FE5222 Homework 1: Due by Thursday, September 5

August 21, 2019

1. (10 Points) Let $W(t)$ be a Brownian motion, $t_1 < t_2 < \ldots < t_n$. Prove that the random vector

$$[W(t_1), W(t_2), \ldots, W(t_n)]$$

is a joint normal distribution. Compute its mean and covariance matrix.

2. (10 Points) Let $W(t)$ be a Brownian motion and $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration for $W(t)$. Prove that the following processes are martingale.
   (a) $W(t)$
   (b) $W^2(t) - t$
   (c) $W^3(t) - 3tW(t)$

3. (15 Points) Let $\Delta(t)$ be a simple process and $I(t) = \int_0^t \Delta(t)dW(t)$. Prove that
   (a) $I(t)$ is $\mathcal{F}_t$-measurable.
   (b) $I(t)$ is a martingale
   (c) **Ito Isometry** $\mathbb{E}[I^2(t)] = \mathbb{E}\left[\int_0^t \Delta^2(s)ds\right]$
   (d) **Quadratic Variation** $[I, I](t) = \int_0^t \Delta^2(s)ds$

4. (10 Points) Let $X(t)$ be an Ito process as

$$X(t) = X(0) + \int_0^t \Delta(s)dW(s) + \int_0^t \Theta(s)ds$$

Prove that

$$[X, X](t) = \int_0^t \Delta^2(s)ds$$

5. (20 Points) Let $W_i(t)$ and $W_j(t)$ be two independent Brownian motions. Prove that

$$\lim_{||\Pi||\to 0} \sum_{k=1}^{n} \left(W_i(t_k) - W_i(t_{k-1})\right)\left(W_j(t_k) - W_j(t_{k-1})\right) = 0$$

where $\Pi : t_0 < t_1 < \ldots < t_n$ is a partition of $[0, T]$ and the limit converges in probability. This limit is called covariation of two processes and denoted by $[W_i, W_j](t)$. This exercise justifies the notation

$$dW_i(t)dW_j(t) = 0$$

Hint: Prove the limit converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ which in turn implies convergence in probability.

6. (10 Points) Let

$$dX(t) = (\alpha - \beta X(t))dt + \sigma dW(t)$$

where α, β and σ are constant. Solve $X(t)$.

7. (15 Points) Let $B_1(t)$ and $B_2(t)$ be two Brownian motions such that

$$dB_1(t)dB_2(t) = \rho(t)dt$$

where $-1 < \rho(t) < 1$ is a stochastic process. Define

$$B_1(t) = W_1(t)$$
$$B_2(t) = \int_0^t \rho(t)dW_1(t) + \int_0^t \sqrt{1-\rho^2(t)}dW_2(t)$$

Prove that W_1 and W_2 are two independent Brownian motions.

8. (10 Points) Let

$$\frac{dX(t)}{X(t)} = \mu_1 dt + \sigma_1 dW(t)$$

and

$$\frac{dY(t)}{Y(t)} = \mu_2 dt + \sigma_2 dW(t)$$

Derive

(a) $d\left(\frac{1}{X(t)}\right)$

(b) $d\left(\frac{X(t)}{Y(t)}\right)$