Answers to End-of-Chapter Problems

CHAPTER 2

2-1. The information from the problem can be summarized in the following table:

	S_1	S_2
A	\$9	\$11
B	−\$5	\$5
Target	\$100	\$100

The answer is to buy 10 units of A and short 2 units of B.

	S_1	S_2
$10 \times$ A	\$90	\$110
$-2 \times$ B	\$10	−\$10
$10 \times$ A $- 2 \times$ B	\$100	\$100

2-2. Because you can borrow at the riskless rate, the Sharpe ratio of your portfolio P will be the same as for ABC stock. Using Equation 2.2 we have:

$$\lambda_P = \frac{\mu_P - r}{\sigma_P} = \frac{\mu_{\text{ABC}} - r}{\sigma_{\text{ABC}}} = \lambda_{\text{ABC}}$$

Rearranging terms,

$$\begin{aligned}\mu_P &= \lambda_{\text{ABC}}\sigma_P + r \\ &= 0.60 \cdot 10\% + 2\% \\ &= 8\%\end{aligned}$$

2-3. After you receive your initial investment of \$100, you would either borrow \$100 or, equivalently, sell \$100 of riskless bonds. You would then invest \$200 in the HSI ETF.

If the market goes up 10%, then your \$200 position in HSI will be worth \$220. The net account value, though, is \$120, \$220 in HSI less your \$100 loan. Looked at another way, this is your initial \$100 plus the \$20 profit from the levered HSI position. The portfolio is now levered only 1.83× (\$220/\$120 = 1.83). In order to bring the portfolio back to 2× leverage, you borrow an additional \$20 and buy an additional \$20 of HSI. Maintaining a constant level of leverage is a form of dynamic replication.

CHAPTER 3

3-1. Using Equation 3.5 and put-call parity, we can re-create the collar by buying a put with a strike at L, selling a put with a strike at U, and buying an amount of riskless bonds, $Ue^{-r(T-t)}$.

$$\begin{aligned}
\text{Collar} &= S + P_L(S,t) - C_U(S,t) \\
&= S + P_L(S,t) - \left[P_U(S,t) + S - Ue^{-r(T-t)}\right] \\
&= P_L(S,t) - P_U(S,t) + Ue^{-r(T-t)}
\end{aligned}$$

3-2. The payoff of this butterfly can be created by buying a call with a strike at \$10, selling two calls with a strike at \$20, and buying a call with a strike at \$30. From Equation 3.7, the intercept and the initial slope are both zero, so there is no need for riskless bonds or the underlying stock. The slopes are then +1, −1, and 0, which gives changes in the slopes of +1, −2, and +1, consistent with our answer.

3-3. To counteract the −0.40 delta of each option, you need to buy 0.40 shares. There are 100 options, so you need to buy 40 shares in total.

Equation 3.12 is true for both calls and puts. If the stock goes up 1%, then dS is \$1, and if it goes down 1%, dS is −\$1. Either way, dS^2 is 1. For each hedged option, we have:

$$\begin{aligned}
dV(S,t) &= \Theta dt + \frac{1}{2}\Gamma dS^2 \\
&= -7.3\frac{1}{365} + \frac{1}{2}0.04 \cdot 1 \\
&= -0.02 + 0.02 \\
&= 0
\end{aligned}$$

The time decay and convexity perfectly cancel over one day for a 1% move. For the entire position, the result is then $100 \times \$0 = \$0$.

The delta-hedged position would not make or lose anything if the stock moved up or down 1%.

3-4. If the stock moves up 4%, then dS is \$4 and dS^2 is 16. For each hedged put, then:

$$
\begin{aligned}
dV(S,t) &= \Theta dt + \frac{1}{2}\Gamma dS^2 \\
&= -7.3\frac{1}{365} + \frac{1}{2}0.04 \cdot 16 \\
&= -0.02 + 0.32 \\
&= 0.30
\end{aligned}
$$

For the entire position of 100 puts, the profit is then $100 \times \$0.30 = \$30$.

If the stock moved up 4%, the position would make \$30.

3-5. Your firm is short \$10,000 of GOOG, which, at \$500 per share, is 20 shares. These 20 shares are delta-hedging 100 call options, so the delta of each call must be $0.20 = 20/100$.

Using put-call parity, Equation 3.4, and assuming interest rates are zero, we have:

$$
\begin{aligned}
C(S,t) &= P(S,t) + S - Ke^{-r(T-t)} \\
&= P(S,t) + S - K
\end{aligned}
$$

We can replace each call by purchasing a put with the same strike and time to expiration, purchasing a share of stock, and selling riskless zero coupon bonds with face value equal to the strike price, \$550. To replace 100 call options, we would need to buy 100 puts, buy \$50,000 worth of GOOG stock, and sell \$55,000 of riskless bonds. The purchase of \$50,000 of GOOG stock, on top of our initial short position, will leave us with \$40,000 of GOOG stock, or 80 shares. The delta of each put must be $-0.80 = -80/100$.

If dividends are zero, put-call parity for a call and put with the same strike and expiration requires $C(S, t) = P(S, t) + S - Ke^{-r(T-t)}$ at any time. Differentiating with respect to the stock price S leads to the result $\Delta_C = \Delta_P + 1$. In our current example, $0.20 = -0.80 + 1$. This is strictly true for European puts on non-dividend-paying stocks. If dividends are nonzero, the formula needs to be amended.

3-6. From the graph, the slopes λ of the piecewise-linear payoff function are given by the table:

S	$V(S)$	λ	Change of Slope
0	20	−1.00	
10	10	0.00	1.00
20	10	1.00	1.00
30	20		
40	30		

According to Equation 3.7, we therefore need to purchase \$20 of riskless bonds (based on the intercept and a riskless rate of 0%), sell one share of the underlying stock (based on λ_0), and buy one call with a strike of 10 and one with a strike of 20 (based on the change in the slopes). Multiplying the prices by these weights gives us the final cost:

	Amount	Unit Price	Cost
Bonds	20.00	1.00	20.00
Stock	−1.00	20.00	−20.00
$C(10)$	1.00	10.09	10.09
$C(20)$	1.00	3.17	3.17
			13.26

3-7. We can similarly replicate the payoff by buying \$10 worth of riskless bonds, a put with a strike at 10, and a call with a strike at 20.

We can use put-call parity to determine the price of the put struck at 10:

$$\begin{aligned} P(S,t) &= C(S,t) - S + Ke^{-r(T-t)} \\ &= 10.09 - 20 + 10 \\ &= 0.09 \end{aligned}$$

	Amount	Unit Price	Cost
Bonds	10	1.00	10
Stock	0.00	20.00	0.00
$P(10)$	1.00	0.09	0.09
$C(20)$	1.00	3.17	3.17
			13.26

The value of this new portfolio is \$13.26, the same as before. Even though they were constructed with different instruments, both portfolios have the same payoff at expiration. By the law of one price, they should have the same value.

CHAPTER 4

4-1. We can use Equations 4.2 and 4.3 to determine the price and vega of the call option. When implied volatility is 20%, the price is ¥845.58 and the vega is ¥4,231.42. If the implied volatility increased to 21%, we would expect the price to increase by ¥42.31 = ¥4,231.42 × 1% to ¥887.89. The actual call price when implied volatility is 21% is ¥887.78.

4-2. To get the notional of the variance contract, we need to divide by twice the strike volatility:

$$\begin{aligned} N_{\text{var}} &= \frac{1}{2\sigma_K} N_{\text{vol}} \\ N_{\text{var}} &= \frac{1}{2(0.25)} 1000000 \\ N_{\text{var}} &= 2000000 \end{aligned}$$

The notional should be €2 million.

The payoff of the hedged position is:

$$\pi = N_{\text{vol}}(\sigma_R - \sigma_K) - N_{\text{var}}\left(\sigma_R^2 - \sigma_K^2\right)$$

When realized volatility is 24%, the payoff is:

$$\begin{aligned} \pi &= 1000000(0.24 - 0.25) - 2000000(0.24^2 - 0.25^2) \\ \pi &= -10000 - (-9800) = -200 \end{aligned}$$

When the realized volatility is 30%, the payoff is:

$$\begin{aligned} \pi &= 1000000(0.30 - 0.24) - 2000000(0.30^2 - 0.25^2) \\ \pi &= 50000 - (55000) = -5000 \end{aligned}$$

Your firm will lose €200 if realized volatility is 24% and €5,000 if realized volatility is 30%. The hedged position loses money if the realized volatility is higher or lower than the strike.

4-3. Because interest rates and dividends are assumed to be zero, we can use Equation 4.3, with $v = 0.15 \times 0.25^{1/2} = 0.075$. Table A4.1 shows value of κ for the options and the weighted average portfolio for a limited number of underlying prices. The chart is based on considerably more points. The weighted average series displays a very stable region around the strike price, but deteriorates quickly below 80 and above 120, where we have no option coverage.

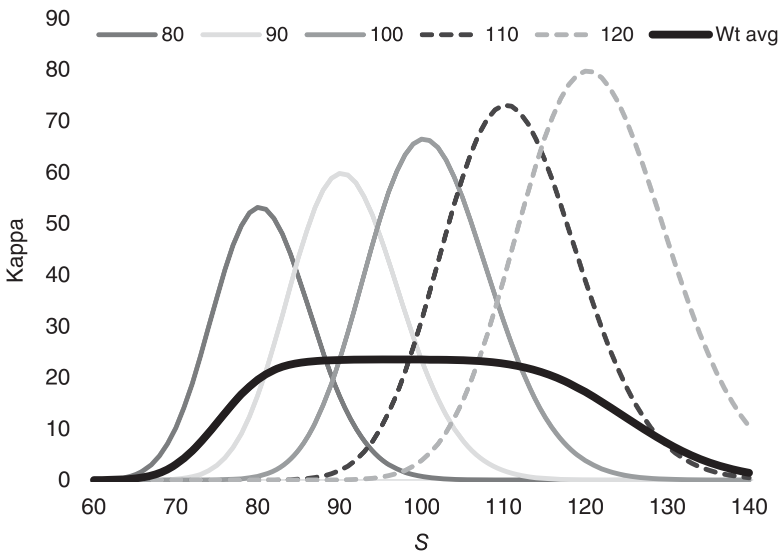


TABLE A4.1 Kappa for Five Options

		S								
K	Wt	60	70	80	90	100	110	120	130	140
80	0.29	0.03	10.19	53.15	16.43	0.71	0.01	0.00	0.00	0.00
90	0.23	0.00	0.19	16.43	59.80	23.50	1.84	0.04	0.00	0.00
100	0.19	0.00	0.00	0.71	23.50	66.44	31.08	3.79	0.17	0.00
110	0.16	0.00	0.00	0.01	1.84	31.08	73.09	38.95	6.65	0.47
120	0.13	0.00	0.00	0.00	0.04	3.79	38.95	79.73	46.96	10.42
Wt avg	**1.00**	**0.01**	**3.04**	**19.57**	**23.42**	**23.48**	**22.72**	**17.19**	**7.20**	**1.43**

4-4. We start with Equation 4.39, setting $r = 0$ and $T = 1$:

$$\pi\left(S_0, S^*, 0, T\right) = \frac{2}{T}\left[rT - (e^{rT} - 1) + e^{rT}\int_0^{S^*}\frac{1}{K^2}P(K)dK \right.$$
$$\left. + e^{rT}\int_{S^*}^{\infty}\frac{1}{K^2}C(K)dK\right]$$
$$\pi\left(S_0, S^*, 0, 1\right) = 2\left[\int_0^{S^*}\frac{1}{K^2}P(K)dK + \int_{S^*}^{\infty}\frac{1}{K^2}C(K)dK\right]$$

Setting S^* to the current stock price, \$10, we have

$$\begin{aligned}
\pi(10, 10, 0, 1) &= 2\left[\int_5^{10}\frac{1}{K^2}\left(\frac{1}{20}K^2 - 0.5K + 1.25\right)dK\right. \\
&\quad \left. + \int_{10}^{15}\frac{1}{K^2}\left(\frac{1}{20}K^2 - 1.5K + 11.25\right)dK\right] \\
&= \frac{1}{10}\left[\int_5^{10}\left(1 - 10\frac{1}{K} + 25\frac{1}{K^2}\right)dK\right. \\
&\quad \left. + \int_{10}^{15}\left(1 - 30\frac{1}{K} + 225\frac{1}{K^2}\right)dK\right] \\
&= \frac{1}{10}\left\{\left[K - 10\ln(K) - 25\frac{1}{K}\right]_5^{10}\right. \\
&\quad \left. + \left[K - 30\ln(K) - 225\frac{1}{K}\right]_{10}^{15}\right\} \\
&= \frac{1}{10}\left[\left(10 - 10\ln(10) - \frac{25}{10}\right) - \left(5 - 10\ln(5) - \frac{25}{5}\right)\right. \\
&\quad \left. + \left(15 - 30\ln(15) - \frac{225}{15}\right) - \left(10 - 30\ln(10) - \frac{225}{10}\right)\right] \\
&= \frac{1}{10}\left(20 + 20\ln(2) - 30\ln(3)\right) \\
&= 2 + 2\ln(2) - 3\ln(3) \\
&\approx 0.0905
\end{aligned}$$

The fair variance strike is approximately 0.0905, which is about $30\%^2$.

4-5. We can use Equation 4.42 to approximate the market price of variance. We begin by calculating the value, $\pi(K_i)$, of the replicating portfolio at each of the available strike prices K_i, using Equation 4.41. Next we calculate the absolute value of the slopes, λ_i, for our piecewise-linear function. We then use these slopes to calculate the weights for the options.

For example, for the first available option, with a strike at \$350, we calculate the slope of the replicating portfolio between \$300 and \$350. For a strike of \$300, we have

$$
\begin{aligned}
\pi(K) &= \frac{2}{T}\left[\left(\frac{S_T - S^*}{S_0}\right) - \ln\left(\frac{S_T}{S_0}\right)\right] \\
\pi(300) &= \frac{2}{0.5}\left[\left(\frac{300 - 500}{500}\right) - \ln\left(\frac{300}{500}\right)\right] \\
&= 4\left[-\frac{2}{5} - \ln\left(\frac{3}{5}\right)\right] \\
&= 0.433
\end{aligned}
$$

Similarly, for a strike of \$350 we have

$$\pi(350) = 0.227$$

The first slope is then

$$
\begin{aligned}
\lambda_i &= \left|\frac{\pi(K_i) - \pi(K_{i-1})}{K_i - K_{i-1}}\right| \\
\lambda_1 &= \left|\frac{0.227 - 0.443}{350 - 300}\right| = 0.004332
\end{aligned}
$$

The second slope can be found in the same way to be 0.002683. The first weight is then

$$
\begin{aligned}
w_1 &= \lambda_1 - \lambda_2 \\
&= 0.004332 - 0.002683 \\
&= 0.001650
\end{aligned}
$$

Repeating this process for each strike, and using Equation 4.42 to calculate the weights, we obtain:

K_i	$\pi(K_i)$	λ_i	w_i	C_i	P_i	$w_i \times O_i$
300	0.443					
350	0.227	0.004332	0.001650		5.81	0.0096
400	0.093	0.002683	0.001260		15.41	0.0194
450	0.021	0.001423	0.000994		32.06	0.0319
500	0.000	0.000429	0.000429		56.23	0.0241
500	0.000	0.000375	0.000375	56.23		0.0211
550	0.019	0.001039	0.000664	37.34		0.0248
600	0.071	0.001597	0.000557	24.15		0.0135
650	0.151	0.002071	0.000475	15.3		0.0073
700	0.254					
					Variance	**0.1516**
					Vol	**0.3893**

The calculated fair variance is 38.93%^{2}. The corresponding volatility is less than the true volatility, 40%. In this case, because the range of strikes used is so narrow, we have underestimated the fair variance.

4-6. If we proceed as we did in the previous question, we get the following values:

K_i	$\pi(K_i)$	λ_i	w_i	C_i	P_i	$w_i \times O_i$
200	1.265					
250	0.773	0.009851	0.003266		0.23	0.0008
300	0.443	0.006586	0.002254		1.53	0.0034
350	0.227	0.004332	0.001650		5.81	0.0096
400	0.093	0.002683	0.001260		15.41	0.0194
450	0.021	0.001423	0.000994		32.06	0.0319
500	0.000	0.000429	0.000429		56.23	0.0241
500	0.000	0.000375	0.000375	56.23		0.0211
550	0.019	0.001039	0.000664	37.34		0.0248
600	0.071	0.001597	0.000557	24.15		0.0135
650	0.151	0.002071	0.000475	15.3		0.0073
700	0.254	0.002481	0.000409	9.53		0.0039
750	0.378	0.002837	0.000356	5.85		0.0021
800	0.520					
					Variance	**0.1618**
					Vol	**0.4022**

The calculated fair variance is 40.22%^{2}. This time, the corresponding volatility is more than the true volatility, 40%. In this case, the upward bias of the piecewise-linear method, which always overestimates the payoff of the swap, dominates the downward bias caused by the limited range of options.

CHAPTER 5

5-1. According to the BSM model, the Sharpe ratios of the call option and the stock must be equal. Starting with Equation 5.10, we have:

$$\frac{(\mu_C - r)}{\sigma_C} = \frac{(\mu_S - r)}{\sigma_S}$$

Rearranging:

$$\begin{aligned}
\mu_C &= (\mu_S - r)\frac{\sigma_C}{\sigma_S} + r \\
&= (\mu_S - r)\frac{S}{C}\frac{|\Delta|\sigma_S}{\sigma_S} + r \\
&= (\mu_S - r)\frac{S}{C}|\Delta| + r \\
&= (12\% - 2\%)\frac{25000}{2500}|0.60| + 2\% \\
&= 62\%
\end{aligned}$$

The expected return of the option is 62%.

Remember, this result is valid over only a small increment of time, dt, over which we can treat Δ as a constant. Over the course of a year, as the time to expiration and the underlying price change, the Δ of the option will change. Options have built-in leverage, and their expected returns can be very high. The leverage is equivalent to $|\Delta|(S/C)$, which, in this case, gives 6× leverage. You should confirm that both the volatility and the excess return of the option are 6× that of the underlying index.

5-2. Initially, the call is worth \$5.64 (based on the implied volatility and 0.5 years to expiration) and increases to \$6.16 (based on the implied volatility, the change in the price of XYZ, and 0.496 years to expiration). The profit on the option is then \$0.52. At the same time, you short \$53.52 of XYZ (based on a delta of 0.5352, computed at the realized volatility). When XYZ increases to \$101, the hedge loses \$0.54. The net profit of the combined position is then a loss of \$0.02.

5-3. Because the riskless rate is assumed to be zero, the value of the P&L from hedging and the present value of the P&L from hedging are equal. The value is equal to the difference in the value of an option valued with the BSM formula using the realized volatility and the implied volatility, that is, $V(S, \tau, \sigma_R) - V(S, \tau, \Sigma)$. In the preceding problem we calculated that $V(S, \tau, \Sigma)$= \$5.64. Similarly, an option priced using the realized volatility 25% would be $V(S, \tau, \sigma_R)$ =\$7.04. Our final answer is the difference, \$7.04 − \$5.64 = \$1.41. (The answer is not \$1.40 because of rounding.)

5-4. The change in P&L is given by

$$\begin{aligned} d\text{P\&L} &= dV_I - \Delta_h dS - \Delta_h SDdt + [(\Delta_h S - V_h) + (V_h - V_I)]rdt \\ &= (dV_h - dV_h) + dV_I - \Delta_h dS - \Delta_h SDdt + \Delta_h Srdt - V_h rdt \\ &\quad + (V_h - V_I)rdt \\ &= dV_h - \Delta_h dS + \Delta_h (r - D)Sdt - V_h rdt + (dV_I - dV_h) \\ &\quad + (V_h - V_I)rdt \end{aligned}$$

We know from Itô's lemma that

$$(dV_h - \Delta_h dS) = \left(\Theta_h + \frac{1}{2}\Gamma_h S^2 \sigma_r^2\right)dt$$

Substituting this into our P&L equation, we obtain

$$\begin{aligned} d\text{P\&L} &= \left(\Theta_h + \frac{1}{2}\Gamma_h S^2 \sigma_r^2\right) dt + \Delta_h (r - D)Sdt - V_h rdt \\ &\quad + (dV_I - dV_h) + (V_h - V_I)rdt \\ &= \left(\Theta_h + \Delta_h (r - D)S + \frac{1}{2}\Gamma_h S^2 \sigma_r^2 - V_h r\right)dt \\ &\quad + (dV_I - dV_h) + (V_h - V_I)rdt \end{aligned}$$

Now the Black-Scholes-Merton solution with the hedged volatility satisfies

$$\Theta_h + \Delta_h (r - D)S + \frac{1}{2}\Gamma_h S^2 \sigma_h^2 - rV_h = 0$$

Therefore,

$$\begin{aligned} d\text{P\&L} &= \frac{1}{2}\Gamma_h S^2 \left(\sigma_r^2 - \sigma_h^2\right) dt + (dV_I - dV_h) + (V_h - V_I)rdt \\ &= \frac{1}{2}\Gamma_h S^2 \left(\sigma_r^2 - \sigma_h^2\right) dt + e^{rt} d\left[e^{-rt}\left(V_h - V_I\right)\right] \end{aligned}$$

Taking present values leads to

$$dPV[P\&L] = e^{-r(t-t_0)} \frac{1}{2}\Gamma_h S^2 \left(\sigma_r^2 - \sigma_h^2\right) dt + e^{rt_0} d\left[e^{-rt}(V_h - V_I)\right]$$

so, integrating over the life of the option, we have

$$PV[P\&L(I, H)] = V_h - V_I + \frac{1}{2}\int_{t_0}^{T} e^{-r(t-t_0)}\Gamma_h S^2 \left(\sigma_R^2 - \sigma_h^2\right)dt$$

Note that, in the limit, if the hedge volatility σ_h is set equal to either the realized volatility σ_R or the implied volatility Σ, then this solution reduces to our previous results.

CHAPTER 6

6-1. From Chapter 4, we know that the price of a European call option when interest rates and dividends are zero is given by

$$\begin{aligned} C(S, K, v) &= SN(d_1) - KN(d_2) \\ d_{1,2} &= \frac{1}{v}\ln\left(\frac{S}{K}\right) \pm \frac{v}{2} \end{aligned}$$

where $v = \sigma\sqrt{\tau}$. We have

$$\begin{aligned} v &= 0.20\sqrt{\frac{1}{4}} \\ &= 0.10 \end{aligned}$$

then

$$\begin{aligned} d_{1,2} &= \frac{1}{0.10}\ln\left(\frac{2000}{2000}\right) \pm \frac{1}{2}0.10 \\ &= \pm 0.05 \end{aligned}$$

The price of the call is then

$$\begin{aligned} C(S, K, v) &= 2000 \times N(0.05) - 2000 \times N(-0.05) \\ &= 2000 \times 0.52 - 2000 \times 0.48 \\ &= 1039.88 - 960.12 \\ &= 79.76 \end{aligned}$$

To calculate the hedging error, we first need to calculate the vega of the call using Equation 6.15:

$$
\begin{aligned}
\frac{\partial C}{\partial \sigma} &= \frac{S\sqrt{\tau}}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \\
&= \frac{2000\sqrt{\frac{1}{4}}}{\sqrt{2\pi}} e^{-\frac{1}{2}0.05^2} \\
&= \frac{1000}{\sqrt{2\pi}} 0.999 \\
&= 398.44
\end{aligned}
$$

So a one-percentage-point change in the implied volatility would change the price of the call by approximately \$4.

Now using Equation 6.12, the standard deviation of the hedging error is approximately

$$
\begin{aligned}
\sigma_{HE} &\approx \sqrt{\frac{\pi}{4}} \frac{\sigma}{\sqrt{n}} \frac{\partial C}{\partial \sigma} \\
&\approx \sqrt{\frac{\pi}{4}} \frac{0.20}{\sqrt{n}} 398.44 \\
&\approx 70.62 \frac{1}{\sqrt{n}}
\end{aligned}
$$

Assuming 21 business days per month, rebalancing weekly, daily, or four times per day corresponds respectively to 63/5 = 12.6, 63, and 252 rebalancings. The standard deviation of the hedging errors in dollars is

$$
\begin{aligned}
\sigma_{HE}(12.6) &\approx 19.90 \\
\sigma_{HE}(63) &\approx 8.90 \\
\sigma_{HE}(252) &\approx 4.45
\end{aligned}
$$

As a percentage of the call price this corresponds to 24.95%, 11.16%, and 5.58%:

$$
\begin{aligned}
\frac{19.90}{79.76} &\approx 24.95\% \\
\frac{8.90}{79.76} &\approx 11.16\% \\
\frac{4.45}{79.76} &\approx 5.58\%
\end{aligned}
$$

6-2. Equation 6.18 gives an approximation for the standard deviation of the hedging error as

$$\frac{\sigma_{HE}}{C} \approx \frac{0.89}{\sqrt{n}}$$

For 12.6, 63, and 252 rebalancings we have

$$\frac{\sigma_{HE}}{C}(12.6) \approx 25.07\%$$
$$\frac{\sigma_{HE}}{C}(63) \approx 11.21\%$$
$$\frac{\sigma_{HE}}{C}(252) \approx 5.61\%$$

These results are only slightly different from the results from the previous problem. Equation 6.18 provides a good approximation to the standard deviation of the hedging error when the option is at-the-money.

6-3. For a non-dividend-paying stock, if the riskless rate is zero, the BSM price of a vanilla European call is

$$C(S, K, v) = SN(d_1) - KN(d_2)$$
$$d_1 = \frac{1}{v}\ln\left(\frac{S}{K}\right) + \frac{v}{2} \quad d_2 = \frac{1}{v}\ln\left(\frac{S}{K}\right) - \frac{v}{2}$$

If the option is at-the-money, then $S = K$, so $d_1 = v/2$ and $d_2 = -v/2$ and

$$C(S, S, v) = S\left[N\left(\frac{v}{2}\right) - N\left(-\frac{v}{2}\right)\right]$$
$$C(S, S, v) = S\left[2N\left(\frac{v}{2}\right) - 1\right]$$

For the last line, we have used the fact that $N(-x) = [1 - N(x)]$.

For small x,

$$N(x) \approx N(0) + N'(0)x$$
$$\approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}x$$

Setting $x = v/2$,

$$N(v/2) \approx \frac{1}{2}\left(1 + \frac{v}{\sqrt{2\pi}}\right)$$

and thus

$$C(S,S,v) \approx \frac{Sv}{\sqrt{2\pi}} = \frac{S\sigma\sqrt{\tau}}{\sqrt{2\pi}}$$

It is common to use the approximation

$$C(S,S,v) \approx 0.4S\sigma\sqrt{\tau}$$

CHAPTER 7

7-1. From Chapter 4, we know that the price of a European call option when interest rates and dividends are zero is given by

$$C(S,K,v) = SN(d_1) - KN(d_2)$$
$$d_{1,2} = \frac{1}{v}\ln\left(\frac{S}{K}\right) \pm \frac{v}{2}$$

where $v = \sigma\sqrt{\tau}$. We have

$$\begin{aligned} v &= 0.20\sqrt{\frac{1}{4}} \\ &= 0.10 \end{aligned}$$

and

$$\begin{aligned} d_{12} &= \frac{1}{0.10}\ln\left(\frac{2000}{2000}\right) \pm \frac{1}{2}0.10 \\ &= \pm 0.05 \end{aligned}$$

The price of the call is

$$\begin{aligned} C(S,K,v) &= 2000 \times N(0.05) - 2000 \times N(-0.05) \\ &= 2000 \times 0.52 - 2000 \times 0.48 \\ &= 1039.88 - 960.12 \\ &= 79.76 \end{aligned}$$

In the absence of transaction costs, the BSM price of the three-month at-the-money call option is \$79.76.

Using Equation 7.19, with a transaction cost of 1 bp and daily rebalancing, we have

$$\begin{aligned}\tilde{\sigma} &\approx \sigma - k\sqrt{\frac{2}{\pi dt}} \\ &= 0.20 - 0.0001\sqrt{\frac{2}{\pi}\frac{256}{1}} \\ &= 0.20 - 0.0001 \times 12.77 \\ &= 0.20 - 0.0013 \\ &= 0.1987\end{aligned}$$

Substituting into the BSM formula we get

$$\tilde{d}_{1,2} = \pm 0.04968$$

and

$$\begin{aligned}\tilde{C}(S,K,v) &= 2000 \times N(0.04968) - 2000 \times N(-0.04968) \\ &= 2000 \times 0.52 - 2000 \times 0.48 \\ &= 1039.62 - 960.38 \\ &= 79.25\end{aligned}$$

The adjusted price for daily rebalancing is \$79.25. Based on the assumptions behind Equation 7.19, the call is worth \$0.51 less, or 0.64% less than it would be in the absence of transaction costs.

Notice that the adjustment to the *implied volatility* does not depend on the time to expiration, only on the frequency of rebalancing, but that the impact on the *price* of the option will depend on the time to expiration. All else being equal, the adjustment to the price will be greater for options with more time to expiration.

7-2. The price of the call option in the absence of transaction costs is still \$79.76. For the short position in the option with a transaction cost of 1 bp and daily rebalancing, using Equation 7.19 we have

$$\begin{aligned}\tilde{\sigma} &\approx \sigma + k\sqrt{\frac{2}{\pi dt}} \\ &= 0.20 + 0.0001\sqrt{\frac{2}{\pi}\frac{256}{1}} \\ &= 0.20 + 0.0001 \times 12.77 \\ &= 0.20 + 0.0013 \\ &= 0.2013\end{aligned}$$

Substituting into the BSM formula we get

$$\tilde{d}_{12} = \pm 0.0503$$

and

$$\begin{aligned}\tilde{C}(S, K, v) &= 2000 \times N(0.0503) - 2000 \times N(-0.0503)\\ &= 2000 \times 0.52 - 2000 \times 0.48\\ &= 1040.13 - 959.87\\ &= 80.26\end{aligned}$$

The adjusted price for daily rebalancing is \$80.26. For the short option position, we would need to charge approximately \$0.50 or 0.64% more to make up for transaction costs. Note that even though the adjustment to implied volatility in Equation 7.19 is symmetric for long versus short positions, the BSM equation is not symmetric to changes in volatility. The adjustment to the price for long and short positions might be similar, but they are not equal.

The difference in the adjusted price for traders who are trying to buy and those who are trying to sell is \$80.26 − \$79.25 = \$1.01. This is 1.28% of the midprice, a nontrivial bid-ask spread induced by a transaction cost of only 1 bp.

7-3. In the absence of transaction costs, the BSM price of the three-month call option with a strike of 2,200 is

$$\begin{aligned}C(S, K, v) &= 2000 \times N(-0.90) - 2200 \times N(-1.00)\\ &= 2000 \times 0.18 - 2200 \times 0.16\\ &= 366.47 - 347.39\\ &= 19.08\end{aligned}$$

The adjustment to implied volatility does not depend on the moneyness of the option, and is therefore the same as in Problem 7-1, that is, $\tilde{\sigma} \approx$ 19.87%. Substituting into the BSM formula, we have

$$\begin{aligned}C(S, K, v) &= 2000 \times N(-0.91) - 2200 \times N(-1.01)\\ &= 2000 \times 0.18 - 2200 \times 0.16\\ &= 363.06 - 344.32\\ &= 18.74\end{aligned}$$

The difference between the unadjusted and adjusted prices is \$0.34 or 1.77%. Compared to Problem 7-1, the adjustment is smaller in dollar terms, but greater in percentage terms.

CHAPTER 8

8-1. Because the riskless rate is zero, we have:

$$C(S, K, v) = SN(d_1) - KN(d_2)$$
$$d_1 = \frac{1}{v}\ln\left(\frac{S}{K}\right) + \frac{v}{2} \quad d_2 = \frac{1}{v}\ln\left(\frac{S}{K}\right) - \frac{v}{2}$$

With $\nu = 0.10$,

$$d_1 = \frac{1}{0.10}\ln\left(\frac{2000}{2100}\right) + \frac{0.10}{2} = -0.4379$$
$$d_2 = \frac{1}{0.10}\ln\left(\frac{2000}{2100}\right) - \frac{0.10}{2} = -0.5379$$

The risk-neutral probability of the call option expiring in the money is

$$N(d_2) = 0.30$$

The delta of the call option is

$$N(d_1) = 0.33$$

In this case, the delta is not exactly equal to the risk-neutral probability, but it is a very good approximation to it.

8-2. Because the riskless rate is not zero, we have

$$C(S, K, \tau, \sigma, r) = SN(d_1) - Ke^{-r\tau}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \quad d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

With $\sigma = 20\%$, $\tau = 1$, and the $r = 2.0\%$,

$$d_1 = \frac{\ln\left(\frac{2000}{2100}\right) + \left(0.02 + \frac{0.20^2}{2}\right)1}{0.20\sqrt{1}} = -0.0440$$

$$d_2 = \frac{\ln\left(\frac{2000}{2100}\right) + \left(0.02 - \frac{0.20^2}{2}\right)1}{0.20\sqrt{1}} = -0.2440$$

The difference between d_1 and d_2 is larger than in the previous example because the implied volatility is larger.

The risk-neutral probability of the call option expiring in the money is

$$N(d_2) = 0.40$$

The delta of the call option is:

$$N(d_1) = 0.48$$

In this case, the delta, 0.48, is close to the probability of expiring in-the-money, 40%, but not as close as in the previous problem.

8-3. Starting with Equation 8.9, we have

$$\Delta \approx \Delta_{\mathrm{ATM}} - \frac{1}{\sqrt{2\pi}}\frac{J}{v}$$

$$\frac{1}{\sqrt{2\pi}}\frac{J}{v} \approx \Delta_{\mathrm{ATM}} - \Delta$$

$$J \approx v\sqrt{2\pi}\left(\Delta_{\mathrm{ATM}} - \Delta\right)$$

As in the sample problem, we know that a one-year at-the-money call with implied volatility of 20% has a delta of approximately 0.54. For this option, then, $v = \sigma\sqrt{\tau} = 0.2 \times \sqrt{1} = 0.20$. Thus,

$$J \approx 0.20 \times \sqrt{2\pi}(0.54 - 0.34)$$
$$J \approx 0.20 \times 2.5 \times 0.20$$
$$J \approx 0.10$$

We need to increase the strike by approximately 10%, which corresponds to an index level of 4,400. If you use the exact BSM formula to compute deltas, you'll find that the 4,400 call actually has a delta of 0.35. To get a delta of 0.34, the strike would have to be closer to 4,430.

8-4. The deltas for the calls are 0.60, 0.58, 0.43, and 0.32, respectively. If you plot the implied volatilities versus the deltas, you will notice a nearly perfect linear relationship (in this case, the implied volatilities are nearly linear in the strike, too, but the fit with delta is slightly better). We can specify the relationship as

$$\Sigma = \alpha + \beta\Delta$$

Using the first and last call, we find the slope

$$\beta = \frac{0.172 - 0.200}{0.32 - 0.60} = \frac{-0.028}{-0.28} = 0.10$$

We then use the first call to find the intercept

$$\alpha = 0.20 - 0.10 \times 0.60 = 0.20 - 0.06 = 0.14$$

The equation for implied volatility in terms of delta is then

$$\Sigma = 0.14 + 0.10\Delta$$

We can check the accuracy of our linear approximation by substituting the deltas and seeing that the original implied volatilities are returned.

8-5. If future annualized volatility is 20%, then, using the square root rule, over three months a +1 standard deviation move corresponds to a return of

$$0.20\sqrt{\frac{1}{4}} = 0.10 = 10\%$$

With the S&P 500 currently at 2,000, a +1 standard deviation move corresponds to a strike of 2,200.

Because interest rates and the dividend yield are zero, the BSM delta of a call option is given by

$$\Delta = \frac{\partial C}{\partial S} = N(d_1)$$

where,

$$d_1 = \frac{1}{v}\ln\left(\frac{S}{K}\right) + \frac{v}{2}$$

For the three-month call option, $v = \sigma\sqrt{\tau} = 0.20\sqrt{\frac{1}{4}} = 0.10$ and

$$\begin{aligned} d_1 &= \frac{1}{0.10}\ln\left(\frac{2000}{2200}\right) + \frac{0.10}{2} \\ &= -1.0031 \end{aligned}$$

and the BSM delta is

$$\begin{aligned} \Delta &= N(-1.0031) \\ &= 0.18 \end{aligned}$$

For the one-year call option, the strike corresponding to a +1 standard deviation is 2,400 and d_1 is

$$\begin{aligned} d_1 &= \frac{1}{0.20}\ln\left(\frac{2000}{2400}\right) + \frac{0.20}{2} \\ &= -1.0116 \end{aligned}$$

And the BSM delta is

$$\begin{aligned} \Delta &= N(-1.0116) \\ &= 0.21 \end{aligned}$$

Notice that even though the strike prices are very different, the deltas are very similar. If we assume constant future volatility, then the BSM deltas will be similar for strikes corresponding to equally likely outcomes. This is one reason that it is convenient to graph implied volatility as a function of delta.

8-6. In order to calculate the implied volatility of the \$110 strike call option, we need to know the delta of the \$110 strike call option, but in order to calculate the delta we need to know the implied volatility. This is the circularity problem we referred to in the chapter when describing implied volatility as a function of delta.

To find the solution to this problem, we need either to proceed by trial and error or to use an optimization function. For pedagogical reasons we proceed by trial and error, choosing different values of delta and then calculating the corresponding strike. To do this, we first need to express the strike as a function of delta. Taking the inverse of our equation for the BSM delta of a call option, we have

$$\Delta = N(d_1)$$

$$N^{-1}(\Delta) = d_1$$

Expanding and rearranging,

$$\begin{aligned}
N^{-1}(\Delta) &= \frac{1}{v}\ln\left(\frac{S}{K}\right) + \frac{1}{2}v \\
\ln\left(\frac{S}{K}\right) &= v\left(N^{-1}(\Delta) - \frac{1}{2}v\right) \\
\frac{S}{K} &= e^{v\left(N^{-1}(\Delta) - \frac{1}{2}v\right)} \\
K &= Se^{-v\left(N^{-1}(\Delta) - \frac{1}{2}v\right)}
\end{aligned}$$

In this particular case, because we are dealing with one-year calls, $v = \Sigma$, and

$$K = Se^{-\Sigma\left(N^{-1}(\Delta) - \frac{1}{2}\Sigma\right)}$$

Because Σ is a function of delta, K is an explicit function of Δ (and the current stock price).

The strike price that we are interested in is not too far out-of-the-money, so a good starting point might be to try a delta of 0.50. For a delta of 0.50, the implied volatility is

$$\begin{aligned}
\Sigma &= 0.20 + 0.30\Delta \\
&= 0.20 + 0.30 \times 0.50 \\
&= 0.35
\end{aligned}$$

Substituting into our equation for the strike, with $N^{-1}(0.5) = 0$, we find

$$\begin{aligned}
K &= Se^{-\Sigma\left(N^{-1}(\Delta) - \frac{1}{2}\Sigma\right)} \\
&= 100e^{-0.35\left(N^{-1}(0.50) - \frac{1}{2}0.35\right)} \\
&= 106.32
\end{aligned}$$

So a delta of 0.50 corresponds to a strike of \$106.32. Clearly, the delta of the \$110 call must be less than 0.50. Next, we try $\Delta = 0.40$:

$$\begin{aligned}
\Sigma &= 0.20 + 0.30\Delta \\
&= 0.32
\end{aligned}$$

and

$$K = 100e^{-0.32\left(N^{-1}(0.40)\ -\frac{1}{2}0.32\right)}$$
$$= 114.14$$

While $\Delta = 0.50$ was too high, $\Delta = 0.40$ is too low. For the next trial, we might just split the interval and try 0.45, or if we want to be a bit fancier, we can interpolate between our first two trial values:

$$\begin{aligned}\Delta_{\text{new}} &= 0.50\left(\frac{114.14 - 110}{114.14 - 106.32}\right) + 0.40\left(\frac{110 - 106.32}{114.14 - 106.32}\right)\\ &= 0.50 \times 0.52 + 0.40 \times 0.48\\ &= 0.4529\end{aligned}$$

Substituting into our formula, we find that this new Δ corresponds to a strike of \$110.09.

If we repeat this process, now interpolating between the 0.50 and 0.4529 values, we arrive at a new trial delta of 0.4541. Substituting, we get an implied volatility of

$$\begin{aligned}\Sigma &= 0.20 + 0.30\Delta\\ &= 0.20 + 0.30 \times 0.4541\\ &= 0.3362\end{aligned}$$

and

$$K = 100e^{-0.3362\left(N^{-1}(0.4541)-\frac{1}{2}0.3362\right)}$$
$$= 110.00$$

This is the desired strike price. The implied volatility for the \$110 strike call is then approximately 33.62%.

CHAPTER 9

9-1. The price of the at-the-money call is \$7.97. For implied volatility of 20.00%, 21.00%, and 21.25%, the price of the \$101 strike call would be \$7.52, \$7.91, and \$8.01, respectively.

Recall from Equation 9.15 that for a 1% increase in the strike, the upper bound on the increase in implied volatility is approximately 1.25%. When the implied volatility is unchanged, the \$101 call price is lower, as expected. Even if volatility increases by 1%, the \$101 price is still lower than the at-the-money price. If we increase the implied volatility by 1.25%, though, the \$101 call price is just slightly higher. Because of the principle of no riskless arbitrage, a call with a higher strike must be worth no more than one with a lower strike; therefore, an implied volatility of 21.25% is too high. In this case, the exact upper bound is closer to 21.13%, just slightly less than our approximation suggested.

9-2. The price of the at-the-money put is \$7.97. For implied volatility of 20.00%, 18.75%, and 18.50%, the price of the \$101 strike put would be \$8.52, \$8.02, and \$7.92, respectively.

When the implied volatility is held constant, the price of the put increases as the strike increases, as expected. If the implied volatility decreases by 1.25%, the price is still higher, but only slightly. If we decrease the implied volatility by 1.50%, we have gone too far, and violate the no-arbitrage constraint. In this case, the exact lower bound is closer to a decrease of 1.38%, slightly lower than our approximation suggested.

9-3. Using the at-the-money call and Equation 9.15, you might guess that the upper bound is close to 26.25% = 20.00% + 6.25%, because

$$d\sigma \leq 1.25/\sqrt{\tau} dK/K = 1.25 \times 5\% = 6.25\%$$

If we calculate the BSM prices of the three calls, we see that 26.25% is a little too high, in that it produces a price \$167.73 for the 2,100 strike that is slightly higher than the price of the at-the-money call, \$159.31. With a little trial and error, we find that an implied volatility of 25.19% gives us a call price of \$159.29, just below the price of the at-the-money call.

K	σ	C
2,000	20.00%	159.31
2,200	15.00%	50.00
2,100	26.25%	167.73
2,100	25.19%	159.29

Using 25.19%, the at-the-money-call is more expensive than the 2,100 call, which is more expensive than the 2,200 call. Using 25.19% does not violate the slope rule for either option.

Though 25.19% does not violate the slope rule, it does violate the curvature rule. You can see this by observing the change in price: from

2,000 to 2,100, the call price decreases by \$0.02 = \$159.31 – \$159.29, but from 2,100 to 2,200 it decreases by \$109.29 = \$159.29 – \$50.00. The slope is getting more negative, but by Equation 9.9 the curvature should be greater than or equal to zero, which means the slope should get less negative. We can see the problem more clearly if we try to price a butterfly using 25.19%:

$$
\begin{aligned}
\pi\_B &= C(K - dK) - 2C(K) + C(K + dK) \\
&= 159.31 - 2 \times 159.29 + 50 \\
&= -109.26
\end{aligned}
$$

By the principle of no riskless arbitrage, the butterfly cannot have a negative value.

To find the upper bound that is consistent with the curvature rule, we can search for the highest implied volatility that returns a butterfly price that is nonnegative. A volatility of 18.29% gives a price of \$104.64 for the 2,100 call. This gives a price for the butterfly of \$0.04, just slightly positive. A volatility of 18.30% would produce a negative value for the butterfly, so, to the nearest basis point, 18.29% is the upper bound for the implied volatility.

CHAPTER 10

10-1. The linear model given in this problem provides a reasonable approximation to call prices for $1{,}700 \leq K \leq 2{,}100$. There are two reasons to be wary of this model. The first is that if we try to extend the model too far it will start to produce very unreasonable prices. In this particular case, past $K = 2{,}239$ the model will actually start to produce negative call prices.

Another problem has to do with butterfly prices. If call prices are linear in strike, then butterfly spreads will always be worth zero. If $C = \alpha + \beta K$, then the price of a butterfly with strikes at $(K - dK)$, K, and $(K + dK)$ is zero because the call prices have no curvature with respect to strike. In this particular case, the price of the three calls with strikes at 1,800, 1,900, and 2,000 were 325, 251, and 177, respectively, and the price of the butterfly spread was $325 - 2 \times 251 + 177 = 0$.

Zero is the lower no-arbitrage limit for the price of a butterfly. Technically this does not violate any of the restrictions discussed in this chapter, but it is unusual. As we'll see in the next chapter, the price of a butterfly option is related to the market price for a security that pays \$1 if the stock ends up between $(K - dK)$ and $(K + dK)$ at expiration. A price of zero for this security suggests that the market perceives a zero probability of the underlying index being between $(K - dK)$ and $(K + dK)$ at expiration. If the S&P 500 call price function was linear between 1,700 and 2,100, this would suggest that there was no probability of the S&P 500 being between 1,700 and 2,100 in 11 months. There is no reason to believe that this should be the case.

CHAPTER 11

11-1. Using Equation 11.3, we calculate the pseudo-probabilities as follows:

$$P[\text{NDX} < 4000] = \$0.28 \times e^{0.05} = 29.44\%$$

$$P[4000 \leq \text{NDX} \leq 4500] = \$0.51 \times e^{0.05} = 53.61\%$$

$$P[\text{NDX} > 4500] = \$0.20 \times e^{0.05} = 21.03\%$$

The three securities cover all possible states of the world: Either the NDX is below 4,000, it is between 4,000 and 4,500, or it is above 4,500. There are no other possibilities, yet the sum of the pseudo-probabilities is 104.08%, not 100%. The securities are not correctly priced.

We should sell short all three of the securities for \$0.99, and invest the \$0.99 at the riskless rate. At the end of the year, our \$0.99 will be worth \$1.04. We can use \$1 to cover the three securities (one will be worth \$1, and the other two will be worth \$0), and keep the difference, \$0.04, as our arbitrage profit.

11-2.

$$
\begin{aligned}
V(S,t) &= \frac{pV(K,T)}{(1+r)^{\frac{1}{2}}} \\
p &= \frac{V(S,t)}{V(K,T)}(1+r)^{\frac{1}{2}} \\
&= \frac{1.00}{10.30}(1.0609)^{\frac{1}{2}} \\
&= \frac{1.00}{10.30}(1.03) \\
&= \frac{1.03}{10.30} \\
&= 10\%
\end{aligned}
$$

11-3. Using Equation 11.6,

$$
\begin{aligned}
V(S,t) &= e^{-r(T-t)} \int_{10}^{12} f(K)V(K,T)dK \\
&= e^{-0.04\times 1} \int_{10}^{12} (-75 + 20K - K^2)\frac{1}{200}(K-10)^3 dK \\
&= \frac{e^{-0.04\times 1}}{200} \int_{10}^{12} (75000 - 42500K + 9250K^2 - 975K^3 \\
&\quad + 50K^4 - K^5)dK \\
&= \frac{e^{-0.04\times 1}}{200} \left[75000K - 21250K^2 + \frac{9250}{3}K^3 - \frac{975}{4}K^4 \right. \\
&\quad \left. + 10K^5 - \frac{1}{6}K^6 \right]_{10}^{12} \\
&= \frac{e^{-0.04\times 1}}{200}(104256 - 104167) \\
&= 0.43
\end{aligned}
$$

The fair present value is \$0.43.

11-4. Denote the annually compounded riskless rate by r. Using the Breeden-Litzenberger formula, as shown in Equation 11.14, the risk-neutral probability density is given by

$$
\rho(S,t,K,T) = (1+r)^2 \left(\frac{20}{21^2} e^{-\frac{K}{21}} \right)
$$

Since we know the probability density for all payoffs, we can value a riskless bond that pays \$1 in every state of the world. The value B is given by integrating the risk-neutral probability density over a constant payoff:

$$
\begin{aligned}
B &= \frac{1}{(1+r)^2} \int_0^\infty 1 \cdot \rho(S,t,K,T)dK \\
&= \frac{1}{(1+r)^2} \int_0^\infty (1+r)^2 \left(\frac{20}{21^2} e^{-\frac{K}{21}} \right) dK \\
&= \frac{20}{21^2} \int_0^\infty e^{-\frac{K}{21}} dK \\
&= \frac{20}{21}
\end{aligned}
$$

Thus, a zero coupon two-year bond is worth 20/21, which is the present value of \$1 discounted by $(1 + r)^2$, which corresponds to a riskless interest rate $r = 2.47\%$.

CHAPTER 12

12-1. The portfolio constructed here is worth 18.69 BRL. This is considerably higher than the price of the barrier option that knocks out at any time along the barrier, because the portfolio knocks out only at two discrete times. The constituents of the replicating portfolio are specified in the following table:

				Value $t = 0$ Months		Value at $t = 6$ Months	
Quantity	**Type**	**Strike**	**Expiration T**	**$S = 6{,}000$**	**$S = 5{,}000$**	**$S = 6{,}000$**	**$S = 5{,}000$**
1.00	Call	5,500	1 year	1,182.67	605.41	926.80	373.39
−1.37	Call	6,000	1 year	−1,306.34	−630.99	−926.80	−331.74
0.18	Call	6,000	6 months	123.68	44.27	0.00	0.00
Portfolio				**0.00**	**18.69**	**0.00**	**41.65**

12-2. We can use barrier in-out parity to construct the replicating portfolio for the up-and-in call. For European options, all else being equal, the price of an up-and-in call should be equal to the price of a standard call minus an up-and-out call. We can construct a replicating portfolio for an up-and-in call by buying a standard call and selling the replicating portfolio for the up-and-out call. Using the result from the previous problem, we have:

				Value at $t = 0$ Months		Value at $t = 6$ Months	
Quantity	**Type**	**Strike**	**Expiration T**	**$S = 6{,}000$**	**$S = 5{,}000$**	**$S = 6{,}000$**	**$S = 5{,}000$**
1.37	Call	6,000	1 year	1,306.34	630.99	926.80	331.74
−0.18	Call	6,000	6 months	−123.68	−44.27	0.00	0.00
Portfolio				**1,182.67**	**586.72**	**926.80**	**331.74**

Notice that the standard call is perfectly canceled by the first call from the original replicating portfolio, leaving us with just two options in our new replicating portfolio.

Just as the value of the up-and-out replicating portfolio is equal to zero on the barrier when $t = 0$ and $t = 6$ months, the value of the up-and-in replicating portfolio is equal to the value of the corresponding standard call on the barrier, 1,182.67 BRL at $t = 0$ and 926.80 BRL at $t = 6$ months. You can verify this by using the BSM formula with the appropriate volatility.

The current value of this replicating portfolio is 586.72 BRL. Its value plus the value of the replicating portfolio for the up-and-out call from the previous problem, 18.69 BRL, is equal to the value of a standard call with the same strike and expiration, 605.41 BRL.

12-3. Let the current time be $t = 0$, and let T denote the expiration of any option. We match the payoff at expiration if the barrier has not been hit by a long position in a European put with a strike of 1,900 and expiration $T = 12$ months.

We then proceed backward from expiration, matching payoffs along the barrier. Two months prior to expiration at an index level of 1,600 and $t = 10$ months, the $T = 12$ months 1,900 strike put is worth \$309.91. To offset this, we must be short a $T = 12$ months 1,600 strike put. Two months prior to expiration, this put is worth \$52.10 on the barrier. By shorting 5.78 = \$300.91/\$52.10 puts, we get a portfolio worth \$0 on the barrier at $t = 10$ months.

Next, at $t = 8$ months, our first two puts are worth −\$119.38 on the barrier. We need to have bought a $T = 10$ months 1,600 strike put to cancel this value. One such put is worth \$52.50, so we need 2.29 = 119.38/52.50 puts.

Continuing in this fashion, we end up with the static replicating portfolio in the following table. The portfolio, which, based on BSM, is worth \$24.60, is only slightly more expensive than the actual down-and-out put that knocks out anywhere along the barrier, \$20.22. This portfolio satisfies all of the constraints specified in the problem; in particular, one can check that its value is zero at $S = 1,600$ at $t = 0$, 2, 4, 6, 8, and 10 months.

				Value $t = 0$	
Quantity	**Type**	**Strike**	**Expiration T**	**$S = 2,000$**	**$S = 1,600$**
1.00	Put	1,900	1 year	110.39	337.60
−5.78	Put	1,600	1 year	−136.98	−736.05
2.29	Put	1,600	10 months	39.99	266.64
0.74	Put	1,600	8 months	8.55	77.40
0.35	Put	1,600	6 months	2.19	32.02
0.21	Put	1,600	4 months	0.43	15.27
0.14	Put	1,600	2 months	0.02	7.12
Portfolio				**24.60**	**0.00**

12-4. Equation 12.4 required that

$$N'\left(\frac{\ln\left(\frac{B}{S}\right)+\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right)-\alpha N'\left(\frac{\ln\left(\frac{S}{B}\right)+\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right)=0$$

Begin by defining two new variables, x and y:

$$x=\frac{\ln\left(\frac{B}{S}\right)+\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\quad y=\frac{\ln\left(\frac{S}{B}\right)+\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$$

We then require that

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}-\alpha\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}=0$$

Write $\alpha=e^{\beta}$. Then,

$$\begin{aligned}
e^{-\frac{1}{2}y^2+\beta}&=e^{-\frac{1}{2}x^2}\\
-\frac{1}{2}y^2+\beta&=-\frac{1}{2}x^2\\
\beta&=\frac{1}{2}(y^2-x^2)
\end{aligned}$$

Substituting for x and y, we obtain

$$\begin{aligned}
\beta&=\frac{1}{2\sigma^2\tau}\left[\left(\ln\left(\frac{S}{B}\right)^2+\ln\left(\frac{S}{B}\right)\sigma^2\tau+\sigma^4\tau^2\right)\right.\\
&\qquad\left.-\left(\ln\left(\frac{B}{S}\right)^2+\ln\left(\frac{B}{S}\right)\sigma^2\tau+\sigma^4\tau^2\right)\right]\\
&=\frac{1}{2\sigma^2\tau}\left[\ln\left(\frac{S}{B}\right)^2-\ln\left(\frac{B}{S}\right)^2+\ln\left(\frac{S}{B}\right)\sigma^2\tau-\ln\left(\frac{B}{S}\right)\sigma^2\tau\right]\\
&=\frac{1}{2}\left[\ln\left(\frac{S}{B}\right)-\ln\left(\frac{B}{S}\right)\right]=\ln(S/B)
\end{aligned}$$

Thus, since $\alpha=\exp(\beta)$,

$$\alpha=\frac{S}{B}$$

CHAPTER 13

13-1. Using Equation 13.6, 13.7, and 13.8, we determine the parameters to be

$$u = \sigma\sqrt{dt} = 0.2\sqrt{\frac{1}{256}} = \frac{0.2}{16} = 0.0125$$

$$d = -\sigma\sqrt{dt} = -u = -0.0125$$

$$p = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{dt} = \frac{1}{2} + \frac{1}{2}\frac{0.1}{0.2}\sqrt{\frac{1}{256}} = \frac{1}{2} + \frac{1}{2}\frac{1}{32} = 0.516$$

If the current stock price is \$75, then after the first step there is a 51.6% probability that the stock price will be \$75.94 = \$75 $\times\, e^{0.0125}$, and a 48.4% probability that the stock price will be \$74.07 = \$75 $\times\, e^{-0.0125}$.

After the second step, there is a 26.6% probability that the stock price will be \$76.90 = \$75.94 $\times\, e^{0.0125}$, a 50.0% probability that the stock price will be \$75 = \$75.94 $\times\, e^{-0.0125}$ = \$74.07 $\times\, e^{0.0125}$, and a 23.5% probability that the stock price will be \$73.15 = \$74.07 $\times\, e^{-0.0125}$. The probabilities appear to add up to 100.1%, but this is only due to rounding. In fact, the stock price must end up at one of these three nodes, and the sum of the probabilities is exactly 100%. Notice that the tree closes: that up-down and down-up lead to the same price. Also notice that the up-down and down-up points return to the initial price, \$75. This is one of the defining features of the Cox-Ross-Rubinstein (CRR) model.

13-2. Using Equation 13.13, we have:

$$u = \mu dt + \sigma\sqrt{dt} = 0.1\frac{1}{256} + 0.2\sqrt{\frac{1}{256}} = 0.0004 + 0.0125$$
$$= 0.0129$$

$$d = \mu dt - \sigma\sqrt{dt} = 0.1\frac{1}{256} - 0.2\sqrt{\frac{1}{256}} = 0.0004 - 0.0125$$
$$= -0.0121$$

In the Jarrow-Rudd convention, $q = (1 - q) = 1/2$. If the current price is \$75, then after the first step there is a 50% probability that the stock will be \$75.97 = \$75 $\times\, e^{0.0129}$, and a 50% probability that the stock will be \$74.10 = \$75 $\times\, e^{-0.0121}$.

After the second step, there is a 25% probability that the stock price will be \$76.96 = \$75.97 $\times\, e^{0.0129}$, a 50.0% probability that the stock price will be at \$75.06 = \$75.97 $\times\, e^{-0.0121}$ = \$74.10 $\times\, e^{0.0129}$, and a 25% probability that the stock price will be \$73.21 = \$74.10 × $e^{-0.0121}$. As with the Cox-Ross-Rubinstein model, the tree closes, but in this case the price at the central node is no longer the same as the initial price.

13-3. The price of a riskless bond with three months to maturity is

$$B = e^{-0.25\times 0.04}\$2{,}100 = \$2{,}079.10$$

The price of SPX in terms of the bond is then

$$S_B = \frac{S}{B} = \frac{2000}{2079.10} = 0.96$$

To find d_1 and d_2, we start with $v = \sigma\sqrt{\tau} = 0.16\sqrt{0.25} = 0.08$. Then,

$$\begin{aligned} d_{1,2} &= \frac{1}{v}\ln(S_B) \pm \frac{v}{2} \\ &= \frac{1}{0.08}\ln(0.96) \pm \frac{0.08}{2} \\ &= -0.48 \pm 0.04 \end{aligned}$$

Finally,

$$\begin{aligned} C_B(S_B, v, r, \tau) &= S_B N(d_1) - N(d_2) \\ &= 0.96 \times N(-0.44) - N(-0.52) \\ &= 0.96 \times 0.33 - 0.30 \\ &= 0.0159 \end{aligned}$$

The call option is worth 1.59% as much as the riskless bond with a face value equal to the strike. We can check this answer against a standard BSM calculator by multiplying C_B by the price of the bond to get 0.0159 × \$2,079.10 = \$33.02.

13-4. From Equation 13.43 with $v = \sigma\sqrt{\tau} = 0.16\sqrt{0.25} = 0.08$, we have

$$\begin{aligned} d_{1,2} &= \frac{1}{v}\ln\left(\frac{Se^{-b\tau}}{K}\right) \pm \frac{v}{2} \\ &= \frac{1}{0.08}\ln\left(\frac{2000e^{-0.04\times 0.25}}{2100}\right) \pm \frac{0.08}{2} \\ &= -0.73 \pm 0.04 \end{aligned}$$

Then

$$\begin{aligned}C(S,K,v,b,\tau) &= Se^{-b\tau}N(d_1) - KN(d_2)\\ &= 2000 \times e^{-0.04\times 0.25}N(-0.69) - 2100 \times N(-0.77)\\ &= 2000 \times 0.99 \times 0.24 - 2100 \times 0.22\\ &= \$21.95\end{aligned}$$

13-5. In order to solve this problem, we need to combine the techniques used in the previous two problems.

From Equation 13.39 for the value of a European option with zero dividends,

$$\begin{aligned}C_B(S_B,v,r,\tau) &= S_BN(d_1) - N(d_2)\\ d_{1,2} &= \frac{1}{v}\ln(S_B) \pm \frac{v}{2}\end{aligned}$$

where $B = Ke^{-r\tau}$ and $S_B = S/B$. Because of the nonzero dividend yield, our problem is equivalent to valuing an option on $e^{-b\tau}$ shares of S, so we must replace S_B with $e^{-b\tau}S_B$ in the previous equations. This leads to the value

$$\begin{aligned}C_B(S_B,v,r,b,\tau) &= e^{-b\tau}S_BN(d_1) - N(d_2)\\ d_{1,2} &= \frac{1}{v}\ln\left(e^{-b\tau}S_B\right) \pm \frac{v}{2} = \frac{1}{v}\left[\ln(S_B) - b\tau \pm \frac{1}{2}v^2\right]\end{aligned}$$

As in Problem 13-3, the riskless bond with three months to maturity has a price of \$2,079.10, the price of SPX in terms of the bond is 0.96, and $v = 0.08$. Substituting into these equations, we have

$$\begin{aligned}d_{1,2} &= \frac{1}{v}\left[\ln(S_B) - b\tau \pm \frac{1}{2}v^2\right]\\ &= \frac{1}{0.08}\left[\ln(0.96) - 0.04 \times 0.25 \pm \frac{1}{2}0.08^2\right]\end{aligned}$$

so that

$$d_1 = -0.57 \quad \text{and} \quad d_2 = -0.65$$

Then,

$$\begin{aligned}C_B(S_B, v, r, b, \tau) &= e^{-b\tau} S_B N(d_1) - N(d_2)\\&= e^{-0.04\times 0.25} 0.9620 N(-0.57) - N(-0.65)\\&= e^{-0.04\times 0.25} 0.9620 \times 0.2844 - 0.2579\\&= 0.2708 - 0.2579\\&= 0.0130\end{aligned}$$

The value of the call option is 1.30% of the value of the riskless bond. We can multiply this value by the value of the bond to get the price of the call option in dollars:

$$C(S_B, v, r, b, \tau) = 0.0130 \times \$2{,}079.10 = \$26.93$$

You can check that this is the correct price in dollars by using a standard BSM calculator.

13-6. The first thing we need to do is calculate the forward rates in each year. If the one-year riskless rate is 5% and the two-year riskless rate is 7.47%, then the forward rate in year 2 is 10%, because

$$(1 + 0.05)(1 + 0.10) = (1 + 0.0747)^2$$

Similarly, the forward rate in year 3 is 15%, because

$$(1 + 0.05)(1 + 0.10)(1 + 0.15) = (1 + 0.0992)^3$$

Next, we calculate the forward volatilities. The total variance over two years is equal to the variance in year 1 plus the forward variance in year 2. The forward volatility in year 2 is then 30%, because

$$20.0\%^2 + 30.0\%^2 \approx 2(25.5\%^2)$$

We can calculate the forward volatility in the third year in a similar fashion. Putting it all together, we have:

	Year 1	**Year 2**	**Year 3**
Riskless rate	5.00%	7.47%	9.92%
Volatility	20.00%	25.50%	31.10%
Forward rate	5.00%	10.00%	15.00%
Forward volatility	20.00%	30.00%	40.00%

In the first year, each time step is $dt_1 = 0.10$. Here the subscript corresponds to the year, not the step. In each of the years, we want $\sigma_i \sqrt{dt_i}$ to be the same, so that

$$\sigma_i \sqrt{dt_i} = \sigma_1 \sqrt{dt_1}$$
$$dt_i = \frac{\sigma_1^2}{\sigma_i^2} dt_1$$

Therefore,

$$dt_2 = \frac{0.20^2}{0.30^2} 0.10 = 0.044$$
$$dt_3 = \frac{0.20^2}{0.40^2} 0.10 = 0.025$$

This corresponds to roughly 23 steps to span year 2 and 40 steps to span year 3. Unfortunately, with this method we are not guaranteed to get an integer number of steps. As we use smaller and smaller steps, this rounding error becomes less of a problem.

Because $\sigma_i \sqrt{dt_i}$ is equal in each year, the up and down parameters in the CRR tree will be the same at every time step. Using Equations 13.6, 13.7, and 13.8,

$$u = \sigma \sqrt{dt} = 0.2\sqrt{0.10} = 0.0632$$
$$d = -\sigma \sqrt{dt} = -u = -0.0632$$

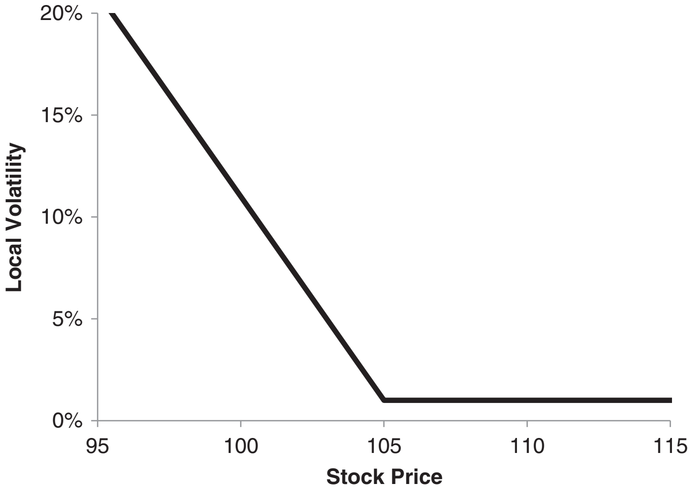
Finally, using Equations 13.28, the q-measure probability in each year is

$$q = \frac{e^{rdt} - e^{-\sigma\sqrt{dt}}}{e^{\sigma\sqrt{dt}} - e^{-\sigma\sqrt{dt}}}$$
$$q_1 = \frac{e^{0.05\times 0.10} - e^{-0.20\sqrt{0.10}}}{e^{0.20\sqrt{0.10}} - e^{-0.20\sqrt{0.10}}} = 0.5238$$
$$q_2 = \frac{e^{0.10\times 0.04} - e^{-0.30\sqrt{0.04}}}{e^{0.30\sqrt{0.04}} - e^{-0.30\sqrt{0.04}}} = 0.5194$$
$$q_3 = \frac{e^{0.15\times 0.02} - e^{-0.40\sqrt{0.02}}}{e^{0.40\sqrt{0.02}} - e^{-0.40\sqrt{0.02}}} = 0.5139$$

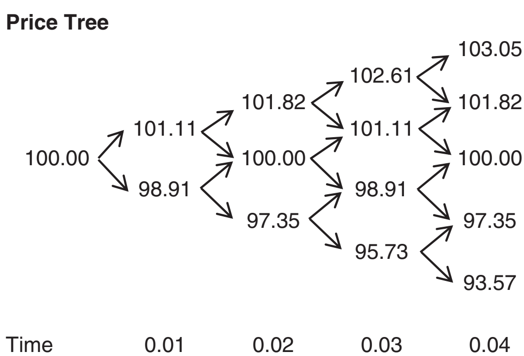
In this case, the q value changes each year not only because the time step is changing, but because the riskless rate is changing, too.

CHAPTER 14

14-1. A graph of the local volatility as a function of stock price is shown here:



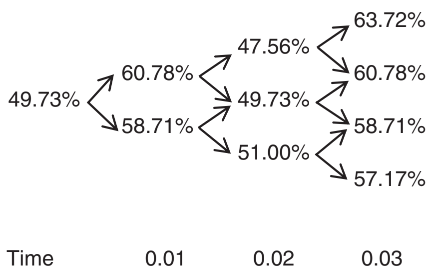
The price tree for the first five levels is:



Compared to the sample problem, the local volatility starts out higher, but decreases more quickly as the stock price increases. Because of this, the highest node on the fifth level is only \$103.05, compared to \$103.34 in the sample problem. Likewise, the local volatility increases more quickly as prices decline, and the lowest node, \$93.57, is lower than \$95.22 in the sample problem.

14-2. Four time steps get us to the fifth level. There is only one price at that level that is greater than the strike price, the uppermost node, \$103.05. At this node, the call will be worth \$1.05. To find the probability of reaching this node, we first construct the *q*-measure transition probability tree:

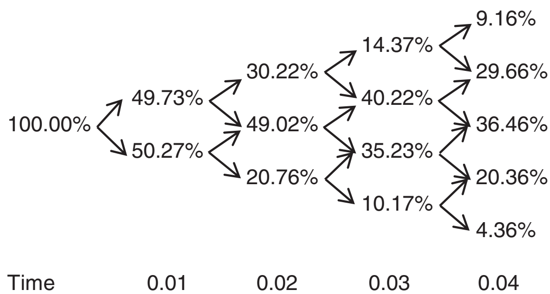
Risk-Neutral Transition Probabilities



The uppermost terminal node can be reached only by moving up in each period. The cumulative probability of reaching the uppermost node is then 49.73% × 60.78% × 47.56% × 63.72% = 9.16%.

We don't need the cumulative probabilities of reaching the other nodes, but calculating these other probabilities can serve as a useful check. For each level the probabilities must add up to 100%. The cumulative probabilities for all of the nodes on the tree are:

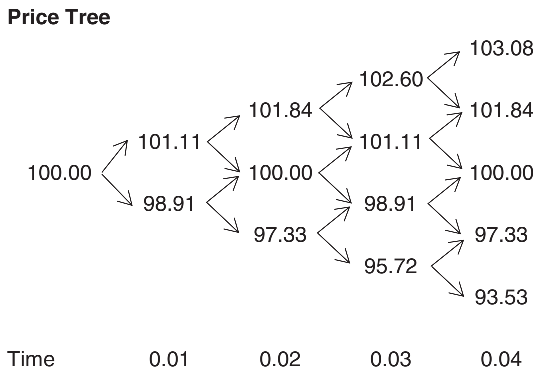
Risk-Neutral Cumulative Probabilities



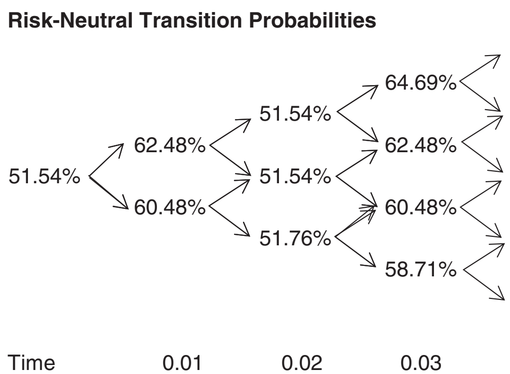
Because the riskless rate is zero, the present value and future value of the option are the same, equal to 9.16% × \$1.05 = \$0.10. This is the same price that we obtained in the sample problem. Even though the price of the uppermost node is less than in the sample problem, the probability of reaching that node is higher. If you calculate to a few more decimal places, you will see that the price of the option in this problem is actually slightly less than in the sample problem ($0.0966 versus $0.1009). The average local volatility between the current price and the strike price is the same in this problem as in the previous problem, 9%, but the rate of change is twice as fast.

14-3. In the Cox-Ross-Rubinstein convention the central spine of the tree remains same, but the riskless rate does affect the outer nodes and the transition probabilities.

The price tree is now



The risk-neutral transition probabilities are



As in the previous problem, at expiration only the uppermost price node, $103.05, will be in the money. The probability of reaching this node is now 51.54% × 62.48% × 51.54% × 64.69% = 10.74%.

The value of the option is then the discounted probability-weighted value of the final uppermost node:

$$C = e^{-4\times 0.01\times 0.04} \times 10.74\% \times (103.08 - 102.00) = \$0.12$$

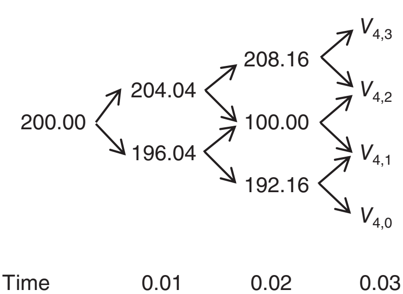
Compared to the previous problem, the value of the call option is higher. Because of the positive riskless rate, the forward prices are greater than before, the expected value of the stock price drifts up over time and the probability of getting to that uppermost node is therefore higher. This is consistent with the fact that increasing the riskless rate increases the value of a call option because the short position in the bond that replicates it is worth less.

14-4. For the first two levels,

$$u = 0.20 \times 0.10 = 0.02$$
$$d = -u = -0.02$$

At the second level, the up price is \$200 × e^u = \$204.04, and the down price is \$200 × e^d = \$196.04. Continuing in this fashion, the first three levels are:

Price Tree



We now need to find the maximum local volatility σ_M for the center node of the third level. As we increase σ_M, $V_{4,2}$ will increase and $V_{4,1}$ will decrease. $V_{4,2}$ cannot be greater than the uppermost node of the third level, \$208.16 = \$200 × e^{2u}. If it were, the node with value \$208.16 would make a transition to two nodes that both have higher prices, which would allow a riskless arbitrage. Similarly, $V_{4,1}$ cannot be lower than the lowermost node of the third level, \$192.16 = \$200 $\times\, e^{2d}$. We have:

$$V_{4,2} = \$200 e^{\sigma_M \sqrt{0.01}} \leq \$200 \times e^{2u} = \$208.16$$
$$V_{4,1} = \$200 e^{-\sigma_M \sqrt{0.01}} \geq \$200 \times e^{2d} = \$192.16$$

Thus,

$$\sigma_M \sqrt{0.01} \leq 2u$$
$$-\sigma_M \sqrt{0.01} \geq 2d$$

Substituting our initial values for u and d, we see that both constraints reduce to

$$\sigma_M \leq 2 \times 20\% = 40\%$$

The maximum local volatility for the center node of the third level is 40%.

CHAPTER 15

15-1. The calendar spread is long a \$1,000 strike call with 1.01 years to expiration and short a \$1,000 strike call with one year to expiration. The butterfly contains three calls, all with one year until expiration: long one call with a strike of \$1,010, short two calls with strikes at \$1,000, and long one call with a strike at \$990. The BSM prices for the options are:

S	K	τ	$\Sigma(K)$	d_1	d_2	$C(K, \tau)$
1,000	990	1.00	10.10%	0.1500	0.0490	45.27
1,000	1,000	1.00	10.00%	0.0500	–0.0500	39.88
1,000	1,010	1.00	9.90%	–0.0510	–0.1500	34.88
1,000	1,000	1.01	10.00%	0.0502	–0.0502	40.08

The prices of the butterfly and calendar spreads are:

$$\text{Butterfly} = \$45.28 - 2 \times \$39.88 + \$34.88 = \$0.40$$
$$\text{Calendar} = \$40.08 - \$39.88 = \$0.20$$

Next we approximate the derivatives needed for Dupire's equation:

$$\frac{\partial C(S,t,K,T)}{\partial T} \approx \frac{\text{Calendar}}{dT} = \frac{\$0.20}{0.01} = 19.87$$

$$\frac{\partial^2 C(S,t,K,T)}{\partial K^2} \approx \frac{\text{Butterfly}}{dK^2} = \frac{\$0.40}{(\$10)^2} = 0.0040$$

From Dupire's equation we therefore have

$$\sigma^2(K,T) = \frac{2\dfrac{\partial C(S,t,K,T)}{\partial T}}{K^2\dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

$$\sigma^2(1{,}000{,}1) = \frac{2 \times 19.87}{1000^2 \times 0.0040}$$

$$= 0.0100$$

The local volatility, to the accuracy we are computing, is simply the square root of this or 10%. Notice that the at-the-money local volatility is almost exactly equal to the at-the-money implied volatility, 10%. This is typical when implied volatility varies only with the strike.

15-2. The relevant call prices are:

S	K	τ	$\Sigma(K)$	d_1	d_2	$C(K,\tau)$
1,000	890	1.00	11.16%	1.0998	0.9881	118.06
1,000	900	1.00	11.05%	1.0086	0.8981	109.53
1,000	910	1.00	10.94%	0.9166	0.8072	101.22
1,000	900	1.01	11.05%	1.0041	0.8931	109.66

The prices of the butterfly and calendar spreads are:

$$\text{Butterfly} = \$118.06 - 2 \times \$109.53 + \$101.22 = \$0.22$$

$$\text{Calendar} = \$109.66 - \$109.53 = \$0.13$$

Next we approximate the derivatives needed for Dupire's equation to obtain

$$\frac{\partial C(S,t,K,T)}{\partial T} \approx \frac{\text{Calendar}}{dT} = \frac{\$0.13}{0.01} = 13.25$$

$$\frac{\partial^2 C(S,t,K,T)}{\partial K^2} \approx \frac{\text{Butterfly}}{dK^2} = \frac{\$0.22}{(\$10)^2} = 0.0022$$

From Dupire's equation we therefore have

$$\sigma^2(K,T) = \frac{2\dfrac{\partial C(S,t,K,T)}{\partial T}}{K^2\dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

$$\sigma^2(900,1) = \frac{2 \times 1.325}{900^2 \times 0.0022}$$
$$= 0.0149$$

The local volatility is the square root of this, 12.2%. Notice that while the at-the-money local volatility and implied volatility were equal, the local volatility at a stock price of \$900, 12.2%, is significantly higher than the implied volatility at a strike of \$900, namely 11.1%. Intuitively, because the implied volatility is an average of the local volatility, the local volatility must change more quickly than the implied volatility. In this case, the local volatility has changed almost twice as quickly. This is another example of the rule of two: 11.1% is the linear average of 10% and 12.2%.

15-3. The relevant call prices at one year near the strike price of \$1,000 are:

S	K	τ	$\Sigma(K)$	d_1	d_2	$C(K,\tau)$
1,000	990	1.00	15.15%	0.1421	−0.0094	65.21
1,000	1,000	1.00	15.00%	0.0750	−0.0750	59.79
1,000	1,010	1.00	14.85%	0.0073	−0.1413	54.62
1,000	1,000	1.01	15.05%	0.0756	−0.0756	60.28

The prices of the butterfly and calendar spreads are:

$$\text{Butterfly} = \$54.62 - 2 \times \$59.79 + \$65.22 = \$0.27$$
$$\text{Calendar} = \$60.28 - \$59.79 = \$0.50$$

As before, we approximate the derivatives needed for Dupire's equation by

$$\frac{\partial C(S,t,K,T)}{\partial T} \approx \frac{\text{Calendar}}{dT} = \frac{\$0.50}{0.01} = 49.75$$
$$\frac{\partial^2 C(S,t,K,T)}{\partial K^2} \approx \frac{\text{Butterfly}}{dK^2} = \frac{\$0.27}{(\$10)^2} = 0.0027$$

From Dupire's equation, we have

$$\sigma^2(K,T) = \frac{2\dfrac{\partial C(S,t,K,T)}{\partial T}}{K^2\dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

$$\sigma^2(1000,1) = \frac{2\times 49.75}{1000^2\times 0.0027}$$
$$= 0.0375$$

The local volatility at one year and \$1,000 is the square root of this, 19.4%.

The relevant call prices at one year near a strike price of \$900 are:

S	K	τ	$\Sigma(K)$	d_1	d_2	$C(K,\tau)$
1,000	890	1.00	16.74%	0.7797	0.6122	132.68
1,000	900	1.00	16.58%	0.7184	0.5527	124.98
1,000	910	1.00	16.41%	0.6567	0.4926	117.47
1,000	900	1.01	16.63%	0.7139	0.5467	125.41

The prices of the butterfly and calendar spreads are:

$$\text{Butterfly} = \$117.47 - 2\times\$12.50 + \$13.27 = \$0.19$$
$$\text{Calendar} = \$125.41 - \$124.979 = \$0.43$$

As before, we approximate the derivatives needed for Dupire's equation by

$$\frac{\partial C(S,t,K,T)}{\partial T} \approx \frac{\text{Calendar}}{dT} = \frac{\$0.427}{0.01} = 42.67$$
$$\frac{\partial^2 C(S,t,K,T)}{\partial K^2} \approx \frac{\text{Butterfly}}{dK^2} = \frac{\$0.19}{(\$10)^2} = 0.0019$$

From Dupire's equation, we have

$$\sigma^2(K,T) = \frac{2\dfrac{\partial C(S,t,K,T)}{\partial T}}{K^2\dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

$$\sigma^2(900,1) = \frac{2\times 42.67}{900^2\times 0.0019}$$
$$= 0.0562$$

The local volatility at one year and \$900 is the square root of this, 23.7%.

We can regard the implied volatility at a strike of \$1,000 as the approximate average of the local volatilities across time at a stock price of \$1,000. At zero time to expiration and $S = K =$ \$1,000, the implied volatility and local volatility are both 10%. At an expiration of one year, the implied volatility for $K =$ \$1,000 is 15%. Regarding 15% as an average of the local volatility at $\tau = 0$, $S = 1{,}000$, and $\tau = 1$, $S = 1{,}000$, we see that the local volatility at $\tau = 1$ must be about 20%, not far from the more precise value of 19.4% found earlier.

Similarly, we can still regard the implied volatility at $\tau = 1$ and $K =$ \$900 as approximately the average of (1) the local volatility of 10% at $\tau = 0$ and $S =$ \$1,000, and (2) the local volatility of 23.7% at $\tau = 1$ and $S =$ \$900, since this is the path the stock has to take from its initial value to the terminal strike price. This average, (10% + 23.7%)/2 = 16.8%, is our guesstimate for the implied volatility of a one-year option struck at \$900, which, from the formula for implied volatility in Problem 15-3, is 16.6%, impressively close to our intuitive estimate. Again, we see that the implied volatility is well approximated by the average of local volatilities over the path between the underlying price at inception and strike at expiration.

15-4. In the case where $S_{T_1} \leq Ke^{r(T_1-t)}$, the first leg expires worthless, so the value of the calendar spread at time T_1 is equal to the value of the second leg, that is, the value of a single call option, which must be greater than or equal to zero.

Next consider the value of the calendar spread $V(t, T_1, T_2)$ at time $t = T_1$, when $S_{T_1} > Ke^{r(T_1-t)}$. The first-leg option is in-the-money and not worthless. Then,

$$V(T_1, T_1, T_2) = C\left(S_{T_1}, T_1, Ke^{r(T_2-t)}, T_2\right) - C\left(S_{T_1}, T_1, Ke^{r(T_1-t)}, T_1\right)$$

We now use the fact that a call is always worth at least as much as a forward with the same strike to show that $V(T_1, T_1, T_2)$ is always greater than or equal to zero, and hence that $V(t, T_1, T_2) \geq 0$.

Because the first call expires in-the-money with $S_{T_1} \geq Ke^{r(T_1-t)}$,

$$V(T_1, T_1, T_2) = C\left(S_{T_1}, T_1, Ke^{r(T_2-t)}, T_2\right) - \left(S_{T_1} - Ke^{r(T_1-t)}\right)$$

Because a call is always worth more than a forward with the same delivery price,

$$\begin{aligned} C\left(S_{T_1}, T_1, Ke^{r(T_2-t)}, T_2\right) &\geq S_{T_1} - e^{-r(T_2-T_1)}\left(Ke^{r(T_2-t)}\right) \\ &\geq S_{T_1} - Ke^{r(T_1-t)} \end{aligned}$$

Thus,

$$V(T_1, T_1, T_2) \geq \left(S_{T_1} - Ke^{r(T_1-t)}\right) - \left(S_{T_1} - Ke^{r(T_1-t)}\right)$$
$$\geq 0$$

When the first leg expires, therefore, the value of the calendar spread is always greater than or equal to zero. Therefore, at any time earlier, the same must be true, and $V(t, T_1, T_2)$ must be greater than or equal to zero, or else there would be an arbitrage.

15-5. We begin by expressing the price of the calendar spread in terms of the BSM implied volatility:

$$V(t, T, T + dT) = C\left(S, t, Ke^{r(T+dT-t)}, T + dT\right) - C(S, t, Ke^{r(T-t)}, T)$$

Notice that in the limit $dT \to 0$, the right-hand side of this equation is related to the total derivative with respect to T, so that

$$\begin{aligned} V(t, T, T + dT) &= \frac{d}{dT}[C(S, t, Ke^{r(T-t)}, T)]dT \\ &= \frac{d}{dT}[C_{\mathrm{BSM}}(S, t, Ke^{r(T-t)}, T, r, \Sigma(S, t, Ke^{r(T-t)}, T))]dT \end{aligned}$$

It is easy to show by substitution in the BSM formula that

$$\begin{aligned} C_{\mathrm{BSM}}\left(S, t, Ke^{r(T-t)}, T, r, \Sigma\left(S, t, Ke^{r(T-t)}, T\right)\right) &= f(S, K, v) \\ &= SN(d_1) - KN(d_2) \end{aligned}$$

where, as we have defined v in this problem,

$$d_{1,2} = \frac{1}{\sqrt{v}} \ln\left(\frac{S}{K}\right) \pm \frac{\sqrt{v}}{2}$$

and $v = (T - t)\Sigma^2$ is the total variance to the forward strike as defined in the question. By the chain rule, since all the T-dependence of C_{BSM} is in the single variable v,

$$\frac{dC_{\mathrm{BSM}}}{dT} = \frac{\partial}{\partial v} f(S, K, v) \frac{\partial v}{\partial T}$$

The first term on the right-hand side is simply proportional to the BSM vega, which we know to be positive from previous chapters; therefore, the requirement that $V(t, T, T + dT) \geq 0$ is equivalent to the requirement that $\partial v/\partial T \geq 0$.

CHAPTER 16

16-1. First we use Equation 16.3 to estimate the implied volatility for the option:

$$\begin{aligned}\Sigma(S,K) &= \sigma_0 + 2\beta S_0 - \beta(S+K)\\ &= 0.25 + 2\times 0.00005 \times 4000 - 0.00005(4000+4200)\\ &= 0.25 + 0.00005(8000-8200)\\ &= 0.25 - 0.00005(200)\\ &= 0.25 - 0.01\\ &= 0.24\end{aligned}$$

To calculate the BSM delta and vega, we first need to calculate d_1:

$$\begin{aligned}v &= \sigma\sqrt{\tau} = 0.24\sqrt{1} = 0.24\\ d_1 &= \frac{1}{v}\ln\left(\frac{S}{K}\right) + \frac{v}{2}\\ &= \frac{1}{0.24}\ln\left(\frac{4000}{4200}\right) + \frac{0.24}{2}\\ &= -0.0833\end{aligned}$$

The BSM Greeks are then given by

$$\begin{aligned}\Delta_{\mathrm{BSM}} &= N(d_1) = 0.47\\ V_{\mathrm{BSM}} &= \frac{S\sqrt{\tau}}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2} = 1590\end{aligned}$$

Substituting into Equation 16.6,

$$\begin{aligned}\Delta &\approx \Delta_{\mathrm{BSM}} - V_{\mathrm{BSM}}\beta\\ &\approx 0.47 - 1590\times 0.00005\\ &\approx 0.39\end{aligned}$$

The correct hedge ratio is approximately 0.39. This is considerably lower than the BSM value of 0.47.

16-2. There are two ways we can approach this problem. The first is to go through the same calculations as in the previous problem. Because the strike of the put is the same as the strike of the call in the previous problem, the correct implied volatility is still the same, 24%. The BSM vega of a call and put with the same expiration and same strike are also equal, so the BSM vega of the put is also 1,590. The only difference is the BSM delta. For a put,

$$\Delta_{\text{BSM}} = -N(-d_1) = -0.53$$

Substituting this into Equation 16.6, which is valid for both calls and puts, we have

$$\begin{aligned}\Delta &\approx \Delta_{\text{BSM}} - V_{\text{BSM}}\beta \\ &\approx -0.53 - 1590 \times 0.00005 \\ &\approx -0.61\end{aligned}$$

The correct hedge ratio is approximately −0.61.

The other way that we could have approached this is to use put-call parity, which must hold independently of any model, so that

$$\begin{aligned}&C - P = S - Ke^{-r\tau} \\ &\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = 1 \\ &\Delta_C - \Delta_P = 1\end{aligned}$$

Using the local volatility call delta from the previous problem, 0.39, we have

$$\begin{aligned}\Delta_P &= \Delta_C - 1 \\ &= 0.39 - 1 \\ &= -0.61\end{aligned}$$

This is exactly the same answer that we arrived at before. Our adjustment equation, Equation 16.6, preserves this relationship because V_{BSM} is the same for both the call and the put.

CHAPTER 18

18-1. We can find the current at-the-money implied volatility as follows:

$$\begin{aligned}\Sigma(K) &= 0.25 - 0.00005(K - 4000) \\ &= 0.25 - 0.00005(4000 - 4000) \\ &= 0.25\end{aligned}$$

If the level of the NDX increased by 10%, the level would be 4,400, and the at-the-money implied volatility would be

$$\begin{aligned}\Sigma(K) &= 0.25 - 0.00005(4400 - 4000)\\ &= 0.25 - 0.00005(400)\\ &= 0.25 - 0.02\\ &= 0.23\end{aligned}$$

Similarly, if the level of the NDX decreased by 10% to 3,600:

$$\begin{aligned}\Sigma(K) &= 0.25 - 0.00005(3600 - 4000)\\ &= 0.25 - 0.00005(-400)\\ &= 0.25 + 0.02\\ &= 0.27\end{aligned}$$

Notice that with the sticky strike rule, as specified, we did not need to know the time to expiration or the index level in order to calculate the implied volatility.

18-2. We start by solving the given equation for Σ_{ATM} when $S = K = 1{,}000$:

$$\begin{aligned}\Sigma_{\text{ATM}} &= 0.18 - 0.02\frac{\ln\left(\frac{K}{S}\right)}{\Sigma\sqrt{\tau}}\\ &= 0.18\end{aligned}$$

Σ_{ATM} is 18% when $S = K = 1{,}000$ for options with both one-year and with three-months to expiration.

For other strikes, we need to solve for Σ using the quadratic equation, as follows:

$$\Sigma = 0.18 - 0.02\frac{\ln\left(\frac{K}{S}\right)}{\Sigma\sqrt{\tau}}$$

$$\Sigma^2 = 0.18\Sigma - 0.02\frac{\ln\left(\frac{K}{S}\right)}{\sqrt{\tau}}$$

$$\Sigma^2 - 0.18\Sigma + 0.02\frac{\ln\left(\frac{K}{S}\right)}{\sqrt{\tau}} = 0$$

so that

$$\Sigma = \frac{0.18 \pm \sqrt{0.18^2 - 4 \times 0.02 \dfrac{\ln\left(\frac{K}{S}\right)}{\sqrt{\tau}}}}{2}$$
$$= 0.09 \pm \frac{1}{2}\sqrt{0.18^2 - 0.08 \frac{\ln\left(\frac{K}{S}\right)}{\sqrt{\tau}}}$$

For a strike of 900 with one year to expiration,

$$\Sigma = 0.09 + \frac{1}{2}\sqrt{0.18^2 - 0.08 \frac{\ln\left(\frac{900}{1000}\right)}{\sqrt{1}}}$$
$$= 0.09 + 0.1010$$
$$= 0.1910$$

where only the positive square root in the quadratic equation solution gives a positive volatility.

Similarly, for a strike of 900 with three months to expiration,

$$\Sigma = 0.09 + \frac{1}{2}\sqrt{0.18^2 - 0.08 \frac{\ln\left(\frac{900}{1000}\right)}{\sqrt{0.25}}}$$
$$= 0.09 + 0.1110$$
$$= 0.2010$$

Notice that for the sticky delta rule, as specified here, implied volatility increases more quickly at shorter expirations for an equal point drop in the strike (but correspondingly greater number of standard deviations).

18-3. The at-the-money equation is still the same, only now it applies to 900 strike options, not 1,000 strike options. The implied volatility for options with strikes of 900 is 18%, irrespective of time to expiration.

For a strike of 1,000 with one year to expiration,

$$\begin{aligned}\Sigma &= 0.09 + \frac{1}{2}\sqrt{0.18 - 0.08\frac{\ln\left(\frac{1000}{900}\right)}{\sqrt{1}}} \\ &= 0.09 + 0.0774 \\ &= 0.1674\end{aligned}$$

For a strike of 1,000 with three months to expiration,

$$\begin{aligned}\Sigma &= 0.09 + \frac{1}{2}\sqrt{0.18^2 - 0.08\frac{\ln\left(\frac{1000}{900}\right)}{\sqrt{0.25}}} \\ &= 0.09 + 0.0623 \\ &= 0.1523\end{aligned}$$

The following table summarizes the results of this problem and the preceding problem:

		Σ	
K	τ	$S = 1{,}000$	$S = 900$
1,000	1.00	18.00%	16.74%
900	1.00	19.10%	18.00%
1,000	0.25	18.00%	15.23%
900	0.25	20.10%	18.00%

Contrary to what we might expect, the implied volatility surface shifted down, not up, when the index dropped. This is a feature of the sticky delta rule with a negative skew.

CHAPTER 19

19-1. We start by rewriting the time series equation as

$$\sigma_{t+1} = \sigma_t + 0.4(20\% - \sigma_t) + \varepsilon_t$$

With $\sigma_0 = 16\%$, $\varepsilon_0 = +3\%$, and $\varepsilon_1 = -3\%$,

$$\begin{aligned}\sigma_1 &= \sigma_0 + 0.4(20\% - \sigma_0) + \varepsilon_0 \\ &= 16\% + 0.4(20\% - 16\%) + 3\% \\ &= 16\% + 1.6\% + 3\% \\ &= 20.6\%\end{aligned}$$

We then feed this value back into the time series equation to get

$$\begin{aligned}\sigma_2 &= \sigma_1 + 0.4(20\% - \sigma_1) + \varepsilon_1 \\ &= 20.6\% + 0.4(20\% - 20.6\%) - 3\% \\ &= 20.6\% - 0.24\% - 3\% \\ &= 17.36\%\end{aligned}$$

Interestingly, in this problem the first shock causes volatility to overshoot the long-run mean.

With the shocks reversed, we have

$$\begin{aligned}\sigma_1 &= \sigma_0 + 0.4(20\% - \sigma_0) + \varepsilon_0 \\ &= 16\% + 0.4(20\% - 16\%) - 3\% \\ &= 16\% + 1.6\% - 3\% \\ &= 14.6\%\end{aligned}$$

and

$$\begin{aligned}\sigma_2 &= \sigma_1 + 0.4(20\% - \sigma_1) + \varepsilon_1 \\ &= 14.6\% + 0.4(20\% - 14.6\%) + 3\% \\ &= 14.6\% + 2.16\% + 3\% \\ &= 19.76\%\end{aligned}$$

With the shocks reversed, we first move away from the mean, and then back toward it. In the sample problem we saw that symmetric shocks did not necessarily produce the same outcome as no shocks. Comparing the results of this problem to the previous problem, we see that the order of the shocks can also affect the final volatility.

19-2. With a mean-reversion parameter of 0.1, in the first case with $\varepsilon_0 = +3\%$, and $\varepsilon_1 = -3\%$, we have

$$\begin{aligned}\sigma_1 &= \sigma_0 + 0.1(20\% - \sigma_0) + \varepsilon_0\\ &= 16\% + 0.1(20\% - 16\%) + 3\%\\ &= 16\% + 0.4\% + 3\%\\ &= 19.4\%\end{aligned}$$

and, feeding this value back into the time-series equation,

$$\begin{aligned}\sigma_2 &= \sigma_1 + 0.1(20\% - \sigma_1) + \varepsilon_1\\ &= 19.4\% + 0.1(20\% - 19.4\%) - 3\%\\ &= 19.4\% + 0.06\% - 3\%\\ &= 16.46\%\end{aligned}$$

With the shocks reversed, with $\varepsilon_0 = -3\%$, and $\varepsilon_1 = +3\%$, we have

$$\begin{aligned}\sigma_1 &= \sigma_0 + 0.1(20\% - \sigma_0) + \varepsilon_0\\ &= 16\% + 0.1(20\% - 16\%) - 3\%\\ &= 16\% + 0.4\% - 3\%\\ &= 13.4\%\end{aligned}$$

and

$$\begin{aligned}\sigma_2 &= \sigma_1 + 0.1(20\% - \sigma_1) + \varepsilon_1\\ &= 13.4\% + 0.1(20\% - 13.4\%) + 3\%\\ &= 13.4\% + 0.66\% + 3\%\\ &= 17.06\%\end{aligned}$$

As in the previous problem, the order of the shocks matters. In contrast to the previous problem, because the mean-reversion parameter is lower (0.1 compared to 0.4), volatility does not move as far toward its long-run mean of 20%, and stays closer to its initial value of 16%.

19-3. Our half-life formula Equation 19.14 is

$$t = \frac{1}{\alpha}\ln(2)$$

When $\alpha = 0.4$,

$$\begin{aligned} t(0.4) &= \frac{1}{0.4}\ln(2) \\ &= 2.5 \times 0.69 \\ &= 1.73 \end{aligned}$$

When $\alpha = 0.1$,

$$\begin{aligned} t(0.1) &= \frac{1}{0.1}\ln(2) \\ &= 10.0 \times 0.69 \\ &= 6.93 \end{aligned}$$

The half-lives are 1.73 and 6.93 periods, respectively. Dividing α by 4 leads to a quadrupling of the half-life.

In the mean reversion sample problem with $\alpha = 0.4$ and a long-run mean of 20%, when there were no shocks volatility moved from 24.00% to 22.40% and then to 21.44%. It passed the halfway mark, 22% = 0.5(24% − 20%), somewhere between the first and second steps, consistent with a half-life of 1.73.

19-4. Assuming no dividends and a zero riskless rate, the price of a European call option is

$$\begin{aligned} C(S, K, \sigma, \tau) &= SN(d_1) - KN(d_2) \\ d_{1,2} &= \frac{1}{v}\ln\left(\frac{S}{K}\right) \pm \frac{v}{2} \end{aligned}$$

where $v = \sigma\sqrt{\tau}$.

Based on the current level (2,000) and volatility (20%) of the SPX, the price of a European call with a strike of 2,000 and 1.01 years to expiration is

$$\begin{aligned} C(2000, 2000, 0.20, 1.01) &= 2000N(0.1005) - 2000N(-0.1005) \\ &= 160.10 \end{aligned}$$

At the four nodes after the first time step:

$$\begin{aligned}C(1900, 2000, 0.15, 1.00) &= 1900N(-0.2670) - 2000N(-0.4170)\\ &= 73.32\\ C(2100, 2000, 0.15, 1.00) &= 2100N(0.4003) - 2000N(0.2503)\\ &= 178.97\\ C(1900, 2000, 0.25, 1.00) &= 1900N(-0.0802) - 2000N(-0.3302)\\ &= 148.03\\ C(2100, 2000, 0.25, 1.00) &= 2100N(0.3202) - 2000N(0.0702)\\ &= 257.78\end{aligned}$$

Because the riskless rate is zero, we calculate the present value of the option as the weighted average of these four possible outcomes. The value of the call in our quadrinomial model is then

$$\begin{aligned}&0.1 \times \$73.32 + 0.4 \times \$178.97 + 0.4 \times \$148.03 + 0.1 \times \$257.78\\ &\quad = \$163.91\end{aligned}$$

This is greater than the BSM value without stochastic volatility.

CHAPTER 20

20-1. We start by rewriting Equation 20.4 assuming it is exactly rather than approximately true:

$$\begin{aligned}\Sigma &= \alpha S^{\beta-1}\left[1 + \frac{(\beta-1)}{2}\ln\left(\frac{K}{S}\right)\right]\\ &= \alpha S^{\beta-1} + \frac{\alpha}{2}(\beta-1)S^{\beta-1}\ln(K) - \frac{\alpha}{2}(\beta-1)S^{\beta-1}\ln(S)\end{aligned}$$

Now take the derivative with respect to K:

$$\frac{\partial\Sigma}{\partial K} = \frac{\alpha}{2}(\beta-1)S^{\beta-1}\frac{1}{K}$$

For at-the-money options, $K = S$, so

$$\frac{\partial\Sigma}{\partial K} = \frac{\alpha}{2}(\beta-1)S^{\beta-2}$$

Now take the derivative with respect to S:

$$\begin{aligned}\frac{\partial \Sigma}{\partial S} &= \alpha(\beta-1)S^{\beta-2} + \frac{\alpha}{2}(\beta-1)^2 S^{\beta-2}\ln(K) \\ &\quad - \frac{\alpha}{2}(\beta-1)^2 S^{\beta-2}\ln(S) - \frac{\alpha}{2}(\beta-1)S^{\beta-2} \\ &= \frac{\alpha}{2}(\beta-1)S^{\beta-2} + \frac{\alpha}{2}(\beta-1)^2 S^{\beta-2}\ln\left(\frac{K}{S}\right)\end{aligned}$$

For at-the-money options, when $K = S$, the last term is zero, and

$$\frac{\partial \Sigma}{\partial S} = \frac{\alpha}{2}(\beta-1)S^{\beta-2}$$

This is the same as the derivative with respect to K, which completes our proof. For at-the-money options, given Equation 20.4,

$$\frac{\partial \Sigma}{\partial K} = \frac{\partial \Sigma}{\partial S} = \frac{\alpha}{2}(\beta-1)S^{\beta-2}$$

CHAPTER 21

21-1. Equation 21.19, reproduced here, is

$$\Sigma \approx \bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\frac{1}{\bar{\bar{\sigma}}^2\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\bar{\bar{\sigma}}^2\tau}{4}\right]$$

Initially, $S = 4{,}000$, $K = 4{,}000$, $\bar{\bar{\sigma}} = 20\%$, and $\tau = 0.5$. Because the standard deviation of path volatility is 16 volatility points, $\mathrm{var}[\bar{\sigma}] = 0.16^2 = 0.0256$. The initial at-the-money volatility is then

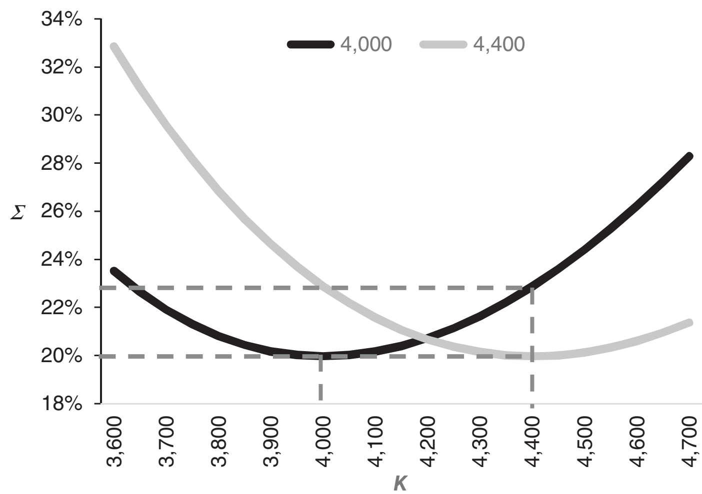
$$\begin{aligned}\Sigma &\approx 0.2 + \frac{1}{2}\times 0.0256 \times \frac{1}{0.2}\left[\frac{1}{0.2^2\times 0.5}\left(\ln\left(\frac{4000}{4000}\right)\right)^2\right. \\ &\quad \left. - \frac{0.2^2\times 0.5}{4}\right] \\ &\approx 0.2 + \frac{1}{2}\times 0.0256\times\frac{1}{0.2}\left[-\frac{0.2^2\times 0.5}{4}\right] \\ &\approx 0.2 - \frac{1}{8}\times 0.0256\times 0.1 \\ &\approx 0.2 - 0.0003 \\ &\approx 0.1997\end{aligned}$$

When the NDX increases to 4,400, six-month volatility for 4,000 strike options becomes

$$\begin{aligned}\Sigma &\approx 0.2 + \frac{1}{2} \times 0.0256 \times \frac{1}{0.2}\left[\frac{1}{0.2^2 \times 0.5}\left(\ln\left(\frac{4400}{4000}\right)\right)^2 \right. \\ &\quad \left. - \frac{0.2^2 \times 0.5}{4}\right] \\ &\approx 0.2 + 0.064[0.4542 - 0.005] \\ &\approx 0.2 + 0.0287 \\ &\approx 0.2287\end{aligned}$$

For the new six-month at-the-money implied volatility, S/K is the same as in the first part of the problem, and because all of the other parameters are the same, the at-the-money volatility is 19.97%, the same as before. This is an example of sticky moneyness.

The before and after smiles are shown in the following chart:



CHAPTER 22

22-1. When the riskless rate and dividends are zero, the Black-Scholes-Merton (BSM) hedge ratio is

$$\Delta_{\mathrm{BSM}} = N(d_1)$$

At-the-money, when $S = K$,

$$\begin{aligned} d_1 &= \frac{1}{\Sigma\sqrt{\tau}} \ln\left(\frac{S}{K}\right) + \frac{1}{2}\Sigma\sqrt{\tau} \\ &= \frac{1}{2} \times 0.16 \times \sqrt{1} \\ &= 0.08 \end{aligned}$$

The BSM hedge ratio is then $N(0.08) = 0.53$.

Under stochastic volatility, the approximate best stock-only hedge is given by Equation 22.12:

$$\Delta = \Delta_{\mathrm{BSM}} + \rho \frac{V_{\mathrm{BS}} q}{\Sigma S}$$

When the riskless rate and dividends are zero,

$$V_{\mathrm{BSM}} = \frac{\partial C_{\mathrm{BSM}}}{\partial \sigma} = \frac{S\sqrt{\tau}}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2}$$

The volatility process specifies $q = 0.25$, and $\rho = -40\%$. Therefore,

$$\begin{aligned} \Delta &= \Delta_{\mathrm{BSM}} + \rho \frac{\sqrt{\tau}}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \frac{q}{\Sigma} \\ &= 0.53 - 0.40 \frac{\sqrt{1}}{\sqrt{2\pi}} e^{-\frac{1}{2}0.08^2} \frac{0.25}{0.16} \\ &= 0.53 - \frac{5}{8} \frac{1}{\sqrt{2\pi}} e^{-0.0032} \\ &= 0.53 - 0.25 \\ &= 0.28 \end{aligned}$$

The best stock-only hedge ratio, 0.28, is significantly lower than the BSM hedge ratio, 0.53, because volatility is stochastic, and the index level and implied volatility are negatively correlated.

22-2. This problem is similar to the sample problem. Notice that the equation for the variance of the path volatility, $\mathrm{var}[\bar{\sigma}]$, behaves as we would expect if instantaneous volatility was mean reverting. When τ is small, it is proportional to τ. When τ is large, it is inversely proportional to τ. For 0. 1-, 0.25-, and 1-year expirations,

$$\mathrm{var}[\bar{\sigma}]_{0.10} = e^{-4\times0.10}0.08\times0.10 + (1-e^{-4\times0.10})\frac{0.02}{0.10} = 0.0713$$
$$\mathrm{var}[\bar{\sigma}]_{0.25} = e^{-4\times0.25}0.08\times0.25 + (1-e^{-4\times0.25})\frac{0.02}{0.25} = 0.0579$$
$$\mathrm{var}[\bar{\sigma}]_{1.00} = e^{-4\times1.00}0.08\times1.00 + (1-e^{-4\times1.00})\frac{0.02}{1.00} = 0.0211$$

The 10% out-of-the-money put corresponds to $\ln(K/S) = -10\%$, or $\ln(S/K) = 10\%$. Using Equation 22.2,

$$\begin{aligned}
\text{Skew} &= \Sigma_{10\%} - \Sigma_{\text{ATM}} \\
&\approx \left(\bar{\bar{\sigma}} + \mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\frac{1}{\bar{\bar{\sigma}}^2\tau}(0.10)^2 - \frac{\bar{\sigma}^2\tau}{4}\right]\right) \\
&\quad - \left(\bar{\bar{\sigma}} - \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\frac{\bar{\sigma}^2\tau}{4}\right]\right) \\
&\approx \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\frac{1}{\bar{\bar{\sigma}}^2\tau}(0.10)^2\right] \\
&\approx \frac{1}{200}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}^3\tau}
\end{aligned}$$

Since $\bar{\bar{\sigma}} = 25\%$,

$$\begin{aligned}
\text{Skew} &\approx \frac{1}{200}\mathrm{var}[\bar{\sigma}]\frac{1}{0.25^3\tau} \\
&\approx \frac{8}{25}\mathrm{var}[\bar{\sigma}]\frac{1}{\tau}
\end{aligned}$$

For the three expirations, then,

$$\text{Skew}_{0.10} \approx \frac{8}{25}\times0.0713\times\frac{1}{0.10} = 0.23 = 23 \text{ volatility points}$$
$$\text{Skew}_{0.25} \approx \frac{8}{25}\times0.0579\times\frac{1}{0.25} = 0.07 = 7 \text{ volatility points}$$
$$\text{Skew}_{1.00} \approx \frac{8}{25}\times0.0211\times\frac{1}{1.00} = 0.01 = 1 \text{ volatility point}$$

In other words, for a 10% drop in strike, implied volatility increases significantly for short expirations, modestly for three-month expirations, and barely at all for one-year expirations.

Notice that the current level of SX5E never entered into the calculation, because of the sticky moneyness. The skew becomes less steep as the time to expiration increases, but, in this model, changes in the level of the index have no impact on the 90–100 strike skew.

CHAPTER 23

23-1. To calculate the probability of two or more jumps, it is tempting to think that we need to calculate the probability of two jumps, three jumps, four jumps, … up to infinity jumps, and add up all of the values. The easier way to calculate the probability of two or more jumps is to realize that there will be either 0 or 1 jumps next year, or two or more:

$$\begin{aligned} \mathrm{P}[n = 0] + \mathrm{P}[n = 1] + \mathrm{P}[n \geq 2] &= 1 \\ \mathrm{P}[n \geq 2] &= 1 - \mathrm{P}[n = 0] - \mathrm{P}[n = 1] \end{aligned}$$

Using Equation 23.12 with $\lambda = 5$/year and $T = 1$ year,

$$\begin{aligned} \mathrm{P}(n, T) &= \frac{(\lambda T)^n}{n!} e^{-\lambda T} \\ &= \frac{(5)^n}{n!} e^{-5} \end{aligned}$$

Then,

$$\begin{aligned} \mathrm{P}(n \geq 2,1) &= 1 - \mathrm{P}(0,1) - \mathrm{P}(1,1) \\ &= 1 - \frac{(5)^0}{0!} e^{-5} - \frac{(5)^1}{1!} e^{-5} \\ &= 1 - (1 + 5)e^{-5} \\ &= 1 - 6e^{-5} \\ &= 1 - 0.0404 \\ &= 0.9596 \end{aligned}$$

The probability of seeing two or more jumps over the coming year is 95.96%.

23-2. The probability of exactly one jump on any given day is 1.6%. For a Poisson process, the probability of n jumps is

$$\mathrm{P}(n, T) = \frac{(\lambda T)^n}{n!} e^{-\lambda T}$$

If we define $p = \mathrm{P}(1, T)$ and $z = \lambda T$, then the probability of exactly one jump is

$$p = \lambda T e^{-\lambda T} = z e^{-z}$$

We could solve this equation for z numerically, but if z is small,

$$\begin{aligned} p &\approx z(1-z) \\ z^2 - z + p &\approx 0 \end{aligned}$$

By the quadratic formula,

$$z = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot p}}{2} = \frac{1 \pm \sqrt{1 - 4p}}{2}$$

Substituting in $p = 1.6\%$, we get z is 0.9837 or 0.0163. The second solution is the one that corresponds to a small probability of a jump. We can verify this by substituting back into the probability equation:

$$\begin{aligned} z e^{-z} &= p \\ 0.0163 e^{-0.0163} &= 0.0160 = 1.6\% \end{aligned}$$

We then have $z = 0.0163$ for $T = 1$ day. In other words, we have:

$$\lambda T = 0.0163 = \frac{0.0163}{\text{day}}(1 \text{ day})$$

The daily frequency is 0.0163 jumps per day. To get the annual frequency x from the daily frequency, we simply multiply the frequency per day by the number of days per year, so that

$$x = 0.0163 \times 256 = 4.16$$

The frequency is 4.16 jumps per year.

Note, if we had simply multiplied 1.6% by 256, we would have gotten 4.10, which is a very close to 4.16. Just as the Taylor expansion was a good approximation for small values of p, the probability of a single jump is a good approximation to the daily frequency when p is small; therefore, multiplying the probability of one jump per day by the number of days per year will also be a good approximation to the annual frequency when p is small.

CHAPTER 24

24-1. Because the jumps are of fixed size, we can use Equation 24.26, replacing the BSM call function C_{BSM} with the BSM put formula P_{BSM}. As before, we express the formula as a weighted average sum:

$$\begin{aligned} P_{\text{JD}} &= e^{-\bar{\lambda}\tau} \sum_{n=0}^{\infty} \frac{(\bar{\lambda}\tau)^n}{n!} P_{\text{BSM}} \left(S, K, \tau, \sigma, r - \lambda(e^J - 1) + \frac{nJ}{\tau} \right) \\ &= \sum_{n=0}^{\infty} w_n P_{\text{BSM}}(n) \end{aligned}$$

where:

$$\begin{aligned} w_n &= e^{-\bar{\lambda}\tau} \frac{(\bar{\lambda}\tau)^n}{n!} \\ P_{\text{BSM}} &= P_{\text{BSM}}(S, K, \tau, \sigma, r_n) \\ r_n &= r - \lambda(e^J - 1) + \frac{nJ}{\tau} \end{aligned}$$

The arguments to the function are $S = 25{,}000$; $K = 24{,}000$; $\tau = 2/52$; $\sigma = 20\%$; $r = 2\%$; $\lambda = 5$/year; and $J = -10\%$.

The BSM formula for the price of a put is

$$P(S, K, \tau, \sigma, r) = Ke^{-r\tau} N(-d_2) - SN(-d_1)$$

where:

$$d_{1,2} = \frac{\ln\left(\frac{S}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

The value of the drift-adjusted puts and the corresponding weights for $n = 1, 2, \ldots, 6$ are:

n	r_n	$P_{BS}(n)$	w_n	$P_{BS}(n) \times w_n$
0	0.4958	26.22	0.8403	22.03
1	−2.1042	1,102.91	0.1462	161.26
2	−4.7042	3,760.00	0.0127	47.83
3	−7.3042	6,784.67	0.0007	5.01
4	−9.9042	10,127.49	0.0000	0.33
5	−12.5042	13,821.88	0.0000	0.02
6	−15.1042	17,904.82	0.0000	0.00

One jump takes the index well past the strike, substantially increasing the value of the put. After one jump, the put differs little from a forward contract. Each additional jump adds to the value of the put. At the same time, the weights are decreasing rapidly. Beyond four jumps, the probabilities are extremely low, and beyond six jumps there is almost no value added. Adding the values for jumps 0–6, we get 236.47.

Without jumps, and with the same diffusion, the BSM value of the put would have been 71.22. The jumps add considerably to the value of the put.

24-2. Because the jumps are normally distributed, we can use Equation 24.29, replacing the BSM call function C_{BSM} with the BSM put formula P_{BS}. As before, we express the formula as a weighted average sum:

$$
\begin{aligned}
P_{\text{JD}} &= e^{-\bar{\lambda}\tau} \sum_{n=0}^{\infty} \frac{(\bar{\lambda}\tau)^n}{n!} P_{\text{BSM}} \left(S, K, \tau, \sqrt{\sigma^2 + \frac{n\sigma_J^2}{\tau}}, r \right. \\
&\qquad \left. - \lambda \left(e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1 \right) + \frac{n\left(\mu_J + \frac{1}{2}\sigma_J^2\right)}{\tau} \right) \\
&= \sum_{n=0}^{\infty} w_n P_{\text{BS}}(n)
\end{aligned}
$$

Here,

$$
\begin{aligned}
w_n &= e^{-\bar{\lambda}\tau} \frac{(\bar{\lambda}\tau)^n}{n!} \\
P_{\text{BS}} &= P_{\text{BS}}(S, K, \tau, \sigma^*, r^*)
\end{aligned}
$$

$$\sigma^* = \sqrt{\sigma^2 + \frac{n\sigma_J^2}{\tau}}$$

$$r^* = r - \lambda\left(e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1\right) + \frac{n\left(\mu_J + \frac{1}{2}\sigma_J^2\right)}{\tau}$$

The parameters are $S = 25{,}000$; $K = 24{,}000$; $\tau = 2/52$; $\sigma = 20\%$; $r = 2\%$; $\lambda = 5$/year; $\mu_J = -10\%$; and $\sigma_J = 5\%$.

The value of the drift- and volatility-adjusted puts and the corresponding weights for $n = 1, 2, \ldots, 6$ are now:

n	σ^*	r^*	$P_{BS}(n)$	w_n	$P_{BS}(n) \times w_n$
0	0.2000	0.4902	26.56	0.8401	22.31
1	0.3240	−2.0773	1,262.87	0.1464	184.84
2	0.4123	−4.6448	3,733.55	0.0128	47.60
3	0.4848	−7.2123	6,678.15	0.0007	4.95
4	0.5477	−9.7798	9,960.69	0.0000	0.32
5	0.6042	−12.3473	13,588.51	0.0000	0.02
6	0.6557	−14.9148	17,593.52	0.0000	0.00

The sum of the values in the last column of the table is 260.04, somewhat higher than the value in the previous problem.

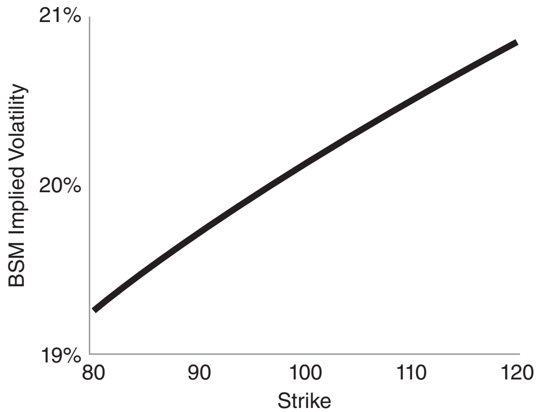
24-3. We use Equation 24.46

$$\Sigma \approx \sigma + \frac{pJ}{S\sqrt{\tau}}\left(\sqrt{\frac{\pi}{2}} + \frac{1}{\sigma\sqrt{\tau}}\ln\left(\frac{K}{S}\right)\right)$$

with the parameters $S = 100$; $p = 10\%$; $J = 15\%$; $\tau = 1/52$; and $\sigma = 20\%$. Then,

$$\begin{aligned}
\Sigma &\approx \left(\sigma + \frac{pJ}{S}\sqrt{\frac{\pi}{2\tau}}\right) + \frac{pJ}{S\sigma\tau}\ln\left(\frac{K}{S}\right) \\
&\approx \left(0.2 + \frac{0.1 \times 0.15}{100}\sqrt{26\pi}\right) + \frac{0.1 \times 0.15 \times 52}{100 \times 0.2}\ln\left(\frac{K}{S}\right) \\
&\approx 0.2014 + 0.039 \times \ln\left(\frac{K}{S}\right) \\
&\approx 0.2014 + 0.039 \times \ln\left(\frac{K}{100}\right)
\end{aligned}$$

The following exhibit shows the volatility smile. Notice that because the potential jump is positive, higher prices have higher BSM implied volatilities and the smile is positively sloped.



If jumps were negative, rather than positive, we would expect the volatility smile to be negatively sloped. Unfortunately, we cannot simply use Equation 24.46 for a negative value of J, since, in deriving the equation, we assumed that large positive jumps would render a call equivalent to a forward (Equation 24.37). To describe the volatility smile in the presence of negative jumps, we could use similar reasoning applied to a put that, after the jump, would have a value close to that of a short position in a forward contract.