Mutual Independence vs Pairwise Independence

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In class we mentioned that pairwise independence does not imply mutual independence. Here is an example for it. Let $\Omega = \{1, 2, 3, 4\}, \mathcal{F} = 2^\Omega$ and $\mathbb{P}(\omega) = \frac{1}{4}, \forall \omega \in \Omega$. Let $A_1 = \{1, 2\}, A_2 = \{2, 3\}, A_3 = \{3, 1\}$. Then

$$\mathbb{P}(A_i) = \frac{1}{2}, \forall\ i = 1, 2, 3$$

$$\mathbb{P}(A_i \cap A_j) = \frac{1}{4}, \forall\ i \neq j$$

, it follows that

$$\mathbb{P}(A_i)\mathbb{P}(A_j) = \mathbb{P}(A_i \cap A_j), \forall\ i \neq j.$$

Hence A_1, A_2 and A_3 are pair-wise independent. However

$$\mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3) = \frac{1}{8}$$

and

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = 0$$

$$\mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3) \neq \mathbb{P}(A_1 \cap A_2 \cap A_3).$$

A_1, A_2, A_3 are thus not mutually independent.

To argue it for σ-algebra, we can take

$$\mathcal{G}_i = \{\emptyset, A_i, A_i^c, \Omega\}.$$

It is easy to verify that $\mathcal{G}_i, i = 1, 2, 3$ are pairwise independent but not mutually independent[1].

[1] If two events A and B are independent, then any combination of A, A^c and B, B^c are independent.