FE5208 Part II Assessment 2

Question 1（**HJM Model**）The following is a model for forward rate evolution by parallel shifts with noises:

$$f(t,T) = g(T-t) + \int_0^t \sigma(s)dW_s,$$

where $g(\cdot)$ and $\sigma(\cdot)$ are functions of one variable. Please work out the following:
(1) the conditions that functions $g(\cdot)$ and $\sigma(\cdot)$ should satisfy for $f(t,T)$ to be arbitrage free,
(2) the dynamics of the short rate, and
(3) the zero bond prices $P(0,T)$ for maturities T.

Question 2 (**LFM under Spot-LIBOR-measure**) Spot-LIBOR-measure is the probability measure Q^d that takes the discrete money account wealth

$$B_d(t) = \prod_{j=0}^{\beta(t)-1}\left(1+\triangle t_j L^j_{T_j}\right)P(t,T_{\beta(t)})$$

as the numeraire. Suppose for every $n,$ the forward LIBOR L^n_t is governed by the following driftless SDE:

$$dL^n_t = \sigma_n(t)L^n_t dW^{Q^{n+1}}_t \text{ under } Q^{n+1},$$

where Q^{n+1} is the forward measure that takes the zero coupon bond $P(1,T_{n+1})$ as the numeraire and $W^{Q^{n+1}}_t$ is a Brownian motion under $Q^{n+1}.$ Using the change-of-drift theorem, prove that the dynamics of L^n_t is

$$dL^n_t = \sigma_n(t)L^n_t \sum_{j=\beta(t)}^{n} \frac{\triangle t_j \rho_{jn}\sigma_j(t)L^j_t}{1+\triangle t_j L^j_t}dt + \sigma_n(t)L^n_t dW^n_t, \text{ under } Q^d,$$

where W^n_t is a Brownian motion under Q^d.