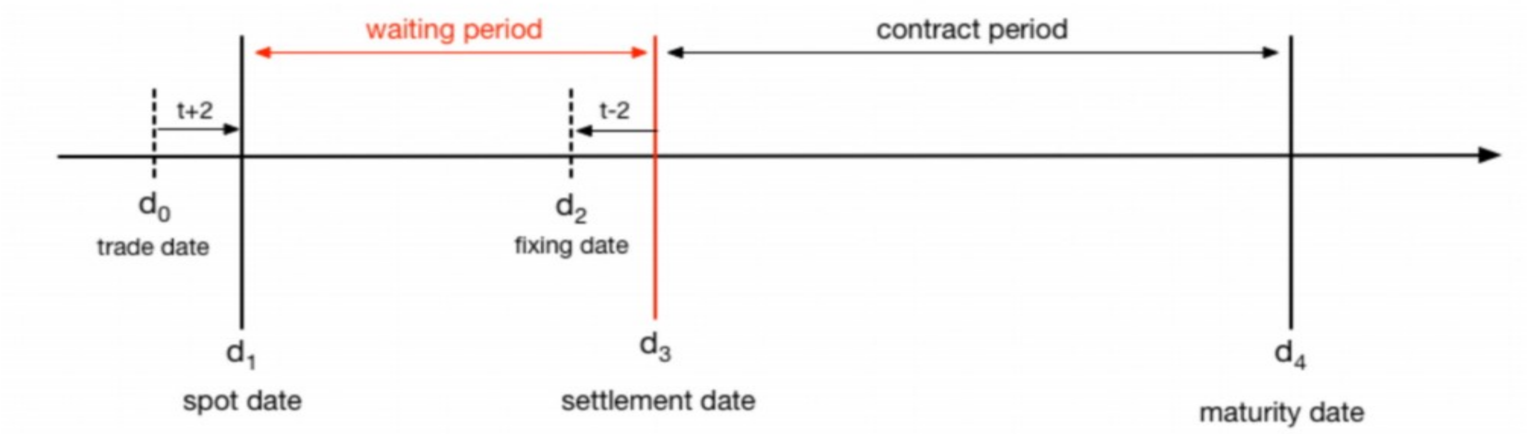
Forward Rate Agreement (FRA) Revisit

OTC contracts between two parties that determine the rate of interest to be paid on the settlement day for a fixed interest period in future. The two parties agree to exchange an interest rate commitment on a notional amount for a given interest period. The settlement amount S is as follows:

$$S = \frac{(R - FRA) \times N \times \Delta}{1 + R \times \Delta}, \quad (3.1)$$

where FRA is a fixed rate, R is the floating rate for the interest period, N is the notional, and Δ is the time fraction calculated by day count convention.

Tips on Dates Related to Interest Rate Trade



From Publicized Source.

The Equation of EH

Let $L_t(t, t+\triangle t)$ be the realized rate at time t of the settlement date for the interest period from t to $t+\Delta$ and $L_0(t, t+\triangle t)$ be the break-even fixed rate for the forward rate agreement. We express the EH as

$$L_0(t, t+\Delta) = E[L_t(t, t+\triangle)]. \quad (3.2)$$

If tests of EH are done empirically, we consider the expectation as being taken under the empirical measure.

Testing the EH

– **Empirical Studies**,

including those of Jarrow, show that it was untrue.

— **Theoretical Investigation**

Let us gain some insight by examining the relationship between the forward instantaneous rate and the expected short rate under the empirical measure. We consider a CIR model under the empirical measure. The expectation of short rate r_T is

$$E(r_T) = r_0 e^{-kT} + k\int_0^T \theta(s)e^{-k(T-s)}ds. \quad (3.3)$$

On the other hand, the forward instantaneous rate

$$f(0,T) = -\frac{\partial \ln P(0,T)}{\partial T}, \quad (3.4)$$

or

$$f(0,T) = r_0 \frac{\partial B(0,T)}{\partial T} + k\int_0^T \theta(s)\frac{\partial B(s,T)}{\partial T} ds. \quad (3.5)$$

Comparing (3.3) and (3.5), we can see that (3.5) involves the short rate volatility σ as in the expression of $B(\cdot,T)$ while (3.3) does not. Therefore,

$$f(0,T) \neq E(r_T). \quad (3.6)$$

3.2 From Short Rate Model to Forward Rate Model

A Shortcoming of Short Rate Models

The calibration of short rate models such as the CIR model requires an "inversion" of the term structure.

(1) It is compositionally difficult.

(2) It can lead to arbitrage.

The Inspiration

The HJM realized that a more self-containing model should impose an **exogenous** stochastic structure upon forward rates $f(t, T)$ and derive other quantities from the forward rates.

Key Relationship

– **Forward Rate-Bond Link**

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}, \ (3.7)$$

$$\text{or } P(t,T) = e^{-\int_t^T f(t,v)dv}. \ (3.8)$$

– **Forward Rate-Short Rate Link**

$$r_t = \lim_{h \to 0} \frac{1 - P(t,t+h)}{hP(t,t+h)} = f(t,t). \ (3.9)$$

(One-Factor) Model Specification

$$f(t,T) - f(0,T) = \int_0^t \alpha(v,T)dv + \int_0^t \sigma(v,T)dW_v, \quad (3.10)$$

where $f(0,T)$ is a fixed deterministic initial forward rate curve.

3.3. Arbitrage-Free Term Structure and the HJM Model

Short Rate Process

$$r_t = f(0,t) + \int_0^t \alpha(v,t)dv + \int_0^t \sigma(v,t)dW_v. \quad (3.11)$$

Interest Accumulation Factor

– Wealth of a money account rolling over at r_t with initial investment of $\$1$ at time 0.

$$B(t) = e^{\int_0^t r_v dv}. \quad (3.12)$$

Bond Price Process

$$\ln P(t,T) = -\int_t^T f(0,y)dy - \int_0^t \left[\int_t^T \alpha(v,y)dy \right] dv - \int_0^t \left[\int_t^T \sigma(v,y)dy \right] dW_v,$$

by Fubini's theorem,

$$= -\int_0^T f(0,y)dy - \int_0^t \left[\int_v^T \alpha(v,y)dy \right] dv - \int_0^t \left[\int_v^T \sigma(v,y)dy \right] dW_v$$

$$+ \int_0^t f(0,y)dy + \int_0^t \left[\int_v^t \alpha(v,y)dy \right] dv + \int_0^t \left[\int_v^t \sigma(v,y)dy \right] dW_v$$

$$= -\int_0^T f(0,y)dy - \int_0^t \left[\int_v^T \alpha(v,y)dy \right] dv - \int_0^t \left[\int_v^T \sigma(v,y)dy \right] dW_v$$

$$+ \int_0^t f(0,y)dy + \int_0^t \left[\int_0^y \alpha(v,y)dv \right] dy + \int_0^t \left[\int_0^y \sigma(v,y)dW_v \right] dy$$

$$= \ln P(0,T) + \int_0^t r_y dy - \int_0^t \left[\int_v^T \alpha(v,y)dv \right] dy$$

$$- \int_0^t \left[\int_v^T \sigma(v,y)dy \right] dW_v, \quad (3.13)$$

by (3.11).

Bond Price SDE

Applying Ito's formula to $\ln P(t,T)$, we have

$$dP(t,T) = [r_t + b(t,T)]P(t,T)dt + a(t,T)P(t,T)dW_t, \quad (3.14)$$

where

$$a(t,T) = -\int_t^T \sigma(t,v)dv, \quad (3.15)$$

$$b(t,T) = -\int_t^T \alpha(t,v)dv + \frac{1}{2}a^2(t,T). \quad (3.16)$$

Discounted Bond Price

$$Z(t,T) = \frac{P(t,T)}{B(t)} \quad (3.17)$$

Transforming $P(t,T)$ into $Z(t,T)$ would removes the portion of the bond's drift due to the short rate process. This is analogous to the removal of risk-free interest rate in pricing equity options.

By Ito's formula, we have

$$\ln Z(t,T) = \ln Z(0,T) + \int_0^t b(v,T)dv - \frac{1}{2}\int_o^t a^2(v,T)dv$$

$$+ \int_0^t a(v,T)dW_v. \quad (3.18)$$

Market Prices for Risk

For a fixed time point T, assume there exists a solutions $\gamma(t;T)$ such that

$$b(t,T) - a(t,T)\gamma(t;T) = 0, \ (3.19)$$

or

$$b(t,T) = a(t,T)\gamma(t;T). \ (3.20)$$

$\gamma(t;T)$ is called the market price for risk. (3.20) is the instantaneous excess expected return above the risk free rate on the bond maturing at T. The right side is the "market price for risk" times the the instantaneous variance of the bond's return.

Equivalent Martingale Probability Measure

A market price for risk $\gamma(t;T)$ exists if and only if there exists an equivalent probability measure $\tilde{Q}$ such that $Z(t,T)$ is a martingale with respect to t.

Uniqueness of the Martingale Measure: across all Bonds

The following are equivalent:

(i) There exists a unique equivalent probability measure $\tilde{Q}$ such that $Z(t,T)$ is a martingale for all T;

(ii) $\gamma(t;T)$ is independent of T.

When $\gamma(t;T)$ is independent of T, we let $\phi(t) = -\gamma(t;T)$ and have

$$b(t,T) = -a(t,T)\phi(t). \quad (3.21)$$

Differentiating (3.21) with respect to T leads to

$$-\alpha(t,T) - a(t,T)\sigma(t,T) = \sigma(t,T)\phi(t), \quad (3.22)$$

or

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,v)dv - \phi(t)\sigma(t,T). \quad (3.23)$$

Remark

(3.23) is a key result of the HJM model. It is called the forward rate drift restriction. This restriction on all drift processes $\{\alpha(\cdot,T) : \forall T\}$ are necessary in order to guarantee the existence of unique equivalent martingale probability measure.

Equivalent Martingale Probability Measure

– A result of **Girsanov's Theorem**:

$$\frac{d\tilde{Q}}{dQ} = \exp\left[\int_0^T \phi(t)dW_t - \frac{1}{2}\int_0^T \phi^2(t)dt\right], \quad (3.24)$$

$$\tilde{W}_t = W_t - \int_0^t \phi(v)dv. \quad (3.25)$$

Dynamics of $f(t,T)$ under $\tilde{Q}$: the HJM Model

It follows from (3.10) and (3.23) that

$$\boxed{df(t,T) = \sigma(t,T)\left(\int_t^T \sigma(t,v)dv\right)dt + \sigma(t,T)d\tilde{W}_t. \quad (3.26)}$$

Martingale Dynamics of $Z(t,T)$ under $\tilde{Q}$

It follows from (3.18) and (3.21) that

$$\frac{dZ(t,T)}{Z(t,T)} = a(t,T)d\tilde{W}_t. \quad (3.27)$$

Bond Price under $\tilde{Q}$

$$P(t,T) = B(t)E_t^{\tilde{Q}}\left[\frac{1}{B(T)}\right] \quad (3.28)$$

$$\text{vs } P(t,T) = B(t)E_t^{Q}\left[\frac{1}{B(T)}\exp\left(\int_0^T \phi(t)dW_t - \frac{1}{2}\int_0^T \phi^2(t)dt\right)\right]. \quad (3.29)$$

3.4. Case Study

– Case I. Drift Term Restriction for Constant Volatility

$$\alpha(t,T) = -\sigma\phi(t) + \sigma^2(T-t). \quad (3.30)$$

Forward Rate Process

$$f(t,T) = f(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) + \sigma\tilde{W}_t \text{ under } \tilde{Q}. \quad (3.31)$$

Short Rate Process

$$r_t = f(0,t) + \frac{1}{2}\sigma^2 t^2 + \sigma\tilde{W}_t \text{ under } \tilde{Q}. \quad (3.32)$$

What short rate model is this?

Bond Price

$$P(t,T) = \frac{P(0,T)}{P(0,t)} e^{-\frac{\sigma^2}{2} Tt(T-t) - \sigma(T-t)\tilde{W}_t}. \quad (3.33)$$

Option Price

Similar to the Black-Scholes formula.

– **Case II. More on the CIR Model**

We continue the work on the calibration of the CIR model. To match an arbitrarily given initial forward rate curve, The calibration of the CIR model is done by "inverting" the following equation to make the spot rate process's parameters implicitly determined by the initial forward rate curve:

$$f(0,T) = r_0 \frac{\partial B(0,T)}{\partial T} + k \int_0^T \theta(s) \frac{\partial B(s,T)}{\partial T} ds \quad (3.34)$$

or

$$\left[\frac{\partial f(0,T)}{\partial T} - r_0 \frac{\partial B^2(0,T)}{\partial T^2} \right] \Big/ k = \theta(T) + k \int_0^T \theta(s) \frac{\partial^2 B(s,T)}{\partial T^2} ds. \quad (3.35)$$

– Volterra integral equation.

CIR did not prove that such an inversion is possible. The model is not compatible with all forward rate curves, as it requires the following condition to ensure positive short rates, or absence of arbitrage:

$$2k\theta(t) \geq \sigma^2 \quad (3.36)$$

or

$$f(0,T) \geq r_0 \frac{\partial B(0,T)}{\partial T} + \frac{\sigma^2}{2} B(0,T), \quad (3.37)$$

which is not always satisfied by a given forward rate curve.