It means that $\Phi_1$ consists of a call option, a zero-coupon bond with face value (par value) $K$ and maturity $T$, and $n$ zero-coupon bonds with face values $D_k$ and maturity times $t_k$, $k = 1, 2, \cdots, n$. $\Phi_2$ consists of a put option and the stock which will pay dividend $D_j$ at $t_j$.

Now, you are asked to **prove** $V_T(\Phi_1) = V_T(\Phi_2)$. Then we can conclude $V_0(\Phi_1) = V_0(\Phi_2)$ which is precisely (1.35) by Corollary 1.30.

10. (No arbitrage delivery price of a forward) A forward contract is an agreement to buy or sell an asset at a certain future time (expiration date) for a certain price (delivery price). One of the parties to a forward contract assumes a long position and agrees to buy the underlying asset on expiration date for certain delivery price $F$. The other party assumes a short position and agrees to sell the asset at the expiration date for the same price. Using arbitrage-free principle to show that the delivery price $F$ on a non-dividend-paying asset with spot price $S_0$ is given by

$$K = S_0 e^{rT}$$

where $r$ is the risk-free interest rate and $T$ is the time to expiry of the forward contract.

**Solution**: Let $F$ denote a forward contract and construct a portfolio

$$\Phi = -B + S - F,$$

with $V_0(B) = S_0$. It means that at $t = 0$, one borrows $S_0$ dollars from the bank to buy the asset $S$ from the market, and one also sell a forward contract. This means he/she agrees to sell $S$ at the expiration date for the delivery price $K$. So, $V_0(F) = 0$ and $V_T(-F) = K - S_T$. The latter equation says that since he/she sold a forward contract, by $T$, he/she has to sell $S$ (whose price is $S_T$) for price $K$ to the buyer of the forward contract. Hence

$$V_0(\Phi) = -S_0 + S_0 + 0 = 0,$$
$$V_T(\Phi) = -S_0 e^{rT} + S_T + K - S_T = K - S_0 e^{rT}.$$

So, if $K > S_0 e^{rT}$, $V_0(\Phi) = 0$ and $V_T(\Phi) > 0$, one has an arbitrage opportunity. On the other hand, if $K < S_0 e^{rT}$, $V_0(-\Phi) = 0$ and $V_T(-\Phi) > 0$, one has an arbitrage opportunity by building a portfolio $-\Phi$ which is $B - S + F$. Since we assume the market is arbitrage-free, we must have $K = S_0 e^{rT}$.

11. Consider a European call $c_1$ with a strike price of $K_1$ and a second European call $c_2$ on the same stock with a strike price of $K_2 > K_1$. Both call options have the same expiration date. Let $c(t, K)$ denote the price of the European call option at time $t$ with strike price $K$. So, $c(t, K_i) = V_t(c_i)$ for $i = 1, 2$. Prove that

$$-e^{-r(T-t)}(K_2 - K_1) < c(t, K_2) - c(t, K_1) < 0.$$

Furthermore, deduce that

$$-e^{-r(T-t)} \leq \frac{\partial c}{\partial K}(t, K) \leq 0. \tag{1.36}$$

In other words, suppose the call price can be expressed as a differentiable function of the strike price, then the derivative must be non-positive and no greater in absolute value than the price of a zero-coupon bond with face value of unity and the same maturity.

**Solution**: Consider the following portfolio at $t = 0$:

$$\Phi = c_2 - c_1 + e^{-rT}(K_2 - K_1).$$

Then

$$V_T(\Phi) = (S_T - K_2)^+ - (S_T - K_1)^+ + (K_2 - K_1) = \begin{cases} (K_2 - K_1) & \text{if } S_T \leq K_1 \\ -S_T + K_2 & \text{if } K_1 < S_T \leq K_2 \\ 0 & \text{if } K_2 < S_T \end{cases}$$

Hence $V_T(\Phi) \geq 0$ and $\mathbb{P}(V_T(\Phi) > 0) > 0$. By Definition 1.1, we must have

$$V_t(\Phi) > 0, \qquad \forall t \in [0, T). \tag{1.37}$$

(If not, then $V_{t^*}(\Phi) \leq 0$ for some $t^* \in [0, T)$, which means $\Phi$ has arbitrage opportunity by Definition 1.1.) Equation (1.37) means

$$V_t(\Phi_1) = c(t, K_2) - c(t, K_1) + e^{-r(T-t)}(K_2 - K_1) > 0.$$

Next, we consider $\Phi = c_1 - c_2$, Then

$$V_T(\Phi) = (S_T - K_1)^+ - (S_T - K_2)^+ = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 < S_T \leq K_2 \\ K_2 - K_1 & \text{if } K_2 < S_T \end{cases}$$

Hence $V_T(\Phi) \geq 0$ and $\mathbb{P}(V_T(\Phi) > 0) > 0$. By Definition 1.1, we must have

$$V_t(\Phi) > 0, \qquad \forall t \in [0, T).$$

This means

$$V_t(\Phi_1) = c(t, K_1) - c(t, K_2) > 0.$$

Finally, by the definition of derivative,

$$\frac{\partial c}{\partial K}(t, K) = \lim_{h \downarrow 0} \frac{c(t, K + h) - c(t, K)}{h}.$$

Since $h > 0$, $-e^{-r(T-t)} < \frac{c(t,K+h)-c(t,K)}{h} < 0$, we get (1.36) by letting $h \downarrow 0$.