FE5222 Advanced Derivative Pricing

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Overview

American Options

An American option gives its buyer the right to exercise the option at any time before or on expiry. An American call option pays $S(t) - K$ and an American put option pays $K - S(t)$ if exercised at time t.

In this lecture, we assume the underlying stock pays no dividend.

Price Bound

Notations:

- c: European call price
- p: European put price
- C: American call price
- P: American put price

Price Bound

European vs American

$$C \geq c$$

$$P \geq p$$

Price Bound

European call

$$\left(S(0) - Ke^{-rT}\right)^+ \leq c \leq S(0)$$

Price Bound

We provide two different arguments: portfolios and risk neutral pricing argument.

Portfolio Argument.

Consider two portfolios

- Portfolio A consists of a European call option with strike K and expiry T and zero-coupon bond pays K at time T.
- Portfolio B consists of a single stock

□

Price Bound

Portfolio Argument - Cont'd.

At time T, portfolio A is worth $\max(S(T), K)$ and portfolio B is worth $S(T)$. Hence portfolio A is worth more than B at time T.

$\Longrightarrow$ Portfolio A is worth more than B at time $t = 0$. Hence

$$c + B(0, T) \geq S(0)$$

$\Longrightarrow$ Since $B(0, T) = Ke^{-rT}$, we have

$$c + Ke^{-rT} \geq S(0)$$

This is equivalent to

$$c \geq S(0) - Ke^{-rT}$$

Price Bound

Portfolio Argument - Cont'd.

Since $c \geq 0$, we have

$$c \geq \left(S(0) - Ke^{-rT}\right)^+$$

The upper bound is obvious. $\square$

Price Bound

Risk Neutral Pricing Argument.

By risk neutral pricing

$$\begin{aligned} c &= e^{-rT}\widetilde{\mathbb{E}}\left[(S(T)-K)^{+}\right] \\ &= \widetilde{\mathbb{E}}\left[\left(e^{-rT}S(T)-e^{-rT}K\right)^{+}\right] \end{aligned}$$

Since $f(x)=(x)^{+}$ is a convex function, by Jensen's inequality

$$\begin{aligned} & \widetilde{\mathbb{E}}\left[\left(e^{-rT}S(T)-e^{-rT}K\right)^{+}\right] \\ \geq & \left(\widetilde{\mathbb{E}}\left[e^{-rT}S(T)-e^{-rT}K\right]\right)^{+} \\ = & \left(S(0)-e^{-rT}K\right)^{+} \end{aligned}$$

This proves the lower bound. □

Price Bound

Risk Neutral Pricing Argument - Cont'd.

The proof for the upper bound is easy

$$\begin{aligned} c &= \widetilde{\mathbb{E}}\left[e^{-rT}\left(S(T)-K\right)^{+}\right] \\ &\leq \widetilde{\mathbb{E}}\left[e^{-rT}S(T)\right] \\ &= S(0) \end{aligned}$$

This completes the proof. $\square$

Price Bound

European put

$$\left(Ke^{-rT} - S(0)\right)^{+} \leq p \leq Ke^{-rT}$$

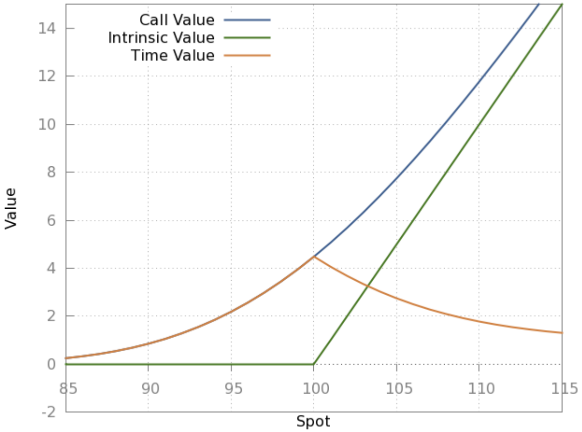
Call-Put Parity

For European options, we have

$$c - p = S(0) - e^{-rT} K$$

Early Exercise Boundary

European call



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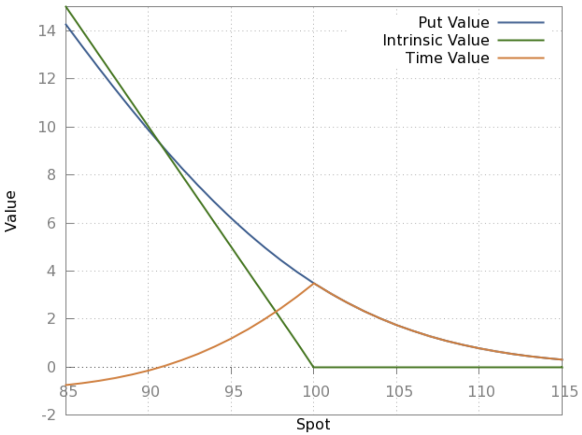
Early Exercise Boundary

European put



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Early Exercise Boundary

It is never optimal to early exercise an American call option (on a non-dividend paying stock).

Early Exercise Boundary

If exercise at $t < T$, the payoff is

$$S(t) - K$$

However,

$$C \geq c \geq \left(S(t) - e^{-r(T-t)}K\right)^{+} > S(t) - K$$

[1] Hence it is not wise to exercise, we shall rather sell the American call option.

[1] We only proved the lower bound for $t = 0$, but it is obvious that this inequality holds for any $t < T$.

Early Exercise Boundary

American put option is a different story as its time value can be negative.

When the stock price drops to a certain level, we shall exercise to lock in cash $K - S$.

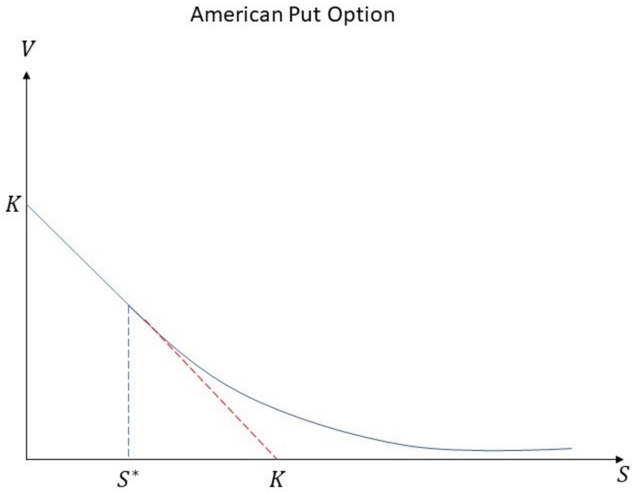
Early Exercise Boundary

Image interest rate r is very high and stock price is virtually 0. If we exercise now, our payoff at T will be

$$(K - S(t))e^{rT} \approx Ke^{rT} > (K - S(T))^+$$

Hence it is optimal to exercise.

Early Exercise Boundary



Early Exercise Boundary

There exists a curve $S^*(t)$ of early exercise boundary. At each time t, it is optimal to exercise the American put if stock price $S(t) \leq S^*(t)$. If $S(t) > S^*(t)$ we shall continue to hold the option.

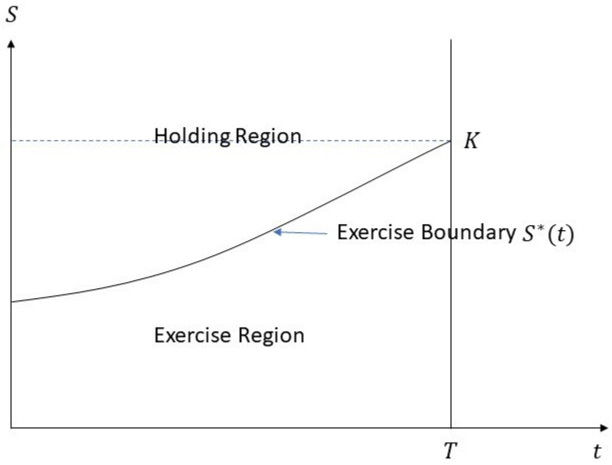
We denote

$$\mathcal{H} = \{(t, s) : 0 \leq t \leq T, s > S^*(t)\}$$

and

$$\mathcal{E} = \{(t, s) : 0 \leq t \leq T, s \leq S^*(t)\}$$

Early Exercise Boundary



Early Exercise Boundary

Properties of early exercise boundary

- It is monotonically increasing and continuous.
- $S^*(T) = K$
- It is convex[2].

[2] This is true for non-dividend paying stock. See Xinfu Chen and John Chadam (2008)

Linear Complementarity Problem

Let $V(t, S)$ be the value of an American put option. Suppose we delta hedge it by shorting $\Delta = V_S$ shares of stocks. The value $\pi(t)$ of this portfolio is

$$\pi(t) = V(t, S) - \Delta S(t)$$

Applying Ito's Lemma to π, we have

$$\begin{aligned} d\pi &= V_t dt + V_S dS(t) + \tfrac{1}{2} V_{SS} dS(t) dS(t) - \Delta dS(t) \\ &= V_t dt + \tfrac{1}{2} \sigma^2 S^2(t) dt \end{aligned}$$

Linear Complementarity Problem

If we are in the holding region, we would expect to earn riskfree rate r. Hence

$$d\pi(t) = r\pi(t)dt$$

when $(t, S(t)) \in \mathcal{F}$.

On the other hand, if we are in the exercise region, holding the option is not optimal. Hence we can earn no more than the riskfree rate. That is

$$d\pi(t) \leq r\pi(t)dt$$

when $(t, S(t)) \in \mathcal{E}$.

Linear Complementarity Problem

Substituting $d\pi(t)$ and $\pi(t)$, we have

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV = 0$$

when $(t, S(t)) \in \mathcal{F}$.

And

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV \leq 0$$

when $(t, S(t)) \in \mathcal{E}$.

Linear Complementarity Problem

The price of American put option also satisfies

$$V(t, S) \geq (K - S)^+$$

Linear Complementarity Problem

If $V(t,S) > (K-S)^+$, we shall not exercise the option. In this case

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV = 0$$

Hence

$$\left(V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV\right)\left(V(t,S) - (K-S)^+\right) = 0$$

Linear Complementarity Problem

The price of an American put option satisfies the so called *linear complementary conditions*

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV \leq 0$$

$$\left(V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV\right)\left(V(t,S) - (K-S)^+\right) = 0$$

$$V(t,S) \geq (K-S)^+$$

Free Boundary Problem

The price of an American put option can also be formulated as a free-boundary problem. We have shown that

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV = 0$$

when $S(t) > S^*(t)$.

Free Boundary Problem

Other conditions to ensure unique solution:

$$V(T,S) = (K-S)^+$$

$$\lim_{S\to\infty} V(t,S) = 0$$

$$\lim_{S\to 0} V(t,S) = K$$

$$\lim_{S\downarrow S^*(t)} V(t,S) = K - S^*(t)$$

$$\lim_{S\downarrow S^*(t)} V_S(t,S) = -1$$

Free Boundary Problem

The last condition $\lim_{S \downarrow S^*(t)} V_S(t, S) = -1$ is called *smooth pasting condition*

Probabilistic Approach

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration. A *stopping time* τ is a non-negative (and possibly ∞) random variable such that

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F} \qquad \forall t \geq 0$$

Remark: It is also called *optional time*.

Probabilistic Approach

Example

Let B be a Borel set and X be a stochastic process. then

$$\tau_B(\omega) = \inf\{s : X(s,\omega) \in B\}$$

is a stopping time, where by convention $\inf \emptyset = \infty$.

Probabilistic Approach

Definition

Let $X(t,\omega)$ be a stochastic process and τ be a stopping time. Then the process $X(t \wedge \tau(\omega), \omega)$ is called a stopped process where $t \wedge \tau = \min\{t, \tau\}$.

Probabilistic Approach

Theorem (Optional Sampling Theorem)

If M is a martingale and τ is a stopping time, then the stopped process $M(t \wedge \tau)$ is a martingale.

Remark: This theorem also holds for sub-martingale and super-martingale.

Probabilistic Approach

Example

A person starts with \$100 and bet \$1 each time for a fair game (50% chance of win and 50% of loss). He will quit the game if either he loses all his money or his capital accumulates to \$500. What is his chance of walking away with \$500?

Probabilistic Approach

Definition

Let $0 \leq t \leq T$ and $x \geq 0$ be given. Assume $S(t) = x$. For $t \leq u \leq T$, let $\mathcal{F}_u^{(t)} = \sigma(X_s : t \leq s \leq u)$ and let $\mathcal{T}_{t,T}$ be the set of all stopping times for the filtration $\left\{\mathcal{F}_u^{(t)}\right\}_{t \leq u \leq T}$ taking values in $[t, T]$ or ∞. The price at time t of the American put with expiry T is defined to be

$$\nu(t, x) = \max_{\tau \in \mathcal{T}_{t,T}} \widetilde{\mathbb{E}}\left[e^{-r(\tau - t)}(K - S(\tau))\,|S(t) = x\right]$$

where $e^{-r(\tau-t)}(K - S(\tau))$ is interpreted as zero if $\tau = \infty$.

Numerical Pricing Methods

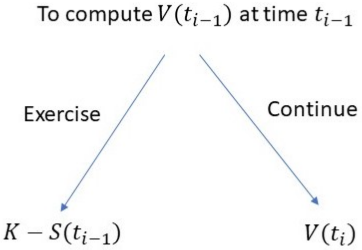
The difficulty with pricing American put option is that early exercise boundary $S^*(t)$ is not known.

At each time t, we need to decide whether to exercise or continue holding it.

$\Longrightarrow$ To compute the value of holding the option, we need to look forward.

Numerical Pricing Methods

The decision is based on the value of option at a later time.



Numerical Pricing Methods

A typical approach is to use dynamic programming (backward)

$$V(0) \leftarrow \cdots \; V(t_{i-1}) \longleftarrow V(t_i) \longleftarrow V(t_{i+1}) \leftarrow \cdots \; V(T)$$

Numerical Methods

- Binomial Tree
- Least Square Monte Carlo (LSM)

Binomial Tree Model

Suppose the stock price follows

$$\frac{dS(t)}{S(t)} = rdt + \sigma d\widetilde{W}(t)$$

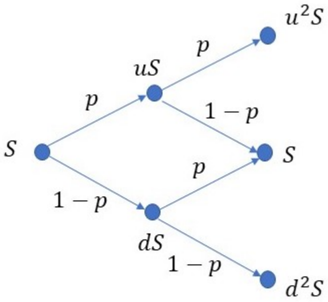
Binomial Tree Model

Fix $T > 0$ and N, let $\Delta t = \frac{T}{N}$. Denote $t_i = i\Delta t, i = 0, \ldots, N$.

Price moves

$$S(t_i) = \begin{cases} uS(t_{i-1}) & \text{probability } p \\ dS(t_{i-1}) & \text{probability } 1-p \end{cases}$$

Binomial Tree Model



$$u > 1, d < 1, ud = 1$$

Binomial Tree Model

We choose $ud = 1$ so that the binomial is recombining. To solve p and u, we match the first two moments which gives

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

and

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

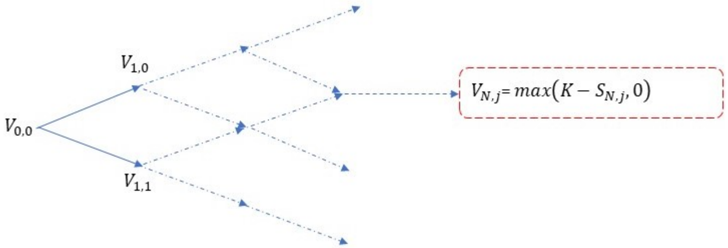
Binomial Tree Model

Denote the stock price for the j-th node at time t_i to be $S_{i,j}$.

To compute the price of an American put option, we start with the leaf nodes, i.e., $i = N$. These nodes correspond to $t_i = T$ and the value of option is

$$V_{N,j} = (K - S_{N,j})^+$$

Binomial Tree Model

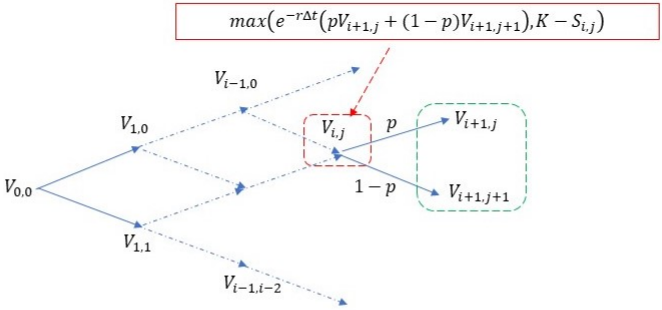


Binomial Tree Model

For $i < N$

$$\begin{aligned} V_{i,j} &= \max\left\{ e^{-r\Delta t}\widetilde{\mathbb{E}}\left[V(S(t_{i+1}) | S(t_i) = S_{i,j} \right], K - S_{i,j} \right\} \\ &= \max\left\{ e^{-r\Delta t}\left(pV_{i+1,j} + (1-p)V_{i+1,j+1} \right), K - S_{i,j} \right\} \end{aligned}$$

Binomial Tree Model



Binomial Tree Model

Improvements

- Binomial Black and Scholes (BBS) method
- BBS method with Richardson extrapolation (BBSR)

BBS Method

Note that at time t_{N-1} the continuation value is equivalent to the price of a European put option.

Hence we can replace it with BS formula for put option. That is

$$V_{N-1,j} = \max\{p(\Delta t, S_{N-1,j}), K - S_{N-1,j}\}$$

where $p(\Delta t, S_{N-1,j})$ is the price of a European put option with time to maturity Δt and spot $S_{N-1,j}$.

BBS Method

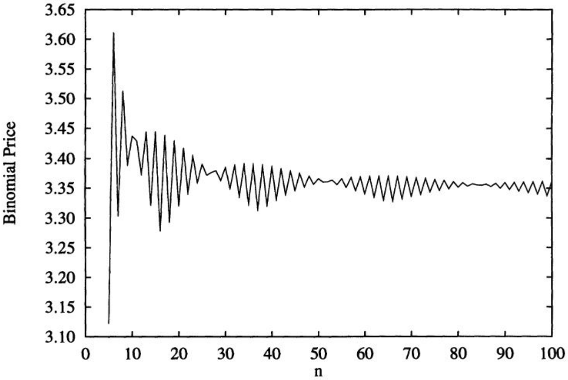


Figure 14
Binomial price versus number of time steps
The put option parameters are $S = 100$, $K = 90$, $r = 0.05$, $\delta = 0$, $\sigma = 0.30$, and $T = 0.5$. The true price is 3.345. The oscillatory convergence of the binomial is quite evident.

Source: Mark Broadie and Jerome Detemple (1996)

BBS Method

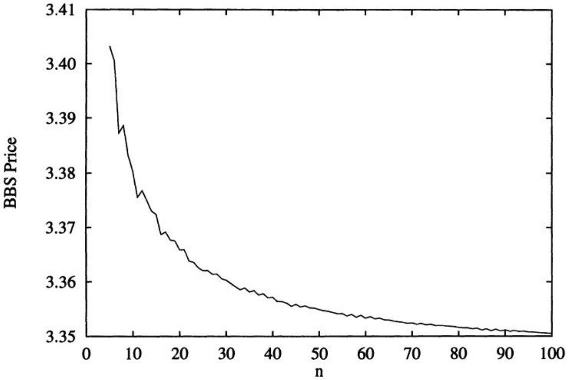


Figure 15
BBS price versus number of time steps
The put option parameters are $S = 100$, $K = 90$, $r = 0.05$, $\delta = 0$, $\sigma = 0.30$, and $T = 0.5$. The true price is 3.345. The convergence of the BBS method is considerably smoother compared to the binomial method.

Source: Mark Broadie and Jerome Detemple (1996)

BBSR Method

Richardson extrapolation method is a simple but powerful method to improve the accuracy of an otherwise mediocre numerical algorithm.

BBSR Method

Suppose we approximate $f(0)$ with $f(h)$ for small h. According to Tayor's expansion

$$f(h) = f(0) + f'(0)h + o(h)$$

The approximation error is of $O(h)$

$$f(h) \approx f(0) + O(h)$$

BBSR Method

Suppose we approximate $f(0)$ with $f(h)$ for small h. According to Tayor's expansion

$$f(h) = f(0) + f'(0)h + o(h)$$

The approximation error is of $O(h)$

$$f(h) \approx f(0) + O(h)$$

BBSR Method

Suppose we can't improve the approximation algorithm f itself. What can we do?

Note that

$$f(\frac{h}{2}) = f(0) + f'(0)\frac{h}{2} + o(h)$$

Combine it with

$$f(h) = f(0) + f'(0)h + o(h)$$

we have

$$2f(\frac{h}{2}) - f(h) = f(0) + o(h)$$

The approximation $2f(\frac{h}{2}) - f(h)$ has improved the accuracy by one order!

BBSR Method

The BBSR Method uses Richardson extrapolation to improve accuracy. Suppose V_1 is the BBS method for American put option when we choose time step to be Δt. And V_2 is the price when we half the time step (double the number of steps). As we discussed above,

$$V = 2V_2 - V_1$$

is a better approximation.

Least Square Monte Carlo

In Monte Carlo method, we usually simulate sample paths

$$S(t_0), \ldots, S(t_i), \ldots, S(t_N)$$

For each sample path we can compute the payoffs of a derivative. The price of this derivative is the (discounted) average of these payoffs.

Least Square Monte Carlo

As in the binomial tree model, to determine whether we want to early exercise or not we need to compute

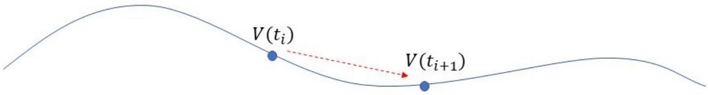
$$\widetilde{\mathbb{E}}\left[V(S(t_{i+1}))|S(t_i) = S_{i,j}\right]$$

It is extremely inaccurate if the conditional expectation is computed using a single sample path.

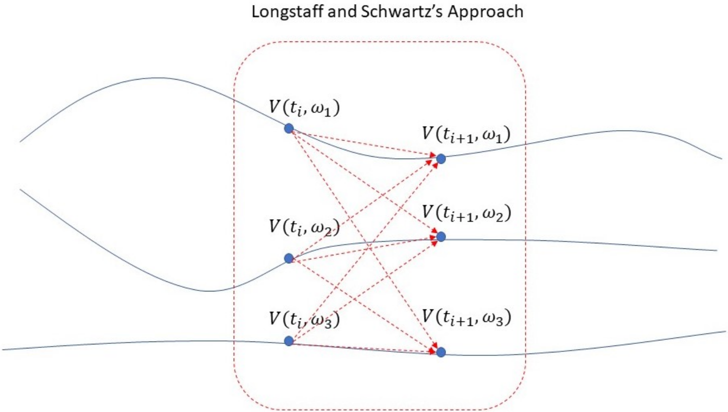
Longstaff and Schwartz (see Longstaff and Schwartz (2001)) proposed a regression based approach and pool the information across sample paths.

Least Square Monte Carlo

Single Sample Path Approach



Least Square Monte Carlo



Least Square Monte Carlo

We take the example in Longstaff and Schwartz (2001) to illustrate the idea of LSM.

Consider an American put option on a non-dividend paying stock. The strike of the put option is 1.10. The expiry is in 3 years and we take a very coarse discretization of time $t = 0, 1, 2, 3$. Assume the riskless rate is $r = 6\%$. For simplicity, only take eight sample paths for the price of the stock.

Least Square Monte Carlo

The eight sample paths are shown in the below table

Stock price paths

Path	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	.93	.97	.92
5	1.00	1.11	1.56	1.52
6	1.00	.76	.77	.90
7	1.00	.92	.84	1.01
8	1.00	.88	1.22	1.34

Least Square Monte Carlo

At time $t = 3$, provided that the option has not been exercised, the cash flow is

Cash-flow matrix at time 3

Path	$t = 1$	$t = 2$	$t = 3$
1	—	—	.00
2	—	—	.00
3	—	—	.07
4	—	—	.18
5	—	—	.00
6	—	—	.20
7	—	—	.09
8	—	—	.00

Least Square Monte Carlo

At time $t = 2$, if the option is in the money, the option buyer needs to decide whether to early exercise or continue to hold the option.

Let X be the stock price at $t = 2$ and Y be the discounted cash flow from $t = 3$ if the option is not exercised. In their original paper, the LSM method assumes the following relationship holds

$$\widetilde{\mathbb{E}}\left[Y|X\right] = \beta_0 + \beta_1 X + \beta_2 X^2$$

Least Square Monte Carlo

Regression at time 2

Path	Y	X
1	$.00 \times .94176$	1.08
2	—	—
3	$.07 \times .94176$	1.07
4	$.18 \times .94176$	.97
5	—	—
6	$.20 \times .94176$	.77
7	$.09 \times .94176$	.84
8	—	—

Least Square Monte Carlo

Given the data shown above, we can use the least square method to find β_0, β_1 and β_2. That gives a function for the expected cash flow of holding the American option to $t = 3$ in terms of stock prices at time $t = 2$:

$$\widetilde{\mathbb{E}}[Y|X] = -1.070 + 2.983X - 1.813X^2$$

Least Square Monte Carlo

Comparing the cash flow from exercise and continue holding, we can see that it's optimal to exercise for the sample paths $4, 6, 7$.

Optimal early exercise decision at time 2

Path	Exercise	Continuation
1	.02	.0369
2	—	—
3	.03	.0461
4	.13	.1176
5	—	—
6	.33	.1520
7	.26	.1565
8	—	—

Least Square Monte Carlo

The cash flow from the last two periods (provided that the option is not exercised in the first period) becomes

Cash-flow matrix at time 2

Path	$t = 1$	$t = 2$	$t = 3$
1	—	.00	.00
2	—	.00	.00
3	—	.00	.07
4	—	.13	.00
5	—	.00	.00
6	—	.33	.00
7	—	.26	.00
8	—	.00	.00

Note that we reset the cash flows to be zero for sample paths 4, 6 and 7.

Least Square Monte Carlo

Proceed to $t = 1$, the sample paths that are in the money are 1, 4, 6, 7, 8. The discounted continuation value and simulated stock prices for those sample paths are given below

Regression at time 1

Path	Y	X
1	$.00 \times .94176$	1.09
2	—	—
3	—	—
4	$.13 \times .94176$	.93
5	—	—
6	$.33 \times .94176$	.76
7	$.26 \times .94176$	.92
8	$.00 \times .94176$	.88

Least Square Monte Carlo

The regression line of continuation value becomes

$$\widetilde{\mathbb{E}}\left[Y|X\right] = 2.038 - 3.335X + 1.356X^2$$

Least Square Monte Carlo

This gives the following continuation value for in the money sample paths

Optimal early exercise decision at time 1

Path	Exercise	Continuation
1	.01	.0139
2	—	—
3	—	—
4	.17	.1092
5	—	—
6	.34	.2866
7	.18	.1175
8	.22	.1533

Hence it is optimal to exercise for the sample paths 4, 6, 7, 8.

Least Square Monte Carlo

Hence it is optimal to exercise for the sample paths 4, 6, 7, 8 at time $t = 1$. We can summarize the early exercise rules for three time periods in the following table

Stopping rule

Path	$t = 1$	$t = 2$	$t = 3$
1	0	0	0
2	0	0	0
3	0	0	1
4	1	0	0
5	0	0	0
6	1	0	0
7	1	0	0
8	1	0	0

Least Square Monte Carlo

The cash flow matrix for three time periods is

Option cash flow matrix

Path	$t = 1$	$t = 2$	$t = 3$
1	.00	.00	.00
2	.00	.00	.00
3	.00	.00	.07
4	.17	.00	.00
5	.00	.00	.00
6	.34	.00	.00
7	.18	.00	.00
8	.22	.00	.00

Least Square Monte Carlo

The price of the option is computed by averaging the discounted cash flows from each sample path as

$$
\begin{aligned}
& \tfrac{1}{8}\left(0.07 \times e^{-0.06\times 3} + (0.17 + 0.34 + 0.18 + 0.22) \times e^{-0.06}\right) \\
= \; & 0.1144
\end{aligned}
$$

Discussion

- Choice of continuation values
  Simulated values vs. regression values
- Choice of basis functions
  Other choices are also recommended in their original paper such as Laguerre polynomials.
- Bias in regression based approach
  1. Upward bias
     Using the same sample paths for determining exercise boundary and the computation of cash flow.
     $\Longrightarrow$ Use a new set of sample paths for computing cash flow and price.
  2. Downward bias
     Sub-optimal early exercise boundary

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Thank you!