FE5222 Homework 5: Due by Thursday, Nov 21

November 16, 2019

1. (30 Points)[1] For a set of positive numbers $x_1, \ldots, x_n$, the harmonic mean is defined as

$$\mu = \frac{n}{\frac{1}{x_1} + \ldots + \frac{1}{x_n}}$$

For a positive function $f(x)$ on the interval $[a, b]$, we can define the harmonic mean as

$$\mu_f = \frac{b-a}{\int_a^b \frac{dx}{f(x)}}$$

In this exercise we will prove that the implied volatility is the harmonic mean of local volatility at short maturities.

As we have discussed in class, the Dupire's equation can be written as a function of implied volatility

$$\sigma^2(T, K) = \frac{\frac{\Sigma}{2T} + \frac{\partial \Sigma}{\partial T} + rK\frac{\partial \Sigma}{\partial K}}{\frac{1}{2}K^2\left(\frac{1}{\Sigma T K^2} + \frac{2d_1}{\Sigma\sqrt{T}K}\frac{\partial \Sigma}{\partial K} + \frac{d_1 d_2}{\Sigma}\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \frac{\partial^2 \Sigma}{\partial K^2}\right)} \tag{1}$$

(a) (15 Points) Prove that

$$\sigma(K) = \lim_{T \to 0} \sigma(T, K) = \frac{\Sigma}{1 + \frac{K}{\Sigma}\ln(\frac{S}{K})\frac{d\Sigma}{dK}} \tag{2}$$

where S is the spot at time $t = 0$.

(b) (15 Points) Use the above result to show that

$$\Sigma(\frac{K}{S}) = \frac{\ln\left(\frac{K}{S}\right)}{\int_0^{\ln\left(\frac{K}{S}\right)} \frac{dx}{\sigma(x)}}$$

This shows that the implied volatility for the option with strike K, expressed as a function of $\ln\left(\frac{K}{S}\right)$ is the harmonic mean of the local volatility at all stock prices between S and K.

2. (20 Points) Assume the implied volatility surface is of the form

$$\Sigma(T, K) = \alpha(T) + \beta(T)x + \gamma(T)x^2$$

where $x = \ln\left(\frac{K}{S}\right)$ is log moneyness and S is the spot price at time $t = 0$. For this particular form of implied volatility surface, derive the local volatility function $\sigma(t, S_t)$ using Dupire's equation in terms of implied volatility (see Equation (1)).

[1] This exercise is from Chapter 5 of Derman et. al.

3. (50 Points) (**Local Time**)[2] Let $W(t)$ be a Brownian motion. Ito's formula says that

$$df(W(t)) = f'(W(t))dt + \frac{1}{2}f''(W(t))dt$$

In integral form, it is

$$f(W(T)) = f(W(0)) + \int_0^T f'(W(t))dt + \frac{1}{2}\int_0^T f''(W(t))dt \tag{3}$$

Ito's formula holds when f'' exists and bounded. Tanaka-Meyer's formula extends it to more general functions. In this exercise we shall show that for a function with discontinuities, a naive application of Ito's formula will lead to wrong answers.

(a) (5 Point) Consider the 'hockey stick' function $f(x) = (x-K)^+$ for some positive number K. f' does not exists at $x = K$ and $f''(x) = 0$ except at $x = K$. If we ignore the value of f'' at $x = K$ to assume $f''(x) \equiv 0$ for all x and substitute it to Ito's formula (3), show that the two sides of the Equation (3) do not equal by computing their expected values.

(b) (5 Point) To get an idea of what is going on here, define a sequence of functions $\{f_n\}_{n=1}^{\infty}$ as

$$f_n(x) = \begin{cases} 0 & \text{if } x \le K - \frac{1}{2n} \\ \frac{n}{2}(x-K)^2 + \frac{1}{2}(x-K) + \frac{1}{8n} & \text{if } K - \frac{1}{2n} \le x \le K + \frac{1}{2n} \\ x - K & \text{if } x \ge K + \frac{1}{2n} \end{cases}$$

Show that

$$f_n'(x) = \begin{cases} 0 & \text{if } x \le K - \frac{1}{2n} \\ n(x-K) + \frac{1}{2} & \text{if } K - \frac{1}{2n} \le x \le K + \frac{1}{2n} \\ 1 & \text{if } x \ge K + \frac{1}{2n} \end{cases}$$

and

$$f_n''(x) = \begin{cases} 0 & \text{if } x < K - \frac{1}{2n} \\ n & \text{if } K - \frac{1}{2n} < x < K + \frac{1}{2n} \\ 0 & \text{if } x > K + \frac{1}{2n} \end{cases}$$

Note that $f_n''(x)$ is not defined at $x = K \pm \frac{1}{2n}$.

(c) (5 Point) Show that

$$\lim_{n\to\infty} f_n(x) = (x-K)^+$$

and

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 0 & \text{if } x < K \\ \frac{1}{2} & \text{if } x = K \\ 1 & \text{if } x > K \end{cases}$$

Since the value of $\lim_{n\to\infty} f_n(x)$ at a single point will not matter in the rest of this exercise, we let

$$\lim_{n\to\infty} f_n'(x) = \mathbb{1}_{(K,\infty)}(x) = \begin{cases} 0 & \text{if } x \le K \\ 1 & \text{if } x > K \end{cases}$$

(d) (5 Point) For each n, since f' is continuous and f'' exists for every x except at at $x = K \pm \frac{1}{2n}$ and it is bounded, Ito's formula holds

$$f_n(W(T)) = f_n(W(0)) + \int_0^T f_n'(W(t))dt + \frac{1}{2}\int_0^T f_n''(W(t))dt$$

[2] This exercise is from Shreve Exercise 4.20. However the original question is missing the coefficient $\frac{1}{2}$ for (d) and (f).

Show that

$$(W(T)-K)^+ = (W(0)-K)^+ + \int_0^T \mathbb{1}_{(K,\infty)}(W(t))dW(t) + \lim_{n\to\infty}\frac{n}{2}\int_0^T \mathbb{1}_{(K-\frac{1}{2n},K+\frac{1}{2n})}(W(t))dt$$

(e) (15 Point) Define local time of the Brownian motion at K to be

$$L_K(T) = \lim_{n\to\infty} n\int_0^T \mathbb{1}_{(K-\frac{1}{2n},K+\frac{1}{2n})}(W(t))dt$$

where the limit holds almost surely. Informally this measure how much time the Brownian motion spends at the infinitesimal interval centered at K.

Show that if the sample path of $W(t)$ stays strictly below K on the time interval $[0,T]$, then

$$L_K(T) = 0$$

[3]

(f) (15 Point) Prove that

$$\frac{1}{2}L_K(T) = (W(T)-K)^+ - \int_0^T \mathbb{1}_{(K,\infty)}(W(t))dW(t)$$

and show that we cannot have

$$L_K(T) = 0$$

almost surely.

Remark: A very good introduction to local time is: *The Pedestrians Guide to Local Time* by Tomas Bjork

[3] Note that the Dirac delta function $\delta_K(x)$ can be informally written as

$$\delta_K(x) = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\mathbb{1}_{(K-\epsilon,K+\epsilon)}(x)$$

Hence

$$L_K(T) = \int_0^T \delta_K(W(t))dt$$