FE5222 Advanced Derivative Pricing

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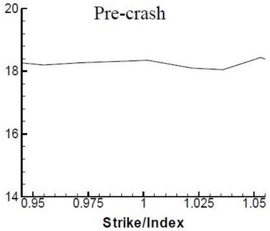
Overview

Stock index volatilities before 1987 market crash



Source: Derman (2008)

Volatility Smiles

On October 22, 1987, The Dow Jones Industrial Average plunged by 22%.

What was the Stock Market Crash Of 1987?

The stock market crash of 1987 was a rapid and severe downturn in stock prices that occurred over several days in late October 1987, affecting stock markets around the globe. In the run-up to the 1987 crash, the Dow Jones Industrial Average (DJIA) more than tripled in the prior 5 years. The Dow then plunged 22% on Black Monday - October 22, 1987. The Federal Reserve and the stock exchanges subsequently intervened to limit the damage by invoking so-called circuit breakers to slow down future plunges.

Source: investopedia

Volatility Smiles

In BSM model, the return of stock is a normal distribution

$$\frac{\Delta S}{S} \approx \sigma\sqrt{\Delta t}\mathcal{N}(0,1)$$

where $\mathcal{N}(0,1)$ is a standard normal distribution.

Volatility Smiles

Assuming the (annualized) implied volatility $\sigma = 50\%$ and 252 trading days per year, the standard deviation of daily return $\frac{\Delta S}{S}$ is about

$$50\% \times \frac{1}{\sqrt{252}} = 3.15\%$$

22% change is about $\frac{22\%}{3.15\%} = 6.8$ standard deviation.

For a normal distribution, the probability that its value is above (or below) 6.8 standard deviation is 5.2×10^{-12}.

Volatility Smiles

In BSM, the lognormal assumption on stock price does not describe the stock dynamics accurately.

In reality, the stock price has a higher probability of big movement than BSM model predicts - fat tailed distribution.

OTM (and/or ITM) options are more expensive than BSM prices.

Volatility Smiles

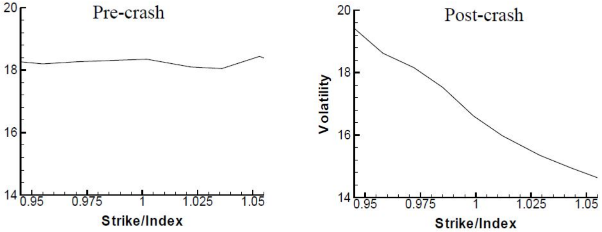
In conclusion, in contrast to BSM model, the implied volatilities in markets are not constant across strikes.

Volatility Smiles

Smile Modeling

Wu Lei

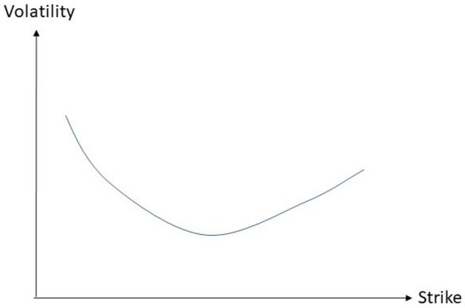
Stock index volatilities before and after 1987 market crash



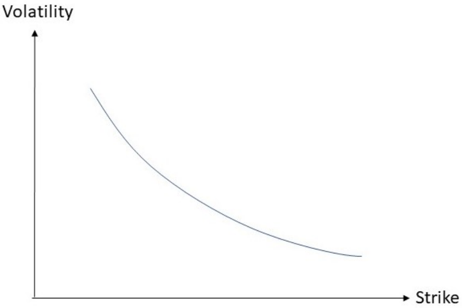
Source: Derman (2008)

Volatility Smiles

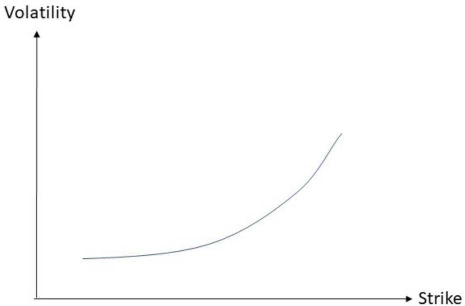
In reality, the volatility curve as a function of strike takes various shapes.



Volatility Smiles



Volatility Smiles



Volatility Smiles

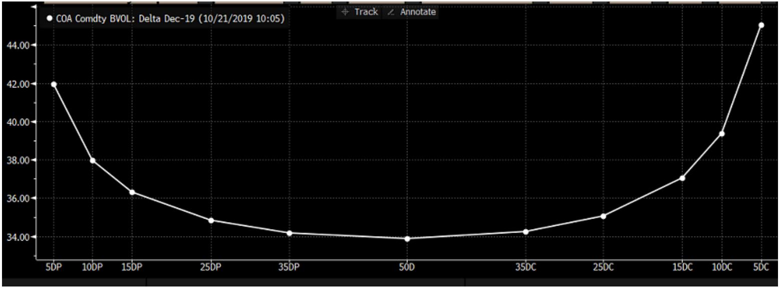
This phenomenon is called volatility smile despite the various shapes of a volatility curve may take.

Volatility Smiles

Reasons for volatility smiles:

1. Market supply and demand
2. Fat tail
3. Others

Volatility Smiles



Source: Bloomberg

Plotting Smiles

As we have seen from the previous slide, volatilities are often plotted against delta as opposed to strike. Why?

Plotting Smiles

As a first attempt, it is natural to plot volatilities as a function of strikes.

However such a volatility curve may not be as useful if we want to compare volatilities between different stocks or even different markets.

Plotting Volatility

A strike at \$20 is high for a stock whose price is around \$10 and extremely low if the stock price is around \$500.

It is better to look at relative strike - moneyness.

Plotting Volatility

Moneyness (or forward moneyness) measures how far OTM/ITM an option is.

Definitions of moneyness

- **moneyness**: $\frac{K}{S}$
- **forward moneyness**: $\frac{K}{F}$ where F is the forward price.
- **log (forward) moneyness** $\ln \frac{K}{S}$ or $\ln \frac{K}{F}$

Plotting Volatility

$\Rightarrow$ There definitions are more of a static view.

Ideally, we need to look at the relative strike $\frac{K}{S_T}$.

Plotting Volatility

An option with a higher implied volatility and longer time to maturity will have a higher chance of moving away from its current price level.

To better gauge how far OTM/ITM an option is, we need to take into account of volatility and time to maturity.

Plotting Volatility

In BSM model,

$$\frac{\ln(K/S_T)}{\sigma\sqrt{T}} \sim \mathcal{N}(\mu, 1)$$

is a normal distribution with mean

$$\mu = \frac{\ln(K/S) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

and unit standard deviation.

Plotting Volatility

In BSM model,

$$\frac{\ln(K/S_T)}{\sigma\sqrt{T}} \sim \mathcal{N}(\mu, 1)$$

is a normal distribution with mean

$$\mu = \frac{\ln(K/S) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

and unit standard deviation.

Plotting Volatility

$\frac{\ln(K/S_T)}{\sigma\sqrt{T}}$ has a standardized variance, hence we can compare it across stocks/markets.

It is a better indicator for moneyness.

Plotting Volatility

The only problem is it is not known as of now!

μ is a maximum likelihood estimate of $\frac{\ln(K/S_T)}{\sigma\sqrt{T}}$.

We may plot volatilities against μ.

Plotting Volatility

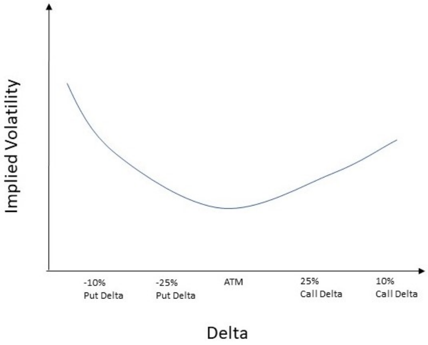
In practice, we are more familiar with delta which is a function of μ.

For example the call delta is $\Phi(d_1) = \Phi(-\mu)$ where

$$d_1 = \frac{\ln \frac{S}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Plotting Volatility

Hence we usually plot volatilities against delta



Plotting Volatility

Advantages:

- standardized x-axis
- delta is (approximately) the risk-neutral probability an option that will expire ITM. It is a better indicator of how far OTM/ITM an option is.

Plotting Volatility

Disadvantages:

- Volatility is a function of delta which in turn is a function of volatility $\Rightarrow$ circularity.
- It is not straightforward to get volatility for a particular strike.

Despite these disadvantages, it is still a common practice to plot volatility as a function of delta.

Volatility for Strike

Let $\Sigma(\Delta)$ be the volatility curve as a function of Δ. How do we find volatility for strike K?

To find volatility for strike K, we need to find delta for strike K.

Volatility for Strike

However delta is a function of volatility.

$$\Delta = \Phi\left(\frac{\ln(F/K) + \frac{1}{2}\Sigma(\Delta)^2 T}{\Sigma(\Delta)\sqrt{T}}\right)$$

$\Rightarrow$ same circularity arises from the way volatility curve is represented.

Volatility for Strike

In practice, we use a root-searching algorithm (such as Newton-Raphson's method) to find Δ that satisfies

$$\Delta = \Phi\left(\frac{\ln(F/K) + \frac{1}{2}\Sigma(\Delta)^2 T}{\Sigma(\Delta)\sqrt{T}}\right)$$

Once Δ is found, we can get volatility from $\Sigma(\Delta)$.

Delta and Smile

A common knowledge among option traders is that delta (absolute value) is approximately the (risk-neutral) probability that an option will expire ITM. We derive this result now.

Delta and Smile

In BSM model, under the risk neutral probability measure

$$S_T = Se^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\mathcal{N}(0,1)}$$

Hence for a call option, it is ITM if

$$\begin{aligned}\mathbb{P}(S_T > K) &= \mathbb{P}(\ln S_T > \ln K) \\ &= \mathbb{P}\left(\mathcal{N}(0,1) \geq -\frac{\ln\frac{S}{K}+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ &= \Phi(d_2)\end{aligned}$$

Delta and Smile

Since $d_1 = d_2 + \sigma\sqrt{T}$, $d_1 \approx d_2$ for small $\sigma\sqrt{T}$. Hence

$$\mathbb{P}\left(S_T > K\right) \approx \Phi(d_1)$$

which is the delta of a call option.

The call delta is approximately the risk-neutral probability of an option expiring ITM.

Delta and Smile

Since

$$\Delta_p = \Delta_c - 1$$

and

$$\begin{aligned} \mathbb{P}\,(\text{a put option ITM}) &= \mathbb{P}\,(\text{a call option OTM}) \\ &= 1 - \Delta_c \\ &= -\Delta_p \end{aligned}$$

$-\Delta_p$ is the probability that a put option will expire in the money.

Consequences of Volatility Smiles

In contrast to the assumption in BSM model of a flat volatility, volatilities exhibit smiles in reality. What are the consequences/implications for trading?

Consequences of Volatility Smiles

1. Pricing for liquid markets

No issue with pricing as BSM is only used as a quoting mechanism.

Consequences of Volatility Smiles

2. Hedging for liquid markets

BSM hedging ratio Δ_{BSM} is not accurate.

Consequences of Volatility Smiles

2. Hedging for liquid markets

Suppose market price of a call option is $C_{BSM}(t, S, K, T, \Sigma)$ where $\Sigma = \Sigma(t, S, K, T)$ and C_{BSM} is the BSM formula.

Hedge ratio with smile is:

$$\Delta = \frac{\partial C_{BSM}}{\partial S} + \frac{\partial C_{BSM}}{\partial \Sigma}\frac{\partial \Sigma}{\partial S}$$

Consequences of Volatility Smiles

2. Hedging for liquid markets

BSM hedge ratio is

$$\Delta_{BSM} = \frac{\partial C_{BSM}}{\partial S}$$

Consequences of Volatility Smiles

2. Hedging for liquid markets

The hedge ratio difference

$$\Delta - \Delta_{BSM} = \frac{\partial C_{BSM}}{\partial \Sigma} \frac{\partial \Sigma}{\partial S}$$

can be substantial.

Consequences of Volatility Smiles

2. Hedging for liquid markets

There are two popular assumptions on the dynamics of volatility surface:

- Sticky-strike volatility: $\Sigma(K)$
- Sticky-delta volatility: $\Sigma(\Delta)$

Under the sticky-strike volatility assumption, hedge ratio will be BSM hedge ratio.

Consequences of Volatility Smiles

3. Pricing Exotic Options

Volatility smile has significant impact on exotic option pricing and hedging.

Consequences of Volatility Smiles

3. Pricing Exotic Options

We look at the impact of smile on the price of a digital call option on stock index with strike $K = 2,000$ and expiry $T = 1$. Assume $S = 2,000$ and $r = 0\%$.

Consequences of Volatility Smiles

3. Pricing Exotic Options

We can replicate the digital call as a call spread

$$D \approx \frac{C(t, S, K, \Sigma(K)) - C(t, S, K + dK, \Sigma(K + dK))}{dK}$$

Consequences of Volatility Smiles

3. Pricing Exotic Options

$$
\begin{aligned}
D &= \lim_{dK \to 0} \frac{C(t,S,K,\Sigma(K)) - C(t,S,K+dK,\Sigma(K+dK))}{dK} \\
&= -\frac{\partial C}{\partial K} - \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial K}
\end{aligned}
$$

Note that this formula does not depend on any model. It is merely based on replication and market quote convention.

Consequences of Volatility Smiles

3. Pricing Exotic Options

Assume $\Sigma(K = 2,000) = 20\%$ and skew

$$\frac{\partial \Sigma}{\partial K}\Big|_{K=2,000} = -0.0001$$

Consequences of Volatility Smiles

3. Pricing Exotic Options

$$
\begin{aligned}
\frac{\partial C}{\partial K} &= -\Phi(d_2) \\
&= -\Phi\left(-\frac{\Sigma\sqrt{T}}{2}\right) \\
&= = -0.46
\end{aligned}
$$

3. Pricing Exotic Options

$$\begin{aligned}\frac{\partial C}{\partial \Sigma} &= \frac{S\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \\ &\approx \frac{S\sqrt{T}}{\sqrt{2\pi}} \\ &= 800.0\end{aligned}$$

Consequences of Volatility Smiles

3. Pricing Exotic Options

Hence, the price of digital call option with smile is

$$\begin{aligned} D &= -\frac{\partial C}{\partial K} - \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial K} \\ &\approx 0.46 + 800 \times 0.0001 \\ &= 0.54 \end{aligned}$$

Consequences of Volatility Smiles

3. Pricing Exotic Options

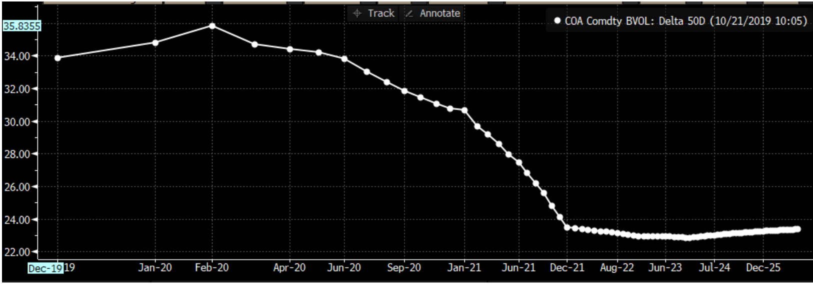
The price of digital call option without smile is

$$D = -\frac{\partial C}{\partial K} \approx 0.46$$

which is about 17% difference compared to the price with smile.

Implied Volatility Surface (IVS)

Volatilities are also not constant across expiries. Volatilities as a function of expiry is called term structure of volatility.



Source: Bloomberg

Implied Volatility Surface (IVS)

We use the notation

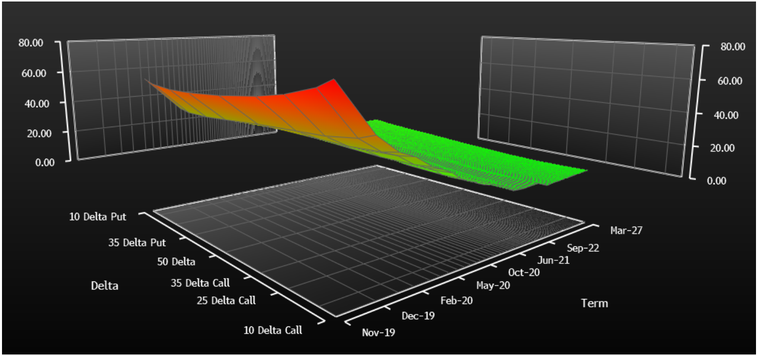
$$\Sigma(t, S, K, T)$$

for the implied volatility for option with strike T and expiry T, seen at time t when the stock price is S.

For a fixed t and S, $\Sigma(t, S, K, T)$ as a function of strike K and expiry T is called implied volatility surface (IVS).

It practice, it is also common to plot IVS as a function of delta Δ and expiry T.

Implied Volatility Surface (IVS)



Source: Bloomberg

Constraints on IVS

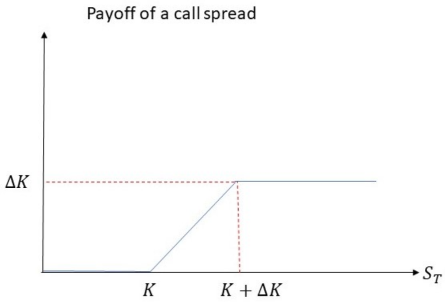
The implied volatility surface needs to admit no (static) arbitrage both in

- Strike dimension: call/put spread arbitrage, butterfly arbitrage
- Time dimension: calendar spread arbitrage

Call/Put Spread Arbitrage

A call spread consists of

- long call at strike K
- short call at strike $K + dK$



Call/Put Spread Arbitrage

Since a call spread has non-negative payoff at expiry, by non-arbitrage principle, at any time t we must have

$$C(K) - C(K + dK) \geq 0$$

This implies

$$\frac{\partial C}{\partial K} \leq 0$$

Call/Put Spread Arbitrage

Similarly, for put options we have

$$\frac{\partial P}{\partial K} \geq 0$$

Call/Put Spread Arbitrage

Let $C = C_{BSM}(t, S, K, \Sigma)$ be the market price. Then

$$\frac{\partial C}{\partial K} = \frac{\partial C_{BSM}}{\partial K} + \frac{\partial C_{BSM}}{\partial \Sigma}\frac{\partial \Sigma}{\partial K}$$

The non-arbitrage constraint on call spread option

$$\frac{\partial C}{\partial K} \geq 0$$

implies

$$\frac{\partial \Sigma}{\partial K} \leq -\frac{\frac{\partial C_{BSM}}{\partial K}}{\frac{\partial C_{BSM}}{\partial \Sigma}}$$

Call/Put Spread Arbitrage

Similarly for put options we have

$$\frac{\partial P_{BSM}}{\partial K} + \frac{\partial P_{BSM}}{\partial \Sigma}\frac{\partial \Sigma}{\partial K} \geq 0$$

Equivalently

$$\frac{\partial \Sigma}{\partial K} \geq -\frac{\frac{\partial P_{BSM}}{\partial K}}{\frac{\partial P_{BSM}}{\partial \Sigma}}$$

Call/Put Spread Arbitrage

Hence

$$-\frac{\frac{\partial P_{BSM}}{\partial K}}{\frac{\partial P_{BSM}}{\partial \Sigma}} \leq \frac{\partial \Sigma}{\partial K} \leq -\frac{\frac{\partial C_{BSM}}{\partial K}}{\frac{\partial C_{BSM}}{\partial \Sigma}}$$

Call/Put Spread Arbitrage

Since

$$\frac{\partial C_{BSM}}{\partial K} = -e^{-rT}\Phi(d_2)$$

and

$$\frac{\partial C_{BSM}}{\partial \Sigma} = K\sqrt{T}e^{-rT}\phi(d_2)$$

we have

$$-\frac{\frac{\partial C_{BSM}}{\partial K}}{\frac{\partial C_{BSM}}{\partial \Sigma}} = \frac{\Phi(d_2)}{K\sqrt{T}\phi(d_2)}$$

Call/Put Spread Arbitrage

For ATM forward (i.e., $K = Se^{rT}$) option

$$d_2 \approx 0$$

hence

$$-\frac{\frac{\partial C_{BSM}}{\partial K}}{\frac{\partial C_{BSM}}{\partial \Sigma}} = \sqrt{\frac{\pi}{2}} \frac{1}{K\sqrt{T}} \approx \frac{1.25}{K\sqrt{T}}$$

which implies

$$\frac{\partial \Sigma}{\partial K} \leq \frac{1.25}{K\sqrt{T}}$$

Call/Put Spread Arbitrage

From

$$\frac{\partial \Sigma}{\partial K} \leq \frac{1.25}{K\sqrt{T}}$$

we can derive an approximate upper bound

$$\Delta\Sigma \leq \frac{1.25}{\sqrt{T}} \frac{\Delta K}{K}$$

Call/Put Spread Arbitrage

Similarly we have

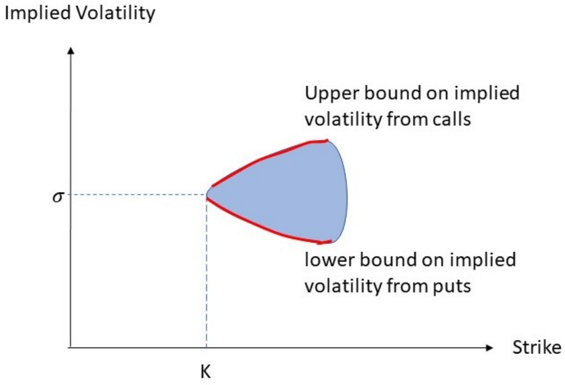
$$\frac{-1.25}{\sqrt{T}}\frac{\Delta K}{K} \leq \Delta\Sigma$$

Combining these two inequalities we have

$$\frac{-1.25}{\sqrt{T}}\frac{\Delta K}{K} \leq \Delta\Sigma \leq \frac{1.25}{\sqrt{T}}\frac{\Delta K}{K}$$

Call/Put Spread Arbitrage

Hence if K is close to ATM forward and volatility for K is σ, in the vicinity of K, the volatility must fall into the shaded areas

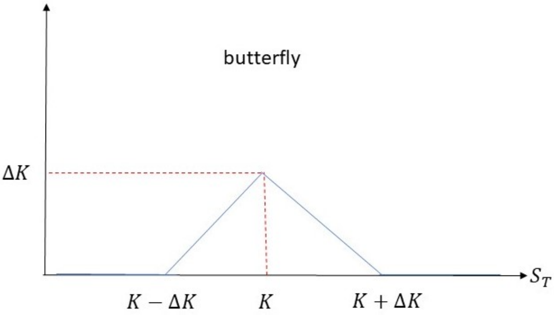


Butterfly Arbitrage

A butterfly

- long a call at strike $K - \Delta K$
- long a call at strike $K + \Delta K$
- short two calls at strike K

Butterfly Arbitrage



Butterfly Arbitrage

The payoff of a butterfly is always non-negative, hence

$$C(K+\Delta K) + C(K-\Delta K) - 2C(K) \geq 0$$

Butterfly Arbitrage

$$\frac{C(K+\Delta K)+C(K-\Delta K)-2C(K)}{\Delta K^2} \geq 0$$

Taking limit as $\Delta K \to 0$, we have

$$\frac{\partial^2 C}{\partial K^2} \geq 0$$

Butterfly Arbitrage

Since

$$\frac{\partial^2 C}{\partial K^2} = \frac{\partial^2 P}{\partial K^2}$$

we also have

$$\frac{\partial^2 P}{\partial K^2} \geq 0$$

Calendar Spread Arbitrage

The IVS also needs to satisfy non-arbitrage condition across time dimension.

Calendar Spread Arbitrage

Fix $T_2 > T_1$ and $K > 0$, consider two call options

- call option C_1 with strike $K_1 = Ke^{rT_1}$ and expiry T_1
- call option C_2 with strike $K_2 = Ke^{rT_2}$ and expiry T_2

We claim that $C_2 \geq C_1$.

Calendar Spread Arbitrage

At time T_1

- The value of the first option

$$C_1(T_1, S_{T_1}) = \max\{S_{T_1} - K_1, 0\}$$

- The value of the second option

$$C_2(T_1, S_{T_1}) \geq \max\left\{S_{T_1} - K_2 e^{-r(T_2 - T_1)}, 0\right\}$$

Calendar Spread Arbitrage

Replacing $K_2 = Ke^{rT_2}$ to the RHS of the inequality, we get

$$
\begin{aligned}
& \max\left\{S_{T_1} - K_2 e^{-r(T_2 - T_1)}, 0\right\} \\
= \;& \max\left\{S_1 - Ke^{rT_1}, 0\right\} \\
= \;& C_1(T_1, S_{T_1})
\end{aligned}
$$

Hence

$$C_2(T_1, S_{T_1}) \geq C_1(T_1, S_{T_1})$$

Calendar Spread Arbitrage

From non-arbitrage principle, at time $t = 0$, we must have

$$C_2(0, S_0) \geq C_1(0, S_0)$$

Calendar Spread Arbitrage

Let $C(T, Ke^{rT})$ be the price of a call option with strike Ke^{rT} and expiry T at time $t = 0$.

The above argument indicates that $C(T, Ke^{rT})$ is an non-decreasing function of T.

It follows that

$$\frac{\partial C(T, Ke^{rT})}{\partial T} \geq 0$$

Calendar Spread Arbitrage

Let

$$\nu^2 = \Sigma(T, Ke^{rT})^2 T$$

be the total variance for forward strike, then

$$\frac{\partial C(T, Ke^{rT})}{\partial T} \geq 0$$

is equivalent to

$$\frac{\partial \nu}{\partial T} \geq 0$$

The total variance for given forward strike must be non-decreasing.

Calendar Spread Arbitrage

Proof.

The market quoted price for the option is

$$\begin{aligned} C(T, Ke^{rT}) &= C_{BSM}\left(S, T, Ke^{rT}, \Sigma(T, K)\right) \\ &= S\Phi(d_1) - K\Phi(d_2) \end{aligned}$$

where

$$d_1 = \frac{\ln \frac{S}{K} + \frac{1}{2}\nu^2}{\nu}$$

and

$$d_2 = d_1 - \nu$$

$\square$

Calendar Spread Arbitrage

Proof.

Hence

$$\frac{\partial C(T, Ke^{rT})}{\partial T} = S\phi(d_1)\frac{\partial d_1}{\partial T} - K\phi(d_2)\frac{\partial d_2}{\partial T}$$

Note that

$$\begin{aligned} \phi(d_2) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(d_1-\nu)^2}{2}} \\ &= \phi(d_1) e^{d_1\nu - \frac{\nu^2}{2}} \\ &= \frac{S}{K}\phi(d_1) \end{aligned}$$

Substituting this into the above equation, we have

$$\frac{\partial C(T, Ke^{rT})}{\partial T} = S\phi(d_1)\frac{\partial(d_1 - d_2)}{\partial T} = S\phi(d_1)\frac{\partial \nu}{\partial T}$$

□

Calendar Spread Arbitrage

Proof.

Hence

$$\frac{\partial C(T, Ke^{rT})}{\partial T} \geq 0$$

is equivalent to

$$\frac{\partial \nu}{\partial T} \geq 0$$

Q.E.D. □

Smile Models

BSM model is inconsistent with volatility smiles observed in the markets.

Many efforts have been made to build models that are consistent with volatility smile.

Smile Models

Smile models:

- Local volatility model
- Stochastic volatility model
- Jump diffusion model

Local Volatility Model

In a local volatility model, instantaneous volatility is a function[1] of t and $S(t)$

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma(t, S(t))dW(t)$$

[1] In Dupire's original paper, it is a deterministic function of t and $S(t)$

Local Volatility Model

Example (Constant Elasticity of Variance (CEV))

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma S(t)^{\beta-1} dW(t)$$

Stochastic Volatility Model

In stochastic volatility model, instantaneous volatility also follows an SDE

$$\frac{dS(t)}{S(t)} = \mu_1 dt + \sigma dW_1(t)$$

$$d\sigma = \mu_2 dt + \nu dW_2(t)$$

where

$$dW_1(t)dW_2(t) = \rho dt$$

Stochastic Volatility Model

Example (Heston Model)

$$\frac{dS(t)}{S(t)} = \mu dt + \sqrt{\nu_t} dW_1(t)$$

$$d\nu_t = -\lambda(\nu_t - \bar{\nu})dt + \eta\sqrt{\nu_t} dW_2(t)$$

$$dW_1(t)dW_2(t) = \rho dt$$

Jump Diffusion Model

A jump diffusion model assumes the rate of return is not continuous and can jump at an instantaneous time interval.

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) + (J-1)S(t)dq$$

where the Poisson process

$$dq = \begin{cases} 0 & \text{with probability } 1-\lambda(t)dt \\ 1 & \text{with probability } \lambda(t)dt \end{cases}$$

References

Investopedia
https://www.investopedia.com/terms/s/stock-market-crash-1987.asp

Enamuel Derman (2008)
The Smile: Constraints and Problems *Lecture 3 (Spring 2008)*

Thank you!