FE5222 Homework 2: Due by Thursday, September 19

September 4, 2019

1. (10 Points) Prove Feynman-Kac Theorem II (discounted) in the lecture notes.

2. (25 Points) (**Chooser Option**) A chooser option gives the owner the right to choose at time $t_0 > 0$ to buy a call or a put option with strike K and expiry $T > t_0$. Hence at time t_0 the value of the chooser option is
$$\max\{C(t_0), P(t_0)\}$$
where $C(t_0)$ and $P(t_0)$ are the value of call and put option with strike K and expiry T respectively. Use risk neutral approach to price this option at time $t = 0$ under the Black-Scholes-Merton model. Hint: use option call-put parity.

3. (25 Points) Let $\alpha(t)$ and $\beta(t)$ be non-random time-dependent functions. $X(0) = 0$ and $X(t)$ satisfies the following SDE
$$dX(t) = \alpha(t)X(t)dt + \beta(t)X(t)dW(t)$$
   (a) (10 Points) Solve $X(t)$
   (b) (15 Points) Show that $X(t)$ is lognormal and derive the mean and variance of $\ln X(t)$.

4. (10 Points) Suppose there are two stocks $S_i(t), i = 1, 2$ in the model that are driven by a single Brownian motion $W(t)$
$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dW(t)$$
where μ_i, σ_i are constant. Let r be the riskless rate. Assume
$$\frac{\mu_1 - r}{\sigma_1} \neq \frac{\mu_2 - r}{\sigma_2}$$
Argue that we can find an arbitrage in this model (note that there is no risk neutral measure in this model).

5. (30 Points) Consider the multidimensional model with m stocks,
$$\frac{dS_i}{S_i} = \alpha_i(t)dt + \sum_{j=1}^{d} \sigma_{i,j}(t)dW_j(t)$$
for $i = 1, \ldots, m$. Assume the riskless rate is a constant r.

   Suppose there exists a solution $\Theta_j(t), j = 1\ldots, d$, for the market price of risk equations (see lecture notes) and let $\widetilde{\mathbb{P}}$ be the risk neutral measure. Then
$$d\widetilde{W}_j(t) = dW_j(t) + \Theta_j(t)dt$$
is a Brownian motion under $\widetilde{\mathbb{P}}$.

(a) (8 Points) Let

$$\sigma_i(t) = \sqrt{\sum_{j=1}^{d} \sigma_{i,j}^2(t)}$$

and

$$dB_i(t) = \frac{1}{\sigma_i(t)} \sum_{j=1}^{d} \sigma_{i,j}(t) dW_j(t)$$

Prove that $B_i(t), i = 1, \ldots, m$, is a Brownian motion under $\mathbb{P}$.

(b) (5 Points) Derive the instantaneous correlation of B_i and B_j for $i \neq j$.

(c) (8 Points) Define $\gamma_i(t) = \frac{1}{\sigma_i(t)} \sum_{j=1}^{d} \sigma_{i,j}(t)\Theta_j(t)$. Show that

$$\widetilde{B}_i(t) = B_i(t) + \int_0^t \gamma_i(u) du$$

is a Brownian motion under $\widetilde{\mathbb{P}}$.

(d) (4 Points) Show that

$$\frac{dS_i(t)}{S_i(t)} = r dt + \sigma_i(t) d\widetilde{B}_i(t)$$

(e) (5 Points) Derive the instantaneous correlation of $\widetilde{B}_i(t)$ and $\widetilde{B}_j(t)$ for $i \neq j$. Compare the result with part (b).