# Lecture 6 - Auctions in Securities Markets

Chen Yi-Chun

NUS

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The Treasury Department auctioned seven-year notes Thursday, closing the door on a record year for sales of longer-term debt in 2019. The auction lifted the total of notes and bonds sold by the U.S. government with maturities ranging from two to 30 years to $2.55 trillion, a 26% increase from 2017, when Congress and President Trump agreed to massive corporate tax cuts.

Aggregate demand at U.S. government debt auctions has remained stable in recent years, with investors and bond dealers submitting bids totaling more than twice the amount of notes and bonds for sale. That comes even though yields on the securities have swung from multiyear highs to multiyear lows during the period.

The share of the debt purchased by investors was the largest since the government began releasing auction data, totaling about three-quarters of securities sold. Bond dealers, conversely, purchased their smallest share of the debt since data became available.

The amount of government debt is expected to rise as the U.S. is forecast to run trillion-dollar budget deficits for the next decade, according to the Congressional Budget Office.

# Treasury Auctions

- A multi-unit auction where each bidder (primary dealer or investor) also demands multiple units.
- Run with various other markets in place (when-issued/forward market, secondary market, repo market, etc..)
- An auction with common values rather than private values.

# Treasury Auctions

- The auctioneer (central banks, Department of Treasury etc.) not only wants to maximize revenue but also to minimize susceptibility of the auction to manipulation (see what Solomon Brothers did in 1991).

- It can be a discriminatory auction (Fed QE auctions) or a uniform-price auction (US and Singapore).
  - Table Summary from 2005.

- Each bidder can submit:
  - one or more competitive bids, each specifying a minimum yield at which the participant is prepared to buy a specified quantity of notes; or
  - a single non-competitive bid specifying the quantity of securities that it is prepared to buy at whatever price is paid by successful competitive bidders (up to $5 million per auction)

# How Does the Auctions work

Suppose the Treasury seeks to raise \$9 million in one-year T-bills with a 5% discount rate and receive the following competitive bids:

\$1 million at 4.79%
\$2.5 million at 4.85%
\$2 million at 4.96%
\$1.5 million at 5%

\$3 million at 5.07% $\longleftarrow$ market-clearing "yield" of a uniform-price auction

\$1 million at 5.1%
\$5 million at 5.5%

# Auction: Single-Unit Setup

- Our discussion here is based on Krishna (2010, Auction Theory, Chapter 6) and originally due to Milgrom and Weber (1982):

- The setup:
  - 1 object (a unit of ZCB) for sale to $n$ potential bidders.
  - Bidders (participants who send competitive bid(s)) are risk-neutral and not subject to any liquidity or budget constraint.
  - Bidders send their bids simultaneously (sealed-bid auctions as opposed to open auctions).
  - The object is allocated to the bidder (the winner) who send the highest bid.
  - Ignore tie.

# First-Price Auction vs Second-Price Auction

- In a first-price auction, the winner pays his own (highest) bid, whereas in a second-price auction the winner pays the second-highest bid.
  - Losers don't need to pay.

- First-price auction is the single-unit counterpart of discriminatory auction and second-price auction is that of uniform-price auction (although what we need is the other way around...).

- This does not mean the auctioneer earns more by running a first-price auction, as the bidders' "equilibrium play" differs.

- Two celebrated results in private-value settings:
  - bidding the true value is optimal in second-price auction, regardless what the others bid;
  - first-price and second-price auctions generate the same revenue.

# Signals

- Each bidder $i$ has only partial information about his value (willingness to pay) from his private signal $X_i$
  - e.g., the ZCB price obtained from his own interest rate forecast and pricing model.
  - Bidder $i$ knows the realization of $X_i$ but not the realization of $X_j$ for other bidders $j \neq i$.

- Let $f$ denote the joint density of the signals $X_1, X_2, \ldots, X_n$. We also assume that signals are (positively) affiliated, i.e., for every $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{x}' = (x_1', x_2', \ldots, x_n')$,

$$f\left(\mathbf{x} \vee \mathbf{x}'\right) f\left(\mathbf{x} \wedge \mathbf{x}'\right) \geq f\left(\mathbf{x}\right) f\left(\mathbf{x}'\right)$$

  where $\mathbf{x} \vee \mathbf{x}' = \left(\max\left(x_1, x_1'\right), \ldots, \max\left(x_n, x_n'\right)\right)$ and $\mathbf{x} \wedge \mathbf{x}' = \left(\min\left(x_1, x_1'\right), \ldots, \min\left(x_n, x_n'\right)\right)$.
  - This means that the bidders signals are positively correlated (in a strong sense).
  - The joint density is assumed to be public information which is known to all (including the auctioneer).

# Common Value

- Assume that the bidders have common values, i.e., each bidder $i$'s value of the object is equal to

$$V = v\left(x_1, x_2, ..., x_n\right)$$

  which can be thought as the common resale value of the ZCB in the secondary market (as opposed to private consumption).
  - Bidder 1, for instance, knows only $X_1 = x_1$ and hence he views $V$ as a random variable $V = v\left(x_1, X_2, ..., X_n\right)$ which dependent on the random variables $X_2, ..., X_n$ whose realizations are unknown to him.

- Assume $v$ is a strictly increasing function in each signal.

- A common-value model a special case of a interdependent-value model where

$$V_i = v_i\left(x_1, x_2, ..., x_n\right).$$

- A common-value model is distinct from a private-value model where

# Symmetric Setup

- Assume that the joint density of signals $f(\cdot)$ and the common value function $v(\cdot)$ are both symmetric, e.g., for three bidders, this means

$$\begin{aligned} f(x_1, x_2, x_3) &= f(x_1, x_3, x_2) = f(x_2, x_1, x_3) \\ &= f(x_2, x_3, x_1) = f(x_3, x_1, x_2) = f(x_3, x_2, x_1). \end{aligned}$$

  and likewise for $v$.

- This implies that each $X_i$ has the same marginal distribution and we may assume $X_i$ takes values in $[0, \omega]$.

- In a symmetric setup, we can focus on an arbitrary bidder, say bidder 1.

# The Winner's Curse

- Define
$$u(x,y) \equiv E[V|X_1 = x, Y_1 < y].$$
where $Y_1 \equiv \max_{i \neq 1} X_i$. By affiliation, $u$ is non-decreasing in $(x,y)$ but we assume that $u(x,y)$ is strictly increasing.
- Bidder 1 should not just bid according to his private signal $X_{1,}$ $E[V|X_1 = x]$ since winning carries a bad news (no one has a signal as high as mine):
$$u(x,x) < u(x,\omega) = E[V|X_1 = x].$$
- It becomes more severe when there are more bidders or when there is more uncertainty about the value.

# The Winner's Curse

- The Winner's Curse does not occur "in equilibrium" since bidders must account for it in forming their equilibrium bids.
- Rule of Thumb: bidders being less susceptible to the Winner's Curse means more revenue to the auctioneer (see Bikhchandani and Huang (1993)).
- This idea suggests that a second-price auction should generate more revenue than a first-price auction.

# Second-Price Auction

- A bidding strategy is a strictly increasing mapping $\beta : [0, \omega] \longrightarrow \mathrm{R}_+$ where $\beta(x)$ means any bidder with signal $x$ bids $\beta(x)$.

- A symmetric equilibrium in an auction is bidding strategy $\beta$ such that if other bidders all follow the bidding strategy $\beta$, bidder $i$'s expected payoff with signal $x$ is maximized at the bid $b = \beta(x)$.

# Second-Price Auction

- Claim. The bidding strategy $\beta^{II}$ with $\beta^{II}(x) \equiv u(x,x)$ for all $x$ forms a symmetric equilibrium in a second-price auction.

- To see this, observe bidder $i$'s expected payoff with signal $x$ and bid $b$ is

$$\begin{aligned}
\Pi^{II}(b,x) &\equiv \int_0^{\left(\beta^{II}\right)^{-1}(b)} \left(u(x,y) - \beta^{II}(y)\right) g(y|x)\, dy \\
&= \int_0^{\left(\beta^{II}\right)^{-1}(b)} \left(u(x,y) - u(y,y)\right) g(y|x)\, dy
\end{aligned}$$

  where $g(\cdot|x)$ is the density of $Y_1$ conditional on $X_1 = x$.

- Thus, $\Pi^{II}(b,x)$ is maximized at $b = \beta^{II}(x)$.

# Example

- Three bidders.

- Suppose that $V$ is uniformly distributed on $[0,1]$ and conditional on $V = v$, $X_1$, $X_2$, and $X_3$ are independently and uniformly distributed on $[0, 2v]$.

- Define $Z = \max\{X_1, X_2, X_3\}$. Conditional on $(X_1, X_2, X_3)$, we know that $V \geq \frac{1}{2}Z$ and hence

$$f(x_1, x_2, x_3) = \int_{\frac{z}{2}}^{1} \frac{1}{8v^3} dv = \frac{4 - z^2}{16z^2}.$$

where $z = \max\{x_1, x_2, x_3\}$.

# Example

- Likewise,

$$
\begin{aligned}
f\left(v|x_1, x_2, x_3\right) &= \frac{1}{8v^3} \times \frac{16z^2}{4-z^2}; \\
E\left[V|Z=z\right] &= \int_{\frac{z}{2}}^{1} v \times \frac{1}{8v^3} \times \frac{16z^2}{4-z^2} dv = \frac{2z}{2+z}.
\end{aligned}
$$

- It follows that

$$
\begin{aligned}
\beta^{II}\left(x\right) &= u\left(x,x\right) = E\left[V|X_1 = x, Y_1 < x\right] \\
&= E\left[V|Z=x\right] \\
&= \frac{2x}{2+x}.
\end{aligned}
$$

# First-Price Auction

- In a first-price auction, the bidder's payoff with signal $x$ by bidding $b$ is

$$\Pi^{I}\left(b,x\right)=\int_{0}^{\beta^{-1}(b)}\left(u\left(x,y\right)-b\right)g\left(y|x\right)dy.$$

- By Leibniz rule, the first-order condition for choosing $b$ to maximize $\Pi^{I}\left(b,x\right)$ is

$$\frac{\partial\Pi^{I}\left(b,x\right)}{\partial b}=\left(u\left(x,\beta^{-1}\left(b\right)\right)-b\right)g\left(\beta^{-1}\left(b\right)|x\right)\frac{1}{\beta'\left(\beta^{-1}\left(b\right)\right)}$$
$$-G\left(\beta^{-1}\left(b\right)|x\right)=0$$

  where $G\left(\cdot|x\right)$ is the CDF of $Y_1$ conditional on $X_1=x$.
- In order for $\beta$ to be a symmetric equilibrium, the first-order condition must be satisfy at $b=\beta\left(x\right)$, i.e.,

$$\left(u\left(x,x\right)-\beta\left(x\right)\right)g\left(x|x\right)\frac{1}{\beta'\left(x\right)}-G\left(x|x\right)=0.$$

# First-Price Auction

- Equivalently, $\beta$ satisfies the differential equation:

$$\beta'(x) = (u(x,x) - \beta(x)) \frac{g(x|x)}{G(x|x)}.$$

- Moreover, $u(x,x) - \beta(x) \geq 0$ for every $x$ (otherwise, bidder $i$ with value $x$ would rather bid 0). Hence, if we assume $u(0,0) = 0$ (or $v(0,...,0) = 0$), there is a boundary condition $\beta(0) = 0$.

- The solution to this differential equation is

$$\beta^I(x) = \int_0^x u(y,y)\, dL(y|x).$$

where

$$L(y|x) = \exp\left(-\int_y^x \frac{g(t|t)}{G(t|t)} dt\right)$$

- Note the first-order condition is only necessary for $\beta^I(x)$ to solve $\max_b \Pi^I(b,x)$. For why $\beta^I(x)$ indeed solves $\max_b \Pi^I(b,x)$, see Krishna (2010, Proposition 6.3).

# Revenue Comparison

- Say CDF $F$ first-order stochastically dominates another CDF $H$ (both defined on $\mathrm{R}$) if $F(x) \leq H(x)$ for all $x$.
- Facts: (**Exercise**)
  - $F$ first-order stochastically dominates $G$ if and only if for every nondecreasing function $w$, we have $\int wdF \geq \int wdH$.
  - Affiliation of signals implies that (Krishna, pp. 285-288)

$$\frac{g(t|t)}{G(t|t)} \leq \frac{g(t|x)}{G(t|x)} \text{ for all } t < x.$$

- We now show that a second-price auction generates no less expected revenue than a first-price auction:
  - In a first-price auction, a bidder with signal realization $x$ pays $\beta^{I}(x)$ upon winning the object;
  - In a second-price auction, a bidder with signal realization $x$ pays in expectation $E\left(\beta^{II}(Y_1)\,|X_1 = x, Y_1 < x\right)$ upon winning the object.

# Revenue Comparison

- We first rewrite

$$E\left(\beta^{II}\left(Y_1\right)|X_1 = x, Y_1 < x\right) = \int_0^x u\left(x, x\right) dK\left(y|x\right)$$

where

$$K\left(y|x\right) = \frac{G\left(y|x\right)}{G\left(x|x\right)}.$$

- Now recall that $u\left(x, x\right)$ is non-decreasing (by affiliation) and

$$\beta^{I}\left(x\right) = \int_0^x u\left(y, y\right) dL\left(y|x\right).$$

- Hence, it suffices to prove that $K\left(\cdot|x\right)$ first-order stochastically dominates $L\left(\cdot|x\right)$.

# Revenue Comparison

- Recall that for all $t < x$, we have (monotone reverse hazard rate)
$$\frac{g(t|t)}{G(t|t)} \leq \frac{g(t|x)}{G(t|x)}.$$
- Hence,
$$\begin{aligned} -\int_y^x \frac{g(t|t)}{G(t|t)} dt &\geq -\int_y^x \frac{g(t|x)}{G(t|x)} dt \\ &= -\int_y^x \frac{d}{dt} \log G(t|x)\, dt \\ &= \log \frac{G(y|x)}{G(x|x)}. \end{aligned}$$
- Applying the exponential function on both sides, we obtain
$$L(y|x) = \exp\left(-\int_y^x \frac{g(t|t)}{G(t|t)}\right) \geq \frac{G(y|x)}{G(x|x)} = K(y|x)$$
namely, $K(\cdot|x)$ first-order stochastically dominates $L(\cdot|x)$.

# English Auction

- An English Auction is an open ascending auction where the winner is the one who remains when the other $n-1$ bidders all drop; moreover, the winner pays the price when the last other bidder drops (again, the losers don't pay).

- For notation simplicity, we only analyze the case with three bidders (see Krishna (2010, Chapter 6 for a complete analysis).

- A bidding strategy is now a pair of functions

$$\begin{aligned} \beta &: [0, \omega] \to \mathrm{R}_+; \\ \alpha &: [0, \omega] \times \mathrm{R}_+ \to \mathrm{R}_+. \end{aligned}$$

  where $\beta(x)$ is the "bid" (recommended price to drop) for signal realization $x$ when no one drops out and $\alpha(x, p)$ is the "bid" for $x$ when one bidder drops out at price $p$ (observed by everyone).

# English Auction

- Define

$$\begin{aligned}\overline{\beta}(x) &= v(x,x,x)\,;\\ \overline{\alpha}(x,p) &= v\left(x,x,\overline{\beta}^{-1}(p)\right)\end{aligned}$$

where we recall that $\overline{\beta}$ is strictly increasing.

- Claim. $(\overline{\beta},\overline{\alpha})$ is a symmetric equilibrium in an English auction.

- Again, let $Y_1 = \max\{X_2, X_3\}$ and $Y_2 = \min\{X_2, X_3\}$. Let $(y_1, y_2)$ be the realization of $(Y_1, Y_2)$. We must have $y_1 \geq y_2$.

- We argue why it is optimal for bidder 1 to follow $(\overline{\beta},\overline{\alpha})$ when other bidders also follow $(\overline{\beta},\overline{\alpha})$.

# English Auction

- Given the realization $(y_1, y_2)$, consider two cases:
  - Case 1. If bidder 1 with signal realization $x$ wins, then $x > y_1$and bidder 1 obtains the payoff

    $$v(x, y_1, y_2) - v(y_1, y_1, y_2) > 0$$

    where the price is $v(y_1, y_1, y_2)$ because the third highest bidder following $\overline{\beta}$ drops at $\overline{\beta}(y_2)$ and the second-highest bidders following $\overline{\alpha}$ will drop at

    $$\overline{\alpha}\left(y_1, \overline{\beta}(y_2)\right) = v(y_1, y_1, y_2).$$

  - Case 2. If bidder 1 with signal realization $x$ loses, then $x < y_1$ and bidder 1, likewise, obtains the payoff

    $$0 > v(x, y_1, y_2) - v(y_1, y_1, y_2).$$

- Hence, bidder 1 will indeed wants to follow $\left(\overline{\beta}, \overline{\alpha}\right)$ to win in the first case and to lose in the second case.

# Revenue Comparison

The revenue of an English auction under the symmetric equilibrium $(\overline{\beta}, \overline{\alpha})$ is

$$
\begin{aligned}
& E\left[R^{\text{Eng}}\right] \\
= \;& E\left[\overline{\alpha}\left(Y_1, Y_2\right)\right] \\
= \;& \int_0^{\omega} \int_0^{x} E\left[v\left(y, y, Y_2\right) | X_1 = x, Y_1 = y\right] g\left(y|x\right) dy f_1\left(x\right) dx \\
\geq \;& \int_0^{\omega} \int_0^{x} E\left[v\left(y, Y_1, Y_2\right) | X_1 = y, Y_1 < y\right] g\left(y|x\right) dy f_1\left(x\right) dx \\
= \;& \int_0^{\omega} \int_0^{x} u\left(y, y\right) g\left(y|x\right) dy f_1\left(x\right) dx \\
= \;& \int_0^{\omega} \int_0^{x} \beta^{II}\left(y\right) g\left(y|x\right) dy f_1\left(x\right) dx \\
= \;& E\left[R^{II}\right]
\end{aligned}
$$

where $f_1$ denotes the marginal density of $X_1$ (induced from $f$) and the inequality follows from affiliation.

# Concluding Remarks

- The single-unit model is meant to approximate the multi-unit reality, but who knows:
  - Uniform-price auction is not really a generalization of second-price auction;
  - Multidimensional types/bids (a demand function) is notoriously difficult to handle.
- Still we learn important insight such as the Winner's Curse and how it affects the revenue ranking of different auctions.

- Bikhchandani and Huang (1993): with the secondary and when-issue markets, the revenue ranking might also be different due to:
  - collective manipulation due to collusion of the bidders;
  - individual manipulation: bid high in order to profit from the short squeeze (corner the market);
  - incentive to bid higher to signal to the buyers in the secondary market that the bidders' private information is favorable.