# Lecture 5 - Continuous-Time Interest Rate Models II

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# The PDE Approach to Bond Pricing

- A general approach taken by Vasicek (1977).
- Similar to the equity-derivative approach developed by Black and Scholes (1973).
- Although the martingale approach is generally thought to be the more powerful and intuitive, the PDE approach still provides a useful tool for the development of numerical methods (see Chapter 10).

# Black-Scholes

- We start with a quick review of the Black-Scholes formula derivation.

- Suppose a stock evolves according to the rule

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- The risk-free interest rate is the constant $r$. We seek to find the price of a call option, which is a function of time and the underlying stock price.

$$C_t = f(t, S_t)$$

- To price the call function, we must determine the unknown function $f$.

# Dynamic Replication

- Form a portfolio consisting of one share of the call option, and $-\Delta_t$ shares of the underlying stock.

- The call option follows the process

$$dC_t = [\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial S_t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 f}{\partial S_t^2}]dt + \sigma S_t \frac{\partial f}{\partial S_t} dW_t$$

- The portfolio consisting of the call option and $-\Delta_t$ shares of the stock follows

$$d[C_t - \Delta_t S_t] = [\frac{\partial f}{\partial t} + \mu S_t (\frac{\partial f}{\partial S_t} - \Delta_t) + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 f}{\partial S_t^2}]dt + \sigma S_t (\frac{\partial f}{\partial S_t} - \Delta_t) dW_t$$

# Hedging

- If we choose
$$\Delta_t = \frac{\partial f}{\partial S_t}$$
then the portfolio is instantaneously hedged,
$$d[C_t - \Delta_t S_t] = [\frac{\partial f}{\partial t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial S_t^2}]dt$$
- Since the portfolio has no risk, it must earn the risk-free rate of return.
$$\begin{aligned} d[C_t - \Delta_t S_t] &= [\frac{\partial f}{\partial t} + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial S_t^2}]dt = r(C_t - \Delta_t S_t)dt \\ &= r(f - \frac{\partial f}{\partial S_t} S_t)dt \end{aligned}$$

# Pricing Partial Differential Equation

- After some rearrangement, this becomes

$$\frac{\partial f}{\partial t} + rS_t \frac{\partial f}{\partial S_t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 f}{\partial S_t^2} = rf$$

- Every derivative (calls, puts, binary options, etc.) security whose price depends only on time and the stock price must satisfy this partial differential equation.

- However, different asset will have different boundary conditions. For a call option with strike price $K$ and maturity $T$, the solution must also satisfy

$$f(T, S_T) = \max(S_T - K, 0)$$

# Call Price Formula

- The call option price is the unique[1] function which solves both the general partial differential equation and the specific boundary condition:

$$f(t, S_t) = S_t \Phi\left(\frac{\ln \frac{S_t}{K} + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}\right)$$
$$- Ke^{-r(T-t)}\Phi\left(\frac{\ln \frac{S_t}{K} + (r - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}\right)$$

where $\Phi(\bullet)$ the cumulative distribution function of a Gaussian random variable,

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{2\pi} e^{-\frac{z^2}{2}} dz$$

[1]Strictly speaking, additional boundary conditions as the stock price approaches 0 and $+\infty$ are needed to specify a unique solution. However, the other solutions which do not satisfy the additional boundary conditions grow at a super fast rate as the stock price approaches 0 and $+\infty$, and are obviously not the correct option prices.

# Bond Pricing

The general principles in this development are that

- $r(t)$ is Markov: $dr(t) = a(t)dt + b(t)dW(t)$;
- $P(t, T)$ depend upon an assessment at time $t$ of how $r(s)$ will vary between $t$ and $T$;
  - the process will be derived and not specified now (short-rate methodology);
- the market is efficient, without transaction costs and all investors are rational.

- The first two principles ensure that $a(t) = a(t, r(t)), b(t) = b(t, r(t))$, and $P(t, T) = P(t, T, r(t))$. Thus, under a one-factor model, price changes for all bonds with different maturity dates are perfectly (but non-linearly) correlated.

# Bond Price Process

- Mimic the option pricing derivation: by Itô's formula

$$dP = \left[\frac{\partial P}{\partial t} + a\frac{\partial P}{\partial r} + b^2\frac{1}{2}\frac{\partial^2 P}{\partial r^2}\right] dt + b\frac{\partial P}{\partial r}dW$$

Equivalently, we write

$$dP = P(t, T, r)[m(t, T, r)dt + S(t, T, r)dW],$$

where

$$\begin{aligned} m(t, T, r) &= \frac{1}{P}\left[\frac{\partial P}{\partial t} + a\frac{\partial P}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 P}{\partial r^2}\right], \\ S(t, T, r) &= \frac{1}{P}b\frac{\partial P}{\partial r} \text{ (the volatility of the price).} \end{aligned} \tag{1}$$

# Risk-Free Portfolio

- Consider two bonds maturing at times $T_1$ and $T_2$ (with $T_1 < T_2$).
- At time $t$ suppose that we hold (dollar not unit) amounts $-V_1(t)$ in the $T_1$-bond (a short position) and $V_2(t)$ in the $T_2$-bond (a long position). The total wealth is $V(t) = V_2(t) - V_1(t)$. We will vary $V_1(t)$ and $V_2(t)$ in such a way that the portfolio is risk-free. The instantaneous investment gain from $t$ to $t + dt$ is

$$
\begin{aligned}
-\frac{V_1(t)}{P(t, T_1)} dP(t, T_1) + \frac{V_2(t)}{P(t, T_2)} dP(t, T_2) \\
&= -V_1(t)(m_1 dt + S_1 dW) + V_2(t)(m_2 dt + S_2 dW) \\
&= (V_2 m_2 - V_1 m_1) dt + (V_2 S_2 - V_1 S_1) dW,
\end{aligned}
$$

where for notational compactness, we write

$$
m_i = m(t, T_i, r(t)) \text{ and } S_i = S(t, T_i, r(t)) \text{ for } i = 1, 2.
$$

# Perfect Hedging

- To make the portfolio risk-free, we set

$$\frac{V_1(t)}{V_2(t)} = \frac{S(t, T_2, r(t))}{S(t, T_1, r(t))} = \frac{S_2}{S_1}.$$

- That is,

$$V_2 S_2 - V_1 S_1 = 0$$

and since $V = V_2 - V_1$,

$$V_2 m_2 - V_1 m_1 = \frac{S_1 V}{S_1 - S_2} m_2 - \frac{S_2 V}{S_1 - S_2} m_1.$$

# Market Price of Risk

- Hence, the instantaneous investment gain is equal to

$$V\left(\frac{m_2 S_1 - m_1 S_2}{S_1 - S_2}\right) dt.$$

- Thus, through our choice of portfolio strategy, we have a risk-free investment strategy.

- Self-financing??
  - NO! See Chapter 12 of Hirsa and Neftci (2014): An Introduction to the Mathematics of Financial Derivatives.

- Since this portfolio is risk-free, the principle of no arbitrage dictates that the portfolio growth rate must equal $r(t)$; that is,

$$\left(\frac{m_2 S_1 - m_1 S_2}{S_1 - S_2}\right) = r(t) \text{ or } \frac{m_1 - r}{S_1} = \frac{m_2 - r}{S_2}.$$

# Market Price of Risk

- This must be true for all maturities. Thus, for all $T > t$,

$$\frac{m(t, T, r(t)) - r(t)}{S(t, T, r(t))} = \gamma(t, r(t)),$$

  where $\gamma(t, r(t))$ is the *market price of risk*; that is, the extra return over $r(t)$ per unit of risk. The key observation here is that $\gamma$ cannot depend on the maturity date $T$.

- $\gamma(t, r(t))$ can often be negative since
  - $P(t, T, r(t))$ is usually a decreasing function of $r(t)$, i.e., $\frac{\partial P}{\partial r} < 0$
  - The volatility, $b(t, r(t))$, of $r(t)$ is usually positive $\Rightarrow$ $S(t, T, r(t)) < 0$.
  - Thus, $\gamma(t, r(t))$ must be negative to ensure that expected returns under $P$, $m(t, T, r(t))$, are greater than the risk-free rate, $r(t)$.

# Bond PDE

- Thus, $m(t, T, r)$ is the risk-free rate plus the risk premium, i.e.,

$$m(t, T, r) = r(t) + \gamma(t, r)S(t, T, r)$$

and (recapping equation (1)):

$$m(t, T, r) = \frac{1}{P}\left[\frac{\partial P}{\partial t} + a\frac{\partial P}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 P}{\partial r^2}\right],$$
$$S(t, T, r) = \frac{1}{P}b\frac{\partial P}{\partial r}.$$

- If we equate the two expressions for $m(t, T, r)$, we find that

$$\frac{\partial P}{\partial t} + (a - \gamma b)\frac{\partial P}{\partial r} + \frac{1}{2}b^2\frac{\partial^2 P}{\partial r^2} - rP = 0 \qquad (2)$$

# Feynman-Kac Formula

- This is of a suitable form to allow us to apply the Feynman-Kac formula (Theorem A.9)

$$\frac{\partial P}{\partial t} + f(t,r)\frac{\partial P}{\partial r} + \frac{1}{2}\rho^2(t,r)\frac{\partial^2 P}{\partial r^2} - R(r)P + h(t,r) = 0, \quad (3)$$

  where

  - $f(t,r) = a(t,r) - \gamma(t,r)b(t,r)$;
  - $\rho(t,r) = b(t,r)$;
  - $R(r) = r$;
  - $h(t,r) = 0$.

# Feynman-Kac Formula

- The boundary condition for this PDE is $P(T, T, r) = \psi(r) = 1$ for all $T, r$.

- By the Feynman-Kac formula there exists a suitable probability triple $(\Omega, \mathcal{F}, Q)$ with filtration $\{\mathcal{F}_t : 0 \leq t < \infty\}$ under which

$$P(t, T, r(t)) = E_Q \left[ \exp \left( - \int_t^T \tilde{r}(s) ds \right) | \mathcal{F}_t \right]. \tag{4}$$

- The process $\tilde{r}(s)$ $(t \leq s \leq T)$ is a Markov diffusion process with $\tilde{r}(t) = r(t)$; moreover, under the measure $Q$, $\tilde{r}(u)$ satisfies the SDE

$$d\tilde{r}(u) = f(u, \tilde{r}(u))du + \rho(u, \tilde{r}(u))d\tilde{W}(u), \tag{5}$$

where $\tilde{W}(u)$ is a standard Brownian motion under $Q$.

# Interest Rate Derivatives

- The Feynman-Kac formula can be applied to interest rate derivative contracts.

- Let $V(t)$ be the price at time $t$ of a derivative which will have a payoff to the holder of $\psi(r(T))$ at time $T$ (which could be described as a function of $P(T, S, r(T))$ if the underlying quantity is $P(t, S, r(t))$). As above we will have

$$\begin{aligned} \frac{\partial V}{\partial t} + f(t,r)\frac{\partial V}{\partial r} + \frac{1}{2}\rho^2(t,r)\frac{\partial^2 V}{\partial r^2} - R(r)V &= 0, \qquad (6) \\ \text{subject to } V(T) &= \psi(r(T)) \end{aligned}$$

where $f(t,r) = a(t,r) - \gamma(t,r)b(t,r), \rho(t,r) = b(t,r)$ and $R(r) = r$.

# Interest Rate Derivatives

- Again by the Feynman-Kac formula we have

$$V(t) = E_Q\left[\exp\left(-\int_t^T \tilde{r}(s)ds\right)\psi(\tilde{r}(T))|\mathcal{F}_t\right], \qquad (7)$$

where $\tilde{r}(s)$ is as in equation (5).

- So the only difference in the PDE problem when compared with the zero-coupon-bond case is in the boundary condition. These formulae are, of course, the same as what we derived from the martingale approach in Lecture 4.

# Remark

- We have developed these results by specifying first the dynamics of the model under $P$ before transferring to the equivalent measure $Q$.

- Practitioners typically start by specifying the dynamics under $Q$ directly (which we will also do below). This immediately gives us the relevant pricing formulae provided we know the parameter values.

- Knowledge of the dynamics under $P$ is not always required but, if they are, the market price of risk, $\gamma(t)$, can then be introduced at this stage. If this approach is taken, modelers must be confident that $\gamma(t)$ satisfies the Novikov condition.

# Gaussian (Markov) Short-Rate Models

- Consider the case where $r(t)$ follows a Gaussian Markov process.
- For Gaussian process we know that $\int_t^T r(u)du$ is a normal random variable. Moreover, for a normal random variable $X \sim N(\mu, \sigma^2)$ we know that
$$E[\exp(X)] = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$
- Hence, to evaluate $E_Q\left[\exp\left(-\int_t^T r(u)du\right)|\mathcal{F}_t\right]$, it suffices to find the mean and variance of $\int_t^T r(u)du$ and apply
$$\begin{aligned} P(t,T) &= E_Q\left[\exp\left(-\int_t^T r(u)du\right)|\mathcal{F}_t\right] \\ &= \exp\begin{pmatrix} -E_Q\left(\int_t^T r(u)du|\mathcal{F}_t\right) \\ +\frac{1}{2}Var_Q\left[\int_t^T r(u)du|\mathcal{F}_t\right] \end{pmatrix} \end{aligned}$$

# Ho and Lee (1986)

- Suppose that $dr(t) = \theta(t)\, dt + \sigma d\tilde{W}(t)$. Hence, we know that
$$r(u) = r(0) + \int_0^u \theta(s)\, ds + \sigma \int_0^u d\tilde{W}(s).$$
- Thus,
$$\begin{aligned} \int_t^T r(u) du &= r(0)(T-t) + \int_t^T \int_0^u \theta(s)\, dsdu \\ &\quad + \sigma \int_t^T \int_0^u d\tilde{W}(s)\, du \end{aligned}$$

- Interchanging the order of integration, we have

$$\begin{aligned}\int_t^T \int_0^u \theta(s)\, dsdu &= \int_0^t \int_t^T \theta(s)\, duds + \int_t^T \int_s^T \theta(s)\, duds \\ &= (T-t)\int_0^t \theta(s)\, ds + \int_t^T \theta(s)(T-s)\, ds\end{aligned}$$

- Likewise,

$$\begin{aligned}\int_t^T \int_0^u d\tilde{W}(s)\, du &= \int_0^t \int_t^T dud\tilde{W}(s) + \int_t^T \int_s^T dud\tilde{W}(s) \\ &= (T-t)\int_0^t d\tilde{W}(s) + \int_t^T (T-s)\, d\tilde{W}(s)\end{aligned}$$

- It follows that

$$\begin{aligned}\int_t^T r(u)du &= (T-t)\, r(t) + \int_t^T \theta(s)(T-s)\, ds \\ &\quad +\sigma \int_t^T (T-s)\, d\tilde{W}(s).\end{aligned}$$

# Ho and Lee (1986)

- Thus,

$$
\begin{aligned}
& E_Q\left[\int_t^T r(u)du|\mathcal{F}_t\right] \\
= & \ (T-t)\,r(t) + \int_t^T \theta(s)\,(T-s)\,ds.
\end{aligned}
$$

- Moreover,

$$
\begin{aligned}
Var\left[\int_t^T r(u)du|\mathcal{F}_t\right] &= \int_t^T \sigma^2\,(T-s)^2\,ds \text{ (Itô isometry)} \\
&= \frac{\sigma^2\,(T-t)^3}{3}.
\end{aligned}
$$

# Vasicek (1977)

- Recall $dr(t) = a(b - r(t))\,dt + \sigma d\tilde{W}(t)$ which is a Ornstein–Uhlenbeck process. Hence, we know that

$$\begin{aligned} r(u) &= e^{-au} r(0) + ab\int_0^u e^{-a(u-s)} ds \\ &\quad + \sigma \int_0^u e^{-a(u-s)} d\tilde{W}(s). \end{aligned}$$

- It follows that

$$\begin{aligned} & \int_t^T r(u)du \\ = \; & \int_t^T e^{-au} r(0)\,du + ab\int_t^T \int_0^u e^{-a(u-s)} ds du \\ & + \sigma \int_t^T \int_0^u e^{-a(u-s)} d\tilde{W}(s)\,du. \end{aligned}$$

# Vasicek (1977)

- Again, we can calculate (**Exercise**)

$$E_Q\left(\int_t^T r(u)du|\mathcal{F}_t\right) \text{ and } Var\left[\int_t^T r(u)du|\mathcal{F}_t\right]$$

- using

$$\begin{aligned}&\int_t^T \int_0^u e^{-a(u-s)}dsdu\\ =\ &\int_0^t \int_t^T e^{-a(u-s)}dsdu + \int_t^T \int_s^T e^{-a(u-s)}dsdu;\end{aligned}$$

and likewise,

$$\begin{aligned}&\int_t^T \int_0^u e^{-a(u-s)}d\tilde{W}(t)du\\ =\ &\int_0^t \int_t^T e^{-a(u-s)}dud\tilde{W}(s) + \int_t^T \int_s^T e^{-a(u-s)}dud\tilde{W}(s)\end{aligned}$$

together with Itô isometry.

# Affine Short-Rate Models

- We have seen in the preceding sections that both the Ho and Lee and Vasicek models have zero-coupon bond prices which are of the *affine* form $P(t, T) = \exp[A(t, T) - B(t, T)r(t)]$ for functions $A$ and $B$ which are specific to each model.

- Are there any other models which give rise to similar affine forms for $P(t, T)$?

# Affine Short-Rate Models

- Consider the general SDE for $r(t)$

$$dr(t) = m(t, r(t))dt + s(t, r(t))d\tilde{W}(t)$$

where $\tilde{W}(t)$ is a standard Brownian motion under the risk-neutral measure $Q$.

- Suppose that

$$P(t, T, r(t)) = \exp[A(t, T) - B(t, T)r(t)].$$

- By Itô's formula, we have

$$dP(t, T) = P(t, T)\left[\begin{array}{c} \left(\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t}r(t) - Bm + \frac{1}{2}Bs^2\right)dt \\ -Bsd\tilde{W}(t) \end{array}\right] \tag{8}$$

(where $m \equiv m(t, r(t))$, etc.).

# Affine Short-Rate Models

- But we also know that, under $Q$,
$$dP(t,T) = P(t,T)[r(t)dt + S(t,T,r(t))d\tilde{W}(t)] \qquad (9)$$
where $S(t,T,r(t))$ is the volatility of $P(t,T)$. This equality comes from the requirement that all tradable assets must have expected growth at the risk-free rate under $Q$.
- It follows that if we define
$$g(t,r) = \frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} r - Bm(t,r) + \frac{1}{2} Bs(t,r)^2 - r,$$
then $g(t,r) = 0$ for all $t$ and $r$.
- Differentiate twice with respect to $r$
$$\frac{\partial^2 g}{\partial r^2} = -B(t,T)\frac{\partial^2 m(t,r)}{\partial r^2} + \frac{1}{2}B(t,T)^2 \frac{\partial^2 (s(t,r)^2)}{\partial r^2} = 0,$$
$$\implies -\frac{\partial^2 m(t,r)}{\partial r^2} + \frac{1}{2}B(t,T)\frac{\partial^2 (s(t,r)^2)}{\partial r^2} = 0.$$

# Affine Short-Rate Models

- Since $B(t, T)$ is a function of $T$ as well as $t$, this identity can only hold if both

$$\frac{\partial^2 (s(t,r)^2)}{\partial r^2} = 0 \text{ and } \frac{\partial^2 m(t,r)}{\partial r^2} = 0.$$

- It is a necessary condition for bond-pricing formulae to be of the form $P(t, T) = \exp[A(t, T) - B(t, T) r(t)]$ that the risk-neutral drift and volatility of $r(t)$ are of the form

  $$m(t, r(t)) = a(t) + b(t) r(t) \text{ and } s(t, r(t)) = \sqrt{\gamma(t) r(t) + \delta(t)},$$

  where $a(t), b(t), \gamma(t)$ and $\delta(t)$ are deterministic functions.

- For general, time-dependent $a(t), b(t), \gamma(t)$ and $\delta(t)$, analytical solutions for $A(t, T)$ and $B(t, T)$ are not normally available. However, we have the following cases.

# Ho and Lee (1986)

$a\,(t) = \theta\,(t)$, $b = 0$, $\gamma = 0$, and $\delta = \sigma^2$ which implies that

$$dr(t) = \theta\,(t)\,dt + \sigma d\tilde{W}(t).$$

This results in

$$\begin{aligned} B(t,T) &= T - t, \\ A(t,T) &= \int_t^T \theta\,(s)\,(s-T)\,ds + \frac{1}{6}\sigma^2(T-t)^3. \end{aligned}$$

The model becomes Merton (1973) when $\theta\,(s) = \theta$, a time-independent version of Ho and Lee (1986).

- (**Exercise**) Verify that this solution satisfies the Bond PDE in (3) with the boundary condition $P(T,T,r) = \psi(r) = 1$.

# Vasicek (1977)

$a = \alpha\mu$, $b = -\alpha$, $\gamma = 0$, and $\delta = \sigma^2$, which implies that $dr(t) = \alpha(\mu - r(t))dt + \sigma d\tilde{W}(t)$. Earlier, for this model, we found that

$$B(t, T) = (1 - e^{-\alpha(T-t)})/\alpha$$

and

$$A(t, T) = (B(t, T) - (T - t))(\mu - \sigma^2/2\alpha^2) - \frac{\sigma^2}{4\alpha} B(t, T)^2.$$

- (**Exercise**) Verify that this solution satisfies the Bond PDE in (3) with the boundary condition $P(T, T, r) = \psi(r) = 1$.

# Affine Short-Rate Models

- Two further cases:
  - Cox, Ingersoll, and Ross (1985) (a time-independent one in which short rate will never become negative);
  - Hull and White (1990) (a time-dependent version of Vasicek)
- In both cases analytical solutions can be found for bond prices.
- Also we don't talk about option pricing here which will be covered in the second half of the module using the so-called forward measure approach.