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Homework V. $=\tilde{E}_2[S_3](HT)P(A_{HT}) = 36q_u^2q_d + 12q_uq_d^2$

4.5.

according to the definition.

$\tilde{E}_2[S_3](HT) = \sum q_u^{H\cdots} q_d^{T\cdots} S_3(\omega_{n+1} \cdots \omega_N) = \sum S_3(\omega) \cdot P(\omega_1 \cdots \omega_n \omega_{n+1} \cdots N)$

$\Rightarrow \tilde{E}_2[S_3](HT)P(A_{HT}) = \sum_{\omega \in A_{HT}} S_3(\omega)P(\omega)$.

where $A_{HT} = \{HTH, HTT\}$. Since $\tilde{E}_2[S_3](\omega)$ does not change value on A_{HT}. we can write.

$$\int_{A_{HT}} \tilde{E}_2[S_3](\omega)\,dP(\omega) = \int_{A_{HT}} S_3(\omega)\,dP(\omega).$$

4.17. we want to prove $X_t = W_t^3 - 3tW_t$ is a martingale.

So we should prove

$$E[W_t^3 - 3tW_t | \mathcal{F}_s] = W_s^3 - 3sW_s.$$

$\Leftrightarrow E[W_t^3 - W_s^3 - 3tW_t + 3sW_s | \mathcal{F}_s] = 0$

$\Leftrightarrow E[(W_t - W_s)^3 + 3(W_t - W_s)^2 W_s - 3(W_t - W_s)W_s^2 + 6(W_t - W_s)W_s^2 -$

$3t(W_t - W_s) + (3s - 3t)W_s + 6W_s^3 | \mathcal{F}_s] = 0.$

because $W_t - W_s$, W_s is normal distribution.

$\therefore E[X^3] = 0$. and W_s is $\mathcal{F}_s$ measurable.

$\Leftrightarrow E[3(t-s)W_s - 0 + 0 + 0 + 3(s-t)W_s] = 0.$

thus we prove $E[W_t^3 - 3tW_t | \mathcal{F}_s] = W_s^3 - 3sW_s$.

5.5

$dX_t = -\alpha X_t dt + \sigma dW_t$.

$d(e^{\alpha t}X_t) = d(g(t, X(t))) = g_t dt + g_x dX_t + \frac{1}{2} g_{xx} dX_t dX_t$

$= \alpha \cdot e^{\alpha t} X_t dt + e^{\alpha t} dX_t + 0$

$= \alpha \cdot e^{\alpha t} X_t dt + e^{\alpha t}(-\alpha X_t dt + \sigma dW_t).$

$= e^{\alpha t} \cdot \sigma \cdot dW_t$.

$\therefore e^{\alpha t} X_t = X_0 + \sigma \int_0^t e^{\alpha s} dW_s$.

$\Rightarrow X_t = e^{-\alpha t} \cdot X_0 + e^{-\alpha t} \cdot \sigma \int_0^t e^{\alpha s} dW_s$.

5.6.

$Y(t) = (X_1^2(t) + X_2^2(t))$.

$\therefore d(Y_t) = d(g(t, X_1(t), X_2(t)))$

$= g_t dt + g_{x_1} dx_1 + g_{x_2} dx_2 + \frac{1}{2} g_{x_1x_1} dx_1 dx_1 + \frac{1}{2} g_{x_2x_2} dx_2 dx_2 + g_{x_1x_2} dx_1 x_2$

$= 0 + 2x_1 dx_1 + 2x_2 dx_2 + 2dx_1 dx_1 + 2dx_2 dx_2 + 0$

because $dX_1(t) = -\alpha X_1(t) dt + \sigma dW_1(t)$

we know $dX_1(t) \cdot dX_1(t) = \sigma^2 dt$.

because $dW_t dt = 0$, $dt\,dt = 0$, $dW_i dW_j = \begin{cases} dt & i = j \\ 0 & i \neq j \end{cases}$.

thus

$d(Y_t) = 2X_1(-\alpha X_1 dt + \sigma dW_1(t)) + 2X_2(-\alpha X_2 dt + \sigma dW_2(t)) + 2\sigma^2 dt$

$= (2\sigma^2 - 2\alpha X_1^2 - 2\alpha X_2^2)dt + 2\sigma X_1 dW_1(t) + 2\sigma X_2 dW_2(t)$.

$= (2\sigma^2 - 2\alpha Y_t)dt + 2\sigma\sqrt{Y_t}\, d\tilde{W}_t$

according to $d\tilde{W}_t = \frac{X_1(t)}{\sqrt{Y_t}} dW_1(t) + \frac{X_2(t)}{\sqrt{Y_t}} dW_2(t)$!

we already have

$d Y_t = (\beta - 2\alpha Y_t)dt + 2\sigma\sqrt{Y_t}\, d\tilde{W}_t$

thus $\beta = 2\sigma^2$.

5.8.

~~$dt, g(t$~~ $d(g(t, X(t))) = d(W_t^3)$

$= 0 + 3W_t^2 dW_t + \frac{1}{2} \cdot 6W_t dW_t dW_t$

$\Rightarrow d(W_t^3) = 0 + 3W_t^2 dW_t + 3W_t dt.$

$\Rightarrow W_t^3 = W_0 + 3\int_0^t W_s^2 dW_s + 3\int_0^t W_s ds$

$\because W_0 = 0$

$\Rightarrow \int_0^t W_s^2 dW_s = \frac{1}{3} W_t^3 - \int_0^t W_s ds$.

Homework VI.

6. $dX_t = b(t, X_t)dt + \sigma(t, W_t)dW_t$.

$\Rightarrow$ take $d(g(t, X(t))) = g_t dt + g_x dx + \frac{1}{2} g_{xx} dx dx$

$= [b(t, X_t)\partial_x g(t, x) + \partial_t g(t, x) + \frac{1}{2}\sigma^2(t, x)\partial^2 g(t, x)]dt + \sigma \partial_x g(t,x) dW_t$

according to the SDE

$\Rightarrow dg(t, x) = [r g(t, x)]dt + \sigma(t, X_t)dW_t$.

$\therefore d(e^{-rt} g(t, X(t))) = -r \cdot e^{-rt} g(t, x)dt + e^{-rt} d(g(t, x))$.

$= -r \cdot e^{-rt} g(t, x)dt + e^{-rt}[r \cdot g(t, x)dt + \sigma(t, x)dW_t]$

$= \sigma(t, X_t)dW_t$.

thus $e^{-rt} g(t, x)$ is a Ito ~~calculus~~ Integral.

so it is a martingale.

thus ~~for~~ we have for $t \leq T$

$E[e^{-rT} g(T, X(T)) | \mathcal{F}_t] \overset{\text{defintion } g(T, X(T)) = f(X(T))}{=\!=} E[e^{-rT} f(X(T)) | \mathcal{F}_t]$

$E[e^{-rT} g(T, X(T)) | \mathcal{F}_t] \overset{\text{martingale}}{=\!=} e^{-rt} g(t, X(t))$.

$= e^{-rt} g(t, x)$.

thus. $e^{-rt} g(t, x) = E^{t,x}[e^{-rT} f(X(T))]$.

because e^{-rt} is $\mathcal{F}_t$ measurable.

$\Rightarrow g(t, x) = E^{t,x}[e^{-r(T-t)} f(X(T))]$.

thus we prove the equation

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8.

a). $dY_t = (r - \beta Y_t)dt + \sigma\sqrt{Y_t}\,dW_t$. $Y_0 = a$.

$\therefore g(t,x) = d(e^{\beta t}Y_t)$

$= \beta \cdot e^{\beta t}Y_t\,dt + e^{\beta t}dY_t + 0$

$\Rightarrow d(e^{\beta t}Y_t) = r \cdot e^{\beta t}dt + e^{\beta t}\sigma\sqrt{Y_t}\,dW_t$.

$\Rightarrow e^{\beta t}Y_t = Y(0) + r\int_0^t e^{\beta s}ds + \sigma\int_0^t e^{\beta s}\sqrt{Y_s}\,dW_s$ -- ①

So integral form is

$Y_t = e^{-\beta t}Y(0) + e^{-\beta t} \cdot r\int_0^t e^{\beta s}ds + e^{-\beta t}\sigma\int_0^t e^{\beta s}\sqrt{Y_s}\,dW_s$.

b). $\mu(t) = E[Y_t]$

we take ①.

the expecation of Ito calculus is 0.

$e^{\beta t}E[Y_t] = e^{\beta t}\mu(t) = E[Y_0] + \frac{r}{\beta}(e^{\beta t} - 1) + 0$.

$\Rightarrow e^{\beta t}\mu(t) = a + \frac{r}{\beta}(e^{\beta t} - 1)$. ... ②

now we see the $\frac{d\mu(t)}{dt} = r - \beta\mu(t)$

we take $dg(t,x) = d(e^{\beta t}\mu(t))$.

$= \beta \cdot e^{\beta t}\mu(t)dt + e^{\beta t}d\mu(t)$

$= r e^{\beta t}dt$.

thus $d(e^{\beta t}\mu(t)) = r \cdot e^{\beta t}dt$.

$e^{\beta t}\mu(t) = r\int_0^t e^{\beta s}ds + \mu(0)$

$= a + \frac{r}{\beta}(e^{\beta t} - 1)$ -- ③

we find ② and ③ are the same.

thus. we can prove the $\mu(t)$ satisfies the differential equation.

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c). take $dg(t,x) = d(Y_t^2)$.

$= g_t dt + 2Y_t dY_t + 2 \cdot \frac{1}{2} dY_t dY_t$.

$= 2Y_t[(r - \beta Y_t)dt + \sigma\sqrt{Y_t}\,dW_t] + (\sigma\sqrt{Y_t})^2 dt$.

$= Y_t(2r + \sigma^2 - 2\beta Y_t)dt + 2\sigma(\sqrt{Y_t})^3 dW_t$.

thus we prove the equation.

d). take $dg(t,x) = d(e^{2\beta t}Y_t^2)$.

$= 2\beta e^{2\beta t}Y_t^2 dt + e^{2\beta t}dY_t^2$

$\Rightarrow d(e^{2\beta t}Y_t^2) = e^{2\beta t}Y_t(2r + \sigma^2)dt + e^{2\beta t}2\sigma(\sqrt{Y_t})^3 dW_t$.

$\Rightarrow e^{2\beta t}Y_t^2 = Y^2(0) + (2r + \sigma^2)\int_0^t e^{2\beta s} \cdot Y_s\,ds + 2\sigma\int_0^t e^{2\beta s}(\sqrt{Y_s})^3 dW_s$.

$\therefore \nu(t) = E[Y_t^2]$

$\Rightarrow e^{2\beta t}\nu(t) = a^2 + (2r + \sigma^2)\int_0^t E(e^{2\beta s} \cdot Y_s)ds + 0$

$= a^2 + (2r + \sigma^2)\int_0^t e^{2\beta s} \cdot E[Y_s]ds$

$= a^2 + (2r + \sigma^2)\int_0^t e^{2\beta s} \cdot \mu(s)\,ds$.

we diffenetial the equal

$\Rightarrow 2\beta \cdot e^{2\beta t}\nu(t)dt + e^{2\beta t}d\nu(t) = (2r + \sigma^2)e^{2\beta t}\mu(t)dt$.

$\Rightarrow \frac{d\nu(t)}{dt} = (2r + \sigma^2)\mu(t) - 2\beta\nu(t)$.

with $\nu(0) = a^2$.

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e). from (b) $\Rightarrow e^{\beta t}\mu(t) = a + \frac{r}{\beta}(e^{\beta t} - 1)$

from d). $\Rightarrow e^{2\beta t}\nu(t) = a^2 + (2r + \sigma^2)\int_0^t e^{2\beta s} \cdot \mu(s)\,ds$.

$= a^2 + (2r + \sigma^2)\int_0^t e^{\beta s}(a + \frac{r}{\beta}(e^{\beta s} - 1))ds$

$= a^2 + \frac{2r + \sigma^2}{\beta}(a - \frac{r}{\beta})(e^{\beta t} - 1) + \frac{2r + \sigma^2}{2\beta} \cdot \frac{r}{\beta}(e^{2\beta t} - 1)$

thus. $Var[Y_t] = E[Y_t^2] - (E[Y_t])^2$

$= \nu(t) - \mu^2(t)$

$= e^{-2\beta t} \cdot [\, a^2 + \frac{2r + \sigma^2}{\beta}(a - \frac{r}{\beta})(e^{\beta t} - 1) + \frac{2r + \sigma^2}{2\beta} \cdot \frac{r}{\beta}(e^{2\beta t} - 1)$

$- (a + \frac{r}{\beta}(e^{\beta t} - 1))^2\,]$.

when $t \to \infty$. we only consider the $e^{\boxed{2\beta t}}$ in this $Var[Y_t]$, or $\geq (2\beta t)$. because $e^{\beta t} \cdot e^{-2\beta t} = e^{-\beta t} \to 0$ when $t \to \infty$.

$\therefore \lim_{t \to \infty} Var[Y_t] = \lim_{t \to \infty}\left(\frac{2r^2 + \sigma^2 r}{2\beta^2}e^{2\beta t} - \frac{r^2}{\beta^2}e^{2\beta t} + (\cdots)e^{\beta t} + \cdots\right) \cdot e^{-2\beta t}$.

$= \frac{\sigma^2 r}{2\beta^2}$.

thus the constant C is $\frac{\sigma^2 r}{2\beta^2}$.

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Homework VII.

2. $f(t,x) = x - Ke^{-r(T-t)}$.

when $t=T$.

$$f(T,x) = x - K.$$

$$\frac{\partial f}{\partial t}(t,x) = -rKe^{-r(T-t)}$$

$$\frac{\partial f}{\partial x}(t,x) = 1. \qquad \frac{\partial^2 f}{\partial x^2} = 0.$$

thus we put these into the equation

$$-rKe^{-r(T-t)} + rx = r(x - Ke^{-r(T-t)}).$$

the SDE is right. thus we prove the forward contract satisfy the SDE and with terminal condition $f(T,x) = x-K$

6. $d\Phi(t) = dc(t, S(t)) - \Delta dS(t)$

$$= \frac{\partial c}{\partial t}dt + \frac{\partial c}{\partial x}dS(t) + \frac{1}{2}\frac{\partial^2 c}{\partial x^2}dS(t)dS(t) - \Delta dS(t).$$

$\because \dfrac{dS(t)}{S(t)} = r\,dt + \sigma dW_t.$

thus.

$$= \left[\frac{\partial c}{\partial t} + \frac{\partial c}{\partial x}r\cdot S(t) + \frac{1}{2}\frac{\partial^2 c}{\partial x^2}\sigma^2 S(t)^2 - \Delta S(t) r\right]dt$$

$$+ \left[\frac{\partial c}{\partial x}\sigma S(t) - \sigma S(t)\Delta\right]dW_t.$$

and ~~thus we~~ we have $\Delta(t) = \dfrac{\partial c}{\partial x}(t, S(t))$.

thus the dW_t goes to 0.

and we have $c(t,S)$ satisfy B-S-M SDE.

so $d\Phi(t) = [rc(t,S(t)) - r\Delta S(t)]dt$

$$= r(c(t,S(t)) - \Delta S(t))dt$$

$$= r\Phi(t)\,dt. \quad \rightsquigarrow \text{then we proved that}$$

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8. $\Phi = \Delta S + B$.

$$d\Phi_t = \Delta_t dS_t + dB_t.$$

$$= rB_t dt + \Delta_t(rS_t dt + \sigma S_t dW_t)$$

$$= r(B_t + \Delta_t S_t)dt + \Delta_t \sigma S_t dW_t.$$

$$= r\Phi_t dt + \Delta_t \sigma S_t dW_t.$$

thus $d(g(t,x)) = d(e^{-rt}\cdot\Phi_t)$

$$= -r\cdot e^{-rt}\Phi_t dt + e^{-rt}d\Phi(t)$$

$$= -r\cdot e^{-rt}\Phi(t)dt + e^{-rt}[r\cdot\Phi_t dt + \Delta_t\sigma S_t dW_t]$$

$$= e^{-rt}\Delta_t\sigma\cdot S_t dW_t. \quad \cdots (1)$$

then we see $d(e^{-rt}S_t)$

$$= -re^{-rt}S_t dt + e^{-rt}dS_t$$

$$= e^{-rt}\sigma S_t dW_t.$$

so. $\Delta_t\cdot d(e^{-rt}S_t) = e^{-rt}\cdot\Delta_t\sigma S_t dW_t \cdots (2)$

we find (2) is the same with (1)

So we prove

$$d(e^{-rt}\Phi(t)) = \Delta_t d(e^{-rt}S_t).$$