FE5222 Advanced Derivative Pricing

Wu Lei

Risk Management Institute, National University of Singapore

rmiwul@nus.edu.sg

September 4, 2019

Overview

Introduction

Two approaches

- Partial Differential Equation (P.D.E.) Approach
- Risk Neutral Approach

P.D.E. Approach in Black-Scholes-Merton Model

Assume the stock price evolves (in the real world) according to the following process

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dW(t)$$

where α and σ are constant.
The quadratic variation of $S(t)$ (in differential form) is

$$dS(t)dS(t) = \sigma^2 S^2(t)dt$$

P.D.E. Approach in Black-Scholes-Merton Model

Let $V(t, S(t))$ be the value of a financial derivative (call/put option etc.) at time t.
By Ito's Lemma, the change of $V(t, S(t))$ from t to $t + dt$ is

$$\begin{aligned} dV(t, S(t)) &= V_t dt + V_S dS(t) + \frac{1}{2} V_{SS} dS(t) dS(t) \\ &= V_t dt + V_S dS(t) + \frac{1}{2}\sigma^2 S^2 V_{SS} dt \end{aligned}$$

P.D.E. Approach in Black-Scholes-Merton Model

If we simultaneously hold $-V_S$ shares of stock at t, the value of our portfolio $\pi(t)$ at time t is

$$\pi(t) = V(t, S(t)) - V_S S(t)$$

P.D.E. Approach in Black-Scholes-Merton Model

The change of portfolio value between t to $t + dt$ is

$$\begin{aligned} d\pi(t) &= dV(t, S(t)) - V_s dS(t) \\ &= V_t dt + \frac{1}{2}\sigma^2 S^2(t) V_{SS} dt \end{aligned}$$

The change of portfolio value is independent of price change!

P.D.E. Approach in Black-Scholes-Merton Model

In other words, this portfolio is not subject to any price risk in the infinitesimal time interval $[t, t + dt]$.

$\Rightarrow$ The portfolio is as safe as holding a riskless asset.

$\Rightarrow$ Its value shall grow at the same rate as a riskless asset (no arbitrage principle).

$\Rightarrow$

$$d\pi(t) = r\pi(t)dt$$

$\Rightarrow$

$$V_t dt + \frac{1}{2}\sigma^2 S^2(t) V_{SS} dt = r\left(V(t, S(t)) - V_S S(t)\right) dt$$

P.D.E. Approach in Black-Scholes-Merton Model

Canceling dt and rearranging it, we get Black-Scholes P.D.E.

$$V_t + rSV_s + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV(t, S) = 0$$

P.D.E. Approach in Black-Scholes-Merton Model

From replicating perspective, at time t if we hold a portfolio $X(t)$ of

- V_S shares of stock
- $\frac{1}{r}\left(V_t + \frac{1}{2}\sigma^2 S^2(t) V_{SS}\right)$ cash

The change of $X(t)$ from t to $t + dt$ is

$$V_S dS(t) + \left(V_t + \frac{1}{2}\sigma^2 S^2(t) V_{SS}\right) dt$$

The is the same as holding the derivative V!.

P.D.E. Approach in Black-Scholes-Merton Model

Example

We can argue that $V(0) = X(0)$. Otherwise there is arbitrage opportunity. Suppose $V(0) > X(0)$.

1. At $t = 0$,
   - Short V
   - Long X
   - Deposit the cash gain $V(0) - X(0)$ at a bank account
2. At expiry T, the value of our positions is:
   - -$V(T)$
   - $X(T)$
   - $V(0) - X(0) +$ interest

   Since $X(T) - X(0) = V(T) - V(0)$, the net value is the amount of interest.

$\Rightarrow$ Lock in riskless gain!

Risk Neutral Approach

Idea:

- Replicate the payoff of a derivative V with a portfolio X consisting of stocks and cash.
- Since the discounted stock prices are martingale under risk-neural measure, the discounted value of X is also a martingale.
- The discounted value $\widetilde{V}$ of V is also a martingale under risk neutral measure. Hence

$$\widetilde{V}(t) = \widetilde{\mathbb{E}}[\widetilde{V}(T)|\mathcal{F}_t]$$

Girsanov's Theorem

Definition

Let Z be a positive random variable such that $\mathbb{E}[Z] = 1$. The Radon-Nikodym derivative process $Z(t)$ is defined as

$$Z(t) = \mathbb{E}[Z|\mathcal{F}_t]$$

Girsanov's Theorem

- $Z(t) > 0$
- $Z(t)$ is a martingale.
- $\mathbb{E}[Z(t)] = 1$.

Girsanov's Theorem

Lemma

Let Z be a positive random variable and $\mathbb{E}[Z] = 1$, $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z$. Let Y be an integrable random variable. Assume that Y is $\mathcal{F}_t$ measurable. Then

$$\widetilde{\mathbb{E}}[Y] = \mathbb{E}[YZ(t)]$$

where $\widetilde{\mathbb{E}}$ is the expectation w.r.t. the probability measure $\widetilde{\mathbb{P}}$.

Girsanov's Theorem

Note that

$$\widetilde{\mathbb{E}}[Y] = \mathbb{E}[YZ]$$

always holds. $Z(t)$ is the estimate of Z given the information $\mathcal{F}_t$. When Y is known at time t, we can refine the expectation on the RHS with available information to use $Z(t)$.

Girsanov's Theorem

Proof.

$$\begin{aligned}\widetilde{\mathbb{E}}[Y] &= \mathbb{E}[YZ] \\ &= \mathbb{E}\left[\mathbb{E}[YZ|\mathcal{F}_t]\right] \\ &= \mathbb{E}\left[Y\mathbb{E}[Z|\mathcal{F}_t]\right] \\ &= \mathbb{E}[YZ(t)]\end{aligned}$$

□

Girsanov's Theorem

Lemma

Let $s < t$ and Y be an $\mathcal{F}_t$ measurable random variable. Then

$$\widetilde{\mathbb{E}}[Y|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]$$

Girsanov's Theorem

- This is the condition expectation version of the previous lemma
- In change of measure, the 'scaling factor' Z needs to be normalized (i.e., $\mathbb{E}[Z] = 1$). However the conditional expectation $\mathbb{E}[Z|\mathcal{F}_s] = Z(t) \neq 1$. Hence we need to rescale it by a factor of $\frac{1}{Z(s)}$ such that $\mathbb{E}[\frac{Z(t)}{Z(s)}|\mathcal{F}_s] = 1$.

Girsanov's Theorem

Proof.

We shall prove that $\frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]$ is the conditional expectation w.r.t. $\widetilde{\mathbb{P}}$ of Y given $\mathcal{F}_s$. To do this, we need to verify

- $\frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]$ is $\mathcal{F}_s$-measurable.
- For any $A \in \mathcal{F}_s$,

$$\widetilde{\mathbb{E}}\left[1_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\right] = \widetilde{\mathbb{E}}\left[1_A Y\right]$$

$\square$

Proof.

Since both $\frac{1}{Z(s)}$ and $\mathbb{E}[YZ(t)|\mathcal{F}_s]$ are $\mathcal{F}_s$-measurable, $\frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]$ is $\mathcal{F}_s$-measurable. $\square$

Proof.

$$
\begin{aligned}
\widetilde{\mathbb{E}}\left[1_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\right] &= \mathbb{E}\left[1_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]Z(s)\right] \\
&= \mathbb{E}\left[1_A \mathbb{E}\left[YZ(t)|\mathcal{F}_s\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[1_A YZ(t)|\mathcal{F}_s\right]\right] \\
&= \mathbb{E}\left[1_A YZ(t)\right] \\
&= \widetilde{\mathbb{E}}\left[1_A Y\right]
\end{aligned}
$$

$\square$

Girsanov's Theorem

Theorem

Let $W(t), 0 \leq t \leq T$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration for the Brownian motion, $\Theta(t)$ is an adapted process. Suppose $\Theta(t)$ satisfies Novikov's condition

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T \Theta^2(s)ds}\right] < \infty$$

Define

$$Z(t) = e^{-\int_0^t \Theta(s)dW(s) - \frac{1}{2}\int_0^t \Theta^2(s)ds}$$

then $Z(t)$ is a martingale and $\mathbb{E}Z(t) = 1, \forall 0 \leq t \leq T$.

Girsanov's Theorem

Theorem (Cont'd)

Furthermore, if we let $Z = Z(T)$,

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z$$

and

$$\widetilde{W}(t) = \int_0^t \Theta(s)ds + W(t)$$

Then $\widetilde{W}(t)$ is a Brownian motion under the measure $\widetilde{\mathbb{P}}$.

Note that we often use differential form

$$d\widetilde{W}(t) = \Theta(t)dt + dW(t)$$

Girsanov's Theorem

Outline of the proof

1. Prove $Z(t)$ is a martingale by showing that $dZ(t)$ has zero drift term
2. Show that $\widetilde{W}$
   - is a martingale (under the probability measure $\widetilde{\mathbb{P}}$)
   - has continuous sample paths; and
   - unit quadratic variation per unit time

$$[\widetilde{W}, \widetilde{W}](t) = t$$

$\Longrightarrow$ By Levy's Theorem $\widetilde{W}$ is a Brownian motion under the probability measure $\widetilde{\mathbb{P}}$

Girsanov's Theorem

Proof.

We first prove $Z(t)$ is a martingale.
Let

$$X(t) = -\int_0^t \Theta(s)dW(s) - \frac{1}{2}\int_0^t \Theta^2(s)ds$$

which written in differential form becomes

$$dX(t) = -\Theta(t)dW(t) - \frac{1}{2}\Theta^2(t)dt$$

Hence

$$dX(t)dX(t) = \Theta^2(t)dt$$

□

Girsanov's Theorem

Proof.

Since $Z(t) = e^{X(t)}$, we can apply Ito's Lemma to the function $f(t,x) = e^x$ and get

$$dZ(t) = f_x dX(t) + \frac{1}{2} f_{xx} dX(t) dX(t)$$

Note that $f_x = f_{xx} = e^x$, we have

$$\begin{aligned} dZ(t) &= Z(t)\left(-\Theta(t)dW(t) - \tfrac{1}{2}\Theta^2(t)dt\right) + \tfrac{1}{2}Z(t)\Theta^2(t)dt \\ &= -\Theta(t)Z(t)dW(t) \end{aligned}$$

Hence

$$Z(t) = Z(0) - \int_0^t \Theta(s)Z(s)dW(s)$$

is a martingale. □

Girsanov's Theorem

Proof.

Now we show $\widetilde{W}(t)$ is a Brownian motion.

- It is trivial that $\widetilde{W}(t)$ has a continuous sample path.
-
$$\begin{aligned} & d\widetilde{W}(t)d\widetilde{W}(t) \\ = \; & (\Theta(t)dt + dW(t))^2 \\ = \; & \Theta^2(t)dtdt + 2\Theta(t)dW(t)dt + dW(t)dW(t) \\ = \; & dt \end{aligned}$$

  Hence $\widetilde{W}(t)$ has unit quadratic variation per unit time
- It's left to show that $\widetilde{W}(t)$ is a martingale under $\widetilde{\mathbb{P}}$

□

Girsanov's Theorem

Proof.

Fix $s < t$, we need to show

$$\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}_s] = \widetilde{W}(s)$$

where $\widetilde{\mathbb{E}}$ is the expectation w.r.t. $\widetilde{\mathbb{P}}$.

We notice that $Z(t)$ is a martingale, hence

$$Z(t) = \mathbb{E}[Z(T)|\mathcal{F}_t] = \mathbb{E}[Z|\mathcal{F}_t]$$

$Z(t)$ is a Radom-Nikodym process.

We can use the change of measure formula for conditional expectation and get

$$\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[\widetilde{W}(t)Z(t)|\mathcal{F}_s]$$

Girsanov's Theorem

Proof.

It is sufficient to show

$$
\begin{aligned}
& \frac{1}{Z(s)}\mathbb{E}[\widetilde{W}(t)Z(t)|\mathcal{F}_s] = \widetilde{W}(s) \\
\iff \quad & \mathbb{E}[\widetilde{W}(t)Z(t)|\mathcal{F}_s] = \widetilde{W}(s)Z(s) \\
\iff \quad & \widetilde{W}(t)Z(t) \text{ is a martingale under } \mathbb{P}
\end{aligned}
$$

□

Girsanov's Theorem

Proof.

We now prove $\widetilde{W}(t)Z(t)$ is a martingale under $\mathbb{P}$.

$$
\begin{aligned}
d\left(\widetilde{W}(t)Z(t)\right) &= Z(t)d\widetilde{W}(t) + \widetilde{W}(t)dZ(t) + d\widetilde{W}(t)dZ(t) \\
&= Z(t)\left(\Theta dt + dW(t)\right) - \widetilde{W}(t)\Theta Z(t)dW(t) \\
&\quad - \left(\Theta dt + dW(t)\right)\Theta Z(t)dW(t) \\
&= Z(t)\Theta dt + Z(t)dW(t) - \widetilde{W}(t)\Theta Z(t)dW(t) \\
&\quad - \Theta^2 Z(t)dW(t)dt - \Theta Z(t)dW(t)dW(t) \\
&= Z(t)\left(1 - \widetilde{W}(t)\Theta\right)dW(t)
\end{aligned}
$$

Since $d\left(\widetilde{W}(t)Z(t)\right)$ has no drift term, it is a martingale under $\mathbb{P}$. This completes the proof. □

Risk Neutral Measure

Model for stock market in the real world measure $\mathbb{P}$

- Stock price process

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \forall 0 \leq t \leq T$$

  where $\alpha(t)$ and $\sigma(t)$ are two adapted processes, and $\sigma(t) > 0$.

- Interest rate process $R(t)$, $R(t)$ is adapted
- Discount process

$$D(t) = e^{-\int_0^t R(s)ds}$$

  Note that

$$dD(t) = -R(t)D(t)dt$$

Risk Neutral Measure

The discounted price $D(t)S(t)$ follows

$$
\begin{aligned}
d(D(t)S(t)) &= S(t)dD(t) + D(t)dS(t) + dD(t)dS(t) \\
&= -R(t)D(t)S(t)dt + \alpha(t)D(t)S(t)dt \\
&\quad +\sigma(t)D(t)S(t)dW(t) \\
&= \sigma(t)D(t)S(t)\left(\frac{\alpha(t)-R(t)}{\sigma(t)}dt + dW(t)\right)
\end{aligned}
$$

Risk Neutral Measure

Let

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}$$

Then

$$\begin{aligned} d(D(t)S(t)) &= \sigma(t)D(t)S(t)\left(\Theta(t)dt + dW(t)\right) \\ &= \sigma(t)D(t)S(t)d\widetilde{W}(t) \end{aligned}$$

where

$$d\widetilde{W}(t) = \Theta(t)dt + dW(t)$$

$\widetilde{W}(t)$ is a Brownian motion under the probability measure $\widetilde{\mathbb{P}}$ defined as

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = e^{-\frac{1}{2}\int_0^T \Theta^2(s)ds - \int_0^T \Theta(s)dW(s)}$$

Risk Neutral Measure

- $\Theta(t)$ is called the *market price of risk*
- $\widetilde{\mathbb{P}}$ is the risk neutral measure
- $$D(t)S(t) = S(0) + \int_0^t \sigma(s)D(s)S(s)d\widetilde{W}(s)$$

  is a martingale under the risk neutral measure $\widetilde{\mathbb{P}}$

Risk Neutral Measure

- Substituting $dW(t) = -\Theta(t)dt + d\widetilde{W}(t)$ to the stock price $dS(t)$, we get

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t)$$

i.e.,

$$\frac{dS(t)}{S(t)} = R(t)dt + \sigma(t)d\widetilde{W}(t)$$

- The mean rate of return for $S(t)$ changes from $\alpha(t)$ to $R(t)$ from real world measure to risk neutral measure
- The instantaneous volatility $\sigma(t)$ does not change. However if $\sigma(t)$ is random, its distribution has changed from real world measure to risk neutral measure.

Risk Neutral Measure

Definition

Let $R(t)$ be the interest rate process. The money market account is defined as

$$M(t) = e^{\int_0^t R(s)ds}$$

- $D(t) = \frac{1}{M(t)}$
- $dM(t) = R(t)M(t)dt$

Value of Portfolio under Risk Neutral Measure

Consider a portfolio X of stocks and a money market account

- Initial capital $X(0)$
- At time t, hold $\Delta(t)$ shares of stock and invest $X(t) - \Delta(t)S(t)$ in a money market account

Value of Portfolio under Risk Neutral Measure

From t to $t + dt$, the change of the value of portfolio is

$$\begin{aligned}
dX(t) &= \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt \\
&= \Delta(t)\left(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\right) \\
&\quad +R(t)(X(t) - \Delta(t)S(t))dt \\
&= R(t)X(t)dt + \sigma(t)\Delta(t)S(t)\left(\Theta(t)dt + dW(t)\right) \\
&= R(t)X(t)dt + \sigma(t)\Delta(t)S(t)d\widetilde{W}(t)
\end{aligned}$$

Value of Portfolio under Risk Neutral Measure

The change of the discounted value of portfolio is

$$\begin{aligned}
d\left(D(t)X(t)\right) &= D(t)dX(t) + X(t)dD(t) + dX(t)dD(t) \\
&= D(t)dX(t) + X(t)dD(t) \\
&= D(t)dX(t) - R(t)D(t)X(t)dt \\
&= D(t)\left(R(t)X(t)dt + \sigma(t)\Delta(t)S(t)d\widetilde{W}(t)\right) \\
&\quad -R(t)D(t)X(t)dt \\
&= \sigma(t)D(t)\Delta(t)S(t)d\widetilde{W}(t)
\end{aligned}$$

Value of Portfolio under Risk Neutral Measure

The discounted value of portfolio is a martingale under the risk neutral measure.

$$\Longrightarrow D(t)X(t) = \widetilde{\mathbb{E}}\left[X(T)D(T)|\mathcal{F}_t\right]$$

$$\Longrightarrow X(t) = \frac{1}{D(t)}\widetilde{\mathbb{E}}\left[X(T)D(T)|\mathcal{F}_t\right]$$

Risk Neutral Pricing

Let $V(T)$ be the payoff of a derivative and $V(T)$ is $\mathcal{F}_T$-measurable. Suppose we can choose a portfolio $X(t)$ of stocks and a money market account with an initial capital $X(0)$ such that

$$X(T,\omega) = V(T,\omega)$$

for all $\omega \in \Omega$.
$\Longrightarrow$ By non-arbitrage argument, we must have

$$X(t) = V(t) \ \forall t$$

From

$$X(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}} [X(T)D(T)|\mathcal{F}_t]$$

we have

$$V(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}} [V(T)D(T)|\mathcal{F}_t]$$

This implies

$$D(t)V(t) = \widetilde{\mathbb{E}} [V(T)D(T)|\mathcal{F}_t]$$

The discounted value $D(t)V(t)$ is a martingale under risk neutral measure.

Risk Neutral Pricing

To summarize, under the assumptions

- $V(T)$ is $\mathcal{F}_T$-measurable
- there is a replicating portfolio of stocks and a money market account with initial capital $X(0)$

we can value the derivative V as

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-\int_t^T R(s)ds} V(T) | \mathcal{F}_t\right]$$

Risk Neutral Pricing

- The condition $V(T)$ is $\mathcal{F}_T$-measurable means the payoff of the derivative must be based on the information available up to time T, including path dependent derivative.
- The existence of a replicating portfolio will be justified later.

Example

We consider the call option price in the Black-Scholes-Merton model (we assume constant interest rate and volatility). Using risk neutral pricing approach, we have

$$c(t, S(t)) = \widetilde{\mathbb{E}}[e^{-r(T-t)}(S(T) - K)^{+}|\mathcal{F}_t]$$

where $\widetilde{\mathbb{E}}$ is the expectation under the risk neutral measure $\widetilde{\mathbb{P}}$.

Example (Cont'd)

Under risk neutral measure, stock price follows

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t)$$

Solving it, we have

$$S(T) = S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))}$$

Substituting it into the pricing formula we have

$$c(0, S(t)) = \widetilde{\mathbb{E}}[(S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))} - K)^+|\mathcal{F}_t]$$

Example (Cont'd)

The conditional expectation can be computed explicitly by noticing that $S(t)$ is known at time t and $W(T) - W(t)$ is independent of $\mathcal{F}_t$ and has a normal distribution $\mathcal{N}(0, T - t)$.

Risk Neutral Pricing

Two outstanding issues

- Does there always exit a replicating portfolio?
- If it exits, how do we find it (in theory)?

Martingale Representation Theorem

Theorem

Let $W(t), 0 \leq t \leq T$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t \geq 0}$ be the filtration generated by $W(t)$. Let $M(t)$ be a martingale w.r.t. $\{\mathcal{F}_t\}$. Then there exists an adapted process $\Gamma(t)$ such that

$$M(t) = M(0) + \int_0^t \Gamma(s) dW(s)$$

Martingale Representation Theorem

- The filtration needs to be generated by $W(t)$. In other words, the only source of uncertainty comes from the Brownian motion.
- From hedging perspective, we shall be able to hedge uncertainty with stock which is driven by the same Brownian motion.

Replicating Portfolio

Given a $\mathcal{F}_T$-measurable payoff $V(T)$ of a financial derivative, we want to find a portfolio X consisting of $\Delta(t)$ shares of stock at time t and initial capital $X(0)$ such that $X(T) = V(T)$

- Define $U(t)$ as

$$U(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}}\left[V(T)D(T)|\mathcal{F}_t\right]$$

  $D(t)U(t)$ is a martingale.

- Since $D(T)V(T)$ is $\mathcal{F}_T$ measurable,

$$U(T) = \frac{1}{D(T)} \widetilde{\mathbb{E}}\left[V(T)D(T)|\mathcal{F}_T\right] = V(T)$$

Replicating Portfolio

- By Martingale Representation Theorem,

$$D(t)U(t) = U(0) + \int_0^t \Gamma(u)d\widetilde{W}(u)$$

- Suppose we have found $\Delta(t)$ to replicate the final payoff $V(T)$ (or $U(T)$). Under risk neutral measure the value of portfolio $X(t)$ is

$$D(t)X(t) = X(0) + \int_0^t \Delta(u)\sigma(u)D(u)S(u)d\widetilde{W}(u)$$

Replicating Portfolio

Comparing the two equations, if we want $X(T) = V(T)$, it suffices to have

$$X(0) = U(0)$$

and

$$\Delta(t) = \frac{\Gamma(t)}{\sigma(t)D(t)S(t)}$$

Replicating Portfolio

Now we have found a portfolio whose payoff at time T is the same as $V(T)$. By our previous argument, $V(t)$ can be valued by

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-\int_t^T R(s)ds} V(T) | \mathcal{F}_t \right]$$

Replicating Portfolio

Two important assumptions

- $\sigma(t)$ is positive
- $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the Brownian motion.

Under these two assumptions, every $\mathcal{F}_T$-measurable derivatives can be hedged. Such as model is said to be complete.

Multidimensional Girsanov's Theorem

Let $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ be two d-dimensional vectors.

- $$x \cdot y = x_1 y_1 + \ldots x_d y_d$$

  is the inner product of the vectors x and y.

- $$||x|| = \sqrt{x_1^2 + \ldots + x_d^2}$$

  is the L_2-norm of the vector x.

Multidimensional Girsanov's Theorem

Theorem

Let $W(t) = (W_1(t), \ldots, W_d(t))$ be a d-dimensional (Standard) Brownian motion, $\Theta(t) = (\Theta_1(t), \ldots, \Theta_d(t))$ be a d-dimensional adapted process. Define

$$Z(t) = exp\left\{-\int_0^t \Theta(t) \cdot dW(t) - \frac{1}{2}\int_0^t ||\Theta(s)||^2 ds\right\}$$

and

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(s) ds$$

Assume

$$\mathbb{E}\left[exp\left\{\frac{1}{2}\int_0^T ||Q(s)||^2 ds\right\}\right] < \infty$$

Multidimensional Girsanov's Theorem

Theorem (Cont'd)

Then

- $Z(t)$ *is a martingale and* $\mathbb{E}Z(t) = 1$
- *Let* $Z = Z(T)$ *and define a probability measure* $\widetilde{\mathbb{P}}$ *as*

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z(\omega)$$

then $\widetilde{W}(t)$ *is a d-dimensional Brownian motion under* $\widetilde{\mathbb{P}}$.

Multidimensional Market Model

We assume m stocks,

$$\frac{dS_i(t)}{S_i(t)} = \alpha_i(t)dt + \sum_{j=1}^{d} \sigma_{i,j}(t)dW_j(t)$$

for $i = 1, \ldots, m$.
The discount process

$$D(t) = e^{-\int_0^t R(s)ds}$$

Multidimensional Market Model

The discounted stock prices

$$\begin{aligned}\frac{dD(t)S_i(t)}{D(t)S_i(t)} &= \frac{dS_i(t)}{S_i(t)} + \frac{dD(t)}{D(t)} + \frac{dS_i(t)}{S_i(t)}\frac{dD(t)}{D(t)} \\ &= \frac{dS_i(t)}{S_i(t)} - R(t)dt \\ &= (\alpha_i(t) - R(t))dt + \sum_{j=1}^{d} \sigma_{i,j}(t)dW_j(t)\end{aligned}$$

Risk Neutral Measure

Given $\Theta_j(t), j = 1, \ldots, d$, from multidimensional Girsanov's Theorem, we can find a measure $\widetilde{\mathbb{P}}$ such that

$$d\widetilde{W}_j(t) = dW_j(t) + \Theta_j(t)dt$$

is a multidimensional Brownian motion.

Risk Neutral Measure

The discounted stock prices

$$\begin{aligned}
\frac{dD(t)S_i(t)}{D(t)S_i(t)} &= (\alpha_i(t) - R(t))\,dt + \textstyle\sum_{j=1}^{d} \sigma_{i,j}(t)dW_j(t) \\
&= \left(\alpha_i(t) - R(t) - \textstyle\sum_{j=1}^{d} \sigma_{i,j}(t)\Theta_j(t)\right) dt \\
&\quad + \textstyle\sum_{j=1}^{d} \sigma_{i,j}(t)d\widetilde{W}_j(t)
\end{aligned}$$

Risk Neutral Measure

If we can choose $\Theta_j(t)$ such that

$$\alpha_i(t) - R(t) - \sum_{j=1}^{d} \sigma_{i,j}(t)\Theta_j(t) = 0, \quad \forall i = 1, \ldots, m$$

Then

$$\frac{dD(t)S_i(t)}{D(t)S_i(t)} = \sum_{j=1}^{d} \sigma_{i,j}(t) d\widetilde{W}_j(t)$$

The discounted prices are martingales under $\widetilde{\mathbb{P}}$.

Risk Neutral Measure

These equations are called *market price of risk equations*

$$\alpha_i(t) - R(t) - \sum_{j=1}^{d} \sigma_{i,j}(t)\Theta_j(t) = 0, \quad \forall i = 1, \dots, m$$

In matrix form

$$\begin{bmatrix} \alpha_1(t) - R(t) \\ \vdots \\ \alpha_m(t) - R(t) \end{bmatrix} = \begin{bmatrix} \sigma_{1,1}(t) & \dots & \sigma_{1,d}(t) \\ \vdots & \ddots & \vdots \\ \sigma_{m,1}(t) & \dots & \sigma_{1,d}(t) \end{bmatrix} \begin{bmatrix} \Theta_1(t) \\ \vdots \\ \Theta_d(t) \end{bmatrix}$$

Risk Neutral Measure

What if there is no solution for these equations?
$\Longrightarrow$ arbitrage! (See Example 5.4.4 in Shreve's book)

Risk Neutral Measure

Definition

A probability measure $\widetilde{\mathbb{P}}$ is said to be a risk neutral measure if

- $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent, i.e., $\widetilde{\mathbb{P}}(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0$.
- The discounted prices $D(t)S_i(t), i = 1, \ldots, m$, are martingales under $\widetilde{\mathbb{P}}$.

Risk Neutral Measure

Lemma

Under risk neutral measure $\widetilde{\mathbb{P}}$, the discounted value of a portfolio X of stock shares and money market account is a martingale.

Risk Neutral Measure

Proof.

Let $X(t)$ be the value of a portfolio with stocks and cash, $\Delta_i(t)$ is the shares of stocks at time t. Then

$$
\begin{aligned}
& dX(t) \\
= \; & \sum_{i=1}^{m} \Delta_i(t) dS_i(t) + R(t)\left(X(t) - \sum_{i=1}^{m} \Delta_i(t) S_i(t)\right) dt \\
= \; & R(t)X(t)dt + \sum_{i=1}^{m} \Delta_i(t)\left(dS_i(t) - R(t)S_i(t)\right) dt \\
= \; & R(t)X(t)dt + \sum_{i=1}^{m} \frac{\Delta_i(t)}{D(t)}\left(D(t)dS_i(t) - R(t)D(t)S_i(t)dt\right) \\
= \; & R(t)X(t)dt + \sum_{i=1}^{m} \frac{\Delta_i(t)}{D(t)} d\left(D(t)S_i(t)\right)
\end{aligned}
$$

□

Risk Neutral Measure

Cont'd.

$\Longrightarrow d\left(D(t)X(t)\right) = \sum_{i=1}^{m} \Delta_i(t) d\left(D(t)S_i(t)\right)$

Under $\widetilde{\mathbb{P}}$, $D(t)S_i(t)$ are martingales (i.e., no drift term), hence $D(t)X(t)$ is a martingale.

$\square$

Arbitrage

Definition

An arbitrage is a portfolio X such that $X(0) = 0$ and for some $T > 0$

$$\mathbb{P}(X(T) \geq = 0) = 1$$

and

$$\mathbb{P}(X(T) > 0) > 0$$

Arbitrage

Remark: we can also replace $\mathbb{P}$ with $\widetilde{\mathbb{P}}$ in the definition.

- $$\mathbb{P}(X(T) \geq= 0) = 1 \Longleftrightarrow \widetilde{\mathbb{P}}(X(T) \geq= 0) = 1$$
- $$\mathbb{P}(X(T) > 0) > 0 \Longleftrightarrow \widetilde{\mathbb{P}}(X(T) > 0) > 0$$

First Fundamental Theorem of Asset Pricing

Theorem

If a market model has a risk neutral probability measure, then it does not admit arbitrage.

The converse is also true under some stronger conditions.

First Fundamental Theorem of Asset Pricing

Proof.

We prove by contradiction. Suppose there exists an arbitrage portfolio X with $X(0) = 0$, by the remark after the definition, we have

$$\widetilde{\mathbb{P}}(X(T) \geq= 0) = 1$$

and

$$\widetilde{\mathbb{P}}(X(T) > 0) > 0$$

Since $D(T) > 0$, we must have

$$\widetilde{\mathbb{P}}(D(T)X(T) \geq= 0) = 1$$

and

$$\widetilde{\mathbb{P}}(D(T)X(T) > 0) > 0$$

□

First Fundamental Theorem of Asset Pricing

Cont'd.

This implies

$$\widetilde{\mathbb{E}}(D(T)X(T)) > 0$$

On the other hand, since $D(t)X(t)$ is a martingale under $\widetilde{\mathbb{P}}$, we must have

$$D(0)X(0) = \widetilde{\mathbb{E}}\left[D(T)X(T)\right] = 0,$$

contradiction.

$\square$

What if there are more than one risk neutral measure?
$\Longrightarrow$ We can construct financial derivatives that can't be fully hedged (replicated).

Let $\widetilde{\mathbb{P}}_1$ and $\widetilde{\mathbb{P}}_2$ be two different risk neutral measures. Since these two probability measures are different, there exists a set A such that $\widetilde{\mathbb{P}}_1(A) \neq \widetilde{\mathbb{P}}_2(A)$.

We design a derivative V whose payoff at time T is

$$V(T) = \frac{1_A(\omega)}{D(T)}$$

If there exists a portfolio X that replicates the payoff of V at time T, then

$$\widetilde{\mathbb{E}}_1(D(T)V(T)) = \widetilde{\mathbb{E}}_1(D(T)X(T)) = X(0)$$

and

$$\widetilde{\mathbb{E}}_2(D(T)V(T)) = \widetilde{\mathbb{E}}_2(D(T)X(T)) = X(0)$$

$\Longrightarrow$

$$\widetilde{\mathbb{E}}_1(D(T)V(T)) = \widetilde{\mathbb{E}}_2(D(T)V(T))$$

Note that

$$\widetilde{\mathbb{P}}_1(A) = \widetilde{\mathbb{E}}_1(D(T)V(T))$$

and

$$\widetilde{\mathbb{P}}_2(A) = \widetilde{\mathbb{E}}_2(D(T)V(T))$$

we have

$$\widetilde{\mathbb{P}}_1(A) = \widetilde{\mathbb{P}}_2(A)$$

This contradicts with the assumption

$$\widetilde{\mathbb{P}}_1(A) \neq \widetilde{\mathbb{P}}_2(A)$$

Second Fundamental Theorem of Asset Pricing

Under what conditions

- there exists a risk neutral measure?
- all derivatives are replicable?

Second Fundamental Theorem of Asset Pricing

Definition

A market model is complete if every financial derivative can be hedged (replicated)

Second Fundamental Theorem of Asset Pricing

Theorem

Assume that in the multidimensional market model, the filtration $\{\mathcal{F}_t\}$ is generated by the Brownian motion, then the following are equivalent

1. *The market model is complete.*
2. $d \leq m$ *and the instantaneous volatility matrix in the market price of risk equations has full rank for a.e.* $t \in [0, T]$.
3. *There exists a unique martingale measure* $\widetilde{\mathbb{P}}$ *for the discounted prices.*

Proof.

See Chapter 10, Marek[1]. □

Introduction

Two pricing approaches:

- PDE
- Risk neutral

How do we connect the two seemingly different approaches?
$\Longrightarrow$ Feynman-Kac Theorem

Stochastic Differential Equation

Definition

A stochastic differential equation (SDE) is an equation of the form

$$dX(t,\omega) = \beta(t, X(t,\omega))dt + \gamma(t, X(t,\omega))dW(t,\omega)$$

where $\beta(t,x)$ and $\gamma(t,x)$ are non-random functions of t and x, and are called *drift* and *diffusion* respectively.

Stochastic Differential Equation

Definition

Given the initial condition

$$X(t) = x$$

a solution to the above SDE is stochastic process $X(T)$, $T \geq t$ such that

$$X(T) = X(t) + \int_t^T \beta(u, X(u))du + \int_t^T \gamma(u, X(u))dW(u)$$

Stochastic Differential Equation

- The process X depends on initial condition. Different initial conditions lead to different stochastic processes.
- The process X starts from initial time t, not before t.
- Under certain (mild) conditions, the stochastic process X is uniquely determined by its initial value $X(t) = x$.

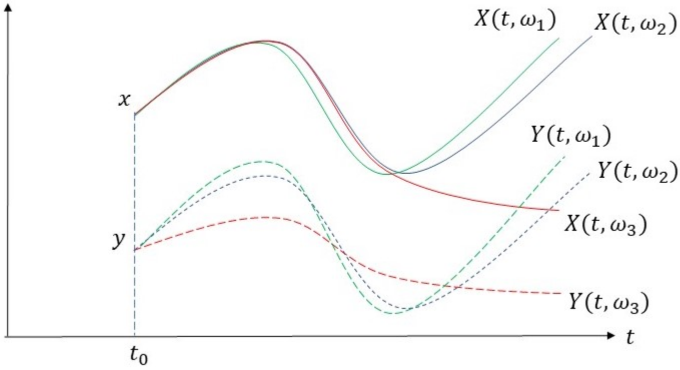
Markov Property

Fix a Borel-function h. Let $X(T)$ be the solution of the SDE with initial condition $X(t) = x$. Define

$$g(t,x) = \mathbb{E}^{t,x} h(X(T))$$

Markov Property

Fix t_0, $g(t,x)$ is determined by stochastic process starting from t_0 with initial value x.



Markov Property

Theorem

Let $X(u), u \geq 0$ be a solution to the SDE with initial condition $X(0)$. Then for $0 \leq t \leq T$

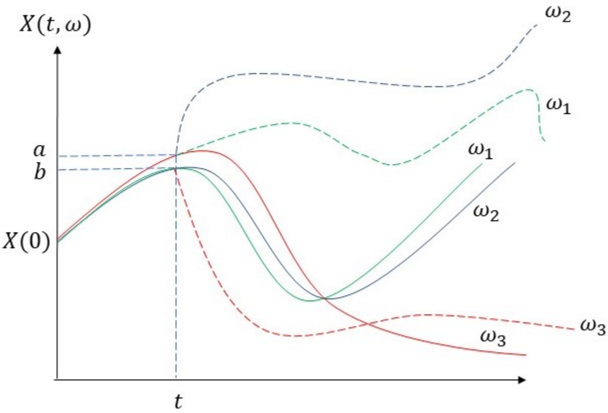
$$\mathbb{E}\left[h(X(T))|\mathcal{F}_t\right] = g(t, X(t))$$

Markov Property

- LHS is for a specific stochastic process $X(t)$
- RHS is for a collection of stochastic processes starting from time t with initial value $X(t,\omega)$ dependent on ω.

$$\begin{array}{ll} & \omega \\ \Rightarrow & X(t,\omega) \\ \Rightarrow & \text{initial condition for SDE} \\ \Rightarrow & \text{stochastic process} \end{array}$$

Markov Property



Markov Property

We can better understand this by looking at the discretizied version of X.
Let $t_0 = 0 < t_1 < \ldots < t_n = T$. Then

$$X(t_{i+1}) \approx X(t_i) + \beta(t_i, X(t_i))dt_i + \gamma(t_i, X(t_i))dW(t_i)$$

where

$$dt_i = t_{i+1} - t_i$$

and

$$dW(t_i) = W(t_{i+1}) - W(t_i)$$

Markov Property

Fix $j > i$, from this discretization we know $X(t_j)$ depends on the value of $X(t_i), dW(t_i), \ldots, dW(t_{j-1})$ through the discretized schema.
In other words, for fixed i and j, there exists a function f such that

$$X(t_j, \omega) \approx f(X(t_i, \omega), dW(t_i, \omega), dW(t_{j-1,\omega}))$$

$X(t_j)$ depends on $X(t_i)$ which is a random variable.

Markov Property

Now consider a stochastic process Y that starts with $Y(t_i) = y$, we have

$$Y(t_j, \omega) \approx f(y, dW(t_i, \omega), dW(t_{j-1}, \omega))$$

Note that y is fixed. That's the difference between $X(t_j)$ and $Y(t_j)$.

Markov Property

The independence of $dW(t_i, \omega), dW(t_{j-1}, \omega)$ on $X(t_i)$ leads to Markov process.
If we let

$$g(t_i, y) = \mathbb{E}\left[Y(t_j, \omega)\right] = \mathbb{E}\left[f(y, dW(t_i, \omega), dW(t_{j-1}, \omega))\right]$$

Then

$$\begin{aligned} \mathbb{E}\left[X(t_j) | \mathcal{F}_{t_i}\right] &= \mathbb{E}\left[f(X(t_i), dW(t_i), \ldots, dW(t_{j-1})) | \mathcal{F}_{t_i}\right] \\ &= g(t_i, X(t_i)) \end{aligned}$$

The last equality follows from the fact that $X(t_i)$ is $\mathcal{F}_{t_i}$-measurable and $dW(t_i), \ldots, dW(t_{j-1})$ are independent of $\mathcal{F}_{t_i}$.

Markov Property

Corollary

The solution to SDE $X(t)$ are Markov processes.

Feynman-Kac Theorem I

Lemma

Let $X(t)$ be the solution to SDE with initial condition at time $t = 0$. Then $g(t, X(t))$ is a martingale

Proof.

Let $s < t$, we want to show

$$\mathbb{E}[g(t, X(t))|\mathcal{F}_s] = g(s, X(s))$$

This follows from

$$\begin{aligned} \mathbb{E}[g(t, X(t))|\mathcal{F}_s] &= \mathbb{E}[\mathbb{E}[h(X(T))|\mathcal{F}_t]|\mathcal{F}_s] \\ &= \mathbb{E}[h(X(T))|\mathcal{F}_s]| \\ &= g(s, X(s)) \end{aligned}$$

$\square$

Feynman-Kac Theorem I

Theorem (Feynmann-Kac)

Consider the stochastic differential equation

$$dX(u) = \beta(u, X(u))dt + \gamma(u, X(u))dW(u)$$

Let h be a Borel function. Fix $T > 0$ and define

$$g(t, x) = \mathbb{E}^{t,x} h(X(T))$$

Then $g(t, x)$ satisfies the partial differential equation

$$g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0$$

with terminal condition

$$g(T, x) = h(x)\ \forall x$$

Feynman-Kac Theorem I

The idea of proof is

1. Find a martingale
2. Take the differential
3. Set the drift term to 0

Feynman-Kac Theorem I

Proof.

Note that $g(t, X(t))$ is martingale and

$$\begin{aligned} dg(t, X(t)) &= g_t dt + g_x dX(t) + \tfrac{1}{2} g_{xx} dX(t) dX(t) \\ &= \left(g_t + g_x \beta + \tfrac{1}{2}\gamma^2 g_{xx}\right) dt + g_x \gamma dW(t) \end{aligned}$$

Since it is a martingale, the drift term must be zero, which implies

$$g_t(t, X(t)) + g_x(t, X(t))\beta(t, X(t)) + \frac{1}{2}\gamma^2(t, X(t)) g_{xx}(t, X(t)) = 0$$

for all t and ω. $\square$

Feynman-Kac Theorem I

Proof.

Hence we have

$$g_t(t,x) + g_x(t,x)\beta(t,x) + \frac{1}{2}\gamma^2(t,x)g_{xx}(t,x) = 0$$

for all (t,x) in the range of $(t, X(t))$. $\square$

Feynman-Kac Theorem II

Theorem (Discounted Feynmann-Kac)

Consider the stochastic differential equation

$$dX(u) = \beta(u, X(u))dt + \gamma(u, X(u))dW(u)$$

Let h be a Borel function. Fix $T > 0$ and define

$$f(t, x) = \mathbb{E}^{t,x}\left[e^{-r(T-t)}h(X(T))\right]$$

Then $f(t, x)$ satisfies the partial differential equation

$$f_t(t, x) + \beta(t, x)f_x(t, x) + \frac{1}{2}\gamma^2(t, x)f_{xx}(t, x) = rf(t, x)$$

with terminal condition

$$f(T, x) = h(x)\ \forall x$$

Feynman-Kac Theorem II

Proof.

HW.
Hint: $e^{rt}f(t, X(t))$ is a martingale. $\square$

Feynman-Kac Theorem - Applications

In the BSM model, under the risk neutral measure stock prices follow

$$\frac{dS(t)}{S(t)} = rdt + \sigma d\widetilde{W}(t)$$

The price of a derivative whose payoff is $h(S(T))$ at time T is

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}h(S(T))|\mathcal{F}_t\right]$$

Note that

$$\widetilde{\mathbb{E}}\left[e^{-r(T-t)}h(S(T))|\mathcal{F}_t\right] = e^{-r(T-t)}g(t, S(t))$$

Hence we can use the (discounted) Feynman-Kac Theorem to derive the BS equation.

$$V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = rV$$

References

Marek Musiela, Marek Rutkowski
Martingale Methods in Financial Modelling, *Springer*

Thank you!