Hint for Q1, HW 3

October 5, 2019

If we start from USD market and choose it as domestic market, we have

$$\frac{dS^{USD}(t)}{S^{USD}(t)} = r_{USD}dt + \sigma_1^{USD} d\widetilde{W}_1^{USD}(t)$$

and

$$\frac{dX(t)}{X(t)} = (r_{USD} - r_{SGD})\, dt + \sigma_2^{USD} d\widetilde{W}_3^{USD}(t)$$

where

$$d\widetilde{W}_3^{USD}(t) = \rho_{USD} d\widetilde{W}_1^{USD}(t) + \sqrt{1-\rho_{USD}^2}\, d\widetilde{W}_2^{USD}(t)$$

and $\widetilde{W}_1^{USD}$ and $\widetilde{W}_2^{USD}$ are two independent Brownian motions under the USD risk neutral measure. Note that

$$d\widetilde{W}_1^{USD}(t) d\widetilde{W}_3^{USD}(t) = \rho_{USD} dt$$

In class we have derived the pricing formula for a Quanto call option with this setting. The option price can be written as a function of $r_{USD}, r_{SGD}, \sigma_1^{USD}, \sigma_2^{USD}$ and ρ_{USD}. We denote it as

$$C_1\left(r_{USD}, r_{SGD}, \sigma_1^{USD}, \sigma_2^{USD}, \rho_{USD}, X(0), S^{USD}(0)\right)$$

Similarly if we use SGD as the domestic market, we have

$$\frac{dS^{SGD}(t)}{S^{SGD}(t)} = r_{SGD}dt + \sigma_1^{SGD} d\widetilde{W}_1^{SGD}(t)$$

and

$$\frac{dY(t)}{Y(t)} = (r_{SGD} - r_{USD})\, dt + \sigma_2^{SGD} d\widetilde{W}_3^{SGD}(t)$$

where

$$d\widetilde{W}_3^{SGD}(t) = \rho_{SGD} d\widetilde{W}_1^{SGD}(t) + \sqrt{1-\rho_{SGD}^2}\, d\widetilde{W}_2^{SGD}(t)$$

$\widetilde{W}_1^{SGD}$ and $\widetilde{W}_2^{SGD}$ are two independent Brownian motions under the SGD risk neutral measure and

$$d\widetilde{W}_1^{SGD}(t) d\widetilde{W}_3^{SGD}(t) = \rho_{SGD} dt$$

In this setting the price for Quanto call option can be written as

$$C_2\left(r_{USD}, r_{SGD}, \sigma_1^{SGD}, \sigma_2^{SGD}, \rho_{SGD}, X(0), S^{SGD}(0)\right)$$

You are asked to prove

$$C_1\left(r_{USD}, r_{SGD}, \sigma_1^{USD}, \sigma_2^{USD}, \rho_{USD}, X(0), S^{USD}(0)\right) = C_2\left(r_{USD}, r_{SGD}, \sigma_1^{SGD}, \sigma_2^{SGD}, \rho_{SGD}, X(0), S^{SGD}(0)\right)$$

To this end we will need to show the relationship between $\sigma_1^{USD}, \sigma_2^{USD}, \rho_{USD}$ and $\sigma_1^{SGD}, \sigma_2^{SGD}, \rho_{SGD}$ by applying Change of Numeraire technique. For example we can start with the first formulation that uses USD as domestic market. Use Change of Numeraire from USD market to SGP market we can derive the SDE for S^{SGD} and $Y = \frac{1}{X(t)}$. Compare it to the the second formulation above, we can derive the relationship between volatilities and correlations. Once we have derived the relationships between these quantities, plugging them into the two pricing formulae, we can show they are equivalent.