FE5222 Advanced Derivative Pricing

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Overview

Overview

What is replication?

Replication is to construct a portfolio of (usually simpler) derivatives to mimic the payoff of another derivative. It is at the core of derivative pricing.

Overview

Two types of replication

- *Static Replication*
  Replicate the payoff of a derivative upfront and don't re-balance positions throughout the lifetime of the derivative
- *Dynamic Replication*
  Continuously adjust positions in response to market changes to replicate the payoff of a derivative

Overview

Hedging vs Replication

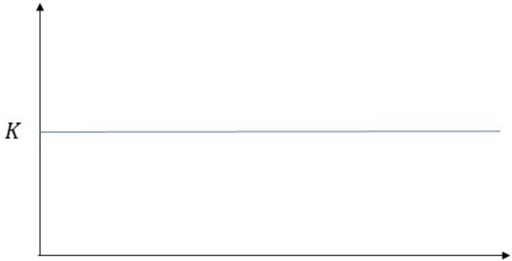
Hedging is to take the opposite positions of a replicating portfolio.

Building Blocks

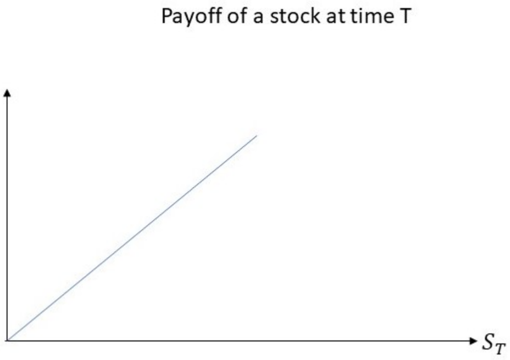
- Zero coupon bond
- Stock
- Call option
- Put option

Zero Coupon Bond

Payoff of a zero coupon bond at expiry T

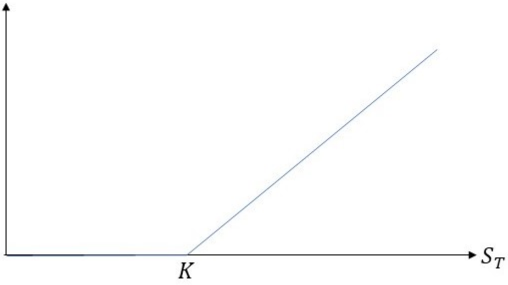


Stock



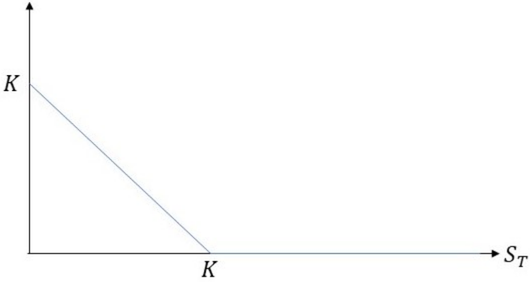
Call Option

Payoff of a call option with strike K at expiry T

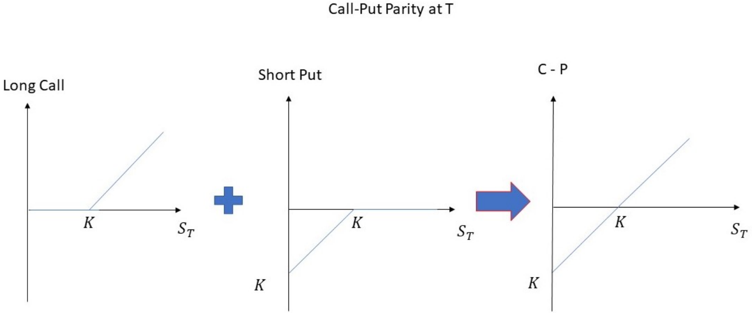


Put Option

Payoff of a put option with strike K at expiry T



Call-Put Parity



Call-Put Parity

At expiry

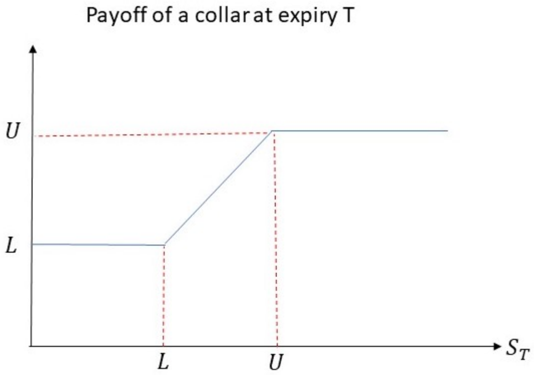
$$C(S_T, K) - P(S_T, K) = S_T - K$$

Collar

To hedge downside price move, the owner of a stock buys a put option at strike L lower than the prevailing stock price. To finance the purchase of put option, he or she sells an out-of-money call option at strike U. Such a portfolio is called a collar whose payoff at expiry is

$$P(S_T, L) + S_T - C(S_T, U)$$

Collar



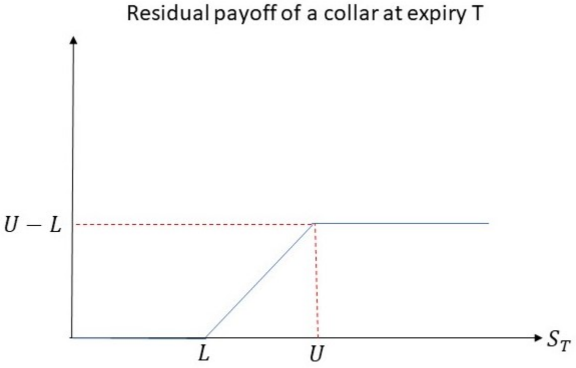
Replicating a Collar

An alternative way to construct a collar:

Since a collar pays a flat price of L when the stock price is below or at L, we can start with a zero-coupon bond that pays L at expiry T.

Replicating a Collar

The residual payoff after a zero-coupon bond that pays L

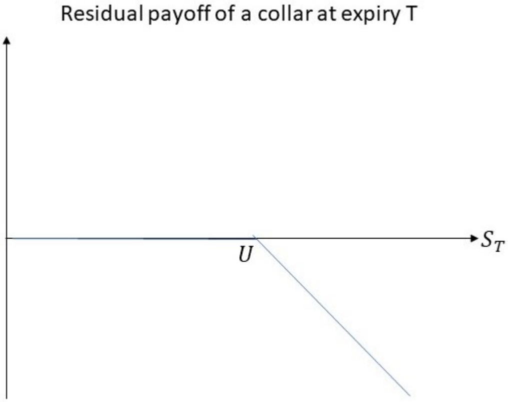


Replicating a Collar

The residual payoff matches the payoff of a call option with strike L for stock price $S_T \leq U$. Hence we can buy a call option with strike L.

Replicating a Collar

After the zero-coupon bond and call option, the residual payoff becomes



Replicating a Collar

The residual payoff is the payoff of a short position of a call option with strike U. Hence we can sell a call option at strike U.

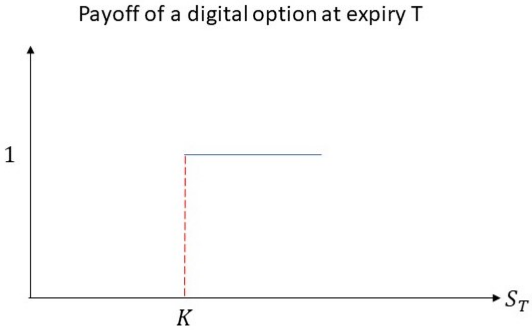
Replicating a Collar

In summary, we can replicate a collar with

$$B(T, L) + C(S_T, L) - C(S_T, U)$$

Replicating a Digital Option

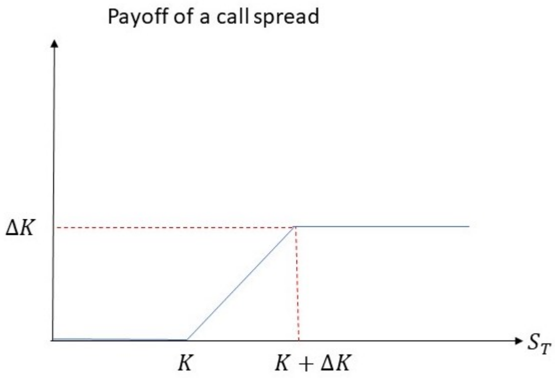
Digital Call



How to replicate this payoff?

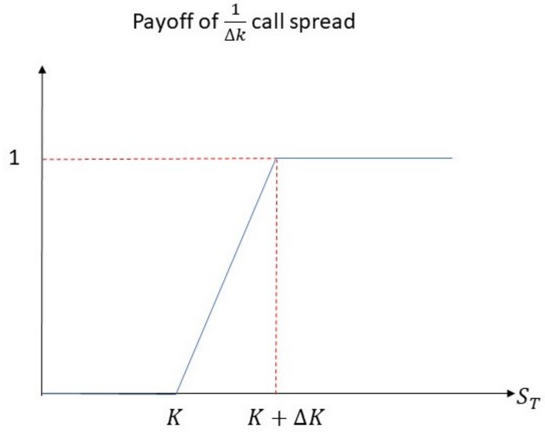
Replicating a Digital Option

Call spread : long a call option at strike K and short a call option at strike $K + \Delta K$



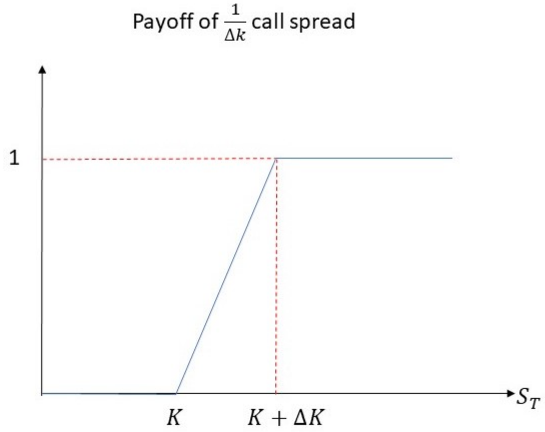
Replicating a Digital Option

Long $\frac{1}{\Delta K}$ call spread : $\frac{1}{\Delta K}(C(S_T, K) - C(S_T, K + \Delta K))$

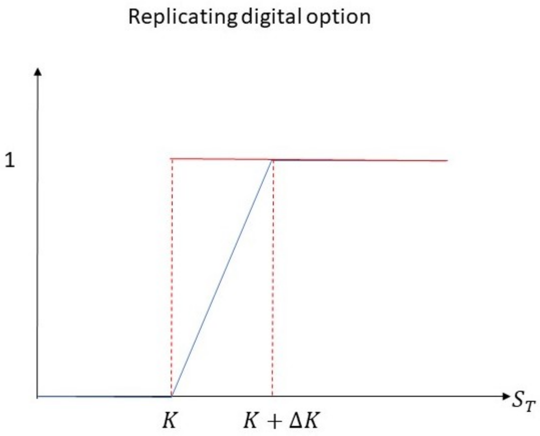


Replicating a Digital Option

Long $\frac{1}{\Delta K}$ call spread : $\frac{1}{\Delta K}\left(C(S_T, K) - C(S_T, K + \Delta K)\right)$

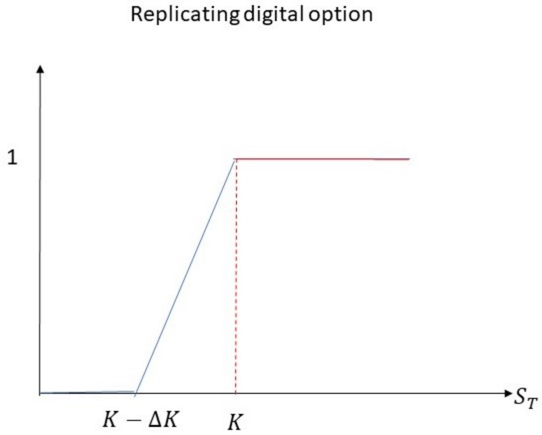


Replicating a Digital Option



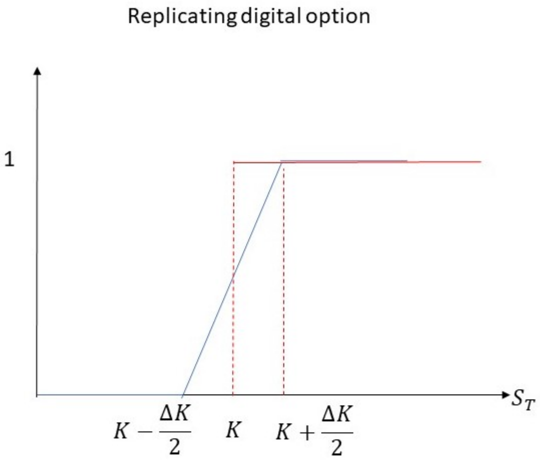
Replicating a Digital Option

Choice of strikes : $C(S_T, K - \Delta K) - C(S_T, K - \Delta K)$



Replicating a Digital Option

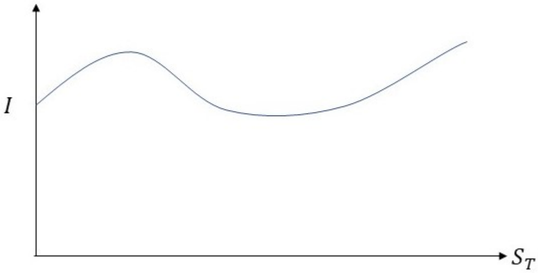
Choice of strikes : $C(S_T, K - \frac{\Delta K}{2}) - C(S_T, K + \frac{\Delta K}{2})$



Replicating a Generic Payoff

We can (approximately) replicate a generic European style payoff with a combination of call/put options, stocks and zero-coupon bonds.

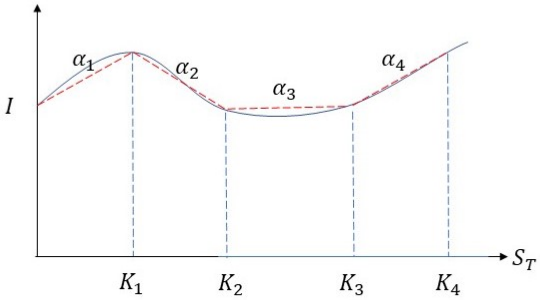
A generic payoff



Replicating a Generic Payoff

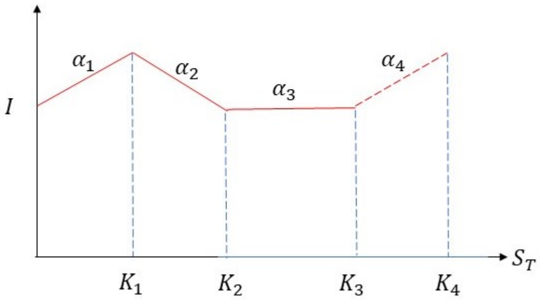
Approximate the generic payoff with a piece-wise linear payoff function

Approximate Replication with Piece-wise Linear Payoff



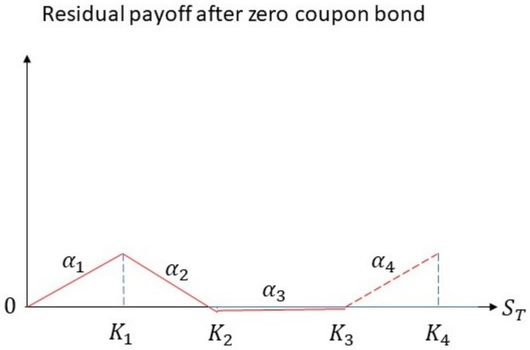
Replicating a Generic Payoff

Piece-wise Linear Payoff



Replicating a Generic Payoff

Buy a zero-coupon that pays l at expiry. The residual payoff to be replicated is

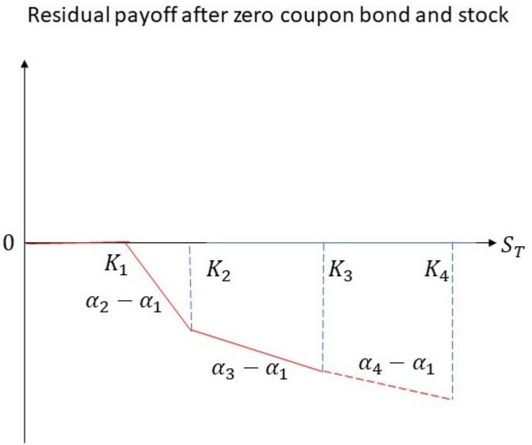


Replicating a Generic Payoff

The payoff between 0 and K_1 is the payoff of α_1 shares of stock, hence we can buy α_1 shares of stock in the replicating portfolio.

Replicating a Generic Payoff

With the zero-coupon bond and stock in the replicating portfolio, the residual payoff becomes



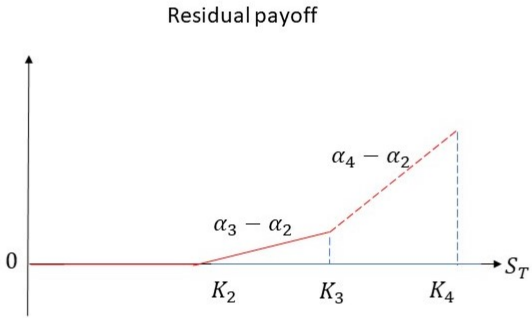
Replicating a Generic Payoff

The part of he residual payoff between 0 and K_2 is the payoff of $\alpha_2 - \alpha_1$ call options with strike K_1[1]

[1] In the graph, $\alpha_2 - \alpha_1$ is negative. Hence we need to short call options.

Replicating a Generic Payoff

The residual payoff becomes



Replicating a Generic Payoff

We can continue with this process. The replicating portfolio is

$$B(T, I) + \alpha_1 S_T + (\alpha_2 - \alpha_1) C(S_T, K_1) + (\alpha_3 - \alpha_2) C(S_T, K_2) + \ldots$$

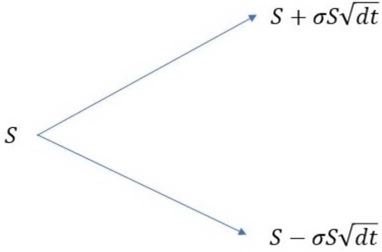
Summary

Static replication

- Simple
- Hard to achieve exact replication
- Lower cost compared to dynamic replication

Dynamic Replication

A simple model



Assume zero interest rate $r = 0$.

Dynamic Replication

Let $C(t,S)$ be the price of a call option

$$
\begin{aligned}
C(t+dt, S+dS) &\approx C(t,S) + \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS)^2 \\
&= C(t,S) + \Theta dt + \Delta dS + \frac{1}{2}\Gamma (dS)^2
\end{aligned}
$$

where $\Theta = \frac{\partial C}{\partial t}$, $\Delta = \frac{\partial C}{\partial S}$ and $\Gamma = \frac{\partial^2 C}{\partial S^2}$.

Dynamic Replication

Call option price change

$$\begin{tikzcd}
 & C(t,S) + \Theta dt + \Delta\sigma S\sqrt{dt} + \frac{1}{2}\Gamma\sigma^2 S^2 dt \\
C(t,S) \arrow[ur] \arrow[dr] & \\
 & C(t,S) + \Theta dt - \Delta\sigma S\sqrt{dt} + \frac{1}{2}\Gamma\sigma^2 S^2 dt
\end{tikzcd}$$

Dynamic Replication

Delta-hedged (short Δ shares of stock) call option price change

$$C(t,S) - \Delta S + \Theta dt + \frac{1}{2}\Gamma\sigma^2 S^2 dt$$

$$C(t,S) - \Delta S$$

$$C(t,S) - \Delta S + \Theta dt + \frac{1}{2}\Gamma\sigma^2 S^2 dt$$

Dynamic Replication

Note that the value of a delta hedged call option is the same whether the stock price S goes up or down.

$\Rightarrow$ A delta hedged call option replicates the payoff of a zero coupon bond in an infinitesimal time interval from t to $t + dt$.

$\Rightarrow$ Δ shares of stocks + zero coupon replicates the payoff of a call option in an infinitesimal time interval from t to $t + dt$.

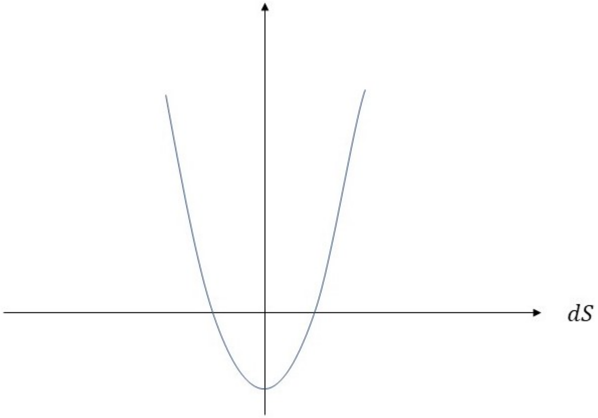
Dynamic Replication

P&L of a delta hedged call option

$$\Theta dt + \frac{1}{2}\Gamma(dS)^2$$

Dynamic Replication

$\Gamma > 0, \Theta < 0$, P&L is a parabolic function of dS



Dynamic Replication

Whether the stock price moves up ($dS > 0$) or down ($dS < 0$), P&L will always increases. This is due to the convexity of option payoff.

Dynamic Replication

What do we pay for the convexity?

Θ: the first term in the P&L equation. Θ is negative, as time advances we lose money from Θ if the price does not move enough to compensate the loss.

Dynamic Replication

From

$$P\&L = \Theta dt + \frac{1}{2}\Gamma(dS)^2$$

we know price has to move at least $\sqrt{\frac{-2\Theta dt}{\Gamma}}$ to break even.

Dynamic Replication

Under zero interest rate, P&L for the hedged portfolio must be zero. That is

$$\Theta dt + \frac{1}{2}\Gamma(dS)^2 = 0$$

Since $dS = \sigma S\sqrt{dt}$, after canceling out dt, the above equality reduces to

$$\Theta + \frac{1}{2}\Gamma\sigma^2 S^2 = 0$$

This is the BS equation for the special case $r = 0$.

Dynamic Replication

Hence

$$P\&L = \frac{1}{2}\Gamma S^2 \left(\left(\frac{dS}{S} \right)^2 - \sigma^2 dt \right)$$

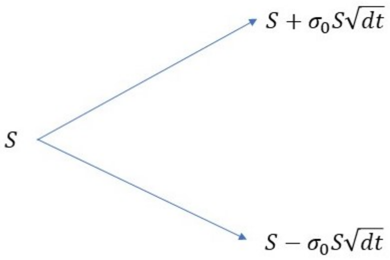
To break even, the price has to move (up or down) at least $S\sigma\sqrt{dt}$ between time t to $t + dt$.

Dynamic Replication

- $S\sigma\sqrt{dt}$ is called *breakeven vol*. It is the minimum amount stock price needs to move (in either direction) to offset the P&L loss from theta.
- In practice, dt is usually one day. $S\sigma\sqrt{dt}$ is the required daily price move to compensate for theta.

Dynamic Replication

If market is pricing the option with volatility σ, but the volatility turns out to be ${\sigma_0}^2$, what is the P&L impact?



[2] σ_0 is called *realized volatility*.

Dynamic Replication

In this case

$$\left(\frac{dS}{S}\right)^2 = \sigma_0^2 dt$$

and the P&L is

$$P\&L = \frac{1}{2}\Gamma S^2 \left(\sigma_0^2 - \sigma^2\right) dt$$

$\Rightarrow$ Delta hedging option means betting on volatility.

Summary

Dynamic replication

- In principle it works in most cases
- Difficult to maintain
- High cost - bid/ask spreads, liquidity, slippage etc.

Weak Static Replication

Difficulty with replicating path-dependent options such as barrier options:

- Static replication does not seem to be feasible.
- Dynamic replication will be too expensive.

$\Rightarrow$ Weak static replication provides a practical way to approximately replicate these exotic options.

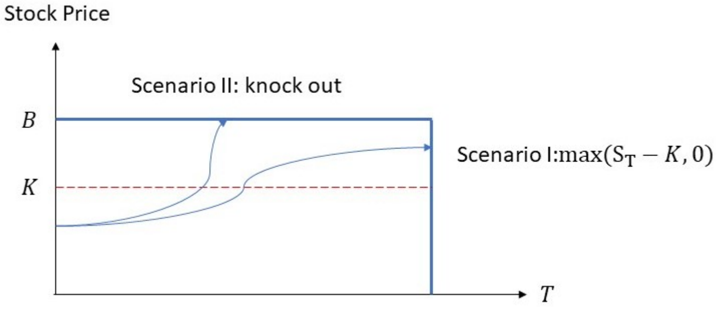
Weak Static Replication

Idea: match the payoff of an exotic option with simple derivatives (vanilla options, zero coupon bonds and stocks etc.) on the boundary and at expiry.

Replicating an Up-and-Out Call

Consider an up-and-out call barrier option with barrier B above the strike K and expires in $T = 1$ year. If at any time t before expiry T, the underlying stock S_t crosses the barrier B, the option knocks out and becomes worthless. Otherwise its payoff is the same as a call option with expiry T.

Replicating an Up-and-Out Call



Replicating an Up-and-Out Call

Ideally if we can replicate the barrier option on the boundary where stock price is B and at expiry where the payoff is $(S_T - K)^+$, with the assumption of continuous stock price (no jump), we shall be able to match the barrier option exactly in all scenarios.

- If the stock price touches the barrier at any time before expiry, we can liquidate the replicating portfolio.
- If the stock price never touches the barrier, by construction the replicating portfolio will have the same payoff as the barrier option.

Replicating an Up-and-Out Call

The reality is we can't match the barrier option payoff at all time t with vanilla (European) options. We have to choose a discrete set of times $t_0 < t_0 < \ldots t_n < T$ and try to match payoff at these points plus expiry.

If we choose enough points the replication shall be reasonably close.

Replicating an Up-and-Out Call

Take the example $K = 100$, $B = 120, S_0 = 100, r = 0$[3] and $\sigma = 20\%$. The value of this barrier option is 1.10.

To make weak static replication work, we need to assume a model which we take to be BSM model.

For simplicity we choose only to match the payoff at expiry $t = 1.0$, six months time $t = 0.5$ and initial time $t = 0$.

[3] The assumption $r = 0$ is for convenience and not essential.

Replicating an Up-and-Out Call

To match the payoff at expiry $t = 1$ provided that the barrier option has not knocked out, we need to buy a call option with strike 100 that expires in 1 year.

Quantity	Type	Strike	Expiry	$t = 0.0$
1.0	Call	100	1 year	7.97

$\Rightarrow$ Note that this is much more expensive than the barrier option.

Replicating an Up-and-Out Call

This portfolio only matches the up-and-out call at expiry but nowhere else. At $t = 0.5$, if the stock price is 120, the call option is worth \$20.72.

				Value at $S_t = 120$
Quantity	Type	Strike	Expiry	$t = 0.5$
1.0	Call	100	1 year	20.72

Replicating an Up-and-Out Call

On the other hand the barrier option is worthless when the underlying stock is 120 at $t = 0.5$ since it knocks out.

To match the payoff on the barrier boundary $S = 120$ at $t = 0.5$, we need to include more options in the portfolio.

Replicating an Up-and-Out Call

We can include call options that expires in 1 year with strike 120. Since these options are worthless when the stock price is below or at 120 at time $t = 1$, including such options does not affect payoff at $t = 1$.

Replicating an Up-and-Out Call

At time $t = 0.5$, a call option with strike 120 and expiry 1 year is worth 6.77. If we short a quantity of -3.06 such options, our portfolio will be worth 0 at time $t = 0.5$ when the stock price is at 120.

Replicating an Up-and-Out Call

				Value at $S_t = 120$	
Quantity	Type	Strike	Expiry	$t = 0.5$	$t = 0$
1.0	Call	100	1 year	20.72	22.15
-3.06	Call	120	1 year	-20.72	-29.28
Portfolio				0	-7.13

Replicating an Up-and-Out Call

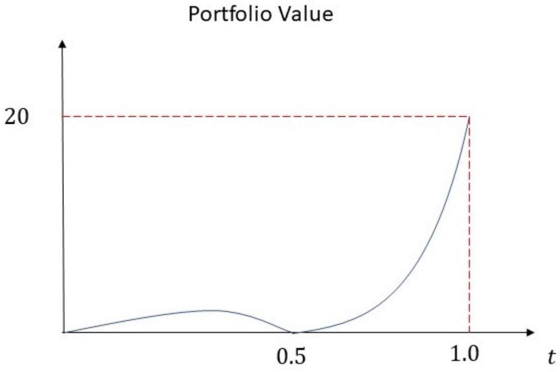
To match the payoff of a barrier option at $t = 0$ for stock price $S = 120$, we can include options with expiry at six months and strike $K = 120$. One such option is worth 6.79 at time $t = 0$ and $S = 120$. Hence we need to long 1.05 such options.

Replicating an Up-and-Out Call

				Value at $S_t = 120$	
Quantity	Type	Strike	Expiry	$t = 0.5$	$t = 0$
1.0	Call	100	1 year	20.72	22.15
-3.06	Call	120	1 year	-20.72	-29.28
1.03	Call	120	6 months	0	7.13
Portfolio				0	0

Replicating an Up-and-Out Call

Portfolio value when stock price $S = 120$



Summary

Weak static replication

- Model based
- Less re-balancing
- Need to unwind positions when exotic options become worthless (e.g., knock out)
- No unique static weak replication portfolio.

Thank you!