FE5208 Part II Assessment 1

Question 1（BK Model）The short rate in the Black–Karasinski (BK) model satisfies the stochastic differential equation

$$dr_t = \left[k\theta(t) + \frac{\sigma^2}{2} - k\ln r_t\right] r_t dt + \sigma r_t dW_t.$$

Please find the closed-form solution r_t.

Question 2 (Vasicek and CIR) For the CIR model

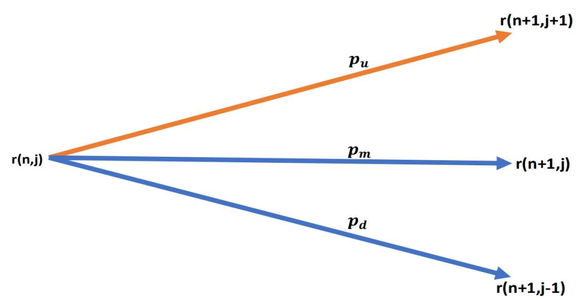
$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}\, dW_t,$$

what is the SDE for the process $y_t = \sqrt{r_t}$? Please find a condition such that the SDE for y_t is a special case of the Vasicek model.

Question 3（The Hull-White Trinomial Tree）Except on the boundaries, $j = -j_{\max}$ and $j = j_{\max}$, a trinomial tree building block for the Hull-White model

$$dr_t = k[\theta(t) - r_t]dt + \sigma dW_t,$$

is shown below:



where

$$r(n+1,j) = r(n+1,0) + \sqrt{3}\,\sigma j, \quad -j_{\max} < j < j_{\max}$$

$$p_u = \frac{1}{6} + \frac{j^2k^2 - jk}{2},$$

$$p_m = \frac{2}{3} - j^2k^2,$$

$$p_d = \frac{1}{6} + \frac{j^2k^2 + jk}{2}.$$

The value of $r(n+1,0)$ is determined by calibrating to a zero coupon bond. Although the branching probabilities p_u, p_m, p_d are state-dependent, i.e. functions of j, please prove that the tree is calibrated to the constant local volatility σ.