## Exercise 2:
## Deriving Zero Coupon Rates

Observe the following market prices for Singapore Government bills and bonds:

| | |
|---|---|
| 6-month bill: | 2.3% discount |
| 1-year bill: | 2.5% discount |
| Old 2.5% bond with 18mo to maturity: | $99.50 |

*(99.50 implies a YTM of 2.8429% s/a)*

To keep day-count from being a distraction, assume the exact half-year, i.e.

- The 6-month period is exactly half a year, 182.5 days/365
- The 12- and 18-month securities have exactly 2 and 3 half-years to maturity

Using the above pricing information and assumptions, give the correct series of discount factors and zero rates expressed on a semi-annual bond-equivalent yield basis:

1) Using market convention for discounting an SGS bill, what is the discount factor of the 6-month bill? (This is easy – it's simply the settlement price factor for $1 of face value) Use this formula:

$df_{6\ month}$ = $SettlePriceFactor_{6\ month\ bill}$ =
1 – (discount rate × days/365) =

1 – (.023 × 1/2) = **0.9885**
**Settle price (PV) of $1m (FV) bill = $988,500**

**"interest" = $1m - $988,500 = $11,500**
**"principal" = $988,500**
**So int/principal = 0.011633789 after 6mo, which is 2.3267 BEY (remember term = exactly ½ year for this example) and remember Discount factor = PV ($1) also called PV factor**

2) What is the corresponding bond-equivalent yield for this bill? Use this formula:

$$\text{BEY (target freq, TF)} = [(1 \div df_{period})^{\wedge(Freq/TF)} - 1] \times TF$$

$$\text{BEY (s/a Act/365)} = [(1 \div df_{6\ month\ bill})^{\wedge(2/2)} - 1] \times 2$$

**$= [(1 \div 0.9885)^{\wedge 1} - 1] \times 2 = 2.3267\%$ equiv. s/a act/365**

3) Using market convention for discounting an SGS bill, what is the discount factor of the 12-month bill? Use this formula:

$$df_{12\ month\ govt\ bill} = 1 - (\text{discount rate} \times \text{days}/365) =$$

$1 - (0.025 \times 365/365) =$ **0.9750**

*Means if you invest \$975,000 spot*
*You receive \$1m 1yr forward and there are no cash flows in between, so the annualised YTM or IRR \$25,000 / 975,000 = 2.5641% annual frequency → need semi-annual equivalent*

4) What is the corresponding bond-equivalent yield for this bill? Use this formula:

Target frequency is what you want = 2
Frequency is what you have = 1
$Freq/TF = TF_{period}/Freq_{period} = ½ = 6mo/12mo$

$$\textbf{BEY for target frequency, TF)} = \textbf{TF x } [(1 \div df_{period})^{\wedge(Freq/TF)} - 1]$$

$$2 \times [(1 \div df_{12mo})^{\wedge(1/2)} - 1] =$$

**$2 \times [(1 \div 0.975)^{\wedge(1/2)} - 1] = 2.5479\%$** s/a BEY (s/a act/365 equiv)

5) Using the discount factors for the 6-month and 1-year bills, calculate the PVs of the cash flows of the 18-month bond in the table.

- Subtract the PV of the first 2 coupons from the price to get the PV of the 18-month payment (PV of $CF_3$).

  99.50 - [(0.9885 + 0.975) x 1.25]=

  **97.045625**

**This 97.045625 is the total cash out of pocket today after buying the bond and stripping off its first 2 coupons and selling them for their PV**

- How do you attain the discount factor from this PV?
  **For zero coupon instrument, df = PV/FV, so 97.045625 / 101.25 = 0.958475**
- What allows us to use the bill rates to discount the bond's coupons? **Identical obligor, equivalent credit, same day of contractual CF, so same discounting rate or factor**

**Don't forget, with single cash-flow debt instuments, FV x df = PV, which means df = PV/FV**

| Cash flow | Future Value (CF) | Discount Factor (df) | Present Value of $CF_n$ |
|---|---|---|---|
| Price $100 par 18-mo bond at t0 | **-99.50** | **1.0000** | -99.50 |
| First Coupon at t0 + 6mo | ~~1.25~~ | **0.9885** | **1.235625** |
| Second Coupon at t0 + 12mo | ~~1.25~~ | **0.9750** | **1.21875** |
| Last Coupon + Par at t0 + 18mo | 101.25 FV | **0.958475** | **-97.045625 PV** |

Notice, that on the first day, you pay $99.50 for the bond, but then sell off two coupons of $1.25 future value collecting their PVs, which means your net cost at inception is $97.045625, and you're left with an instrument whose cash flows look just like a bill, with a PV outflow at the beginning and a FV inflow at the end, with no CFs in between. This means **97.045625 is the PV of 101.25!**

**df = PV/FV, so 97.045625 ÷ 101.25 = 0.958475**

*In this process we've synthetically created a 18-month zero-coupon instrument from a coupon-bearing instrument with f=0.66667 (i.e. 2/3 of a payment per year) but TF = 2 (payment frequency of semi-annual). We want this rate on s/a basis equiv, so on to question 6…*

6) What is the 18-month zero rate in semi-annual act/365 equivalent? Use this formula:

**TF x** $[(1 \div df_{period})^{\wedge(Freq/TF)} - 1]$ = yield s/a act/365

$2 \times [(1 \div df_{18\ month})^{\wedge 0.6666/2} - 1] =$

$2 \times [(1 \div \mathbf{0.958475})^{0.3333} - 1] = 2.8475\%$ s/a

**Also can calculate:**

**TF x** $[(1 \div df_{18mo})^{\wedge(6mo/18mo)} - 1] =$

$2 \times [(1 \div \mathbf{0.958475})^{0.3333} - 1] = 2.8475\%$ s/a

7) Now, if you saw a new 2-year SGS bond with a 3.0% coupon trading at 99.90

Calculate YTM in excel or on a financial calculator. Did you get 3.0519% s/a?),

Now use the above calculated information to figure the 2-year zero rate

| Cash flow | Future Value ($CF_n$) | Discount Factor ($df_n$) | Present Value of $CF_n$ |
|---|---|---|---|
| 24-month bond. Buy at t0 | -99.9 | 1.0 | **-99.90** |
| First Coupon at t0 + 6mo | **~~1.50~~** | **0.9885** | **1.48275** |
| Second Coupon at t0 + 12mo | **~~1.50~~** | **0.975** | **1.4625** |
| Third Coupon at t0 + 18mo | **~~1.50~~** | **0.958475** | **1.4377** |
| Last Coupon + Par at t0 + 24mo | **101.50** | **.94105** | **-95.517** |

Just to finish up, review this table and see if you understand how these numbers were attained and if you have attained the same ones. *The BEY rates were convertible from the price information given you. The Zero Coupon discount factors and rates were derivable in a process generally known as* yield-curve boot-strapping.

| Maturity | ZC Disc.Factor | S/A ZC BEY | S/A YTM (BEY)* |
|---|---|---|---|
| ½ Year | **0.9885** | 2.3267% | 2.3267% |
| 1 Year | **0.9750** | 2.5479% | 2.5479% |
| 1½ Years | **0.95848** | 2.8475% | 2.8429% |
| 2 Years | **0.94105** | 3.0611% | 3.0519% |

*You may have slightly different answers due to rounding*

*All BEYs are on semi-annual Act/365 in this analysis. These are the BEYs and YTMs (same freq and day-count) from the visible prices in the market for bills and bonds (Together known as the “PAR curve”)

## Exercise 3: Valuing Bonds Using Zero Coupon Rates

We use the Zero Coupon Rates and Discount Factors we derived in Exercise 2 for this exercise:

| Maturity | ZC Rate | Discount Factor* |
|---|---|---|
| ½ Year | 2.3267% | 0.9885 |
| 1 Year | 2.5479% | 0.9750 |
| 1½ Years | 2.8475% | 0.958475 |
| 2 Years | 3.0611% | 0.94105 |

* To keep it simplest, don't worry about leap (366 days) years and assume ½ a year is exactly 0.5 of a year (182½ days)

1) Find the value of (based on $100 of par) the following 2-year Singapore government securities (s.a. act/act) to the above zero-curve. Follow the example of pricing the 10% coupon bond, and then do the same for the 0% and 6% bonds:

   **Value of 10% coupon** (i.e. $100 of par & $5 coupons)
   **= $\Sigma(CF_n \times df_n)$**
   = (5 x .9885)+(5 x .9750)+(5 x .958475)+(105 x .94105)
   = $113.371 per $100 of par
   YTM (-113.42 PV at t0, s/a coupon = $5, Par repayment of $100) = 3.0336% s/a

   **Value of 6% coupon bond**
   = [3 x(.9885 +.9750 +.958475 +.94105)]+(100 x.94105)= $105.694 per $100 of par
   YTM (-105.694 PV at t0, s/a coupon = $3, Par repayment of $100) = 3.0438% s/a

   **Value of 0-coupon bond** = 0.94105 x $100 = 94.105.
   YTM = 3.0611%

| Coupon | Value | YTM(IRR) |
|---|---|---|
| 0% | 94.105 | 3.0611% |
| 6% | 105.694 | 3.0438% |
| 10% | 113.42 | 3.0336% |

2. Can you think of any reason that the 3 securities of equal contractual maturity and obligations of the same borrower would offer 3 different yields?
   **Their cash flows are not the same. The higher the coupon, the more cash in-flows the shorter the average maturity**

## Exercise 4:
## Deriving Forward Rates from Zero Rates

We use the Zero Coupon Rates and Discount Factors we derived in Exercise 1 for this exercise:

| Maturity | ZC Disc.Factor | S/A ZC Rate |
|---|---|---|
| ½ Year | **0.9885** | 2.3268% |
| 1 Year | **0.9750** | 2.5479% |
| 1½ Years | **0.958475** | 2.8475% |
| 2 Years | **0.94105** | 3.0611% |

1) Each of the zero-coupon rates you see above is one for which interest accruals begin immediately (spot, usually T+1), and end at their respective maturities. But what can the 1yr zero and 6mo zero tell you about what a forward-starting funding should cost? With the above information we can calculate the so-called forward implied rates. Follow this procedure to calculate the first one.

   Implied Forward Rate =
   [($df_{shortdate}$ ÷ $df_{longdate}$)-1] X 1/(date diff. in years)

   [($Df_{6month}$ ÷ $DF_{12month}$)-1] X (1/0.5) = **2.7692% s/a act/365**

*By the way, the other (more common) way to calculate implied forward rates is to use the spot-starting zero rates in the following equation. F = Frequency = 2 in this case.*

$$Rate_{ShortxLong}\% = \left( \frac{(1+Rate_{0xLong}/F)^{nLong}}{1+Rate_{0xShort}/F)^{nShort}} - 1 \right) \times F$$

Where both rates already on same frequency F and day-count
And nLong = number of periods from spot to long maturity (0 x Long)
And nShort = number of periods from spot to short maturity (0 X Short)

$$Rate_{6x12}\% = \left( \frac{(1+.025479/2)^2}{1+.023267/2)^1} - 1 \right) \times 2 = 2.7693\%$$ s/a act/365

*There is a slight difference in answer between the two formulas (if you try both), due to rounding. Keep in mind we've simplified this away from day-count, by assuming even 6-mo periods.*

2. Now do the same for the 12 x 18mo and the 18 x 24mo forward rates and complete the following table:

*You should get these answers within a rounding error:*

| Forward Period | Implied Forward 6-mo Rate |
|---|---|
| 0mo X 6mo (spot-6mo) | **2.3267% act/365** |
| 6mo X 12mo | **2.7692% act/365** |
| 12mo X 18mo | **3.4482% act/365** |
| 18mo X 24mo | **3.7033% act/365** |