FE5222 Homework 4: Due by Thursday, Oct 24

1. (25 Points) Your firm owns 100 puts. Each put has a delta of -0.40, gamma of 0.04 and theta of -7.3. The underlying price is \$100.0.

   (a) How many shares should you buy or short in order to delta-hedge this position?

   (b) After you have delta hedged the position, how much would you expect to make if, by the end of the next day, the stock moved up 1%. Down 1%? Assume 365 days a year (hence $dt = \frac{1}{365.0}$ for 1 day) and 0% interest rate.

   (c) If the stock moves up 4% (on the same day), how many more shares of stock should you buy or short to keep your position delta neutral?

   Solution

   See Exercise 3-3 and 3-4, The Volatility Smile. Solutions for exercises are given a the back of the book.

2. (35 Points) Replicate the payoff of a one-year down-and-out European put with a strike of 80 and a barrier of 60. The current stock price is 100. The stock pays no dividends, and the riskless rate is zero. Assume BSM and an implied volatility of 20%.

   (a) Use three vanilla European options to match the payoff of the down-and-out put a) at expiration when the barrier has not been hit, b) six months prior to expiration, at barrier and c) today, at barrier.

   (b) What is the value of replication portfolio?

   Solution

   See p. 221 - 223, The Volatility Smile.

3. (25 Points) Let $C(t, S(t))$ be the price of a call option at time t when the stock price is $S(t)$ in the BSM model. Assume interest rate r is zero. Let

$$\Gamma(t) = \frac{\partial^c C(t,S)}{\partial S^2} |_{S=S(t)}$$

   be the gamma at time t when the stock price is $S(t)$.

   Show that

$$\mathbb{E}\left[\left(\Gamma(t)S^2(t)\right)^2\right] \approx \Gamma^2(0)S^4(0)\sqrt{\frac{T^2}{T^2-t^2}}$$

   Solution For $r = 0$, we have

$$\Gamma(t) = K\frac{\phi\left(d_2(t)\right)}{S^2(t)\sigma\sqrt{T-t}}$$

where

$$d_2(t) = \frac{\ln\left(\frac{S(t)}{K}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

Multiplying both sides by $S^2(t)$, we have

$$\Gamma(t)S^2(t) = K\frac{\phi\left(d_2(t)\right)}{\sigma\sqrt{T-t}}$$

Hence

$$\frac{\Gamma(t)S^2(t)}{\Gamma(0)S^2(0)} = \sqrt{\frac{T}{T-t}}\frac{\phi\left(d_2(t)\right)}{\phi\left(d_2(0)\right)}$$

Note that

$$\frac{\phi\left(d_2(t)\right)}{\phi\left(d_2(0)\right)} = e^{\frac{1}{2}\left(d_2^2(0)-d_2^2(t)\right)}$$

we have

$$\left(\frac{\Gamma(t)S^2(t)}{\Gamma(0)S^2(0)}\right)^2 = \frac{T}{T-t}e^{d_2^2(0)-d_2^2(t)}$$

Taking expectation we have

$$\mathbb{E}\left[\left(\frac{\Gamma(t)S^2(t)}{\Gamma(0)S^2(0)}\right)^2\right] = \frac{T}{T-t}\mathbb{E}\left[e^{d_2^2(0)-d_2^2(t)}\right]$$

Under BSM model,

$$S(t) = S(0)e^{-\frac{1}{2}\sigma^2 t+\sigma W(t)}$$

For at-the-money option,

$$d_2(0) = -\frac{1}{2}\sigma\sqrt{T}$$

and

$$d_2(t) = \frac{-\frac{1}{2}\sigma T + W(t)}{\sqrt{T-t}}$$

Hence

$$e^{d_2^2(0)-d_2^2(t)} = e^{\frac{\sigma^2 T}{4}-\frac{1}{T-t}\left(-\frac{\sigma T}{2}+W(t)\right)^2}$$

and

$$\mathbb{E}\left[e^{d_2^2(0)-d_2^2(t)}\right] = e^{\frac{\sigma^2 T}{4}}\mathbb{E}\left[e^{-\frac{1}{T-t}\left(-\frac{\sigma T}{2}+W(t)\right)^2}\right]$$

For small σ,

$$\begin{aligned}
\mathbb{E}\left[e^{d_2^2(0)-d_2^2(t)}\right] &\approx \mathbb{E}\left[e^{-\frac{1}{T-t}(W(t))^2}\right] \\
&= \tfrac{1}{\sqrt{2\pi}}\int e^{-\frac{t}{T-t}x^2}e^{-\frac{x^2}{2}}dx \\
&= \tfrac{1}{\sqrt{2\pi}}\int e^{-\frac{1}{2}\frac{T+t}{T-t}x^2}dx \\
&= \sqrt{\tfrac{T-t}{T+t}}
\end{aligned}$$

It then follows that

$$\mathbb{E}\left[\left(\frac{\Gamma(t)S^2(t)}{\Gamma(0)S^2(0)}\right)^2\right] \approx \sqrt{\frac{T^2}{T^2-t^2}}$$

which is equivalent to

$$\mathbb{E}\left[\left(\Gamma(t)S^2(t)\right)^2\right] \approx \left(\Gamma(0)S^2(0)\right)^2\sqrt{\frac{T^2}{T^2-t^2}}$$

4. (15 Points) Let $V(S,K) = (S-K)^2 1_{[S \geq K]}$, derive the second-order derivative $\frac{\partial V(S^2,K)}{\partial S^2}$ using Heaviside function and/or Dirac Delta function.

Solution

Note that

$$\begin{aligned} V(S,K) &= (S-K)^2 \mathbb{1}_{[S \geq K]} \\ &= (S-K)(S-K)^+ \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial V(S,K)}{\partial S} &= \frac{\partial (S-K)}{\partial S}(S-K)^+ + (S-K)\frac{\partial (S-K)^+}{\partial S} \\ &= (S-K)^+ + (S-K)\,H(S-K) \\ &= 2\,(S-K)^+ \end{aligned}$$

Taking partial derivative w.r.t. S on the both sides of the above equation, we have

$$\frac{\partial^2 V(S,K)}{\partial S^2} = 2H(S-K)$$