FE5222 Advanced Derivative Pricing

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Overview

Hedging P&L Analysis

We consider the P&L of a hedged European call option. Let

- $0 = t_0 < \ldots < t_n = T$ be equally spaced time intervals where T is the expiry of a derivative, $\delta t = t_i - t_{i-1}$
- S_i: stock price at time t_i
- $C_i = C(t_i, S_i)$: price of call option at time t_i when the stock price is S_i
- $\Delta_i = \Delta(t_i, S_i)$: hedge ratio for time t_i when stock price S_i. It is the number of shares of stock we short.

Note that Δ_i is an arbitrary hedging strategy. It does not have to be the BSM hedge ratio.

Hedging P&L Analysis

Suppose we start with a call option. At time t_0, we short Δ_0 shares of stocks and deposit the cash amount $\Delta_0 S_0$ into a bank account. Our portfolio at time t_0 consists of

1. A call option
2. $-\Delta_0$ shares of stock
3. $\Delta_0 S_0$ cash

The total value is C_0.

Hedging P&L Analysis

At time t_1, before we re-hedge our positions. The value of assets in our portfolio is

1. Call option: C_1
2. Stock: $-\Delta_0 S_1$
3. Cash: $\Delta_0 S_0 e^{r\delta t}$

The total value is $C_1 - \Delta_0 S_1 + \Delta_0 S_0 e^{r\delta t}$.

Hedging P&L Analysis

At time t_1, we short another $\Delta_1 - \Delta_0$ stocks to re-hedge. Our portfolio is

1. A call option
2. Stock: $-\Delta_1$
3. Cash: $\Delta_0 S_0 e^{r\delta t} + (\Delta_1 - \Delta_0) S_1$

The total value is $C_1 - \Delta_1 S_1 + \Delta_0 S_0 e^{r\delta t} + (\Delta_1 - \Delta_0) S_1$.

Hedging P&L Analysis

At time t_2, before we re-hedge, the value of our assets is

1. Call option: C_2
2. Stock: $-\Delta_1 S_2$
3. Cash: $\Delta_0 S_0 e^{2r\delta t} + (\Delta_1 - \Delta_0) S_1 e^{r\delta t}$

The total value is $C_2 - \Delta_1 S_2 + \Delta_0 S_0 e^{2r\delta t} + (\Delta_1 - \Delta_0) S_1 e^{r\delta t}$.

Hedging P&L Analysis

At time t_2, we short $\Delta_2 - \Delta_1$ to re-hedge, our portfolio is

1. A call option
2. Stock: $-\Delta_2$
3. Cash: $\Delta_0 S_0 e^{2r\delta t} + (\Delta_1 - \Delta_0) S_1 e^{r\delta t} + (\Delta_2 - \Delta_1) S_2$

The total value is

$$C_2 - \Delta_2 S_2 + \Delta_0 S_0 e^{2r\delta t} + (\Delta_1 - \Delta_0) S_1 e^{r\delta t} + (\Delta_2 - \Delta_1) S_2.$$

Hedging P&L Analysis

Continue this process, at time t_n, after re-hedge our portfolio is

1. A call option
2. Stock: $-\Delta_n$
3. Cash: $\Delta_0 S_0 e^{nr\delta t} + (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\delta t} + \ldots + (\Delta_n - \Delta_{n-1}) S_n$

The total value is

$$C_n - \Delta_n S_n + \Delta_0 S_0 e^{nr\delta t} + (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\delta t} + \ldots + (\Delta_n - \Delta_{n-1}) S_n$$

Hedging P&L Analysis

In the limit when $\delta t \to 0$, we can replace the summation with integral

$$C_T - \Delta_T S_T + \Delta_0 S_0 e^{rT} + \int_0^T e^{r(T-t)} S(t) \left[d\Delta(t) \right]_b$$

Note that for an integrand

$$\int_0^T \alpha(t, \omega) \left[d\Delta(t) \right]_b = \lim \sum_i \alpha(t_{i+1}, \omega)(\Delta(t_{i+1}) - \Delta(t_i))$$

is the backward Ito's Integral.

Hedging P&L Analysis

Denote

$$V = e^{-rT}\left(C_T - \Delta_T S_T + \Delta_0 S_0 e^{rT} + \int_0^T e^{r(T-t)} S(t)\left[d\Delta(t)\right]_b\right)$$

- V is the value of a hedged option at time 0
- In general V is random and path-dependent
- In BSM model, if we choose hedge ratio $\Delta(t)$ to be BSM delta, then V is deterministic and equal to the price of call option at time $t = 0$.

Hedging P&L Analysis

From

$$\begin{aligned} & d(e^{r(T-t)}S(t)\Delta(t)) \\ = \; & -re^{r(T-t)}S(t)\Delta(t)dt + e^{r(T-t)}\Delta(t)dS(t) \\ & +e^{r(T-t)}S(t)[d\Delta(t)]_b \end{aligned}$$

we can solve

$$\begin{aligned} \int_0^T e^{r(T-t)}S(t)[d\Delta(t)]_b &= S_T\Delta_T - e^{rT}S_0\Delta_0 \\ &- \int_0^T e^{r(T-t)}\Delta(t)\left(dS(t) - rS(t)dt\right) \end{aligned}$$

Hedging P&L Analysis

Substituting this into V we have

$$V = e^{-rT} C_T - \int_0^T e^{-rt} \Delta(t) \left(dS(t) - rS(t) dt \right)$$

Hedging P&L Analysis

If we work with BSM model

$$dS(t) - rS(t)dt = \sigma S(t)dW(t)$$

In this case

$$V = e^{-rT} C_T - \int_0^T e^{-rt} \Delta(t) \sigma S(t) dW(t)$$

Hedging P&L Analysis

- Note that
$$\mathbb{E}[V] = e^{-rt}\mathbb{E}[C_T]$$
which is exactly risk neutral pricing formula.
- If the stock price follows Geometric Brownian Motion with drift r, the expectation is irrelevant to hedging strategy $\Delta(t)$. Even if we don't hedge, the expected value is still the same.

Hedging with Realized Volatility

Assume

- The stock price follows

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma_r dW(t)$$

  in the real world measure.

- Market is pricing option with σ_i
- We predict the realized volatility σ_r will be greater than σ_i.

How do we make a profit from this misprice in the market?

$\Rightarrow$ We buy the option from market and delta hedge it according to BSM hedge ratio $\Delta(\sigma_r)$.

Hedging with Realized Volatility

Suppose our prediction is correct and the realized volatility turns out to be σ_r. We would expect to see P&L

$$V^r(0) - V^i(0)$$

- $V^r(0)$ is the price of option based on σ_r. That what the fair value we think should be.
- $V^i(0)$ is the price of option based on σ_i. That is what market is pricing.
- We use superscript i and r to denote the values based on realized volatility σ_r and implied volatility σ_i respectively.

Hedging with Realized Volatility

Is our expectation justified?

If it is true, how is this future known profit realized over time?

Hedging with Realized Volatility

Suppose at time t, our portfolio consists of

- Long an option whose market value is $V^i(t)$
- Short $\Delta^r(t)$ shares of stock
- Cash $\Delta^r(t)S - V^i(t)$

Hedging with Realized Volatility

P&L from t to $t + dt$ is

$$\begin{aligned} & dP\&L(t) \\ = \; & dV^i(t) - \Delta^r(t)dS + r\left(\Delta^r(t)S - V^i(t)\right)dt \end{aligned}$$

The total value of portfolio is $dP\&L(t)$ and its PV is $e^{-rt}dP\&L(t)$.

Hedging with Realized Volatility

After re-hedging at $t + dt$, our portfolio

- Long a option whose market value is $V^i(t + dt)$
- Short $\Delta^r(t + dt)$ shares of stock
- Cash $\Delta^r(t + dt)S - V^i(t + dt) + dP\&L(t)$

Hedging with Realized Volatility

If we exclude the cash amount $dP\&L(t)$ from our portfolio, the value of our portfolio will be zero at time $t + dt$.

For this reduced portfolio, P&L from $t + dt$ to $t + 2dt$ is

$$
\begin{aligned}
& dP\&L(t+dt) \\
= \; & dV^i(t+dt) - \Delta^r(t+dt)dS \\
& + r\left(\Delta^r(t+dt)S - V^i(t+dt)\right)dt
\end{aligned}
$$

Hedging with Realized Volatility

The total P&L from $t + dt$ to $t + 2dt$ is

$$dP\&L(t + dt) + dP\&L(t)(e^{rdt} - 1)$$

Hedging with Realized Volatility

The 2-period P&L from t to $t + 2dt$ is

$$dP\&L(t + dt) + dP\&L(t)e^{rdt}$$

Its PV is

$$e^{-r(t+dt)}dP\&L(t + dt) + e^{-rt}dP\&L(t)$$

Hedging with Realized Volatility

Hence the PV of P&L from 0 to T is

$$P\&L = \int_0^T e^{-rt} dP\&L(t)$$

where

$$dP\&L(t) = dV^i(t) - \Delta^r(t)dS + r\left(\Delta^r(t)S - V^i(t)\right)dt$$

Hedging with Realized Volatility

From the derivation of BSM equation, we know that

$$dV^r(t) - \Delta^r(t)dS = r(V^r(t) - \Delta^r(t)S)dt$$

Using this to simplify $dP\&L(t)$, we have

$$\begin{aligned} dP\&L(t) &= dV^i(t) - dV^r(t) - r(V^i(t) - V^r(t))dt \\ &= e^{rt}d\left[e^{-rt}\left(V^i(t) - V^r(t)\right)\right] \end{aligned}$$

Hedging with Realized Volatility

The P&L formula becomes

$$\begin{aligned} P\&L &= \int_0^T e^{-rt} dP\&L(t) \\ &= \int_0^T d\left[e^{-rt}\left(V^i(t) - V^r(t)\right)\right] \\ &= \left[V^i(t) - V^r(t)\right]_{t=0}^T \\ &= V^r(0) - V^i(0) \end{aligned}$$

since $V^r(T) = V^i(T)$.

This is what we expected to see.

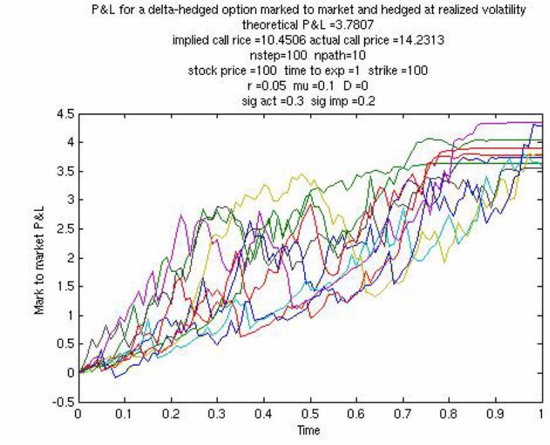
Hedging with Realized Volatility

If we know the (future) realized volatility and hedge continuously, the final P&L at the expiry is deterministic and equal to the difference between the value of option based on realized volatility and implied volatility.

Hedging with Realized Volatility

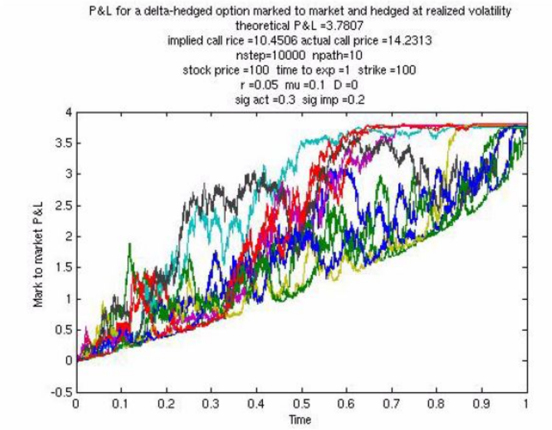
How does the P&L vary over the whole hedging process?

Hedging with Realized Volatility



Source: Derman (2008)

Hedging with Realized Volatility



Source: Derman (2008)

Hedging with Realized Volatility

It appears that

- P&L shows some degree of randomness.
- P&L converges eventually as time approaches to expiry
- For sufficiently large number of simulation steps, P&L converges to theoretical value $V^r(0) - V^i(0)$.
- P&L seems to have a time-dependent lower bound.

Hedging with Realized Volatility

To see that P&L is random, we notice that the P&L from time 0 to t is

$$\left(V^i(t) - V^r(t)\right) - \left(V^i(0) - V^r(0)\right)$$

The first term $V^i(t) - V^r(t)$ depends on the stock price $S(t)$ which is random.

Hedging with Realized Volatility

To better understand how P&L changes, let's have a closer look at

$$dP\&L(t) = dV^i - \Delta^r dS + r\left(\Delta^r S - V^i\right) dt$$

Hedging with Realized Volatility

Applying Ito's Lemma to $V^i(t)$, we have

$$dV^i(t) = \Theta^i dt + \Delta^i dS + \frac{1}{2}\Gamma^i S^2 \sigma_r^2 dt$$

where Θ^i, Δ^i and Γ^i are BSM theta, delta and gamma respectively based on implied volatility.

Note that the volatility in last term is σ_r since that is the real volatility for the return of stock price.

Hedging with Realized Volatility

The P&L from t to $t + dt$ is

$$
\begin{aligned}
dP\&L(t) &= dV^i - \Delta^r dS + r\left(\Delta^r S - V^i\right) dt \\
&= \left[\Theta^i dt + \Delta^i dS + \tfrac{1}{2}\Gamma^i S^2 \sigma_r^2 dt\right] \\
&\quad -\Delta^r dS + r\left(\Delta^r S - V^i\right) dt \\
&= \left[\Theta^i + \tfrac{1}{2}\Gamma^i S^2 \sigma_r^2\right] dt + \left(\Delta^i - \Delta^r\right) dS \\
&\quad +r\left(\Delta^r S - V^i\right) dt
\end{aligned}
$$

Hedging with Realized Volatility

From BSM equation, we have

$$\Theta^i + \Delta^i rS + \frac{1}{2}\Gamma^i S^2 \sigma_i^2 = rV^i$$

Hence

$$\begin{aligned} dP\&L(t) &= \tfrac{1}{2}\Gamma^i S^2 \left(\sigma_r^2 - \sigma_i^2\right) dt \\ &+ \left(\Delta^i - \Delta^r\right) \left[(\mu - r)Sdt + \sigma_r SdW\right] \end{aligned}$$

Note that

- $dP\&L(t)$ is random even in the infinitesimal time interval $(t, t+dt)$ unless $\Delta^i = \Delta^r$.
- When $\Delta^i = \Delta^r$, we get the P&L equation we derived in last class.

Hedging with Implied Volatility

Realized volatility is not known, in practice, traders usually hedge with implied volatility. How does P&L look like in this case?

Hedging with Implied Volatility

If we hedge with implied volatility, i.e., use hedge ratio Δ_i, P&L from t to $t + dt$ is

$$dP\&L(t) = \frac{1}{2}\Gamma^i S^2 \left(\sigma_r^2 - \sigma_i^2\right) dt$$

$\Rightarrow$ Non-random in the infinitesimal interval.

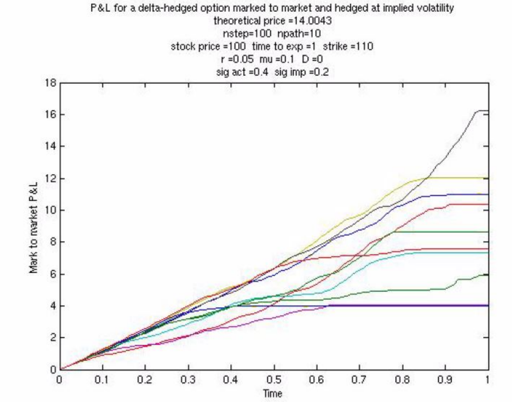
Hedging with Implied Volatility

PV of P&L from time 0 to T is

$$P\&L = \frac{1}{2}\int_0^T e^{-rt}\Gamma^i S^2\left(\sigma_r^2 - \sigma_i^2\right) dt$$

- Γ^i is random as it depends on stock price.
- In contrast to delta hedging with realized volatility, P&L is not deterministic and highly path-dependent!
- When option is deep ITM or OTM, Γ^i is small, in this case P&L is insensitive to volatility.

Hedging with Implied Volatility



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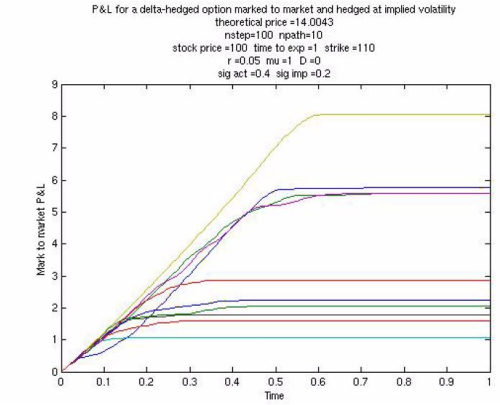
Hedging with Implied Volatility



Source: Derman (2008)

Effect of Discrete Hedging

Our previous analysis is based on continuous hedging. In reality traders hedge

- at equally spaced time intervals
- when change in risks exceeds a threshold

How does hedging frequency impact P&L?

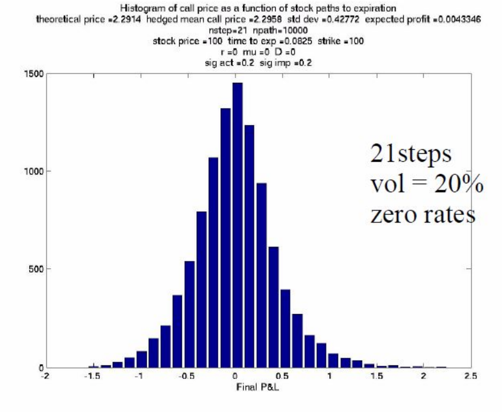
Monte Carlo Simulations

We assume

- BSM model
- Hedge at regular time intervals.
- Hedge with implied volatility σ_i

Monte Carlo Simulations

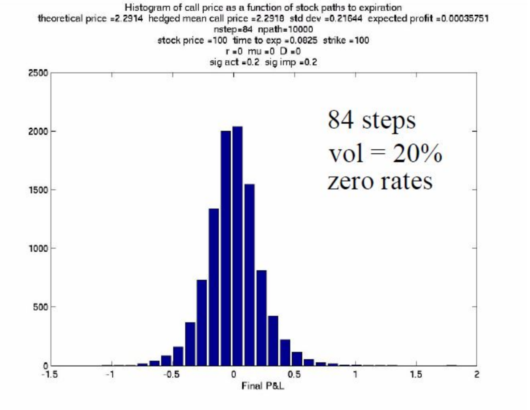
Case 1: $\sigma_i = \sigma_r$, # of steps $= 21$



Source: Derman (2008)

Monte Carlo Simulations

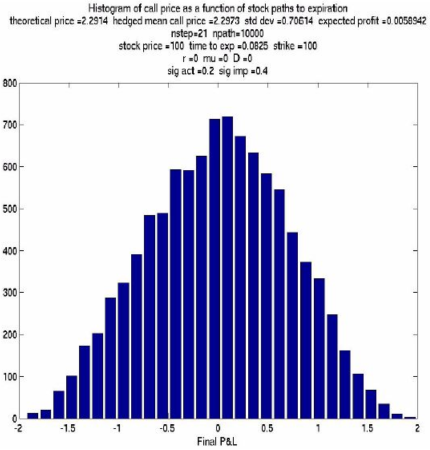
Case 2 : $\sigma_i = \sigma_r$, # of steps = 84



Source: Derman (2008)

Monte Carlo Simulations

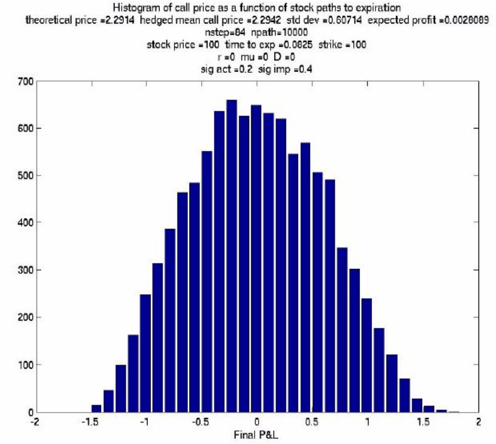
Case 3: $\sigma_i = 40\%, \sigma_r = 20\%$, $\#$ of steps $= 21$



Source: Derman (2008)

Monte Carlo Simulations

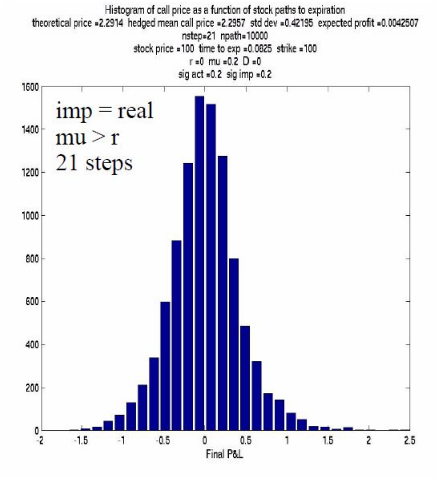
Case 4 : $\sigma_i = 40\%, \sigma_r = 20\%$, # of steps = 84



Source: Derman (2008)

Monte Carlo Simulations

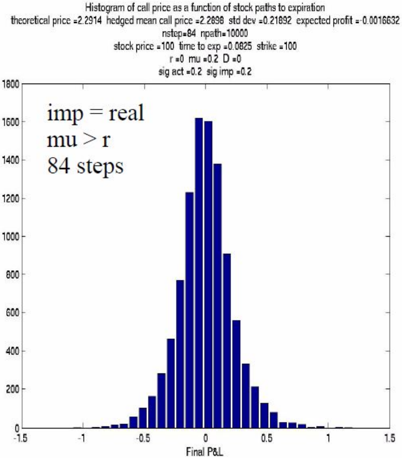
Case 5: $\mu = 20\%, r = 0\%, \sigma_i = \sigma_r = 20\%$, $\#$ of steps $= 21$



Source: Derman (2008)

Monte Carlo Simulations

Case 6 : $\mu = 20\%, r = 0\%, \sigma_i = \sigma_r = 20\%$, $\#$ of steps $= 84$



Source: Derman (2008)

Monte Carlo Simulations

Observations

- When $\sigma_i = \sigma_r$, quadruple the number of hedging $\Rightarrow$ halve the standard deviation.
- When $\sigma_i \neq \sigma_r$, increasing the number of hedging does not seem to reduce the standard deviation much.

Discrete Hedging Error

Hedging portfolio

- Long one call option
- Short $\frac{\partial C}{\partial S}$ shares of stock

Portfolio value

$$\pi = C - \frac{\partial C}{\partial S} S$$

Discrete Hedging Error

Between t to $t + \Delta t$, if fully hedged, P&L will be

$$r\pi\Delta t$$

Discrete hedged P&L using realized volatility is

$$C(t + \Delta t, S + \Delta S) - \frac{\partial C}{\partial S}\Delta S - C$$

Hedging error is

$$\begin{aligned} HE &= C(t + \Delta t, S + \Delta S) - \tfrac{\partial C}{\partial S}\Delta S - C - r\pi\Delta t \\ &= C(t + \Delta t, S + \Delta S) - \tfrac{\partial C}{\partial S}\Delta S - C - r\left(C - \tfrac{\partial C}{\partial S}S\right)\Delta t \end{aligned}$$

Discrete Hedging Error

Using Tayor's expansion, we have

$$\begin{aligned}
HE &= C(t+\Delta t, S+\Delta S) - \tfrac{\partial C}{\partial S}\Delta S - C - r\pi\Delta t \\
&\approx \left[C + \tfrac{\partial C}{\partial t}\Delta t + \tfrac{\partial C}{\partial S}\Delta S + \tfrac{1}{2}\Gamma\left(\Delta S\right)^2\right] \\
&\quad -\tfrac{\partial C}{\partial S}\Delta S - C - r\left(C - \tfrac{\partial C}{\partial S}S\right)\Delta t \\
&= \left[\tfrac{\partial C}{\partial t} - r\left(C - \tfrac{\partial C}{\partial S}S\right)\right]\Delta t + \tfrac{1}{2}\Gamma\left(\Delta S\right)^2 \\
&\approx \left[\tfrac{\partial C}{\partial t} - r\left(C - \tfrac{\partial C}{\partial S}S\right)\right]\Delta t + \tfrac{1}{2}\Gamma\sigma^2 S^2\left(\Delta W(t)\right)^2
\end{aligned}$$

Discrete Hedging Error

From BSM equation,

$$\frac{\partial C}{\partial t} - r\left(C - \frac{\partial C}{\partial S}S\right) = -\frac{1}{2}\Gamma\sigma^2 S^2$$

hence

$$\begin{aligned} HE &= \tfrac{1}{2}\Gamma\sigma^2 S^2((\Delta W(t))^2 - \Delta t) \\ &= \tfrac{1}{2}\Gamma\sigma^2 S^2(Z^2 - 1)\Delta t \end{aligned}$$

where

$$Z = \frac{\Delta W(t)}{\Delta t}$$

is a standard normal distribution and independent of Γ and S.

Discrete Hedging Error

Let $t_0 < t_1 < \ldots < t_n$ be equally spaced time intervals for hedging, total hedging error is

$$HE = \frac{1}{2}\frac{\sigma^2 T}{n}\sum_i \Gamma_i S_i^2 (Z_i^2 - 1)$$

where Z_i are i.i.d. standard normal and independent of $\Gamma_i S_i^2$.

Discrete Hedging Error

Hedge P&L Analysis

Wu Lei

P&L of Hedged Options

Effect of Different Hedging Strategies

Effect of Discrete Hedging

Effect of Transaction Cost

Note that

$$\begin{aligned}\mathbb{E}[HE] &= \frac{1}{2}\frac{\sigma^2 T}{n}\sum_i \mathbb{E}\left[\Gamma_i S_i^2 (Z_i^2 - 1)\right] \\ &= \frac{1}{2}\frac{\sigma^2 T}{n}\sum_i \mathbb{E}\left[\Gamma_i S_i^2\right]\mathbb{E}\left[(Z_i^2 - 1)\right] \\ &= 0\end{aligned}$$

The variance of hedging error is

$$\begin{aligned}\sigma_{HE}^2 &= \mathbb{E}[HE^2] \\ &= \frac{1}{4}\frac{\sigma^4 T^2}{n^2}\sum_i \mathbb{E}\left(\Gamma_i S_i^2\right)^2 \mathbb{E}\left(Z_i^2 - 1\right)^2 \\ &= \frac{1}{2}\frac{\sigma^4 T^2}{n^2}\sum_i \mathbb{E}\left(\Gamma_i S_i^2\right)^2\end{aligned}$$

Discrete Hedging Error

We can prove that (HW)

$$\mathbb{E}\left(\Gamma_i S_i^2\right)^2 \approx \Gamma_0^2 S_0^4 \sqrt{\frac{T^2}{T^2 - t_i^2}}$$

Hence, the variance of hedging error is

$$\sigma_{HE}^2 \approx \frac{1}{2}\frac{\sigma^4 T^2}{n^2}\Gamma_0^2 S_0^4 \sum_i \sqrt{\frac{T^2}{T^2 - t_i^2}}$$

Discrete Hedging Error

The variance of hedging error is

$$
\begin{aligned}
\sigma_{HE}^2 &\approx \frac{1}{2}\frac{\sigma^4 T^2}{n^2}\Gamma_0^2 S_0^4 \sum_i \sqrt{\frac{T^2}{T^2-t_i^2}} \\
&= \frac{1}{2}\frac{\sigma^4 T}{n}\Gamma_0^2 S_0^4 \sum_i \sqrt{\frac{T^2}{T^2-t_i^2}}\Delta t_i \\
&\approx \frac{1}{2}\frac{\sigma^4 T}{n}\Gamma_0^2 S_0^4 \int_0^T \sqrt{\frac{T^2}{T^2-t^2}}dt \\
&= \frac{\pi}{4}\frac{\sigma^4 T^2}{n}\Gamma_0^2 S_0^4
\end{aligned}
$$

Discrete Hedging Error

The standard deviation of hedging error is

$$\sigma_{HE} \approx \sqrt{\frac{\pi}{4}} \frac{\sigma^2 T}{\sqrt{n}} \Gamma_0 S_0^2$$

Using the fact that

$$\Gamma_0 S_0^2 = \frac{1}{\sigma T} \frac{\partial C}{\partial \sigma}$$

we have

$$\sigma_{HE} \approx \sqrt{\frac{\pi}{4}} \frac{\sigma}{\sqrt{n}} \frac{\partial C}{\partial \sigma}$$

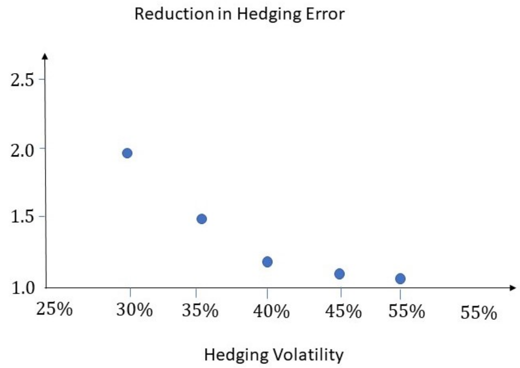
Quadruple the number of steps $\Rightarrow$ halve the standard deviation

Discrete Hedging Error

In the analysis we assume hedging with realized volatility.
What if hedging volatility is not equal to realized volatility?

Discrete Hedging Error

Assume realized volatility is 30%, the following graph shows the reduction of hedging error when we increase the number of hedging from 100 to 400 using MC.



Discrete Hedging Error

Assuming BSM model, we have

- If we estimate future realized volatility correctly and hedge continuously, the P&L will be the value of option.
- If we hedge discretely at realized volatility, P&L is random (but centered around BSM value). The standard deviation is proportional to $\frac{1}{\sqrt{n}}$.
- If the implied volatility is not equal to realized volatility and we hedge continuously at implied volatility, P&L is path-dependent and un predictable.
- If we hedge discretely at implied volatility, P&L is path-dependent and unpredictable. There is also a random component due to it.

Modeling Transaction Cost

Assume the transaction cost is a fraction k of share prices.

If we long (or short) N shares of stock, the transaction cost will be $kS|N|$.

Modeling Transaction Cost

Suppose we hedge with BSM hedge ratio $\Delta(t, S(t))$. From t to $t + dt$, the additional shares of stock we need to buy (or sell) is

$$\Delta(t + dt, S(t + dt)) - \Delta(t, S(t))$$

Modeling Transaction Cost

$$
\begin{aligned}
& \Delta(t+dt, S(t+dt)) - \Delta(t, S(t)) \\
= \quad & \frac{\partial \Delta}{\partial S} dS + O(dt) \\
= \quad & \frac{\partial \Delta}{\partial S} \sigma S dW(t) + O(dt) \\
= \quad & \frac{\partial \Delta}{\partial S} \sigma S \sqrt{dt} Z + O(dt) \\
\approx \quad & \frac{\partial^2 V}{\partial S^2} \sigma S \sqrt{dt} Z
\end{aligned}
$$

where

$$
Z = \frac{dW(t)}{\sqrt{dt}}
$$

is a standard normal distribution.

Modeling Transaction Cost

The transaction cost from t to $t + dt$ with delta hedging is

$$k\sigma S^2 \sqrt{dt} \left| \frac{\partial^2 V}{\partial S^2} \right| |Z|$$

The (conditional) expectation of transaction cost is

$$k\sigma S^2 \sqrt{dt} \left| \frac{\partial^2 V}{\partial S^2} \right| \mathbb{E}\left[|Z| \mid \mathcal{F}_t \right] = k\sigma S^2 \sqrt{dt} \left| \frac{\partial^2 V}{\partial S^2} \right| \sqrt{\frac{2}{\pi}}$$

Modeling Transaction Cost

The transaction cost at an infinitesimal time interval is $O(\sqrt{dt})$.

If we choose $dt = \frac{T}{n}$ where n is the number of intervals, the total transaction cost is

$$n \times O(\sqrt{dt}) = O(\sqrt{n}) \to \infty \qquad \text{as } n \to \infty$$

$\Rightarrow$ Continuous hedging implies infinite transaction cost.

Pricing with Transaction Cost

Let V be the price of a derivative, consider delta hedged portfolio π

- long V
- short Δ shares of stocks

Pricing with Transaction Cost

The change of value of the portfolio between t to $t + dt$ including transaction cost is

$$
\begin{aligned}
d\pi &= dV - \Delta dS - \text{transaction cost} \\
&= dV - \Delta dS - k\sigma S^2\sqrt{dt}\left|\frac{\partial^2 V}{\partial S^2}\right| |Z| \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt - k\sigma S^2\sqrt{dt}\left|\frac{\partial^2 V}{\partial S^2}\right| |Z|
\end{aligned}
$$

This is not a perfect hedge as the last term is not deterministic!

Pricing with Transaction Cost

The (conditional) expectation of return $\mathbb{E}\left[d\pi|\mathcal{F}_t\right]$ is

$$\left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt-k\sigma S^2\sqrt{dt}\left|\frac{\partial^2 V}{\partial S^2}\right|\sqrt{\frac{2}{\pi}}$$

Pricing with Transaction Cost

We assume to earn riskless rate of return r

$$\mathbb{E}\left[d\pi|\mathcal{F}_t\right] = r\left(V - \Delta S\right)dt$$

$\Longrightarrow$

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt - k\sigma S^2\sqrt{dt}\left|\frac{\partial^2 V}{\partial S^2}\right|\sqrt{\frac{2}{\pi}} = r\left(V - \Delta S\right)dt$$

Pricing with Transaction Cost

Canceling dt we can derive a pricing PDE similar to BSM equation

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - k\sigma S^2 \left|\frac{\partial^2 V}{\partial S^2}\right| \sqrt{\frac{2}{\pi dt}} = rV$$

Pricing with Transaction Cost

For call/put option, gamma is positive, we have

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} + \frac{1}{2}\hat{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

where

$$\hat{\sigma}^2 = \sigma^2 - 2k\sigma\sqrt{\frac{2}{\pi dt}}$$

Pricing with Transaction Cost

- For a long position in call/put option, the effect of including transaction cost is equivalent to reducing volatility. This makes option cheaper.
- dt is the hedging interval.
- Increasing hedging frequency $\Leftrightarrow$ higher transaction cost $\Leftrightarrow$ lower $\hat{\sigma}$.

Pricing with Transaction Cost

For a short position in call/put option,

$$d\pi = -dV + \Delta dS - \text{transaction cost}$$

We can show

$$\hat{\sigma}^2 = \sigma^2 + 2k\sigma\sqrt{\frac{2}{\pi dt}}$$

$\Rightarrow$ This increases the price of call/put option we short

$\Rightarrow$ Reduce the value of our portfolio.

References

Enamuel Derman (2008)
Dynamic Replication: Realities and Myths of Option Pricing, *Lecture 2 (Spring 2008)*

Thank you!