Binomial Representation Theorem

- Recall the *discrete stochastic integral*: if $\{X_n\}_{n\geq 0}$ is a $(\mathbb{P}, \{\mathcal{F}_n\}_{n\geq 0})$-martingale and $\{\phi_n\}_{n\geq 1}$ is $\{\mathcal{F}_n\}_{n\geq 0}$-previsible, then

$$Z_n = Z_0 + \sum_{j=0}^{n-1} \phi_{j+1}\left(X_{j+1} - X_j\right),$$

  where Z_0 is a constant, is also a $(\mathbb{P}, \{\mathcal{F}_n\}_{n\geq 0})$-martingale.

- In a binary tree model, the converse is also true.

- Suppose that the measure $\mathbb{Q}$ is such that the discounted price process $\{\tilde{S}_n\}$ is a $(\mathbb{Q}, \{\mathcal{F}_n\}_{n\geq 0})$-martingale. If $\{\tilde{V}_n\}$ is any other $(\mathbb{Q}, \{\mathcal{F}_n\}_{n\geq 0})$-martingale, then there exists a $(\mathbb{Q}, \{\mathcal{F}_n\}_{n\geq 0})$-predictable process $\{\phi_n\}_{n\geq 1}$ such that

$$\tilde{V}_n = \tilde{V}_0 + \sum_{j=0}^{n-1} \phi_{j+1} \left(\tilde{S}_{j+1} - \tilde{S}_j\right).$$

- To prove this, we must show that

$$\tilde{V}_{i+1} - \tilde{V}_i = \phi_{i+1} \left(\tilde{S}_{i+1} - \tilde{S}_i\right)$$

where ϕ_{i+1} is $\mathcal{F}_i$-measurable.

- For a given node at $t = i\delta t$, write $\left\{\tilde{S}_{i+1}(u), \tilde{S}_{i+1}(d)\right\}$ for the two possible values of $\tilde{S}_{i+1}$, and $\left\{\tilde{V}_{i+1}(u), \tilde{V}_{i+1}(d)\right\}$ for the corresponding values of $\tilde{V}_{i+1}$.

- We can solve

$$\tilde{V}_{i+1}(u) - \tilde{V}_i = \phi_{i+1}\left(\tilde{S}_{i+1}(u) - \tilde{S}_i\right) + k_{i+1}$$

and

$$\tilde{V}_{i+1}(d) - \tilde{V}_i = \phi_{i+1}\left(\tilde{S}_{i+1}(d) - \tilde{S}_i\right) + k_{i+1}$$

to get

$$\phi_{i+1} = \frac{\tilde{V}_{i+1}(u) - \tilde{V}_{i+1}(d)}{\tilde{S}_{i+1}(u) - \tilde{S}_{i+1}(d)}.$$

- Because $\left\{\tilde{S}_{i+1}(u), \tilde{S}_{i+1}(d)\right\}$ and $\left\{\tilde{V}_{i+1}(u), \tilde{V}_{i+1}(d)\right\}$ are known at time $t = i\delta t$, this ϕ_{i+1} is $\mathcal{F}_i$-measurable.

- Also

$$k_{i+1} = \tilde{V}_{i+1} - \tilde{V}_i - \phi_{i+1}\left(\tilde{S}_{i+1} - \tilde{S}_i\right)$$

  and because k_{i+1} is also $\mathcal{F}_i$-measurable,

$$\begin{aligned} k_{i+1} &= \mathbb{E}\left[\tilde{V}_{i+1} - \tilde{V}_i - \phi_{i+1}\left(\tilde{S}_{i+1} - \tilde{S}_i\right)\middle| \mathcal{F}_i\right] \\ &= \mathbb{E}\left[\tilde{V}_{i+1} - \tilde{V}_i\middle| \mathcal{F}_i\right] - \phi_{i+1}\mathbb{E}\left[\tilde{S}_{i+1} - \tilde{S}_i\middle| \mathcal{F}_i\right] \\ &= 0 \end{aligned}$$

  because both $\{\tilde{S}_n\}$ and $\{\tilde{V}_n\}$ are $(\mathbb{Q}, \{\mathcal{F}_n\}_{n\geq 0})$-martingales.

- That completes the proof.

- Note that the tree did *not* need to be recombining, only binary.

- We did, however, tacitly assume that $\tilde{S}_{i+1}(u) \neq \tilde{S}_{i+1}(d)$ at this node, and hence at all nodes.

- Self-financing: we can build a dynamic portfolio of stock and cash with discounted value $\{\tilde{V}_n\}$, by holding ϕ_{i+1} shares in $[i\delta t, (i+1)\delta t)$ and keeping the balance $V_i - \phi_{i+1} S_i$ in cash.

Continuous Time Limit

- Fix a time $t > 0$, and let $\delta t = t/N$; if N is large, and hence δt is small, the binary tree should approximate a continuous time model.

- At a node with stock price s, assume that the successor nodes are $s \exp\left(\nu \delta t \pm \sigma \sqrt{\delta t}\right)$ for a *drift* ν and *volatility* σ.

- Suppose that under the *market* measure $\mathbb{P}$, these are equally likely.

- Then the expected value at the next step, conditionally on being at this node, is

$$se^{\nu\delta t} \times \frac{1}{2}\left(e^{\sigma\sqrt{\delta t}} + e^{-\sigma\sqrt{\delta t}}\right) \approx s\left[1 + \left(\nu + \frac{1}{2}\sigma^2\right)\delta t\right],$$

and the conditional variance is

$$\frac{1}{4}\left(se^{\nu\delta t}\right)^2\left(e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}\right)^2 \approx s^2\sigma^2\delta t.$$

whence the conditional standard deviation is $s\sigma\sqrt{\delta t}$.

- Suppose that by time $t = N\delta t$, the price has moved up X_N times, and therefore down $N - X_N$ times.

- Then

$$
\begin{aligned}
S_t &= S_0 \exp\left[N\nu\delta t + X_N \sigma\sqrt{\delta t} - (N - X_N)\,\sigma\sqrt{\delta t}\right] \\
&= S_0 \exp\left[\nu t + \sigma\sqrt{t}\left(\frac{2X_N - N}{\sqrt{N}}\right)\right].
\end{aligned}
$$

- X_N is binomial, so, by the Central Limit Theorem, $Z_N \triangleq (2X_N - N)/\sqrt{N}$ is approximately standard normal for large N, so we can write this as

$$
S_t = S_0 \exp\left(\nu t + \sigma\sqrt{t} Z_N\right),
$$

where Z_N is approximately $N(0,1)$.

- So, under the market measure $\mathbb{P}$, S_t is approximately log-normally distributed.

- Under the *martingale* measure $\mathbb{Q}$, the probability of an up-jump is

$$p = \frac{e^{r\delta t} - e^{\nu\delta t - \sigma\sqrt{\delta t}}}{e^{\nu\delta t + \sigma\sqrt{\delta t}} - e^{\nu\delta t - \sigma\sqrt{\delta t}}}$$
$$\approx \frac{1}{2}\left[1 - \sqrt{\delta t}\left(\frac{\nu + \frac{1}{2}\sigma^2 - r}{\sigma}\right)\right].$$

- So X_N is also binomial under $\mathbb{Q}$, but with parameter $p \neq \frac{1}{2}$.

- Again using the CLT, under $\mathbb{Q}$, $(2X_N - N)/\sqrt{N}$ is approximately $N\left[-\sqrt{t}\left(\nu + \frac{1}{2}\sigma^2 - r\right)/\sigma, 1\right]$.

- So now we can write

$$S_t = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}Z_N^*\right],$$

where $Z_N^* \stackrel{\triangle}{=} Z_N + \sqrt{t}\left(\nu + \frac{1}{2}\sigma^2 - r\right)/\sigma$ is, now under $\mathbb{Q}$, approximately $N(0, 1)$.

- Note that $Z_N^* \neq Z_N$:
  - Z_N is approximately $N(0, 1)$ under $\mathbb{P}$;
  - Z_N^* is approximately $N(0, 1)$ under $\mathbb{Q}$.

- Option pricing: if a European option with maturity T has payoff $C(S_T)$, then its arbitrage-free price is $\mathbb{E}^{\mathbb{Q}}\Big[e^{-rT}C(S_T)\Big]$.

- We would expect this to be approximately
$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}C\left(S_0 \exp\left[\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z^*\right]\right)\right]$$
(although this follows from the convergence argument only for bounded, continuous $C(\cdot)$).

- Here, under $\mathbb{Q}$, Z^* is approximately $N(0,1)$.

- For the special case of a European call with strike K and $C(S) = (S - K)_+$, we find the classic *Black-Scholes* price

$$S_0 \Phi \left[\frac{\log \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2 \right) T}{\sigma\sqrt{T}} \right] - K e^{-rT} \Phi \left[\frac{\log \frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2 \right) T}{\sigma\sqrt{T}} \right]$$

- Here $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$