Lecture 4 - Continuous-Time Interest Rate Models

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The Martingale Approach

- A prevailing approach to the pricing of bonds and interest rate derivatives that uses the theory of martingales to establish prices and hedging strategies.
- Some of the earliest descriptions of this approach can be found in the papers by Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983).

The Martingale Approach

- Consider a term-structure model with a one-dimensional Brownian motion as the only source of randomness (one factor).

- Suppose that we have the stochastic differential equations (SDEs) for the risk-free rate of interest, $r(t)$, and the price at time t of a zero-coupon bond $P(t, T)$ which matures at time T, then

$$dr(t) = a(t)dt + b(t)dW(t) \tag{1}$$

$$dP(t, T) = P(t, T)[m(t, T)dt + S(t, T)dW(t)] \tag{2}$$

  where

  - $a(t)$, $b(t)$, $m(t, T)$, and $S(t, T)$ are previsible functions (so they can be stochastic).
  - a short-rate model takes $a(t)$ and $b(t)$ as given and determines $m(t, T)$ and $S(t, T)$ accordingly.

The Martingale Approach

- Associated with these processes we have the risk-free cash account, $B(t)$, which satisfies the SDE

$$dB(t) = r(t)B(t)dt$$

to which the solution is

$$B(t) = B(0) \exp \left[\int_0^t r(u)du \right].$$

- Define the market price of risk to be the previsible process

$$\gamma(t) = \frac{m(t,T) - r(t)}{S(t,T)}$$

which represents the excess expected return per unit of volatility.

Fundamental Theorem

- Consider an interest rate derivative contract which pays X_S ($\mathcal{F}_S$-measurable) at time $S < T$.
- What is the no-arbitrage price, $V(t)$, at time $t < S$ for this contract, given the short rate process $r(t)$ and bond price process $P(t, T)$?
- **FT**: There exists a measure Q equivalent to P (i.e., $P(A) > 0 \iff Q(A) > 0$) with

$$V(t) = E_Q\left[\exp\left(-\int_t^S r(u)du\right) X_S | \mathcal{F}_t\right]$$

  where

$$dr(t) = (a(t) - \gamma(t)b(t))dt + b(t)d\tilde{W}(t)$$

  and $\tilde{W}(t)$ is a standard Brownian motion under Q.

Math Review: Product Rule

- Suppose $X(t)$ and $Y(t)$ are one-dimensional diffusion processes:

$$dX(t) = a_X(t)dt + b_X(t)dW(t);$$
$$dY(t) = a_Y(t)dt + b_Y(t)dW(t).$$

- Let $R(t) = X(t)Y(t)$. Then,

$$dR(t) = X(t)dY(t) + Y(t)dX(t) + d\langle X, Y\rangle(t).$$

where

$$d\langle X, Y\rangle(t) = b_X(t)b_Y(t)dt$$

- $d\langle X, Y\rangle(t)$ is often written as $dX(t)dY(t)$ and we apply "box calculus".

Step 1

- Define the discounted price process

$$Z(t,T) = \frac{P(t,T)}{B(t)} = P(t,T)\exp\left(-\int_0^t r(u)du\right).$$

- As in discrete time, the key step is to establish the probability measure Q equivalent to P under which the discounted price process, $Z(t,T)$, is a martingale.

- By the product rule we have

$$\begin{aligned} dZ(t,T) = {} & B(t)^{-1}dP(t,T) + P(t,T)d(B(t)^{-1}) \\ & + d\left\langle B^{-1}, P\right\rangle(t). \end{aligned}$$

Step 1

- By Itô's formula

$$
\begin{aligned}
d(B(t)^{-1}) &= -\frac{1}{B(t)^2}dB(t) + \frac{1}{2}\frac{2}{B(t)^3}d\langle B\rangle(t) \\
&= -\frac{r(t)dt}{B(t)} \ (dB\,(t) \ \text{ has no volatility})
\end{aligned}
$$

- Hence,

$$
\begin{aligned}
dZ(t,T) &= \frac{P(t,T)}{B(t)}(m(t,T)dt + S(t,T)dW(t)) \\
&\quad - \frac{r(t)P(t,T)dt}{B(t)} \ (d\left\langle B^{-1}, P\right\rangle(t) = 0 \text{ by product rule}) \\
&= Z(t,T)[(m(t,T) - r(t))dt + S(t,T)dW(t)]
\end{aligned}
$$

- $Z(t,T)$ is not a martingale under the real world probability P.

Example: Probability Density Function

- Given $X \sim \text{Uniform}[0,1]$ under P, define a random variable
$$Y = 2X.$$
Then we can easily show that
$$Y > 0, \;\; \mathrm{E}[Y] = 1.$$
- Define a new probability $\tilde{\mathrm{P}}$ such that for any set A
$$\tilde{\mathrm{P}}(A) = \mathrm{E}(1_A \cdot Y).$$
- For instance,
$$\begin{aligned} \tilde{\mathrm{P}}[(0,1/2)] &= \frac{1}{4} \text{ and } \tilde{\mathrm{P}}[(1/2,1)] = \frac{3}{4}, \text{ whereas} \\ \mathrm{P}[(0,1/2)] &= \frac{1}{2} \text{ and } \mathrm{P}[(1/2,1)] = \frac{1}{2}. \end{aligned}$$
- In fact, $\tilde{\mathrm{P}}$ is the distribution of $\sqrt{X}$ whose CDF is $F(x) = x^2$ and pdf is $2x$.

Change of Probabilities for a Random Variable

- Y is a random variable with $\mathrm{P}(Y \geq 0) = 1$ and $\mathrm{E}(Y) = 1$ (think about Y being a density). Then for any set A we can define
$$\tilde{\mathrm{P}}(A) = \mathrm{E}(1_A \cdot Y),$$
- Then it can be shown that $\tilde{\mathrm{P}}$ is a new probability.
- It can be shown that the transform that goes from E to $\tilde{\mathrm{E}}$ is given by
$$\tilde{\mathrm{E}}[X] = \mathrm{E}[X \cdot Y]. \tag{3}$$

Going Back and Forth

- When $Y > 0$ P-almost surely, for any random variable X, we can introduced a new random variable $\frac{X}{Y}$. Now by (3)

$$\tilde{\mathrm{E}}\left[\frac{X}{Y}\right] = \mathrm{E}\left[\frac{X}{Y} \cdot Y\right] = \mathrm{E}\left[X\right],$$

which give the transform from $\tilde{\mathrm{E}}$ to E:

$$\mathrm{E}\left[X\right] = \tilde{\mathrm{E}}\left[\frac{X}{Y}\right]. \tag{4}$$

- Such Y is called the Radon-Nikodym derivative, denoted as

$$\frac{d\tilde{\mathrm{P}}}{d\mathrm{P}}.$$

Example: Normal Random Variables

- Given $X \sim N(0,1)$ under P, define a random variable

$$Y = \exp\left\{-\gamma X - \frac{1}{2}\gamma^2\right\}.$$

Then we can easily show that

$$Y > 0, \quad \mathrm{E}\left[Y\right] = 1 \tag{5}$$

- Define a new probability $\widetilde{\mathrm{P}}$ such that for any set A

$$\widetilde{\mathrm{P}}\left(A\right) = \mathrm{E}\left(1_A \cdot Y\right).$$

Now consider

$$\tilde{X} = X + \gamma.$$

Then, under P, $\tilde{X}$ has a distribution of $N(\gamma, 1)$, and under $\widetilde{\mathrm{P}}$, $\tilde{X}$ has a distribution of $N(0,1)$.

- The probability density function of X is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, x \in \mathbb{R}.$$

- For any constant $\gamma \in \mathbb{R}$, the probability density function of $X - \gamma$ is

$$f(x+\gamma) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(x+\gamma)^2}{2}} = f(x) e^{-\gamma x - \frac{\gamma^2}{2}}, \quad x \in \mathbb{R}.$$

- Then, for every bounded function ψ, we have

$$\begin{aligned} \mathrm{E}[\psi(X-\gamma)] &= \int \psi(x) f(x) e^{-\gamma x - \frac{\gamma^2}{2}} dx \\ &= \mathrm{E}[\psi(X) e^{-\gamma X - \frac{\gamma^2}{2}}] = \tilde{\mathrm{E}}[\psi(X)], \end{aligned}$$

which means that !! the P-distribution of $X - \gamma$ coincides with the $\tilde{\mathrm{P}}$-distribution of X, i.e.

under $\tilde{\mathrm{P}}$, $X + \gamma$ is distributed as $N(0, 1)$.

Change of Measure for Stochastic Processes

- First, we introduce a $\mathcal{F}_T$-measurable random variable Y such that
$$Y > 0, \quad \mathrm{E}\left[Y\right] = 1.$$
- Second, we define a Radon-Nikodym derivative process
$$Y(t) = \mathrm{E}\left[Y|\mathcal{F}_t\right], \quad 0 \leq t \leq T.$$
  - $Y(t)$ is a martingale with respect to P (Tower Property);
  - For any $X_T \in \mathcal{F}_T$, define
$$\tilde{\mathrm{E}}\left(X_T\right) \equiv \mathrm{E}\left(X_T \cdot Y\left(T\right)\right)$$
  - Again, denote the corresponding new measure by $\tilde{\mathrm{P}}$ (i.e., $\tilde{\mathrm{P}}\left(A\right) = \mathrm{E}\left(1_A \cdot Y\right)$ for each $A \in \mathcal{F}_T$).

Radon-Nikodym Derivative Process

- Now suppose that $X_t \in \mathcal{F}_t$. We have
$$\begin{aligned}\widetilde{\mathrm{E}}(X_t) &= \mathrm{E}(X_t \cdot Y(T)) \\ &= \mathrm{E}(\mathrm{E}(X_t \cdot Y(T)\,|\mathcal{F}_t)) \\ &= \mathrm{E}(X_t \cdot \mathrm{E}(Y(T)\,|\mathcal{F}_t)) \\ &= \mathrm{E}(X_t \cdot Y(t)).\end{aligned}$$
  - This means $Y(t)$ can be used to evaluate the expectation of $X_t \in \mathcal{F}_t$ under the new measure $\widetilde{\mathrm{P}}$.
- Thus, (**Exercise**) for $X \in \mathcal{F}_t$, we have
$$\widetilde{\mathrm{E}}[X|\mathcal{F}_s] = \mathrm{E}\left[X \cdot \frac{Y(t)}{Y(s)}|\mathcal{F}_s\right], \quad 0 \le s \le t \le T. \tag{6}$$
- This means we could more generally view
$$\frac{Y(t)}{Y(s)} = \frac{d\widetilde{\mathrm{P}}}{d\mathrm{P}}(s,t)$$
as the Radon-Nikodym derivative to evaluate the expectation of $X \in \mathcal{F}_t$ and conditional on $\mathcal{F}_s$ under the new measure $\widetilde{\mathrm{P}}$.

Cameron and Martin Formula

- Suppose that under probability P, $W(t)$ is a standard Brownian motion.
- For any constant γ, and $T > 0$ and $T \geq t \geq 0$, we can choose a special Radon-Nikodym derivative process
$$Y(t) := \exp(-\gamma W(t) - \frac{1}{2}\gamma^2 t).$$
- Define a new probability $\tilde{\mathrm{P}}(\cdot)$ on $\mathcal{F}_T$ by
$$\tilde{\mathrm{P}}(A) := \mathrm{E}(1_A \cdot Y(T)), \quad A \in \mathcal{F}_T.$$
- Then the two probability measures $\tilde{\mathrm{P}}$ and P are equivalent when restricted on $\mathcal{F}_T$.
- Cameron and Martin (1944): under $\tilde{\mathrm{P}}$, $\tilde{W}(t) = W(t) + \gamma t$ is a standard Brownian motion, and therefore, $W(t) = \tilde{W}(t) - \gamma t$ is a Brownian motion with drift $-\gamma$ and variance one.

Girsanov Theorem

- Let $W(t)$ be a standard Brownian motion under P. For a process $\gamma(s) \in \mathcal{F}_s$ consider the process

$$Y(t) = \exp\left\{-\int_0^t \gamma(s)dW(s) - \frac{1}{2}\int_0^t \gamma(s)ds\right\}$$

- Assume that Novikov's condition holds:

$$\mathrm{E}\left[\exp\left\{\frac{1}{2}\int_0^T \gamma(t)^2 dt\right\}\right] < \infty.$$

- Then, $Y(t)$ is a martingale with $\mathrm{E}[Y(t)] = 1$ (i.e., a Radon-Nikodym derivative process), and

$$\tilde{W}(t) := W(t) + \int_0^t \gamma(s)ds, \quad 0 \le t \le T$$

is a standard Brownian motion under $\tilde{\mathrm{P}}$.

Step 1

Define a new process $\tilde{W}(t) = W(t) + \int_0^t \gamma(u)du$ where we recall

$$\gamma(t) = \frac{m(t,T) - r(t)}{S(t,T)}.$$

Then we have

$$\begin{aligned} dZ(t,T) &= Z(t,T)[(m(t,T) - r(t) - \gamma(t)S(t,T))dt \\ &\quad + S(t,T)(dW(t) + \gamma(t)dt)] \\ &= Z(t,T)S(t,T)d\tilde{W}(t). \end{aligned} \tag{7}$$

Provided $\gamma(s)$ satisfies the *Novikov* condition

$$E_P\left[\exp\left(\frac{1}{2}\int_0^T \gamma(u)^2 du\right)\right] < \infty,$$

by Girsanov Theorem, there exists a measure Q equivalent to P with Radon-Nikodym derivative:

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \gamma(u) dW(u) - \frac{1}{2}\int_0^T \gamma(u)^2 du\right)$$

under which $\tilde{W}(t)$ is a standard Brownian motion.

Step 1

- Note that under the same change of measure we have

$$dP(t, T) = P(t, T)[r(t)dt + S(t, T)d\tilde{W}(t)]. \tag{8}$$

- In particular, under Q, the prices of all tradable assets have drift equal to the current price times the risk-free rate.

- We will see later that this feature holds for all assets (not just zero-coupon bonds) and in fact beyond one-factor model.

Step 1

- Now we note that the SDE for $Z(t,T)$ under Q (equation (7)) has zero drift (that is, no dt term).

- It follows that $Z(t,T)$ is a martingale under Q if one of the following (sufficient) technical conditions is satisfied:

$$E_Q\left[\left(\int_0^T S(t,T)^2 Z(t,T)^2 dt\right)^{\frac{1}{2}}\right] < \infty$$

  or

$$E_Q\left[\exp\left(\frac{1}{2}\int_0^T S(t,T)^2 dt\right)\right] < \infty$$

  (A *necessary* condition for $Z(t,T)$ to be a martingale is that it has zero drift in the SDE.)

- We are done if $X = 1$ (ZCB).

Steps 2 and 3

- For $t < S < T$ define $D(t) = E_Q[B(S)^{-1}X_S | \mathcal{F}_t]$. This is a martingale under Q (Tower Property).
- Since $Z(t, T)$ and $D(t)$ are both Q-martingales, by the Martingale Representation Theorem (Theorem A.10 in the textbook) there exists a previsible process $\phi(t)$ such that

$$D(t) = D(0) + \int_0^t \phi(u) dZ(u, T).$$

  Note that this requires $S(t, T)$ to be non-zero for all $t < S$ almost surely.

Step 4: Replicating Portfolio

- Consider the portfolio which holds at time t
  - $\phi(t)$ units of $P(t, T)$;
  - $\psi(t) \equiv D(t) - \phi(t)Z(t, T)$ units of $B(t)$.

- The value at time t of this portfolio is

$$\begin{aligned} V(t) &= \phi(t)P(t, T) + \psi(t)B(t) \\ &= B(t)[\phi(t)Z(t, T) + \psi(t)] = B(t)D(t). \end{aligned}$$

- The instantaneous investment gain is

$$\phi(t)dP(t, T) + \psi(t)dB(t).$$

Step 4: Self-Financing

- The corresponding instantaneous change in the portfolio value is

$$
\begin{aligned}
dV(t) &= d[B(t)D(t)] \\
&= B(t)dD(t) + D(t)dB(t) \text{ (product rule)} \\
&= B(t)\phi(t)dZ(t,T) + D(t)r(t)B(t)dt \\
&= \phi(t)B(t)S(t,T)Z(t,T)d\tilde{W}(t) \\
&\quad + [\phi(t)Z(t,T) + \psi(t)]\, r(t)B(t)dt \;\; (\text{def of } \psi(t)) \\
&= \phi(t)P(t,T)(r(t)dt + S(t,T)d\tilde{W}(t)) + \psi(t)r(t)B(t)dt \\
&= \phi(t)dP(t,T) + \psi(t)dB(t).
\end{aligned}
$$

which is equal to the instantaneous investment gain over the same period.

Step 5: Replicating

- The portfolio replicates X since

$$\begin{aligned} V(S) &= B(S)\,D(S) \\ &= B(S)E_Q[B(S)^{-1}X_S|\mathcal{F}_s] \\ &= X_S. \end{aligned}$$

- It follows that

$$\begin{aligned} V(t) = B(t)D(t) &= E_Q\left[\frac{B(t)}{B(S)}X_S|\mathcal{F}_t\right] \\ &= E_Q\left[\exp\left(-\int_t^S r(u)du\right)X_S|\mathcal{F}_t\right]. \end{aligned}$$

- As a result, for all S such that $0 < S < T$,

$$P(t,S) = E_Q\left[\exp\left(-\int_t^S r(u)du\right)|\mathcal{F}_t\right].$$

Other Asset

- Recall

$$\begin{aligned} dV(t) &= \phi(t)P(t,T)[r(t)dt + S(t,T)d\tilde{W}(t)] + \psi(t)B(t)r(t)dt \\ &= [\phi(t)P(t,T) + \psi(t)B(t)]r(t)dt \\ &+ \phi(t)P(t,T)S(t,T)d\tilde{W}(t) \\ &= V(t)[r(t)dt + \sigma_V(t)d\tilde{W}(t)] \end{aligned}$$

  where

$$V(t)\sigma_V(t) = \phi(t)P(t,T)S(t,T).$$

- Thus, under Q, the prices of all assets have the risk-free rate of interest as the expected growth rate.

- In contrast, under the real-world measure P, we have

$$\begin{aligned} dV(t) &= V(t)[r(t)dt + \sigma_V(t)(dW(t) + \gamma(t)dt)] \\ &= V(t)[(r(t) + \gamma(t)\sigma_V(t))dt + \sigma_V(t)dW(t)]. \end{aligned}$$

Risk Premium

- The excess expected growth rate under P on the bond or derivative, $\gamma(t)\sigma_V(t)$, is called the risk premium.
- Risk premiums on different assets are closely linked (through their dependence on $\gamma(t)$) and can differ (in a one-factor model) only through the volatility in the tradable asset, e.g., $\sigma_V(t)$ in the derivative or $S(t, T)$ for a zero-coupon bond.

Risk Premium

- In general, we anticipate that ZCB will have a positive risk premium (that is, $\gamma(t)S(t, T) > 0$ for all $T > t$) to reward investors for the extra (future interest-rate) risk they are taking on.

- It follows that derivatives $V(t)$ for which $\sigma_V(t)$ has the same sign as $S(t, T)$, e.g., call options on $P(t, T)$, have a positive risk premium.

- derivatives $V(t)$ for which $\sigma_V(t)$ has the opposite sign as $S(t, T)$, e.g., put options on $P(t, T)$, have a negative risk premium.