Lecture notes on Stochastic Calculus and Quantitative Methods

Instructor: Jie LIU[1]

The lecture notes are based on chapters from the following books

- G. H. Choe, "Stochastic Analysis for Finance with Simulations", Springer, 2016. (electronic book available online from NUS library).
- Lishang Jiang, "Mathematical Modeling and Methods of Option Pricing", Translated by Canguo Li, World Scientific, 2005.
- B. Oksendal, "Stochastic Differential Equations: an Introduction with Applications", 6th edition, Springer, 2005.
- S. E. Shreve, "Stochastic Calculus for Finance I: The Binomial Asset Pricing Model", Springer, 2004.
- S. E. Shreve, "Stochastic Calculus for Finance II: Continuous-Time Models", Springer, 2004.

Approximate Schedule

	Date	Topics
1.	Aug 14	Option pricing and arbitrage-free principle
2.	Aug 19	Binomial tree method and Black-Scholes-Merton formula
3.	Aug 26	Finite probability space
4.	Sep 2	Brownian motion
5.	Sep 9	The Itô integral
6.	Sep 16	Midterm
7.	Sep 30	The Itô formula
8.	Oct 7	Stochastic differential equations
9.	Oct 14	Black-Scholes-Merton equation
10.	Oct 21	Risk neutral pricing
11.	Nov 4	Monte Carlo method
12.	Nov 11	Importance sampling
13.	Nov 30	Final

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Grading policy:

- In-class quizzes 13% (The lowest grade will be dropped.)
- Homework 12% (4 homework assignments. No late homework will be accepted.)
- Midterm 25%, Final 50%. (Both are in-class, closed-book exams. Some of the exam problems may be taken from the homework or quizzes. The exams cannot be taken at any time other than the regularly scheduled time. Only scientific calculator is allowed in the exams.)

Guildline for the letter grades:

z-score $(= \frac{x-\mu}{\sigma})$	grade
≥ 0.90	A,A+
$[0.40, 0.90)$	A-
$[-1.0, 0.40)$	B+
$[-1.5, -1.0)$	B
$[-2.0, -1.5)$	B-
$[-2.5, -2.0)$	C+
$[-3.0, -2.5)$	C
$[-3.5, -3.0)$	D+
$[-4.0, -3.5)$	D
< -4.0	F

A tentative schedule for the 4 homework assignments are:

1) Assignment 1: Due Aug 31 (Saturday), 11:55 pm. Homework I and Homework II (cover Chapters 1 and 2).

2) Assignment 2: Due Sep 14 (Saturday), 11:55 pm. Homework III and Homework IV (cover Chapters 3 and 4). Since I will upload the solution in the early morning of Sep 15 so that one can use it to prepare for one's midterm on Sep 16, no late homework is accepted.

3) Assignment 3: Due Oct 21 (Monday), 11:55 pm. Homework V, VI, VII (cover Chapters 5, 6, 7).

4) Assignment 4: Due Nov 16 (Saturday), 11:55 pm. Homework VIII, IX (cover Chapters 8 and 9).

Many homework problems already contain solutions. Those are for your own practice and you do not need to submit them.

Please solve homework problems that originally do not contain solutions. And put your solutions (photos, scanned pages, or other types of electronic files) into a **single** pdf file and name it as StudentID_Yourname_AssignmentNumber.pdf, before you submit it through LumiNUS.

Contents

1 Option pricing and arbitrage-free principle (1 lecture)

Chapter 1 of the lecture notes mainily follows Chapter 2 of "Mathematical Modeling and Methods of Option Pricing" by Lishang Jiang. Some table and examples are taken from "Options, Futures, and Other Derivatives" (10th edition) by John Hull and "Principles of Corporate Finance" (12th edition) by Brealey, Myers, and Allen.

1.1 Options and option pricing

An option is a contact that the holder can buy from, or sell to, the seller of the option a certain amount of underlying asset at a specified price (strike price) at any time on or before a given date (expiration date).

A call option gives the holder the right to buy the underlying asset. A put option gives the holder the right to sell.

The action to perform the buying or selling of the underlying asset according to the option contract is called exercise. European options can be exercised only at the expiration date. American options can be exercised on or prior to the expiration date. As a matter of fact, most equity options are American style while many index options are European style.

Let K be the strike price and T be the expiration date. Then an option's payoff (value) $\textcircled{p}_T$ at expiration date is

$$\textcircled{p}_T = (S_T - K)^+, \qquad \text{(call option)} \tag{1.1}$$

$$\textcircled{p}_T = (K - S_T)^+, \qquad \text{(put option)} \tag{1.2}$$

where S_T denotes the price of the underlying asset at maturity time $t = T$ and $a^+ = \max(a, 0)$.

Example 1.1 *Suppose the price of certain stock is 66.6 USD on April 30, and the stock may go up in August. The investor pays 39,000 USD to purchase a call option to buy 10,000 shares at the strike price 68.0 USD per share on August 22.*

We consider two scenarios

A) *The stock goes up to 73.0 USD on August 22. The investor exercises the option to receive a payoff* $= (73 - 68) \times 10,000 = 50,000$ *USD. The return is*

$$\textit{return} = \frac{50,000 - 39,000}{39,000} = 28.2\%.$$

B) *The stock goes down to 66.0 USD on August 22. The investor lost the entire invested 39,000 USD.*

As a derived security, the price of an option, denoted by $\textcircled{p}_t$, varies with the price of its underlying asset (S_t). We want to find the function $V(t, S)$ of two variables such that

$$\textcircled{p}_t = V(t, S_t) \qquad \text{for } t \in [0, T].$$

We know the value of $\textcircled{p}_t$ when $t = T$:

$$\textcircled{p}_T = V(T, S_T) = \begin{cases} (S_T - K)^+, & \text{(call option)} \\ (K - S_T)^+, & \text{(put option)} \end{cases} \tag{1.3}$$

The problem of option pricing is hence a backward problem.

Example 1.2 *A bull call spread can be made by buying a call option with a certain exercise price and selling a call option on the same stock with a higher exercise price. Both call options have the same expiration date. Consider a European call with an exercise price of K_1 and a second European call with an exercise price of $K_2 > K_1$. The following is the payoff table for this strategy if we ignore the premium (=option price) paid.*

strategy	$S_T \leq K_1$	$K_1 < S_T \leq K_2$	$K_2 < S_T$
A long call at K_1	*0*	$S_T - K_1$	$S_T - K_1$
A short call at K_2	*0*	0	$K_2 - S_T$
Total	*0*	$S_T - K_1$	$K_2 - K_1$

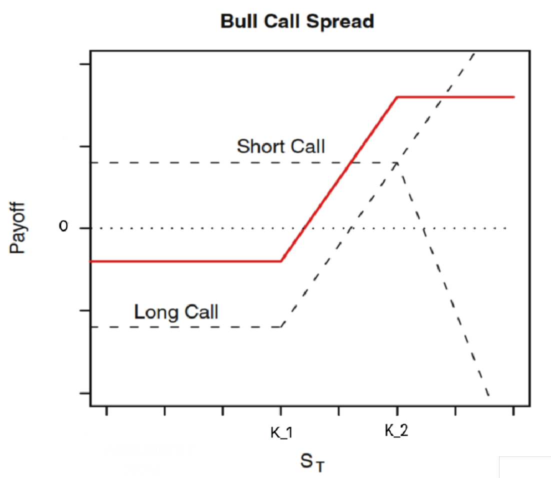

Figure 1.1: Payoff v.s. S_T. Note that the payoff is lowered by the option price paid. We will study option pricing later.

Please read "Options Markets" by John Cox and Mark Rubinstein if you want to learn more about how to use options as building blocks to create interesting payoff structures.

In the table in Example 1.2, longing an option (or stock) simply means that you have purchased the option (or stock). You can also short a stock by borrowing shares of stock (in most cases from a brokerage house), and sells these shares. Later, you will cover your short position by purchasing the same number of shares you have shorted and deliver these shares to your brokerage house to replace the shares you borrowed. If you establish a short option position you may be required to margin this position. This means that you must provide funds or securities as a guarantee since a short option position may entail an obligation and you must show that you are financially able to do so.

Options are traded both on exchanges and in the over-the-counter market. The following table gives the bid and ask quotes for some of the call options trading on Google (ticker symbol: GOOG) from Chicago Board Options Exchange website `http://www.cboe.com/delayedquote/quote-table`

GOOG (GOOGLE INC.)

Aug 08, 2019 @ 13:40 ET (Delayed) BID 1196.46008 ASK 1197.24011 VOL 726,635 LAST 1196.85669 CHANGE +22.8667 (+1.95%)

Filters by:

Volume: All Expiration Type: All Options Range: Near the Money Expiration: 2019 September View Chain

Options Chain

Total Records: 16

Calls 09/06/2019 Puts

Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int	Strike	Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int
0	0	31.8	36.5	0	0.2322	0.5418	0.0051	0	GOOG 1192.500	0	0	25.3	28.5	0	0.2207	-0.4564	0.0053	0
25.6	0	31.7	33	0	0.2279	0.5297	0.0052	5	GOOG 1195.000	19.1	0	25.6	30.4	0	0.2201	-0.4693	0.0053	3
32	+7.3	30.4	31.5	1	0.2267	0.5167	0.0052	1	GOOG 1197.500	31.1	0	26.7	32.4	0	0.2228	-0.4819	0.0053	5
29.42	+5.82	28.7	30.2	9	0.2248	0.5032	0.0052	14	GOOG 1200.000	31.4	-14.7	29.9	32.4	2	0.2245	-0.4958	0.0053	19

Calls 09/13/2019 Puts

Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int	Strike	Last	Net	Bid	Ask	Vol	IV	Delta	Gamma	Int
0	0	35	40.5	0	0.2296	0.5446	0.0046	0	GOOG 1192.500	0	0	28	31.9	0	0.2213	-0.454	0.0048	0
0	0	33	38	0	0.2226	0.5338	0.0047	0	GOOG 1195.000	34.2	0	28.8	32.4	0	0.2175	-0.4659	0.0049	4
27.6	0	32.1	37.5	0	0.2261	0.5222	0.0047	4	GOOG 1197.500	0	0	30.2	35.9	0	0.226	-0.477	0.0047	0
32.15	+5.1	32.8	34	3	0.2249	0.5104	0.0047	23	GOOG 1200.000	34.58	-13.77	33.7	37.2	1	0.2329	-0.4891	0.0045	0

The column Net reports the difference from the current business day's last reported sales price, and a previous business day's last reported sales price. Since the Google's stock price $\uparrow$, the call price $\uparrow$, while the put price $\downarrow$. Since the options are American style which allows early exercise, it is obvious that as $T \uparrow$, both prices should $\uparrow$ which is also confirmed from the above table.

The Bid and Ask prices are the prices for you to sell or buy an option respectively. The trading Volume (Vol) is the number of contacts (usually 100 shares per contract) traded in that day. Open Interest (Int) is the number of contact outstanding, that is, the number of long positions or, equivalently, the number of short positions. Open interest decreases as open trades are closed. The higher the volume and the open interest, the more liquid the option is thought to be. We will learn the meanings of Implied Volatility (IV), Delta, and Gamma later.

1.2 Financial market and arbitrage-free principle

There are two basic laws of economics: (1) there can be no reward without risk, and (2) gaining an advantage over skilled and knowledgeable competitors in a free market is extraordinarily difficult.

Consider a financial **market** consisted of a risk-free asset B (e.g. bond or money market account) and n risky assets (stocks, options, ...) S_i, $i = 1, ..., n$. Let

$$V_t(B) = B_t, \qquad V_t(S_i) = S_{it}$$

be the values of the bound and risky asset i at time t $(i = 1, ..., n)$. In time period $[t_0, t_1]$. The corresponding payoffs are $B_{t_1} - B_{t_0}$ and $S_{it_1} - S_{it_0}$, and returns are $\frac{B_{t_1} - B_{t_0}}{B_{t_0}}$ and $\frac{S_{it_1} - S_{it_0}}{S_{it_0}}$, $(i = 1, ..., n)$.

A portfolio Φ is

$$\Phi = \alpha B + \sum_{i=1}^{n} \phi_i S_i, \tag{1.4}$$

where $(\alpha, \phi_1, ..., \phi_n) \in \mathbb{R}^{n+1}$ and is called the investment strategy. In general, $\alpha, \phi_1, ..., \phi_n$ are functions of time t, as the investor may adjust the investment strategy over the time. However, $\alpha, \phi_1, ..., \phi_n$ are not to be changed between two adjacent transaction times $t = t_m$, $t = t_{m+1}$.

The value of the portfolio Φ at time t, which is denoted by $V_t(\Phi)$ or Φ_t, is

$$\Phi_t = V_t(\Phi) = \alpha_t B_t + \sum_{i=1}^{n} \phi_{it} S_{it}. \tag{1.5}$$

$V_{t_{m+1}}(\Phi) - V_{t_m}(\Phi)$ is the portfolio Φ's profit over time period $[t_m, t_{m+1}]$ under the investment strategy $(\alpha_{t_m}, \phi_{1t_m}, ..., \phi_{nt_m})$:

$$V_{t_{m+1}}(\Phi) - V_{t_m}(\Phi) = \alpha_{t_m}[B_{t_{m+1}} - B_{t_m}] + \sum_{i=1}^{n} \phi_{it_m}[S_{it_{m+1}} - S_{it_m}]. \tag{1.6}$$

If during the entire transaction period $[0, T]$, the investor does not add or withdraw fund, then the entire transaction process is said to be self-financing. It means the current portfolio value is precisely the initial investment plus the any trading gains.

Definition 1.1 *A self-financing investment strategy Φ is said to have arbitrage opportunity in $[0, T]$, if there exists $T^* \in [0, T)$, such that*

$$V_{T^*}(\Phi) = 0$$

and

$$V_T(\Phi) \geq 0,$$

and

$$\mathbb{P}(V_T(\Phi) > 0) > 0,$$

where $\mathbb{P}(A)$ denotes the probability of event A.

Definition 1.2 *If there exists no arbitrage opportunity for any self-financing investment strategy Φ in $[0, T]$, then the market is said to be arbitrage-free in the time period $[0, T]$.*

Theorem 1.1 *If the market is arbitrage-free in a time period $[0, T]$, and Φ_1 and Φ_2 are portfolios satisfying*

$$V_T(\Phi_1) \geq V_T(\Phi_2), \tag{1.7}$$

and

$$\mathbb{P}(V_T(\Phi_1) > V_T(\Phi_2)) > 0, \tag{1.8}$$

then

$$V_t(\Phi_1) > V_t(\Phi_2) \qquad \text{for any } t \in [0, T). \tag{1.9}$$

Remark: In particular, if $V_T(\Phi_1) > V_T(\Phi_2)$, then $V_t(\Phi_1) > V_t(\Phi_2)$ for any $t \in [0, T]$.
Proof: We prove by contradiction. If (1.9) is false, then there exists a $t^* \in [0, T)$ such that

$$V_{t^*}(\Phi_1) \leq V_{t^*}(\Phi_2).$$

Now, we define

$$E = V_{t^*}(\Phi_2) - V_{t^*}(\Phi_1) \geq 0 \tag{1.10}$$

and construct a portfolio Φ_c at $t = t^*$

$$\Phi_c = \Phi_1 - \Phi_2 + B$$

where B is the risk-free asset (bond) of the market that satisfies $B_{t^*} = E$. We now claim that Φ_c has arbitrage opportunity in $[t^*, T]$. This leads to contradiction since the market is assumed to be arbitrage-free. Hence our initial assumption is not correct and we proves (1.9) by contradiction.

We are left to prove the claim that Φ_c has arbitrage opportunity. Note that

$V_{t^*}(\Phi_c) = V_{t^*}(\Phi_1) - V_{t^*}(\Phi_2) + B_{t^*} \overset{(1.10) \text{ and } B_{t^*}=E}{=} 0$

$V_T(\Phi_c) = V_T(\Phi_1) - V_T(\Phi_2) + B_T \overset{(1.7)}{\geq} 0$

"if event $V_T(\Phi_1) > V_T(\Phi_2)$ happens, then event $V_T(\Phi_c) > 0$ happens" $\overset{(1.8)}{\Rightarrow} \mathbb{P}(V_T(\Phi_c) > 0) > 0.$ □

By a similar argument, we can prove (Question 3 of Homework I) the following theorem

Theorem 1.2 *If the market is arbitrage-free in time period* $[0, T]$*, and* Φ_1 *and* Φ_2 *are portfolios satisfying*

$$V_T(\Phi_1) \geq V_T(\Phi_2), \tag{1.11}$$

then

$$V_t(\Phi_1) \geq V_t(\Phi_2) \qquad \textit{for any } t \in [0, T). \tag{1.12}$$

Corollary 1.1 *If the market is arbitrage-free in time period* $[0, T]$*, and at time* T

$$V_T(\Phi_1) = V_T(\Phi_2),$$

then

$$V_t(\Phi_1) = V_t(\Phi_2) \qquad \textit{for any } t \in [0, T). \tag{1.13}$$

Proof: Apply Theorem 1.2 to $V_T(\Phi_1) \geq V_T(\Phi_2)$ to get $V_t(\Phi_1) \geq V_t(\Phi_2)$. Apply Theorem 1.2 to $V_T(\Phi_2) \geq V_T(\Phi_1)$ to get $V_t(\Phi_2) \geq V_t(\Phi_1)$. Hence we have (1.13). □

1.3 European option pricing and put-call parity

We now assume

1. The market is arbitrage-free.
2. All transactions are free of charge.
3. The risk-free interest rate r is a constant.
4. The underlying asset pays no dividends.

Consider the following notation

- S_t: the risky asset price.
- c_t: European call option price [2] .
- p_t: European put option price.
- C_t: American call option price.
- P_t: American put option price.
- K: the option's strike price.
- T: the option's expiration date.

[2] Here, the option price is the price to buy or sell 1 share of stock.

- r: the risk-free interest rate [3].

For a zero-coupon bond with face value $K_0 = Ke^{-rT}$ and compound interest rate r, its value, denoted by B_t, satisfies $\frac{dB_t}{dt} = rB_t$ [4], or

$$B_t = K_0 e^{rt} = Ke^{-r(T-t)}. \tag{1.14}$$

Theorem 1.3 *For European option pricing, the following valuations are true when $t \in [0, T)$:*

$$(S_t - Ke^{-r(T-t)})^+ < c_t < S_t, \tag{1.15}$$

$$(Ke^{-r(T-t)} - S_t)^+ < p_t < Ke^{-r(T-t)}. \tag{1.16}$$

Example 1.3 *($c_0 > (S_0 - Ke^{-rT})^+$). Suppose that $S_0 = \$20$, $K = \$18$, $r = 10\%$ per annum, and $T = 1$ year. If the European call price is \$3, an arbitrageur can short the stock and long the call. His cash inflow is $\$20 - \$3 = \$17$. If invested for 1 year at 10% per annum, the \$17 grows to $17e^{0.1} = \$18.79$. At the end of the year, the option expires. If the stock price is greater than \$18, the arbitrageur exercises the option for \$18, closes the short position in the stock, and makes a profit of*

$$\$18.79 - \$18 = \$0.79.$$

If the stock price is less than \$18, say, \$17, a stock is bought in the market and the short position is closed. The arbitrageur's profit is

$$\$18.79 - \$17 = \$1.79.$$

One can use a portfolio to describe the above process: the arbitrageur build a portfolio $\Phi = c - S + B$ with $B_0 = 17$ which ensures $\Phi_0 = 0$. Then one can show that $\Phi_T > 0$ no matter $S_T \geq 18$ or $S_T < 18$.

Proof of Theorem 1.3: Let us first prove $\underbrace{S_t}_{\Phi_2} < \underbrace{c_t + Ke^{-r(T-t)}}_{\Phi_1}$. We consider two portfolios at $t = 0$:

$$\Phi_1 = c + B, \qquad \Phi_2 = S$$

[3] Instead of the interest rates implies by Treasury bills, traditionally derivatives dealers use LIBOR (London Interbank Offered Rate) rates as risk-free interest rate. But LIBOR rates are not totally risk-free. Following the credit crisis that started in 2007, many dealers switched to using overnight indexed swap (OIS) rates as risk-free rates, at least for collateralized transactions. An OIS allows overnight borrowing or lending between financial institutions for a period to be swapped for borrowing or lending at a fixed rate for the period. The fixed rate in an OIS is referred to as the OIS rate. See John Hull "Options, Futures and Other Derivatives", 10th edition, §4.2, 4.3 for more details.

[4] which can be written as $\frac{dB_t}{B_t} = rdt$, saying that the return $\frac{B_{t+\Delta t} - B_t}{B_t}$ is proportional to Δt with a constant coefficient r.

where the value of the bond B at time t is $B_t = Ke^{-r(T-t)}$. Then

$$V_T(\Phi_1) = V_T(c) + V_T(B) = (S_T - K)^+ + K = \left\{ \begin{array}{ll} S_T, & \text{if } S_T \geq K \\ K, & \text{if } S_T < K \end{array} \right\} \geq S_T = V_T(\Phi_2).$$

Moreover,

$$\mathbb{P}(V_T(\Phi_1) > V_T(\Phi_2)) = \mathbb{P}(S_T < K) > 0.$$

Hence by Theorem 1.1, for all $t \in [0, T)$,

$$V_t(\Phi_1) > V_t(\Phi_2),$$

i.e., $c_t + Ke^{-r(T-t)} > S_t$ or $c_t > S_t - Ke^{-r(T-t)}$. But it is obvious that $c_t > 0$, hence

$$c_t > \max(S_t - Ke^{-r(T-t)}, 0) = (S_t - Ke^{-r(T-t)})^+.$$

This proves the first half of (1.15). The rest can be proved similarly and is skipped. □

Theorem 1.4 *(put-call parity)*

$$c_t + Ke^{-r(T-t)} = p_t + S_t. \tag{1.17}$$

Proof: Construct two portfolios at $t = 0$,

$$\Phi_1 = c + Ke^{-rT}, \qquad \Phi_2 = p + S,$$

and then consider their values at $t = T$:

$$\begin{aligned} V_T(\Phi_1) &= V_T(c) + V_T(Ke^{-rT}) = (S_T - K)^+ + K = \max(K, S_T), \\ V_T(\Phi_2) &= V_T(p) + V_T(S) = (K - S_T)^+ + S_T = \max(K, S_T). \end{aligned}$$

By Corollary 1.30, we have

$$V_t(\Phi_1) = V_t(\Phi_2) \qquad \forall t \leq T.$$

This proves (1.17). □

Example 1.4 *(put-call parity and capital structure) (This part is taken from §11.5 of John Hull's "Options, Futures and Other Derivatives", 10th edition.) Fischer Black, Myron Scholes* [5]*, and Robert Merton were the pioneers of option pricing. In the early 1970s, they also showed that options can be used to characterize the capital structure of a company. Today this analysis is widely used by financial institutions to assess a company's credit risk.*

To illustrate the analysis, consider a simple situation where a company has assets that are financed with zero-coupon bonds and equity. The bonds mature in five years at which time a principal payment of K is required. The company pays no dividends. If the assets are

[5] The Pricing of Options and Corporate Liabilities, The Journal of Political Economy, Vol. 81, 1973, 637–654.

worth more than K in five years, the equity holders choose to repay the bond holders. If the assets are worth less than K, the equity holders choose to declare bankruptcy and the bond holders end up owning the company.

$$\text{\textit{The value of the equity in five years is therefore }} \max(A_T - K, 0),$$

where A_T is the value of the company's assets at that time. This shows that the equity holders have a five-year European call option on the assets of the company with a strike price of K. What about the bond holders? They get $\min(A_T, K)$ in five years. Simple mathematics shows that $\min(A_T, K) = K - \max(K - A_T, 0)$. Hence we can consider the value of the bonds in five years as $V_T(\Phi)$ with $\Phi =$ a risk-free asset$-$ a European put option p. This shows that ***today*** *the bonds are worth $V_0(\Phi)$ which is the present value of K minus the value of a five-year European put option on the assets with a strike price of K.*

To summarize, if c and p are the values, respectively, of five-year call and put options on the company's assets with strike price K, then

$$\begin{aligned}
&\textit{Value of company's equity} = c \\
&\textit{Value of company's debt} = \textit{PresentValue}(K) - p = Ke^{-rT} - p.
\end{aligned}$$

Denote the value of the assets of the company today by A_0. The value of the assets must equal the total value of the instruments used to finance the assets. This means that it must equal the sum of the value of the equity and the value of the debt, so that

$$A_0 = c + [Ke^{-rT} - p].$$

Rearranging this equation, we have

$$c + Ke^{-rT} = p + A_0.$$

This is the put-call parity result for call and put options on the assets of the company.

1.4 American option pricing and early exercise

Theorem 1.5 *If the market is arbitrage-free, then for all $t \in [0, T]$, there must be*

$$C_t \geq (S_t - K)^+, \tag{1.18}$$
$$P_t \geq (K - S_t)^+. \tag{1.19}$$

Proof: Because the proofs are similar, we only prove (1.18). Suppose (1.18) is not true. Then, there is a $t \in [0, T]$ such that

$$\underbrace{0 < C_t}_{\text{easy to prove}} < \max(S_t - K, 0).$$

From $\max(S_t - K, 0) > 0$, we conclude $S_t - K > 0$. Hence

$$C_t < S_t - K.$$

Then a trader can borrow money C_t from the bank to buy the American call option at time t, and then immediately exercise the option. The cash flow at time t is $S_t - K - C_t > 0$. Thus the trader gains a riskless profit immediately. This contradict with the assumption that the market is arbitrage-free. Hence (1.18) is true. $\square$

Theorem 1.6 *If a stock S does not pay dividend, then*

$$C_t = c_t \qquad \forall t \in [0, T], \tag{1.20}$$

i.e., the "early exercise" term is of no use for American call option on a non-dividend-paying stock.

Proof: Since American option can be early exercised, its gaining opportunity must be no less than that of the European option. Therefore, $C_t \geq c_t$. [6] By the first half of (1.15), if $t \in [0, T)$,

$$C_t \geq c_t > (S_t - Ke^{-r(T-t)})^+ \geq \underbrace{(S_t - K)^+}_{\text{profit if exercise } C \text{ at } t}$$

(recall $-Ke^{-r(T-t)} \geq -K$). This indicates that it is unwise to **early** exercise the American call option C. $\square$

Remark: For the put option, we do not have $P_t = p_t$. For example, if at time t, $S_t < K(1 - e^{-r(T-t)})$, then the holder should exercise the put option immediately. The immediate gain is

$$K - S_t > K - K(1 - e^{-r(T-t)}) = Ke^{-r(T-t)},$$

and by depositing the gain in a saving's account, the total payoff will exceed K at $t = T$. This should be compared with the payoff at the option's expiration date T, which will never exceed K in any case.

Recall the put-call parity (1.17) which says that $c_t - p_t = S_t - Ke^{-r(T-t)}$. We have the following relation for C_t and P_t (Theorem 2.6 of Jiang, and (11.7) of John Hull, 10th editon):

Theorem 1.7 *If C, P are non-dividend-paying American call option and put option respectively, then*

$$S_t - K \leq C_t - P_t \leq S_t - Ke^{-r(T-t)} \tag{1.21}$$

Proof: The right hand side of (1.21) is easy to prove:

$$\underbrace{P_t \geq p_t}_{\text{obvious}} = c_t + Ke^{-r(T-t)} - S_t \overset{(1.20)}{=} C_t + Ke^{-r(T-t)} - S_t. \tag{1.22}$$

To prove the left hand side of (1.21), we construct two portfolios at time t

$$\Phi_1 = C + K, \qquad \Phi_2 = P + S.$$

[6] For the same reason, we have $P_t \geq p_t$.

If during the period $[t, T)$, the American put option P is not early exercised, then

$$\begin{aligned} V_T(\Phi_1) &= (S_T - K)^+ + Ke^{r(T-t)}, \\ V_T(\Phi_2) &= (K - S_T)^+ + S_T. \end{aligned}$$

Therefore,

$$\begin{cases} \text{when } K \geq S_T, & V_T(\Phi_1) = Ke^{r(T-t)} > K = V_T(\Phi_2) \\ \text{when } K < S_T, & V_T(\Phi_1) = S_T + K(e^{r(T-t)} - 1) > S_T = V_T(\Phi_2) \end{cases}$$

If the American put option P is early exercised at some time $\tau \in [t, T)$, then

$$V_\tau(\Phi_1) = C_\tau + Ke^{r(\tau - t)} \overset{(1.18)}{\geq} (S_\tau - K)^+ + Ke^{r(\tau-t)} = \begin{cases} S_\tau + K(e^{r(\tau-t)} - 1) & \text{if } S_\tau > K \\ Ke^{r(\tau-t)} & \text{if } S_\tau < K \end{cases},$$

$$V_\tau(\Phi_2) = \underbrace{(K - S_\tau)^+}_{\text{by early exercise assumption}} + S_\tau = \begin{cases} S_\tau & \text{if } S_\tau > K \\ K & \text{if } S_\tau < K \end{cases}.$$

Hence $V_\tau(\Phi_1) \geq V_\tau(\Phi_2)$.

In any case, by Theorem 1.1,

$$V_t(\Phi_1) \geq V_t(\Phi_2).$$

This proves the left hand side of (1.21). □

Example 1.5 *An American call option on a non-dividend-paying stock with strike price \$20.00 and maturity in 5 months is worth \$1.50. Suppose that the current stock price is \$19.00 and the risk-free interest rate is 10% per annum. From (1.21) with $t = 0$, we have*

$$19 - 20 \leq C_0 - P_0 \leq 19 - 20e^{-0.1 \times 5/12}$$

or $1 \geq P_0 - C_0 \geq 0.18$. With C_0 at \$1.50, P_0 must lie between \$1.68 and \$2.50.

Now we mention, without proving, two theorems that show the dependence of option pricing on the strike price. They are stated for your information only and will not be tested.

Theorem 1.8 *If $K_1 > K_2$, $\theta \in [0, 1]$ and $\alpha > 0$, then*

$$0 \leq c_t(K_2) - c_t(K_1) \leq K_1 - K_2, \tag{1.23}$$
$$0 \leq p_t(K_1) - p_t(K_2) \leq K_1 - K_2, \tag{1.24}$$
$$c_t(\theta K_1 + (1-\theta)K_2) \leq \theta c_t(K_1) + (1-\theta)c_t(K_2), \tag{1.25}$$
$$p_t(\theta K_1 + (1-\theta)K_2) \leq \theta p_t(K_1) + (1-\theta)p_t(K_2). \tag{1.26}$$

The same is true with c_t replaced by C_t and p_t replaced by P_t.

Theorem 1.9

$$c_t(\alpha S_t, \alpha K) = \alpha c_t(S_t, K), \tag{1.27}$$
$$p_t(\alpha S_t, \alpha K) = \alpha p_t(S_t, K). \tag{1.28}$$

The same is true with c_t replaced by C_t and p_t replaced by P_t.

Remark: A stock split occurs when the existing shares are "split" into more shares. For example, in a 3-for-1 stock split, three new shares are issued to replace each existing share. The 3-for-1 stock split should cause the stock price to go down to 1/3 of its previous value. By (1.27) and (1.28) with $\alpha = 1/3$, we should reduce the strike price to 1/3 of its previous value, and the number of shares covered by one contract should be increased to 3 times of its previous amount.

In practice, after an n-for-m stock split, the stock price as well as the strike price are reduced to m/n of their previous values ($\alpha = m/n$), and the number of shares covered by one contract is increased to n/m of its previous value.

1.5 Real options and why do we need a pricing formula

Material in this section is taken from §22-1, The Value of Follow-On Investment Opportunities, from "Principles of Corporate Finance" 12th edition by Brealey, Myers, and Allen. It won't be tested. It is presented merely to convince you why we need a pricing formula even though the price of a stock option is actually determined by the supply and demand in the market. It is 1982. You are assistant to the chief financial officer (CFO) of Blitzen Computers, an established computer manufacturer casting a profit-hungry eye on the rapidly developing personal computer market. You are helping the CFO evaluate the proposed introduction of the Blitzen Mark I Micro.

Firms can best help their shareholders by accepting all projects that are worth more than they cost. In other words, they need to seek out projects with positive net present values (NPV). To find net present value we have to calculate present value of future cash flows. Just discount future cash flows by an appropriate rate r, usually called the discount rate, hurdle rate, or opportunity cost of capital. It is really an opportunity cost of capital because it depends on the investment opportunities available to investors in financial markets. Whenever a corporation invests cash in a new project, its shareholders lose the opportunity to invest the cash on their own.

$$NPV = C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \ldots\ldots$$

Here, $(C_0, C_1, C_2, \cdots)$ is the stream of cash flows at (year 0, year 1, year 2, $\cdots$).

The discount rate r is determined by rates of return prevailing in financial markets. If the future cash flow is absolutely safe, then the discount rate is the interest rate on safe securities such as U.S. government debt. If the future cash flow is uncertain, then the expected cash flow should be discounted at the expected rate of return offered by equivalent-risk securities.

TABLE 22.1 Summary of cash flows and financial analysis of the Mark I microcomputer ($ millions).

	Year					
	1982	1983	1984	1985	1986	1987
After-tax operating cash flow (1)		+110	+159	+295	+185	0
Capital investment (2)	450	0	0	0	0	0
Increase in working capital (3)	0	50	100	100	–125	–125
Net cash flow (1) – (2) – (3)	–450	+60	+59	+195	+310	+125
NPV at 20% = –$46.45, or about –$46 million						

Cash flows are discounted for two simple reasons: because (1) a dollar today is worth more than a dollar tomorrow and (2) a safe dollar is worth more than a risky one.

For our example,

$$NPV = -450 + \frac{60}{1.2} + \frac{59}{1.2^2} + \frac{195}{1.2^3} + \frac{310}{1.2^4} + \frac{125}{1.2^5} \approx -46.45.$$

TABLE 22.2 Valuing the option to invest in the Mark II microcomputer.

Assumptions

1. The decision to invest in the Mark II must be made after three years, in 1985.
2. The Mark II investment is double the scale of the Mark I (note the expected rapid growth of the industry). Investment required is $900 million (the exercise price), which is taken as fixed.
3. Forecasted cash inflows of the Mark II are also double those of the Mark I, with present value of $807 million in 1985 and $807/(1.2)^3 = \$467$ million in 1982.
4. The future value of the Mark II cash flows is highly uncertain. This value evolves as a stock price does with a standard deviation of 35% per year. (Many high-technology stocks have standard deviations higher than 35%.)
5. The annual interest rate is 10%.

Interpretation

The opportunity to invest in the Mark II is a three-year call option on an asset worth $467 million with a $900 million exercise price.

Valuation

$$\text{PV(exercise price)} = \frac{900}{(1.1)^3} = 676$$

$$\text{Call value} = [N(d_1) \times P] - [N(d_2) \times \text{PV(EX)}]$$

$$d_1 = \log[P/\text{PV(EX)}]/\sigma\sqrt{t} + \sigma\sqrt{t}/2$$

$$= \log[.691]/.606 + .606/2 = -.3072$$

$$d_2 = d_1 - \sigma\sqrt{t} = -.3072 - .606 = -.9134$$

$$N(d_1) = .3793,\ N(d_2) = .1805$$

$$\text{Call value} = [.3793 \times 467] - [.1805 \times 676] = \$55.1 \text{ million}$$

Here the option is to buy a nontraded real asset, the Mark II. We cannot observe the Mark II's value; we have to compute it. The Mark II's forecasted cash flows are set out in Table 22.3. The project involves an initial outlay of $900 million in 1985. The cash inflows start in the following year and have a present value of $807 ($= \frac{120}{1.2} + \frac{118}{1.2^2} + \frac{390}{1.2^3} + \frac{620}{1.2^4} + \frac{250}{1.2^5}$) million in 1985, equivalent to $467 ($= \frac{807}{1.2^3}$ or equivalently $= \frac{120}{1.2^4} + \frac{118}{1.2^5} + \frac{390}{1.2^6} + \frac{620}{1.2^7} + \frac{250}{1.2^8}$)

million in 1982 as shown in Table 22.3. So the real option to invest in the Mark II amounts to a three-year call on an underlying asset worth \$467 million, with a \$900 million exercise price. We evaluate it in Table 22.2 using the Black-Scholes formula that we will learn in Chapter 2 and then in Chapter 6 again.

Table 22.2 uses a standard deviation of 35% per year. Where does that number come from? We recommend you look for comparables, that is, traded stocks with business risks similar to the investment opportunity. For the Mark II, the ideal comparables would be growth stocks in the personal computer business, or perhaps a broader sample of high-tech growth stocks. Use the average standard deviation of the comparable companies' returns as the benchmark for judging the risk of the investment opportunity.

The NPV of the Mark I project is −\$46 million, but it creates the expansion option for the Mark II. The expansion option is worth \$55 million, so

$$\text{Adjusted Present Value} = -46 + 55 = \$9 \text{ million}.$$

	Year							
	1982		1985	1986	1987	1988	1989	1990
After-tax operating cash flow				+220	+318	+590	+370	0
Increase in working capital				100	200	200	−250	−250
Net cash flow				+120	+118	+390	+620	+250
Present value at 20%	+467	←	+807					
Investment, PV at 10%	676	←	900					
	(PV in 1982)							
Forecasted NPV in 1985			−93					

TABLE 22.3 Cash flows of the Mark II microcomputer, as forecasted from 1982 (\$ millions).

1.6 Homework I

(Only submit solutions to Questions 1,4,5,9.)

1. A straddle is a portfolio with long positions in a European call and a European put with the same strike price, maturity, and underlying. The straddle is seen to benefit from a movement in either direction away from the strike price. Show that the payoff of a straddle is $|S_T - K|$ if we ignore the premium by constructing the payoff table as in Example 1.2.

2. Suppose that an amount A is invested for n years at an interest rate of R per annum. If the rate is compounded m times per annum, the terminal value of the investment is

$$A\left(1+\frac{R}{m}\right)^{mn}.$$

If the interest rate is 10% and is measured with semiannual compounding, What is the value of \$100 at end of 1 year?

Determine the limit

$$\lim_{m\to\infty} A\left(1+\frac{R}{m}\right)^{mn}.$$

Solution:

$$\$100 \times (1+0.05)^2 = \$110.25.$$

Recall $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$ and $\lim_{x\to\infty}(f(x))^k = (\lim_{x\to\infty} f(x))^k$. Hence

$$\lim_{m\to\infty} A\left(1+\frac{R}{m}\right)^{mn} = A\lim_{\frac{m}{R}\to\infty}\left(1+\frac{R}{m}\right)^{\frac{m}{R}Rn} = A\left(\lim_{\frac{m}{R}\to\infty}\left(1+\frac{R}{m}\right)^{\frac{m}{R}}\right)^{Rn} = Ae^{Rn}.$$

3. Prove Theorem 1.2. [Hint: Prove by contradiction. If the conclusion is false, one can construct a portfolio $\Phi_c = \Phi_1 - \Phi_2 + B$ which has arbitrage opportunity. You need to specify the value of B_t at some $t = t^*$.]

Proof: We prove by contradiction. If the conclusion is false, then there exists a $t^* \in [0, T)$ such that

$$V_{t^*}(\Phi_1) < V_{t^*}(\Phi_2).$$

Now, we define

$$E = V_{t^*}(\Phi_2) - V_{t^*}(\Phi_1) > 0 \tag{1.29}$$

and construct a portfolio Φ_c at $t = t^*$

$$\Phi_c = \Phi_1 - \Phi_2 + B$$

where B is the risk-free asset (bond) of the market that satisfies $B_{t^*} = E$. We now claim that Φ_c has arbitrage opportunity in $[t^*, T]$. This proves (1.12).

To prove the claim, note that

$$V_{t^*}(\Phi_c) = V_{t^*}(\Phi_1) - V_{t^*}(\Phi_2) + B_{t^*} = 0 \text{ by (1.29) and } B_{t^*} = E$$
$$V_T(\Phi_c) = V_T(\Phi_1) - V_T(\Phi_2) + B_T > 0 \text{ by (1.11).} \qquad \square$$

4. Consider a European put option on a non-dividend-paying stock when the stock price is \$38, the strike price is \$40, the time to maturity is 3 months, and the risk-free rate of interest is 10% per annum. Find a lower bound for the option price.

5. The price of a non-dividend paying stock is \$19 and the price of a three-month European call option on the stock with a strike price of \$20 is \$1. The risk-free rate is 4% per annum. What is the price of a three-month European put option with a strike price \$20?

6. We want to prove (1.25): For any $K_0, K_1 \geq 0$, if $\theta \in [0,1]$, $c_t(\theta K_1 + (1-\theta)K_0) \leq \theta c_t(K_1) + (1-\theta)c_t(K_0)$. To do that, we construct two portfolios at $t = 0$:

$$\Phi_1 = \theta c(K_1) + (1-\theta)c(K_0), \qquad \Phi_2 = c(\theta K_1 + (1-\theta)K_0).$$

Prove that $V_T(\Phi_1) \geq V_T(\Phi_2)$. Then (1.25) follows from Theorem 1.2.

Proof: On the expiration date $t = T$,

$$V_T(\Phi_1) = \theta(S_T - K_1)^+ + (1-\theta)(S_T - K_0)^+, \tag{1.30}$$
$$V_T(\Phi_2) = (S_T - \theta K_1 - (1-\theta)K_0)^+. \tag{1.31}$$

Without loss of generality, we can assume $K_1 \geq K_0$. Let $K_\theta = \theta K_1 + (1-\theta)K_0$. $K_1 \geq K_\theta \geq K_0$. There are 4 cases:

1. $S_T \geq K_1$: $V_T(\Phi_1) = S_T - K_\theta = V_2(\Phi_2)$.
2. $K_\theta \leq S_T < K_1$: $V_T(\Phi_1) = (1-\theta)(S_T - K_2)$. $V_T(\Phi_2) = (S_T - K_\theta) = \theta(S_T - K_1) + (1-\theta)(S_T - K_2) \leq V_T(\Phi_1)$.
3. $K_0 \leq S_T < K_\theta$: $V_T(\Phi_1) = (1-\theta)(S_T - K_2)$. $V_T(\Phi_2) = 0 \leq V_T(\Phi_1)$.
4. $S_T < K_0$: $V_T(\Phi_1) = 0 = V_2(\Phi_2)$.

Hence $V_T(\Phi_1) \geq V_T(\Phi_2)$. $\square$

7. A 1-month European put option on a non-dividend-paying stock is currently selling for \$2.50. The stock price is \$47, the strike price is \$50, and the risk-free interest rate is 6% per annum. What opportunities are there for an arbitrageur?

Solution: For European put option, if the market is arbitrage-free, we should have (1.16)

$$(Ke^{-rT} - S_0)^+ < p_0 < Ke^{-rT}$$

which implies

$$(50e^{-0.06\times(1/12)} - 47)^+ < p_0 < 50e^{-0.06\times(1/12)},$$

$$49.75 - 47 < p_0 < 49.75 \quad \Rightarrow \quad 2.75 < p_0 < 49.75.$$

Hence $(Ke^{-rT} - S_0)^+ < p_0$ is violated. Since the European put option is undervalued, the arbitrageur should buy the put option. Then he also need to buy the stock so that he can sell the stock to the option seller to close out. Hence an arbitrageur can build a portfolio

$$\Phi = p + S - B$$

by buying the put option, buying the stock, and borrowing B_0 dollars from the money market at $t = 0$. (This portfolio can also be seen from the condition $p_0 + S_0 - Ke^{-rT} < 0$.) The value of B_0 is determined by the requirement that

$$V_0(\Phi) = 0 = p_0 + S_0 - B_0.$$

Then

$$V_T(\Phi) = (K - S_T)^+ + S_T - B_0 e^{rT} = (K - S_T)^+ + S_T - (p_0 + S_0)e^{rT}$$

Right now, $p_0 < (Ke^{-rT} - S_0)$. $-p_0 e^{rT} > -(Ke^{-rT} - S_0)e^{rT} = -(K - S_0 e^{rT})$. Hence

$$V_T(\Phi) > (K - S_T)^+ + S_T - (K - \cancel{S_0 e^{rT}}) - \cancel{S_0 e^{rT}} \geq 0.$$

In the last step, we have used the fact that $a^+ - a = \max(a, 0) - a \geq 0$.

If one uses $p_0 = (Ke^{-rT} - S_0) - \varepsilon$ with $\varepsilon = \$0.25$ for this problem, then $V_T(\Phi) \geq \varepsilon e^{rT}$.

8. A European call option and put option on a stock both have strike price of \$20 and an expiration date in three months. Both sell for \$3. The risk-free interest rate is 10% per annum, the current stock price is \$18. Identity the arbitrage opportunity open to a trader.

Solution: By (1.17),

$$c_0 + Ke^{-rT} = p_0 + S_0. \tag{1.32}$$

$$3 + 20e^{-0.1\times(3/12)} = p_0 + 18 \quad \Rightarrow \quad p_0 = 4.51$$

The put is then undervalued relative to the call. So, the arbitrageur should buy the put and sell the call. He also need to buy a stock so that he can sell it to the the buyer of the call if the stock price is high at the end of three months. This portfolio can also be seen from the violation of (1.32):

$$p_0 + S_0 - c_0 - Ke^{-rT} < 0. \tag{1.33}$$

So, the arbitrageur should build a portfolio

$$\Phi = p + S - c - B$$

by borrowing money B from the money market to buy the put, buy the stock, and sell the call. B_0 is determined by

$$0 = V_0(\Phi) = p_0 + S_0 - c_0 - B_0.$$

Since $p_0 + S_0 - c_0 < Ke^{-rT}$ by (1.33), we able to choose an $\varepsilon > 0$ so that $p_0 + S_0 - c_0 = Ke^{-rT} - \varepsilon$. For this problem, $\varepsilon = 20e^{-0.1\times 0.25} - 18 = 1.51$.

$$\begin{aligned}
V_T(\Phi) =&(K - S_T)^+ + S_T - (S_T - K)^+ - (p_0 + S_0 - c_0)e^{rT}\\
=&(K - S_T)^+ + S_T - (S_T - K)^+ - (Ke^{-rT} - \varepsilon)e^{rT}\\
=&\left\{\begin{array}{ll} K & \text{if } S_T \le K \\ K & \text{if } S_T > K \end{array}\right\} - K + \varepsilon e^{rT}\\
=&\varepsilon e^{rT} > 0.
\end{aligned}$$

We know exactly the gain of the arbitrageur.

Remark: The above solution indeed proves why the put-call parity $p_0+S_0-c_0-Ke^{-rT} = 0$ should be valid in an arbitrage-free market.

9. (Dividend Put-Call Parity Formula) Note that by the arbitrage-free principle, it is easy to show that when a stock pays a dividend D_1 at t_1, the stock's value is immediately reduced by the amount of the dividend. In other words, $\lim_{t\uparrow t_1} S_t - D_1 = \lim_{t\downarrow t_1} S_t$.

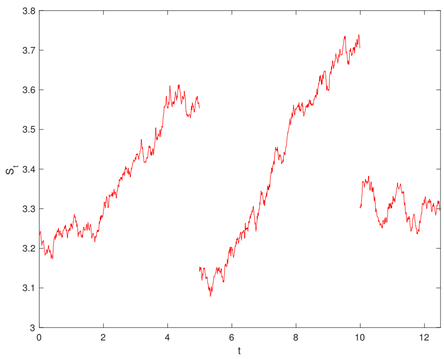

Figure 1.2: $t_1 = 5$, $t_2 = 10$, $D_1 = 0.4$, $D_2 = 0.4$.

Assume that a dividend D_j is paid at time t_j, where $0 < t_1 < t_2 < \cdots < t_n \le T$. Let D denote the present value of the dividend stream:

$$D = e^{-rt_1}D_1 + e^{-rt_2}D_2 + \cdots e^{-rt_n}D_n. \tag{1.34}$$

Consider an European call option and an European put option, each with strike price K, maturity T, and underlying one share of S, assumed to be dividend-paying. We want to prove

$$c_0 + Ke^{-rT} + D = p_0 + S_0. \tag{1.35}$$

The idea is to consider two portfolios at $t = 0$:

$$\begin{aligned}
\Phi_1 &= c + Ke^{-rT} + D,\\
\Phi_2 &= p + S.
\end{aligned}$$

It means that Φ_1 consists of a call option, a zero-coupon bond with face value (par value) K and maturity T, and n zero-coupon bonds with face values D_k and maturity times t_k, $k = 1, 2, \cdots, n$. Φ_2 consists of a put option and the stock which will pay dividend D_j at t_j.

Now, you are asked to **prove** $V_T(\Phi_1) = V_T(\Phi_2)$. Then we can conclude $V_0(\Phi_1) = V_0(\Phi_2)$ which is precisely (1.35) by Corollary 1.30.