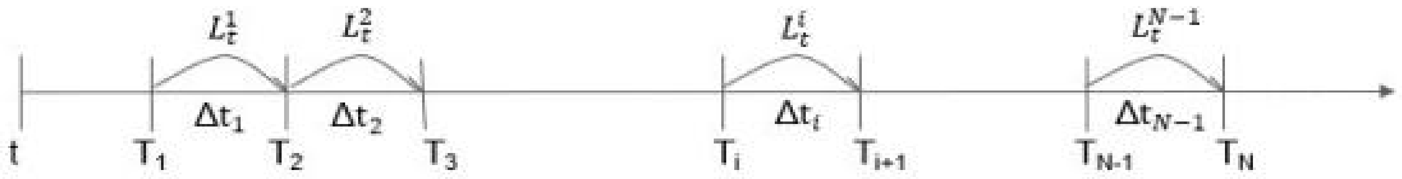
Section 4. BGM Model: the Evolution of Forward LIBOR Rates

- The Industrial Standard for Interest Rate Derivatives

4.1 Caplets and Forward LIBOR Dynamics under Separate Measures

Tenor Structure and Interest Periods

The construction of the LFM is based on a discretization of the yield curve on a set of consecutive interest periods:



Pricing Caplets

Caplet is a call option on a LIBOR with the payoff $\triangle t_n[L^n_{T_n} - X]^+$ settled at T_{n+1} for a notional amount of $\$1$, where X is the strike.

The LFM starts from a series of linear SDEs for pricing such caplets. For instance, we consider the price at time 0 of a T_2-maturing caplet reset at time T_1 with strike X and a notional amount of $\$1$. The payoff of this caplet is $\triangle t_1[L^1_{T_1} - X]^+$.

Denote by Q^2 the T_2-forward measure, which is the measure with the zero bond

$P(\cdot, T_2)$ being the numeraire. We have

$$L^1_t = \frac{P(t,T_1) - P(t,T_2)}{\triangle t_1 P(t,T_2)}. \quad (4.1)$$

(4.1) with $t = 0$ gives the initial value of forward LIBOR L^1_0.

The discounted bond prices $P(t,T_1)/P(t,T_2)$ and $P(t,T_2)/P(t,T_2)$ are both martingale under Q^2. Hence the process of LIBOR L_t^1 is also a martingale under Q^2. So we model the dynamics of L_t^1 under the Q^2 as follows:

$$dL_t^1 = \sigma_1(t)L_t^1 dW_t^{Q^2}. \quad (4.2)$$

where $W_t^{Q^2}$ is a Brownian motion under Q^2. This is a driftless lognormal process with time dependent volatility $\sigma_2(t)$.

Driftless Dynamics of LIBOR L_t^n under T_{n+1}-forward measure Q^{n+1}

$$\begin{cases} dL_t^n = \sigma_n(t) L_t^n dW_t^{Q^{n+1}} \\ L_0^n = \frac{P(0,T_n)-P(0,T_{n+1})}{\Delta t_n P(0,T_{n+1})} \end{cases}, \quad (4.3)$$

where $\sigma_n(t)$ are time dependent functions and $W_t^{Q^{n+1}}$ are Brownian motions under Q^{n+1} respectively.

Caplet Prices

The caplet prices produced by the LFM coincide with the results of the Black formula:

$$Cpl^{LFM}(0, T_n, T_{n+1}, K) = Cpl^{Black}(0, T_n, T_{n+1}, K, v_n)$$

$$= P(0, T_{n+1})\triangle t_n Bl(K, L_0^n, v_n), \ (4.4)$$

$$Bl(K, L_0^n, v_n) = L_0^n \phi(d_1(K, L_0^n, v_n)) - K\phi(d_2(K, L_0^n, v_n)), \ (4.5)$$

$$d_1(K, L_0^n, v_n) = \frac{\ln \frac{L_0^n}{K} + \frac{v_n^2 T_n}{2}}{v_n}, \ (4.6) \qquad d_2(K, L_0^n, v_n) = \frac{\ln \frac{L_0^n}{K} - \frac{v_n^2 T_n}{2}}{v_n}, \ (4.7)$$

where

$$v_n^2 = \frac{1}{T_n}\int_0^{T_n} \sigma_n^2(t)dt. \ (4.8)$$

4.2 LFM/BGM Model under a Unified Measure

What is the dynamics of L_t^1 under Q^1?

We solve this problem with the purpose of explaining the mathematical theory and techniques needed for deriving the LFM.

Some Notations

S_t : process of numeraire.
U_t : process of numeraire.
Q^S : martingale measure induced by S_t.
Q^U : martingale measure induced by U_t.
$W_t^{Q^S}$: Brownian motion under Q^S.
$W_t^{Q^U}$: Brownian motion under Q^U.

Change of Drift Theorem

If a process X_t follows the SDE:

$$dX_t = m_t^S X_t dt + \sigma_t(X_t) dW_t^{Q^S}, \quad (4.9)$$

under Q^S, then the dynamics of X_t under Q^U is given as follows:

$$dX_t = m_t^U X_t dt + \sigma_t(X_t) dW_t^{Q^U} \quad (4.10)$$

with

$$m_t^U dt = m_t^S dt - d(\ln X_t) \cdot d\left(\ln \frac{S_t}{U_t}\right). \quad (4.11)$$

Remarks

– This result is derived based on the Girsanov theorem with

$$\frac{dQ^U}{dQ^S} = \frac{U_T S_0}{U_0 S_T}. \quad (4.12)$$

– Unlike the drift term, the volatility term in a dynamics does not vary with a choice of numeraire.

Dynamics of L_t^1 under Q^1

We are going to derive the drift for L_t^1 as in the following SDE under Q^1 :

$$dL_t^1 = m_t^{Q^1} L_t^1 dt + \sigma_1(t) L_t^1 dW_t^{Q^1}. \quad (4.13)$$

Consider two numeraires S_t and U_t :

$$S_t = P(t, T_2), \quad (4.14)$$

$$U_t = P(t, T_1). \quad (4.15)$$

Since

$$U_t \cdot \frac{1}{1+\triangle t_1 L_t^1} = S_t, \ (4.16)$$

we have

$$\ln \frac{S_t}{U_t} = -\ln(1+\triangle t_1 L_t^1), \ (4.17)$$

$$d\left(\ln \frac{S_t}{U_t}\right) = \cdots dt - \frac{\triangle t_1 \sigma_1(t) L_t^1}{1+\triangle t_1 L_t^1} dW_t^{Q^2} \ (4.18) \text{ under } Q^2,$$

$$d(\ln L_t^1) = \cdots dt + \sigma_1(t) dW_t^{Q^2} \ (4.19) \text{ under } Q^2,$$

$$d(\ln L_t^1) \cdot d\left(\ln \frac{S_t}{U_t}\right) = -\frac{\triangle t_1 \sigma_1^2(t) L_t^1}{1+\triangle t_1 L_t^1} dt. \ (4.20)$$

By (4.10) – (4.11),

$$m_i^{Q^1} = 0 - \left(-\frac{\Delta t_1 \sigma_1(t) L_i^1}{1 + \Delta t_1 L_i^1}\right)$$

$$= \frac{\Delta t_1 \sigma_1(t) L_i^1}{1 + \Delta t_1 L_i^1}, \quad (4.21)$$

or

$$dL_i^1 = \frac{\Delta t_1 \sigma_1(t) L_i^1}{1 + \Delta t_1 L_i^1} L_i^1 dt + \sigma_1(t) L_i^1 dW_i^1 \quad (4.22) \text{ under } Q^1,$$

where we use W_i^1 to stand for $W_i^{Q^1}$ for simplicity.

Discrete Money Account

$$B_d(0) = 1, \quad (4.23)$$

$$B_d(t) = \prod_{j=0}^{\beta(t)-1} \left(1 + \Delta t_j L_{T_j}^j\right) P(t, T_{\beta(t)}), \quad (4.24)$$

where $\beta(t)$ is the first index i such that the forward LIBOR L_t^i expires after time t, i.e.

$$\beta(t) = \min_{i=1,\cdots,N} \{i : t < T_i\}. \quad (4.25)$$

Investment Operation for this Money Account

$B_d(t)$ is obtained by starting from \$1 and re-invest at each tenor date T_i in the zero-coupon bond maturing at the next tenor T_{i+1}.

Spot (LIBOR) Measure

The Measure Q^d with $B_d(t)$ being the numeraire is called the spot measure or spot LIBOR measure.

LFM under the Spot LIBOR Measure

$$\boxed{dL_t^n = \sigma_n(t)L_t^n \sum_{j=\beta(t)}^{n} \frac{\triangle t_j \rho_{jn}\sigma_j(t)L_t^j}{1+\triangle t_j L_t^j}dt + \sigma_n(t)L_t^n dW_t^n,\ n = 1,\cdots,N,} \quad (4.26)$$

under Q^d, where W_t^n are Brownian motions under Q^d and ρ_{jn} is the correlation coefficient between W_t^j and W_t^n, i.e.

$$dW_t^j \cdot dW_t^n = \rho_{jn}dt. \quad (4.27)$$

– Similar to the derivation of (4.22), the derivation of (4.26) is done by using the charges-of-measure technique.

– The LFM is also referred to as the Brace-Gatarex-Musiela (**BGM**) model.

Correlation Matrix

The determination of the correlation matrix (ρ_{jn}) can done by estimation from historical data or by calibration to the swaption market. We will further address this issue.

4.3. Monte Carlo Method for LFM

– The LFM (4.26) is used to price exotic interest rate derivative products by Monte Carlo simulations.

– For very long dated products, the simulated paths of forward LIBOR could blow up due to numerical instability in computing the positive drift term. The higher the volatilities are, the worse the numerical approximation is.

– To avoid explosion, one can consider using the LFM under the forward measure.

LFM under the T_{N+1}-Forward/Terminal Measure

$$dL_t^n = -\sigma_n(t)L_t^n \sum_{j=n+1}^{N} \frac{\Delta t_j \rho_{jn} \sigma_j(t) L_t^j}{1 + \Delta t_j L_t^j} dt + \sigma_n(t) L_t^n dW_t^n, \; n = 1, \cdots, N, \quad (4.28)$$

under Q^{N+1}, where Q^{N+1} is the martingale measure with $P(t, T_{N+1})$ being the numeraire. Q^{N+1} is also called the T_{N+1}-forward measure or the terminal measure.

Remark

The drift terms of the LFM under the terminal measure is negative. So it overcomes the problem of explosion in Monte Carlo simulations.

A Guideline

A good numerical solution is one that preserves the critical properties that the true solution possesses in the real world.

Deflated Bond Prices

$$D_n(t) = \frac{P(t, T_n)}{P(t, T_{N+1})}$$

$$= \prod_{j=n}^{N} \left(1 + \delta L_t^j\right), \quad (4.29)$$

where we set $\triangle t_j = \delta, \ \forall j,$ for ease of expression and consider the case of one-factor model, i.e. $W_t^k = W_t, \ \forall k.$

Differences of Deflated Bonds

$$X_t^n = \frac{1}{\delta}(D_n(t) - D_{n+1}(t))$$

$$= L_t^n \prod_{i=n+1}^{N}(1 + \delta L_t^i). \quad (4.30)$$

Martingale Processes X_t^n

$$\frac{dX_t^n}{X_t^n} = \left(\sigma_n + \sum_{j=n+1}^{N} \frac{\delta X_t^j \sigma_j}{1 + \delta X_t^j + \cdots + \delta X_t^N}\right) dW, \quad (4.31)$$

noting that

$$L_t^n = \frac{X_t^n}{1 + \delta X_t^{n+1} + \cdots + \delta X_t^N}. \quad (4.32)$$

The Method of No Arbitrage Monte Carlo Simulation

(1) Simulate of $\hat{X}^n$ with time step size h :

$$\hat{X}^n_{(i+1)h} = \hat{X}^n_{ih} \exp\left[-\frac{1}{2}\hat{V}^2_{\hat{X}_n}(ih)h + \sqrt{h}\,\hat{V}_{\hat{X}_n}(ih)\xi_{i+1}\right], \quad (4.33)$$

$$\hat{V}_{\hat{X}_n}(ih) = \hat{\sigma}_n(ih) + \sum_{j=n+1}^{N} \frac{\delta \hat{X}^j_{ih}\hat{\sigma}_j(ih)}{1 + \delta \hat{X}^j_{ih} + \cdots + \delta \hat{X}^N_{ih}}, \quad (4.34)$$

$$\hat{\sigma}_j(ih) = \sqrt{\frac{1}{h}\int_0^h \sigma_j^2(ih+u)du}\,. \quad (4.35)$$

where ξ_{i+1} are simples of iid $\mathcal{N}(0,1)$. This numerical scheme naturally ensures that the discrete process $\hat{X}^n_{(i+1)h}$ is a martingale. Sometimes we can take $\hat{\sigma}_j(ih) = \sigma_j(ih)$ for approximation.

(2) Compute the LIBOR rates:

$$\hat{L}^n = \frac{\hat{X}^n}{1 + \delta \hat{X}^{n+1} + \cdots + \delta \hat{X}^N} \quad (4.36)$$

and the deflated bond prices:

$$\hat{D}_n = 1 + \delta \sum_{j=n}^{N} \hat{X}^j. \quad (4.37)$$

(3) Compute the payout and price of a product.

Advantages of the Method

(1) This Monte Carlo method makes the deflated bond prices $\hat{D}$ discrete-time positive martingales, i.e. is arbitrage free.

(2) The last caplet price produced by the method has no discretization error, i.e. is identical to the result of $(4.4)-(4.7)$, if exact intregrals are obtained for (4.35).

4.4. Forward LIBOR Correlations and Swaption

Swaption Is a Correlation Product

While the volatility functions $\sigma_n(t)$ in LFM solely determine the prices of caplets and hence caps, $\sigma_n(t)$ and correlation coefficient matrix (ρ_{ij}) jointly determine the prices of swaptions.

Forward Swap Rate as a Function of LIBORs

$$S^{\alpha,\beta}(t) = \sum_{i=\alpha}^{\beta-1} \omega_i(t) L_t^i, \quad (4.39)$$

$$\omega_i(t) = \frac{\triangle t_i P(t, T_{i+1})}{\sum_{k=\alpha}^{\beta-1} \triangle t_k P(t, T_{k+1})}. \quad (4.40)$$

Linear Approximation of Forward Swap Rate

$$S^{\alpha,\beta}(t) \approx \sum_{i=\alpha}^{\beta-1} \omega_i(0) L_t^i. \quad (4.41)$$

Empirical studies Monte Carlo simulations shows that the variability of the ω's is much smaller than the variability of the F's.

Rebonato's Formula

The swaption volatilities $v_{\alpha,\beta}^{LFM}$ can be approximated as follows:

$$(v_{\alpha,\beta}^{LFM})^2 \approx \sum_{i,j=\alpha}^{\beta-1} \frac{\omega_i(0)\omega_j(0)L_0^i L_0^j \rho_{ij}}{S^{\alpha,\beta}(0)^2} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt. \quad (4.42)$$

The Derivation (Practitioner's Style)

It follows from (4.41) that

$$dS^{\alpha,\beta}(t) \approx \sum_{i=\alpha}^{\beta-1} \omega_i(0) dL_t^i$$

$$= \cdots dt + \sum_{i=\alpha}^{\beta-1} \omega_i(0)\sigma_i(t) L_t^i dW_t^i. \quad (4.43)$$

The quadratic variation

$$dS^{\alpha,\beta}(t) dS^{\alpha,\beta}(t) \approx \sum_{i,j=\alpha}^{\beta-1} \omega_i(0)\omega_j(0) L_t^i L_t^j \rho_{ij} \sigma_i(t)\sigma_j(t) dt. \quad (4.44)$$

The percentage quadratic variation is

$$d\ln S^{\alpha,\beta}(t)\, d\ln S^{\alpha,\beta}(t) = \frac{dS^{\alpha,\beta}(t)}{S^{\alpha,\beta}(t)} \frac{dS^{\alpha,\beta}(t)}{S^{\alpha,\beta}(t)}$$

$$= \frac{1}{S^{\alpha,\beta}(t)^2} \sum_{i,j=\alpha}^{\beta-1} \omega_i(0)\omega_j(0) L_t^i L_t^j \rho_{ij} \sigma_i(t) \sigma_j(t) dt$$

$$\approx \frac{1}{S^{\alpha,\beta}(0)^2} \sum_{i,j=\alpha}^{\beta-1} \omega_i(0)\omega_j(0) L_0^i L_0^j \rho_{ij} \sigma_i(t) \sigma_j(t) dt. \quad (4.45)$$

The par percentage volatility $\sigma_{\alpha,\beta}^{LFM}$ of $S^{\alpha,\beta}(T_\alpha)$ is given by

$$(\sigma_{\alpha,\beta}^{LFM})^2 T_\alpha = \int_0^{T_\alpha} d\ln S^{\alpha,\beta}(t)\, dS^{\alpha,\beta}(t)$$

$$\approx \frac{1}{S^{\alpha,\beta}(0)^2} \sum_{i,j=\alpha}^{\beta-1} \frac{\omega_i(0)\omega_j(0)L_0^i L_0^j \rho_{ij}}{} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt. \quad (4.46)$$

(4.46) can be used for calibrating the LIBOR correlation matrix (ρ_{ij}) to the market prices of swaptions.