FE5222 Solutions to Homework 2

October 5, 2019

1. (Q2) (**Chooser Option**) A chooser option gives the owner the right to choose at time $t_0 > 0$ to buy a call or a put option with strike K and expiry $T > t_0$. Hence at time t_0 the value of the chooser option is

$$\max\{C(t_0), P(t_0)\}$$

where $C(t_0)$ and $P(t_0)$ are the value of call and put option with strike K and expiry T respectively. Use risk neutral approach to price this option at time $t = 0$ under the Black-Scholes-Merton model. Hint: use option call-put parity.

Solution:

Let $C(t,T,K)$ be the price of a call option with strike K and expiry T at time t, and $P(t,T,K)$ be the price of a put option with strike K and expiry T at time t.

Using risk neutral pricing, we have

$$C(t,T,K) = e^{-r(T-t)}\mathbb{E}\left[(S(T)-K)^+ \,|\mathcal{F}_t\right]$$

and

$$P(t,T,K) = e^{-r(T-t)}\mathbb{E}\left[(K-S(T))^+ \,|\mathcal{F}_t\right]$$

where $\mathbb{E}[\bullet]$ is the expectation w.r.t. risk neutral measure.

From

$$(S(T)-K)^+ - (K-S(T))^+ = S(T) - K$$

and using the fact that discounted stock price is a martingale under risk neutral measure, we can derive the call-put parity

$$C(t,T,K) - P(t,T,K) = S(t) - e^{-r(T-t)}K$$

Since $\max\{x,y\} = \max\{x-y,0\} + y$, the value of a chooser option at time t_0 is

$$\begin{aligned}
\max\{C(t_0,T,K), P(t_0,T,K)\} &= C(t_0,T,K) + \max\{0, P(t_0,T,K) - C(t_0,T,K)\} \\
&= C(t_0,T,K) + \max\left\{0, e^{-r(T-t_0)}K - S(t_0)\right\}
\end{aligned}$$

Hence the price of the chooser option at time $t = 0$ is

$$\begin{aligned}
V &= e^{-rt_0}\mathbb{E}\left[\max\{C(t_0,T,K), P(t_0,T,K)\}\right] \\
&= e^{-rt_0}\mathbb{E}\left[C(t_0,T,K)\right] + e^{-rt_0}\mathbb{E}\left[\max\left\{0, e^{-r(T-t_0)}K - S(t_0)\right\}\right]
\end{aligned}$$

Note that

$$\begin{aligned} e^{-rt_0}\mathbb{E}\left[C(t_0,T,K)\right] &= e^{-rt_0}\mathbb{E}\left[e^{-r(T-t_0)}\mathbb{E}\left[(S(T)-K)^+ \,|\mathcal{F}_t\right]\right] \\ &= e^{-rT}\mathbb{E}\left[\mathbb{E}\left[(S(T)-K)^+ \,|\mathcal{F}_t\right]\right] \\ &= e^{-rT}\mathbb{E}\left[(S(T)-K)^+\right] \end{aligned}$$

which is the price of a call option with strike K and expiry T.

$$e^{-rt_0}\mathbb{E}\left[\max\left\{0, e^{-r(T-t_0)}K - S(t_0)\right\}\right]$$

is the price of a put option with strike $e^{-r(T-t_0)}K$ and expiry t_0.

Hence

$$V = C(0,T,K) + P(0,t_0,e^{-r(T-t_0)}K)$$

Q.E.D.

2. (Q3) Let $\alpha(t)$ and $\beta(t)$ be non-random time-dependent functions. $X(0) = 1$ and $X(t)$ satisfies the following SDE

$$dX(t) = \alpha(t)X(t)dt + \beta(t)X(t)dW(t)$$

Solution:

Note that $\alpha(t)$ and $\beta(t)$ are non-random, but time-dependent. A common mistake in the homework is to assume these two are constant.

Applying Ito's formula to $\ln X(t)$ we have

$$\begin{aligned} d\ln X(t) &= \tfrac{dX(t)}{X(t)} - \tfrac{1}{2}\tfrac{dX(t)}{X(t)}\tfrac{dX(t)}{X(t)} \\ &= \alpha(t)dt + \beta(t)dW(t) - \tfrac{1}{2}\beta^2(t)dt \\ &= \left(\alpha(t) - \tfrac{1}{2}\beta^2(t)\right)dt + \beta(t)dW(t) \end{aligned}$$

Integrating it from 0 to t and using the fact that $\ln X(0) = 0$, we have

$$\ln X(t) = \int_0^t \left(\alpha(s) - \frac{1}{2}\beta^2(s)\right)ds + \int_0^t \beta(s)dW(s)$$

Hence

$$X(t) = e^{\int_0^t \left(\alpha(s)-\frac{1}{2}\beta^2(s)\right)ds + \int_0^t \beta(s)dW(s)}$$

Since β is deterministic (i.e., non-random), we have proven in class

$$\int_0^t \beta(s)dW(s) \sim \mathcal{N}\left(0, \int_0^t \beta^2(s)ds\right)$$

Hence

$$\ln X(t) \sim \mathcal{N}\left(\int_0^t \left(\alpha(s) - \frac{1}{2}\beta^2(s)\right)ds, \int_0^t \beta^2(s)ds\right)$$

Q.E.D.

3. (Q5) Consider the multidimensional model with m stocks,

$$\frac{dS_i}{S_i} = \alpha_i(t)dt + \sum_{j=1}^{d}\sigma_{i,j}(t)dW_j(t)$$

for $i = 1, \dots, m$. Assume the riskless rate is a constant r.

Suppose there exists a solution $\Theta_j(t), j = 1 \dots, d$, for the market price of risk equations (see lecture notes) and let $\widetilde{\mathbb{P}}$ be the risk neutral measure. Then

$$d\widetilde{W}_j(t) = dW_j(t) + \Theta_j(t)dt$$

is a Brownian motion under $\widetilde{\mathbb{P}}$.

(a) (8 Points) Let

$$\sigma_i(t) = \sqrt{\sum_{j=1}^{d} \sigma_{i,j}^2(t)}$$

and

$$dB_i(t) = \frac{1}{\sigma_i(t)} \sum_{j=1}^{d} \sigma_{i,j}(t) dW_j(t)$$

Prove that $B_i(t), i = 1, \dots, m$, is a Brownian motion under $\mathbb{P}$.

(b) (5 Points) Derive the instantaneous correlation of B_i and B_j for $i \neq j$.

(c) (8 Points) Define $\gamma_i(t) = \frac{1}{\sigma_i(t)} \sum_{j=1}^{d} \sigma_{i,j}(t)\Theta_j(t)$. Show that

$$\widetilde{B}_i(t) = B_i(t) + \int_0^t \gamma_i(u) du$$

is a Brownian motion under $\widetilde{\mathbb{P}}$.

(d) (4 Points) Show that

$$\frac{dS_i(t)}{S_i(t)} = r dt + \sigma_i(t) d\widetilde{B}_i(t)$$

(e) (5 Points) Derive the instantaneous correlation of $\widetilde{B}_i(t)$ and $\widetilde{B}_j(t)$ for $i \neq j$. Compare the result with part (b).

Solution:

(a) We will use Levy's Theorem to show that $B_i(t)$ is a Brownian motion. We will have to prove that $B_i(t)$ is continuous, a martingale and $[B_i, B_i](t) = t$

Note that

$$B_i(t) = \int_0^t \frac{1}{\sigma_i(s)} \sum_{j=1}^{d} \sigma_{i,j}(s) dW_j(s) = \sum_{j=1}^{d} \int_0^t \frac{1}{\sigma_i(s)} \sigma_{i,j}(s) dW_j(s)$$

Since each $\int_0^t \frac{1}{\sigma_i(s)} \sigma_{i,j}(s) dW_j(s)$ is an Ito's integral, it is continuous and a martingale. Hence $B_i(t)$, as the sum of continuous martingales, is also a continuous martingale.

$$\begin{aligned}
dB_i(t)dB_i(t) &= \left(\tfrac{1}{\sigma_i(t)} \textstyle\sum_{j=1}^{d} \sigma_{i,j}(t) dW_j(t)\right) \left(\tfrac{1}{\sigma_i(t)} \textstyle\sum_{j=1}^{d} \sigma_{i,j}(t) dW_j(t)\right) \\
&= \tfrac{1}{\sigma_i^2(t)} \textstyle\sum_{j,k} \sigma_{i,j}(t)\sigma_{i,k}(t) dW_j(t) dW_k(t) \\
&= \tfrac{1}{\sigma_i^2(t)} \textstyle\sum_{j} \sigma_{i,j}^2(t) dt
\end{aligned}$$

where the last equality follows from the fact that $dW_j(t)dW_k(t) = 0, j \neq k$ and $dW_j(t)dW_k(t) = dt, j = k$.

By definition $\frac{1}{\sigma_i^2(t)} \sum_j \sigma_{i,j}^2(t)dt = dt$, hence $dB_i(t)dB_i(t) = dt$. That is

$$[B_i, B_i](t) = t.$$

We can conclude that $B_i(t)$ is a Brownian motion.

(b) The instantaneous correlation is defined as

$$\frac{\mathbb{E}[dX(t)dY(t)|\mathbb{F}_t]}{\sqrt{Var(dX(t)|\mathbb{F}_t)Var(dY(t)|\mathbb{F}_t)}}$$

where $dX(t)$ and $dY(t)$ shall be interpreted as change of $X(t)$ and $Y(t)$ respectively at an infinitesimal time interval $(t, t+dt)$ and $\mathbb{E}[\bullet|\mathcal{F}_t]$ is the conditional expectation and $Var(\bullet|\mathcal{F}_t)$ is the conditional variance. A less formal definition is

$$\frac{dX(t)dY(t)}{\sqrt{dX(t)dX(t)}\sqrt{dY(t)dY(t)}}$$

Since

$$\begin{array}{rcl} dB_i(t)dB_j(t) & = & \left(\frac{1}{\sigma_i(t)} \sum_{k=1}^d \sigma_{i,k}(t)dW_k(t)\right)\left(\frac{1}{\sigma_j(t)} \sum_{l=1}^d \sigma_{j,l}(t)dW_l(t)\right) \\ & = & \frac{1}{\sigma_i(t)\sigma_j(t)} \sum_{k,l} \sigma_{i,k}(t)\sigma_{j,l}(t)dW_k(t)dW_l(t) \\ & = & \frac{1}{\sigma_i(t)\sigma_j(t)} \sum_k \sigma_{i,k}(t)\sigma_{j,k}(t)dt \end{array}$$

$dB_i(t)dB_i(t) = dt$ and $dB_j(t)dB_j(t) = dt$, the instantaneous correlation is

$$\frac{dB_i(t)dB_j(t)}{\sqrt{dB_i(t)dB_i(t)}\sqrt{dB_j(t)dB_j(t)}} = \frac{1}{\sigma_i(t)\sigma_j(t)} \sum_k \sigma_{i,k}(t)\sigma_{j,k}(t)$$

(c) Note that

$$\begin{array}{rcl} d\widetilde{B}_i(t) & = & dB_i(t) + \gamma_i(t)dt \\ & = & \frac{1}{\sigma_i(t)} \sum_{j=1}^d \sigma_{i,j}(t)dW_j(t) + \frac{1}{\sigma_i(t)} \sum_{j=1}^d \sigma_{i,j}(t)\Theta_j(t)dt \\ & = & \frac{1}{\sigma_i(t)} \sum_{j=1}^d \sigma_{i,j}(t)\left(dW_j(t) + \Theta_j(t)dt\right) \\ & = & \frac{1}{\sigma_i(t)} \sum_{j=1}^d \sigma_{i,j}(t)d\widetilde{W}_j(t) \end{array}$$

Hence the proof is the same as that in (a) with $\widetilde{W}_j$ in place of W_j.