Lecture 2 - Discrete-Time Binomial Models

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Spring 2020

Ho and Lee Model (1986)

- Continuous-time short-rate process:

$$dr(t) = \theta(t)\, dt + \sigma dW_t.$$

- Features:
  - Symmetric ("bell shaped") distribution of rates in the future;
  - Time-dependent;
  - More flexible than Vasicek (lectures 4 and 5) and can be calibrated to market data by implying the form of $\theta(t)$ from market prices;
  - No mean-reverting;
  - Negative interest rate occurs with positive probability (Hull-White will take care of this).

- This lecture focuses on a discrete-time version.

Setup

- Dynamics of bond prices $P(t, T)$: $t = 1, 2, ...$ and $T = t, t+1, t+2, ...t+N$ where $P(t, t) = 1$.
- Data: A set of observed prices $P(0, T)$: $T = 1, 2, ...N$.
- Continuously compounding but the implied risk-free interest rate adjusts only at moment/date $t = 1, 2, ...$:

  $$r(t+s) = -\log P(t, t+1) \text{ for } 0 \leq s < 1,$$

  which is $\mathcal{F}_t$-measurable.

Cash Account

- Investing one dollar at $t = 0$:

$$B(0) = 1.$$

- Moreover,

$$B(t+1) = \exp\left[\int_0^{t+1} r(s)\, ds\right] = \exp\left[\sum_{s=0}^{t} r(s)\right].$$

- Hence,

$$\frac{B(t+1)}{B(t)} = \exp(r(t)) = \frac{1}{P(t, t+1)}.$$

Goal

- Develop a model of bond price process which is arbitrage-free.

- Trivial when the interest rate is deterministic.

$$
\begin{aligned}
F(0, T, T+1) &\equiv \log\left[\frac{P(0,T)}{P(0,T+1)}\right]; \\
P(t,T) &\equiv \exp\left[-\sum_{s=t}^{T-1} F(0,s,s+1)\right] = \frac{P(0,T)}{P(0,t)}.
\end{aligned}
$$

- We will model stochastic interest rate process:
  - First, a general approach from bond price process to short rate process.
  - Second, a reverse approach from (time-homogeneous) short rate process to bond price process.

Binomial Tree

- A natural candidate from binomial tree of stock prices:

$$P(1, T) = \begin{cases} u(0, T)\, \frac{P(0,T)}{P(0,1)}, & \text{“up”}; \\ d(0, T)\, \frac{P(0,T)}{P(0,1)}, & \text{“down”}. \end{cases}$$

- More generally,

$$P(t+1, T) = \begin{cases} u(t, T-t)\, \frac{P(t,T)}{P(t,t+1)}, & \text{“up”}; \\ d(t, T-t)\, \frac{P(t,T)}{P(t,t+1)}, & \text{“down”}. \end{cases}$$

- $u(t, s) > d(t, s)$ and both are assumed to be known at time t and independent of t.

- Write $u(s)$ and $d(s)$ and s denote the outstanding term to maturity. Set $u(1) = d(1) = 1$ so that $P(t, t) = 1$.

Fundamental Theorem of Asset Pricing

- FT(i): If the model is arbitrage-free, then,

  $$u(T) > 1 > d(T) > 0, \forall T \geq 2;$$

  moreover, there is some $q \in (0, 1)$ such that

  $$\frac{1 - d(T)}{u(T) - d(T)} \equiv q(T) = q$$

  where q defines the risk-neutral probability (denoted by Q) of "up".
- FT(ii): If there is some $q \in (0, 1)$ which is the probability of "up" such that

  $$E_Q\left(\frac{P(1, T)}{B(1)}\right) = \frac{P(0, T)}{B(0)} \text{ for all } T,$$

  then there is no arbitrage between time 0 and 1.

- From q, we have a linear relation between $u(T)$ and $d(T)$.

Proof of FT(ii)

- Suppose that

$$E_Q\left(\frac{P(1,T)}{B(1)}\right)=\frac{P(0,T)}{B(0)} \text{ for all } T,$$

- For any portfolio $\{x_T\}_{T=1}^{N}$, we have

$$\begin{aligned}
E_Q\left[\sum_{T=1}^{N} x_T P(1,T)\right] &= \sum_{T=1}^{N} x_T \frac{1}{P(0,1)} E_Q\left(\frac{P(1,T)}{B(1)}\right) \\
&= \sum_{T=1}^{N} x_T \frac{1}{P(0,1)} \frac{P(0,T)}{B(0)} \\
&= \frac{1}{P(0,1)} \sum_{T=1}^{N} x_T P(0,T).
\end{aligned}$$

- Hence, there is no arbitrage.

Proof of FT(i)

- We cannot have $u(T) > d(T) > 1$:

$$P(1,T) \geq d(T)\frac{P(0,T)}{P(0,1)} > \frac{P(0,T)}{P(0,1)} = P(0,T)\,B(1) \Leftrightarrow$$

$$\frac{P(1,T)}{P(0,T)} > B(1) \ \left(= \frac{B(1)}{B(0)} = P(0,1) \right),$$

  i.e., T-bond for sure pays out more than the cash account -> can short $P(0,1)$ and long $P(0,T)$ to arbitrage.

- Similarly, we can't have $1 > u(T) > d(T)$.

Proof of (i)

Define

$$P_{Q_T}\left(\text{“up”}\right) = \frac{1-d\left(T\right)}{u\left(T\right)-d\left(T\right)} = q\left(T\right), \text{ i.e.,}$$
$$q\left(T\right)u\left(T\right) + \left(1-q\left(T\right)\right)d\left(T\right) = 1.$$

Thus, we have

$$E_{Q_T}\left[P\left(1,T\right)|\mathcal{F}_0\right] = \frac{P\left(0,T\right)}{P\left(0,1\right)}.$$

Since $P\left(0,1\right) = \frac{B(0)}{B(1)}$, this means that

$$E_{Q_T}\left[\frac{P\left(1,T\right)}{B\left(1\right)}|\mathcal{F}_0\right] = \frac{P\left(0,T\right)}{B\left(0\right)}.$$

Similarly, $\frac{P(t,T)}{B(t)}$ is a Q_T-martingale.

Proof of FT(i)

$\underline{q(T) = q(2) \text{ for all } T \geq 3}$

Replicate $P(1,2)$ by using T-bond and cash: $xB(1) + yP(1,T)$:
At $t = 1$,

$$\begin{array}{rcl} \text{up} & \Rightarrow & xB(1) + yu(T)P(0,T)B(1) = u(2)P(0,2)B(1); \\ \text{down} & \Rightarrow & xB(1) + yd(T)P(0,T)B(1) = d(2)P(0,2)B(1); \end{array}$$

(recall $B(1) = \frac{1}{P(0,1)}$).

Proof of FT(i)

Thus, (typo corrected)

$$
\begin{aligned}
x^* &= \frac{\left(u(T)\,d(2) - d(T)\,u(2)\right) P(0,2)}{u(T) - d(T)};\\
y^* &= \frac{\left(u(2) - d(2)\right) P(0,2)}{\left(u(T) - d(T)\right) P(0,T)}.
\end{aligned}
$$

Moreover,

$$
x^* + y^* P(0,T) = \underbrace{\left[u(2)\,\frac{1 - d(T)}{u(T) - d(T)} + d(2)\,\frac{u(T) - 1}{u(T) - d(T)}\right]}_{=1} P(0,2)
$$

We have got

$$
q(T) = q(2) \text{ for each } T \geq 3.
$$

Lattice Rather Than Tree

- Goal: To make the bond price today depend only on the number of "up"s in the past but not the order.
- Computational efficiency.
- $P(t, T, i)$ denotes the bond price after having i "down" steps.
- Recombining binomial model for $t = 1$:

$$
\begin{aligned}
P(1, T, 0) &= u(T)\,\frac{P(0, T, 0)}{P(0, 1, 0)};\\
P(1, T, 1) &= d(T)\,\frac{P(0, T, 0)}{P(0, 1, 0)}.
\end{aligned}
$$

Recombining Binomial Model

- For $t = 2$,

$$
\begin{aligned}
P(2, T, 1) &= d(T-1)\left(u(T)\frac{P(0, T, 0)}{P(0,1,0)}\right)\frac{1}{P(1,2,0)} \text{ up-down} \\
&= u(T-1)\left(d(T)\frac{P(0, T, 0)}{P(0,1,0)}\right)\frac{1}{P(1,2,1)} \text{ down-up}
\end{aligned}
$$

- It follows that

$$
\frac{d(T-1)\,u(T)}{P(1,2,0)} = \frac{u(T-1)\,d(T)}{P(1,2,1)}.
$$

- That is,

$$
\frac{d(T)}{u(T)} = k\frac{d(T-1)}{u(T-1)} \text{ where } k = \frac{P(1,2,1)}{P(1,2,0)} = \frac{d(2)}{u(2)} \in (0,1).
$$

Recombining Binomial Model

To sum up,

$$\begin{aligned}\frac{d(T)}{u(T)} &= k^{T-1};\\ u(T) &= \frac{1}{(1-q)\,k^{T-1}+q};\\ d(T) &= \frac{k^{T-1}}{(1-q)\,k^{T-1}+q}\end{aligned}$$

where the latter two follow because $qu(T)+(1-q)\,d(T)=1$.

Forward Rate Curve

- "Instantaneous" forward rate

$$
\begin{aligned}
F(t, T-1, T) &= \log \frac{P(t, T-1)}{P(t, T)} \\
&= F(0, T-1, T) + \log \frac{u(T-t)}{u(T)} - D(t) \log k
\end{aligned}
$$

  where

$$D(t) = \#\text{ of "down"s between date 1 and date } t.$$

- $D(t)$ does not depend on T.

Forward Rate Curve

- Argue by induction. If $D(t+1) = D(t)$ ("up" at $t+1$), then

$$\begin{aligned}
& F(t+1, T-1, T, D(t+1)) \\
= \; & \log \frac{P(t+1, T-1, D(t+1))}{P(t+1, T, D(t+1))} \\
= \; & \log \frac{u(T-t-1)\, P(t, T-1, D(t)) \,/\, P(t, t+1, D(t))}{u(T-t)\, P(t, T, D(t)) \,/\, P(t, t+1, D(t))} \\
= \; & \log \frac{u(T-t-1)}{u(T-t)} + F(t, T-1, T, D(t)) \\
= \; & \log \frac{u(T-t-1)}{u(T-t)} + F(0, T-1, T) + \log \frac{u(T-t)}{u(T)} \\
& -D(t) \log k.
\end{aligned}$$

- **Exercise**: Derive the case with $D(t+1) = D(t) + 1$ ("down" at $t+1$).

Risk-Free Interest Rate

- $r(t)$ is a random walk with constant volatility but time-varying drift:

$$
\begin{aligned}
r(t) &= F(t, t, t+1) \\
&= F(0, t, t+1) + \log \frac{u(1)}{u(t+1)} - D(t) \log k \\
&= F(0, t, t+1) - \log u(t+1) - D(t) \log k \\
&= F(0, t, t+1) - \log d(t+1) + U(t) \log k
\end{aligned}
$$

(where $U(t) = \#$ of "up"s between date 1 and date t.)

- General framework but...
  - No mean-reverting;
  - Over a specific period of time it is necessary to place constraints on q and k so that $r(t)$ remains positive;
  - Still for each q and k, for all large enough t, $r(t)$ becomes negative with positive probability;
  - The model need not be time-homogeneous.

Example

	$T=1$	$T=2$	$T=3$	$T=4$
$P(0,T)$	0.94	0.9	0.87	0.84
$P(1,T)$	...	0.94 or 0.965	...	...

$$
\begin{aligned}
u(2) &= \frac{P(1,2,0)\,P(0,1)}{P(0,2)} = 1.007889; \\
d(2) &= \frac{P(1,2,1)\,P(0,1)}{P(0,2)} = 0.981778; \\
q &= \frac{1-d(2)}{u(2)-d(2)} = 0.697872; \\
k &= \frac{d(2)}{u(2)} = 0.974093.
\end{aligned}
$$

Example

- According to

$$
\begin{aligned}
u(T) &= \frac{1}{(1-q)\,k^{T-1}+q}; \\
d(T) &= \frac{k^{T-1}}{(1-q)\,k^{T-1}+q}.
\end{aligned}
$$

- We can compute

	$T=1$	$T=2$	$T=3$	$T=4$
$u(T)$	1	1.007889	1.015694	1.023414
$d(T)$	1	0.981778	0.963749	0.945917

Example

- Also recall

$$P\left(t,\,T,x\right)=\begin{cases} u\left(T-t+1\right)\frac{P(t-1,T,x)}{P(t-1,t,x)}, & \text{“up”};\\ d\left(T-t+1\right)\frac{P(t-1,T,x-1)}{P(t-1,t,x-1)}, & \text{“down”}.\end{cases}$$

- For instance,

$$\begin{aligned} P\left(1,4,0\right) &= u\left(4\right)\frac{P\left(0,4,0\right)}{P\left(0,1,0\right)}=1.023414\times\frac{0.84}{0.94}=0.914\,54;\\ P\left(1,4,1\right) &= d\left(4\right)\frac{P\left(0,4,0\right)}{P\left(0,1,0\right)}=0.945917\times\frac{0.84}{0.94}=0.84529.\end{aligned}$$

- **Exercise**: Compute $P\left(t,4,x\right)$ for $t=2,3$ and $x=0,1,2,3$.

Time Homogeneity

- The drift is time-dependent in the previous binomial model and hence the current $r(t)$ does not fully pin down future term structure.
- We now consider a time-homogeneous Markov model of short rate.
- Denote the set of possible interest rates by

$$A = \{..., r_{-2}, r_{-1}, r_0, r_1, r_2, ...\}$$

- Suppose that

$r(t+1) = r_{i-1}$ or r_{i+1} with (*real world*) probability one given $r(t)$

Time Homogeneity

- Recall when the interest rate process is deterministic, we have

$$P(t, T) \equiv \exp\left[-\sum_{s=t}^{T-1} F(0, s, s+1)\right].$$

- Now it is stochastic yet Markov (depending only on r_t). Hence, if $r(t) = r_i$, we may set

$$\begin{aligned} P(t, t+1, r(t)) &= e^{-r_i}; \\ P(t, t+2, r(t)) &= e^{-r_i}\left[q_i e^{-r_{i+1}} + (1-q_i)\, e^{-r_{i-1}}\right] \\ &\cdots \end{aligned}$$

for some $q_i \in (0, 1)$.

Fundamental Theorem Revisited

- More generally, we can prove

$$
\begin{aligned}
P(t,T) &= E_Q\left[\exp\left(-\sum_{s=t}^{T-1} r(s)\right)|\mathcal{F}_t\right] \\
&= E_Q\left[\exp\left(-\sum_{s=t}^{T-1} r(s)\right)|r(t)\right] \\
&= P(t,t+1)\,E_Q\left[P(t+1,T)\,|r(t)\right]
\end{aligned}
$$

where the (risk neutral) transition probability is

$$
\begin{aligned}
\Pr{}_Q\left[r(t+1)=r_{i+1}|r(t)=r_i\right] &= q_i; \\
\Pr{}_Q\left[r(t+1)=r_{i-1}|r(t)=r_i\right] &= 1-q_i.
\end{aligned}
$$

- The value of $P(t,T)$ given the value of $r(t)$ will be denoted as $P(t,T,r(t))$.

Random Walk

- An important special case where $q_i = q$ for each i is the probability of "up" with a "tick" size $\delta > 0$; moreover, each "up" or "down" is drawn independently of the history.
- How does the feature manifest itself in a Markov transition matrix?
- We have

$$\begin{aligned} P(0,2) &= P(0,1)\, E_Q\left[P(1,2)\,|\,r(0)\right] \\ &= e^{-r(0)}\left[q e^{-(r(0)+\delta)} + (1-q)\, e^{-(r(0)-\delta)}\right]. \end{aligned}$$

- Hence,

$$q = \frac{e^{-(r(0)-\delta)} - P(0,2)\, e^{r(0)}}{e^{-(r(0)-\delta)} - P(0,2)\, e^{-(r(0)+\delta)}}.$$

Fundamental Theorem Revisited

- We aim to show

$$P(t,T) = E_Q\left[\exp\left(-\sum_{s=t}^{T-1} r(s)\right)|r(t)\right] = E_Q\left[\frac{B(t)}{B(T)}|r(t)\right].$$

- This is equivalent to showing $Z(t,T) = D(t,T)$ where

$$\begin{aligned} Z(t,T) &= \frac{P(t,T)}{B(t)} \text{ is a martingale;} \\ D(t,T) &= E_Q\left[\frac{1}{B(T)}|r(t)\right] \text{ is also a martingale.} \end{aligned}$$

- Martingale representation theorem:

$$D(t,T) = D(0,T) + \sum_{s=1}^{t} \phi(s,T)\,\Delta Z(s,s+1)$$

where $\phi(s,T)$ is predictable process and $\Delta Z(s,s+1) \equiv Z(s,s+1) - Z(s-1,s+1)$.

A Portfolio which Replicates ZCB

- Buy at time $t-1$
  - $\phi(t,T)$ units of $P(t-1,t+1)$ and
  - $\psi(t,T)$ units of $B(t-1)$ such that:

$$\phi(t+1,T)\,P(t,t+2)+\psi(t+1,T)\,B(t)=B(t)\,D(t,T).$$

- The portfolio is self-financing and replicating $P(t,T)$ (since $B(T)\,D(T,T)=1$). Thus,

$$P(t,T)=B(t)\,D(t,T).$$