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Homework III.

1. a).

$\varphi(u) = E[e^{uX}] = \int_R e^{ux} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx.$

$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{ux - \frac{x^2}{2t}} dx = e^{\frac{1}{2}u^2 t} \cdot \underline{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \cdot e^{-\frac{(x-ut)^2}{2t}} dx}.$

this is a variable with ut mean and t variance.
So this will be 1.

$\Rightarrow \varphi(u) = e^{\frac{1}{2}u^2 t}.$

b). $E[e^X] = e^u E[e^{X-u}]$

$= e^u \cdot \int_{-\infty}^{\infty} e^x \cdot \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} dx.$

$= e^u \cdot e^{\frac{1}{2}\sigma^2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \cdot e^{\frac{-(x-\sigma)^2}{2\sigma^2}} dx.$

$= e^{u+\frac{1}{2}\sigma^2}.$

c). $E[(e^X)^2] = E[e^{2X}] = e^{2u} \cdot E(e^{\underbrace{2X-2u}_{\sim N(0,\, 4\sigma^2)}}).$

$\because E(e^{2X-2u}) = \int_{-\infty}^{\infty} e^x \cdot \frac{1}{\sqrt{8\pi\sigma^2}} \frac{-x^2}{8\sigma^2} dx.$

$= \frac{e^{2\sigma^2}}{\sqrt{8\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{(x-4\sigma^2)^2}{8\sigma^2}} dx$ $\Rightarrow E(e^{2X}) = e^{2u+2\sigma^2}$

$= e^{2\sigma^2} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{8\pi\sigma^2}} e^{-\frac{(x-4\sigma^2)^2}{8\sigma^2}} dx}_{N(4\sigma^2,\, 4\sigma^2)}.$

$= e^{2\sigma^2}.$

$\therefore Var[e^X] = E[(e^X)^2] - E^2[e^X]$

$= e^{2\sigma^2 + 2u} - e^{2u+\sigma^2}$

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4. $e^{-rT} EX = e^{-rT} P(\log S_T \geq \log K).$

$\because \log S_T \sim N(\log S_0 + (r - \frac{\sigma^2}{2})T,\ \sigma^2 T).$

$\therefore P(\log S_T \geq \log K)$ we can have a standard normal distribution

$\Leftrightarrow P(X \geq \frac{\log K - \log S_0 - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}})$ where $X \sim N(0,1)$.

$\Leftrightarrow P(X \geq \frac{\log\frac{K}{S_0} - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}})$

$\Leftrightarrow P(X \geq \frac{-\log\frac{S_0}{K} - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}) \Leftrightarrow P(X \geq -d_-).$

$= N(d_-).$

$\therefore e^{-rT} EX = e^{-rT} N(d_-).$

6.

a). $E[[W,W]_{\Pi,T}] = \sum_{j=0}^{n-1} E[(W_{(j+1)} - W_j)^2]$

$= \sum_{j=0}^{n-1} Var[W_{(j+1)} - W_{(j)}] = \sum_{j=0}^{n-1} (t_{j+1} - t_j).$

$= T.$

(b) $Var[[W,W]_{\Pi,t_j}] = E[([W,W]_{\Pi,t_j} - (t_{j+1} - t_j))^2].$

$= E[(W_{(j+1)} - W_{(j)})^4] - 2\underline{E[(W_{(j+1)} - W_j)^2]}(t_{j+1} - t_j) + (t_{j+1} - t_j)^2$

$\because E[(W_{j+1} - W_j)^4] = 3\,Var^2[W_{j+1} - W_j]$

$= 3(t_{j+1} - t_j)^2.$

$\therefore = 3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2$

$= 2(t_{j+1} - t_j)^2 \leq 2 \cdot \max_{j=0,\dots,n-1}(t_{j+1} - t_j) \cdot (t_{j+1} - t_j).$

$\leq 2 \cdot \Pi \cdot (t_{j+1} - t_j)$

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$$\therefore \sum_{j=0}^{n-1} Var[(W_{t_{j+1}} - W_{t_j})^2] \le 2 \cdot \|\Pi\| \cdot T.$$

when $\|\Pi\| \to 0$ $\sum_{j=0}^{n-1} Var[(W_{j+1} - W_{t_j})^2] \to 0$

9. $$P_Y = \int_{-\infty}^{\infty} P_{X,Y}\, dx$$

$$= \int_{-\infty}^{\infty} \sqrt{\frac{3}{8\pi}} \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{(x-\frac{1}{2}y)^2 + \frac{3}{4}y^2}{-2}} dx$$

$$= \sqrt{\frac{3}{8\pi}} \cdot e^{-\frac{3}{8}y^2} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{(x-\frac{1}{2}y)^2}{-2}} dx}_{N(\frac{1}{2}y,\,1)}.$$

$$= \sqrt{\frac{3}{8\pi}} \cdot e^{-\frac{3}{8}y^2}$$

$$P_{X|Y} = \frac{P_{X,Y}}{P_Y}$$

$$= \frac{\frac{\sqrt{3}}{4\pi} e^{-\frac{1}{2}(x^2 - xy + y^2)}}{\sqrt{\frac{3}{8\pi}} e^{-\frac{3}{8}y^2}} = \frac{1}{\sqrt{2\pi}} e^{\frac{(x-\frac{1}{2}y)^2}{-2}}$$

$$\therefore E(X|Y) = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{(x-\frac{1}{2}y)^2}{-2}} dx$$

we assume that $x - \frac{1}{2}y = s \Rightarrow x = s + \frac{1}{2}y$ and $\underline{s \sim N(0,1)}$

$$\therefore E(X|Y) = \int_{-\infty}^{\infty} (s + \tfrac{1}{2}y) \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds.$$

$$= \int_{-\infty}^{\infty} s \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds + \frac{1}{2}y \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds$$

$$= E(S) + \frac{1}{2}y = \frac{1}{2}y$$

$$\therefore E(X|Y) = \frac{1}{2}Y.$$

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14. $$P(X_3 \le 3) = P(\log X_3 \le \log 3) = P(W_3 - \tfrac{3}{2} \le \log_e 3).$$

$$= P(W_3 \le \log_e 3 + \tfrac{3}{2})$$

又 $\because W_3 \sim N(0,3)$.

$$= P\left(\frac{W_3}{\sqrt{3}} \le \frac{\ln 3 + \frac{3}{2}}{\sqrt{3}}\right).$$

$\therefore \frac{W_3}{\sqrt{3}} \sim N(0,1)$.

Hence $P(X_3 \le 3) = P\left(\frac{W_3}{\sqrt{3}} \le \frac{\ln 3 + \frac{3}{2}}{\sqrt{3}}\right) \approx \Phi(1.5) \approx 0.9332$

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Homework IV.

6. note we have $E[g(x)] = \int g(x) f(x)\,dx$.

$$\therefore \quad E[X_t^2] = \int_{-\infty}^{\infty} e^{2x^2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\,dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(1-4t)x^2}{2t}}\,dx.$$

$$\underline{\underline{\text{set } y=\sqrt{1-4t}\,x}}\ \frac{1}{\sqrt{1-4t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{\frac{-y^2}{2t}}\,dy.$$

$$= \frac{1}{\sqrt{1-4t}}.$$

Thus we prove the question.

9.

① note that $E[M_t] = E[E[M_t \mid G]]$

$$\underline{\underline{G=F_0}}\ E[E[M_t \mid F_0]] = E[M_0] = 0$$

② The variance of $M_t = \int_0^t e^{-\alpha s}\,dW_s$

$$Var[M_t] = E[(M_t - 0)^2] = E[(\int_0^t e^{-\alpha s}\,dW_s)^2].$$

$$\underline{\underline{\text{Ito Isomery}}}\ E\int_0^t (e^{-\alpha s})^2\,ds$$

$$= \int_0^t E(e^{-2\alpha s})\,ds$$

$$= \int_0^t e^{-2\alpha s}\,ds$$

$$= \frac{1}{2\alpha}(1 - e^{-2\alpha t}).$$

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10. $Y_t = \int_0^t \sqrt{|W_s|}\,dW_s$.

because Y_t is a Ito Integral.

$\therefore E(Y_t) = 0$.

$$Var(Y_t) = E[(\int_0^t \sqrt{|W_s|}\,dW_s - 0)^2] = E[(\int_0^t \sqrt{|W_s|}\,dW_s)^2]$$

$$= E\int_0^t |W_s|\,ds.$$

$$= \int_0^t E|W_s|\,ds.$$

for $E|W_s| = 2\int_0^{\infty} x \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}\,dx$.

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}\,dx^2 = \frac{1}{\sqrt{2\pi s}} \cdot -2s \cdot e^{-\frac{x^2}{2s}} \Big|_0^{\infty}$$

$$= \sqrt{\frac{2s}{\pi}}.$$

$$\therefore Var(Y_t) = \int_0^t \sqrt{\frac{2s}{\pi}}\,ds = \sqrt{\frac{2}{\pi}} \int_0^t s^{\frac{1}{2}}\,ds = \sqrt{\frac{2}{\pi}}\, \frac{2}{3} \cdot s^{\frac{3}{2}} \Big|_0^t$$

$$= \frac{2}{3}\sqrt{\frac{2}{\pi}}\, t^{\frac{3}{2}}.$$