Section 2. Short Rate Models

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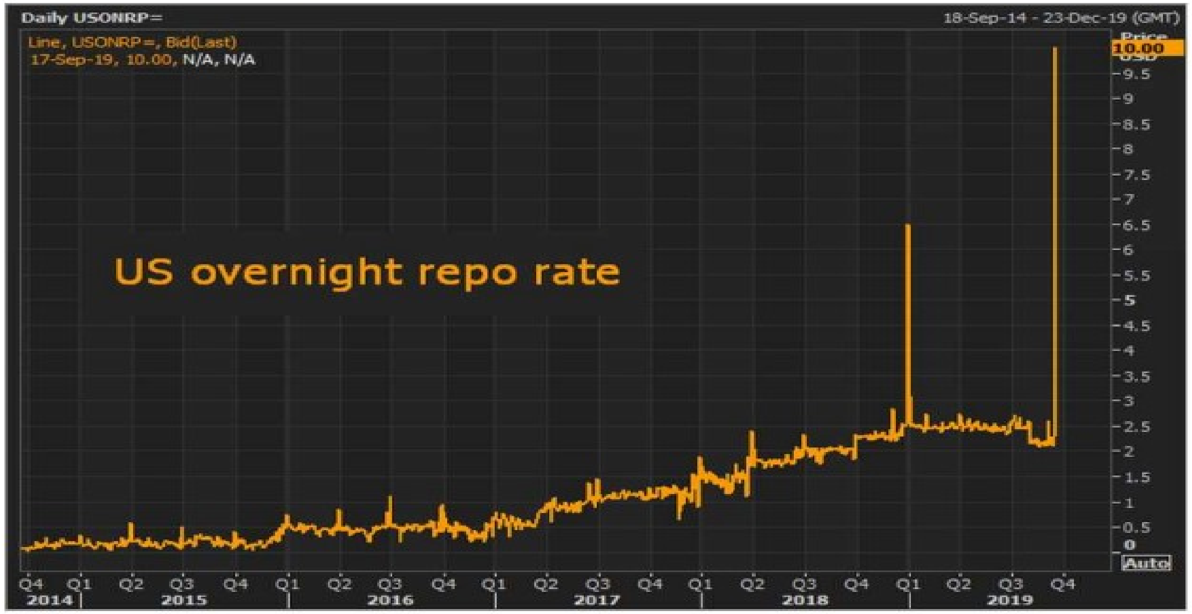
2.1 What Is Short Rate r_t?

– An Idealized Variable. Overnight Rate as a Proxy.

Type	Settlement date	Rate (%)
Cash	Overnight rate	5.58675
Cash	Tomorrow next rate	5.59375
Cash	1m	5.625
Cash	3m	5.71875
Future	Dec-97	5.76
Future	Mar-98	5.77
Future	Jun-98	5.82
Future	Sep-98	5.88
Future	Dec-98	6
Swap	2y	6.01253
Swap	3y	6.10823
Swap	4y	6.16
Swap	5y	6.22
Swap	7y	6.32
Swap	10y	6.42
Swap	15y	6.56
Swap	20y	6.56
Swap	30y	6.56

From Publicized Source.

It is more volatile than longer term rates



From Publicized Source.

Case Study of Overnight Rate Hike

Overnight lending cost in Hong Kong hits new post-crisis high, Financial Times, JULY 31 2017.

From Publicized Source.

- Month-end liquidity drop.

- HSBC share buy-back.

- Li Ka Shing sold Hutchison Telecommunications fixed-line phone business for HKD14.5 billions.

Relationship with Bond Prices and Forward Rates

With Zero Coupon Bond Price:

$$Z(t,T) = E_t\left(e^{-\int_t^T r_s ds}\right). \quad (2.1)$$

With Forward Rates:

$$f_t(T) = F_t(T,T), \quad (2.2)$$

$$r_t = f_t(t). \quad (2.3)$$

2.2 Guidelines for Interest Rate Modeling

- Does the dynamics imply positive rates, i.e., $r_t > 0 \ \forall t$?

- What distribution does the dynamics imply for r_t? Is it, for instance, a fat-tailed distribution?

- Are bond prices (and therefore spot rates, forward rates and swap rates) explicitly computable from the dynamics?

- Are bond-option (and cap, floor, swaption) prices explicitly computable from the dynamics?

- Is the model mean reverting, in the sense that the expected value of the short rate tends to a constant value as time grows towards infinity, while its variance does not explode?

- How do the volatility structures implied by the model look like?

- Does the model allow for explicit short-rate dynamics under the forward measures?

- How suited is the model for Monte Carlo simulation?

- How suited is the model for building recombining lattices?

- Does the chosen dynamics allow for historical estimation techniques to be used for parameter estimation purposes?

2.3 The Vasicek Model

$$dr_t = k(\theta - r_t)dt + \sigma dW_t. \quad (2.4)$$

The model attractiveness lies in the simplicity of interest rate distribution and the availability of close form solutions for rates, bonds and other related quantities.

Drawback

The model admits negative rates.

Analytical Results

$$r_t = r_s e^{-k(t-s)} + \theta[1 - e^{-k(t-s)}] + \sigma \int_s^t e^{-k(t-u)} dW_u. \quad (2.5)$$

The technique for obtaining the solution is similar to that for solving a similar ODE. This is a normally distributed random variable with

$$E\{r_t|\mathcal{F}_s\} = r_s e^{-k(t-s)} + \theta[1 - e^{-k(t-s)}], \quad (2.6)$$

$$Var\{r_t|\mathcal{F}_s\} = \frac{\sigma^2}{2k}[1 - e^{-2k(t-s)}]. \quad (2.7)$$

Zero Coupon Bond Price:

$$P(t, r_t, T) = A(t, T)e^{-B(t,T)r_t}, \quad (2.8)$$

where

$$A(t, T) = \exp\left\{\left(\theta - \frac{\sigma^2}{2k^2}\right)[B(t, T) - T + t] - \frac{\sigma^2}{4k}B(t, T)^2\right\}, \quad (2.9)$$

$$B(t, T) = \frac{1}{k}[1 - e^{-k(T-t)}]. \quad (2.10)$$

Derivation

(1) Brutal: hard integration based on (2.1).

(2) Elegant: by means of PDE.

Let M_t be the discounted bond price

$$M_t = \exp\left(-\int_0^t r_u du\right) P(t, r_t, T). \quad (2.11)$$

By Ito's formula,

$$
\begin{aligned}
dM_t = &-r_t \exp\left(-\int_0^t r_u du\right) P(t, r_t, T) dt \\
&+ \exp\left(-\int_0^t r_u du\right) \frac{\partial P(t, r_t, T)}{\partial t} dt \\
&+ \exp\left(-\int_0^t r_u du\right) \frac{\partial P(t, r_t, T)}{\partial r_t} [k(\theta - r_t) dt + \sigma dW_t] \\
&+ \frac{1}{2} \exp\left(-\int_0^t r_u du\right) \frac{\partial^2 P(t, r_t, T)}{\partial r_t^2} \sigma^2 dt. \quad (2.12)
\end{aligned}
$$

Since M_t is a martingale, we have the dt term to be zero, or

$$\frac{\partial P(t,r_t,T)}{\partial t} + k(\theta - r)\frac{\partial P(t,r_t,T)}{\partial r_t} + \frac{\sigma^2}{2}\frac{\partial^2 P(t,r_t,T)}{\partial r_t^2} - r_t P(t,r_t,T) = 0. \quad (2.13)$$

SDE for Bond Price

$$\frac{d_t P(t,T)}{P(t,T)} = r_t dt - \sigma B(t,T) dW_t. \quad (2.14)$$

Option Price

At time T for zero coupon bond maturing at time S with strike X :

$$ZBO(t,T,S,X) = P(t,S)N(h) - XP(t,T)N(h-\sigma_p) \text{ for call, } \quad (2.15)$$

$$= -P(t,S)N(-h) + XP(t,T)N(\sigma_p - h) \text{ for put, } \quad (2.16)$$

standard normal cumulative distribution function, and

$$\sigma_p = \sigma\sqrt{\frac{1-e^{-2k(T-t)}}{2k}}\,B(T,S), \quad (1.17)$$

$$h = \frac{1}{\sigma_p}\ln\frac{P(t,S)}{P(t,T)X} + \frac{\sigma_p}{2}. \quad (1.18)$$

The Black-Scholes Option Pricing Formula

$$c = SN(d_1) - Xe^{-rT}N(d_2)$$

$$p = Xe^{-rT}N(-d_2) - SN(-d_1),$$

$$d_1 = \frac{\ln(S/X) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S/X) + (r-\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

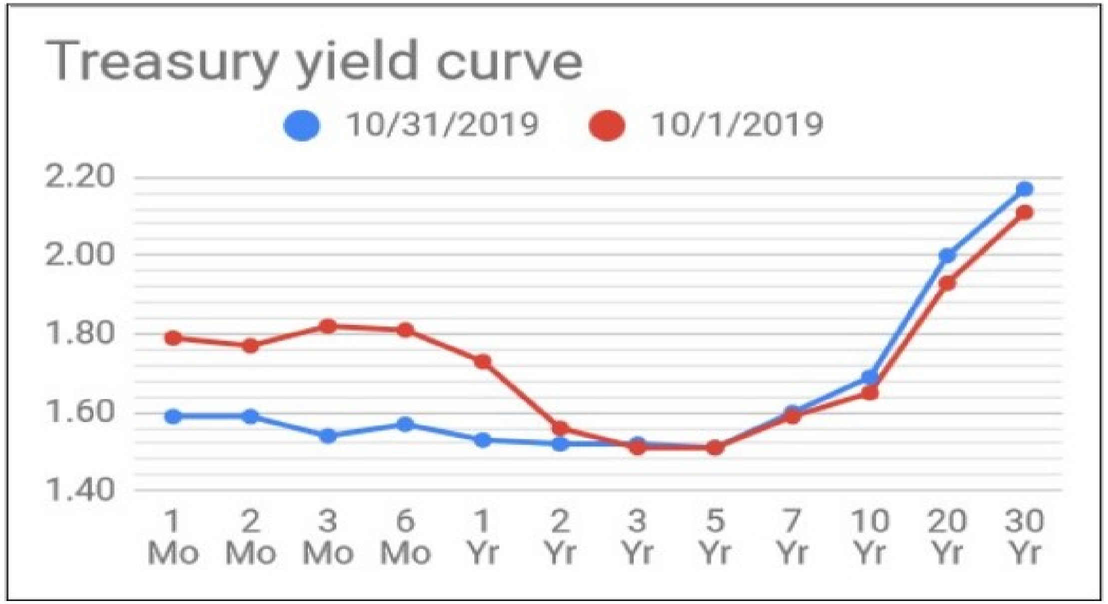
S = Stock price.
X = Strike price of option.
r = Risk-free interest rate.
T = Time to expiration in years.
σ = Volatility of the relative price change of the underlying stock price.
$N(x)$ = The cumulative normal distribution function.

Key Point

The Black-Scholes kind of formula is resulted from the fact that $P(t,T)$ is lognomal because r_t is normal by (2.1).

A Challenge

In real life, the yield curve, or interest rate term structure, is backed by tradable securities with their prices moving up and down.



From Publicized Source.

Hence equations (2.8) – (2.10) have to be satisfied for multiple buckets of T, which is impossible for all the parameters k, θ, and σ to be fixed as constants.

2.4 The Hull-White Model

$$dr_t = k[\theta(t) - r_t]dt + \sigma dW_t, \quad (2.19)$$

where $\theta(t)$ is a deterministic function of time t.

Assuming that the market prices of zero bonds $P^M(0,T)$ are given and the market instantaneous forward rates

$$f^M(0,T) = -\frac{\partial lnP^M(0,T)}{\partial T} \quad (2.20)$$

are available. We are going to calibrate the Hull-White model to this market information.

Solution of the Hull-White Model

$$r_t = r_s e^{-k(t-s)} + k \int_s^t \theta(u) e^{-k(t-u)} du + \sigma \int_s^t e^{-k(t-u)} dW_u. \quad (2.21)$$

Mean and Variance

$$E\{r_t | \mathcal{F}_s\} = r_s e^{-k(t-s)} + k \int_s^t \theta(u) e^{-k(t-u)} du, \quad (2.22)$$

$$Var\{r_t | \mathcal{F}_s\} = \frac{\sigma^2}{2k}[1 - e^{-2k(t-s)}]. \quad (2.23)$$

Zero Coupon Bond Price

$$P(t,T) = \bar{A}(t,T)e^{-r_t B(t,T)}, \quad (2.24)$$

$$\bar{A}(t,T) = A(t,T)e^{-k\int_t^T \theta(u)B(u,T)du}. \quad (2.25)$$

Intantaneous Forward Rate

$$f(t,T) = k\int_t^T \theta(u)e^{-k(T-u)}du - \frac{\sigma^2}{2}B^2(t,T) + r_t e^{-k(T-t)}. \quad (2.26)$$

Calibration to Market Yield Curve

It follows from (2.26) that

$$\theta(t) = f^M(0,t) + \frac{1}{k}\frac{\partial f^M(0,t)}{\partial t} + \frac{\sigma^2}{2k^2}(1 - e^{-2kt}). \quad (2.27)$$

Probability Negative Rates

$$Prob(r_t < 0) = N\left(-\frac{r_0 e^{-kt} + k\int_0^t \theta(u) e^{-k(t-u)} du}{\sqrt{\frac{\sigma^2}{2k}(1 - e^{-2kt})}} \right). \quad (2.28)$$

But how bad is it in practice?

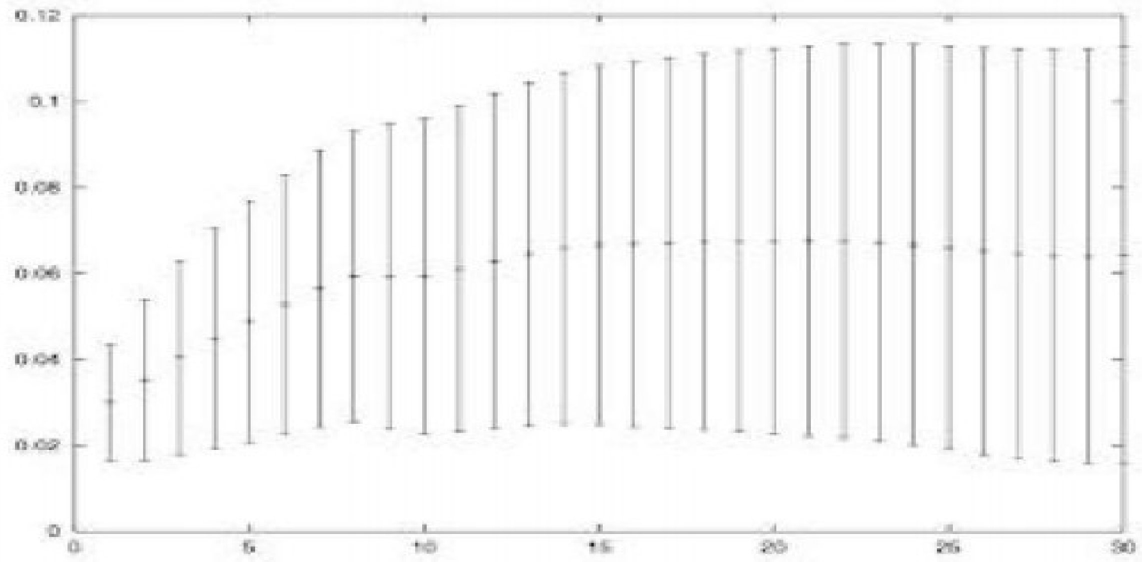


Fig. 3.2. The two standard-deviation window for the instantaneous short rate r as time goes by (market data as of 2 June 1999).

From Textbook of Brigo and Mercurio.

Volatility Calibration

A more general Hull-White model, with the other parameters a and σ also formulated as time dependent functions, can be used to calibrate to market quoted interest rate volatilities. However the resulting future volatility structures are likely to be unrealistic, as was remarked by Hull and White themselves.

The volatility calibrated is better carried out by another short rate model, the DBT model developed by practitioners of Goldman Sachs.

2.5 The Cox, Ingersoll and Ross（CIR）Model

– Preclusion of Negative Rates

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}\,dW_t. \quad (2.29)$$

r_t is

(1) mean reverting, towards θ with speed k.

(2) positive, i.e. $r_t > 0$, if $2k\theta \geq \sigma^2$.

Intuitively, when r_t approaches 0, the standard deviation $\sigma\sqrt{r_t}$ becomes very small, which dampens the magnitude of random shock dw_t. So the evolution of r_t is dominated by the drift term, which pushes r_t upwards towards θ. The condition $2k\theta \geq \sigma^2$ quantifies the enforcement of such dominance in the parameters.

(3) Mean and variance are

$$E\{r_t|\mathcal{F}_s\} = r_s e^{-k(t-s)} + \theta[1 - e^{-k(t-s)}], \text{ (2.30) approaching } \theta \text{ as } t \to +\infty,$$

$$Var\{r_t|\mathcal{F}_s\} = \frac{\sigma^2 r_s}{k}[e^{-k(t-s)} - e^{-2k(t-s)}] + \frac{\theta\sigma^2}{2k}[1 - e^{-k(t-s)}]^2. \text{ (2.31)}$$

Bond Prices

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}, \quad (2.32)$$

where

$$A(t,T) = \left[\frac{2he^{\frac{(k+h)(T-t)}{2}}}{2h + (k+h)(e^{h(T-t)} - 1)}\right]^{\frac{2k\theta}{\sigma^2}}, \quad (2.33)$$

$$B(t,T) = \frac{2[e^{h(T-t)} - 1]}{2h + (k+h)[e^{h(T-t)} - 1]}, \quad (2.34)$$

$$h = \sqrt{k^2 + 2\sigma^2}. \quad (2.35)$$

In order to fit a market yield curve, we extend the parameter θ from a constant to a time dependent function $\theta(t)$. The bond prices become

$$P(t,T) = \hat{A}(t,T)e^{-B(t,T)r_t}, \quad (2.36)$$

$$\hat{A}(t,T) = e^{-k\int_t^T \theta(s)B(s,T)ds}. \quad (2.37)$$

Function $\theta(t)$ can be worked out by calibrating $P(t,T)$ to the market.

2.6 The Black-Derman-Toy (BDT) Model and Tree Method

Though it can be formulated by an SDE as follows:

$$d\ln r_t = \left[\theta(t) + \frac{\sigma'(t)}{\sigma(t)}\ln r_t\right]dt + \sigma(t)dW_t, \quad (2.38)$$

the BDT model was originally developed and is often used based on a binomial tree.

Model Input

- Risk free zero-coupon bond yields or prices $P(0,T)$.
- Data related to short rate volatilities σ_n.

BDT Binomial Tree

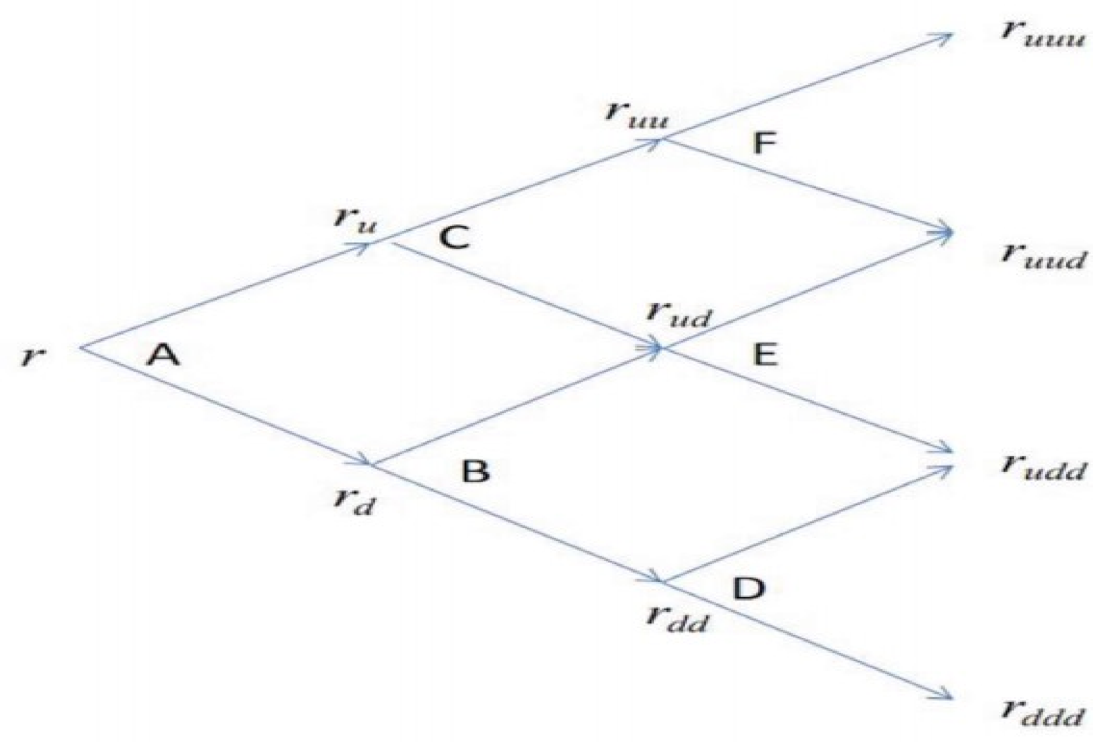
Example

Suppose we have the following market date:

T_i	R_i	V_i
$1y$	10.0	
$2y$	11.0	19.0%
$3y$	12.0	18.0%

T_i are the maturities of zero coupon bonds,
R_i are the annually compounded spot rates for maturities T_i,
V_i are the volatilities applied to the time interval $[0, 1y]$ for the yield of the zero bonds maturing at T_i.

We construct the following recombining binomial tree of the forward rate:



(1) To fit the $2y$ zero bond price and the forward rate volatility, r_u and r_d should satisfy

$$\frac{1}{1.10}\left(\frac{1}{2}\frac{1}{1+r_d}+\frac{1}{2}\frac{1}{1+r_u}\right)=\frac{1}{1.11^2}, \quad (2.39)$$

$$\frac{1}{2}\ln\frac{r_u}{r_d}=0.19. \quad (2.40)$$

The results are $r_u = 0.1432$ and $r_d = 0.0979$. (2.40) can be verified by the following calculation for a two-point distribution:

$$\frac{1}{2}\ln^2 r_d + \frac{1}{2}\ln^2 r_u - \left[\frac{1}{2}(\ln r_d + \ln r_u)\right]^2$$

$$= \left[\frac{1}{2}(\ln r_d - \ln r_u)\right]^2$$

$$= 0.19^2, \ (2.41)$$

noting that r_d and r_u are the yields observed at node B and node C of the bond maturing at $2y$.

(2) r_{dd}, r_{ud} and r_{uu} satisfy

$$\frac{1}{1.10}\left[\frac{1}{2}\frac{1}{1.0979}\left(\frac{1}{2}\frac{1}{1+r_{dd}}+\frac{1}{2}\frac{1}{1+r_{ud}}\right)\right.$$

$$\left.+\frac{1}{2}\frac{1}{1.1432}\left(\frac{1}{2}\frac{1}{1+r_{ud}}+\frac{1}{2}\frac{1}{1+r_{uu}}\right)\right]=\frac{1}{1.12^3}, \quad (2.42)$$

$$\frac{1}{2}\ln\frac{r_{uu}}{r_{ud}}=\sigma_2, \quad (2.43)$$

$$\frac{1}{2}\ln\frac{r_{ud}}{r_{dd}}=\sigma_2, \quad (2.44)$$

where σ_2 is the volatility of the forward rate for the interest period $[1y, 2y]$.

Let B_d and B_u be the prices at node B and at node C respectively of the zero bond maturing at $3y$. Then

$$B_d = \frac{1}{1.0979}\left[\frac{1}{2}\frac{1}{1+r_{dd}} + \frac{1}{2}\frac{1}{1+r_{ud}}\right] \quad (2.45)$$

$$B_u = \frac{1}{1.1432}\left[\frac{1}{2}\frac{1}{1+r_{ud}} + \frac{1}{2}\frac{1}{1+r_{uu}}\right]. \quad (2.46)$$

Their yields are

$$y_d = \sqrt{\frac{1}{B_d}} - 1, \quad (2.47)$$

$$y_u = \sqrt{\frac{1}{B_u}} - 1. \quad (2.48)$$

Although σ_2 is not known directly, it can be determined by an iterative search so that

$$\frac{1}{2}\ln\frac{y_u}{y_d} = 0.18. \quad (2.49)$$

The solutions of r_{uu}, r_{ud}, r_{dd} are

$$r_{uu} = 0.1942, \quad (2.50)$$

$$r_{ud} = 0.1377, \quad (2.51)$$

$$r_{dd} = 0.0976, \quad (2.52)$$

corresponding to the result of $\sigma_2 = 0.172$.

Remark

The original work of BDT takes bond yield volatilities as model input as the original purpose was to price bond options. In fact, interest rates volatility information is also contained in the prices of caps and floors, which have greater market liquidity than bond options. Volatilities of forward rates, σ_n, can be derived by bootstrapping the market quotes on caps and floors, and used as direct input for the BDT model.

N-period BDT Tree

Construct a recombining binomial tree for r_t with up and down probabilities

$$p_u,\ p_d = \frac{1}{2}\ (2.53)$$

and the values on the nodes at T_n given as follows:

$$r(n+1,1) = r_{n+1},$$

$$r(n+1,i+1) = r(n+1,i)e^{-2\sigma_n\sqrt{\Delta t}},\ i = 1,\cdots,n,\ (2.54)$$

$$
\begin{array}{ccccccc}
 & & & & & & r_4 \\
 & & & & & \nearrow & \\
 & & & & r_3 & & \\
 & & & \nearrow & & \searrow & \\
 & & r_2 & & & & r_4 v_4 \\
 & \nearrow & & \searrow & & \nearrow & \\
r_1 & & & & r_3 v_3 & & \\
 & \searrow & & \nearrow & & \searrow & \\
 & & r_2 v_2 & & & & r_4 v_4^2 \\
 & & & \searrow & & \nearrow & \\
 & & & & r_3 v_3^2 & & \\
 & & & & & \searrow & \\
 & & & & & & r_4 v_4^3
\end{array}
$$

$$r(n+1,1) = r_{n+1}, \; v_n = e^{-2\sigma_n \sqrt{\triangle t}}.$$

Calibration

(1) **To Volatilities**:

The variance of the logarithm of the rate is

$$\frac{1}{2}\ln^2 r(n,i) + \frac{1}{2}\ln^2 r(n,i-1) - \left[\frac{1}{2}(\ln r(n,i) + \ln r(n,i-1))\right]^2$$

$$= \left[\frac{1}{2}(\ln r(n,i) - \ln r(n,i-1))\right]^2$$

$$= \sigma_n^2 \,\triangle t. \;\; (2.55)$$

This means (2. 54) has natually calibrated the BDT tree to market caplet volatilities

(2) **To Zero Coupon Bonds**:

Suppose the tree has been calibrated up to the time T_{n-1}. To meet the price $P(T_{n+1})$ of the zero coupon bond maturing at time T_{n+1}, the values of r_{n+1} at the next time step T_n should satisfy the following :

$$\sum_{i=1}^{n+1} \frac{D(n+1,i)}{1+r_{n+1}e^{-2\sigma(n)(i-1)\sqrt{\triangle t}}\triangle t} = P(T_{n+1}), \quad (2.56)$$

where $D(n+1,i)$, $i=1,2,\cdots$, or $n+1$, is the Arrow-Debreu price for the contingent claim that, at time T_n, pays \$1 for state $r(n+1,i)$ and pays \$0 for all other states $r(n+1,j)$, $j \neq i$.

$D(n+1,i)$ can be computed recursively:

$$D(n+1,1) = \frac{1}{2}D(n,1)\frac{1}{1+r(n,1)\Delta t}, \quad (2.57)$$

$$D(n+1,i) = \frac{1}{2}D(n,i-1)\frac{1}{1+r(n,i-1)\Delta t} + \frac{1}{2}D(n,i)\frac{1}{1+r(n,i)\Delta t}, \quad (2.58)$$

$$D(n+1,n+1) = \frac{1}{2}D(n,n)\frac{1}{1+r(n,n)\Delta t}. \quad (2.59)$$

$\sigma(t)$: **Implied Volatility or Local Volatility**?

It is surprising enough that $\sigma(t)$ is actually the implied volatility of the short process r_t for maturity t, although $\sigma(t)$ appears as if it is in the position of local volatility of r_t in (2.38). This is because the solution to (2.38) can be expressed as

$$r_t = U(t)e^{\sigma(t)W_t}, \quad (2.60)$$

a lognormal distribution with **implied** volatility $\sigma(t)$.

Proof. Since $\ln r_t = \ln U(t) + \sigma(t)W_t$, it follows from the Ito's formula that

$$d\ln r_t = \frac{U'(t)}{U(t)}dt + \sigma'(t)W_t dt + \sigma(t)dW_t$$

$$= \frac{U'(t)}{U(t)}dt + \sigma'(t)\frac{\ln r_t - \ln U(t)}{\sigma(t)}dt + \sigma(t)dW_t$$

$$= \left[\theta(t) + \frac{\sigma'(t)}{\sigma(t)}\ln r_t\right]dt + \sigma(t)dW_t, \quad (2.61)$$

where

$$\theta(t) = \frac{U'(t)}{U(t)} - \frac{\sigma'(t)}{\sigma(t)}\ln U(t) \quad (2.62)$$

is a deterministic function of t.