8 Risk-neutral pricing (1 lecture)

In the previous chapter, we first derived the Black-Scholes equation (7.11) and then solve it by the Feynman-Kac formula (Question 6 of Homework VI) to derive (7.19):

$$c(t,x) = \mathbb{E}^{t,x}[e^{-r(T-t)}(X_T - K)^+]$$

with X_t satisfying $dX_s = rX_s ds + \sigma X_s dW_s$ and $X_t = x$. σ is the volatility for the stock price S_t.

In this chapter, we we consider stochastic risk-free interest rate r[64] for the money market account and stochastic drift coefficient and stochastic volatility for the stock price (see (8.3)). We give a more direct derivation of the above pricing formula with $e^{-r(T-t)}$ replaced by $e^{-\int_t^T r(s,\omega)ds}$. Like in the last chapter, we will still use the replicating portfolio $\Delta S + B$ to replicate the option. The difference lies in how to find Δ. In the last chapter, we use Black-Scholes equation + Feynman-Kac formula and obtain $\Delta(t) = \partial c/\partial x(t, S(t))$ with c being the solution to Black-Scholes equation. In this chapter, we find Δ to be (8.43) using Girsanov theorem + martingale representation theorem. These two theorems are very importance results in stochastic calculus that one needs to know.

In order for you to appreciate the derivation that we will present later, let us consider what happens if we still use the Black-Scholes equation + Feynman-Kac formula approach when r, α and σ in (8.3) are stochastic. For simplicity, let us only consider stochastic σ. Then in addition to the stochastic differential equation of S, there is another stochastic differential equation of σ. At time t, $\mathcal{F}(t)$ tells you current values of both $S(t)$ and $\sigma(t)$. Option price c at time t depends on both S and σ at t. So, it is $c(t, S(t), \sigma(t))$ with $c(t,x,y)$ being a function depending on three variables. Then, instead of the single-variate Itô formula in (7.6), we have to use multi-variate Itô formula (5.20)

$$dc(t,S(t),\sigma(t)) = \frac{\partial c}{\partial t}dt + \frac{\partial c}{\partial x}dS_t + \frac{\partial c}{\partial y}d\sigma(t) + \frac{1}{2}\left(\frac{\partial^2 c}{\partial x^2}(dS_t)^2 + \frac{\partial^2 c}{\partial y^2}(d\sigma(t))^2 + 2\frac{\partial^2 c}{\partial x \partial y}dS(t)d\sigma(t)\right).$$

As a result, the Black-Scholes type equation is much more complicated than (7.11). Those who are interested in what we may obtain, can read Section 3.5.4 of Yue-Kuen Kwok, "Mathematical Models of Financial Derivatives", 2nd edition, where the stochastic σ case is considered.

Recall that in the discrete stock price model, we have found that under the risk neutral probability $\tilde{\mathbb{P}}(H) = q_u$, $\tilde{\mathbb{P}}(T) = q_d$, the discounted option price and the discounted option price are martingales (2.35), (2.36):

$$\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{e^{r(n+1)\delta t}}\right] = \frac{S_n}{e^{rn\delta t}}, \tag{8.1}$$

$$\tilde{\mathbb{E}}_n\left[\frac{Ⓓ_{n+1}}{e^{r(n+1)\delta t}}\right] = \frac{Ⓓ_n}{e^{rn\delta t}}. \tag{8.2}$$

[64]risk-free interest rate is the coefficient in $dB(t) = r(t)B(t)$ with B being the money market account.

We now consider the continuous stock price model

$$dS(t,\omega) = \alpha(t,\omega)S(t,\omega)dt + \sigma(t,\omega)S(t,\omega)dW(t,\omega). \quad {}^{65} \tag{8.3}$$

Note that we have allowed the mean rate of return α and the volatility σ to be adapted stochastic processes. See (5.22), (5.23), (5.26) for the meaning of the coefficients.

Example 8.1 *Show that the solution of (8.3) is*

$$S(t) = S(0)e^{\int_0^t \left(\alpha(s,\omega)-\frac{1}{2}\sigma^2(s,\omega)\right)ds+\int_0^t \sigma(s,\omega)dW_s}. \tag{8.4}$$

Proof: *Let $X(t,\omega) = \int_0^t \left(\alpha(s,\omega) - \frac{1}{2}\sigma^2(s,\omega)\right) ds+\int_0^t \sigma(s,\omega)dW_s$. By the equivalence between integral form and differential form, we know $dX_t = \left(\alpha(t,\omega) - \frac{1}{2}\sigma^2(t,\omega)\right) dt+\sigma(t,\omega)dW_t$. By Itô formula,*

$$\begin{aligned} dS_t = de^{X_t} &= e^{X_t}dX_t + \frac{1}{2}e^{X_t}(dX_t)^2 \\ &= e^{X_t}\left(\left(\alpha_t - \frac{1}{2}\sigma_t^2\right)dt + \sigma_t dW_t + \frac{1}{2}\sigma_t^2 dt\right) \\ &= \alpha_t e^{X_t}dt + \sigma_t e^{X_t}dW_t = \alpha_t S_t dt + \sigma_t S_t dW_t. \end{aligned}$$

We hope to find the risk-neutral probability measure $\tilde{\mathbb{P}}$ under which the discounted stock price and the discounted option price are martingales:

$$\tilde{\mathbb{E}}[e^{-\int_t^T r(s,\omega)ds}S_T|\mathcal{F}_t] = S_t, \qquad 0 \le t \le T, \tag{8.5}$$

$$\tilde{\mathbb{E}}[e^{-\int_t^T r(s,\omega)ds}Ⓓ_T|\mathcal{F}_t] = Ⓓ_t, \qquad 0 \le t \le T, \tag{8.6}$$

where r is the interest rate of the money market which is now allowed to be random (i.e., money market account satisfies $dB(t,\omega) = r(t,\omega)B(t,\omega)dt)$, and $\mathcal{F}_t$ is given by Definition 4.1 which represents all the information up to t.

Here are the main steps and the main ideas in our following discussion which follows Sections 5.2 and 5.3 of Shreve II.

(a) We introduce

$$D(t,\omega) = e^{-\int_0^t r(s,\omega)ds}$$

and prove that for a self-financing portfolio $\Phi = \Delta S + B$, we always have

$$d(D(t)\Phi(t)) = \Delta(t)\sigma(t)D(t)S(t)d\tilde{W}(t)$$

where $\tilde{W}$ is a Brownian motion plus a drift.

[65] which is another way to write $S(t,\omega) = S_0(\omega) + \int_0^t \alpha(s,\omega)S(s,\omega)ds + \int_0^t \sigma(s,\omega)S(s,\omega)dW(s,\omega)$.

(b) Consider the old probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the Brownian motion W is defined. By the Girsanov's theorem that we will present, we can introduce a new measure $\tilde{\mathbb{E}}$ on Ω so that the $\tilde{W}$ above, which is W plus a drift, becomes a Brownian motion when a new probability $\tilde{\mathbb{P}}(A)$ are assigned to every event $A \in \Omega$. Now $D(t)\Phi(t)$ is a martingale and therefore $\Phi(t) = \frac{1}{D(t)}\tilde{\mathbb{E}}[D(T)\Phi(T)|\mathcal{F}(t)]$.

(c) We hope that we can choose Δ so that $\Phi(T) = \textcircled{p}_T$, where $\textcircled{p}_T = (S_T - K)^+$ if it's European call option for example. If we can do that, by the arbitrage free argument of Chapter 1, we know that $\Phi(t)$, which is $\frac{1}{D(t)}\tilde{\mathbb{E}}[D(T)\textcircled{p}_T|\mathcal{F}(t)]$ by part (b) and $\Phi(T) = \textcircled{p}_T$, can be considered as the price of the option at time t.

(d) It takes two steps to achieve $\Phi(T) = \textcircled{p}_T$.

Step I: we define $\textcircled{p}(t)$ by (this uses the new measure $\tilde{\mathbb{E}}$)

$$D(t)\textcircled{p}(t) = \tilde{\mathbb{E}}[D(T)\textcircled{p}_T|\mathcal{F}(t)].$$

Then we can show by iterated conditioning that $D(t)\textcircled{p}(t)$ is a martingale (this is a simple generalization of Question 11 of Homework II). Now, note that at this stage, we can not call $\textcircled{p}(t)$ the price of the option at time t because we can not directly exercise/sell/buy $\textcircled{p}(t)$ in the market. This $\textcircled{p}(t)$ only exits in our mind. We have to reply on a tradable or realizable $\Phi(t)$ to determine the value of the option at time t. So the following step is conceptually crucial.

Step II: By the martingale representation theorem that we will present later, we know that since $D(t)\textcircled{p}(t)$ is a martingale under $\tilde{\mathbb{E}}$, there is a $\tilde{\Gamma}$ so that

$$D(t)\textcircled{p}(t) = D(0)\textcircled{p}(0) + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u).$$

Hence we can choose Δ and $\Phi(0)$ properly so that the two martingales $D(t)\Phi(t)$ and $D(t)\textcircled{p}(t)$ agree with each other. Since $\Delta(t)$ exists, $\Phi(t)$ is realizable (at least conceptually). Since $\Phi(T)$ agrees with the value of an option at time T and $\textcircled{p}(t) = \Phi(t)$ no matter what happens, $\Phi(t)$ can now be called the price of this option at time t.

Remark: : I hope now you can appreciate the martingale argument: as long as Δ exists, no matter how complicated it is, its expression does not enter explicitly into the evaluation formula (8.6). So, step (d) above won't enter into the final evaluation formula.

8.1 Discount process, market price of risk, and portfolio process

Note that (8.5) is equivalent to $\tilde{\mathbb{E}}[e^{-\int_0^T r(s)ds} S_T|\mathcal{F}_t] = e^{\int_0^t -r(s)ds} S_t$. So we introduce

$$D(t, \omega) = e^{-\int_0^t r(s,\omega)ds} \tag{8.7}$$

which can be called as the discount process. By Itô formula, $dD(t,\omega) = -r(t,\omega)D(t,\omega)dt$ [66]. By the product rule (Question 9 of Homework V), we have

$$
\begin{aligned}
d(D(t)S(t)) =& -r(t)D(t)S(t)dt + D(t)[\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)] \\
=& \sigma(t)D(t)S(t)\left(\frac{\alpha(t)-r(t)}{\sigma(t)}dt + dW(t)\right) \\
=& \sigma(t)D(t)S(t)d\tilde{W}(t) \qquad (8.8)
\end{aligned}
$$

where

$$
\tilde{W}(t) = W(t) + \int_0^t \Theta(s)ds, \qquad \Theta(t) = \frac{\alpha(t)-r(t)}{\sigma(t)}. \qquad (8.9)
$$

Θ is called the market price of risk since σ is the risk and $\alpha - r$ is the risk premium.

We will show that there is a probability measure $\tilde{\mathbb{P}}$ which makes $\tilde{W}$ a Brownian motion under this measure. Then the discounted stock price $D(t)S(t)$ is a martingale in this $\tilde{\mathbb{P}}$ world and we will then have (8.5). How about (8.6)?

To get (8.6), we need to repeat the argument we have gone through in Section 7.1. Like (7.3), consider again the agent who begins with initial capital $\Phi(0)$ and at each time $t \in [0,T]$ holds $\Delta(t)$ shares of stock, investing or borrowing at the interest rate $r(t,\omega)$ as necessary to finance this:

$$
\Phi = \Delta S + B. \qquad (8.10)
$$

The differential of this agent's portfolio value at each time t is due to two factors, the capital gain $\Delta(t)dS(t)$ on the stock position and the interest earning $r(t)[\Phi(t) - \Delta(t)S(t)]dt$ on the cash position (recall $dB_t(\omega) = r(t,\omega)B_t(\omega)dt$ for the money market account B_t)

$$
\begin{aligned}
d\Phi(t) =& \Delta(t)dS(t) + r(t)[\Phi(t) - \Delta(t)S(t)]dt \\
=& \Delta(t)[\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)] + r(t)[\Phi(t) - \Delta(t)S(t)]dt \\
=& r(t)\Phi(t)dt + \Delta(t)\sigma(t)S(t)[\Theta(t)dt + dW(t)], \qquad (8.11)
\end{aligned}
$$

with $\Theta(t) = \frac{\alpha(t)-r(t)}{\sigma(t)}$ defined in (8.9). Multiplying both sides of (8.11) by $D(t,\omega) = e^{-\int_0^t r(s,\omega)ds}$, we obtain

$$
\begin{aligned}
d(D(t)\Phi(t)) =& \Delta(t)\sigma(t)D(t)S(t)[\Theta(t)dt + dW(t)] \\
\overset{(8.9)}{=}& \Delta(t)\sigma(t)D(t)S(t)d\tilde{W}(t). \qquad (8.12)
\end{aligned}
$$

Please note that we are repeating the calculation of (7.15) where we used to assume constant r, Θ, and σ. So, **if** $\tilde{W}$ is a Brownian motion under probability measure $\tilde{\mathbb{P}}$, then $D(t)\Phi(t)$ is an Itô integral and hence a martingale. Let Ⓟ$_T$ be any $\mathcal{F}(T)$-measurable random variable. It represents the payoff at time T of a derivative security. We allow this payoff to be path-dependent (i.e., to depend on anything that occurs between times 0 and T), which is what

[66] Let $X(t,\omega) = \int_0^t -r(s,\omega)ds$. Then $dX_t(\omega) = -r(t,\omega)dt$ because of the equivalence between (5.5) and (5.6). Then we apply Itô formula to $D(t) = e^{X_t}$: $de^{X_t} = e^{X_t}dX_t + \frac{1}{2}e^{X_t}(dX_t)^2 = -r(t,\omega)D(t,\omega)dt$.

$\mathcal{F}(T)$-measurability means. For example, $ⓟ_T = \left(\frac{1}{T}\int_0^T S(t)dt - K\right)^+$ for a fixed-strike Asian call (See equation (7.5.3) of Shreve II). $ⓟ_T = (S(T)-K)^+$ for European call. $ⓟ_T = 1$ for zero-coupon bond. See also Question 2 and 7 in Homework VIII.

We wish to know what initial capital $\Phi(0)$ and the portfolio process $\Delta(t)$ an agent would need in order to hedge a short position in this derivative security, i.e., in order to have

$$\Phi(T) = ⓟ_T \qquad \text{almost surely.} \tag{8.13}$$

The selection of Δ requires martingale representation theory (Theorem 8.4). It turns out we do not need to know how to construct Δ if we are only interested in the price of the derivative security. To derive the pricing formula, we only need to know the existence of such a Δ. This is an example where abstract philosophical question (about existence) can be useful for practice.

If (8.13) is true, the fact that $D(t)\Phi(t)$ is a martingale under $\tilde{\mathbb{P}}$ implies

$$D(t)\Phi(t) = \tilde{\mathbb{E}}[D(T)\Phi(T)|\mathcal{F}(t)] = \tilde{\mathbb{E}}[D(T)ⓟ_T|\mathcal{F}(t)]. \tag{8.14}$$

The value $\Phi(t)$ of the hedging portfolio in (8.14) is the capital needed at time t in order to successfully complete the hedge of the short position in the derivative security with payoff $ⓟ_T$. Hence we can call this the price $ⓟ(t)$ of the derivative security at time t, and (8.14) becomes

$$D(t)ⓟ(t) = \tilde{\mathbb{E}}[D(T)ⓟ_T|\mathcal{F}(t)], \qquad 0 \le t \le T. \tag{8.15}$$

This is the continuous-time analogue of the risk-neutral pricing formula in the binomial model. Dividing (8.15) by $D(t)$, which is $\mathcal{F}(t)$ measurable and hence can be moved inside the conditional expectation on the right hand side of (8.15), and recalling the definition of $D(t)$, we may write (8.15) as

$$ⓟ(t) = \tilde{\mathbb{E}}[e^{-\int_t^T r(s,\omega)ds}ⓟ_T|\mathcal{F}(t)], \qquad 0 \le t \le T \tag{8.16}$$

which is (8.6).

(8.5) follows from (8.8) and the fact that $\tilde{W}_t$ is a Brownian motion under $\tilde{\mathbb{P}}$: By (8.8), $D(T)S(T) = D(t)S(t) + \int_t^T \sigma(t)D(t)S(t)d\tilde{W}_s$. Taking $\tilde{\mathbb{E}}[\cdot|\mathcal{F}(t)]$ on both sides, we obtain

$$S(t) = D^{-1}(t)\tilde{\mathbb{E}}[D(T)S(T)|\mathcal{F}(t)] = \tilde{\mathbb{E}}[D^{-1}(t)D(T)S_T|\mathcal{F}(t)] = \tilde{\mathbb{E}}[e^{-\int_t^T r(s,\omega)ds}S_T|\mathcal{F}(t)]. \tag{8.17}$$

8.2 Girsanov's theorem for a single Brownian motion

Before we present the martingale representation theory, we first discuss Girsanov theorem on how to change the probability measure from $\mathbb{P}$ to $\tilde{\mathbb{P}}$ so that the stochastic process $\tilde{W}(t) = W(t) + \int_0^t \Theta(s)ds$ becomes Brownian motion under the new measure $\tilde{\mathbb{P}}$.

It is helpful to image that Brownian motion $W(\omega, t)$ is the motion of a small particle ω (could be dust or pollen) in a bottle of water. Different ω corresponds to different particle. Standard Brownian motion means that the path on average has no drift (mean 0). But if you put the water bottle on a driving car, all the paths suddenly have a drift. In physics, you can change to another reference frame with does not move relative to the water bottle, and then the drifted Brownian motion becomes standard Brownian motion. In probability, we do not change the reference frame. Instead, we change the probability assigned to ω to eliminate the drift. The point is that you should not be surprised when you find that we can treat a drifted Brownian motion as a zero-drift Brownian motion when viewing it from another angle.

Recall that we have already seen examples of change of probability measure in Example 2.5 ($\mathbb{E}[\textcircled{D}_T] = p_u \textcircled{D}^u + p_d \textcircled{D}^d$) and Example 2.6 ($\tilde{\mathbb{E}}[\textcircled{D}_T] = q_u \textcircled{D}^u + q_d \textcircled{D}^d$). See also Question 13 of Homework IV.

Our discussion on $\tilde{\mathbb{P}}$ contains three levels: Let us start with a systematic presentation on how to change the expectation of a random variable on a finite probability space. Then we move on to change the expectation of a random variable on a general probability space. Finally we discuss how to change the expectation of a stochastic process (which is a sequence of random variables) on a general probability space.

Suppose that on a finite sample space Ω, there are two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ (see Definition 2.1) . Let us assume that $\mathbb{P}$ and $\tilde{\mathbb{P}}$ both give positive probability to every element of Ω, so that we can form the quotient

$$Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}. \tag{8.18}$$

Because it depends on ω, Z is a random variable. It is called the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$ (on a finite probability space). Z has three important properties which we state as a theorem:

Theorem 8.1 *(Theorem 3.1.1 of Shreve I) Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be probability measures on a finite sample space Ω, assume that $\mathbb{P}(\omega) > 0$ and $\tilde{\mathbb{P}}(\omega) > 0$ for every $\omega \in \Omega$, and define the random variable Z by (8.18). Then*

i) $\mathbb{P}(Z > 0) = 1$;

ii) $\mathbb{E}Z = 1$;

iii) for any random variable Y,
$$\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]. \tag{8.19}$$

Proof: (i) is obvious since for every $\omega \in \Omega$, $\tilde{\mathbb{P}}(\omega) > 0$, $\mathbb{P}(\omega) > 0$. (ii), (iii) follow direct from computations:

$$\mathbb{E}Z = \sum_{\omega\in\Omega} Z(\omega)\mathbb{P}(\omega) = \sum_{\omega\in\Omega} \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}\mathbb{P}(\omega) = \sum_{\omega\in\Omega} \tilde{\mathbb{P}}(\omega) = 1.$$

$$\tilde{\mathbb{E}}[Y] = \sum_{\omega\in\Omega} Y(\omega)\tilde{\mathbb{P}}(\omega) = \sum_{\omega\in\Omega} Y(\omega)\frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}\mathbb{P}(\omega) = \sum_{\omega\in\Omega} Y(\omega)Z(\omega)\mathbb{P}(\omega) = \mathbb{E}[ZY]. \quad \square$$

If we are given a random variable Z with $\mathbb{E}[Z] = 1$, we can change from $\mathbb{P}$ to $\tilde{\mathbb{P}}$ by reassigning probabilities in Ω using Z. Now we state this result for the more general probability space.

Theorem 8.2 *(Theorem 1.6.1 of Shreve II) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely nonnegative random variable with $\mathbb{E}[Z] = 1$. For $A \in \mathcal{F}$, define*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega). \tag{8.20}$$

Then $\tilde{\mathbb{P}}$ is a probability measure with $d\tilde{\mathbb{P}}(\omega) = Z(\omega)\mathbb{P}(\omega)$. If Y is a random variable, then

$$\tilde{\mathbb{E}}[Y] = \int_\Omega Y(\omega)d\tilde{\mathbb{P}}(\omega) = \int_\Omega Y(\omega)Z(\omega)d\mathbb{P}(\omega) = \mathbb{E}[YZ]. \tag{8.21}$$

Example 8.2 *(Example 1.6.6 of Shreve II) Let X be a standard normal random variable $\sim N(0,1)$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let Θ be a constant and define $Y = X + \Theta$. Then $\mathbb{E}[Y] = \Theta$ and Var$[Y] = 1$. We want to change the distribution of Y without changing the random variable Y. Namely, we won't change the mapping $Y : \omega \to Y(\omega)$, we only chance the $\mathbb{P}$ in $(\Omega, \mathcal{F}, \mathbb{P})$.* [67].

We first define the positive random variable

$$Z(\omega) = e^{-\Theta X(\omega) - \frac{1}{2}\Theta^2}. \tag{8.22}$$

[68] *Note that $\mathbb{E}[Z] = 1$ as*

$$\mathbb{E}[Z] = \int_{\mathbb{R}} e^{-\Theta x - \frac{1}{2}\Theta^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\Theta)^2}{2}} dx = 1.$$

[69] *Now we use Z to create a new probability measure $\tilde{\mathbb{P}}$ by adjusting the probabilities of events in Ω. We do this by defining*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)d\mathbb{P}(\omega) \qquad \forall A \in \mathcal{F}. \tag{8.23}$$

[67] The change of the actual to the risk-neutral probability measure changes the distribution of asset prices without changing the asset prices themselves.

[68] This makes sense because, let's say, $\Theta > 0$, the $\tilde{\mathbb{P}}$ defined in (8.23) gives more weights to negative X than positive X. That reduces the expectation of X and eventually makes $\tilde{\mathbb{E}}Y = 0$.

[69] Note how we choose the exponent so that we can exactly complete the squares.

We claim that $\tilde{\mathbb{P}}(Y \le b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{y^2}{2}} dy$. *Then* Y *is a standard normal random variable under* $\tilde{\mathbb{P}}$. *The claim follows from direct computation:*

$$\begin{aligned}
\tilde{\mathbb{P}}(Y \le b) &= \int_{\{\omega: Y(\omega) \le b\}} Z(\omega) d\mathbb{P}(\omega) = \int_{\Omega} \mathbb{I}_{\{X(\omega)+\Theta \le b\}} e^{-\Theta X(\omega) - \frac{1}{2}\Theta^2} d\mathbb{P}(\omega) \\
&\overset{(3.10)}{=} \int_{-\infty}^{\infty} \mathbb{I}_{\{x+\Theta \le b\}} e^{-\Theta x - \frac{1}{2}\Theta^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \int_{-\infty}^{\infty} \mathbb{I}_{\{x+\Theta \le b\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\Theta)^2}{2}} dx \\
&= \int_{-\infty}^{\infty} \mathbb{I}_{\{y \le b\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\frac{y^2}{2}} dy.
\end{aligned}$$

Recall the change of variable formula in calculus $\int_{I_x} f(\phi(x)) \frac{d\phi}{dx} dx = \int_{I_y} f(y) dy$, where $y = \phi(x)$ maps interval I_x to interval I_y. Letting $f = 1$, we get

$$\int_{I_y} dy = \int_{I_x} \frac{d\phi}{dx} dx = \int_{I_x} \frac{dy}{dx} dx.$$

Since (8.20) can be rewritten as

$$\int_A d\tilde{\mathbb{P}}(\omega) = \tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \qquad \forall A \in \mathcal{F}, \tag{8.24}$$

we denote $Z(\omega)$ as $\frac{d\tilde{\mathbb{P}}(\omega)}{d\mathbb{P}(\omega)}$ and call it the Randon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$.

So far we only discussed the change of measure for random variables. To find the $\tilde{\mathbb{P}}$ so that $\tilde{W} = W(t) + \int_0^t \Theta(s) ds$ is a martingale, we have to deal with stochastic processes (which are sequences of random variables) and need the Girsanov theorem.

Theorem 8.3 *(Girsanov theorem) Let* $W(t)$, $0 \le t \le T$, *be a Brownian motion on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, *and let* $\mathcal{F}(t)$, $0 \le t \le T$, *be a filtration for this Brownian motion (Definition 4.1) . Let* $\Theta(t)$, $0 \le t \le T$, *be an* $\mathcal{F}_t$*-adapted process (Definition 2.9) . Define*

$$Z(t) = e^{-\int_0^t \Theta(s,\omega) dW(s) - \frac{1}{2}\int_0^t \Theta^2(s,\omega) ds}, \tag{8.25}$$

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(s,\omega) ds, \tag{8.26}$$

and assume that $\mathbb{E}\int_0^T \Theta^2(s) Z^2(s) ds < \infty$. *Set*

$$Z = Z(T). \tag{8.27}$$

Then

$$\mathbb{E}Z = 1$$

and under the probability measure $\tilde{\mathbb{P}}$ given by

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \qquad \forall A \in \mathcal{F}, \tag{8.28}$$

the process $\tilde{W}(t)$, $0 \le t \le T$, is a Brownian motion.

Example 8.3 *Now, let us look at Question 10 of Homework III from Girsanov's point of view. Recall that in that problem, we have a standard Brownian motion W_t and its shift $B_t = W_t + \theta t$. We have proved*

$$\mathbb{E}[h(B_{[0,T]})] = \mathbb{E}[e^{\theta W_T - \frac{1}{2}\theta^2 T} h(W_{[0,T]})]$$

with $h(W_{[0,T]}) \stackrel{def}{=} \max_{0 \le t \le T} W_t - \min_{0 \le t \le T} W_t - W_T$. The proof over there is very elementary and does not use any big theorem.

But that factor $e^{\theta W_T - \frac{1}{2}\theta^2 T}$ should look familiar to you now. So, let us try Girsanov. Define $Z = e^{-\theta W_T - \frac{1}{2}\theta^2 T}$ and $\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$. By Girsanov Theorem, B_t is a Brownian motion under the new measure $\tilde{\mathbb{P}}$. Furthermore, $\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$ implies $d\tilde{\mathbb{P}}(\omega) = Z(\omega) d\mathbb{P}(\omega)$. Hence for any X,

$$\mathbb{E}[X] = \int_\Omega X(\omega) d\mathbb{P}(\omega) = \int_\Omega X(\omega) Z^{-1} d\tilde{\mathbb{P}}(\omega) = \tilde{\mathbb{E}}[X Z^{-1}]$$

which is the companion formula of (8.21). Now, let $X = h(B_{[0,T]})$. $Z^{-1} = e^{\theta W_T + \frac{1}{2}\theta^2 T} = e^{\theta(B_T - \theta T) + \frac{1}{2}\theta^2 T} = e^{\theta B_T - \frac{1}{2}\theta^2 T}$.

$$\mathbb{E}[h(B_{[0,T]})] = \tilde{\mathbb{E}}[e^{\theta B_T - \frac{1}{2}\theta^2 T} h(B_{[0,T]})] = \mathbb{E}[e^{\theta W_T - \frac{1}{2}\theta^2 T} h(W_{[0,T]})].$$

In the last step, we have used the fact that B_t is a Brownian motion under $\tilde{\mathbb{P}}$ hence has the same distribution as W_t under $\mathbb{P}$.

Proof of Girsanov Theorem: (The proof won't be tested.) Using Itô formula, it is easy to verify that

$$dZ(t) = -\Theta(t) Z(t) dW(t).$$

Hence

$$Z(t) = Z(0) - \int_0^t \Theta(u) Z(u) dW(u).$$

Since Itô integral is a martingale (Theorem 4.4) , so is $Z(t)$. Hence

$$Z(t) = \mathbb{E}[Z(T) | \mathcal{F}(t)] \tag{8.29}$$

and $\mathbb{E}Z = Z(0) = 1$.

Next, we show that $\tilde{W}(t)Z(t)$ is a martingale under $\mathbb{P}$: By the product rule,

$$\begin{aligned}
&d(\tilde{W}(t)Z(t))\\
=&\tilde{W}(t)dZ(t) + Z(t)d\tilde{W}(t) + d\tilde{W}(t)dZ(t)\\
=&-\tilde{W}(t)\Theta(t)Z(t)dW(t) + Z(t)(dW(t) + \Theta(t)dt)\\
&+ (dW(t) + \Theta(t)dt)(-\Theta(t)Z(t)dW(t))\\
=&(-\tilde{W}(t)\Theta(t) + 1)Z(t)dW(t).
\end{aligned} \tag{8.30}$$

Hence $\tilde{W}(t)Z(t)$ is a martingale under $\mathbb{P}$.

We will use Levy's criteria , Theorem 3.5, to prove that $\tilde{W}$ is a Brownian motion. For that, we need to verify two things: Firstly, we need to prove that $\tilde{W}(s)$ is a martingale under $\tilde{\mathbb{P}}$, i.e., when $0 \le s \le t \le T$,

$$\tilde{W}(s) = \tilde{\mathbb{E}}[\tilde{W}(t)|\mathcal{F}_s] \tag{8.31}$$

where $\tilde{\mathbb{E}}$ is induced by the $\tilde{\mathbb{P}}$ in (8.28).

Secondly (recall the quadratic variation $[\tilde{W}, \tilde{W}]$ defined in Question 6 of Homework III), we need to check that

$$[\tilde{W}, \tilde{W}](t) \overset{\text{def}}{=} \lim_{\max_j |t_{j+1}-t_j|\to 0} \sum_{j=0}^{n-1} |\tilde{W}(t_{j+1}, \omega) - \tilde{W}(t_j, \omega)|^2 = t \text{ almost surely.} \tag{8.32}$$

Proof of (8.31): First, we use iterated conditioning: since $s \le t$ and $Z \overset{(8.27)}{=} Z(T)$,

$$\begin{aligned}
\mathbb{E}[\tilde{W}(t)Z|\mathcal{F}(s)] &\overset{\text{iterated conditioning}}{=} \mathbb{E}[\mathbb{E}[\tilde{W}(t)Z(T)|\mathcal{F}(t)]|\mathcal{F}(s)]\\
&\overset{\tilde{W}(t) \text{ is } \mathcal{F}(t) \text{ measurable}}{=} \mathbb{E}[\tilde{W}(t)\mathbb{E}[Z(T)|\mathcal{F}(t)]|\mathcal{F}(s)]\\
&\overset{(8.29)}{=} \mathbb{E}[\tilde{W}(t)Z(t)|\mathcal{F}(s)]\\
&\overset{(8.30)}{=} \tilde{W}(s)Z(s).
\end{aligned} \tag{8.33}$$

Next, we use the conditional expectation formula (4.53) in Question 13 of Homework IV:

$$\tilde{\mathbb{E}}[\tilde{W}(t)|\mathcal{F}(s)] \overset{(4.53)}{=} \frac{1}{\mathbb{E}[Z|\mathcal{F}(s)]}\mathbb{E}[\tilde{W}(t)Z|\mathcal{F}(s)] \tag{8.34}$$

$$\overset{(8.33),(8.29)}{=} \frac{1}{Z(s)}\tilde{W}(s)Z(s) = \tilde{W}(s). \tag{8.35}$$

Proof of (8.32): (8.32) is often expressed shortly as $d\tilde{W}(t)d\tilde{W}(t) = dt$[70]. By Theorem 5.1, we know $dtdW(t) = 0 = dtdt$. Hence

$$d\tilde{W}(t)d\tilde{W}(t) = (dW(t) + \Theta(t)dt)(dW(t) + \Theta(t)dt) = dW(t)dW(t) = dt.$$

This proves (8.32) and hence finishes the proof of Theorem 8.3.

[70] see (4.53) in the proof of Itô formula, Theorem 5.1, which explain the real meaning of $(dW_t)^2 = dt$

8.3 Value of portfolio process under the risk-neutral measure

Now, we show that we can choose Δ and $\Phi(0)$ so that the Φ in (8.10) satisfies (8.13). For that, we need the following martingale representation theorem whose proof is beyond the scope of this course. But I hope you can remember the result which is not hard because it simply says that Itô integral $\Leftrightarrow$ martingale.

Theorem 8.4 *(Theorem 5.3.1 of Shreve II) Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \le t \le T$ be the filtration generated by this Brownian motion. Let $M(t)$ be a martingale with respect to this filtration (i.e., for every t, $M(t)$ is $\mathcal{F}(t)$ measurable and for $0 \le s \le t \le T$, $\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s)$). Then there is an adapted process $\Gamma(u)$, $0 \le u \le T$, such that*

$$M(t) = M(0) + \int_0^t \Gamma(u)dW(u), \qquad 0 \le t \le T. \tag{8.36}$$

When combining Theorems 8.3 and 8.4 (there is a small technical issue we did not mention but can be found in the remark after Corollary 5.3.2 of Shreve II) , we get

Corollary 8.1 *(Corollary 5.3.2 of Shreve II) Let $W(t)$, $0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \le t \le T$ be the filtration generated by this Brownian motion. Let $\Theta(t)$, $0 \le t \le T$, be an adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right\}, \tag{8.37}$$

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)du, \tag{8.38}$$

and assume that $\mathbb{E}\int_0^T \Theta^2(u)Z^2(u)du < \infty$. Set $Z = Z(T)$. Then $\mathbb{E}[Z] = 1$, and under the probability measure $\tilde{\mathbb{P}}$ given by

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega)dP(\omega) \qquad \text{for all } A \in \mathcal{F},$$

the process $\tilde{W}(t)$, $0 \le t \le T$, is a Brownian motion. If $\tilde{M}(t)$, $0 \le t \le T$, is a martingale under $\tilde{\mathbb{P}}$, then there is an adapted process $\tilde{\Gamma}(u)$, $0 \le u \le T$, such that

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u), \qquad 0 \le t \le T. \tag{8.39}$$

Example 8.4 *Let $\{W_t : t \ge 0\}$ be standard Brownian motion with $W_0 = 0$. Suppose M_t is a martingale with respect to filtration $\mathcal{F}_t$ defined in Definition 4.1. By the martingale representation Theorem 8.4, there is an adapted process Γ_u such that*

$$M_t = M_0 + \int_0^t \Gamma_u dW_u.$$

Find Γ_u for the following M_t:

a) $M_t = W_t^2 - t$.

b) $M_t = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$.

Solution *Note that* $\Gamma_t dW_t = dM_t$. *At the same time, we can use Itô formula to compute* dM_t.

(a) $dM_t = d(W_t^2 - t) = 2W_t dW_t + (dW_t)^2 - dt = 2W_t dW_t$. *(By the way, this calculation also shows that* M_t *is a martingale.) Hence* $\Gamma_t = 2W_t$ *or* $\Gamma_u = 2W_u$.

(b) $X_t = \sigma W_t - \frac{1}{2}\sigma^2 t$. *Apply Itô formula to* de^{X_t}. $dM_t = de^{\sigma W_t - \frac{1}{2}\sigma^2 t} = e^{X_t} dX_t + \frac{1}{2}e^{X_t}(dX_t)^2 = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}(\sigma dW_t - \frac{1}{2}\sigma^2 dt) + \frac{1}{2}e^{\sigma W_t - \frac{1}{2}\sigma^2 t}(\sigma dW_t - \frac{1}{2}\sigma^2 dt)^2 = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\sigma dW_t$. *Hence* $\Gamma_u = \sigma e^{\sigma W_u - \frac{1}{2}\sigma^2 u}$.

Now, following the idea of Section 5.2.3 of Shreve II and Section 16.1 of Choe, we move back to (8.13) and show that we can choose Δ and $\Phi(0)$ so that the Φ in (8.10) satisfies (8.13), i.e., $\Phi(T) = ⓟ_T$ almost surely.

On the one hand, by (8.12), the discounted value of the portfolio is a martingale under $\tilde{\mathbb{P}}$ and

$$D(t)\Phi(t) = \Phi(0) + \int_0^t \Delta(u)\sigma(u)D(u)S(u)d\tilde{W}(u), \tag{8.40}$$

where $\tilde{W}$ is a Brownian motion under $\tilde{\mathbb{P}}$.

On the other hand, recall that we are given an $\mathcal{F}(T)$-measurable random variable $ⓟ_T$. Define $ⓟ(t)$ by

$$D(t)ⓟ(t) = \tilde{\mathbb{E}}[D(T)ⓟ_T|\mathcal{F}(t)]. \tag{8.41}$$

Then, by iterated conditioning, we know that for any $s \le t \le T$,

$$\begin{aligned}\tilde{\mathbb{E}}[D(t)ⓟ(t)|\mathcal{F}(s)] &\overset{(8.41)}{=} \tilde{\mathbb{E}}[\tilde{\mathbb{E}}[D(T)ⓟ(T)|\mathcal{F}(t)]|\mathcal{F}(s)] \\ &\overset{\text{iterated conditioning}}{=} \tilde{\mathbb{E}}[D(T)ⓟ(T)|\mathcal{F}(s)] = D(s)ⓟ(s).\end{aligned}$$

Hence $D(t)ⓟ(t)$ is a martingale. By Corollary 8.1, we know there is an $\mathcal{F}_t$-adapted process $\tilde{\Gamma}(t)$ such that

$$D(t)ⓟ(t) = ⓟ(0) + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u). \tag{8.42}$$

Comparing (8.42) and (8.40), we see that if we choose $\Delta(u)$ so that

$$\tilde{\Gamma}(u) = \Delta(u)\sigma(u)D(u)S(u), \qquad 0 \le u \le T, \tag{8.43}$$

and if we choose $\Phi(0) = ⓟ(0)$, then we have $\Phi(T) = ⓟ_T$.

Then we can continue with (8.14) and derive (8.16)

$$ⓟ(t) = \tilde{\mathbb{E}}[e^{-\int_t^T r(u)du}ⓟ_T|\mathcal{F}(t)], \qquad 0 \le t \le T, \tag{8.44}$$

which is (8.6), the continuous-time analogue of (2.36).

Note that since $\Theta(t) = \frac{\alpha(t)-r(t)}{\sigma(t)}$ and $d\tilde{W}(t) = \Theta(t)dt + dW(t)$, the $S(t)$ in (8.3) satisfies

$$dS(t) = S(t)\left(\alpha(t)dt + \sigma(t)dW(t)\right) = S(t)\left(r(t)dt + \sigma(t)d\tilde{W}\right). \tag{8.45}$$

The drift in (8.3) is irrelevant to the asset price dynamics under the risk-neutral measure. This explains the name "risk-neutral". In a risk-neutral world, the rate of return on risky assets would be the same as the risk-free rate.

Example 8.5 *Recall that we have mentioned after equation (8.13) that* ⓟ$_T$ *does not have to be the pay-off of an option at time T. It can be any $\mathcal{F}_T$ measurable random variable. In particular, it can be* 1 *at time T. Correspondingly, we obtain the price at time t of a zero-coupon bound pay 1 dollar at time T:*

$$B(t,T) = \tilde{\mathbb{E}}[e^{-\int_t^T r(u)du}|\mathcal{F}(t)], \qquad 0 \le t \le T \tag{8.46}$$

by (8.44).

One should also note that if we choose ⓟ$_T = S(T)$, *(8.44) leads to (8.17):*

$$S(t) = \tilde{\mathbb{E}}[e^{-\int_t^T r(s,\omega)ds}S(T)|\mathcal{F}(t)]. \tag{8.47}$$

8.4 Homework VIII

(Only submit solutions to Questions 2,5,7.)

1. (Black-Scholes-Merton formula for time-varying, nonrandom interest rate and volatility) Consider a stock whose price differential is

$$dS(t) = r(t)S(t)dt + \sigma(t)S(t)d\tilde{W}(t),$$

where $r(t)$ and $\sigma(t)$ are nonrandom functions of t and $\tilde{W}$ is a Brownian motion under the risk-neutral measure $\tilde{\mathbb{P}}$. Let $T > 0$ be given, and consider a European call, whose value at time zero is

$$c(0, S(0)) = \mathbb{E}\left[e^{-\int_0^T r(t)dt}(S(T) - K)^+\right].$$

a) Show that $S(T)$ is of the form $S(0)e^X$, where X is a normal random variable, and determine the mean and variance of X.

b) Define

$$\begin{aligned} BSM(T, x; K, R, \Sigma) =& xN\left(\frac{1}{\Sigma\sqrt{T}}\left[\log\frac{x}{K} + (R + \frac{\Sigma^2}{2})T\right]\right) \\ &- e^{-RT}KN\left(\frac{1}{\Sigma\sqrt{T}}\left[\log\frac{x}{K} + (R - \frac{\Sigma^2}{2})T\right]\right) \end{aligned} \tag{8.48}$$

denote the value at time zero of a European call expiring at time T when the underlying stock has constant volatility Σ and the interest rate R is constant. Show that

$$c(0, S(0)) = BSM\left(T, S(0); K, \frac{1}{T}\int_0^T r(t)dt, \left(\frac{1}{T}\int_0^T \sigma^2(t)dt\right)^{1/2}\right). \tag{8.49}$$

Solution: a)

$$d\log S_t = \frac{dS_t}{S_t} - \frac{1}{2}\frac{1}{S_t^2}(dS_t)^2 = \left(r(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)d\tilde{W}_t$$

So,

$$S_T = S_0 \exp\left(\int_0^T \left(r(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)d\tilde{W}_t\right) = S_0e^X,$$

where $X = \int_0^T \left(r(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)d\tilde{W}_t$ is a normal random variable. With the risk-neutral measure,

$$\tilde{\mathbb{E}}X = \int_0^T \left(r(t) - \frac{1}{2}\sigma^2(t)\right)dt, \qquad \tilde{\mathrm{Var}}[X] \overset{\text{Itô isometry}}{=} \int_0^T \sigma^2(t)dt.$$

b) It follows from the same calculation as in (3.13).

2. (Every strictly positive asset is a generalized geometric Brownian motion) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which is defined a Brownian motion $W(t)$, $0 \leq t \leq T$. Let $\mathcal{F}(t)$, $0 \leq t \leq T$, be the filtration generated by this Brownian motion. Introduce the $\tilde{W}$ and $\tilde{\mathbb{P}}$ as in (8.9) and (8.28). Now, let V_T (called Ⓓ$_T$ in Section 8.3) be an almost surely positive $\mathcal{F}(T)$-measurable random variable. By (8.16), the price at time t of a security paying V_T at time T is

$$V(t) = \tilde{\mathbb{E}}\left[e^{-\int_t^T r(u)du} V_T | \mathcal{F}(t)\right], \tag{8.50}$$

where $r(t)$ is the risk-free interest rate which can be a stochastic process. In this problem, we drop all the ω's in the expression and write $V(t, \omega)$, $r(t, \omega)$ simply as $V(t)$, $r(t)$.

a) Show that there exists an adapted process $\tilde{\Gamma}(t)$, $0 \leq t \leq T$, such that

$$dV(t) = r(t)V(t)dt + \frac{\tilde{\Gamma}(t)}{D(t)} d\tilde{W}(t), \qquad 0 \leq t \leq T, \tag{8.51}$$

where $D(t) = e^{-\int_0^t r(s)ds}$ is the discount process [Hint: First, show that $D(t)V(t) = \tilde{\mathbb{E}}\left[D(T)V(T)|\mathcal{F}(t)\right]$ and then show that $D(t)V(t)$ is a martingale by iterated conditioning. Apply the martingale representation theorem or Corollary 8.1 to $D(t)V(t)$.]

b) By (8.50) and $V_T > 0$ almost surely, one can prove that $V(t) > 0$ almost surely. You do not need to prove this fact. Use this fact and (8.51), prove that there exists a process $\sigma(t)$, $0 \leq t \leq T$, such that

$$dV(t) = r(t)V(t)dt + \sigma(t)V(t)d\tilde{W}(t), \qquad 0 \leq t \leq T. \tag{8.52}$$

[Hint: Can you rewrite $\frac{\tilde{\Gamma}(t)}{D(t)}$ as $\sigma(t)V(t)$ for some $\sigma(t)$?] (You do not need to prove, but it is clear that $\sigma(t)$ is an adapted stochastic process.) In other words, prior to time T, the price of every asset with almost surely positive price at time T follows a generalized (because the volatility may be random) geometric Brownian motion.

3. (Hedging a cash flow) Let $W(t)$, $0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \leq t \leq T$, be the filtration generated by this Brownian motion. Let the mean rate of return $\alpha(t)$, the interest rate $r(t)$, and the volatility $\sigma(t)$ be adapted stochastic processes, and assume that $\sigma(t)$ is never zero. Consider a stock price process whose differential is given by

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \qquad 0 \leq t \leq T. \tag{8.53}$$

Suppose an agent must pay a cash flow at rate $C(t)$ at each time t, where $C(t)$, $0 \leq t \leq T$, is an adapted stochastic process. He wants to replicate the cash flow with a portfolio $X(t) = \Delta(t)S(t) + B(t)$ where $\Delta(t)$ is the shares of stock the agent hold at each time t and $B(t)$ is the money market account. The agent will withdraw at rate $C(t)$ dollars per unit time. The differential of the agent's portfolio value will be

$$dX(t) = \underbrace{\Delta(t)dS(t) + r(t)[X(t) - \Delta(t)S(t)]dt}_{\text{gain process}} - \underbrace{C(t)dt}_{\text{wealth process}}. \tag{8.54}$$

Show that there is a nonrandom value of $X(0)$ and a stochastic process $\Delta(t)$, $0 \le t \le T$, such that $X(T) = 0$ almost surely. (Hint: You should try to understand and then mimic the steps of how to choose $\Delta(t)$ in (8.43): Define the risk-neutral measure and apply Corollary 8.1 of the Martingale Representation Theorem to the process

$$\tilde{M}(t) = \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du|\mathcal{F}(t)\right], \qquad 0 \le t \le T, \tag{8.55}$$

where $D(t)$ is the discount process (8.7).)

Solution: We introduce Θ and $\tilde{W}$ as in (8.9). Then define the risk-neutral measure $\tilde{\mathbb{P}}$ as in (8.28). Then $\tilde{W}$ is a Brownian motion under $\tilde{\mathbb{P}}$ and (8.53) becomes $dS(t) = r(t)S(t)dt + \sigma S(t)d\tilde{W}(t)$. Given $C(t)$, if we can find $\Delta(t)$ and $B(0)$ so that

$$X(t) = \Delta(t)S(t) + B(t)$$

is a solution to (8.54) with terminal condition $X(T) = 0$, then we can use $X(0)$ (with $B(0) = X(0) - \Delta(0)S(0)$) dollars to build up a portfolio which exactly generates the cash flow $C(t)$ over the time interval $[0, T]$. Hence $X(0)$ is the no-arbitrage price of the cash flow $C(t)$. Like in (8.12), we know that (8.54) is equivalent[71] to

$$\begin{aligned} d(D(t)X(t)) &= D(t)dX(t) - X(t)r(t)D(t)dt \\ &\overset{(8.54)}{=} D(t)\Delta(t)dS(t) - D(t)r(t)\Delta(t)S(t)dt - D(t)C(t)dt \\ &= D(t)\Delta(t)\sigma(t)S(t)d\tilde{W} - D(t)C(t)dt, \end{aligned} \tag{8.56}$$

or, in integral form (with $X(T) = 0$), $-D(0)X(0) = \int_0^T D(t)\Delta(t)\sigma(t)S(t)d\tilde{W} - D(t)C(t)dt$,

$$\int_0^T D(t)C(t)dt = D(0)X(0) + \int_0^T D(t)\Delta(t)\sigma(t)S(t)d\tilde{W}. \tag{8.57}$$

So, as long as we can find $\Delta(t)$, $0 \le t \le T$ and $X(0)$ so that (8.57) is true, we have a solution to (8.54) with $X(T) = 0$.

(8.57) tells us that $\int_0^T D(t)C(t)dt$ has an Itô integral or martingale representation. So, we introduce $\tilde{M}(t)$ by (8.55)

$$\tilde{M}(t) = \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du|\mathcal{F}(t)\right], \qquad 0 \le t \le T,$$

[71] we will show the equivalence in the subsequent proof.

which is a martingale by the iterated conditioning property $\tilde{\mathbb{E}}[\tilde{M}(t)|\mathcal{F}(s)] = \tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}\Big[\int_0^T D(u)C(u)du|\mathcal{F}(t)\Big]|\mathcal{F}(s)\Big] = \tilde{\mathbb{E}}\left[\int_0^T D(u)C(u)du|\mathcal{F}(s)\right] = \tilde{M}(s)$ when $s \le t$. By Corollary 8.1, we can find an adapted process $\tilde{\Gamma}(u)$ so that

$$\tilde{M}(t) = \tilde{M}(0) + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u), \qquad 0 \le t \le T.$$

Then we define

$$\Delta(t) = \frac{\tilde{\Gamma}(t)}{D(t)\sigma(t)S(t)}.$$

One can check that the $X(t)$ defined by $X(0) = \tilde{M}(0) = \mathbb{E}[\int_0^T D(u)C(u)du]$ (does the price makes sense to you for random cash flow $C(t)$?) and

$$X(t) = \frac{1}{D(t)}\left(X(0)\underbrace{D(0)}_{=1} + \int_0^t \tilde{\Gamma}(u)d\tilde{W}(u) - \int_0^t D(u)C(u)du\right)$$

satisfies (8.56),

$$d(D(t)X(t)) = D(t)\Delta(t)\sigma(t)S(t)d\tilde{W} - D(t)C(t)dt.$$

The last equation is equivalent to

$$dX(t) = r(t)X(t)dt + \Delta(t)\underbrace{\sigma(t)S(t)d\tilde{W}(t)}_{dS(t)-r(t)S(t)dt} - C(t)dt,$$

or,

$$dX(t) = \Delta(t)dS(t) + r(t)[X(t) - \Delta(t)S(t)]dt - C(t)dt$$

which is precisely (8.53). Finally,

$$\begin{aligned} X(T) &= \frac{1}{D(T)}\left(\tilde{M}(0) + \int_0^T \tilde{\Gamma}(u)d\tilde{W}(u) - \int_0^T D(u)C(u)du\right) \\ &= \frac{1}{D(T)}\left(\tilde{M}(T) - \int_0^T D(u)C(u)du\right) = 0. \end{aligned}$$

4. Consider a Ho-Lee model where the risk-free interest rate $r(t)$ satisfies the stochastic differential equation

$$dr_t = \theta_t dt + \sigma d\tilde{W}_t$$

under the risk-neutral probability measure. Assume $\tilde{W}_0 = 0$, σ is a constant, and θ_t is a deterministic function. Compute the price of a zero-coupon bond with payoff \$1 and maturity T. In other words, by (8.16) with $ⓓ_T = 1$ or (8.50) with $V_T = 1$, you are asked to compute

$$V_0 = \tilde{\mathbb{E}}[e^{-\int_0^T r_t dt}|\mathcal{F}(0)] = \tilde{\mathbb{E}}[e^{-\int_0^T r_t dt}].$$

Solution: Observe that

$$\int_0^T r_t dt = \int_0^T \left(r_0 + \int_0^t \theta_s ds + \sigma \tilde{W}_s \right) dt = \mu + \sigma Y$$

where

$$\mu \stackrel{\text{def}}{=} r_0 T + \int_0^T \left(\int_0^t \theta_s ds \right) dt, \qquad Y \stackrel{\text{def}}{=} \int_0^T \tilde{W}_s ds.$$

Apparently, since θ_t is deterministic function, μ a number. Next, we claim that Y is normally distributed with mean zero and variance $\frac{1}{3}T^3$. (That is something that we have proved in the midterm.) From the definition of Riemann integral, we can see that $\int_0^T \tilde{W}_s ds = \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} \tilde{W}_{t_i} \Delta t$ with $N = \frac{T}{\Delta t}$. Since the sum of jointly normally distributed random variables is still normally distributed, and the limit remains to be normally distributed, $\int_0^T \tilde{W}_s ds$ is normally distributed. Or we can use the integration by parts formula (5.11) (with $\tilde{W}_0 = 0$)

$$\int_0^t f_s d\tilde{W}_s = f_t \tilde{W}_t - \int_0^t \tilde{W}_s df_s.$$

We get

$$Y = \int_0^T \tilde{W}_s ds = T\tilde{W}_T - \int_0^T s d\tilde{W}_s. \tag{8.58}$$

Hence Y is normally distributed with mean zero. To find the variance, we can use $\int_0^T \tilde{W}_s ds$:

$$\begin{aligned}
\tilde{\text{Var}}[Y] = \tilde{\mathbb{E}}[Y^2] &= \tilde{\mathbb{E}}\left[\left(\int_0^T \tilde{W}_s ds \right) \left(\int_0^T \tilde{W}_t dt \right) \right] = \tilde{\mathbb{E}}\left[\int_0^T \int_0^T \tilde{W}_s \tilde{W}_t ds dt \right] \\
&= \int_0^T \int_0^T \tilde{\mathbb{E}}\left[\tilde{W}_s \tilde{W}_t \right] ds dt = \int_0^T \int_0^T \min(s,t) ds dt \\
&= \int_0^T \left(\int_0^t s ds + \int_t^T t ds \right) dt = \int_0^T \left(\frac{t^2}{2} + (T-t)t \right) dt \\
&= T\frac{T^2}{2} - \frac{1}{2}\frac{T^2}{3} = \frac{T^3}{3}.
\end{aligned}$$

Alternative, we can use (8.58) and write $\tilde{W}_T = \int_0^T d\tilde{W}_s$ to get

$$\begin{aligned}
\tilde{\text{Var}}[Y] = \tilde{\mathbb{E}}[Y^2] &= \tilde{\mathbb{E}}\left[(T\tilde{W}_T)^2 \right] + \tilde{\mathbb{E}}\left[\left(\int_0^T s d\tilde{W}_s \right)^2 \right] - 2\tilde{\mathbb{E}}\left[T \left(\int_0^T d\tilde{W}_s \right) \left(\int_0^T s d\tilde{W}_s \right) \right] \\
&\stackrel{(4.13)}{=} T^3 + \int_0^T s^2 ds - 2T \int_0^T s ds = \frac{T^3}{3}.
\end{aligned}$$

By Question 1 of Homework III, we find that

$$V_0 = \tilde{\mathbb{E}}[e^{-\mu - \sigma Y}] = e^{-\mu + \frac{1}{2}\sigma^2 \frac{T^3}{3}} = e^{-\mu + \frac{\sigma^2 T^3}{6}}.$$

5. Consider the $\{W_t\}$, $\mathbb{P}$, $\{\tilde{W}_t\}$, $Z_t = e^{-\int_0^t \Theta(s,\omega)dW_s - \frac{1}{2}\int_0^t \Theta^2(s,\omega)ds}$, and $\tilde{\mathbb{P}}$ in Theorem 8.3. Assume that $\tilde{M}_t = \int_0^t a(s,\omega)d\tilde{W}_s$ for some $\mathcal{F}_s$-adapted process $a(s,\omega)$. This implies that $\tilde{M}_t$ is a martingale under $\tilde{\mathbb{P}}$. Show that $\tilde{M}_t Z_t$ is a martingale under $\mathbb{P}$.

   [Hint: Show that $d(\tilde{M}_t Z_t) = \text{something} \times dW_t$.]

6. (Forward price) By definition, a forward contact is an agreement to pay a specified delivery price K at a delivery date T for the asset whose price at time t is $S(t)$. The T-forward price for this asset at time t, is the value of K that makes the forward contract have no-arbitrage price zero at time t. Determine K.

   Solution: We set $ⓟ_T = S(T) - K$ in (8.13) which is the pay-off of the forward contract at time T. Hence by (8.44), the value[72] of the forward contract at time t is $\tilde{\mathbb{E}}[e^{-\int_t^T r(s,\omega)ds}(S(T)-K)|\mathcal{F}_t]$. We need to choose K so that this value is zero. Hence

$$0 = \tilde{\mathbb{E}}[e^{-\int_t^T r(s,\omega)ds}(S(T)-K)|\mathcal{F}_t] = \tilde{\mathbb{E}}[e^{-\int_t^T r(s,\omega)ds}S(T)|\mathcal{F}_t] - K\tilde{\mathbb{E}}[e^{-\int_t^T r(s,\omega)ds}|\mathcal{F}_t].$$

   This gives

$$K = \frac{\tilde{\mathbb{E}}[e^{-\int_t^T r(s,\omega)ds}S(T)|\mathcal{F}_t]}{\tilde{\mathbb{E}}[e^{-\int_t^T r(s,\omega)ds}|\mathcal{F}_t]} = \frac{S(t)}{B(t,T)} \tag{8.59}$$

   by (8.46) and (8.47) in Example 8.5.

   The following is for your information only and won't be tested: An investor who buy a futures in an exchange is requested to deposit fund into a margin account to safeguard against the possibility of default. At the end of each trading day, the futures holder will pay to or receive from the writer the full amount of the change in the futures price from the previous day through the margin account. The futures price at delivery date T is set to be $S(T)$, being the value of the underlying financial asset or commodity at time T. This asset can be commodity, stock, currency, or bond. It turns out, the futures price at time t is

$$\tilde{\mathbb{E}}[S(T)|\mathcal{F}(t)]. \tag{8.60}$$

   Note that if $r(s,\omega)$ is a constant, forward price = futures price. Otherwise, since different interest rates applied to the intermediate payments, they might be different. If you are interested in the derivation, you can read Shreve II Page 244 or Page 172 of Duffie's "Dynamic Asset Pricing Theory".

7. (Chooser option) Consider a model with constant interest rate r. According to the risk-neutral pricing formula (8.44), for $0 \le t \le T$, the price at time t of a European call expiring at time T is

$$c(t) = \tilde{\mathbb{E}}[e^{-r(T-t)}(S(T)-K)^+|\mathcal{F}(t)]$$

[72] We have used the terms "price" and "value" interchangeably for options. But "forward price" and "forward value" are different quantifies for forward contracts.

where $S(T)$ is the underlying asset price at time T and K is the strike price of the call. Similarly, the price at time t of a European put expiring at time T is

$$p(t) = \tilde{\mathbb{E}}[e^{-r(T-t)}(K - S(T))^+|\mathcal{F}(t)].$$

Finally, because $e^{-rt}S(t)$ is a martingale under $\tilde{\mathbb{P}}$ ((8.47)), the price at time t of a forward contract for delivery of one share of stock at time T in exchange for a payment of K at time T is

$$f(t) = \mathbb{E}[e^{-r(T-t)}(S(T) - K)|\mathcal{F}(t)] = S(t) - e^{-r(T-t)}K.$$

Because

$$(S(T) - K)^+ - (K - S(T))^+ = S(T) - K,$$

we have the put-call parity relationship

$$\begin{aligned} c(t) - p(t) &= \tilde{\mathbb{E}}[e^{-r(T-t)}(S(T) - K)^+ - e^{-r(T-t)}(K - S(T))^+|\mathcal{F}(t)] \\ &= \tilde{\mathbb{E}}[S(T) - K|\mathcal{F}(t)] = f(t). \end{aligned}$$

Now consider a date t_0 between 0 and T, and consider a chooser option, which gives the right at time t_0 to choose to own either the call or the put.

a) $(t = t_0)$ Show that the value of the chooser option at time t_0 is

$$c(t_0) + \max(0, -f(t_0)) \quad \text{which is} \quad c(t_0) + \left(e^{-r(T-t_0)}K - S(t_0)\right)^+. \tag{8.61}$$

[Hint: For chooser option, the value at t_0 is $\max\{c(t_0), p(t_0)\}$.]

b) $(t = 0)$ Show that the value of the chooser option at time 0 is the sum of the value of a call expiring at time T with strike price K and the value of a put expiring at time t_0 with strike price $e^{-r(T-t_0)}K$.

[Hint: Use (8.44) with $T = t_0$, $Ⓟ_T$ = result from Part a).]