Lecture 1 - Introduction to Bond Market

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Term Structure

- An synonym of what we call a yield curve.
- A graph that plots the yields/interest rates of (zero coupon) bonds in the Y-axis with their maturities, or time, in the X-axis.
  - Libor rates, Treasury yield rates, and Treasury yield curve
- "Usually" positive-sloped.

Term Structure

- Commonly used yield curves are the LIBOR curve and the US Treasury yield curve.

Important in two major areas:

- Macroeconomics: Term Structure is a major indicator of economic activity. Models can be used to learn/forecast macroeconomy, understand and help Monetary policy.

- Financial Economics: Term Structure affects valuation of all assets (discounting). Dynamic models are useful to value term structure derivatives, and manage interest rate risk (i.e., volatility).

Bond

- A securitized form of loan.
- Coupon bond vs Zero Coupon Bond; Government Bond vs Corporate bond etc..
- Bond prices are quoted in two different forms:
  - The dirty price is the actual amount paid in return for the right to the full amount of each future coupon payment and redemption proceeds.
  - The clean price = the dirty price - the accrued interest.
- Coupon may be paid monthly, quarterly, or semi-annually.

Zero Coupon Bond

- Zero Coupon Bond (ZCB): A financial instrument that pays a fixed amount of money (\$1) at some fixed maturity, with no coupon/interest being paid before the maturity.
- $P(t, T)$ denotes the price of a zero coupon bond (T-bond) which matures at time T at time $t \leq T$.
- Coupon bond can be written as a summation of multiple ZCBs.
- Assume $P(T, T) = 1$, $P(t, T) > 0$, and zero probability of default.

Spot Rate

- $R(t, T)$: yield (to maturity) of the T-bond. Hence,

$$R(t, T) = -\frac{\log P(t, T)}{T - t},$$

i.e.,

$$P(t, T) = e^{-(T-t)R(t,T)}.$$

- One-to-one correspondence between $P(t, T)$ and $R(t, T)$ (we can trade $P(t, T)$ but not $R(t, T)$).

- A proxy of "risk-free rate" in theory.

Forward Rate

- Arises within the term of a forward contract: we agree at time t that we will invest \$1 at some future time $T > t$ in return for $e^{(S-T)f(t,T,S)}$ at time $S > T$.

- A forward loan is engineered like any forward contract, except what is being transacted is not a currency or commodity, but a loan.

- Day-count adjustment, say ACT/360:

$$\delta(S,T) = \frac{S-T}{360}$$

Why Forward Loan

- For a borrower (resp. lender) to lock in the current low (resp. high) borrowing rates from money market

- A business may face a floating-rate liability at time t_1 and would like to hedge the liability by securing a future loan with a known cost.

- Forward contracts are often settled by two parties in over-the-counter (OTC) markets (instead of in exchange as the future contract).

Cash Flow of Borrowing a Forward Loan

- Receive \$1 at time T.
- Pay $1 + F(t, T, S) \times \delta(S, T)$ at time S.
- Replicating a forward loan: at time t buy \$$x$ of T-bond and sell \$$x$ of S-bond. What is x?

Forward Rate and Bond Price

- One-Time interest payment:

$$1 + F(t, T, S) \times \delta(S, T) = \frac{P(t, T)}{P(t, S)}.$$

- Continuously compounding:

$$F(t, T, S) \times (S - T) = \log \frac{P(t, T)}{P(t, S)}.$$

Instantaneous Forward Rate

- Define the instantaneous forward rate $f(t, T)$ as

$$f(t, T) = \lim_{S \to T} F(t, T, S) = -\frac{\partial}{\partial T} \log P(t, T).$$

- Hence,

$$P(t, T) = \exp\left[-\int_t^T f(t, u)\, du\right].$$

- $f(t, T) > 0$ iff the bond price is decreasing in T.

Short Rate models vs Forward Rate models

- The short rate is denoted as $r(t)$:

$$r(t) = \lim_{T \to t} R(t, T) = R(t, t) = f(t, t).$$

- A short-rate model, in the context of interest rate derivatives, is a mathematical model that describes the future evolution of interest rates by describing the future evolution of the short rate $r(t)$.

- A forward-rate model instead models $f(t, T)$ as a stochastic process.

Why Stochastic Interest Rate

- Why are interest rate derivatives of interest otherwise?
- Even Risk-free interests are not constant and subject to unpredictable shocks.
- The Black-Scholes assumption becomes problematic with a longer-dated derivatives.

Short Rate Models

- Vasicek model (1977): $dr(t) = a(b - r(t))\,dt + \sigma dW_t$;
- Cox-Ingersoll-Ross model (1985):
  $dr(t) = a(b - r(t))\,dt + \sigma\sqrt{r(t)}dW_t$;
- Ho and Lee model (1986): $dr(t) = \theta(t)\,dt + \sigma dW_t$;
- Hull and White model (1990):
  $dr(t) = (\theta(t) - \alpha r(t))\,dt + \sigma(t)\,dW_t$.

Short Rate Models

- These are also called affine short rate models: $\log P(t, T)$ ($\approx$ the yield) is a linear function of $r(t)$.
- One-factor short rate models vs (multi-factor short rate models).
- Time-homogeneous short rate models vs time-dependent short rate models.

Par Yield

- Par Yield curve $\rho(t, T)$ specifies the coupon rates $\rho(t, T)$ at which the new bonds (issued at t and matured at T) should be priced if they are to issued at par (their nominal value), i.e., if the coupon will be paid at each $s = t + 1, t + 2, ..., T$

$$1 = \rho(t, T) \sum_{s=t+1}^{T} P(t, s) + P(t, T).$$

- Equivalently,

$$\rho(t, T) = \frac{1 - P(t, T)}{\sum_{s=t+1}^{T} P(t, s)}.$$

- Each curve uniquely determines the other three: $P(t, T)$, $f(t, T)$, $R(t, T)$, and $\rho(t, T)$.

Example

T	1	2	3	4	5
$F(0, T-1, T)$	0.0420	0.0500	0.0550	0.0560	0.0530

Calculate

$$\begin{aligned} P(0, T) &= \exp\left[-\sum_{t=1}^{T} F(0, t-1, t)\right]. \\ R(0, T) &= -\frac{\log P(0, T)}{T}; \\ \rho(0, T) &= \frac{1 - P(t, T)}{\sum_{s=1}^{T} P(0, s)}. \end{aligned}$$

Arbitrage: Example

- Consider the ZCB price

$$P(0, T) = \exp\left[-\int_0^T f(0, u)\, du\right].$$

- Suppose that

$$f(1, T) = f(0, T) + \varepsilon$$

where ε is a real-valued random variable.

- Consider three ZCBs with maturity $1 < T_1 < T_2 < T_3$.

Example

Suppose that

$$P\left(0,t\right)=e^{-0.08t},\forall t>0.$$

$$P\left(1,t+1\right)=\begin{cases} e^{-0.1t}, & \text{with probability } p; \\ e^{-0.06t}, & \text{with probability } 1-p. \end{cases}$$

Arbitrage

Find a portfolio at time 0 of (x_1, x_2, x_3) such that

$$
\begin{aligned}
V_0 &= \sum_{i=1}^{3} x_i P(0, T_i) = 0; \\
V_1(\varepsilon) &= \sum_{i=1}^{3} x_i P(1, T_i) \\
\text{``} &\geq 0\text{''} \text{ with probability one;} \\
\text{``} &> 0\text{''} \text{ with positive probability.}
\end{aligned}
$$

Arbitrage

Observe that

$$\begin{aligned} P(1,T) &= \exp\left[-\int_1^T f(1,u)\,du\right] \\ &= \exp\left[-\int_1^T (f(0,u)+\varepsilon)\,du\right] = \frac{P(0,T)}{P(0,1)}e^{-\varepsilon(T-1)}. \end{aligned}$$

Hence,

$$V_1(\varepsilon) = \frac{e^{-\varepsilon(T_2-1)}}{P(0,1)}g(\varepsilon) \text{ where } g(\varepsilon) = \sum_{i=1}^{3} x_i P(0,T_i)\,e^{-\varepsilon(T_i-T_2)}.$$

Arbitrage

- Calculate

$$
\begin{aligned}
g(0) &= \sum_{i=1}^{3} x_i P(0, T_i) = V_0. \\
g'(0) &= \sum_{i=1}^{3} x_i T_i P(0, T_i) \ \text{ if } g(0) = 0 \\
g''(\varepsilon) &= \sum_{i=1}^{3} x_i (T_2 - T_i)^2 P(0, T_i)\, e^{-\varepsilon(T_i - T_2)} \\
&> 0 \Rightarrow_{g(0)=g'(0)=0} \ V_1(\varepsilon) > 0.
\end{aligned}
$$

- Arbitrage with a butterfly strategy: short 1 unit of T_2-bond and long $x_1 > 0$ units of T_1-bond and $x_3 > 0$ unit of T_3-bond such that $g(0) = g'(0) = 0$ and $g''(\varepsilon) > 0$.

Cash Account

- Suppose that the (instantaneous) interest rate is stochastic and the process is pinned down by a probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

- Define
$$B(t) = B(0)\exp\left(\int_0^t r(s)\,ds\right) \text{ (which is } \mathcal{F}_t \text{ measurable).}$$

- Equivalently,
$$dB(t) = r(t)\,B(t)\,dt.$$

Fundamental Theorem of Asset Pricing

- (i) Bond prices evolve in a way that is arbitrage free if and only if there is a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which for each T, the discounted price process $P(t, T) / B(t)$ is a martingale for all t: $0 < t < T$;

- (ii) if (i) holds, then the market is complete if and only if the probability measure $\mathbb{Q}$ is unique.

- The probability measure $\mathbb{Q}$, as we all know, is called the equivalent martingale measure or the risk-neutral probability.

Implications

- Since $P(t,T)/B(t)$ is a martingale, we have

$$
\begin{aligned}
P(t,T)/B(t) &= E_{\mathrm{Q}}\left[P(T,T)/B(T)\,|\mathcal{F}_t\right] \Leftrightarrow \\
P(t,T) &= E_{\mathrm{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)|\mathcal{F}_t\right]
\end{aligned}
$$

- Hence, if a derivative pays X which is $\mathcal{F}_T$-measurable, then its value at time t is

$$
V(t) = E_{\mathrm{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)X|\mathcal{F}_t\right].
$$

Forward Pricing

- Suppose that a contract is arranged such that price K is paid at time T in return of \$1 being paid at time S

- The contract has value

$$V(t) = E_{\mathrm{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)(P(T,S)-K)\,|\mathcal{F}_t\right].$$

- Since

$$P(T,S) = E_{\mathrm{Q}}\left[\exp\left(-\int_T^S r(s)\,ds\right)|\mathcal{F}_T\right],$$

we obtain

$$\begin{aligned} V(t) &= E_{\mathrm{Q}}\left[\exp\left(-\int_t^S r(s)\,ds\right)|\mathcal{F}_t\right] \\ &\quad -K\times E_{\mathrm{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)|\mathcal{F}_t\right] \\ &= P(t,S) - KP(t,T). \end{aligned}$$

Put-Call Parity

- European options with the same exercise date T, strike price K, and the S-bond price $P(t,S)$ as the underlying with $S > T$.

- Two portfolios:
  - A: One call option plus K units of the T-bond, $P(t,T)$.
  - B: One put option plus 1 unit of S-bond, $P(s,T)$.

- At time T, both has value $\max\{P(T,S), K\}$:
  - A has value $\max\{P(T,S) - K, 0\} + K$.
  - B has value $\max\{K - P(T,S), 0\} + P(T,S)$.
- Hence, for each $t < T$, we have

$$c(t) + KP(t,T) = p(t) + P(t,S).$$