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Problem set 1.

1. ~~Calculate~~ ~ .

we have

T	1	2	3	4	5
$F(0, T-1, T)$	0.042	0.05	0.055	0.056	0.053

use. $P(0,T) = \exp\left[-\sum_{t=1}^{T} F(0,t-1,t)\right]$

and $R(0,T) = -\frac{\log P(0,T)}{T}$.

We have.

T	1	2	3	4	5
$P(0,T)$	0.95887	0.91211	0.86329	0.81628	0.77444.
$R(0,T)$	0.0420	0.0460	0.0490	0.05075	0.0512.

~ interest rate

2. consider the case $D(t+1) = D(t)+1$.

$$F(t+1, T-1, D(t+1))$$

$$= \log \frac{d(T-t-1)P(t, T-1, D(t))/P(t, t+1, D(t))}{d(T-t)P(t, T, D(t))/P(t, t+1, D(t))}$$

$$= \log \frac{d(T-t-1)}{d(T-t)} + F(t, T-1, T, D(t))$$

$$= \log \frac{u(T-t-1)\cdot k^{T-t-2}}{u(T-t)k^{T-t-1}} + F(0, T-1, T) + \log \frac{u(T-t)}{u(T)} - D(t)\log k.$$

$$= \log \frac{u(T-t-1)}{u(T-t)} - \log k + F(0, T-1, T) + \log \frac{u(T-t)}{u(T)} - D(t)\log k.$$

$$= F(0, T-1, T) + \log \frac{u(T-t-1)}{u(T-t)} - D(t+1)\log k.$$

3. According to the $u(T)$ and $d(T)$ we have already computed.

apply

$$p(t, T, x) = \begin{cases} u(T-t+1)\dfrac{P(t-1, T, x)}{P(t-1, t, x)} \\ d(T-t+1)\dfrac{P(t-1, T, x)}{P(t-1, t, x)}. \end{cases}$$

we can have. $P(t, 4, x)$ for $t = 2, 3$. $x = 0, 1, 2, 3$.

$x \backslash t$	0	1	2	3.
0	0.8400	0.91454	0.96258	0.98812
1		0.84529	0.91336	0.96253
2			0.86664	0.93759
3				0.91330

4.

Because $D(t) = E_Q[\frac{f(P(T,S))}{B(T)} | \mathcal{F}_t]$ is a martingale under Q.

By Martingale-Representation Theorem

$$D(t) = D(0) + \sum_{m=1}^{t} \phi(m) \Delta Z(m,S).$$

Define $\psi(t) = D(t-1) - \phi(t) Z(t-1,S)$, consider the portfolio holds $\phi(t)$ units of S-bond and $\psi(t)$ units of risk-free bond from t-1 to t.

$$V(t) = \phi(t+1) P(t,S) + \psi(t+1) B(t) = B(t)[\phi(t+1) Z(t,S) + \psi(t+1)]$$

$$= B(t) D(t)$$

$$= B(t)[D(t-1,T) + \phi(t,T) \Delta Z(t,S)]$$

$$= B(t)[\phi(t) Z(t-1,S) + \psi(t) + \phi(t,T) \Delta Z(t,S)]$$

$$= B(t)[\phi(t,T) Z(t,S) + \psi(t,T)]$$

$$= \phi(t,T) P(t,S) + \psi(t,T) B(t)$$ which is the value of portfolio at t just before rebalancing.

5.

table ①

$$P(3.4.1) = e^{-r(3.1)} \times [q P(4.4.2) + (1-q) P(4.4.1)] = e^{-0.05}(\frac{1}{2} \times 100 + \frac{1}{2} \times 100) = 95.1229.$$

$$P(3.4.0) = e^{-r(3.0)} \times [q P(4.4.1) + (1-q) P(4.4.0)] = e^{-0.03}(\frac{1}{2} \times 100 + \frac{1}{2} \times 100) = 97.0446.$$

$$P(2.4.1) = e^{-r(2.1)} \times [q P(3.4.2) + (1-q) P(3.4.1)] = e^{-0.06}(\frac{1}{2} \times 93.2394 + \frac{1}{2} \times 95.1229)$$

$$= 88.0965$$

$$P(2.4.0) = e^{-0.04} \times (\frac{1}{2} \times 95.1229 + \frac{1}{2} \times 97.0446) = 92.3163.$$

$$P(1.4.1) = e^{-0.07} \times (\frac{1}{2} \times 85.2186 + \frac{1}{2} \times 88.0965) = 81.0787.$$

$$P(1.4.0) = e^{-0.05} \times (\frac{1}{2} \times 88.0965 + \frac{1}{2} \times 92.3163) = 86.0923.$$

$$P(0.4.0) = e^{-0.06} \times (\frac{1}{2} \times 81.0787 + \frac{1}{2} \times 86.0923) = 78.7197.$$

thus we can get the table in Slide 19.

table ②. For every step. consider early executed.

$$V(t,x) = \min\{100 e^{-0.055(4-t)}, e^{-r(t,x)}(q V(t+1,x+1) + (1-q) V(t+1,x))\}.$$

$$\therefore V(3,3) = \min\{100 e^{-0.055}, 91.3931\} = 91.3931$$

$$V(3,2) = \min\{100 e^{-0.055}, 93.2394\} = 93.2394$$

$$V(3,1) = \min\{100 e^{-0.055}, 95.1229\} = 94.6485$$

$$V(3,0) = \min\{100 e^{-0.055}, 97.0446\} = 94.6485.$$

thus.

$V(2.2) = \min\{100 \cdot e^{-0.055\times2}, e^{-r(2.2)}(\frac{1}{2}\times 91.2931 + \frac{1}{2}\times 93.2394)\} = 85.2186.$

$V(2.1) = \min\{100 \cdot e^{-0.055\times2}, e^{-r(2.1)}(\frac{1}{2}\times 93.2394 + \frac{1}{2}\times 94.6485)\} = 88.4731.$

$V(2.0) = \min\{100 \times e^{-0.058\times2}, e^{-r(2.0)}(\frac{1}{2}\times 94.6485 + \frac{1}{2}\times 94.6485)\} = 89.5834$

$V(1.1) = \min\{100 \times e^{-0.055\times3}, e^{-r(1.1)}(\frac{1}{2}\times 85.2186 + \frac{1}{2}\times 88.4731)\} = 80.9745$

$V(1.0) = \min\{100 \times e^{-0.055\times3}, e^{-r(1.0)}(\frac{1}{2}\times 88.4731 + 89.5834\times\frac{1}{2})\} = 84.6863$

$V(0.0) = \min\{100 \times e^{-0.055\times4}, e^{-r(0.0)}(\frac{1}{2}\times 80.9745 + \frac{1}{2}\times 84.6863)\} = 78.0067.$

we can get the table in slide 22.

6. $\frac{B(0)}{B(2)} = \exp(-r(0) - r(1))$

$P(2.3) = \exp(-r(2))$

$\because$ $r(0)$ is known and

$$r(2) = \begin{cases} r(1) + 0.01 \\ r(1) - 0.01. \end{cases}$$

$\Rightarrow$ $\frac{B(0)}{B(2)}$ and $P(2.3)$ are positively correlated.

7. show $D(t,T) = D(0,T) + \sum_{s=1}^{t} \phi(s,T)\Delta Z(s,s+1)$

equals to $\tilde{D}(t+1,T) - \hat{D}(t,T) = \phi_{t+1}(\tilde{Z}_{t+1} - \tilde{Z}_t)$

solve
$$\tilde{D}_{(t+1,T)}(u) - \hat{D}_t = \phi_{t+1}(\tilde{Z}_{t+1}(u) - \hat{Z}_t) + k_{t+1}. \Rightarrow \phi_{t+1} = \frac{\tilde{D}_{t+1}(u) - \tilde{D}_{t+1}(d)}{\tilde{Z}_{t+1}(u) - \hat{Z}_t(d)}.$$
$$\tilde{D}_{(t+1)}(d) - \hat{D}_t = \phi_{t+1}(\tilde{Z}_{t+1}(d) - \tilde{Z}_t) + k_{t+1}$$

and apply ϕ_{t+1} into the equation we can get $k_{t+1} = 0$.

because $\{\tilde{D}_{t+1}(u), \tilde{D}_{t+1}(d)\}$ and $\{\tilde{Z}_{t+1}(u), \tilde{Z}_{t+1}(d)\}$. is $\mathcal{F}_t$ measurable.

and $\tilde{D}_t$ and $\tilde{Z}_t$ are both Q-martingales (from t, to t+1.) so we can proof the first equation

$\tilde{D}(t,T)$ is martingale under Q by conditional expectation

$Z(t,t+2) = \frac{P(t,t+2)}{B(t)} = E_Q[Z(t+1,t+2)|\mathcal{F}_t]$ is a martingale under Q from t to t+1.

the Answer is no, because $Z(s,t)$ is a martingale from t to t+1, ~~which means it~~ if we switch D and
Z in the formula, we should have ~~the~~ requirement $Z(t,T)$ is a martingale under Q, which is what we want to proof