FE5222 Advanced Derivative Pricing

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Overview

The assumption of constant volatility in BSM model is inconsistent with market observations.

Is there a BSM-like model that can price European options in the market consistently?

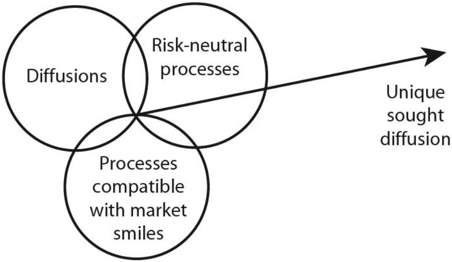
Local Volatility Model

- Dupire (1994) developed a local volatility model for continuous time and showed that there exits a unique risk neutral diffusion process that is consistent with European option prices.
- Derman & Iraj Kani (1994) developed a tree model which is consistent with market prices for European options.

Local Volatility Model

1. A unique diffusion process

If we restrict ourselves to diffusions, there is a unique risk-neutral (drift equal to the short-term rate) process for the spot which is compatible with European option prices:



Source: Dupire (1994)

In LVM, the instantaneous volatility is a deterministic function of t and S_t

$$\frac{dS_t}{S_t} = rdt + \sigma(t, S_t)dW_t$$

Local Volatility Model

In LVM, the pricing PDE is

$$V_t + rSV_s + \frac{1}{2}\sigma^2(t,S)S^2V_{SS} - rV = 0 \tag{1}$$

Dupire's Equation

In this section, we will discuss

- Kolmogorov Backward Equation
- Kolmogorov Forward Equation
- Dupire's Equation

Kolmogorov Backward Equation

Consider the stochastic differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

Kolmogorov Backward Equation

Fix $T > t$, let $p(t, T, x, y)$ be the transition probability density for the solution to this equation.

It is the probability density function of X_T if we solve the equation with initial condition $X_t = x$.

Kolmogorov Backward Equation

Theorem

The transition density function $p(t, T, x, y)$ satisfies the Kolmogorov backward equation

$$p_t(t, T, x, y) + \mu(t, x) p_x(t, T, x, y) + \frac{1}{2}\sigma^2(t, x) p_{xx}(t, T, x, y) = 0$$

Kolmogorov Backward Equation

In the following proof, we will need the concept of a smooth function and compact support.

- A smooth function is a function that has derivatives of all orders.
- The support of a function f is

$$supp\ f = \overline{\{x : f(x) \neq 0\}}$$

- A compact set in $\mathbb{R}^n$ is a closed and bounded set.

Kolmogorov Backward Equation

One example for a smooth function with compact support is the so called *bump function* defined as

$$f(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \forall -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Kolmogorov Backward Equation

Lemma

Let f be an integrable function such that

$$\int f(x)h(x)dx = 0$$

for all smooth and compact function h. Then $f(x) = 0$ for (almost surely) all x.

Remark: In fact, smoothness is not necessary for this lemma. However we will need smoothness for the derivation of Komogorov Forward Equation.

Kolmogorov Backward Equation

Proof.

For any smooth function $h(x)$ with compact support, let

$$\begin{aligned} g(t,x) &= \mathbb{E}\left[h(X_T)\right] \\ &= \int h(y)p(t,T,x,y)dy \end{aligned}$$

$\square$

Kolmogorov Backward Equation

Proof.

Taking partial derivatives w.r.t. t and x respectively, we have

$$g_t(t,x) = \int h(y) p_t(t, T, x, y) dy \tag{2}$$

$$g_x(t,x) = \int h(y) p_x(t, T, x, y) dy \tag{3}$$

$$g_{xx}(t,x) = \int h(y) p_{xx}(t, T, x, y) dy \tag{4}$$

□

Kolmogorov Backward Equation

Proof.

From Feynman-Kac Theorem, we have

$$g_t(t,x) + \mu(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) = 0$$

Replacing Equation (2), (3) and (4) into the above equation, we have

$$\int h(y)p_t dy + \mu(t,x)\int h(y)p_x dy + \frac{1}{2}\sigma^2(t,x)\int h(y)p_{xx} dy = 0$$

$\square$

Kolmogorov Backward Equation

Proof.

Hence

$$\int h(y)\left[p_t + \mu(t,x)p_x + \frac{1}{2}\sigma^2(t,x)p_{xx}\right]dy = 0$$

$\square$

Kolmogorov Backward Equation

Proof.

Since this holds for all smooth function h with compact support, we must have

$$p_t + \mu(t,x)p_x + \frac{1}{2}\sigma^2(t,x)p_{xx} = 0 \qquad \text{Q.E.D.} \tag{5}$$

$\square$

Kolmogorov Forward Equation

Theorem

The transition density function $p(t, T, x, y)$ satisfies the Kolmogorov forward equation

$$\frac{\partial p}{\partial T} + \frac{\partial}{\partial y}\left(\mu(T, y)p\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T, y)p\right) = 0$$

Remark: It is also called Fokker-Planck equation.

Kolmogorov Forward Equation

Proof.

Let h be a smooth function with compact support. By Ito's Lemma, we have

$$\begin{aligned} dh(X_s) &= h_x dX_s + \frac{1}{2} h_{xx} d\left[X, X\right](s) \\ &= \left[\mu(s, X_s) h_x(X_s) + \frac{1}{2}\sigma^2(s, X_s) h_{xx}(X_s)\right] ds \\ & \quad + h_x(X_s)\sigma(s, X_s) dW_s \end{aligned} \tag{6}$$

□

Kolmogorov Forward Equation

Proof.

Consider the process X that starts from t with initial condition $X(t) = x$.

Integrating Equation (6) from t to T, we obtain

$$h(X_T) = h(X_t) + I_1 + I_2$$

where

$$I_1 = \int_t^T \left[\mu(s, X_s)h_x(X_s) + \frac{1}{2}\sigma^2(s, X_s)h_{xx}(X_s)\right] ds$$

and

$$I_2 = \int_t^T h_x(X_s)\sigma(s, X_s)dW_s$$

Kolmogorov Forward Equation

Proof.

Since I_2 is an Ito's integral whose mean is zero, we have

$$\mathbb{E}\left[h(X_T)\right] = h(x) + \mathbb{E}\left[I_1\right] \tag{7}$$

$\square$

Kolmogorov Forward Equation

Proof.

The LHS of Equation (7) is

$$\mathbb{E}\left[h(X_T)\right] = \int_{-\infty}^{\infty} h(y)p(t, T, x, y)dy$$

$\square$

Kolmogorov Forward Equation

Proof.

$$
\begin{aligned}
& \mathbb{E}\left[I_1\right] \\
= \; & \mathbb{E}\left[\int_t^T \left[\mu(s, X_s) h_x(X_s) + \frac{1}{2}\sigma^2(s, X_s) h_{xx}(X_s)\right] ds\right] \\
= \; & \int_t^T \mathbb{E}\left[\mu(s, X_s) h_x(X_s) + \frac{1}{2}\sigma^2(s, X_s) h_{xx}(X_s)\right] ds \\
= \; & \int_t^T \int_{-\infty}^{\infty} p(t, s, x, y)\left[\mu(s, y) h_x(y) + \frac{1}{2}\sigma^2(s, y) h_{xx}(y)\right] dyds \\
= \; & \int_t^T \left[\int_{-\infty}^{\infty} p(t, s, x, y)\mu(s, y) h_x(y) dy\right] ds \\
& + \frac{1}{2}\int_t^T \left[\int_{-\infty}^{\infty} p(t, s, x, y)\sigma^2(s, y) h_{xx}(y) dy\right] ds
\end{aligned}
$$

□

Kolmogorov Forward Equation

Proof.

Now we evaluate

$$\int_{-\infty}^{\infty} p(t,s,x,y)\mu(s,y)h_x(y)dy$$

and

$$\int_{-\infty}^{\infty} p(t,s,x,y)\sigma^2(s,y)h_{xx}(y)dy$$

□

Kolmogorov Forward Equation

Proof.

Integrating by parts and using the fact that $h(x)$, $h_x(x)$ and $h_{xx}(x)$ vanish when $|x|$ is large enough, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} p(t,s,x,y)\mu(s,y)h_x(y)dy \\ = & -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left(p(t,s,x,y)\mu(s,y)\right)h(y)dy \end{aligned} \tag{8}$$

and

$$\begin{aligned} & \int_{-\infty}^{\infty} p(t,s,x,y)\sigma^2(s,y)h_{xx}(y)dy \\ = & \int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2}\left(p(t,s,x,y)\sigma^2(s,y)\right)h(y)dy \end{aligned} \tag{9}$$

□

Kolmogorov Forward Equation

Proof.

Hence

$$
\begin{aligned}
& \mathbb{E}\left[I_1\right] \\
= \quad & -\int_t^T \left[\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left(p(t,s,x,y)\mu(s,y)\right) h(y)dy\right] ds \\
& +\frac{1}{2}\int_t^T \left[\int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2}\left(p(t,s,x,y)\sigma^2(s,y)\right) h(y)dy\right] ds
\end{aligned}
$$

□

Kolmogorov Forward Equation

Proof.

Substituting these to Equation (7), we obtain

$$\begin{aligned}
& \int_{-\infty}^{\infty} h(y)p(t,T,x,y)dy \\
= \quad & h(x) - \int_t^T \left[\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left(p(t,s,x,y)\mu(s,y)\right) h(y)dy\right] ds \\
& + \frac{1}{2}\int_t^T \left[\int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2}\left(p(t,s,x,y)\sigma^2(s,y)\right) h(y)dy\right] ds
\end{aligned}$$

$\square$

Kolmogorov Forward Equation

Proof.

Taking derivative w.r.t. to T, we have

$$
\begin{aligned}
& \int_{-\infty}^{\infty} h(y)\frac{\partial}{\partial T}p(t,T,x,y)dy \\
= \quad & -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left(p(t,T,x,y)\mu(T,y)\right)h(y)dy \\
& +\frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2}\left(p(t,T,x,y)\sigma^2(T,y)\right)h(y)dy
\end{aligned}
$$

$\square$

Kolmogorov Forward Equation

Proof.

Re-arranging it, we have

$$\int_{-\infty}^{\infty} h(y)\left[\frac{\partial p}{\partial T} + \frac{\partial}{\partial y}\left(\mu(T,y)p\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)p\right)\right] dy = 0$$

which implies

$$\frac{\partial p}{\partial T} + \frac{\partial}{\partial y}\left(\mu(T,y)p\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)p\right) = 0 \qquad \text{Q.E.D.}$$

□

Kolmogorov Backward/Forward Equation

Kolmogorov Backward Equation

$$p_t(t,T,x,y)+\mu(t,x)p_x(t,T,x,y)+\frac{1}{2}\sigma^2(t,x)p_x(t,T,x,y)=0$$

Kolmogorov Forward Equation

$$\frac{\partial p}{\partial T}+\frac{\partial}{\partial y}\left(\mu(T,y)p\right)-\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)p\right)=0$$

Kolmogorov Backward/Forward Equation

- Kolmogorov backward equation
  Fix T, it is an PDE of initial condition $X_t = x$.
- Kolmogorov forward equation
  Fix initial condition $X(t) = x$ and it is an PDE w.r.t. T and $X_T = y$.

Dupire's Equation

Consider the SDE

$$\frac{dS_t}{S_t} = rdt + \sigma(t, S_t)dW_t$$

where r is a constant and W_t is a standard Brownina motion. Let $C(T, K)$ be the price of a call option with expiry T and strike K, given $S(0) = S_0$.

Dupire's Equation

Theorem

Let $C(T, K)$ be the price of a call option with strike K and expiry T. Then the following so called Dupire's equation holds

$$C_T + rKC_K - \frac{1}{2}\sigma^2(T, K)K^2 C_{KK} = 0$$

Dupire's Equation

Remark: Dupire's equation is often written as

$$\sigma^2(T, K) = \frac{C_T + rKC_K(T, K)}{\frac{1}{2}K^2 C_{KK}(T, K)} \tag{10}$$

The RHS of Equation (10) can be used to define the notion of local volatility.

Dupire's Equation

- Dupire's equation is a forward equation. It is PDE for the option price with different expiry T and strike K.
- On the contrary, the PDE Equation (1) is the option price with fixed expiry T and strike K, but different time t and spot price S_t.
- In practice, Dupire's equation is often used for model calibration and the PDE Equation (1) is used for pricing.

Dupire's Equation

Proof.

Let $p(0, T, S_0, y)$ be the transition density function for the stock price process that starts at time $t = 0$ with S_0. For notational simplicity, we write it as $p(T, y)$.

The call price is

$$C(T, K) = e^{-rT} \int (y - K)^+ p(T, y) dy$$

Taking partial derivative w.r.t. T, we have

$$\begin{aligned} & C_T(T, K) \\ = \quad & -rC(T, K) + e^{-rT} \int (y - K)^+ \frac{\partial p}{\partial T} dy \end{aligned}$$

$\square$

Dupire's Equation

Proof.

By Kolmogorov forward equation

$$\frac{\partial p}{\partial T} = -\frac{\partial}{\partial y}\left(ryp(T,y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2p(T,y)\right)$$

Hence

$$\begin{aligned}&\int (y-K)^+\frac{\partial p}{\partial T}dy \\ =\ & -\int (y-K)^+\frac{\partial}{\partial y}\left(ryp(T,y)\right)dy \\ &+\frac{1}{2}\int (y-K)^+\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2p(T,y)\right)dy\end{aligned}$$

$\square$

Dupire's Equation

Proof.

Integrating by parts, we have

$$
\begin{aligned}
& \int (y-K)^+ \frac{\partial}{\partial y}\left(ryp(T,y)\right) dy \\
= \;& \int_K^\infty (y-K) \frac{\partial}{\partial y}\left(ryp(T,y)\right) dy \\
= \;& -r \int_K^\infty yp(T,y) dy
\end{aligned}
$$

with the assumption

$$
\lim_{y\to\infty} (y-k) yp(T,y) = 0
$$

□

Dupire's Equation

Proof.

Similarly

$$\begin{aligned} & \int (y-K)^+ \tfrac{\partial^2}{\partial y^2} \left(\sigma^2(T,y) y^2 p(T,y)\right) dy \\ = \; & \int_K^\infty (y-K) \tfrac{\partial^2}{\partial y^2} \left(\sigma^2(T,y) y^2 p(T,y)\right) dy \\ = \; & \sigma^2(T,K) K^2 p(T,K) \end{aligned}$$

with the assumption

$$\lim_{y\to\infty} (y-K) \frac{\partial}{\partial y} \left(\sigma^2(T,y) y^2 p(T,y)\right)$$

and

$$\lim_{y\to\infty} \sigma^2(T,y) y^2 p(T,y) = 0$$

□

Dupire's Equation

Proof.

Hence

$$
\begin{aligned}
& \int (y-K)^+ \tfrac{\partial}{\partial T} p(T,y)dy \\
= \; & r\int_K^\infty yp(T,y)dy + \tfrac{1}{2}\sigma^2(T,K)K^2 p(T,K)
\end{aligned}
$$

□

Dupire's Equation

Proof.

It follows that

$$
\begin{aligned}
& C_T(T,K) \\
= \quad & -rC(T,K) + e^{-rT}\int (y-K)^+ \frac{\partial}{\partial T} p dy \\
= \quad & -rC(T,K) + re^{-rT}\int_K^\infty yp(T,y)dy \\
& +\frac{1}{2}e^{-rT}\sigma^2(T,K)K^2 p(T,K) \\
= \quad & rKe^{-rT}\int_K^\infty p(T,y)dy + \frac{1}{2}e^{-rT}\sigma^2(T,K)K^2 p(T,K) \\
= \quad & -rKC_K(T,K) + \frac{1}{2}\sigma^2(T,K)K^2 C_{KK}(T,K)
\end{aligned}
$$

where in the last equality we use the identities for implied risk-neutral probability density. Q.E.D. □

Understanding Local Volatility

In this section we will look at some facts and properties of local volatility.

- Local variance as the conditional expectation of instantaneous variance
- Local volatility in terms of implied volatility
- Implied variance as the average of local variance over the life of option when there is no skew.

Local Variance as Conditional Expectation of Instantaneous Variance

Theorem (Tanaka-Meyer Formula)

Let X_t be an Ito process such that

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dW_t$$

and K is a real number. Then

$$\begin{aligned} & (X_t - K)^+ \\ = \; & (X_0 - K)^+ + \int_0^t H_K(X_s)dX_s + \frac{1}{2}\int_0^t \delta_K(X_s)d\left[X, X\right](s) \end{aligned}$$

where $H_K(\cdot)$ is the Heaviside function and $\delta_K(\cdot)$ is the Dirac delta function.

Local Variance as Conditional Expectation of Instantaneous Variance

In differential form, we have

$$d\left(X_t - K\right)^+ = H_K(X_t)dX_t + \frac{1}{2}\delta_K(X_t)d\left[X, X\right](t)$$

Local Variance as Conditional Expectation of Instantaneous Variance

If we let $f(x) = (x - K)^+$, then

$$f'(x) = H_K(x)$$

and

$$f''(x) = \delta_K(x)$$

Hence Tanaka-Meyer formula is

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X, X](t)$$

which is a generalization of Ito's formula.

Local Variance as Conditional Expectation of Instantaneous Variance

Now we assume the stock price follows

$$\frac{dS_t}{S_t} = rdt + \sigma(t,\omega)dW_t$$

where σ is an arbitrary adapted-process.

We want to investigate how the instantaneous volatility $\sigma(t,\omega)$ is related to local volatility as defined in Equation (10).

Local Variance as Conditional Expectation of Instantaneous Variance

The price for the call option with expiry T and strike K is

$$C(T, K) = e^{-rT}\mathbb{E}\left[(S_T - K)^+\right]$$

Taking derivative w.r.t. T we have

$$C_T = -rC + e^{-rT}\frac{\partial}{\partial T}\mathbb{E}\left[(S_T - K)^+\right] \tag{11}$$

Local Variance as Conditional Expectation of Instantaneous Variance

To evaluate

$$\frac{\partial}{\partial T}\mathbb{E}\left[(S_T - K)^+\right]$$

we will use Tanaka-Meyer formula.

Local Variance as Conditional Expectation of Instantaneous Variance

Applying Tanaka-Meyer formula, we have

$$
\begin{aligned}
& d(S_T - K)^+ \\
= \ & H_K(S_T)dS_T + \tfrac{1}{2}\delta_K(S_T)d\left[S,S\right](T) \\
= \ & \left[H_K(S_T)rS_T + \tfrac{1}{2}S_T^2\sigma^2(T,\omega)\delta_K(S_T)\right]dT \\
& + H_K(S_T)\sigma(T,\omega)dW_T
\end{aligned}
$$

Taking expectation on both sides, we have

$$
\begin{aligned}
& \mathbb{E}\left[d(S_T - K)^+\right] \\
= \ & \mathbb{E}\left[H_K(S_T)rS_T + \tfrac{1}{2}S_T^2\sigma^2(T,\omega)\delta_K(S_T)\right]dT \\
& + \mathbb{E}\left[H_K(S_T)\sigma(T,\omega)dW_T\right]
\end{aligned}
$$

Local Variance as Conditional Expectation of Instantaneous Variance

Since $H_K(S_T)$ and $\sigma(T,\omega)$ are $\mathcal{F}_T$-measurable, using iterated property of conditional expectation, we have

$$\begin{aligned} & \mathbb{E}\left[H_K(S_T)\sigma(T,\omega)dW_T\right] \\ = & \mathbb{E}\left[\mathbb{E}\left[H_K(S_T)\sigma(T,\omega)dW_T|\mathcal{F}_T\right]\right] \\ = & \mathbb{E}\left[H_K(S_T)\sigma(T,\omega)\mathbb{E}\left[dW_T|\mathcal{F}_T\right]\right] \\ = & 0 \end{aligned}$$

where in the last equality we use the fact that

$$\mathbb{E}\left[dW_T|\mathcal{F}_T\right] = 0$$

Local Variance as Conditional Expectation of Instantaneous Variance

Hence

$$
\begin{aligned}
& \mathbb{E}\left[d(S_T - K)^+\right] \\
= \;& \mathbb{E}\left[H_K(S_T) r S_T + \tfrac{1}{2} S_T^2 \sigma^2(T,\omega)\delta_K(S_T)\right] dT
\end{aligned}
$$

Dividing both sides by dT, we have

$$
\begin{aligned}
& \frac{\partial}{\partial T}\mathbb{E}\left[(S_T - K)^+\right] \\
= \;& \mathbb{E}\left[H_K(S_T) r S_T + \tfrac{1}{2} S_T^2 \sigma^2(T,\omega)\delta_K(S_T)\right] \\
= \;& r\mathbb{E}\left[H_K(S_T) S_T\right] + \tfrac{1}{2}\mathbb{E}\left[S_T^2 \sigma^2(T,\omega)\delta_K(S_T)\right]
\end{aligned} \tag{12}
$$

Local Variance as Conditional Expectation of Instantaneous Variance

To evaluate $\mathbb{E}\left[H_K(S_T)S_T\right]$, we notice that

$$\begin{aligned} & \mathbb{E}\left[H_K(S_T)S_T\right] \\ = \; & \mathbb{E}\left[(S_T - K)H_K(S_T)\right] + K\mathbb{E}\left[H_K(S_T)\right] \end{aligned} \tag{13}$$

Local Variance as Conditional Expectation of Instantaneous Variance

Note that

$$\begin{aligned} \mathbb{E}\left[(S_T - K)H_K(S_T)\right] &= \mathbb{E}\left[(S_T - K)^+\right] \\ &= e^{rT} C \end{aligned} \tag{14}$$

Since

$$\begin{aligned} \frac{\partial}{\partial K}\mathbb{E}\left[(S_T - K)^+\right] &= \mathbb{E}\left[\frac{\partial (S_T - K)^+}{\partial K}\right] \\ &= -\mathbb{E}\left[H_K(S_T)\right] \end{aligned}$$

we have

$$\begin{aligned} \mathbb{E}\left[H_K(S_T)\right] &= -\frac{\partial}{\partial K}\mathbb{E}\left[(S_T - K)^+\right] \\ &= -e^{rT} C_K \end{aligned} \tag{15}$$

Substituting Equation (14) and (15) into Equation (13), we have

$$\mathbb{E}\left[H_K(S_T)S_T\right] = e^{rT}\left(C - KC_K\right) \tag{16}$$

Local Variance as Conditional Expectation of Instantaneous Variance

Now we evaluate $\mathbb{E}\left[S_T^2\sigma^2(T,\omega)\delta_K(S_T)\right]$.

Using tower property of conditional expectation, we have

$$\begin{aligned}
& \mathbb{E}\left[S_T^2\sigma^2(T,\omega)\delta_K(S_T)\right] \\
= \; & \mathbb{E}\left[\mathbb{E}\left[S_T^2\sigma^2(T,\omega)\delta_K(S_T)|S_T\right]\right] \\
= \; & \mathbb{E}\left[\mathbb{E}\left[\sigma^2(T,\omega)|S_T\right]S_T^2\delta_K(S_T)\right]
\end{aligned}$$

Local Variance as Conditional Expectation of Instantaneous Variance

Let $p(x)$ be the probability density function of S_T, then

$$
\begin{aligned}
& \mathbb{E}\left[\mathbb{E}\left[\sigma^2(T,\omega)|S_T\right] S_T^2\delta_K(S_T)\right] \\
= & \int \mathbb{E}\left[\sigma^2(T,\omega)|S_T = x\right] x^2\delta_K(x)p(x)dx \\
= & K^2 p(K)\mathbb{E}\left[\sigma^2(T,\omega)|S_T = K\right]
\end{aligned}
$$

Note that in the last equality we use the following property of Dirac delta function

$$
\int f(x)\delta_K(x)dx = f(K)
$$

Local Variance as Conditional Expectation of Instantaneous Variance

Since

$$\begin{aligned} p(K) &= \frac{\partial^2}{\partial K^2}\mathbb{E}\left[(S_T-K)^+\right] \\ &= e^{rT}C_{KK} \end{aligned}$$

we have

$$\begin{aligned} & \mathbb{E}\left[S_T^2\sigma^2(T,\omega)\delta_K(S_T)\right] \\ = \; & e^{rT}K^2C_{KK}\mathbb{E}\left[\sigma^2(T,\omega)|S_T=K\right] \end{aligned} \tag{17}$$

Local Variance as Conditional Expectation of Instantaneous Variance

Substituting Equation (16) and Equation (17) into Equation (12), we have

$$\begin{aligned} & \frac{\partial}{\partial T}\mathbb{E}\left[(S_T - K)^+\right] \\ = \; & e^{rT}\left(r\left(C - KC_K\right) + \frac{1}{2}K^2 C_{KK}\mathbb{E}\left[\sigma^2(T,\omega)|S_T = K\right]\right) \end{aligned} \tag{18}$$

Local Variance as Conditional Expectation of Instantaneous Variance

Substituting the above equation into Equation (11), we have

$$C_T = -rKC_K + \frac{1}{2}K^2 C_{KK}\mathbb{E}\left[\sigma^2(T,\omega)|S_T = K\right]$$

which implies

$$\mathbb{E}\left[\sigma^2(T,\omega)|S_T = K\right] = \frac{C_T + rKC_K}{\frac{1}{2}K^2 C_{KK}} \tag{19}$$

Local Variance as Conditional Expectation of Instantaneous Variance

Comparing Equation (19) with Equation (10), we can see that local variance is the risk-neutral expectation of the instantaneous variance conditional on the final stock price S_T being equal to strike K.

Local Variance as Conditional Expectation of Instantaneous Variance

We can see that in general the solution for Equation (19) is not unique, there are two instantaneous volatility process $\sigma = \sigma'$ such that

$$\mathbb{E}\left[\sigma(T,\omega)^2 | S_T = K\right] = \mathbb{E}\left[\sigma'(T,\omega)^2 | S_T = K\right]$$

Knowing the vanilla option prices is not enough to find σ in general.

Local Variance as Conditional Expectation of Instantaneous Variance

However if we restrict ourselves to the case $\sigma(t, S_t)$ as a deterministic function of t and S_t, we can uniquely determine $\sigma(t, S_t)$ from vanilla option prices from Dupire's equation.

Local Volatility from Implied Volatility

In practice we often work in terms of implied volatilities as opposed to price. In the following we will derive the local volatility in terms of implied volatilities.

Local Volatility from Implied Volatility

Recall that Dupire's equation

$$\sigma(T, K)^2 = \frac{C_T + rKC_K}{\frac{1}{2}K^2 C_{KK}}$$

We assume

$$C(T, K) = C_{BSM}(T, K, \Sigma(T, K))$$

where C_{BSM} is the BSM pricing formula and $\Sigma(T, K)$ is the implied volatility.

Local Volatility from Implied Volatility

The numerator is

$$\begin{aligned} & C_T + rKC_K \\ = \quad & \frac{\partial C_{BSM}}{\partial T} + \frac{\partial C_{BSM}}{\partial \Sigma}\frac{\partial \Sigma}{\partial T} + rK\left(\frac{\partial C_{BSM}}{\partial K} + \frac{\partial C_{BSM}}{\partial \Sigma}\frac{\partial \Sigma}{\partial K}\right) \end{aligned} \tag{20}$$

Local Volatility from Implied Volatility

From

$$\frac{\partial C_{BSM}}{\partial T} = \frac{S\Sigma\phi(d_1)}{2\sqrt{T}} + re^{-rT}K\Phi(d_2)$$

$$\frac{\partial C_{BSM}}{\partial K} = -e^{-rT}\Phi(d_2)$$

and

$$\frac{\partial C_{BSM}}{\partial \Sigma} = S\sqrt{T}\phi(d_1)$$

we have

$$\frac{\partial C_{BSM}}{\partial T} + rK\frac{\partial C_{BSM}}{\partial K} = \frac{\partial C_{BSM}}{\partial \Sigma}\frac{\Sigma}{2T}$$

Local Volatility from Implied Volatility

Substituting this into Equation (20), the numerator becomes

$$\frac{\partial C_{BSM}}{\partial \Sigma}\left(\frac{\Sigma}{2T}+\frac{\partial \Sigma}{\partial T}+rK\frac{\partial \Sigma}{\partial K}\right) \tag{21}$$

For the denominator, we have

$$
\begin{aligned}
& C_{KK} \\
= \quad & \frac{\partial^2 C_{BSM}}{\partial K^2} + 2\frac{\partial^2 C_{BSM}}{\partial K \partial \Sigma}\frac{\partial \Sigma}{\partial K} + \frac{\partial^2 C_{BSM}}{\partial \Sigma^2}\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \frac{C_{BSM}}{\partial \Sigma}\frac{\partial^2 \Sigma}{\partial K^2}
\end{aligned}
$$

Local Volatility from Implied Volatility

From

$$\frac{\partial^2 C_{BSM}}{\partial K^2} = \frac{\partial C_{BSM}}{\partial \Sigma} \frac{1}{\Sigma T K^2}$$

$$\frac{\partial^2 C_{BSM}}{\partial K \partial \Sigma} = \frac{\partial C_{BSM}}{\partial \Sigma} \frac{d_1}{\Sigma \sqrt{T} K}$$

and

$$\frac{\partial^2 C_{BSM}}{\partial \Sigma^2} = \frac{\partial C_{BSM}}{\partial \Sigma} \frac{d_1 d_2}{\Sigma}$$

we have

$$C_{KK} = \frac{C_{BSM}}{\partial \Sigma} \left(\frac{1}{\Sigma T K^2} + \frac{d_1}{\Sigma \sqrt{T} K} \frac{\partial \Sigma}{\partial K} + \frac{d_1 d_2}{\Sigma} \left(\frac{\partial \Sigma}{\partial K} \right)^2 + \frac{\partial^2 \Sigma}{\partial K^2} \right) \tag{22}$$

Local Volatility from Implied Volatility

Substituting Equation (21) and (22) into Dupire's Equation we have

$$\sigma^2(T,K) = \frac{\frac{\Sigma}{2T} + \frac{\partial\Sigma}{\partial T} + rK\frac{\partial\Sigma}{\partial K}}{\frac{1}{2}K^2\left(\frac{1}{\Sigma T K^2} + \frac{d_1}{\Sigma\sqrt{T}K}\frac{\partial\Sigma}{\partial K} + \frac{d_1 d_2}{\Sigma}\left(\frac{\partial\Sigma}{\partial K}\right)^2 + \frac{\partial^2\Sigma}{\partial K^2}\right)} \tag{23}$$

Implied Variance as the Average of Local Variance

Implied volatility is often interpreted as the market expectation of the average of volatility throughout the life of an option. This is in general not true. However it can be justified when there is no skew.

Implied Variance as the Average of Local Variance

Assume that there is no skew, i.e., $\Sigma(T, K)$ does not depend on K. In this case, the local volatility σ does not depend on K either. From Equation (23), we have

$$\sigma^2(T) = \Sigma^2 + 2T\Sigma\frac{\partial\Sigma}{\partial T}$$

which implies

$$\Sigma^2(T) = \frac{1}{T}\int_0^T \sigma^2(t)dt$$

Implied Variance as the Average of Local Variance

This shows

- implied variance as the average of instantaneous variance for the life of option
- implied volatility is a global measure of volatility
- local volatility is a local measure of volatility for a particular pair of T and K.

Features of Local Volatility Model

Local volatility model

1. A model to value European options in the market consistently.
2. A snapshot of current market rather than a model for the dynamics of volatility.
3. Smiles in local volatility model tend to flat as time advances.

Model Calibration

In practice the models we build usually come with parameters. The process of choosing model parameters to fit market prices (or volatility) is called calibration.

Model Calibration

For a local volatility model, calibration is about finding the local volatility function $\sigma(t, S_t)$.

Two common approaches

- Parametric Approach
- Non-parametric Approach
  - Match prices

$$C_T + rKC_K - \frac{1}{2}\sigma^2(T,K)K^2C_{KK} = 0 \tag{24}$$

  - Match implied volatilities

$$\sigma^2(T,K) = \frac{\frac{\Sigma}{2T} + \frac{\partial \Sigma}{\partial T} + rK\frac{\partial \Sigma}{\partial K}}{\frac{1}{2}K^2\left(\frac{1}{\Sigma T K^2} + \frac{d_1}{\Sigma\sqrt{T}K}\frac{\partial \Sigma}{\partial K} + \frac{d_1 d_2}{\Sigma}\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \frac{\partial^2 \Sigma}{\partial K^2}\right)} \tag{25}$$

Parametric Approach

In the parametric approach, we assume the local volatility surface takes a particular functional form.

Examples

1. CEV model

$$\sigma(t, S_t) = \alpha S_t^{\beta-1}$$

2. Quadratic local volatility

$$\sigma(t, S_t) = \gamma(t)\left(a + bS_t + cS_t^2\right)$$

Parametric Approach

We will use a particular functional form for local volatility surface to show the whole process of model calibration.

We assume the local volatility function takes the form

$$\sigma(t, S_t) = a(t) + b(t)x + c(t)x^2$$

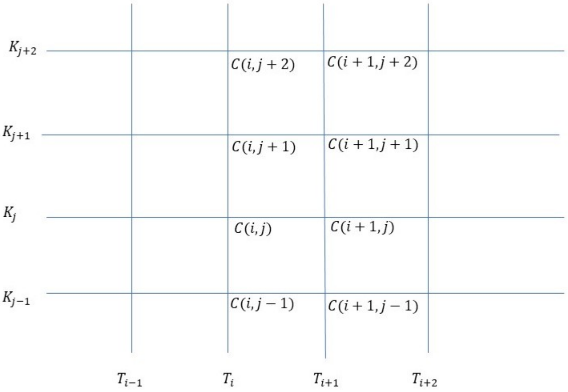
where x is the log-moneyness

$$x = \ln\left(\frac{S_t}{S_0}\right)$$

and $a(t), b(t)$ and $c(t)$ are time-dependent functions.

Parametric Approach

Suppose that we can obtain option prices for expiries $T_1 < T_2 < \ldots < T_m$. And for each expiry T_i, there are n options with strikes $K_{i,j}, j = 1, \ldots, n$.



Parametric Approach

In practice we usually look at the discretized form of $a(t), b(t)$ and $c(t)$. The common choices are piece-wise constant or piece-wise linear.

1. Piece-wise constant

$$a(t) = a_i, b(t) = b_i, c(t) = c_i, \forall t \in [T_{i-1}, T_i)$$

   where a_i, b_i and c_i are constants to be determined

2. Piece-wise linear
   $a(t), b(t)$ and $c(t)$ are linear on each interval $[T_{i-1}, T_i]$

Parametric Approach

In this example, we assume $a(t), b(t)$ and $c(t)$ are piece-wise constant functions.

The model parameters are a_i, b_i and c_i for $1 \leq i \leq m$.

We use a bootstrap process to solve these parameters.

Parametric Approach

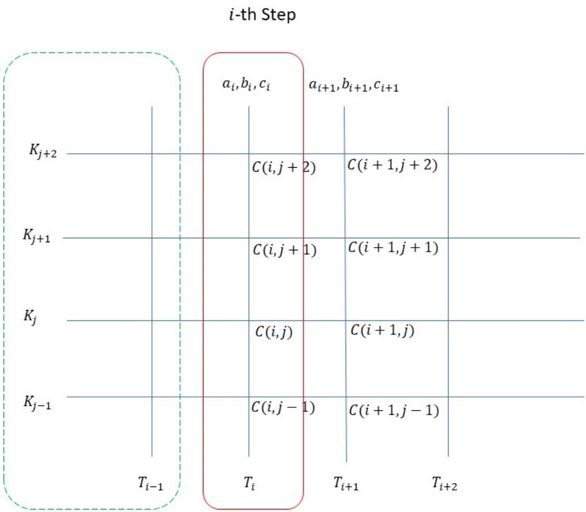
Suppose a_s, b_s and c_s are known for all $s = 1, \ldots, i-1$.

At i-th step, we solve

$$\min_{a_i, b_i, c_i} \sum_{j=1}^{n} \left(C^M(T_i, K_j) - C(T_i, K_j) \right)^2 \tag{26}$$

where $C^M(T_i, K_j)$ are model prices by solving Dupire's forward equation (24).

Parametric Approach



Parametric Approach

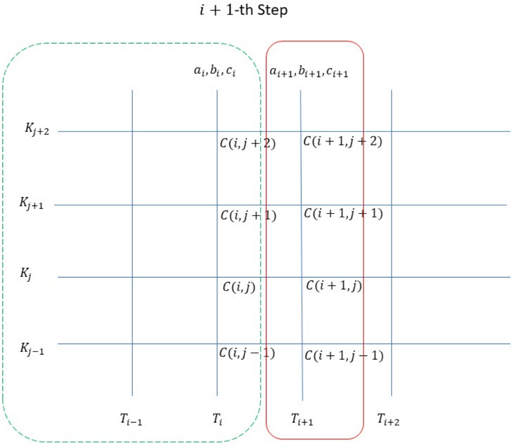
The minimization problem (26) is usually solved by an numerical iteration schema as illustrated below

Algorithm 1: Finding a_i, b_i and c_i

Initialize a_i, b_i, c_i
for $s \leftarrow 1$ **to** S **do**
  Using Dupire's forward equation to solve $C^M(T_i, K_j)$ for all $j = 1, \ldots, n$
  if *converges* **then**
    return a_i, b_i, c_i
  try new values for a_i, b_i, c_i
return a_i, b_i, c_i

Parametric Approach

Once we have solved values for parameter a_i, b_i and c_i, we can move to next layer to solve a_{i+1}, b_{i+1} and c_{i+1}.



Parametric Approach

We can repeat this process until we reach the final (m-th) layer.

$$a_1, b_1, c_1 \to \ldots \to a_m, b_m, c_m$$

Non-parametric Approach

The non-parametric approach does not assume a functional form for the local volatility surface. Instead it aims to recover the whole local volatility surface (in practice a fine grid of strikes and expiries).

There are several ways to do this. We present two of them

- Optimization based
- Interpolation/extrapolation based

Non-parametric Approach

In the optimization based method, we typically have a grid for strike and expiry space. Let $\sigma_{i,j} = \sigma(T_i, K_j)$ be the local volatility for the point (T_i, K_j) on the grid. The calibration becomes the following minimization problem

$$\min_{\sigma_{i,j}} \sum_{i,j} w_{i,j} \left(C^M(T_i, K_j) - C(T_i, K_j) \right)^2$$

where $w_{i,j}$ are weights.

Note that we attempt to solve all $\sigma_{i,j}$ on one go.

Non-parametric Approach

This minimization problem is often ill-posed since the number of parameters $\sigma_{i,j}$ are usually larger than the number of options in the market. Hence the solution is not stable. A small perturbation in the input data will result in very different results.

Non-parametric Approach

To stabilize the calibration, regularization methods are usually employed. One popular method is Tikhonov regularization. For example, the second order regularization is

$$\min_{\sigma_{i,j}} \sum_{i,j} w_{i,j} \left(C^M(T_i, K_j) - C(T_i, K_j) \right)^2 + \lambda \left\| \frac{\partial^2 \sigma}{\partial s^2} + \frac{\partial^2 \sigma}{\partial s \partial t} + \frac{\partial^2 \sigma}{\partial t^2} \right\|$$

where $||\cdot||$ is usually L_2 norm.

Non-parametric Approach

Note that in this approach, $\sigma_{i,j}$ is discrete. In case local volatility is needed for a point not on the grid, interpolation or extrapolation will be needed.

Non-parametric Approach

Another popular non-parametric approach is to use the implied volatility formulation of Dupire's equation (25). In this approach, we interpolate/extrapolate an implied volatility surface. The local volatilities can then be calculated from Equation (25).

$$IVS \Longrightarrow LVS$$

Non-parametric Approach

The implied volatility surface can be constructed in various ways (see for example Borovkova & Permanab (2009)).

Examples:

- Parametric

$$\sigma(x, T) = b_1 + b_2 x + b_3 x^2 + b_4 T + b_5 x T$$

  where $x = \ln\left(\frac{S_t}{S_0}\right) / T$.

- Semi-parametric

$$\sigma(x, T) = b_{1,T} + b_{2,T} x + b_{3,T} x^2$$

- Non-parametric
  Cubic spline and its variant etc.

Comparison Between Different Approaches

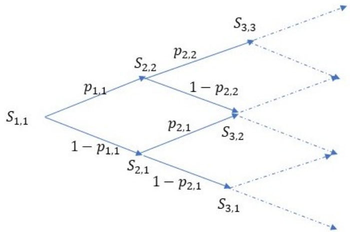
- Parametric approach usually gives more smooth local volatility surface. However in the case the number of parameters are larger than available market option prices. There may be an identifiability issue.
- The optimization based non-parametric approach is usually a high dimension optimization problem and slow to solve.
- It is easy to implement the implied volatility approach. However the extrapolation becomes difficult for strikes far from ATM. There is no guarantee that the local variance is positive.

Implied Binomial Tree

Binomial tree model is often used as a prototype model. How do we incorporate smile in binomial tree?

Implied Binomial Tree

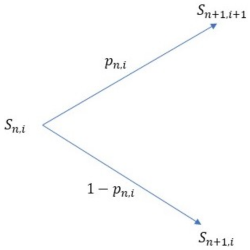
Let $\Delta t = \frac{T}{N}$ and $t_n = n\Delta t$. We build a binomial tree as below



- The node (n, i) corresponds to the i-th stock price level at time t_n
- At the node (n, i), the (implied risk neutral) probability of the stock price moves up to $S_{n+1,i+1}$ is $p_{n,i}$, and moves down to $S_{n+1,i}$ is $1 - p_{n,i}$.
- The initial price $S_{1,1} = S_0$.

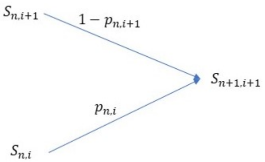
Implied Binomial Tree

Structure of implied binomial tree



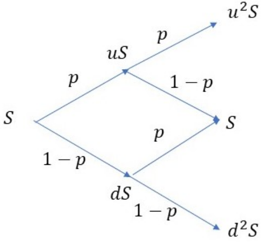
Implied Binomial Tree

Structure of implied binomial tree for interior nodes



Implied Binomial Tree

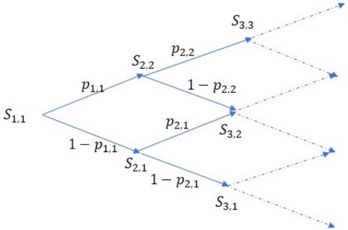
For BSM model, the binomial tree is homogeneous where the price goes up (or down) by a factor of u (or d) and the transition probability is p for all nodes.



$$u > 1, d < 1, ud = 1$$

Implied Binomial Tree

The implied binomial tree is inhomogeneous as we need to adjust S_u, S_d and p for each node to match market prices of options.



Implied Binomial Tree

In binomial tree, at each node, we choose stock price S_u, S_d and transition probability p for the next period to match the first two moments of the stock price dynamics.

In implied binomial tree, for each node, we choose S_u, S_d and p to match the first moment and option prices.

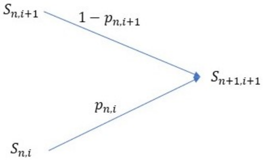
Implied Binomial Tree

Let $\lambda_{n,i}$ be the risk neutral probability that the stock price reaches to the node (n, i).

Note that $e^{-rt_n}\lambda_{n,i}$ is the Arrow-Debreu price of a derivative that pays out 1 if the stock price is $S_{n,i}$ at time t_n and zero otherwise.

Implied Binomial Tree

From



we have

$$\lambda_{n+1,i+1} = \lambda_{n,i+1}(1 - p_{n,i+1}) + \lambda_{n,i} p_{n,i} \tag{27}$$

where $1 \leq i \leq n-1$ for interior nodes.

Implied Binomial Tree

For boundary nodes, we have

$$\lambda_{n+1,n+1} = \lambda_{n,n} p_{n,n} \tag{28}$$

and

$$\lambda_{n+1,1} = \lambda_{n,1}(1 - p_{n,1}) \tag{29}$$

Implied Binomial Tree

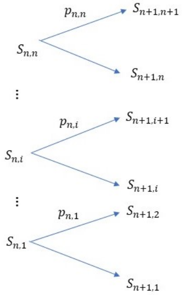
Together with the initial condition that $\lambda_{1,1} = 1$, Equation (27), (28) and (29) provide an iterative schema to compute $\lambda_{n,i}$ provided that we know $p_{n,i}$.

We compute $p_{n,i}$ and $S_{n,i}$ by forward induction.

Implied Binomial Tree

Suppose we have obtained values of $\lambda_{n,i}$, $p_{n,i}$ and $S_{n,i}$ for all i at n-th level. To move to level $n+1$, what are the parameters we need to compute?

Implied Binomial Tree



- Transition probability $p_{n,i}$ for $1 \leq i \leq n$
- $S_{n+1,i}$ for $1 \leq i \leq n+1$.

There are a total of $2n+1$ parameters.

Implied Binomial Tree

We impose two constraints

- Match the first moment
- Match the market prices of call/put options

Implied Binomial Tree

To match the first moment, we must have

$$S_{n,i} = e^{-r\Delta t}(p_{n,i}S_{n+1,i+1} + (1 - p_{n,i})S_{n+1,i})$$

for $i = 1, \ldots, n$.

If we denote $F_{n,i} = e^{r\Delta t}S_{n,i}$, the constraint can be written as

$$F_{n,i} = p_{n,i}S_{n+1,i+1} + (1 - p_{n,i})S_{n+1,i}$$

Solving the above equation for $p_{n,i}$, we have

$$p_{n,i} = \frac{F_{n,i} - S_{n+1,i}}{S_{n+1,i+1} - S_{n+1,i}} \tag{30}$$

Note that $S_{n+1,i}$ and $S_{n+1,i+1}$ are unknown.

Implied Binomial Tree

Let $C(T,K)$ be the undiscounted price of a call option with strike K and expiry T. Using risk neutral pricing formula, we have

$$C(t_{n+1},K)=\sum_{j=1}^{n+1}\lambda_{n+1,j}\left(S_{n+1,j}-K\right)^{+}$$

Note that the summation is over all the nodes at time t_{n+1}

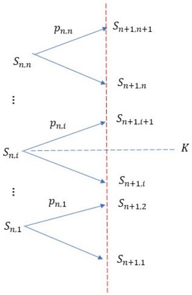
Implied Binomial Tree

Note that the price of a call option with strike $S_{n,i}$ and expiry t_{n+1} can reduce to

$$C(t_{n+1}, S_{n,i}) = \sum_{j=i+1}^{n+1} \lambda_{n+1,j} \left(S_{n+1,j} - S_{n,i}\right)$$

Implied Binomial Tree

This is illustrated in the figure below



Implied Binomial Tree

Since

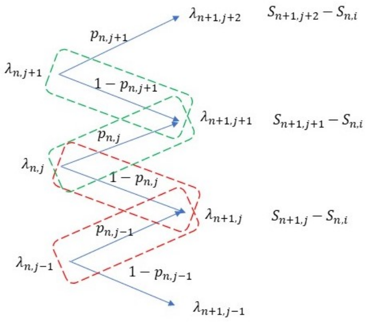
$$\lambda_{n+1,j} = p_{n,j-1}\lambda_{n,j} + (1 - p_{n,j})\lambda_{n,j}$$

we have

$$C(t_{n+1}, S_{n,i}) = \sum_{j=i+1}^{n+1} \left(p_{n,j-1}\lambda_{n,j} + (1 - p_{n,j})\lambda_{n,j}\right)\left(S_{n+1,j} - S_{n,i}\right) \tag{31}$$

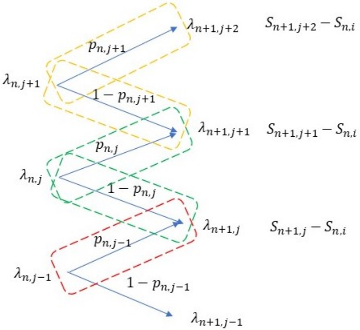
with the convention that $\lambda_{n,n+1} = 0$.

Implied Binomial Tree



Implied Binomial Tree

We can reorganize the terms in the summation as below



Implied Binomial Tree

Note that

$$\begin{aligned}
& \lambda_{n,j} p_{n,j} \left(S_{n+1,j+1} - S_{n,i}\right) + \lambda_{n,j}(1 - p_{n,j}) \left(S_{n+1,j} - S_{n,i}\right) \\
= & \lambda_{n,j} \left[\left(p_{n,j} S_{n+1,j+1} + (1 - p_{n,j}) S_{n+1,j}\right) - S_{n,i}\right] \\
= & \lambda_{n,j} \left(F_{n,j} - S_{n,i}\right)
\end{aligned}$$

Using this to simplify the Equation (31), we have

$$C(t_{n+1}, S_{n,i}) = \lambda_{n,i} p_{n,i} \left(S_{n+1,i+1} - S_{n,i}\right) + \sum_{j=i+1}^{n} \lambda_{n,j} \left(F_{n,j} - S_{n,i}\right) \tag{32}$$

Implied Binomial Tree

Combing it with Equation (30), we can solve $S_{n+1,i+1}$ as a function of $S_{n+1,i}$

$$
\begin{aligned}
& S_{n+1,i+1} \\
= \quad & \frac{S_{n+1,i}\left[C(t_{n+1},S_{n,i})-\sum_{j=i+1}^{n}\lambda_{n,j}\left(F_{n,j}-S_{n,i}\right)\right]-S_{n,i}\lambda_{n,i}\left(F_{n,i}-S_{n+1,i}\right)}{\left[C(t_{n+1},S_{n,i})-\sum_{j=i+1}^{n}\lambda_{n,j}\left(F_{n,j}-S_{n,i}\right)\right]-\lambda_{n,i}\left(F_{n,i}-S_{n+1,i}\right)}
\end{aligned}
$$

Note that $C(t_{n+1}, S_{n,i})$ is obtained from market and $\lambda_{n,j}, (F_{n,j}$ and $S_{n,i}$ are known by induction, hence if we know the value for $S_{n+1,i}$, we can compute $S_{n+1,i+1}$.

Implied Binomial Tree

If we know $S_{n+1,i}$ for some i, then we know the value of $S_{n+1,j}$ for all $j \geq i$

$$S_{n+1,i} \longrightarrow S_{n+1,i+1} \longrightarrow \dots \longrightarrow S_{n+1,n+1}$$

Implied Binomial Tree

We choose i to be around the center

- If $n+1$ is odd, we choose $i = \frac{n+2}{2}$

$$S_{n+1,i} = S_0$$

- If $n+1$ is even, we choose $i = \frac{n+1}{2}$ and let

$$S_{n+1,i} = \frac{S_0^2}{S_{n+1,i+1}}$$

Implied Binomial Tree

For the nodes below the center node, we can similarly use put options to compute $S_{n+1,i}$ and $p_{n,i}$, using the formula

$$
\begin{aligned}
& S_{n+1,i} \\
= \quad & \frac{S_{n+1,i+1}\left[P(t_{n+1},S_{n,i})-\sum_{j=i+1}^{n}\lambda_{n,j}\left(S_{n,i}-F_{n,j}\right)\right]-S_{n,i}\lambda_{n,i}\left(F_{n,i}-S_{n+1,i+1}\right)}{\left[P(t_{n+1},S_{n,i})-\sum_{j=i+1}^{n}\lambda_{n,j}\left(S_{n,i}-F_{n,j}\right)\right]+\lambda_{n,i}\left(F_{n,i}-S_{n+1,i}\right)}
\end{aligned}
$$

We can solve $S_{n+1,i}$ in the reverse order

$$
S_{n+1,i} \longrightarrow S_{n+1,i-1} \longrightarrow \ldots \longrightarrow S_{n+1,1}
$$

Implied Binomial Tree

To avoid arbitrage, we must have

$$0 \leq p_{n,i} \leq 1$$

Hence the stock price must satisfy

$$F_{n,i} \leq S_{n+1,i+1} \leq F_{n,i+1}$$

In case the resulting prices from the above process don't satisfy this condition, we can may choose the prices as

$$\ln\left(\frac{S_{n+1,i+1}}{S_{n+1,i}}\right) = \ln\left(\frac{S_{n,i}}{S_{n,i-1}}\right)$$

or

$$S_{n+1,i+1} = \frac{F_{n,i} + F_{n,i+1}}{2}$$

This does not guarantee the absence of negative probability.

Implied Binomial Tree

What is the local volatility for the node (n, i)?

$$\sigma^2(t_n, S_{n,i})\Delta t = p_{n,i}\left(1 - p_{n,i}\right)\ln\left(\frac{S_{n+1,i+1}}{S_{n+1,i}}\right)$$

This can be obtained by using the fact

$$Var\left[\Delta \ln S(t)|\mathcal{F}_t\right] \approx \sigma^2(t, S_t)\Delta t$$

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Thank you!