Q1 : Estimation of volatility (30 marks)

(i) Calculate the unweighted average daily variance for the time series. Explain any assumptions or simplifications you have made, and the working for each step. (3 marks)

Unweighted average daily variance = average of the squared daily deviations where the 'daily deviation' is the difference between that day's return and the mean daily return. Assumptions/simplifications:

1. $\text{daily return} = Ln\left(\frac{S_n}{S_{n-1}}\right)$ (in financial markets we want to equate moves to zero and infinity; further, for valuing options with BS the assumption is that the log-returns are normally distributed; hence use the log change rather than % change)
2. $\text{mean return} = 0$ (in fact it is very very close to zero; by assuming zero, the results for GARCH are greatly simplified)

For the data provided, the unweighted average daily variance is **0.0000989**.

(ii) For a lambda of 0.94, show the exponential weights for the five newest data points. (1 mark)

The newest datapoint's weight will be (1-lambda), with earlier datapoints incrementally multiplied by lambda.

0.04684
0.04984
0.05302
0.05640
0.06000

(iii) Calculate the exponentially-weighted average daily variance for the time series. (1 mark)

The exponential weights are applied to the squared deviations before the average is taken.
exponentially-weighted average daily variance = **0.0000761**

(iv) Explain the qualitative and quantitative difference between the results for (i) and (iii). (2 marks)

Observation : unweighted (0.0000989) > weighted (0.0000761)

Quantitative : The exponentially-weighted average is lower than the unweighted average, which means that in the recent past the daily variance has been relatively low.

Qualitative : The exponentially-weighted average has a bigger contribution from more recent datapoints and is therefore more reflective of variance in the recent past.

(v) What is the meaning of each of the three parameters? (2 marks)

alpha : is the model weight given to the most recent squared return
beta : is the model weight given to the most recent variance estimate

$\gamma = 1 - \alpha - \beta$ (where γ is the model weight given to the long term average variance VL)

$\omega = \gamma \cdot V_L$

(vi) Calculate the implied long-term average daily variance. (1 marks)

$V_L = \dfrac{\omega}{1-\alpha-\beta}$ = **0.0000200**

(vii) Show the GARCH(1,1) variance estimates and log likelihoods for the five oldest data points. Explain any assumptions you have made. (3 marks)

The GARCH(1,1) variance estimate is calculated as follows: $\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$

The log likelihoods are: $-\ln(v_i) - \dfrac{u_i^2}{v_i}$

Since each estimate depends on the previous period's estimate, the first estimate must be seeded with some value. This is assumed in the leftmost table below to be the first squared deviation, and in the rightmost table to be the unweighted average daily variance.

Date	GARCH var estimate	likelihood	GARCH var estimate	likelihood
03-Jan-2006	8.29E-06	10.7010	9.89E-05	9.1376
04-Jan-2006	1.28E-05	8.4502	9.44E-05	8.8860
05-Jan-2006	1.83E-05	10.2850	9.17E-05	9.1726
06-Jan-2006	2.20E-05	7.2122	8.80E-05	8.4609
09-Jan-2006	2.86E-05	7.0509	8.80E-05	8.2301

(viii) Calibrate the three GARCH(1,1) parameters using the Maximum Likelihood Method. Explain the steps you took. (5 marks)

The three parameters alpha beta and omega are calibrated such that the sum total of all the likelihoods is maximized. Three dummy parameters, each linked to a model parameter and all initially set to 100, were used as inputs to the Solver to enable it to function optimally. Results are shown for the two different seed estimates as per (vii).

	using first squared deviation	using unweighted average daily variance
alpha	0.1177049	0.1062404
beta	0.8599220	0.8755859
omega	0.0223731	0.0181737

Both results look quite palatable – the solver constraints are not hit in either case (no minimum constraint for beta was applied); also the results, while not exactly the same, are pretty similar, so not much to be concerned about from a modelling perspective.

(ix) Calibrate the GARCH(1,1) parameters using the Maximum Likelihood Method and Variance Targeting. Explain the steps you took. What is the benefit of using Variance Targeting? (3 marks)

In this case, VL is fixed as the unweighted average daily variance. Since $V_L = \frac{\omega}{1-\alpha-\beta}$ then omega becomes a function of alpha and beta, leaving these as the two free parameters. The two parameters alpha and beta are calibrated such that the sum total of all the likelihoods is maximized as before. Two dummy parameters, each linked to a model parameter and all initially set to 100, were used as inputs to the Solver to enable it to function optimally. Results are shown for the two different seed estimates as per (vii).

	using first squared deviation	using unweighted average daily variance
alpha	0.1124882	0.1024271
beta	0.8605487	0.8762433
omega	0.0269631	0.0213295

The benefit of using Variance Targeting is that there are only two parameters instead of three, meaning the calibration process gives more stable results. This matters when the underlying data is not well behaved, for example the variance is not mean-reverting over the time period being observed, in which case there may be multiple maxima or even none at all. In this case, VL is fixed by design, which has the effect of constraining the ratio of omega to (1-alpha-beta), i.e. not allowing gamma to get too close to zero.

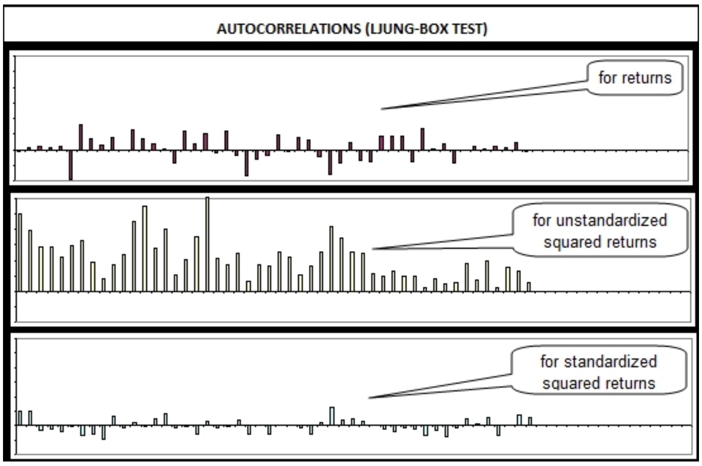
(x) Calculate, for lags of 1 – 50 trading days, the autocorrelations for: (3 marks)

a. the daily returns
b. the squared returns (or actual variance)
c. the actual variance standardized to the GARCH(1,1) variance estimates based on (ix).

These lags are calculated using Variance Targetting, and with the initial variance estimate set to the unweighted average variance.

Lag, k	rk for Rt	rk for Rt2	rk for Rt2 /σt2
1	-0.004	0.251	0.049
2	0.007	0.197	0.050
3	0.011	0.145	-0.017
4	0.005	0.141	-0.013
5	0.010	0.110	-0.022
…			
46	0.003	0.098	0.028
47	0.008	0.011	-0.036
48	0.005	0.076	0.002
49	0.022	0.066	0.034
50	-0.005	0.028	0.028

(xi) Plot the results for a, b and c, graphically. Calculate the Ljung-Box statistics for each, and comment on these results and what it means that the calibrated GARCH(1,1) model has achieved. (6 marks)



The LB statistic shows the degree of autocorrelation in a time series taking into account a range of lags, in this case from 1 – 50 days.

The LB statistic for the returns is relatively low, i.e. the autocorrelation is statistically insignificant, as one would expect (if this were not the case, one could make some predictions about future returns meaning the market was grossly inefficient).

The LB statistic for the squared returns, or variance, is high, i.e. the autocorrelation is statistically very significant, meaning that there is a lot of clustering (non-randomness, predictability) in the variance.

The LB statistic for the standardized squared returns is very low, i.e. the autocorrelation is statistically insignificant. meaning that the GARCH(1,1) model is successfully accounting for essentially all of the clustering in the variance and that its estimates are meaningful.

Q2 : Smile calibration (30 marks)

(i) Using the volatilities above : for each of the three market structures, calculate the volatility, strike and price of each of its components, and the net cost of buying the structure. (5 marks)

Buying a delta-neutral straddle consists of:

Straddle Strike for Delta Neutral (DN)

P		C	
30.00%	vol	30.00%	
99.6308	strike	99.6308	total
3.2424	price	3.6116	6.854

Buy a Call (strike X, vol 30.00%)
Buy a Put (strike X, vol 30.00%)
where X is such that the total delta net of the premiums paid is zero.
i.e. $\left(\text{Delta}_{\text{Call}} - \text{Price}_{\text{Call}}\right) + \left(\text{Delta}_{\text{Put}} - \text{Price}_{\text{Put}}\right) = 0$

There is a formulaic solution for X: $X = F \cdot e^{-\frac{1}{2}\sigma^2 (T-t)}$. Here, X=99.6308 and the DN Straddle has a net premium of 6.854 DEN per NUM payable.

Buying a 25-delta Risk Reversal consists of:

Market price of market RR

25dP		25dC	
37.500%	vol	31.500%	
25%	raw delta	25%	
93.5442	strike	106.7147	total
1.694	price	1.288	-0.406

Buy a Call (strike for 25% delta, vol 30.00%+4.50%+(-6.00%)/2 = 31.50%)
Sell a Put (strike for 25% delta, vol 30.00%+4.50%-(-6.00%)/2 = 37.50%)
The strikes can be solved by guessing using the BS formula for the option delta, and regressing using e.g. Newton-Raphson. Here the Call strike is 106.7147 and the Put Strike is 93.5442. The RR has a net premium of 0.406 DEN per NUM receivable.

Buying a 25-delta Strangle consists of:

Market price of market STGL

25dP		25dC	
34.500%	vol	34.500%	
25%	raw delta	25%	
94.0050	strike	107.4229	total
1.552	price	1.405	2.957

Buy a Call (strike for 25% delta, vol 30.00%+4.50% = 34.50%)
Buy a Put (strike for 25% delta, vol 30.00%+4.50% = 34.50%)
The strikes can be solved by guessing using the BS formula for the option delta, and regressing using e.g. Newton-Raphson. Here the Call strike is 107.4229 and the Put Strike is 94.0050. The STGL has a net premium of 2.957 DEN per NUM payable.

All the price outputs are from the BS formula and are in the same units as S (and K).

(ii) Using the smile function : for each of the market structures, calculate the volatility and price of each of its components, and the net cost of buying the structure. (5 marks)

Smile Function σ(K)

25dP	ATM	25dC	
93.5442	99.6308	106.7147	strike
-6.67E-02	-3.70E-03	6.50E-02	ln K/F
37.500%	30.000%	31.500%	vol
0.2982	-0.4368	10.6902	polynomial

To value the market structures using the smile function σ(K), three points are first required, and then a polynomial function is solved to pass through these points. The three points are:
a Call (strike for 25% delta, vol 31.50%)
a Call (or Put) (strike and vol of the DN straddle)
a Put (strike for 25% delta, vol 37.50%)
The polynomial uses the log strike rather than the strike as its x-axis, as this function is much smoother. Then for the market structures, the same strikes are used as before (i.e. they are fixed not solved). Each strike is provided to the polynomial function as a log strike and the output is the smile volatility used for valuation. The results are:

Straddle Strike for Delta Neutral (DN)

P		C	
30.00%	vol	30.00%	
99.6308	strike	99.6308	total
3.2424	price	3.6116	6.854

Smile price of market RR (as two separate options)

93.5442	strike	106.7147	
-0.0667	log strike	0.0650	
37.500%	smile vol	31.500%	total
1.694	price	1.288	-0.406

Smile price of market STGL (as 2 separate options)

94.0050	strike	107.4229	
-0.0618	log strike	0.0716	
36.610%	smile vol	32.177%	total
1.746	price	1.198	2.944

Straddle : cost 6.854
Risk Reversal : cost -0.406
Strangle : cost 2.944

(iii) Compare these net prices with those from (i) and give a brief explanation of any differences. (2 marks)

prices	using market vols	using smile vols	difference
DN straddle	6.854	6.854	nil, as the equivalent straddle vol/strike and the market straddle vol/strike are the same by definition
25d risk reversal	-0.406	-0.406	nil, as the smile function is manually set to match the market RR strikes and vols (initially)
25d strangle	2.957	2.944	material difference as the curvature of the smile function does not match the curvature implied by the market smile

(iv) Solve for (calibrate) the smile function. Show the calibrated 25 delta Put option and 25 delta Call option volatilities and explain the steps you made in calibration. (3 marks)

Smile Function σ(K)

25dP	ATM	25dC	
93.5442	99.6308	106.7147	strike
-6.67E-02	-3.70E-03	6.50E-02	ln K/F
37.572%	30.000%	31.572%	vol
0.2982	-0.4365	10.8554	polynomial

The smile function inputs (the 25dP and 25dC volatilities) are adjusted in such a way as to give a smile function σ(K) that results in smile valuations of the market structures that exactly match those derived directly from the market vols. This can be done by e.g. goalseek or solver.
25dP Vol = 37.572%
25dC Vol = 31.572%

(v) Construct a vega-neutral butterfly as a ratio spread of delta-neutral straddles and market strangles. (1 mark)
Butterfly is a spread of DN Straddles against 25d Strangles (buy one and sell the other). For VN Butterfly, ratio the amounts such that the net vega is zero.
Straddle Vega = 11.395 + 11.395 = 22.790
Strangle Vega = 8.557 + 8.375 = 16.931
Weight = ratio of vegas = 1.3460
Buy 1 Straddle and Sell 1.3460 Strangles for net zero vega.

(vi) Calculate the vega, vanna and volga of each of the delta neutral straddle, the market risk reversal, and the vega-neutral butterfly from (v). Evaluate the 'Black-Scholes price' for each using the atm volatility for all components. (3 marks)
The vega, volga and vanna are all calculated using the DN Straddle vol (i.e. the 'atm' vol) rather than the smile vols.

	DNSTDL	25DRR	VNBF
BSV VEGA	22.790	0.694	0.000
BSV VOLGA	0.000	0.468	-45.739
BSV VANNA	0.000	1.521	-0.243

(vii) Calculate the weights of each of the three structures in (vi) required, to build three portfolios that have: (3 marks)

	DNS	25DRR	VNBF
a. only vega, and no vanna or volga	1.0000	0.0000	0.0000
b. only vanna, and no vega or volga	-0.0304	1.0000	0.0102
c. only volga, and no vega or vanna	-0.0049	0.1595	1.0000

(viii) For each of these three portfolios, calculate the differences between their respective weighted prices and weighted Black-Scholes prices. Calculate the implied change in price per unit of each of vega, vanna and volga. (5 marks)

	Weighted BSV Price	Weighted MKT Price	Change in price	Quantum of risk	Change in price per unit of risk
a. only vega, and no vanna or volga	6.854	6.854	0.0000	vega = +22.79	nil
b. only vanna, and no vega or volga	-0.0573	-0.5850	-0.5276	vanna = +1.5184	-0.3475
c. only volga, and no vega or vanna	+3.9251	+2.7759	-1.1493	volga = -45.6645	+0.0252

(ix) For a Call option of tenor 30 days and strike 104.50, calculate the smile price using the adjustments from (viii). (3 marks)
The BS value of the option, using the DN Straddle or atm vol of 30.00% is 1.706.
Its vanna is +0.661, and its volga is +8.883. Using the adjustments above, its smile price would be:
1.706 + (+0.661 x -0.3475) + (+8.883 x +0.0252) = 1.7004