NATIONAL UNIVERSITY OF SINGAPORE

FE5112 - Stochastic Calculus and Quantitative Methods

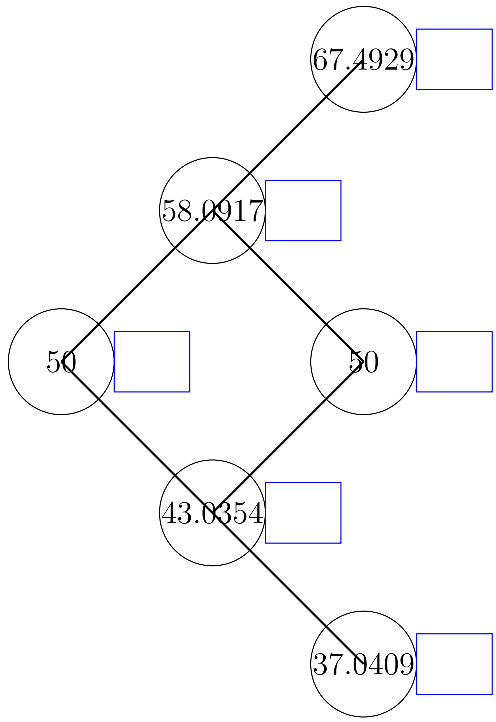
(Semester 1 : AY2018/2019)

Time allowed : $2\frac{1}{2}$ hours

INSTRUCTIONS TO CANDIDATES

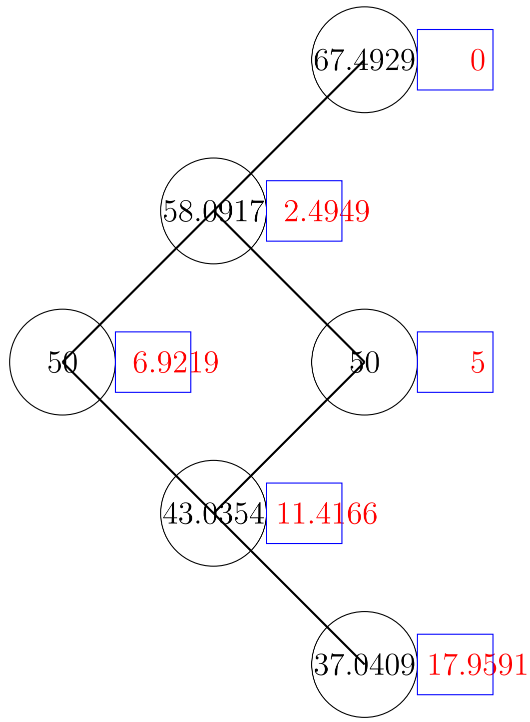
1. Please write your student number only. Do not write your name.
2. This assessment paper contains **SIX** questions and comprises **FOUR** printed pages.
3. The total mark for this paper is 100.
4. Answer **ALL** questions.
5. Please start each question on a new page.
6. This is a CLOSED BOOK examination. However, students are allowed to bring an A4 sized help sheet which can be written on both sides.
7. Students are allowed to use scientific calculators.
8. Students should lay out systematically the various steps in the calculations.
9. Students are not allowed to take this assessment paper away from the examination hall.

Question 1 [20 marks] Consider the problem of using binomial tree method to calculate the European and American put options with $S_0 = 50$, $K = 55$, $\mathtt{T} = 0.5$, $r = 0.04$, $\sigma = 0.3$. We set $\delta t = \mathtt{T}/2$ and construct a two step binomial tree with $u = e^{\sigma\sqrt{\delta t}} \approx 1.16183$, $d = e^{-\sigma\sqrt{\delta t}} \approx 0.860708$, $\rho = e^{r\delta t} \approx 1.01005$. Evaluate the European and American put option prices based on the binomial tree. Keep at least 5 significant digits in your calculation.

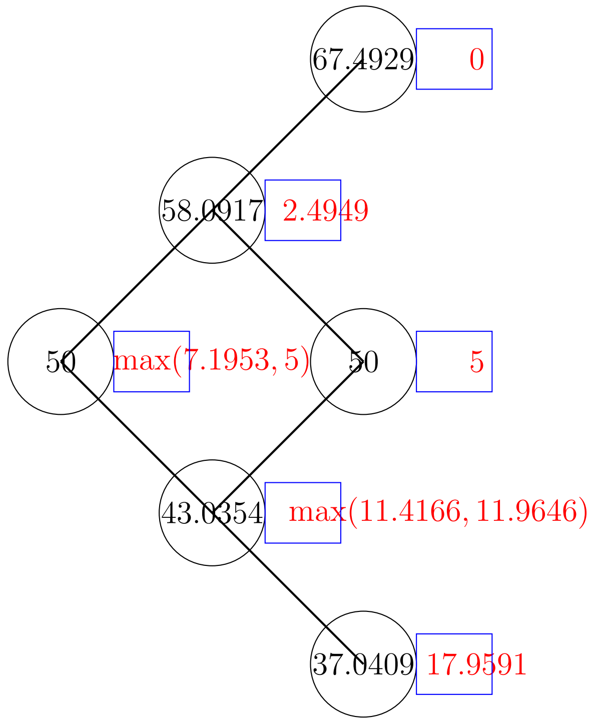


Solution:

$q_u = \frac{\rho - d}{u - d} = \frac{1.01005 - 0.860708}{1.16183 - 0.860708} \approx 0.4960$, $q_d = 1 - q_u$.




```
Put_Eu = 6.9219
```


67.4929
0
58.0917
2.4949
50
max(7.1953, 5)
50
5
43.0354
max(11.4166, 11.9646)
37.0409
17.9591

`Put_Am = 7.1953`

Question 2 [10 marks] Let W_t be the standard Brownian motion with $W_0 = 0$. Determine the mean and variance of

$$\int_0^1 e^{W_t} dW_t.$$

You can directly use the fact that $\mathbb{E}[e^{\theta W_t}] = e^{\frac{1}{2}\theta^2 t}$ which has been proved in your homework.

Solution:

$$\mathbb{E}\left[\int_0^1 e^{W_t} dW_t\right] = 0.$$

$$\operatorname{Var}\left[\int_0^1 e^{W_t} dW_t\right] = \mathbb{E}\left[\left(\int_0^1 e^{W_t} dW_t\right)^2\right] = \mathbb{E}\int_0^1 (e^{W_t})^2 dt = \int_0^1 \mathbb{E}(e^{2W_t}) dt = \int_0^1 e^{2t} dt = \frac{e^2 - 1}{2}.$$

Question 3 [20 marks] Consider

$$X_t = \sigma(T-t)\int_0^t \frac{1}{T-s}dW_s, \qquad t \in [0,T].$$

a) Compute $\mathbb{E}[X_t]$ and prove that $\mathrm{Var}[X_t] = \sigma^2 \frac{t(T-t)}{T}$.

b) Use Itô formula to show that X_t satisfies the stochastic differential equation

$$dX_t = -\frac{X_t}{T-t}dt + \sigma dW_t, \qquad X_0 = 0.$$

[Hint: Recall $d\left(\int_0^t f(s,\omega)dW_s\right) = f(t,\omega)dW_t$.]

Solution:

(a) $\mathbb{E}[X_t] = 0$ by the property of Itô integral.

$$\mathrm{Var}[X_t] = \mathbb{E}[X_t^2] = \sigma^2(T-t)^2\int_0^t \frac{1}{(T-s)^2}ds = \sigma^2(T-t)^2 \left.\frac{1}{T-s}\right|_{s=0}^{s=t} = \sigma^2\frac{t(T-t)}{T}.$$

(b) It is obvious that $X_0 = 0$.

$$\begin{aligned} dX_t &= -\sigma\left(\int_0^t \frac{1}{T-s}dW_s\right)dt + \sigma(T-t)\frac{1}{T-t}dW_t \\ &= -\frac{X_t}{T-t}dt + \sigma dW_t. \end{aligned}$$

Question 4 [20 marks] Consider the CIR model

$$dY_t = (\gamma - \beta Y_t)dt + \sigma\sqrt{Y_t}dW_t, \qquad Y_0 = a, \tag{1}$$

where γ, β, σ, a are positive constants.

a) Write down equation (1) in integral form.

b) Let $u(t) = \mathbb{E}[Y_t]$. Use the integral form of (1) to show that $u(t)$ satisfies the differential equation

$$\frac{d}{dt}u(t) = \gamma - \beta u(t), \qquad u(0) = a.$$

c) By Itô formula, show that

$$dY_t^2 = Y_t(2\gamma + \sigma^2 - 2\beta Y_t)dt + 2\sigma\left(\sqrt{Y_t}\right)^3 dW_t. \tag{2}$$

d) Let $v(t) = \mathbb{E}[Y_t^2]$. Use the integral form of (2) to find a differential equation that is satisfied by $v(t)$. This equation can contain $u(t)$ and you do not need to solve for $u(t)$ from b).

e) Use b) and d) to determine the constant C so that

$$\mathrm{Var}[Y_t] = \mathbb{E}[Y_t^2] - (\mathbb{E}[Y_t])^2 \to C \qquad \text{when} \qquad t \to \infty. \tag{3}$$

[Hint: You do not have to solve the differential equations in b) and d). Suppose you know $u(t) \to U$ and $v(t) \to V$ for some constants U and V when $t \to \infty$, what can you say about U and V?]

Solution:

(a)

$$Y_t - Y_0 = \int_0^t (\gamma - \beta Y_s)ds + \int_0^t \sigma\sqrt{Y_s}dW_s.$$

(b) Taking $\mathbb{E}$ on both sides of the above equation, we get

$$u(t) - u(0) = \int_0^t (\gamma - \beta u(s))ds.$$

Taking $\frac{d}{dt}$ of the above equation, we get

$$\frac{d}{dt}u(t) = \gamma - \beta u(t).$$

$u(0) = \mathbb{E}[Y_0] = \mathbb{E}[a] = a.$

(c)

$$
\begin{aligned}
dY_t^2 &= 2Y_t dY_t + (dY_t)^2 = 2Y_t(\gamma - \beta Y_t)dt + 2Y_t\sigma\sqrt{Y_t}dW_t + \sigma^2 Y_t dt \\
&= Y_t(2\gamma + \sigma^2 - 2\beta Y_t)dt + 2\sigma\left(\sqrt{Y_t}\right)^3 dW_t.
\end{aligned}
$$

(d)

$$
Y_t^2 - Y_0^2 = \int_0^t Y_s(2\gamma + \sigma^2 - 2\beta Y_s)ds + \int_0^t 2\sigma\left(\sqrt{Y_s}\right)^3 dW_s.
$$

$$
\mathbb{E}[Y_t^2] - \mathbb{E}[Y_0^2] = \int_0^t (2\gamma + \sigma^2)\mathbb{E}[Y_s]ds - 2\beta\int_0^t \mathbb{E}[Y_s^2]ds.
$$

$$
\frac{d}{dt}v(t) = -2\beta v(t) + (2\gamma + \sigma^2)u(t).
$$

(e) When $t \to \infty$, $u(t) \to$ a constant which is denoted as U. Then U satisfies $\gamma - \beta U = 0$. Hence $U = \frac{\gamma}{\beta}$. $v(t)$ also converges to a constant, which is called V. Then $-2\beta V + (2\gamma+\sigma^2)U = 0$. $V = \frac{2\gamma+\sigma^2}{2\beta}U = \frac{2\gamma^2+\sigma^2\gamma}{2\beta^2}$.

$$
\mathrm{Var}[Y_t] = \mathbb{E}[Y_t^2] - \left(\mathbb{E}[Y_t]\right)^2 \to V - U^2 = \frac{\gamma\sigma^2}{2\beta^2}.
$$

So, $C = \frac{\gamma\sigma^2}{2\beta^2}$.

Question 5 [15 marks] Recall that a portfolio $\Phi = \Delta S + B$ is call self-financing if it satisfies both $d\Phi_t = \Delta_t dS_t + dB_t$ and $\Phi_t = \Delta_t S_t + B_t$. Here S_t is the stock price, Δ_t is the number of shares of stock, B_t is the amount in the money market account at time t. Prove that for any stock price model, a self-financing portfolio $\Phi = \Delta S + B$ satisfies

$$d\left(e^{-rt}\Phi_t\right) = \Delta_t d\left(e^{-rt}S_t\right) \tag{4}$$

which means that change in the discounted portfolio value is solely due to change in the discounted stock price. The parameter r in (4) comes from the interest rate of the money market account B whose value satisfies $dB_t = rB_t dt$. Note that S_t may not satisfy the geometric Brownian motion model.

Proof: By Itô formula with $g(t,x) = e^{-rt}x$,

$$\begin{aligned} d(e^{-rt}S_t) =& dg(t,S_t) = \frac{\partial g}{\partial x}dt + \frac{\partial g}{\partial x}dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dS_t)^2 \\ =& -re^{-rt}S_t dt + e^{-rt}dS_t, \end{aligned}$$

$$\begin{aligned} d(e^{-rt}\Phi_t) =& dg(t,\Phi_t) = \frac{\partial g}{\partial x}dt + \frac{\partial g}{\partial x}d\Phi_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(d\Phi_t)^2 \\ =& -re^{-rt}\Phi(t)dt + e^{-rt}d\Phi(t) \\ =& -re^{-rt}\left(\Delta_t S_t + B_t\right)dt + e^{-rt}\left(\Delta_t dS_t + dB_t\right) \\ =& -re^{-rt}\Delta_t S_t dt + e^{-rt}\Delta_t dS_t \\ =& \Delta_t\left(-re^{-rt}S_t dt + e^{-rt}dS_t\right) \\ =& \Delta_t d(e^{-rt}S_t). \end{aligned}$$

Question 6 [15 marks] Given constant r, functions b, σ, and f, consider the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$$

and the deterministic partial differential equation

$$\partial_t g(t,x) + b(t,x)\partial_x g(t,x) + \frac{1}{2}\sigma^2(t,x)\partial_x^2 g(t,x) = rg(t,x) \tag{5}$$

with terminal condition

$$g(T,x) = f(x) \qquad \text{for all } x. \tag{6}$$

Fix $T > 0$, show that $e^{-rt}g(t, X(t))$ is a martingale, and the solution $g(t,x)$ of (5) and (6) can be represented as

$$g(t,x) = \mathbb{E}^{t,x}[e^{-r(T-t)}f(X(T))], \tag{7}$$

where $\mathbb{E}^{t,x}$ denotes the conditional expectation under the condition that $X(t) = x$.
[Hint: First show that $d(e^{-rt}g(t, X(t)))$ equals something times dW_t. Then integrate the resulting identity from t to T on both sides before taking $\mathbb{E}^{t,x}$.]
Proof: By Itô formula

$$\begin{aligned}
d(e^{-rt}g(t,X_t)) &= \partial_t(e^{-rt}g)dt + \partial_x(e^{-rt}g)dX + \frac{1}{2}\partial_x^2(e^{-rt}g(t,x))(dX)^2 \\
&= (e^{-rt}\partial_t g - re^{-rt}g)dt + e^{-rt}\partial_x g(bdt + \sigma dW_t) + e^{-rt}\frac{1}{2}\partial_x^2 g(t,x)\sigma^2 dt \\
&= e^{-rt}\left(\partial_t g + b\partial_x g + \frac{1}{2}\sigma^2\partial_x^2 g - rg\right)dt + e^{-rt}\sigma\partial_x g dW_t \\
&\overset{(5)}{=} e^{-rt}\sigma\partial_x g dW_t.
\end{aligned}$$

Hence $e^{-rt}g(t,X_t)$ is a martingale. Moreover, integrating $d(e^{-rt}g(t,X_t)) = e^{-rt}\sigma\partial_x g dW_t$ from t to T, we get

$$e^{-rT}g(T,X_T) - e^{-rt}g(t,X_t) = \int_t^T e^{-rs}\sigma(s,X_s)\partial_x g(s,X_s)dW_s.$$

Hence

$$\begin{aligned}
\mathbb{E}^{t,x}[e^{-rT}g(T,X(T))] &= \mathbb{E}^{t,x}[e^{-rt}g(t,X(t) + \int_t^T e^{-rs}\sigma(s,X_s)\partial_x g(s,X_s)dW_s] \\
&= \mathbb{E}^{t,x}[e^{-rt}g(t,X(t)].
\end{aligned}$$

But $\mathbb{E}^{t,x}[e^{-rT}g(T,X(T))] = \mathbb{E}^{t,x}[e^{-rT}f(X(T))]$ while $\mathbb{E}^{t,x}[e^{-rt}g(t,X(t)] = e^{-rt}g(t,x)$ as $\mathbb{E}^{t,x}$ requires $X(t) = x$. This proves

$$\mathbb{E}^{t,x}[e^{-rT}g(T,X(T))] = e^{-rt}g(t,x)$$

which leads to (7).

END OF PAPER