FE5222 Advanced Derivative Pricing

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Overview

1 Introduction

2 Dupire's Equation

3 Understanding Local Volatility

Local Volatility Model

The assumption of constant volatility in BSM model is inconsistent with market observations.

Is there a BSM-like model that can price European options in the market consistently?

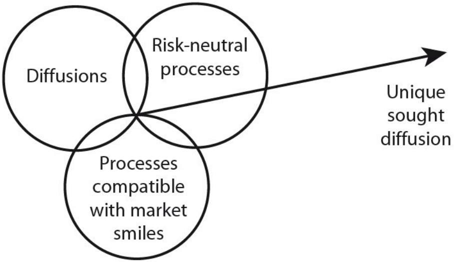
Local Volatility Model

- Dupire (1994) developed a local volatility model for continuous time and showed that there exits a unique risk neutral diffusion process that is consistent with European option prices.
- Derman & Iraj Kani (1994) developed a tree model which is consistent with market prices for European options.

Local Volatility Model

1. A unique diffusion process

If we restrict ourselves to diffusions, there is a unique risk-neutral (drift equal to the short-term rate) process for the spot which is compatible with European option prices:



Source: Dupire (1994)

Local Volatility Model

In LVM, the instantaneous volatility is a deterministic function of t and S_t

$$\frac{dS_t}{S_t} = rdt + \sigma(t, S_t)dW_t$$

Local Volatility Model

In LVM, the pricing PDE is

$$V_t + rSV_s + \frac{1}{2}\sigma^2(t,S)S^2V_{SS} - rV = 0 \tag{1}$$

Dupire's Equation

In this section, we will discuss

- Kolmogorov Backward Equation
- Kolmogorov Forward Equation
- Dupire's Equation

Kolmogorov Backward Equation

Consider the stochastic differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

Kolmogorov Backward Equation

Fix $T > t$, let $p(t, T, x, y)$ be the transition probability density for the solution to this equation.

It is the probability density function of X_T if we solve the equation with initial condition $X_t = x$.

Kolmogorov Backward Equation

Local Volatility Model

Wu Lei

Introduction

Dupire's Equation

Understanding Local Volatility

Theorem

The transition density function $p(t, T, x, y)$ satisfies the Kolmogorov backward equation

$$p_t(t, T, x, y) + \mu(t, x) p_x(t, T, x, y) + \frac{1}{2}\sigma^2(t, x) p_x(t, T, x, y) = 0$$

Kolmogorov Backward Equation

In the following proof, we will need the concept of a smooth function and compact support.

- A smooth function is a function that has derivatives of all orders.
- The support of a function f is

$$supp\ f = \overline{\{x : f(x) \neq 0\}}$$

- A compact set in $\mathbb{R}^n$ is a closed and bounded set.

Kolmogorov Backward Equation

One example for a smooth function with compact support is the so called *bump function* defined as

$$f(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \forall -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Kolmogorov Backward Equation

Lemma

Let f be an integrable function such that

$$\int f(x)h(x)dx = 0$$

for all smooth and compact function h. Then $f(x) = 0$ for (almost surely) all x.

Remark: In fact, smoothness is not necessary for this lemma. However we will need smoothness for the derivation of Komogorov Forward Equation.

Kolmogorov Backward Equation

Proof.

For any smooth function $h(x)$ with compact support, let

$$
\begin{aligned}
g(t,x) &= \mathbb{E}\left[h(X_T)\right] \\
&= \int h(y)p(t,T,x,y)dy
\end{aligned}
$$

$\square$

Kolmogorov Backward Equation

Proof.

Taking partial derivatives w.r.t. t and x respectively, we have

$$g_t(t,x) = \int h(y)p_t(t,T,x,y)dy \tag{2}$$

$$g_x(t,x) = \int h(y)p_x(t,T,x,y)dy \tag{3}$$

$$g_{xx}(t,x) = \int h(y)p_{xx}(t,T,x,y)dy \tag{4}$$

□

Kolmogorov Backward Equation

Proof.

From Feynman-Kac Theorem, we have

$$g_t(t,x) + \mu(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) = 0$$

Replacing Equation (2), (3) and (4) into the above equation, we have

$$\int h(y)p_t dy + \mu(t,x)\int h(y)p_x dy + \frac{1}{2}\sigma^2(t,x)\int h(y)p_{xx} dy = 0$$

$\square$

Kolmogorov Backward Equation

Proof.

Hence

$$\int h(y)\left[p_t + \mu(t,x)p_x + \frac{1}{2}\sigma^2(t,x)p_{xx}\right] dy = 0$$

$\square$

Kolmogorov Backward Equation

Proof.

Since this holds for all smooth function h with compact support, we must have

$$p_t + \mu(t,x)p_x + \frac{1}{2}\sigma^2(t,x)p_{xx} = 0 \qquad \text{Q.E.D.} \tag{5}$$

$\square$

Kolmogorov Forward Equation

Theorem

The transition density function $p(t, T, x, y)$ satisfies the Kolmogorov forward equation

$$\frac{\partial p}{\partial T} + \frac{\partial}{\partial y}\left(\mu(T, y)p\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T, y)p\right) = 0$$

Remark: It is also called Fokker-Planck equation.

Kolmogorov Forward Equation

Proof.

Let h be a smooth function with compact support. By Ito's Lemma, we have

$$\begin{aligned} dh(X_s) &= h_x dX_s + \frac{1}{2} h_{xx} d\left[X, X\right](s) \\ &= \left[\mu(s, X_s) h_x(X_s) + \frac{1}{2}\sigma^2(s, X_s) h_{xx}(X_s)\right] ds \\ &\quad + h_x(X_s)\sigma(s, X_s) dW_s \end{aligned} \tag{6}$$

□

Kolmogorov Forward Equation

Proof.

Consider the process X that starts from t with initial condition $X(t) = x$.

Integrating Equation (6) from t to T, we obtain

$$h(X_T) = h(X_t) + I_1 + I_2$$

where

$$I_1 = \int_t^T \left[\mu(s, X_s) h_x(X_s) + \frac{1}{2}\sigma^2(s, X_s) h_{xx}(X_s) \right] ds$$

and

$$I_2 = \int_t^T h_x(X_s)\sigma(s, X_s) dW_s$$

Kolmogorov Forward Equation

Proof.

Since I_2 is an Ito's integral whose mean is zero, we have

$$\mathbb{E}\left[h(X_T)\right] = h(x) + \mathbb{E}\left[I_1\right] \tag{7}$$

□

Kolmogorov Forward Equation

Proof.

The LHS of Equation (7) is

$$\mathbb{E}\left[h(X_T)\right] = \int_{-\infty}^{\infty} h(y)p(t, T, x, y)dy$$

$\square$

Kolmogorov Forward Equation

Proof.

$$
\begin{aligned}
& \mathbb{E}\left[I_1\right] \\
= \ & \mathbb{E}\left[\int_t^T \left[\mu(s, X_s) h_x(X_s) + \frac{1}{2}\sigma^2(s, X_s) h_{xx}(X_s)\right] ds\right] \\
= \ & \int_t^T \mathbb{E}\left[\mu(s, X_s) h_x(X_s) + \frac{1}{2}\sigma^2(s, X_s) h_{xx}(X_s)\right] ds \\
= \ & \int_t^T \int_{-\infty}^{\infty} p(t, s, x, y)\left[\mu(s, y) h_x(y) + \frac{1}{2}\sigma^2(s, y) h_{xx}(y)\right] dyds \\
= \ & \int_t^T \left[\int_{-\infty}^{\infty} p(t, s, x, y)\mu(s, y) h_x(y) dy\right] ds \\
& + \frac{1}{2}\int_t^T \left[\int_{-\infty}^{\infty} p(t, s, x, y)\sigma^2(s, y) h_{xx}(y) dy\right] ds
\end{aligned}
$$

$\square$

Kolmogorov Forward Equation

Proof.

Now we evaluate

$$\int_{-\infty}^{\infty} p(t,s,x,y)\mu(s,y)h_x(y)dy$$

and

$$\int_{-\infty}^{\infty} p(t,s,x,y)\sigma^2(s,y)h_{xx}(y)dy$$

□

Kolmogorov Forward Equation

Proof.

Integrating by parts and using the fact that $h(x)$, $h_x(x)$ and $h_{xx}(x)$ vanish when $|x|$ is large enough, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} p(t,s,x,y)\mu(s,y)h_x(y)dy \\ = \quad & -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left(p(t,s,x,y)\mu(s,y)\right)h(y)dy \end{aligned} \tag{8}$$

and

$$\begin{aligned} & \int_{-\infty}^{\infty} p(t,s,x,y)\sigma^2(s,y)h_{xx}(y)dy \\ = \quad & \int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2}\left(p(t,s,x,y)\sigma^2(s,y)\right)h(y)dy \end{aligned} \tag{9}$$

$\square$

Kolmogorov Forward Equation

Proof.

Hence

$$
\begin{aligned}
& \mathbb{E}\left[I_1\right] \\
= \quad & -\int_t^T \left[\int_{-\infty}^{\infty} \frac{\partial}{\partial y} \left(p(t,s,x,y)\mu(s,y)\right) h(y)dy\right] ds \\
& +\frac{1}{2}\int_t^T \left[\int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2} \left(p(t,s,x,y)\sigma^2(s,y)\right) h(y)dy\right] ds
\end{aligned}
$$

$\square$

Kolmogorov Forward Equation

Proof.

Substituting these to Equation (7), we obtain

$$
\begin{aligned}
& \int_{-\infty}^{\infty} h(y)p(t,T,x,y)dy \\
= \; & h(x) - \int_t^T \left[\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left(p(t,s,x,y)\mu(s,y)\right) h(y)dy \right] ds \\
& + \frac{1}{2}\int_t^T \left[\int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2}\left(p(t,s,x,y)\sigma^2(s,y)\right) h(y)dy \right] ds
\end{aligned}
$$

$\square$

Kolmogorov Forward Equation

Proof.

Taking derivative w.r.t. to T, we have

$$
\begin{aligned}
& \int_{-\infty}^{\infty} h(y)\frac{\partial}{\partial T}p(t,T,x,y)dy \\
= \quad & -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left(p(t,T,x,y)\mu(T,y)\right) h(y)dy \\
& +\frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2}\left(p(t,T,x,y)\sigma^2(T,y)\right) h(y)dy
\end{aligned}
$$

$\square$

Kolmogorov Forward Equation

Proof.

Re-arranging it, we have

$$\int_{-\infty}^{\infty} h(y)\left[\frac{\partial p}{\partial T} + \frac{\partial}{\partial y}\left(\mu(T,y)p\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,yp)\right)\right] dy = 0$$

which implies

$$\frac{\partial p}{\partial T} + \frac{\partial}{\partial y}\left(\mu(T,y)p\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)p\right) = 0 \qquad \text{Q.E.D.}$$

□

Kolmogorov Backward/Forward Equation

Kolmogorov Backward Equation

$$p_t(t, T, x, y) + \mu(t, x) p_x(t, T, x, y) + \frac{1}{2}\sigma^2(t, x) p_x(t, T, x, y) = 0$$

Kolmogorov Forward Equation

$$\frac{\partial p}{\partial T} + \frac{\partial}{\partial y}\left(\mu(T, y)p\right) - \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T, y)p\right) = 0$$

Kolmogorov Backward/Forward Equation

- Kolmogorov backward equation
  Fix T, it is an PDE of initial condition $X_t = x$.
- Kolmogorov forward equation
  Fix initial condition $X(t) = x$ and it is an PDE w.r.t. T and $X_T = y$.

Dupire's Equation

Consider the SDE

$$\frac{dS_t}{S_t} = rdt + \sigma(t, S_t)dW_t$$

where r is a constant and W_t is a standard Brownina motion. Let $C(T, K)$ be the price of a call option with expiry T and strike K, given $S(0) = S_0$.

Dupire's Equation

Theorem

Let $C(T, K)$ be the price of a call option with strike K and expiry T. Then the following so called Dupire's equation holds

$$C_T + rKC_K - \frac{1}{2}\sigma^2(T, K)K^2 C_{KK} = 0$$

Dupire's Equation

Remark: Dupire's equation is often written as

$$\sigma^2(T,K) = \frac{C_T + rKC_K(T,K)}{\frac{1}{2}K^2C_{KK}(T,K)} \tag{10}$$

The RHS of Equation (10) can be used to define the notion of local volatility.

Dupire's Equation

- Dupire's equation is a forward equation. It is PDE for the option price with different expiry T and strike K.
- On the contrary, the PDE Equation (1) is the option price with fixed expiry T and strike K, but different time t and spot price S_t.
- In practice, Dupire's equation is often used for model calibration and the PDE Equation (1) is used for pricing.

Dupire's Equation

Proof.

Let $p(0, T, S_0, y)$ be the transition density function for the stock price process that starts at time $t = 0$ with S_0. For notational simplicity, we write it as $p(T, y)$.

The call price is

$$C(T, K) = e^{-rT} \int (y - K)^+ p(T, y) dy$$

Taking partial derivative w.r.t. T, we have

$$\begin{aligned} & C_T(T, K) \\ = \quad & -rC(T, K) + e^{-rT} \int (y - K)^+ \tfrac{\partial p}{\partial T} dy \end{aligned}$$

$\square$

Dupire's Equation

Proof.

By Kolmogorov forward equation

$$
\begin{aligned}
& \frac{\partial p}{\partial T} \\
= \quad & -\frac{\partial}{\partial y}\left(ryp(T,y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2 p(T,y)\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \int (y-K)^+ \frac{\partial p}{\partial T} dy \\
= \quad & -\int (y-K)^+ \frac{\partial}{\partial y}\left(ryp(T,y)\right) dy \\
& +\frac{1}{2}\int (y-K)^+ \frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2 p(T,y)\right) dy
\end{aligned}
$$

$\square$

Dupire's Equation

Proof.

Integrating by parts, we have

$$
\begin{aligned}
& \int (y-K)^+ \frac{\partial}{\partial y}\left(ryp(T,y)\right) dy \\
= \; & \int_K^\infty (y-K) \frac{\partial}{\partial y}\left(ryp(T,y)\right) dy \\
= \; & -r \int_K^\infty yp(T,y) dy
\end{aligned}
$$

with the assumption

$$
\lim_{y \to \infty} (y-k) yp(T,y) = 0
$$

$\square$

Dupire's Equation

Proof.

Similarly

$$\begin{aligned} & \int (y-K)^{+} \frac{\partial^2}{\partial y^2} \left(\sigma^2(T,y) y^2 p(T,y) \right) dy \\ = & \int_K^{\infty} (y-K) \frac{\partial^2}{\partial y^2} \left(\sigma^2(T,y) y^2 p(T,y) \right) dy \\ = & \sigma^2(T,K) K^2 p(T,K) \end{aligned}$$

with the assumption

$$\lim_{y \to \infty} (y-K) \frac{\partial}{\partial y} \left(\sigma^2(T,y) y^2 p(T,y) \right)$$

and

$$\lim_{y \to \infty} \sigma^2(T,y) y^2 p(T,y) = 0$$

□

Dupire's Equation

Proof.

Hence

$$
\begin{aligned}
& \int (y-K)^+ \tfrac{\partial}{\partial T} p(T,y)dy \\
= \quad & r\int_K^\infty yp(T,y)dy + \tfrac{1}{2}\sigma^2(T,K)K^2 p(T,K)
\end{aligned}
$$

$\square$

Dupire's Equation

Proof.

It follows that

$$
\begin{aligned}
& C_T(T,K) \\
= \; & -rC(T,K) + e^{-rT}\int (y-K)^+ \frac{\partial}{\partial T} p dy \\
= \; & -rC(T,K) + re^{-rT}\int_K^\infty yp(T,y)dy \\
& +\frac{1}{2}e^{-rT}\sigma^2(T,K)K^2 p(T,K) \\
= \; & -rKe^{-rT}\int_K^\infty p(T,y)dy + \frac{1}{2}e^{-rT}\sigma^2(T,K)K^2 p(T,K) \\
= \; & -rKC_K(T,K) + \frac{1}{2}\sigma^2(T,K)K^2 C_{KK}(T,K)
\end{aligned}
$$

where in the last equality we use the identities for implied risk-neutral probability density. Q.E.D. □

Understanding Local Volatility

In this section we will look at some facts and properties of local volatility.

- Local variance as the conditional expectation of instantaneous variance
- Local volatility in terms of implied volatility
- Implied variance as the average of local variance over the life of option when there is no skew.

Local Variance as Conditional Expectation of Instantaneous Variance

Theorem (Tanaka-Meyer Formula)

Let X_t be an Ito process such that

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dW_t$$

and K is a real number. Then

$$\begin{aligned} & (X_t - K)^+ \\ = \; & (X_0 - K)^+ + \int_0^t H_K(X_s)dX_s + \frac{1}{2}\int_0^t \delta_K(X_s)d\,[X, X]\,(s) \end{aligned}$$

where $H_K(\cdot)$ is the Heaviside function and $\delta_K(\cdot)$ is the Dirac delta function.

Local Variance as Conditional Expectation of Instantaneous Variance

In differential form, we have

$$d\left(X_t - K\right)^+ = H_K(X_t)dX_t + \frac{1}{2}\delta_K(X_t)d\left[X, X\right](t)$$

Local Variance as Conditional Expectation of Instantaneous Variance

If we let $f(x) = (x - K)^+$, then

$$f'(x) = H_K(x)$$

and

$$f''(x) = \delta_K(x)$$

Hence Tanaka-Meyer formula is

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X,X](t)$$

which is a generalization of Ito's formula.

Local Variance as Conditional Expectation of Instantaneous Variance

Now we assume the stock price follows

$$\frac{dS_t}{S_t} = rdt + \sigma(t,\omega)dW_t$$

where σ is an arbitrary adapted-process.

We want to investigate how the instantaneous volatility $\sigma(t,\omega)$ is related to local volatility as defined in Equation (10).

Local Variance as Conditional Expectation of Instantaneous Variance

The price for the call option with expiry T and strike K is

$$C(T,K) = e^{-rT}\mathbb{E}\left[(S_T - K)^+\right]$$

Taking derivative w.r.t. T we have

$$C_T = -rC + e^{-rT}\frac{\partial}{\partial T}\mathbb{E}\left[(S_T - K)^+\right] \tag{11}$$

Local Variance as Conditional Expectation of Instantaneous Variance

To evaluate

$$\frac{\partial}{\partial T}\mathbb{E}\left[(S_T - K)^+\right]$$

we will use Tanaka-Meyer formula.

Local Variance as Conditional Expectation of Instantaneous Variance

Applying Tanaka-Meyer formula, we have

$$\begin{aligned} & d(S_T - K)^+ \\ = \; & H_K(S_T)dS_T + \tfrac{1}{2}\delta_K(S_T)d\left[S,S\right](T) \\ = \; & \left[H_K(S_T)rS_T + \tfrac{1}{2}S_T^2\sigma^2(T,\omega)\delta_K(S_T)\right]dT \\ & + H_K(S_T)\sigma(T,\omega)dW_T \end{aligned}$$

Taking expectation on both sides, we have

$$\begin{aligned} & \mathbb{E}\left[d(S_T - K)^+\right] \\ = \; & \mathbb{E}\left[H_K(S_T)rS_T + \tfrac{1}{2}S_T^2\sigma^2(T,\omega)\delta_K(S_T)\right]dT \\ & + \mathbb{E}\left[H_K(S_T)\sigma(T,\omega)dW_T\right] \end{aligned}$$

Local Variance as Conditional Expectation of Instantaneous Variance

Since $H_K(S_T)$ and $\sigma(T,\omega)$ are $\mathcal{F}_T$-measurable, using iterated property of conditional expectation, we have

$$\begin{aligned} & \mathbb{E}\left[H_K(S_T)\sigma(T,\omega)dW_T\right] \\ = & \mathbb{E}\left[\mathbb{E}\left[H_K(S_T)\sigma(T,\omega)dW_T|\mathcal{F}_T\right]\right] \\ = & \mathbb{E}\left[H_K(S_T)\sigma(T,\omega)\mathbb{E}\left[dW_T|\mathcal{F}_T\right]\right] \\ = & 0 \end{aligned}$$

where in the last equality we use the fact that

$$\mathbb{E}\left[dW_T|\mathcal{F}_T\right] = 0$$

Local Variance as Conditional Expectation of Instantaneous Variance

Hence

$$\begin{aligned} & \mathbb{E}\left[d(S_T-K)^+\right] \\ = \ & \mathbb{E}\left[H_K(S_T)rS_T + \tfrac{1}{2}S_T^2\sigma^2(T,\omega)\delta_K(S_T)\right]dT \end{aligned}$$

Dividing both sides by dT, we have

$$\begin{aligned} & \tfrac{\partial}{\partial T}\mathbb{E}\left[(S_T-K)^+\right] \\ = \ & \mathbb{E}\left[H_K(S_T)rS_T + \tfrac{1}{2}S_T^2\sigma^2(T,\omega)\delta_K(S_T)\right] \\ = \ & r\mathbb{E}\left[H_K(S_T)S_T\right] + \tfrac{1}{2}\mathbb{E}\left[S_T^2\sigma^2(T,\omega)\delta_K(S_T)\right] \end{aligned} \tag{12}$$

Local Variance as Conditional Expectation of Instantaneous Variance

To evaluate $\mathbb{E}\left[H_K(S_T)S_T\right]$, we notice that

$$\begin{aligned} & \mathbb{E}\left[H_K(S_T)S_T\right] \\ = \quad & \mathbb{E}\left[(S_T - K)H_K(S_T)\right] + K\mathbb{E}\left[H_K(S_T)\right] \end{aligned} \tag{13}$$

Local Variance as Conditional Expectation of Instantaneous Variance

Note that

$$\begin{aligned}\mathbb{E}\left[(S_T - K)H_K(S_T)\right] &= \mathbb{E}\left[(S_T - K)^+\right] \\ &= e^{rT}C\end{aligned} \tag{14}$$

Since

$$\begin{aligned}\frac{\partial}{\partial K}\mathbb{E}\left[(S_T - K)^+\right] &= \mathbb{E}\left[\frac{\partial (S_T - K)^+}{\partial K}\right] \\ &= -\mathbb{E}\left[H_K(S_T)\right]\end{aligned}$$

we have

$$\begin{aligned}\mathbb{E}\left[H_K(S_T)\right] &= -\frac{\partial}{\partial K}\mathbb{E}\left[(S_T - K)^+\right] \\ &= -e^{rT}C_K\end{aligned} \tag{15}$$

Local Variance as Conditional Expectation of Instantaneous

Substituting Equation (14) and (15) into Equation (13), we have

$$\mathbb{E}\left[H_K(S_T)S_T\right] = e^{rT}\left(C - KC_K\right) \tag{16}$$

Local Variance as Conditional Expectation of Instantaneous Variance

Now we evaluate $\mathbb{E}\left[S_T^2\sigma^2(T,\omega)\delta_K(S_T)\right]$.

Using tower property of conditional expectation, we have

$$
\begin{aligned}
& \mathbb{E}\left[S_T^2\sigma^2(T,\omega)\delta_K(S_T)\right] \\
= \; & \mathbb{E}\left[\mathbb{E}\left[S_T^2\sigma^2(T,\omega)\delta_K(S_T)|S_T\right]\right] \\
= \; & \mathbb{E}\left[\mathbb{E}\left[\sigma^2(T,\omega)|S_T\right]S_T^2\delta_K(S_T)\right]
\end{aligned}
$$

Local Variance as Conditional Expectation of Instantaneous Variance

Let $p(x)$ be the probability density function of S_T, then

$$
\begin{aligned}
& \mathbb{E}\left[\mathbb{E}\left[\sigma^2(T,\omega)|S_T\right] S_T^2\delta_K(S_T)\right] \\
= & \int \mathbb{E}\left[\sigma^2(T,\omega)|S_T = x\right] x^2\delta_K(x)p(x)dx \\
= & K^2 p(K)\mathbb{E}\left[\sigma^2(T,\omega)|S_T = K\right]
\end{aligned}
$$

Note that in the last equality we use the following property of Dirac delta function

$$\int f(x)\delta_K(x)dx = f(K)$$

Local Variance as Conditional Expectation of Instantaneous Variance

Since

$$\begin{aligned} p(K) &= \frac{\partial^2}{\partial K^2}\mathbb{E}\left[(S_T - K)^+\right] \\ &= e^{rT} C_{KK} \end{aligned}$$

we have

$$\begin{aligned} & \mathbb{E}\left[S_T^2 \sigma^2(T,\omega)\delta_K(S_T)\right] \\ = \ & e^{rT} K^2 C_{KK} \mathbb{E}\left[\sigma^2(T,\omega) | S_T = K\right] \end{aligned} \tag{17}$$

Local Variance as Conditional Expectation of Instantaneous Variance

Substituting Equation (16) and Equation (17) into Equation (12), we have

$$\begin{aligned} & \frac{\partial}{\partial T}\mathbb{E}\left[(S_T - K)^+\right] \\ = \; & e^{rT}\left(r\left(C - KC_K\right) + \frac{1}{2}K^2 C_{KK}\mathbb{E}\left[\sigma^2(T,\omega)|S_T = K\right]\right) \end{aligned} \tag{18}$$

Local Variance as Conditional Expectation of Instantaneous Variance

Substituting the above equation into Equation (11), we have

$$C_T = -rKC_K + \frac{1}{2}K^2 C_{KK}\mathbb{E}\left[\sigma^2(T,\omega)|S_T = K\right]$$

which implies

$$\mathbb{E}\left[\sigma^2(T,\omega)|S_T = K\right] = \frac{C_T + rKC_K}{\frac{1}{2}K^2 C_{KK}} \tag{19}$$

Local Variance as Conditional Expectation of Instantaneous Variance

Comparing Equation (19) with Equation (10), we can see that local variance is the risk-neutral expectation of the instantaneous variance conditional on the final stock price S_T being equal to strike K.

Local Variance as Conditional Expectation of Instantaneous Variance

We can see that in general the solution for Equation (19) is not unique, there are two instantaneous volatility process $\sigma = \sigma'$ such that

$$\mathbb{E}\left[\sigma(T,\omega)|S_T = K\right] = \mathbb{E}\left[\sigma'(T,\omega)|S_T = K\right]$$

Knowing the vanilla option prices is not enough to find σ in general.

Local Variance as Conditional Expectation of Instantaneous Variance

However if we restrict ourselves to the case $\sigma(t, S_t)$ as a deterministic function of t and S_t, we can uniquely determine $\sigma(t, S_t)$ from vanilla option prices from Dupire's equation.

Local Volatility from Implied Volatility

In practice we often work in terms of implied volatilities as opposed to price. In the following we will derive the local volatility in terms of implied volatilities.

Local Volatility from Implied Volatility

Recall that Dupire's equation

$$\sigma^(T, K) = \frac{C_T + rKC_K}{\frac{1}{2}K^2 C_{KK}}$$

We assume

$$C(T, K) = C_{BSM}(T, K, \Sigma(T, K))$$

where C_{BSM} is the BSM pricing formula and $\Sigma(T, K)$ is the implied volatility.

Local Volatility from Implied Volatility

The numerator is

$$\begin{aligned} & C_T + rKC_K \\ = \quad & \frac{\partial C_{BSM}}{\partial T} + \frac{\partial C_{BSM}}{\partial \Sigma}\frac{\partial \Sigma}{\partial T} + rK\left(\frac{\partial C_{BSM}}{\partial K} + \frac{\partial C_{BSM}}{\partial \Sigma}\frac{\partial \Sigma}{\partial K}\right) \end{aligned} \tag{20}$$

Local Volatility from Implied Volatility

From

$$\frac{\partial C_{BSM}}{\partial T} = \frac{S\Sigma\phi(d_1)}{2\sqrt{T}} + re^{-rT}K\Phi(d_2)$$

$$\frac{\partial C_{BSM}}{\partial K} = -e^{-rT}\Phi(d_2)$$

and

$$\frac{\partial C_{BSM}}{\partial \Sigma} = S\sqrt{T}\phi(d_1)$$

we have

$$\frac{\partial C_{BSM}}{\partial T} + rK\frac{\partial C_{BSM}}{\partial K} = \frac{\partial C_{BSM}}{\partial \Sigma}\frac{\Sigma}{2T}$$

Local Volatility from Implied Volatility

Substituting this into Equation (20), the numerator becomes

$$\frac{\partial C_{BSM}}{\partial \Sigma}\left(\frac{\Sigma}{2T}+\frac{\partial \Sigma}{\partial T}+rK\frac{\partial \Sigma}{\partial K}\right) \tag{21}$$

Local Volatility from Implied Volatility

For the denominator, we have

$$
\begin{aligned}
& C_{KK} \\
= \quad & \frac{\partial^2 C_{BSM}}{\partial K^2} + 2\frac{\partial^2 C_{BSM}}{\partial K \partial \Sigma}\frac{\partial \Sigma}{\partial K} + \frac{\partial^2 C_{BSM}}{\partial \Sigma^2}\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \frac{C_{BSM}}{\partial \Sigma}\frac{\partial^2 \Sigma}{\partial K^2}
\end{aligned}
$$

Local Volatility from Implied Volatility

From

$$\frac{\partial^2 C_{BSM}}{\partial K^2} = \frac{C_{BSM}}{\partial \Sigma} \frac{1}{\Sigma T K^2}$$

$$\frac{\partial^2 C_{BSM}}{\partial K \partial \Sigma} = \frac{C_{BSM}}{\partial \Sigma} \frac{d_1}{\Sigma \sqrt{T} K}$$

and

$$\frac{\partial^2 C_{BSM}}{\partial \Sigma^2} = \frac{C_{BSM}}{\partial \Sigma} \frac{d_1 d_2}{\Sigma}$$

we have

$$C_{KK} = \frac{C_{BSM}}{\partial \Sigma} \left(\frac{1}{\Sigma T K^2} + \frac{d_1}{\Sigma \sqrt{T} K} \frac{\partial \Sigma}{\partial K} + \frac{d_1 d_2}{\Sigma} \left(\frac{\partial \Sigma}{\partial K} \right)^2 + \frac{\partial^2 \Sigma}{\partial K^2} \right) \tag{22}$$

Local Volatility from Implied Volatility

Substituting Equation (21) and (22) into Dupire's Equation we have

$$\sigma^2(T,K) = \frac{\left(\frac{\Sigma}{2T} + \frac{\partial \Sigma}{\partial T} + rK\frac{\partial \Sigma}{\partial K}\right)}{\frac{1}{2}K^2\left(\frac{1}{\Sigma T K^2} + \frac{d_1}{\Sigma\sqrt{T}K}\frac{\partial \Sigma}{\partial K} + \frac{d_1 d_2}{\Sigma}\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \frac{\partial^2 \Sigma}{\partial K^2}\right)} \tag{23}$$

Implied Variance as the Average of Local Variance

Implied volatility is often interpreted as the market expectation of the average of volatility throughout the life of an option. This is in general not true. However it can be justified when there is no skew.

Implied Variance as the Average of Local Variance

Assume that there is no skew, i.e., $\Sigma(T, K)$ does not depend on K. In this case, the local volatility σ does not depend on K either. From Equation (23), we have

$$\sigma^2(T) = \Sigma^2 + 2T\Sigma\frac{\partial \Sigma}{\partial T}$$

which implies

$$\Sigma^2(T) = \frac{1}{T}\int_0^T \sigma^2(t)dt$$

Implied Variance as the Average of Local Variance

This shows

- implied volatility as the average of volatility for the life of option
- implied volatility is a global measure of volatility
- local volatility is a local measure of volatility for a particular pair of T and K.

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Thank you!