NATIONAL UNIVERSITY OF SINGAPORE

FE5112 - Stochastic Calculus and Quantitative Methods

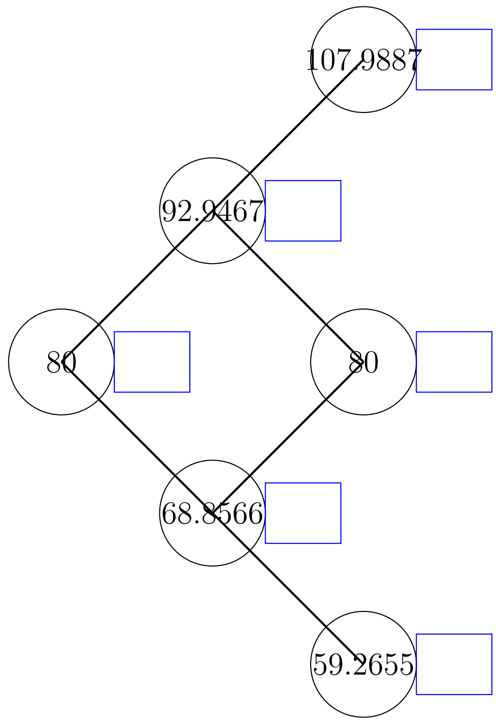
(Semester 1 : AY2017/2018)

Time allowed : $2\frac{1}{2}$ hours

INSTRUCTIONS TO CANDIDATES

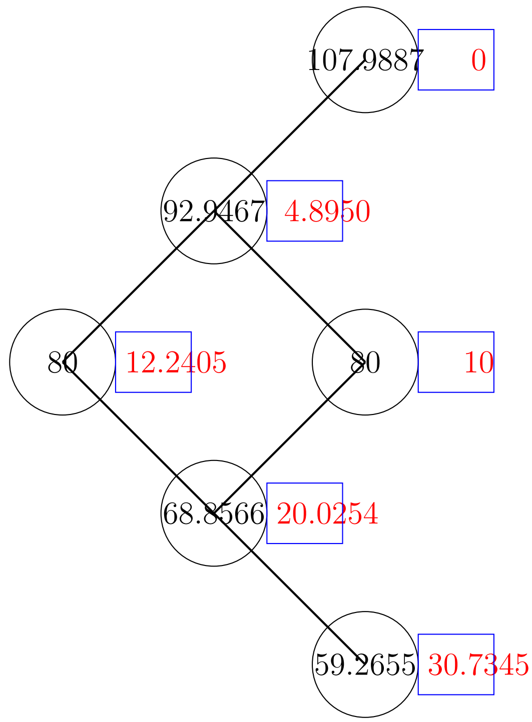
1. Please write your student number only. Do not write your name.
2. This assessment paper contains **SIX** questions and comprises **THREE** printed pages.
3. The total mark for this paper is 100.
4. Answer **ALL** questions.
5. Please start each question on a new page.
6. This is a CLOSED BOOK examination. However, students are allowed to bring an A4 sized help sheet which can be written on both sides.
7. Students are allowed to use scientific calculators.
8. Students should lay out systematically the various steps in the calculations.
9. Students are not allowed to take this assessment paper away from the examination hall.

Question 1 [20 marks] Consider the problem of using binomial tree method to calculate the European and American put options with $S_0 = 80$, $K = 90$, $\mathtt{T} = 0.5$, $r = 0.05$, $\sigma = 0.3$. We set $\delta t = \mathtt{T}/2$ and construct a two step binomial tree with $u = e^{\sigma\sqrt{\delta t}} \approx 1.16183$, $d = e^{-\sigma\sqrt{\delta t}} \approx 0.860708$, $\rho = e^{r\delta t} \approx 1.01258$. Evaluate the European and American put option prices based on the binomial tree. Keep at least 5 significant digits in your calculation.



Solution:

$q_u = \frac{\rho - d}{u - d} \approx 0.504342$, $q_d = 1 - q_u$.

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Put_Eu = 12.2405
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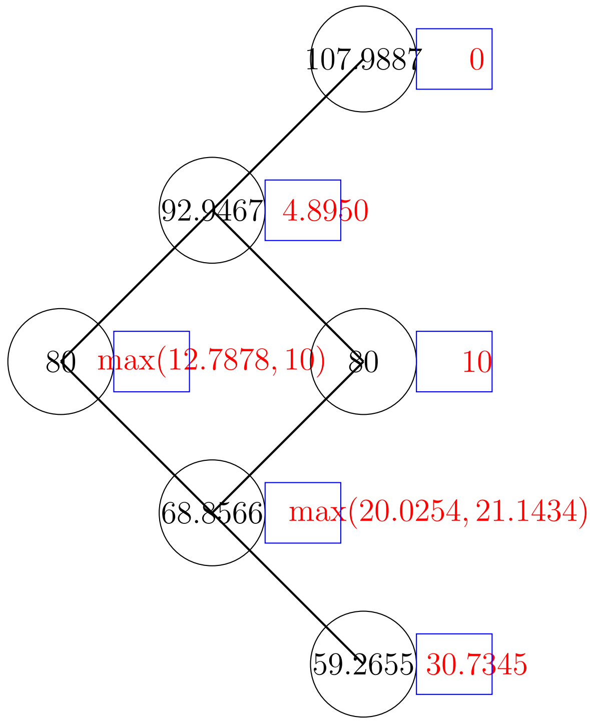

`Put_Am = 12.7878`

Question 2 [20 marks] Let W_t be a one-dimensional Brownian motion with $W_0 = 0$.

a) Compute dW_t^4 and then write W_T^4 as the sum of an ordinary integral with respect to time and an Itô integral.

b) Take expectations on both sides of the formula you obtained in a), use the fact that $\mathbb{E}[W_t^2] = t$, and derive the formula $\mathbb{E}[W_T^4] = 3T^2$.

c) Use the method of a) and b) to derive a formula for $\mathbb{E}[W_T^6]$.

Solution: a)

$$dW_t^4 = 4W_t^3 dW_t + \frac{1}{2} 12 W_t^2 dt.$$

$$\begin{aligned} W_T^4 =& W_0^4 + 4\int_0^T W_t^3 dW_t + 6\int_0^T W_t^2 dt \\ =& 4\int_0^T W_t^3 dW_t + 6\int_0^T W_t^2 dt. \end{aligned}$$

b)

$$\mathbb{E}[W_T^4] = 6\int_0^T \mathbb{E}[W_t^2] dt = 6\int_0^T t dt = 6\frac{T^2}{2} = 3T^2.$$

c)

$$dW_t^6 = 6W_t^5 dW_t + \frac{1}{2} 30 W_t^4 dt.$$

$$W_T^6 = W_0^6 + 6\int_0^T W_t^5 dW_t + 15\int_0^T W_t^4 dt.$$

$$\mathbb{E}[W_T^6] = 15\int_0^T \mathbb{E}[W_t^4] dt = 15\int_0^T 3t^2 dt = 15T^3.$$

Question 3 [15 marks] Suppose that X_t is the Ornstein-Uhlenheck process $dX_t = -\alpha X_t dt + \sigma dW_t$. Find the constant β so that $Y_t = X_t^2$ satisfies the CIR model

$$dY_t = (\beta - 2\alpha Y_t)dt + 2\sigma\sqrt{Y_t}d\tilde{W}_t,$$

where $\tilde{W}_t = \int_0^t \frac{X_s}{|X_s|}dW_s$. You do not need to prove, but it is good to know that by Theorem 4.5 of the lecture notes (Levy's criteria of Brownian motion), $\tilde{W}_t$ is a Brownian motion.

Solution: Note that $\sqrt{Y_t} = |X_t|$ and $d\tilde{W}_t = \frac{X_t}{|X_t|}dW_t$.

$$\begin{aligned} dY_t =& 2X_t dX_t + \frac{1}{2}2(dX_t)^2 \\ =& -2\alpha(X_t)^2 dt + 2\sigma X_t dW_t + \sigma^2 dt \\ =& \left(\sigma^2 - 2\alpha Y_t\right) dt + 2\sigma|X_t|\frac{X_t}{|X_t|}dW_t \\ =& \left(\sigma^2 - 2\alpha Y_t\right) dt + 2\sigma\sqrt{Y_t}d\tilde{W}_t. \end{aligned}$$

Hence $\beta = \sigma^2$.

Question 4 [15 marks] Recall that a portfolio $\Phi = \Delta S + B$ is call self-financing portfolio if it satisfies both $d\Phi_t = \Delta_t dS_t + dB_t$ and $\Phi_t = \Delta_t S_t + B_t$. Here S_t is the stock price, Δ_t is the number of shares of stock, B_t is the amount in the money market account at time t. Prove that for any stock price model, a self-financing portfolio $\Phi = \Delta S + B$ satisfies

$$d\left(e^{-rt}\Phi_t\right) = \Delta_t d\left(e^{-rt}S_t\right) \tag{1}$$

which means that change in the discounted portfolio value is solely due to change in the discounted stock price. The parameter r in (1) comes from the interest rate of the money market account B whose value satisfies $dB_t = rB_t dt$. [If you can only prove (1) for the geometric Brownian motion stock price model, you can only get 10 marks.]

Proof: By Itô formula with $g(t,x) = e^{-rt}x$,

$$\begin{aligned} d(e^{-rt}S_t) =& dg(t,S_t) = \frac{\partial g}{\partial x}dt + \frac{\partial g}{\partial x}dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dS_t)^2 \\ =& -re^{-rt}S_t dt + e^{-rt}dS_t, \end{aligned}$$

$$\begin{aligned} d(e^{-rt}\Phi_t) =& dg(t,\Phi_t) = \frac{\partial g}{\partial x}dt + \frac{\partial g}{\partial x}d\Phi_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(d\Phi_t)^2 \\ =& -re^{-rt}\Phi(t)dt + e^{-rt}d\Phi(t) \\ =& -re^{-rt}\left(\Delta_t S_t + B_t\right)dt + e^{-rt}\left(\Delta_t dS_t + dB_t\right) \\ =& -re^{-rt}\Delta_t S_t dt + e^{-rt}\Delta_t dS_t \\ =& \Delta_t\left(-re^{-rt}S_t dt + e^{-rt}dS_t\right) \\ =& \Delta_t d(e^{-rt}S_t). \end{aligned}$$

Question 5 [15 marks] Determine the constant α so that

$$X_t = e^{\alpha t}\cos(W_t)$$

is a martingale. (Hint: Show that the differential dX_t has no dt term. Why does this imply that X_t is a martingale?)

Solution:

$$\begin{aligned} dX_t =& \alpha e^{\alpha t}\cos(W_t)dt + e^{\alpha t}(-\sin(W_t))dW_t + \frac{1}{2}e^{\alpha t}\left(-\cos(W_t)\right)dt \\ =& \left(\alpha - \frac{1}{2}\right)e^{\alpha t}\cos(W_t)dt - e^{\alpha t}\sin(W_t)dW_t. \end{aligned}$$

Hence when $\alpha = \frac{1}{2}$, $X_t = X_0 - \int_0^t e^{\alpha s}\sin(W_s)dW_s$. Because Itô integral is a martingale, X_t is a martingale.

Question 6 [15 marks] First show that the function $f(t,x) = x - e^{-r(T-t)}K$ satisfies

$$\frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 f}{\partial x^2}(t,x) - rf(t,x) = 0. \tag{2}$$

Then use this result to show that if $c(t,x)$ satisfies (2) with f replaced by c and the terminal condition $c(T,x) = (x-K)^+$, then $p(t,x) = c(t,x) - x + e^{-r(T-t)}K$ satisfies (2) with f replaced by p and the terminal condition $p(T,x) = (K-x)^+$.
Proof:

$$\begin{aligned} &\frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 f}{\partial x^2}(t,x) - rf(t,x) \\ =& -re^{-r(T-t)}K + rx - r\left(x - e^{-r(T-t)}K\right) = 0. \end{aligned}$$

Since (2) is a linear equation in f, it is clear that if both $f_1(t,x)$ and $f_2(t,x)$ satisfy (2), then for any constants λ_1 and λ_2, $c(t,x) \stackrel{\text{def}}{=} \lambda_1 f_1(t,x) + \lambda_2 f_2(t,x)$ satisfies (2). (This is the so-called superposition principle for linear equations.) Hence p satisfies (2) with f replaced by p. Finally,

$$p(T,x) = c(T,x) - x + K = (x-K)^+ - x + K = (K-x)^+.$$

END OF PAPER