# FE5222 Advanced Derivative Pricing

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# Overview

# Introduction

Under risk neutral measure $\widetilde{\mathbb{P}}$, the discounted stock price $D(t)S(t)$ is a martingale.

$D(t)S(t)$ is in fact the stock price in the money market account numeraire as $D(t)S(t) = \frac{S(t)}{M(t)}$

The risk neutral measure $\widetilde{\mathbb{P}}$ is the probability measure under which stock prices in the unit of $M(t)$ is a martingale.

# Introduction

In this lecture, we will

- Extend to other numeraires
- For a given numeraire, find a probability measure under which stock prices in this particular numeraire are martingale
- Apply change of numeraire technique to price exotic options such as exchange option and Quanto options

# Introduction

Reasons for different numeraires

- Financial considerations
- Mathematically more convenient to work with.

# Model

In real world,

- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- $W(t) = (W_1(t), \ldots, W_d(t))$ Brownian motion
- $\{\mathcal{F}\}_{t \geq 0}$ filtration generated by Brownian motion $W(t)$
- Stock prices

$$\frac{dS_i(t)}{S_i(t)} = \alpha_i(t)dt + \sum_{j=1}^{d} \sigma_{i,j}(t)dW_j(t), i = 1, \ldots, m$$

- Discount process

$$D(t) = e^{-\int_0^t R(s)ds}$$

# Model

In risk neutral world,

- Probability measure $\widetilde{\mathbb{P}}$
- Assume there exists a unique $\Theta_j(t), j = 1, \ldots, d$

$$d\widetilde{W}_j(t) = dW_j(t) + \Theta_j(t)$$

such that

$$\widetilde{W}(t) = \left(\widetilde{W}_1(t), \ldots, \widetilde{W}_d(t)\right)$$

is a $d$-dimensional Brownian motion under $\widetilde{\mathbb{P}}$.

- Stock prices

$$\frac{dS_i(t)}{S_i(t)} = R(t)dt + \sum_{j=1}^{d} \sigma_{i,j}(t)d\widetilde{W}_j(t)$$

# Change of Numeraire

## Theorem

*Let $N(t)$ be a strictly positive price process of a non-dividend paying asset (stock or derivatives), then there exists a vector volatility process*

$$\nu(t) = (\nu_1(t), \ldots, \nu_d(t))$$

*such that*

$$\frac{dN(t)}{N(t)} = R(t)dt + \nu(t) \cdot d\widetilde{W}(t)$$

# Change of Numeraire

**Proof.**

Under risk neutral measure $\widetilde{\mathbb{P}}$, the discounted price process $D(t)N(t)$ is martingale. By Martingale Representation Theorem, there exists an adapted process

$$\widetilde{\Gamma}(t) = \left(\widetilde{\Gamma}_1(t), \ldots, \widetilde{\Gamma}_d(t)\right)$$

such that

$$d\left(D(t)N(t)\right) = \widetilde{\Gamma}(t) \cdot d\widetilde{W}(t)$$

□

# Change of Numeraire

**Proof.**

Hence

$$\frac{d\left(D(t)N(t)\right)}{D(t)N(t)} = \nu(t) \cdot d\widetilde{W}(t)$$

where

$$\nu(t) = \frac{1}{D(t)N(t)}\widetilde{\Gamma}(t)$$

This implies

$$\frac{dN(t)}{N(t)} = R(t)dt + \nu(t) \cdot d\widetilde{W}(t) \quad Q.E.D.$$

# Change of Numeraire

Solving $N(t)$, we have

$$N(t) = N(0)e^{\int_0^t \left(R(s) - \frac{1}{2}||\nu(s)||^2\right)ds + \int_0^t \nu(s)\cdot d\widetilde{W}(s)}$$

Let $Z(t) = \frac{D(t)N(t)}{N(0)}$, then

$$Z(t) = e^{-\frac{1}{2}\int_0^t ||\nu(s)||^2 ds + \int_0^t \nu(s)\cdot d\widetilde{W}(s)}$$

is a Radon-Nikodym derivative process.

$\Longrightarrow$ We can apply Girsanov's Theorem

## Change of Numeraire



Fix $T > 0$, let $Z = Z(T) = \frac{D(T)N(T)}{N(0)}$. Define a new probability measure $\widetilde{\mathbb{P}}^{(N)}$ as

$$\frac{d\widetilde{\mathbb{P}}^{(N)}}{d\widetilde{\mathbb{P}}} = Z = \frac{D(T)N(T)}{N(0)}$$

Define

$$d\widetilde{W}_j^{(N)} = -\nu_j(t)dt + d\widetilde{W}_j(t)$$

Then

$$\widetilde{W}^{(N)} = \left(\widetilde{W}_1^{(N)}, \ldots, \widetilde{W}_d^{(N)}\right)$$

is a $d$-dimensional Brownian motion under the measure $\widetilde{\mathbb{P}}^{(N)}$.

# Change of Numeraire

We also have

- For any $X$

$$\widetilde{\mathbb{E}}^{(N)}[X] = \frac{1}{N(0)}\widetilde{\mathbb{E}}[XD(T)N(T)]$$

- If $X$ is $\mathcal{F}_t$-measurable, then

$$\widetilde{\mathbb{E}}^{(N)}[X] = \frac{1}{N(0)}\widetilde{\mathbb{E}}[XD(t)N(t)]$$

- For any $s < t$ and $\mathcal{F}_t$-measurable random variable $Y$

$$\widetilde{\mathbb{E}}^{(N)}[Y|\mathcal{F}_s] = \frac{1}{D(s)N(s)}\widetilde{\mathbb{E}}[YD(t)N(t)|\mathcal{F}_s]$$

# Change of Numeraire

We often use these identities in the reverse way. For example, from the last identity we have

$$\widetilde{\mathbb{E}}\left[YD(t)N(t)|\mathcal{F}_s\right] = D(s)N(s)\widetilde{\mathbb{E}}^{(N)}\left[Y|\mathcal{F}_s\right]$$

To calculate

$$\widetilde{\mathbb{E}}\left[YD(t)N(t)|\mathcal{F}_s\right]$$

we can change measure to $\widetilde{\mathbb{P}}^{(N)}$ and calculate

$$\widetilde{\mathbb{E}}^{(N)}\left[Y|\mathcal{F}_s\right]$$

which often leads to simpler computation as we have moved $D(s)N(s)$ out of the expectation.

# Change of Numeraire

The last identity also comes in the following form

$$\widetilde{\mathbb{E}}\left[V|\mathcal{F}_s\right] = D(s)N(s)\widetilde{\mathbb{E}}^{(N)}\left[\frac{V}{D(t)N(t)}|\mathcal{F}_s\right]$$

for $\mathcal{F}_t$-measurable random variable $V$

# Change of Numeraire

## Theorem

*Let $S(t)$ and $N(t)$ be the prices of two assets*

$$\frac{d\left(D(t)S(t)\right)}{D(t)S(t)} = \sigma(t) \cdot d\widetilde{W}(t)$$

*and*

$$\frac{d\left(D(t)N(t)\right)}{D(t)N(t)} = \nu(t) \cdot d\widetilde{W}(t)$$

*where $\sigma(t) = (\sigma_1(t), \ldots, \sigma_d(t))$ and $\nu(t) = (\nu_1(t), \ldots, \nu_d(t))$ are their vector volatility processes respectively.*

# Change of Numeraire



## Theorem (Cont'd)

*Take $N(t)$ as the numeraire and denote $S^{(N)}(t) = \frac{S(t)}{N(t)}$. Under the probability measure $\widetilde{\mathbb{P}}^{(N)}$, $S^{(N)}(t)$ is a martingale. Moreover*

$$\frac{dS^{(N)}(t)}{S^{(N)}(t)} = (\sigma(t) - \nu(t)) \cdot d\widetilde{W}^{(N)}(t)$$

# Change of Numeraire

*Remarks*

- The instantaneous volatilities of $S(t)$ and $N(t)$ are $||\sigma(t)||_2$ and $||\nu(t)||_2$ respectively
- The instantaneous volatility of $S^{(N)}(t)$ is $||\sigma(t) - \nu(t)||_2$ which is not the same as $||\sigma(t)||_2 - ||\nu(t)||_2$.
- In general, if $S(t)$ and $N(t)$ are positively correlated, $S^{(N)}(t)$ will have a lower volatility compared to $S(t)$. A special case is $S(t) = N(t)$, the volatility of $S^{(N)}(t)$ is zero.
- If we take money market account $M(t)$ as numeraire, since $M(t)$ has zero vector volatility process , the vector volatility process (hence the instantaneous volatility) of $S(t)$ does not change from real-world measure to risk neutral measure.

# Change of Numeraire



**Proof.**

$$
\begin{aligned}
d\left(\frac{1}{D(t)N(t)}\right) &= -\frac{d(D(t)N(t))}{(D(t)N(t))^2} + \frac{d(D(t)N(t))d(D(t)N(t))}{(D(t)N(t))^3} \\
&= -\frac{\nu(t)\cdot d\widetilde{W}(t)}{D(t)N(t)} + \frac{||\nu(t)||^2 dt}{D(t)N(t)}
\end{aligned}
$$

$\Longrightarrow$

$$
\frac{d\left(1/(D(t)N(t))\right)}{1/(D(t)N(t))} = -\nu(t)\cdot d\widetilde{W}(t) + ||\nu(t)||^2 dt
$$

$\square$

# Change of Numeraire

## Cont'd.

Let $X(t) = D(t)S(t)$ and $Y(t) = \frac{1}{D(t)N(t)}$, we have

$$\frac{dX(t)}{X(t)} = \sigma(t) \cdot d\widetilde{W}(t)$$

and

$$\frac{dY(t)}{Y(t)} = -\nu(t) \cdot d\widetilde{W}(t) + ||\nu(t)||^2 dt$$

$\square$

# Change of Numeraire

## Cont'd.

Using the Ito's product rule

$$\frac{d\,(XY)}{XY} = \frac{dX}{X} + \frac{dY}{Y} + \frac{dX}{X}\frac{dY}{Y}$$

we have

$$\begin{aligned} & \frac{d(XY)}{XY} \\ = \; & \sigma(t)\cdot d\widetilde{W}(t) - \nu(t)\cdot d\widetilde{W}(t) + \|\nu(t)\|^2 dt - \sigma(t)\cdot\nu(t)dt \\ = \; & (\sigma(t)-\nu(t))\cdot\left(d\widetilde{W}(t) - \nu(t)dt\right) \\ = \; & (\sigma(t)-\nu(t))\cdot d\widetilde{W}^{(N)}(t) \end{aligned}$$

$\square$

# Change of Numeraire

## Cont'd.

Since $S(t)^{N(t)} = X(t)Y(t)$, we have

$$\frac{dS^{(N)}(t)}{S^{(N)}(t)} = (\sigma(t) - \nu(t)) \cdot d\widetilde{W}^{(N)}(t) \qquad Q.E.D.$$

$\square$

# Change of Numeraire

In summary, we can choose any positively priced asset $N(t)$ as numeraire. We have

- Probability measure

$$\frac{d\widetilde{\mathbb{P}}^{(N)}}{d\widetilde{\mathbb{P}}} = \frac{D(T)N(T)}{N(0)} \tag{1}$$

- Brownian motion under $\widetilde{\mathbb{P}}^{(N)}$

$$d\widetilde{W}^{(N)}(t) = -\nu(t)dt + d\widetilde{W}(t) \tag{2}$$

  where $\nu(t)$ is the vector volatility process for $N(t)$

- Asset prices in the new numeraire $S^{(N)}(t) = \frac{S(t)}{N(t)}$ are martingales under the measure $\widetilde{\mathbb{P}}$ and

$$\frac{dS^{(N)}(t)}{S^{(N)}(t)} = (\sigma(t) - \nu(t)) \cdot d\widetilde{W}^{(N)}(t) \tag{3}$$

# Change of Numeraire

- Pricing formulae
  1. For any $X$
  $$\widetilde{\mathbb{E}}^{(N)}[X] = \frac{1}{N(0)}\widetilde{\mathbb{E}}[XD(T)N(T)] \tag{4}$$
  2. If $X$ is $\mathcal{F}_t$-measurable, then
  $$\widetilde{\mathbb{E}}^{(N)}[X] = \frac{1}{N(0)}\widetilde{\mathbb{E}}[XD(t)N(t)] \tag{5}$$
  3. For any $s < t$ and $\mathcal{F}_t$-measurable random variable $Y$
  $$\widetilde{\mathbb{E}}^{(N)}[Y|\mathcal{F}_s] = \frac{1}{D(s)N(s)}\widetilde{\mathbb{E}}[YD(t)N(t)|\mathcal{F}_s] \tag{6}$$

# Applications



## Example (Exchange Option)

Consider two stock prices under risk neutral measure $\widetilde{\mathbb{P}}$

$$\frac{dS_1(t)}{S_1(t)} = rdt + \sigma_1 d\widetilde{W}_1(t)$$

and

$$\frac{dS_2(t)}{S_2(t)} = rdt + \sigma_2 \left(\rho d\widetilde{W}_1(t) + \sqrt{1-\rho^2} d\widetilde{W}_2(t)\right)$$

where $r, \rho, \sigma_2$ and $\sigma_2$ are constant, $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$ are Brownian motion under $\widetilde{\mathbb{P}}$.

# Applications

## Example (Exchange Option - Cont'd)

In vector form

$$\frac{dS_1(t)}{S_1(t)} = rdt + \sigma \cdot d\widetilde{W}(t)$$

and

$$\frac{dS_2(t)}{S_2(t)} = rdt + \nu \cdot d\widetilde{W}(t)$$

where

$$\sigma = (\sigma_1, 0)$$

$$\nu = \left(\sigma_2 \rho, \sigma_2 \sqrt{1-\rho^2}\right)$$

and

$$d\widetilde{W}(t) = \left(d\widetilde{W}_1(t), d\widetilde{W}_2(t)\right)$$

# Applications

## Example (Exchange Option - Cont'd)

An exchange option gives the owner the right but not obligation to exchange $S_2(t)$ for $S_1(t)$ at expiry $T$. The payoff at time $T$ is

$$(S_1(T) - S_2(T))^+$$

$\Rightarrow$ How do we value this option?

By risk neutral pricing, we have

$$V = \widetilde{\mathbb{E}}\left[e^{-rT}(S_1(T) - S_2(T))^+\right]$$

This involves two dimensional integration ($\widetilde{W}_1(T)$ and $\widetilde{W}_2(T)$) and seems difficult!

# Applications

## Example (Exchange Option - Cont'd)

Rewrite $V$ as

$$\begin{aligned} V &= \widetilde{\mathbb{E}}\left[e^{-rT}\left(S_1(T) - S_2(T)\right)^+\right] \\ &= \widetilde{\mathbb{E}}\left[e^{-rT} S_2(T)\left(\frac{S_1(T)}{S_2(T)} - 1\right)^+\right] \end{aligned}$$

Apply the pricing formula (4),

$$\begin{aligned} V &= \widetilde{\mathbb{E}}\left[e^{-rT} S_2(T)\left(\frac{S_1(T)}{S_2(T)} - 1\right)^+\right] \\ &= S_2(0)\widetilde{\mathbb{E}}^{(S_2)}\left[\left(S_1^{(S_2)}(T) - 1\right)^+\right] \end{aligned}$$

# Applications

## Example (Exchange Option - Cont'd)

By the Change of Numeraire Theorem

$$\frac{dS_1^{(S_2)}(t)}{S_1^{(S_2)}(t)} = (\sigma - \nu) \cdot d\widetilde{W}^{(S_2)}(t)$$

Let

$$d\widetilde{B}(t) = \frac{1}{||\sigma - \nu||_2}(\sigma - \nu) \cdot d\widetilde{W}^{(S_2)}(t)$$

which is a Brownian motion. Then

$$\frac{dS_1^{(S_2)}(t)}{S_1^{(S_2)}(t)} = ||\sigma - \nu||_2 d\widetilde{B}(t)$$

# Applications

## Example (Exchange Option - Cont'd)

Let

$$\hat{\sigma} = ||\sigma - \nu||_2 = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

we have

$$\frac{dS_1^{(S_2)}(t)}{S_1^{(S_2)}(t)} = \hat{\sigma}\, d\widetilde{B}(t)$$

$S_1^{(S_2)}$ is a Geometric Brownian motion under $\widetilde{\mathbb{P}}^{(S_2)}$

# Applications

## Example (Exchange Option - Cont'd)

We can then apply the familiar BS formula to compute

$$V = S_2(0)\mathbb{E}^{(S_2)}\left[\left(S_1^{(S_2)}(T) - 1\right)^+\right]$$

which gives

$$V = S_1(0)\Phi(d_1) - S_2(0)\Phi(d_2)$$

where

$$d_{1,2} = \frac{\ln\left(\frac{S_1(0)}{S_2(0)}\right) \pm \frac{1}{2}\hat{\sigma}^2 T}{\hat{\sigma}\sqrt{T}}$$

and $\Phi$ is the cumulative density function of a standard normal distribution.

# Cross-Currency Market Model

Consider a model with two markets: domestic market and foreign market. Under the real world measure $\mathbb{P}$,

- $W(t) = (W_1(t), W_2(t))$ Brownian motion
- Stock price $S(t)$ in domestic currency

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma_1 dW_1(t)$$

- *Interest rates $r_d$ and $r_f$*
  1. $r_d$ interest rate for domestic market
  2. $r_f$ interest rate for for foreign market

  We assume $r_d$ and $r_f$ are constant.
- *Money market accounts $M_d(t)$ and $M_f(t)$*
  1. $M_d(t) = e^{r_d t}$ for domestic market
  2. $M_f(t) = e^{r_f t}$ for foreign market

# Cross-Currency Market Model

- Foreign exchange rate $X(t)$
  $X(t)$ is the units of domestic currency per unit of foreign currency. In other words, it is the price (in domestic currency) of one unit of foreign currency.

$$\frac{dX(t)}{X(t)} = \gamma dt + \sigma_2 \left( \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t) \right)$$

# Cross-Currency Market Model

*Remarks*

- For notational simplicity, we assume $\alpha, r_d, r_f, \sigma_1, \sigma_2, \rho$ and $\gamma$ are all constant.
- $\rho$ is the instantaneous correlation between the stock $S(t)$ and foreign exchange rate $X(t)$, as shown below

$$\frac{dX(t)}{X(t)}\frac{dS(t)}{S(t)} = \rho\sigma_1\sigma_2 dt$$

## Domestic Risk Neutral Measure



There are three primary (tradable) assets in this model:

- Stock $S(t)$
- Domestic money market account $M_d(t)$
- Foreign money market account $M_f(t)$

Under the domestic risk neutral measure, the discounted prices of these assets (denominated in domestic currency) must be martingale.

The foreign exchange rate $X$ is not tradable!

# Domestic Risk Neutral Measure

The discounted price of the domestic money market account is

$$e^{-r_d t} M_d(t) = 1$$

which is a martingale

The discounted stock price follows

$$\frac{d\left(e^{-r_d t} S(t)\right)}{e^{-r_d t} S(t)} = \sigma_1 \left( \frac{\alpha - r_d}{\sigma_1} dt + dW_1(t) \right)$$

# Domestic Risk Neutral Measure



The discounted price of $M_f(t)$ in domestic currency is

$$e^{-r_d t} M_f(t) X(t) = e^{(r_f - r_d)t} X(t)$$

Its SDE is

$$\begin{aligned} & \frac{d\left(e^{(r_f - r_d)t} X(t)\right)}{e^{(r_f - r_d)t} X(t)} \\ = \quad & \sigma_2 \left( \frac{(r_f - r_d + \gamma)}{\sigma_2} dt + \left( \rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t) \right) \right) \end{aligned}$$

# Domestic Risk Neutral Measure



Let

$$\Theta_1(t) = \frac{\alpha - r_d}{\sigma_1}$$

$$\Theta_2(t) = \frac{1}{\sqrt{1-\rho^2}} \left( \frac{r_f - r_d + \gamma}{\sigma_1} - \rho \Theta_1(t) \right)$$

and

$$d\widetilde{W}_1(t) = \Theta_1(t)dt + dW_1(t)$$

$$d\widetilde{W}_2(t) = \Theta_2(t)dt + dW_1(t)$$

By Girsanov's Theorem, there exists a probability measure $\widetilde{\mathbb{P}}$ such that $\widetilde{W}(t) = \left(\widetilde{W}_1(t), \widetilde{W}_2(t)\right)$ is a Brownian motion.

## Domestic Risk Neutral Measure



We have

$$\frac{d\left(e^{-r_d t}S(t)\right)}{e^{-r_d t}S(t)} = \sigma_1 d\widetilde{W}_1(t)$$

and

$$\frac{d\left(e^{(r_f - r_d)t}X(t)\right)}{e^{(r_f - r_d)t}X(t)} = \sigma_2\left(\rho d\widetilde{W}_1(t) + \sqrt{1-\rho^2}\,d\widetilde{W}_2(t)\right)$$

The discounted stock price and foreign money market account are both martingales under $\widetilde{\mathbb{P}}$.

$\widetilde{\mathbb{P}}$ is the (unique) risk neutral measure for the domestic market.

# Domestic Risk Neutral Measure

From the above SDEs, we have

- $$\frac{d\left(S(t)\right)}{S(t)} = r_d dt + \sigma_1 d\widetilde{W}_1(t)$$

- $$\frac{dX(t)}{X(t)} = (r_d - r_f)dt + \sigma_2\left(\rho d\widetilde{W}_1(t) + \sqrt{1-\rho^2}\, d\widetilde{W}_2(t)\right)$$

- $$\frac{d\left(S(t)\right)}{S(t)}\frac{dX(t)}{X(t)} = \rho\sigma_1\sigma_2 dt$$

Note that the drift term for $X(t)$ is $r_d - r_f$ under domestic risk neutral measure. It is similar to a dividend-paying stock.

# Foreign Risk Neutral Measure

If we choose $N(t) = M_f(t)X(t)$ as a numeraire, from Change of Numeraire Theorem we can define a probability measure

$$\frac{d\widetilde{\mathbb{P}}^f}{d\widetilde{\mathbb{P}}} = \frac{D(T)N(T)}{N(0)} = \frac{e^{(r_f - r_d)T}X(T)}{X(0)}$$

Under this measure, $\frac{S(t)}{M_f(t)X(t)}$ and $\frac{M_d(t)}{M_f(t)X(t)}$ are martingales.

Note that $\frac{S(t)}{M_f(t)X(t)}$ and $\frac{M_d(t)}{M_f(t)X(t)}$ are the discounted price of stock and money market account in the unit of foreign currency respectively.

$\widetilde{\mathbb{P}}^f$ is called *foreign risk neutral measure*.

# Foreign Risk Neutral Measure



Note that

$$\begin{aligned}
\frac{d(D(t)N(t))}{D(t)N(t)} &= \frac{d\left(e^{(r_f-r_d)t}X(t)\right)}{e^{(r_f-r_d)t}X(t)} \\
&= \sigma_2\left(\rho d\widetilde{W}_1(t)+\sqrt{1-\rho^2}d\widetilde{W}_2(t)\right) \\
&= \left(\rho\sigma_2, \sqrt{1-\rho^2}\sigma_2\right)\cdot d\widetilde{W}(t)
\end{aligned}$$

The vector volatility process for the numeraire is

$$\nu = \left(\sigma_2\rho, \sigma_2\sqrt{1-\rho^2}\right)$$

# Foreign Risk Neutral Measure



The Brownian motion under the change of numeraire is defined as

$$d\widetilde{W}_1^f(t) = -\sigma_2 \rho dt + d\widetilde{W}_1(t)$$

and

$$d\widetilde{W}_2^f(t) = -\sigma_2 \sqrt{1-\rho^2} dt + d\widetilde{W}_2(t)$$

$$\widetilde{W}^f(t) = \left(\widetilde{W}_1^f(t), \widetilde{W}_2^f(t)\right)$$

is a Brownian motion under the measure $\widetilde{\mathbb{P}}^f$.

# Foreign Risk Neutral Measure

From Change of Numeraire Theorem, we have

$$
\begin{aligned}
& d\left(\frac{M_d(t)}{M_f(t)X(t)}\right) \\
= \quad & \frac{M_d(t)}{M_f(t)X(t)}\left(-\sigma_2 \rho d\widetilde{W}_1^f(t) - \sigma_2\sqrt{1-\rho^2}d\widetilde{W}_1^f(t)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& d\left(\frac{S(t)}{M_f(t)X(t)}\right) \\
= \quad & \frac{S(t)}{M_f(t)X(t)}\left((\sigma_1 - \sigma_2\rho)d\widetilde{W}_1^f(t) - \sigma_2\sqrt{1-\rho^2}d\widetilde{W}_1^f(t)\right)
\end{aligned}
$$

# Foreign Risk Neutral Measure

Hence

$$\frac{d\,(1/X(t))}{1/X(t)} = (r_f - r_d)dt - \sigma_2 \left(\rho d\widetilde{W}_1^f(t) + \sqrt{1-\rho^2} d\widetilde{W}_1^f(t)\right)$$

and

$$\frac{d\,(S(t)/X(t))}{S(t)/X(t)} = r_f dt + (\sigma_1 - \sigma_2\rho) d\widetilde{W}_1^f(t) - \sigma_2\sqrt{1-\rho^2} d\widetilde{W}_1^f(t)$$

# Foreign Risk Neutral Measure

Pricing formulae

$$\widetilde{\mathbb{E}}^f[V] = \frac{1}{X(0)}\widetilde{\mathbb{E}}\left[Ve^{(r_f - r_d)T}X(T)\right]$$

When $V$ is $\mathcal{F}_t$-measurable

$$\widetilde{\mathbb{E}}^f[V] = \frac{1}{X(0)}\widetilde{\mathbb{E}}\left[Ve^{(r_f - r_d)t}X(t)\right]$$

and the conditional expectation is

$$\widetilde{\mathbb{E}}^f[V|\mathcal{F}_s] = \frac{1}{X(s)}\widetilde{\mathbb{E}}\left[Ve^{(r_f - r_d)(t-s)}X(t)|\mathcal{F}_s\right]$$

# Summary

In domestic risk neutral measure $\widetilde{\mathbb{P}}$,

$$\frac{d\left(S(t)\right)}{S(t)} = r_d dt + \sigma_1 d\widetilde{W_1}(t) \tag{7}$$

$$\frac{dX(t)}{X(t)} = (r_d - r_f)dt + \sigma_2\left(\rho d\widetilde{W_1}(t) + \sqrt{1-\rho^2}d\widetilde{W_2}(t)\right) \tag{8}$$

$$\frac{d\left(S(t)\right)}{S(t)}\frac{dX(t)}{X(t)} = \rho\sigma_1\sigma_2 dt \tag{9}$$

# Summary



Change of measure

$$N(t) = e^{r_f t} X(t) \tag{10}$$

$$\frac{d\widetilde{\mathbb{P}}^f}{d\widetilde{\mathbb{P}}} = \frac{e^{(r_f - r_d)T} X(T)}{X(0)} \tag{11}$$

$$d\widetilde{W}_1^f(t) = -\sigma_2 \rho dt + d\widetilde{W}_1(t) \tag{12}$$

$$d\widetilde{W}_2^f(t) = -\sigma_2 \sqrt{1-\rho^2} dt + d\widetilde{W}_2(t) \tag{13}$$

# Summary

In foreign risk neutral measure $\widetilde{\mathbb{P}}^f$

$$\frac{d\left(1/X(t)\right)}{1/X(t)} = (r_f - r_d)dt - \sigma_2\left(\rho d\widetilde{W}_1^f(t) + \sqrt{1-\rho^2}d\widetilde{W}_2^f(t)\right) \tag{14}$$

$$\frac{d\left(S(t)/X(t)\right)}{S(t)/X(t)} = r_f dt + (\sigma_1 - \sigma_2\rho)d\widetilde{W}_1^f(t) - \sigma_2\sqrt{1-\rho^2}d\widetilde{W}_2^f(t) \tag{15}$$

$$\frac{d\left(S(t)/X(t)\right)}{S(t)/X(t)}\frac{d\left(1/X(t)\right)}{1/X(t)} = \left(\sigma_2^2 - \rho\sigma_1\sigma_2\right)dt \tag{16}$$

# Summary

Pricing formulae

$$\widetilde{\mathbb{E}}^f\left[V\right] = \frac{1}{X(0)}\widetilde{\mathbb{E}}\left[Ve^{(r_f - r_d)T}X(T)\right] \tag{17}$$

When $V$ is $\mathcal{F}_t$-measurable

$$\widetilde{\mathbb{E}}^f\left[V\right] = \frac{1}{X(0)}\widetilde{\mathbb{E}}\left[Ve^{(r_f - r_d)t}X(t)\right] \tag{18}$$

and the conditional expectation is

$$\widetilde{\mathbb{E}}^f\left[V|\mathcal{F}_s\right] = \frac{1}{X(s)}\widetilde{\mathbb{E}}\left[Ve^{(r_f - r_d)(t-s)}X(t)|\mathcal{F}_s\right] \tag{19}$$

# Applications



**Example (Quanto Option)**

Suppose $S^{USD}(t)$ is the price of a stock in USD. An SGD investor owns a call option that pays $\bar{Q}\left(S^{USD}(T) - K\right)^+$ at expiry $T$, where $\bar{Q}$ is a pre-agreed exchange rate, e.g., $\bar{Q} = 1.38$. What is the price of this option in SGD at time $t = 0$?

# Applications

## Example (Quanto Option)

There are two ways to price this option:

- USD as the domestic market
- SGD as the domestic market

# Applications

## Example (Quanto Option - USD as Domestic Market)

Using USD as the domestic market, the option price is

$$V = \bar{Q} e^{-r_{SGD} T} \widetilde{\mathbb{E}}^f \left( S^{USD}(T) - K \right)^+$$

Suppose

$$\frac{dS^{USD}(t)}{S^{USD}(t)} = r_{USD} dt + \sigma_1 d\widetilde{W}_1(t)$$

and

$$\frac{dX(t)}{X(t)} = (r_{USD} - r_{SGD}) dt + \sigma_2 \left( \rho d\widetilde{W}_1(t) + \sqrt{1 - \rho^2} d\widetilde{W}_2(t) \right)$$

under the USD risk neutral measure.

# Applications



## Example (Quanto Option - USD as Domestic Market)

From the change of measure formula (12)

$$\frac{dS^{USD}(t)}{S^{USD}(t)} = (r_{USD} + \rho\sigma_1\sigma_2)dt + \sigma_1 d\widetilde{W}_1^f(t)$$

Using the familiar BS formula, we have

$$V = \bar{Q}\left(e^{r_{USD} - r_{SGD} + \rho\sigma_1\sigma_2} S^{USD}(0)\Phi(d_1) - e^{-r_{SGD}T} K\Phi(d_2)\right)$$

where

$$d_{1,2} = \frac{\ln(S^{USD}(0)/K) + (r_{USD} + \rho\sigma_1\sigma_2 \pm \frac{1}{2}\sigma_1^2)T}{\sigma_1\sqrt{T}}$$

# Applications

**Example (Quanto Option - SGD as Domestic Market)**

Now we look at how we can value the option using SGD market as the domestic market.

Using SGD as the domestic market, the option price is

$$V = \bar{Q}e^{-r_{SGD}T}\widetilde{\mathbb{E}}\left(\frac{S^{SGD}(T)}{Y(T)} - K\right)^{+}$$

where $Y(t)$ is the exchange rate measured as units of SGD per USD.

# Applications

## Example (Quanto Option - SGD as Domestic Market)

Under SGD market measure,

$$\frac{dS^{SGD}(t)}{S^{SGD}(t)} = r_{SGD}dt + \sigma_1 d\widetilde{W}_1(t)$$

and

$$\frac{dY(t)}{Y(t)} = (r_{SGD} - r_{USD})dt + \sigma_2 \left(\rho d\widetilde{W}_1(t) + \sqrt{1-\rho^2} d\widetilde{W}_2(t)\right)$$

under the SGD risk neutral measure.

# Applications

## Example (Quanto Option - SGD as Domestic Market)

We can solve $S^{SGD}(T)$ and $Y(T)$ and have

$$S^{SGD}(T) = S^{SGD}(0)e^{(r_{SGD}-\frac{1}{2}\sigma_1^2)T+\sigma_1\widetilde{W}_1(T)}$$

and

$$Y(T) = Y(0)e^{(r_{SGD}-r_{USD}-\frac{1}{2}\sigma_2^2)T+\rho\sigma_2\widetilde{W}_1(T)+\sqrt{1-\rho^2}\sigma_2\widetilde{W}_2(T)}$$

Hence

$$\frac{S^{SGD}(T)}{Y(T)} = \frac{S^{SGD}(0)}{Y(0)}e^{(r_{USD}+\frac{1}{2}\sigma_2^2-\frac{1}{2}\sigma_1^2)T+(\sigma_1-\rho\sigma_2)\widetilde{W}_1(T)-\sqrt{1-\rho^2}\sigma_2\widetilde{W}_2(T)}$$

# Applications

## Example (Quanto Option - SGD as Domestic Market)

Let

$$\hat{\sigma}^2 = (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)$$

and

$$d\widetilde{W}_3(t) = \frac{1}{\hat{\sigma}}\left((\sigma_1 - \rho\sigma_2)d\widetilde{W}_1(t) - \sqrt{1-\rho^2}\sigma_2\widetilde{W}_2(t)\right)$$

Then $\widetilde{W}_3(t)$ is a Brownian motion. And

$$\frac{d\left(S^{SGD}(t)/Y(t)\right)}{dS^{SGD}(t)/Y(t)} = \left(r_{USD} + \sigma_2^2 - \rho\sigma_1\sigma_2\right)dt + \hat{\sigma}\, d\widetilde{W}_3(t)$$

# Applications

## Example (Quanto Option - SGD as Domestic Market)

As before, we can use the familiar BS formula to get

$$\begin{aligned} & \widetilde{\mathbb{E}}\left(\frac{S^{SGD}(T)}{Y(T)} - K\right)^{+} \\ = \quad & e^{\left(r_{USD}+\sigma_2^2-\rho\sigma_1\sigma_2\right)T}\left(\frac{S^{SGD}(0)}{Y(0)}\right)\Phi(d_1) - K\Phi(d_2) \end{aligned}$$

where

$$d_{1,2} = \frac{\ln\left(\frac{S^{SGD}(0)}{Y(0)K}\right) + \left(r_{USD} + \sigma_2^2 - \rho\sigma_1\sigma_2 \pm \frac{1}{2}\hat{\sigma}^2\right)T}{\hat{\sigma}\sqrt{T}}$$

# Applications



## Example (Quanto Option - SGD as Domestic Market)

Hence

$$
\begin{aligned}
V \;=\; & \bar{Q}\left(e^{\left(r_{USD}-r_{SGD}+\sigma_2^2-\rho\sigma_1\sigma_2\right)T}\left(\frac{S^{SGD}(0)}{Y(0)}\right)\Phi(d_1)\right. \\
& \left.-e^{-r_{SGD}T}K\Phi(d_2)\right)
\end{aligned}
$$

# Applications

**Example (Quanto Option)**

Question: the two pricing formulae for quanto option look quite different, how do you reconcile them? (HW)

# Applications

**Example (Option Stuck in Foreign Currency)**

A US bank offers a call option on a US stock $S(t)$ to an SGD investor. At expiry, the payoff of the call option payoff is

$$\left(\frac{S(T)}{X(T)} - K\right)^+$$

where $X(T)$ is the foreign exchange rate measured as the units of USD per SGD and $K$ is the strike in SGD. How shall the bank value this option?

# Applications

## Example (Option Stuck in Foreign Currency)

From the perspective of a US bank, the payoff in USD is

$$\left(\frac{S(T)}{X(T)} - K\right)^{+} X(T)$$

Using Risk Neutral Pricing Formula, the price in USD is

$$C = e^{-r_{USD}T}\widetilde{\mathbb{E}}\left[\left(\frac{S(T)}{X(T)} - K\right)^{+} X(T)\right]$$

where $\widetilde{\mathbb{E}}$ is the expectation under the risk neutral measure for USD market.

# Applications

**Example (Option Stuck in Foreign Currency)**

We use USD market as the domestic market and SGD market as the foreign market. Under the USD market

$$\frac{dS(t)}{S(t)} = r_{USD}dt + \sigma_1 d\widetilde{W}_1(t)$$

and

$$\frac{dX(t)}{X(t)} = (r_{USD} - r_{SGD})\, dt + \sigma_2 \left(\rho d\widetilde{W}_1(t) + \sqrt{1-\rho^2} d\widetilde{W}_2(t)\right)$$

# Applications

## Example (Option Stuck in Foreign Currency)

From Equation (17), we have

$$\widetilde{\mathbb{E}}\left[VX(T)\right] = X(0)e^{(r_{USD}-r_{SGD})T}\widetilde{\mathbb{E}}^{f}\left[V\right]$$

Let

$$V = \left(\frac{S(T)}{X(T)} - K\right)^{+}$$

Then

$$C = X(0)e^{-r_{SGD}T}\widetilde{\mathbb{E}}^{SGD}\left[\left(\frac{S(T)}{X(T)} - K\right)^{+}\right]$$

# Applications

## Example (Option Stuck in Foreign Currency)

From Equation (15), we have

$$\begin{aligned} & \frac{d(S(t)/X(t))}{S(t)/X(t)} \\ = \quad & r_{SGD}dt + (\sigma_1 - \sigma_2\rho)d\widetilde{W}_1^{SGD}(t) - \sigma_2\sqrt{1-\rho^2}d\widetilde{W}_2^{SGD}(t) \\ = \quad & r_{SGD}dt + \hat{\sigma}d\widetilde{W}_3^{SGD}(t) \end{aligned}$$

where

$$\hat{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$$

and

$$\begin{aligned} & d\widetilde{W}_3^{SGD}(t) \\ = \quad & \frac{1}{\hat{\sigma}}\left((\sigma_1 - \sigma_2\rho)d\widetilde{W}_1^{SGD}(t) - \sigma_2\sqrt{1-\rho^2}d\widetilde{W}_2^{SGD}(t)\right) \end{aligned}$$

is a Brownian motion under SGD risk neutral measure.

# Applications

## Example (Option Stuck in Foreign Currency)

From this, we have

$$\widetilde{\mathbb{E}}\left[\left(\frac{S(T)}{X(T)} - K\right)^{+} X(T)\right] = e^{r_{SGD}T}\frac{S(0)}{X(0)}\Phi(d_1) - K\Phi(d_2)$$

where

$$d_{1,2} = \frac{\ln\left(\frac{S(0)}{X(0)K}\right) + \left(r_{SGD} \pm \frac{1}{2}\hat{\sigma}^2\right)T}{\hat{\sigma}\sqrt{T}}$$

# Applications

## Example (Option Stuck in Foreign Currency)

Hence

$$C = S(0)\Phi(d_1) - e^{-r_{SGD}T} X(0) K \Phi(d_2)$$

where

$$d_{1,2} = \frac{\ln\left(\frac{S(0)}{X(0)K}\right) + \left(r_{SGD} \pm \frac{1}{2}\hat{\sigma}^2\right) T}{\hat{\sigma}\sqrt{T}}$$

# Thank you!