7 Black-Scholes-Merton equation (1 lecture)

In Section 2, we have discussed how to price an option based on the assumption that the market is arbitrage-free. We have obtained upper and lower bounds for the option price. To obtain more precise formula for the option price, we need to know how the underlying asset price (stock price) changes.

In 1900, Louis Bachelier proposed a model for stock price

$$S(t) = S(0) + \alpha t + \sigma W(t) \quad \text{which implies} \quad dS(t) = \alpha dt + \sigma dW(t). \tag{7.1}$$

In Bachelier's thesis, he is able to derive a option price formula based on (7.1). See Page 736 of Essentials of Stochastic Finance: Facts, Models, Theory by A. N. Shiryaev. The problem of (7.1) is that $S(t)$ can be negative.

In 1965[57], Paul Samuelson, the first American Nobel Economics Prize winner (1970), replaced the change of the stock price $S(t+\Delta t) - S(t)$ by the arithematic return $\frac{S(t+\Delta t)-S(t)}{S(t)}$ in Bachelier's model, and proposed the following model

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dW(t). \tag{7.2}$$

We have seen from Question 3 of Homework V that the $S(t) = S(0)e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W_t}$ which is always positive. Using (7.2), Samuelson and others studied option pricing problem. However, their results contain parameters that describe the risk preference of individual investors, and hence not applicable in real world trading.

In 1973, using model (7.2), Fischer Black and Myron Scholes proposed their famous option pricing formula that we will discuss later. The novelty of this formula is that it is independent of the risk preference of individual investors. As we will see later that it puts all investors in a risk-neutral world where the expected return equals the risk-free interest rate.

7.1 Derivation of the Black-Scholes-Merton equation

Consider a European call (or put) option that pays $(S(T) - K)^+$ (or $(K - S(T))^+$) at time T. The stock price $S(t)$ is modelled by (7.2).

Let $c(t, x)$ denote the value of the European call (or put) option at time t if the stock price at t is $S(t) = x$. There is nothing random about the function $c(t, x)$. However, $c(t, S(t))$ is random since $S(t)$ is random. We want to derive the equation satisfied by $c(t, x)$. This equation is the so called Black-Scholes-Merton equation.

The following argument on how to determine the Black-Scholes-Merton equation for $c(t, x)$ is a generalization of Questions 4 and 5 of Homework II. It follows Section 4.5 of Shreve II by introducing the idea of hedging.

[57] P. A. Samuelson, Rational theory of warrant pricing, Industrial Management Review 6 (1965) 13–31.

Consider a portfolio Φ consisting of Δ shares of underlying stock and a money market account of B dollars with interest rate r

$$\Phi = \Delta S + B. \tag{7.3}$$

$\Delta = \Delta(t)$ denotes the number of stock at time t. We want to determine $\Delta(t)$ so that the portfolio value $\Phi(t)$ at each time $t \in [0, T]$ agrees with $c(t, S(t))$. The value of $\Delta(t)$ can be adjusted with respect to time, but must be adapted to the filtration $\mathcal{F}_t$ defined in Definition 4.1. It means that all decisions can only be based on the current information $\mathcal{F}_t$, without anticipation of the future. See (2.45) for an example in the discrete case.

The remainder of the portfolio value, $\Phi(t) - \Delta(t)S(t)$, is invested in the money market account $B(t)$ which satisfies $dB(t) = rB(t)dt$. This equation says that the infinitesimal[58] increment (or differential) dB_t is $rB_t dt$ when t increases by dt. The increment (or differential) $d\Phi(t)$ for the investor's portfolio value at each time t is due to two factors, the capital gain $\Delta(t)dS(t)$ on the stock position and the interest earnings $rB(t)dt = r(\Phi(t) - \Delta(t)S(t))dt$ on the cash position[59]. In other words, [60]

$$\begin{aligned} d\Phi(t) =& \Delta(t)dS(t) + dB(t) \\ =& \Delta(t)dS(t) + r[\Phi(t) - \Delta(t)S(t)]dt \end{aligned} \tag{7.4}$$

The derivation of the Black-Scholes-Merton equation contains 3 steps:

1. First, we consider $d(e^{-rt}\Phi(t))$. The reason we should consider $e^{-rt}\Phi(t)$ is because of the $r\Phi(t)$ term in (7.4). Like Question 6 of Homework VI, to get rid of this term, we should consider the discounted portfolio value $e^{-rt}\Phi(t)$. We claim that with Itô formula, we have

$$d(e^{-rt}\Phi(t)) = \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t). \tag{7.5}$$

Proof: By Itô formula with $g(t, x) = e^{-rt}x$,

$$\begin{aligned} d(e^{-rt}\Phi(t)) =& dg(t, \Phi(t)) = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}d\Phi(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(d\Phi(t))^2 \\ =& -re^{-rt}\Phi(t)dt + e^{-rt}d\Phi(t) \\ \overset{(7.4)}{=}& -re^{-rt}\cancel{\Phi(t)}dt + e^{-rt}\Delta(t)dS(t) + e^{-rt}r[\cancel{\Phi(t)} - \Delta(t)S(t)]dt \\ \overset{(7.2)}{=}& \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t). \end{aligned}$$

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[58] meaning infinitely small

[59] Portfolio of this kind, which satisfies both $d\Phi(t) = \Delta(t)dS(t) + dB(t)$ and $\Phi(t) = \Delta(t)S(t) + B(t)$ are called self-financing portfolio. In another words, we do not need to infuse or withdraw cash when changing $\Delta(t)$.

[60] For your information, note that the discrete-time analogue of (7.4) is

$$\Phi_{n+1} - \Phi_n = \Delta_n(S_{n+1} - S_n) + (e^{r\delta t} - 1)(\Phi_n - \Delta_n S_n)$$

which follows from $\Phi_{n+1} = e^{r\delta t}(\Phi_n - \Delta_n S_n) + \Delta_n S_{n+1}$ in (2.46).

2. Next, we consider $d\left(e^{-rt}c(t,S(t))\right)$ since we want to enforce $\Phi(t) = c(t,S(t))$ later on. We let $g(t,x) = e^{-rt}x$ (hence $\frac{\partial^2 g}{\partial x^2} = 0$) and use Itô formula with $X_t = c(t,S(t))$ to get

$$d\left(e^{-rt}c(t,S(t))\right) = -re^{-rt}c(t,S(t))dt + e^{-rt}dc(t,S(t)).$$

Then we apply Itô formula to $dc(t,S(t))$ with $X_t = S(t)$ to get

$$\begin{aligned}
&d\left(e^{-rt}c(t,S(t))\right)\\
&= -re^{-rt}c(t,S(t))dt + e^{-rt}\left[\frac{\partial c}{\partial t}(t,S(t))dt + \frac{\partial c}{\partial x}(t,S(t))dS(t) + \frac{1}{2}\frac{\partial^2 c}{\partial x^2}(t,S(t))(dS(t))^2\right]\\
&\overset{(7.2)}{=} e^{-rt}\Big[-rc(t,S(t)) + \frac{\partial c}{\partial t}(t,S(t)) + \alpha S(t)\frac{\partial c}{\partial x}(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c}{\partial x^2}(t,S(t))\Big]dt\\
&\quad + e^{-rt}\sigma S(t)\frac{\partial c}{\partial x}(t,S(t))dW(t).
\end{aligned} \tag{7.6}$$

3. Finally, we want to determine $\Delta(t)$ so that the portfolio value $\Phi(t)$ at each time $t \in [0,T]$ agrees with $c(t,S(t))$. Hence we want to have

$$e^{-rt}\Phi(t) = e^{-rt}c(t,S(t)) \tag{7.7}$$

for all t. By (7.7), we know the right hand sides of (7.5) and (7.6) should equal. This leads to

$$\begin{aligned}
&\Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t)\\
&= \left[-rc(t,S(t)) + \frac{\partial c}{\partial t}(t,S(t)) + \alpha S(t)\frac{\partial c}{\partial x}(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c}{\partial x^2}(t,S(t))\right]dt\\
&\quad + \sigma S(t)\frac{\partial c}{\partial x}(t,S(t))dW(t)
\end{aligned} \tag{7.8}$$

Equating the coefficients for dt and $dW(t)$, we get

$$\Delta(t) = \frac{\partial c}{\partial x}(t,S(t)), \tag{7.9}$$

$$\frac{\partial c}{\partial t}(t,S(t)) + rS(t)\frac{\partial c}{\partial x}(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c}{\partial x^2}(t,S(t)) - rc(t,S(t)) = 0. \tag{7.10}$$

Please note how the parameter α is removed from (7.10). Instead of requiring (7.10) to be true at $(t,S(t))$, we indeed ask more, and require (7.10) to be true for all (t,x):

$$\frac{\partial c}{\partial t}(t,x) + rx\frac{\partial c}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 c}{\partial x^2}(t,x) - rc(t,x) = 0. \tag{7.11}$$

The above equation is the Black-Scholes-Merton equation. This equation has appeared before in (6.23) if we choose $\sigma(t,x) = \sigma x$ and $b(t,x) = rx$.

For European call option, c satisfies the terminal condition

$$c(T, x) = (x - K)^+ \tag{7.12}$$

and the boundary conditions at $x = 0$ and $x = \infty$

$$c(t, 0) = 0, \qquad \lim_{x\to\infty} [c(t, x) - (x - e^{-r(T-t)}K)] = 0 \qquad \forall t \in [0, T]. \tag{7.13}$$

Why (7.13)? For large x, the call option is deep in the money ($S_t > K$) and very likely to end in the money. In this case, the price of the call is almost as much as the price of the forward contract f, whose payoff at expiration date T is $S(T) - K$. If $\Phi = S - Ke^{-rT}$, then Φ satisfies $V_T(\Phi) = V_T(f) = S(T) - K$. Hence by Corollary 1.1 , $V_t(f) = V_t(\Phi) = S_t - Ke^{-rT}e^{rt} = S_t - Ke^{-r(T-t)}$. The value of the forward contract f is set to be the price of the call option when $S_t \to \infty$. That is why we enforce (7.13). See Question 2 of Homework VII.

Remark: Suppose we have found this function c. If an investor starts with initial capital $\Phi(0) = c(0, S(0))$ and uses the hedge $\Delta(t) = \frac{\partial c}{\partial x}(t, S(t))$, then no matter which of its possible paths the stock price follows, when the option expires, the agent hedging the short position has a portfolio whose value agrees with the option payoff.

Remark: (risk neutral measure) If we continue, and introduce

$$\tilde{W}(t) = \theta t + W(t) \tag{7.14}$$

with $\theta = \frac{\alpha - r}{\sigma}$, we get

$$\begin{aligned} d(e^{-rt}c(t, S(t))) &\overset{(7.7)}{=} d(e^{-rt}\Phi(t)) \\ &\overset{(7.5)}{=} \Delta(t)\sigma e^{-rt}S(t)\left(\frac{\alpha - r}{\sigma}dt + dW(t)\right) = \Delta(t)\sigma e^{-rt}S(t)d\tilde{W}(t). \end{aligned} \tag{7.15}$$

The above equation is an equivalent way to write

$$e^{-rt}c(t, S(t)) = c(0, S(0)) + \int_0^t f(s, \omega)d\tilde{W}(s) \tag{7.16}$$

with $f(s, \omega) = \Delta(t)\sigma e^{-rt}S(t)$. It turns out that one can change the probability measure from $\mathbb{P}$ to $\tilde{\mathbb{P}}$ so that $\tilde{W}$ is a Brownian motion in this new probability measure $\tilde{\mathbb{P}}$. Then $\int_0^t f(s, \omega)d\tilde{W}(s)$ is a martingale under $\tilde{\mathbb{P}}$. In particular, it has zero mean for every t. This provides a shortcut and avoid dealing with the complicated function $f(s, \omega)$ in option pricing. If we have such a $\tilde{\mathbb{P}}$, with $\tilde{\mathbb{E}}$ being the expectation under $\tilde{\mathbb{P}}$, taking $\tilde{\mathbb{E}}$ on both sides of (7.16), we get (we can set $t = T$)

$$\tilde{\mathbb{E}}[e^{-rt}c(t, S(t))] = c(0, S(0)). \tag{7.17}$$

This $\tilde{\mathbb{P}}$ is called risk-neutral measure. (7.16) then says that $e^{-rt}c(t, S(t))$ is a martingale in this risk-neutral world $\tilde{\mathbb{P}}$. (7.17) gives the option price $c(0, S(0))$. One can then use Monte Carlo method to find the value of $c(0, S(0))$.

In the discrete case, we have learned in the binomial tree model that in the risk-neutral world, the probability that how $S(t)$ evolves has changed from the real world probability p_u, p_d to risk-neutral probability q_u, q_d. So, what happens here? Well, (7.2) has now become

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dW(t) = rdt + \sigma d\left(\theta t + W(t)\right) = rdt + \sigma d\tilde{W}(t), \tag{7.18}$$

and that is how we should model the option price in the risk-neutral world.

7.2 Solution to the Black-Scholes-Merton equation

The Black-Scholes-Merton equation (7.11) does not involve probability. It is a partial differential equation, and the arguments t and x are dummy variables, not random variables.

By the Feynman-Kac formula in Question 6 of Homework VI ($b(t, X_t) = rX_t$, $\sigma(t, X_t) = \sigma X_t$, $g(T, x) = (x - K)^+$), we know the solution of (7.11)+(7.12) can be written as

$$c(t, x) = \mathbb{E}^{t,x}[e^{-r(T-t)}(X_T - K)^+] \tag{7.19}$$

where X_T satisfies $dX_s = rX_s ds + \sigma X_s dW_s$, $X_t = x$, i.e. $X_T = xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W_T-W_t)}$ (the expression of X_T/X_t is derived in Question 3 (d) of Homework V). Please note that we introduce this new stochastic process X_s so that $c(t, x)$ can be written as an expectation. This X_t is simply (7.18), being the S_t in the risk-neutral world driven by $d\tilde{W}$. Then we can say that to evaluate $\mathbb{E}^{t,x}[e^{-r(T-t)}(X_T - K)^+]$, we just need to take the expectation of $e^{-r(T-t)}(S_T - K)^+$ in the risk-neutral world where $\tilde{W}$ instead of W are the Brownian motion and S satisfies (7.18). (7.19) will be the starting point for using Monte Carlo simulation to determine option price.

With $Y \sim N\left((r - \frac{1}{2}\sigma^2)(T - t), \sigma^2(T - t)\right)$, we know

$$c(t, x) = e^{-r(T-t)} x\mathbb{E}^{t,x}\left[\left(e^Y - \frac{K}{x}\right)\right]$$

Then by Example 3.3 or Equation (3.16), we get

$$c(t, x) = xN(d_+(T - t, x)) - Ke^{-r(T-t)}N(d_-(T - t, x)) \tag{7.20}$$

where

$$d_\pm(\tau, x) \overset{(3.16)}{=} \frac{\log \frac{1}{K/x} + \mu}{\gamma} = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]$$

and

$$N(y) \overset{(3.15)}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}} dz.$$

We can check that the condition (7.13) is satisfied as well. If you think the above derivation is too cumbersome, feel free to use calculus to directly verify that (7.20) solves (7.11)+(7.12).

Remark: Yet another way to obtain (7.20) is to first use Question 5 of Homework VII to change the Black-Schole equation to a heat equation $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2}$, which can then be solved exactly (according to any textbook on partial differential equation). See Section 15.2 of Choe's book for details.

By put-call parity (1.17) , $p(t, S(t)) = c(t, S(t)) - (S(t) - e^{-r(T-K)}K)$. Hence

$$\begin{aligned} p(t,x) =& c(t,x) - x + e^{-r(T-t)}K \\ =& x[N(d_+(T-t,x)) - 1] - Ke^{-r(T-t)}[N(d_-(T-t,x)) - 1] \\ =& Ke^{-r(T-t)}N(-d_-(T-t,x)) - xN(-d_+(T-t,x)). \end{aligned} \tag{7.21}$$

Example 7.1 *Recall the real option example in Section 1.5. We want to find the value of having the option to spend \$900 million to invest in Mark II in 1985.* $K = \$900$ *million. (This is the value at the expiration date, not right now.) Right now, it is 1982.* $T = 3$ *years.* $\sigma = 0.35$. $r = \log 1.1$[61]. $S_0 = \$467$ *million which is determined by first forecast the cash flow of Mark II and then discount it back to 1982 with discount rate 0.2:* $467 = \frac{120}{1.2^4} + \frac{118}{1.2^5} + \frac{390}{1.2^6} + \frac{620}{1.2^7} + \frac{250}{1.2^8}$, *where* $120, 118, ..., 250$ *are the forecasted net cash flows in years 1986,1987,...,1990 respectively. The discount rate is higher than the risk-free interest rate since the forecasted net cashflows are risky.*

Hence $d_+ = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right] = -0.3075$, $d_- = d_+ - \sigma\sqrt{T} = -0.9137$. $c(0, S_0) = S_0N(d_+) - Ke^{-rT}N(d_-) = 55.0956$.

Here is the Matlab code for the computation:

```
sig = 0.35; T = 3; K = 900; r = log(1.1); S0 = 467;
d1=(log(S0/K)+(r+0.5*sig^2)*T)/(sig*sqrt(T));
d2=d1-sig*sqrt(T);
c=S0*normcdf(d1)-K*exp(-r*T)*normcdf(d2)
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The derivatives of the function $c(t, x)$ with respect to various variables are called the Greeks. One can verify that (see Theorem 15.3 of Choe)

$$\Delta = \frac{\partial c}{\partial x}(t,x) = N(d_+(T-t,x)), \tag{7.22}$$

$$\Theta = \frac{\partial c}{\partial t}(t,x) = -\frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)) - rKe^{-r(T-t)}N(d_-(T-t,x)), \tag{7.23}$$

$$\Gamma = \frac{\partial^2 c}{\partial x^2}(t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t,x)), \tag{7.24}$$

$$\rho = \frac{\partial c}{\partial r} = (T-t)Ke^{-r(T-t)}N(d_-(T-t,x)), \tag{7.25}$$

$$\text{Vega} = \frac{\partial c}{\partial \sigma} = x\sqrt{T-t}N'(d_+(T-t,x)). \tag{7.26}$$

[61] Please note that the risk free interest rate 0.1 in the book "principles of corporate finance" is compounded annually while our r is compounded continuously. Hence $e^r = 1 + 0.1$. The book uses the equivalent form $d_+ = \frac{1}{\sigma\sqrt{T}}\log\frac{S_0}{Ke^{-rT}} + \frac{1}{2}\sigma\sqrt{T}$.

Note that $\Delta > 0$ and $\Gamma > 0$ which implies that for fixed t, the function $c(t,x)$ is increasing and convex in the variables x, as shown in the following Figure. We also have $\rho > 0$, Vega > 0 and $\Theta < 0$.

Proof of (7.22) (the others are similar):

$$\frac{\partial c}{\partial x}(t,x) = N(d_+) + xN'(d_+)\frac{1}{x\sigma\sqrt{T-t}} - Ke^{-r(T-t)}N'(d_-)\frac{1}{x\sigma\sqrt{T-t}}.$$

So, we are left to verify

$$xN'(d_+) - Ke^{-r(T-t)}N'(d_-) = 0. \tag{7.27}$$

Note that $d_+(\tau,x) = d_-(\tau,x) + \sigma\sqrt{\tau}$.

$$\begin{aligned} xN'(d_+) =& x\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_+^2} = x\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_-^2}e^{-\frac{1}{2}\sigma^2(T-t)}e^{-\sigma\sqrt{T-t}d_-} \\ =& xN'(d_-)e^{-\frac{1}{2}\sigma^2(T-t)}e^{-\left[\log(x/K)+(r-\sigma^2/2)(T-t)\right]} = KN'(d_-)e^{-r(T-t)}. \qquad \square \end{aligned}$$

Hence (7.20) tells us $c(t,x) = x\frac{\partial c}{\partial x}(t,x) - Ke^{-r(T-t)}N(d_-(T-t,x))$ or

$$c(t,S_t) = S_t\frac{\partial c}{\partial x}(t,S_t) - Ke^{-r(T-t)}N(d_-(T-t,S_t))$$

for any S_t. It says that to hedge a short position for the option, one should hold $\frac{\partial c}{\partial x}(t,S_t)$ shares of stock, whose value is $S_t\frac{\partial c}{\partial x}$, and borrow $Ke^{-r(T-t)}N(d_-(T-t,S_t))$ dollars from the money market. That is what we have seen in (7.3), (7.7), (7.9).

Example 7.2 *(continuous version of Question 6 of Homework II) Prove that for European call option on a nondividend paying asset, the elasticity $e_c = \frac{\partial c}{\partial S}\frac{S}{c} > 1$. For European put option on a nondividend paying asset, the elasticity $e_p = \frac{\partial p}{\partial S}\frac{S}{p} < 0$.*

Proof*:*

$$e_c = \frac{\partial c}{\partial S}\frac{S}{c} = \frac{SN(d_+)}{SN(d_+) - Ke^{-r(T-t)}N(d_-)} > 1.$$

$p(t,S) = c(t,S) - S + Ke^{-r(T-t)}$. $\partial p/\partial S = \partial c/\partial S - 1$.

$$e_p = \frac{\partial p}{\partial S}\frac{S}{p} = \frac{SN(d_+) - S}{p} < 0$$

since $N < 1$ unless $d_+ = +\infty$.

Example 7.3 *Take $y = \log x$ and $v(t,y) = e^{-rt}c(t,x)$ in the Black-Scholes-Merton equation (7.11). First, show that $v(t,y)$ satisfies*

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial y^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial v}{\partial y} = 0. \tag{7.28}$$

(By the way, you should recall Questions 5 and 6 of Homework VI to see how to remove the rc term in (7.11).) To solve (7.28), one way is to use finite difference. We want to find v_i^n which approximates $v(t^n, y_i)$ with $t^n = n\delta t$ and $y_i = ih$. Note that $\frac{\partial v}{\partial t}(t^{n+1}, y_i) \approx \frac{v(t^{n+1},y_i)-v(t^n,y_i)}{\delta t}$, $\frac{\partial v}{\partial y}(t^{n+1}, y_i) \approx \frac{v(t^{n+1},y_{i+1})-v(t^{n+1},y_{i-1})}{2h}$, $\frac{\partial^2 v}{\partial y^2}(t^{n+1}, y_i) \approx \frac{v(t^{n+1},y_{i+1})-2v(t^{n+1},y_i)+v(t^{n+1},y_{i-1})}{h^2}$. Hence we can require $\{v_i^n, i = 0, \pm 1, \pm 2, \cdots, n = 0, 1, 2, \cdots\}$ to satisfy

$$\frac{v_i^{n+1} - v_i^n}{\delta t} + \frac{1}{2}\sigma^2 \frac{v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}}{h^2} + \left(r - \frac{1}{2}\sigma^2\right) \frac{v_{i+1}^{n+1} - v_{i-1}^{n+1}}{2h} = 0. \tag{7.29}$$

Once we know $\{v_i^{n+1}, i = 0, \pm 1, \pm 2, ...\}$, we can compute $\{v_i^n, i = 0, \pm 1, \pm 2, ...\}$ by the above equation. Show that if we take $h^2 = \sigma^2 \delta t$, the resulting numerical method is very close to the binomial tree method we have learned before. In the binomial tree method $q_u = \frac{\rho - d}{u-d} = \frac{e^{r\delta t} - e^{-\sigma\sqrt{\delta t}}}{e^{\sigma\sqrt{\delta t}} - e^{-\sigma\sqrt{\delta t}}}$. By Taylor expansion $e^z = 1 + z + \frac{1}{2}z^2 + \cdots$, one can verify that $q_u \approx \tilde{q}_u = \frac{(r - \frac{1}{2}\sigma^2)\delta t + \sigma\sqrt{\delta t}}{2\sigma\sqrt{\delta t}}$. Please explain how u and d are related to h.

Solution*: (7.28) is derived by chain rule, like Questions 4 and 5 of Homework VII. We skip the details since it is much simpler than Question 5 . When $h^2 = \sigma^2 \delta t$, from (7.29) we obtain*

$$\begin{aligned} v_i^n =& \cancel{v_i^{n+1}} + \frac{1}{2}\left(v_{i+1}^{n+1} - 2\cancel{v_i^{n+1}} + v_{i-1}^{n+1}\right) + \delta t \frac{r - \frac{1}{2}\sigma^2}{2\sigma\sqrt{\delta t}} \left(v_{i+1}^{n+1} - v_{i-1}^{n+1}\right) \\ =& \tilde{q}_u v_{i+1}^{n+1} + (1 - \tilde{q}_u) v_{i-1}^{n+1}. \end{aligned} \tag{7.30}$$

v_i^n approximates $v(t^n, y_i)$ which by definition, is $e^{-rt^n} c(t^n, e^{y_i})$. Let $S = e^{y_i}$, $u = e^h = e^{\sigma\sqrt{\delta t}}$, $u = e^{-h} = e^{-\sigma\sqrt{\delta t}}$. Then $e^{y_{i+1}} = Su$ and $e^{y_{i-1}} = Sd$. Hence, (7.30) implies the binomial tree option pricing formula $c(t^n, S) = e^{-r\delta t}\left(q_u c(t^{n+1}, uS) + q_d c(t^{n+1}, dS)\right)$.

7.3 Implied volatility, volatility smile and stochastic volatility

This subsection is based on Chapter 20 of John Hull's book and Chapter 8 of Duffie's book.

The one parameter in the Black-Scholes-Merton pricing formulas that cannot be directly observed is the volatility of the stock price. In the Monte Carlo chapter, we will discuss how this can be estimated from a history of the stock price. In practice, traders usually work with what are known as implied volatilities. These are the volatilities implied by option prices observed in the market.

To illustrate how implied volatilities are calculated, suppose that the market price of a European call option on a non-dividend-paying stock is 1.875 when $S_0 = 21$, $K = 20$, $r = 0.1$, and $T = 0.25$. The implied volatility is the value of σ that, when substituted into Black-Scholes formula (7.20), gives $c = 1.875$. We will learn how to find σ numerically in the Monte Carlo chapter as well.

Note that by (7.26), the Black-Scholes formula (7.20) satisfies $\frac{\partial c}{\partial \sigma} = \sqrt{\frac{T}{2\pi}} S_0 e^{-\frac{d_+^2}{2}} > 0$. So the option price c is an increasing function of σ when other parameters K, r, and S_0 are fixed. Hence given a market price c_{mkt}, we can always find one and only one value of σ so that the c given by (7.20) equals c_{mkt}.

Implied volatilities are used to monitor the market's opinion about the volatility of a particular stock. Whereas historical volatilities are backward looking, implied volatilities are forward looking. Traders often quote the implied volatility of an option rather than its price. This is convenient because the implied volatility tends to be less variable than the option price. The implied volatilities of actively traded options on an asset are often used by traders to estimate appropriate implied volatilities for other options on the asset.

If the assumptions underlying the Black-Scholes formula are correct, the implied volatility should not depend on the strike price, the time to maturity, and whether we are using call or put option to compute the volatility.

We claim that the implied volatility of a European call option is the same as that of a European put option when they have the same strike price and time to maturity. We leave the proof as an exercise (Question 9 of Homework VII).

However, it has been widely noted that actual market prices for European options on the same underlying asset have associated Black-Scholes implied volatilities that vary with both strike price and time to maturity. For example, for equity options (both on individual stocks and on stock indices), implied volatilities depend on strike prices in the manner illustrated in Figure 7.1, which is often termed a smile curve: The volatility decreases as the strike price increases. The volatility used to price a low-strike-price option (i.e., a deep-out-of-the-money put or a deep-in-the-money call) is significantly higher than that used to price a high-strike-price option (i.e., a deep-in-the-money put or a deep-out-of-themoney call).

In the Black-Scholes-Merton model, the asset price $S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}$ has lognormal distribution. The volatility smile for equity options corresponds to the implied probability distribution of the asset price given by the solid line in the right plot of Figure 7.1 (see the next example on how implied probability distribution of the asset price is determined by the volatility smile). A lognormal distribution with the same mean and standard deviation as the implied distribution is shown by the dotted line. It can be seen that the implied distribution has a heavier left tail and a less heavy right tail than the lognormal distribution.

The two plots in Figure 7.1 are consistent. For example, consider a deep-out-of-the-money call with a strike price of $K_2 > S_0$. According to the right plot, it going to have a lower price when the implied distribution is used than when the lognormal distribution is used. This is because the option pays off only if the stock price S_T goes above K_2, and the probability of this is lower for the implied probability distribution than for the lognormal distribution. Therefore, we expect the implied distribution to give a relatively low price for the option. A relatively low price leads to a relatively low implied volatility[62] and this is exactly what we observe in the left plot when $K_2/S_0 > 1$. The above argument is taken from Section 20.3 of John Hull's book.

[62] Recall Vega $= \frac{\partial c}{\partial \sigma} > 0$ which means if $\sigma \uparrow$, $c \uparrow$; if $\sigma \downarrow$, $c \downarrow$.

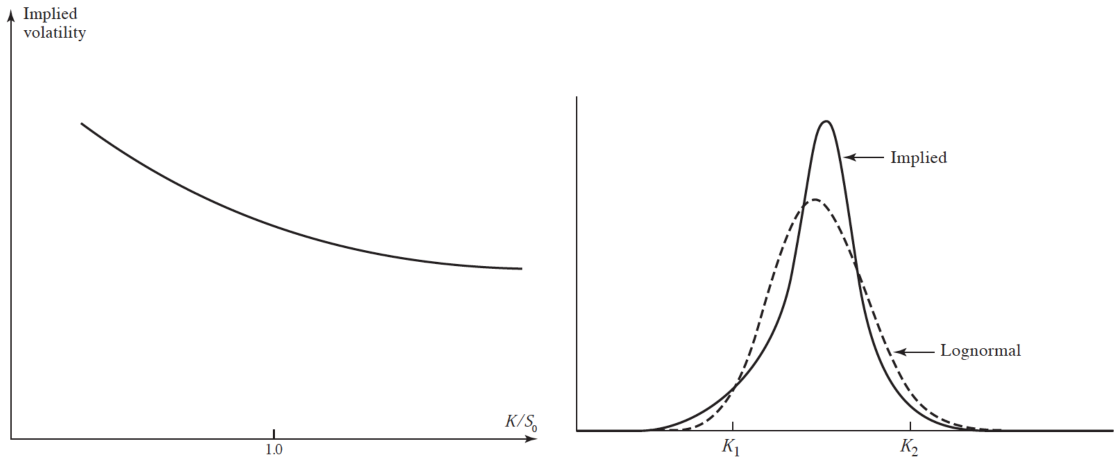


Figure 7.1: From John Hull's book: Left: volatility smile for stock option or stock indices options. The implied volatility values are obtained by averaging options of different maturities. Right: implied distribution and lognormal distribution of the underlying stock price.

Example 7.4 *(Use volatility smiles to estimate risk-neutral probability distribution. Appendix of Chapter 20 of John Hull's book.) We know* $c = c(0, S_0) = \tilde{\mathbb{E}}[e^{-rT}(S_T - K)^+]$. *Suppose the risk-neutral probability density of* S_T *is* g. *Show that*

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT} g(K). \tag{7.31}$$

This result allows one to use volatility smiles to estimate risk-neutral probability distribution. For example, suppose c_1, c_2, *and* c_3 *are the price of* T*-years European call options with strike prices of* $K - \delta$, K, *and* $K + \delta$ *respectively. Assuming* δ *is small, an estimation of* $g(K)$ *is*

$$e^{-rT} \frac{c_1 - 2c_2 + c_3}{\delta^2}.$$

Proof: *Let* $Y = S_T$. *The pdf of* Y *is* $g(y)$. *Then*

$$\begin{aligned} c = c(K) &= \tilde{\mathbb{E}}[e^{-rT}(S_T - K)^+] \\ &= e^{-rT} \int_{-\infty}^{\infty} (y - K)^+ g(y) dy = e^{-rT} \int_K^{\infty} (y - K) g(y) dy. \end{aligned}$$

$$
\begin{aligned}
c'(K) = \lim_{h\to 0}\frac{c(K+h)-c(K)}{h} &= e^{-rT}\lim_{h\to 0}\frac{\int_{K+h}^{\infty}(y-K-h)g(y)dy - \int_{K}^{\infty}(y-K)g(y)dy}{h}\\
&= e^{-rT}\lim_{h\to 0}\Bigg(\frac{\int_{K+h}^{\infty}(y-K-h)g(y)dy - \int_{K+h}^{\infty}(y-K)g(y)dy}{h}\\
&\qquad + \frac{\int_{K+h}^{\infty}(y-K)g(y)dy - \int_{K}^{\infty}(y-K)g(y)dy}{h}\Bigg)\\
&= e^{-rT}\lim_{h\to 0}\left(\frac{\int_{K+h}^{\infty}-hg(y)dy}{h} - \frac{\int_{K}^{K+h}(y-K)g(y)dy}{h}\right)\\
&= -e^{-rT}\int_{K}^{\infty} g(y)dy - e^{-rT}\left.(y-K)g(y)\right|_{y=K}\\
&= -e^{-rT}\int_{K}^{\infty} g(y)dy.
\end{aligned}
$$

$$
c''(K) = e^{-rT}g(K).
$$

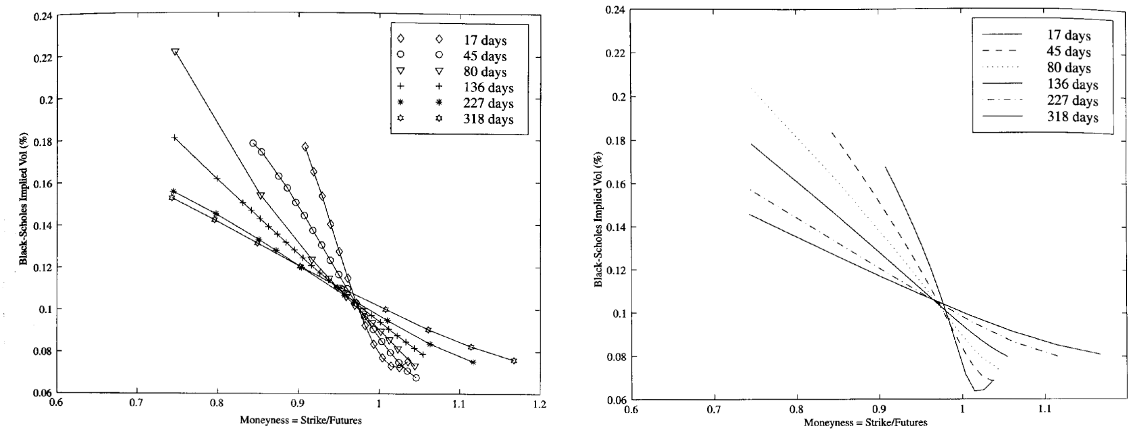


Figure 7.2: From Duffie's book: Smile curves implied by S&P 500 index options of six different times to expiration, from market data for Nov 2, 1993. Left: Market data. Right: Calibrated Heston model. Futures price $= \tilde{\mathbb{E}}[S(T)]$ (see Shreve II Page 244 or Duffie Page 172).

A similar Figure 7.2 is mentioned in Duffie's book. According to Section 8.E of Duffie's book, the right plot of Figure 7.2 is produced with the stochastic volatility model

$$
dS_t = rS_t dt + S_t\sqrt{V_t}dW_t^s, \tag{7.32}
$$

$$
dV_t = \kappa(\bar{v} - V_t)dt + \sigma_v\sqrt{V_t}dW_t^v, \tag{7.33}
$$

where $W^s, W^v \in \mathbb{R}$ are generated by $W^s = C^s \cdot W$, $W^v = C^v \cdot W$, $W = (W^1, W^2) \in \mathbb{R}^2$ being standard 2-dimensional Brownian motion. $C^s = (C_1^s, C_2^s)$, $C^v = (C_1^v, C_2^v)$ are 2-dimensional unit[63] vectors. In other words, $dW_t^s = C_1^s dW_t^1 + C_2^s dW_t^2$, $dW_t^v = C_1^v dW_t^1 + C_2^v dW_t^2$. $r = 0.0319$. Other parameters, including the initial value of V_t, are estimated by minimizing mean squared errors. A total of 87 options, observed on November 2, 1993, are used. As a result, $C^s \cdot C^v = -0.66$, $\kappa = 19.66$, $\bar{v} = 0.017$, $\sigma_v = 1.516$, $\sqrt{V_0} = 0.094$. During this estimation process, one needs to repeatedly compute the option price based on some chosen parameters. Even though the usual pricing formula for the call option is mentioned in Section 8E of Duffie's book,

$$c(0, S_0, V_0) = \tilde{\mathbb{E}}[e^{-rT}(S_T - K)^+],$$

Duffie indeed computed the option price by the more efficient transformation method proposed in the paper Darrell Duffie, Jun Pan and Kenneth Singleton, Transform Analysis and Asset Pricing for Affine Jump-Diffusions, Econometrica, 68 (2000) 1343–1376. `https://www.darrellduffie.com/uploads/pubs/DuffiePanSingleton2000.pdf`. A simplified presentation of the paper (without jumps) is in Section 8.F of Duffie's book.

7.4 Computer experiments

This part discusses discrete time hedge and is taken from Section 15.5 of Choe. Recall that our Φ in (7.3) that replicates the option. Suppose we adjust Δ at $t_j = j\delta t$ for $j = 1, 2, 3, \ldots$. Before making the adjustment at t_i, consider the time interval $[t_{i-1}, t_i)$. We hold $\Delta_{t_{i-1}}$ shares of stock during this interval and put $B_{t_{i-1}}$ in the bank account at t_{i-1}. At t_i, we have

$$c_{t_i} = \Delta_{t_{i-1}} S_{t_i} + B_{t_{i-1}} e^{r\delta t}. \tag{7.34}$$

Immediately after we making the adjustment at t_i, we have

$$c_{t_i} = \Delta_{t_i} S_{t_i} + B_{t_i}. \tag{7.35}$$

Hence

$$B_{t_i} = B_{t_{i-1}} e^{r\delta t} + \left(\Delta_{t_{i-1}} - \Delta_{t_i}\right) S_{t_i}. \tag{7.36}$$

The following code is from Section 15.5 of Choe. Unlike what we have done in the lecture notes, where we take B_0 so that $-c+\Delta S+B = 0$ initially, Choe's portfolio is $\Pi = -c+\Delta S+B$ with $B_0 = 0$. It is risk-free with the same Δ as ours.

The last plot of the computer result, which is entitled Π, shows that his portfolio is very close to the curve $\Pi(0)e^{rt}$ for all t, and is indeed risk-free.

```
T = 5;
r = 0.10; % interest rate
mu = 0.15; % drift coefficient
sigma = 0.3; % volatility
S0 = 100; % asset price at time t=0
K = 110; % strike price
```

[63] means length $= 1$.

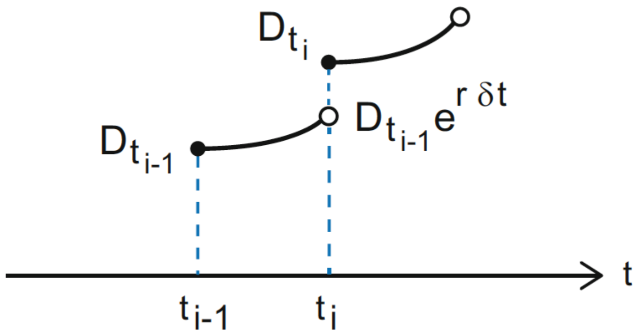
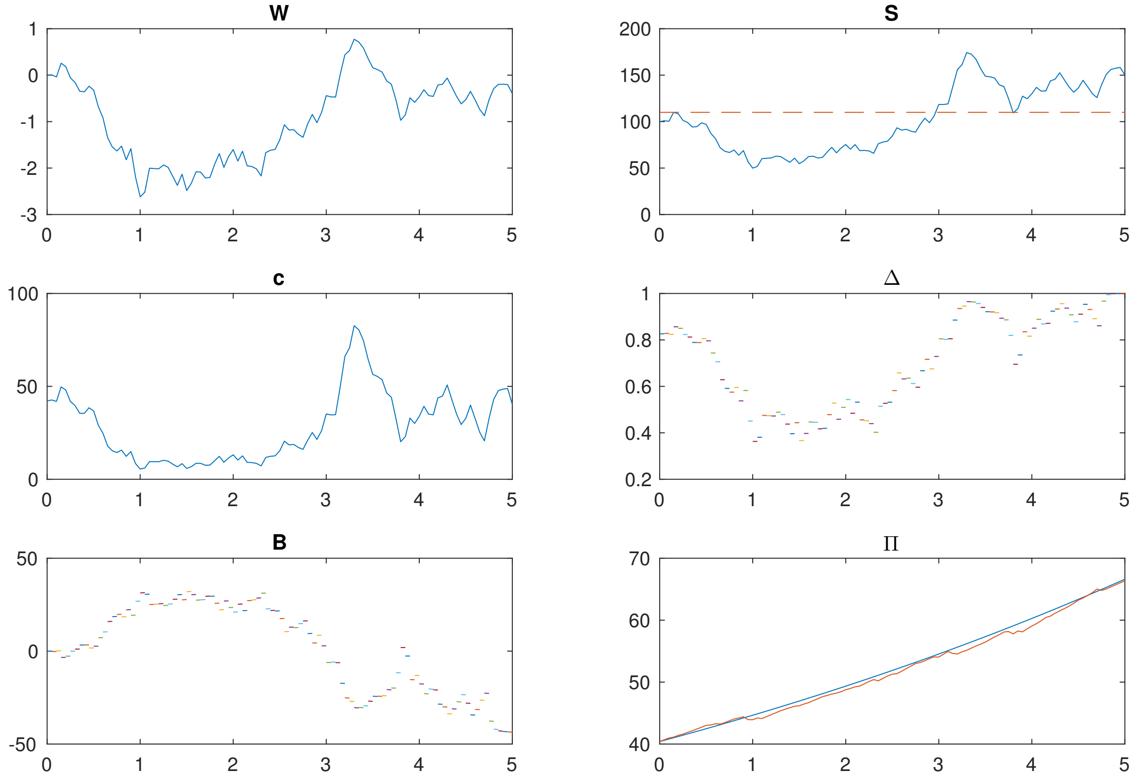


Figure 7.3: Figure from Choe. He uses D_{t_i} while I uses B_{t_i}.

```
N = 100 ; % number of time steps
dt = T/N;
t_value = [0:dt:T];
W = zeros(1,N+1); % Brownian motion
S = zeros(1,N+1); % asset price
c = zeros(1,N+1); % option price
Delta = zeros(1,N+1); % Delta
B = zeros(1,N+1); % bank deposit
Pi = zeros(1,N+1); % portfolio
B(1) = 0.0; % Choose any number for the initial cash amount.
S(1)= S0;
for i=2:N+1
    dW = sqrt(dt)*randn;
    W(i) = W(i-1) + dW;
    S(i) = S(i-1) + mu*S(i-1)*dt + sigma*S(i-1)*dW;
end
for i=1:N+1
    tau = T-(i-1)*dt;
    d1 = (log(S(i)/K) + (r+0.5*sigma^2)*tau)/sigma/sqrt(tau);
    d2 = d1 - sigma*sqrt(tau);
    c(i)= S(i)*normcdf(d1) - K*exp(-r*tau)*normcdf(d2);
    Delta(i)=normcdf(d1); % from Black-Scholes formula
end

% Unlike what we have done in the lecture notes, where we take
% B(1) so that -c + Delta S + B =0 initially,
% Choe choose B(1)=0. Hence Pi = -c + Delta S + B may not be
% zero initially.
```

```
for i = 1:N
    B(i+1) = exp(r*dt)*B(i) + (Delta(i)-Delta(i+1))*S(i+1);
    % self-financing
end
for i = 1:N+1
    Pi(i) = -c(i) + B(i) + Delta(i)*S(i);
end
subplot(3,2,1);
plot([0:dt:T],W)
title('W')
subplot(3,2,2);
plot([0:dt:T],S)
title('S')
hold on
plot([0:dt:T],K*ones(N+1,1),'--')
subplot(3,2,3);
plot([0:dt:T],c)
title('c')
subplot(3,2,4);
for i=1:N
    x = (i-1)*dt:dt/(500/N):i*dt;
    y = Delta(i)*exp(0*x);
    plot(x,y) % Plot the graph on each subinterval.
    hold on
end
title('\Delta')
subplot(3,2,5);
for i=1:N
    x = (i-1)*dt:dt/(500/N):i*dt;
    y = B(i);
    plot(x,y*exp(r*(x-(i-1)*dt))) % Plot the graph on each subinterval.
    hold on
end
title('B')
subplot(3,2,6);
plot([0:dt:T],Pi(1)*exp(r*[0:dt:T]))
hold on
plot([0:dt:T],Pi)
title('\Pi')
```


W
S
c
Δ
B
Π

7.5 Homework VII

(Only submit solutions to Questions 2,6,8.)

1. Use put-call parity $c_0 + Ke^{-rT} = p_0 + S_0$ to explain why the option price should not depend on the drift coefficient α in (7.2).

   Solution: We know that at expiration date T, $c_T = (S_T - K)^+$ while $p_T = (K - S_T)^+$. If option price depends on α. How would the call option price changes when α increases? One may say that if α is larger, S_T has higher chance to $> K$. Hence call option price c_0 should increase. But at the same time, as S_T is less likely to be $< K$, put option price p_0 should decrease. Then the put-call parity

   $$c_0 + Ke^{-rT} = p_0 + S_0$$

   can no longer hold. Then there would be arbitrage opportunity, as we have discussed before. Hence by put-call parity, option price should not depend on α.

2. A forward contract f on a non-dividend-paying stock S is an agreement to buy or sell a stock S at price K at expiration date T. Hence its pay-off at time T is $S_T - K$. From the discussion after (7.13), we have learned that its price at time t is $f_t = S_t - Ke^{-r(T-t)}$. Since it is a derivative depending on the underlying stock, it should satisfy (7.11). Verify that $f(t,x) = x - Ke^{-r(T-t)}$ does satisfy

   $$\frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t,x) - rf(t,x) = 0. \tag{7.37}$$

   with terminal condition $f(T,x) = x - K$.

3. (Verify the put-call parity) Use Question 1 to show that if $c(t,x)$ satisfies (7.11) with $c(T,x) = (x-K)^+$, then $p(t,x) = c(t,x) - x + Ke^{-r(T-t)}$ satisfies the same equation

   $$\frac{\partial p}{\partial t}(t,x) + rx\frac{\partial p}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 p}{\partial x^2}(t,x) - rp(t,x) = 0 \tag{7.38}$$

   with $p(T,x) = (K-x)^+$.

   Proof: Since (7.11) is a linear equation in c, it is clear that if both $c_1(t,x)$ and $c_2(t,x)$ satisfy (7.11), then for any constants λ_1 and λ_2, $c(t,x) \stackrel{\text{def}}{=} \lambda_1 c_1(t,x) + \lambda_2 c_2(t,x)$ satisfies (7.11). (This is the so-called superposition principle for linear equations.) Since both c and f satisfies (7.11), $p(t,x) = c(t,x) - f(t,x)$ also satisfies (7.11), which is (7.38). Moreover,

   $$p(T,x) = c(T,x) - x + K = (x-K)^+ - x + K = (K-x)^+.$$

4. Verify (1.27) (which can be rewritten as $c_t(s_t, \alpha K) = \alpha c_t(s_t/\alpha, K)$) which says if $c(t,x)$ satisfies (7.11) with $c(T,x) = (x-K)^+$, then $v(t,x) = \alpha c(t, x/\alpha)$ satisfies

   $$\frac{\partial v}{\partial t}(t,x) + rx\frac{\partial v}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2}(t,x) - rv(t,x) = 0. \tag{7.39}$$

with $v(T,x) = (x-\alpha K)^+$.

Proof: (I hope you can understand chain rule from this example.) To evaluate $\frac{\partial c}{\partial x}(t,x)$, we first take the function $c(t,x)$, take its partial derivative with respect to x, and then stick in (t,x) to get a number. Note the different roles played by the two x's in $\frac{\partial c}{\partial x}(t,x)$. The first one denotes the partial derivative of a function c with respect to its second variable. The second one is a number x. The condition that

$$\frac{\partial c}{\partial t}(t,x) + rx\frac{\partial c}{\partial x}(t,x) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 c}{\partial x^2}(t,x) - rc(t,x) = 0$$

for any t and x means that if we take the function $c(t,x)$, take its partial derivatives, and then evaluate those derivatives at point (t,x), the combination $\frac{\partial c}{\partial t}(t,x) + rx\frac{\partial c}{\partial x}(t,x) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 c}{\partial x^2}(t,x) - rc(t,x)$ is zero. This is true for any point (t,x), and hence is true at point $(t,x/\alpha)$. It means that

$$\frac{\partial c}{\partial t}(t,x/\alpha) + r(x/\alpha)\frac{\partial c}{\partial x}(t,x/\alpha) + \frac{1}{2}\sigma^2(x/\alpha)^2\frac{\partial^2 c}{\partial x^2}(t,x/\alpha) - rc(t,x/\alpha) = 0. \tag{7.40}$$

Since $v(t,x) = \alpha c(t,x/\alpha)$, taking partial derivatives with respect to t or x, we get

$$\frac{\partial v}{\partial t}(t,x) = \alpha\frac{\partial c}{\partial t}(t,x/\alpha), \quad \frac{\partial v}{\partial x}(t,x) = \alpha\frac{\partial c}{\partial x}(t,x/\alpha)\frac{1}{\alpha} = \frac{\partial c}{\partial x}(t,x/\alpha).$$

Since $\frac{\partial v}{\partial x}(t,x) = \frac{\partial c}{\partial x}(t,x/\alpha)$, taking partial derivatives with respect to x, we get

$$\frac{\partial^2 v}{\partial x^2}(t,x) = \frac{\partial^2 c}{\partial x^2}(t,x/\alpha)\frac{1}{\alpha}.$$

Hence (7.40) implies

$$\frac{1}{\alpha}\frac{\partial v}{\partial t}(t,x) + r(x/\alpha)\frac{\partial v}{\partial x}(t,x) + \frac{1}{2}\sigma^2(x/\alpha)^2\frac{\partial^2 v}{\partial x^2}(t,x)\alpha - r\frac{1}{\alpha}v(t,x) = 0. \tag{7.41}$$

This is (7.39). Since $v(t,x) = \alpha c(t,x/\alpha)$,

$$v(T,x) = \alpha c(T,x/\alpha) = \alpha(x/\alpha - K)^+ = (x-\alpha K)^+.$$

5. Suppose $c(t,x)$ satisfies

$$\frac{\partial c}{\partial t}(t,x) + rx\frac{\partial c}{\partial x}(t,x) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 c}{\partial x^2}(t,x) - rc(t,x) = 0, \quad \forall (t,x) \in [0,T]\times[0,\infty), \tag{7.42}$$

and terminal condition

$$c(T,x) = (x-K)^+ \qquad \forall x \in [0,\infty). \tag{7.43}$$

Let

$$u(\tau,y) = \frac{1}{K}e^{ay+b\tau}c(T-\frac{2\tau}{\sigma^2}, Ke^y). \tag{7.44}$$

Determine a and b so that $u(\tau, y)$ satisfies

$$\frac{\partial u}{\partial \tau}(\tau, y) = \frac{\partial^2 u}{\partial y^2}(\tau, y). \tag{7.45}$$

Solution: (This problem will not be tested. I simply want to show you that Black-Schole-Merton equation can be simplified.) We know (7.42) is true for any (t, x). Hence it is true at point $(T - \frac{2\tau}{\sigma^2}, Ke^y)$:

$$\frac{\partial c}{\partial t}(T-\frac{2\tau}{\sigma^2}, Ke^y)+r(Ke^y)\frac{\partial c}{\partial x}(T-\frac{2\tau}{\sigma^2}, Ke^y)+\frac{1}{2}\sigma^2(K^2e^{2y})\frac{\partial^2 c}{\partial x^2}(T-\frac{2\tau}{\sigma^2}, Ke^y)-rc(T-\frac{2\tau}{\sigma^2}, Ke^y) = 0. \tag{7.46}$$

Taking $\frac{\partial}{\partial \tau}$ on both sides of (7.44). By chain rule,

$$\frac{\partial u}{\partial \tau}(\tau, y) = \frac{e^{ay+b\tau}}{K}\frac{\partial c}{\partial t}(T - \frac{2\tau}{\sigma^2}, Ke^y)\left(-\frac{2}{\sigma^2}\right) + \frac{be^{ay+b\tau}}{K}c(T - \frac{2\tau}{\sigma^2}, Ke^y)$$

$$\frac{\partial u}{\partial y} = \frac{e^{ay+b\tau}}{K}\frac{\partial c}{\partial x}(T - \frac{2\tau}{\sigma^2}, Ke^y)Ke^y + \frac{ae^{ay+b\tau}}{K}c(T - \frac{2\tau}{\sigma^2}, Ke^y)$$

$$\frac{\partial^2 u}{\partial y^2} = Ke^{(a+2)y+b\tau}\frac{\partial^2 c}{\partial x^2}(T-\frac{2\tau}{\sigma^2}, Ke^y)+(2a+1)e^{(a+1)y+b\tau}\frac{\partial c}{\partial x}(T-\frac{2\tau}{\sigma^2}, Ke^y)+\frac{a^2e^{ay+b\tau}}{K}c(T-\frac{2\tau}{\sigma^2}, Ke^y).$$

Hence $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial y^2}$ leads to

$$\frac{\partial c}{\partial t} + \frac{\sigma^2}{2}K^2e^{2y}\frac{\partial^2 c}{\partial x^2} + \frac{\sigma^2}{2}(2a+1)Ke^y\frac{\partial c}{\partial x} + \frac{\sigma^2}{2}(a^2 - b)c = 0.$$

Note that now $y \in (-\infty, \infty)$ and $\tau \in [0, \frac{\sigma^2}{2}T]$. $u(0, y) = e^{ay}(e^y - 1)^+$. Comparing with (7.46), we know

$$\frac{\sigma^2}{2}(2a+1) = r, \qquad \frac{\sigma^2}{2}(a^2 - b) = -r.$$

Hence

$$a = \frac{r}{\sigma^2} - \frac{1}{2}, \qquad b = \left(\frac{r}{\sigma^2} + \frac{1}{2}\right)^2.$$

One can solve (7.45) and get $c(t, x)$. See Section 15.2 of Choe's book.

6. We now present the derivation of Black-Scholes-Merton equation in Lishang Jiang Section 5.2. Let $c(t, x)$ denote the value of an option at time t if the stock price at that time is $S(t) = x$. We want to derive an equation for $c(t, x)$ and hence obtain its formula. Like in (2.2), construct a portfolio

$$\Phi = c - \Delta S, \tag{7.47}$$

where Δ denotes the number of shares of underlying stock. Suppose that

$$\Phi(t) = \Phi(0) + \int_0^t dc(s, S(s)) - \int_0^t \Delta(s) dS(s) \tag{7.48}$$

which means (by the equivalence between (5.5) and (5.6))

$$d\Phi(t) = dc(t, S(t)) - \Delta dS(t). \tag{7.49}$$

As in (2.3), we can manage to choose Δ so that Φ is not random. By the arbitrage-free principle, we expect that the resulting Φ behaves like a bank deposit, i.e., (same as (4.1))

$$d\Phi(t) = r\Phi(t)dt. \tag{7.50}$$

Here r is the risk-free interest rate for the money market.

Show that if we choose

$$\Delta(t) = \frac{\partial c}{\partial x}(t, S(t)) \tag{7.51}$$

and let $c(t, S)$ satisfy

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial x} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial x^2} - rc = 0, \tag{7.52}$$

then we have (7.50). [Hint: Use Itô formula to calcualte dc and then stick the result into the right hand side of (7.49). If we want to enforce (7.50), we should set the right hand side of (7.49) equal to $r\Phi dt$ which is $rc(t, S(t))dt - r\Delta(t)S(t)dt$.]

7. Prove (7.23)

   Proof:

$$\begin{aligned}
\frac{\partial c}{\partial t}(t,x) =& xN'(d_+)\left[\log\frac{x}{K}\sigma^{-1}(-1/2)(T-t)^{-3/2}(-1) + \sigma^{-1}\left(r+\frac{\sigma^2}{2}\right)\frac{1}{2}(T-t)^{-1/2}(-1)\right] \\
&- rKe^{-r(T-t)}N(d_-) \\
&- Ke^{-r(T-t)}N'(d_-)\left[\log\frac{x}{K}\sigma^{-1}(-1/2)(T-t)^{-3/2}(-1) + \sigma^{-1}\left(r-\frac{\sigma^2}{2}\right)\frac{1}{2}(T-t)^{-1/2}(-1)\right.
\end{aligned}$$

Now, we use (7.27) and get

$$\begin{aligned}
\frac{\partial c}{\partial t}(t,x) =& xN'(d_+)\left[\frac{1}{2}\log\frac{x}{K}\sigma^{-1}(T-t)^{-3/2} - \sigma^{-1}\left(r+\frac{\sigma^2}{2}\right)\frac{1}{2}(T-t)^{-1/2}\right] \\
&- rKe^{-r(T-t)}N(d_-) \\
&- xN'(d_+)\left[\frac{1}{2}\log\frac{x}{K}\sigma^{-1}(T-t)^{-3/2} - \sigma^{-1}\left(r-\frac{\sigma^2}{2}\right)\frac{1}{2}(T-t)^{-1/2}\right] \\
=& - rKe^{-r(T-t)}N(d_-) - xN'(d_+)\frac{\sigma}{2}(T-t)^{-1/2}.
\end{aligned}$$

8. Recall that a portfolio $\Phi = \Delta S + B$ is call self-financing portfolio if it satisfies both $d\Phi_t = \Delta_t dS_t + dB_t$ and $\Phi_t = \Delta_t S_t + B_t$. Here S_t is the stock price, Δ_t is the number of shares of stock, B_t is the amount in the money market account at time t. Prove that for any stock price model, a self-financing portfolio $\Phi = \Delta S + B$ satisfies

$$d\left(e^{-rt}\Phi_t\right) = \Delta_t d\left(e^{-rt}S_t\right) \tag{7.53}$$

which means that change in the discounted portfolio value is solely due to change in the discounted stock price. The parameter r in (7.53) comes from the interest rate of the money market account B whose value satisfies $dB_t = rB_t dt$.

9. Prove that the implied volatility of a European call option is the same as that of a European put option when they have the same strike price and time to maturity.

   Proof: The proof of this claim is based on put-call parity. Suppose that, for a particular value of the volatility, p_{BS} and c_{BS} are the values of European put and call options calculated using the Black-Scholes-Merton formulas (7.20) and (7.21). Suppose further that p_{mkt} and c_{mkt} are the market values of these options. Because put-call parity holds for the Black-Scholes-Merton model (Question 3 of Homework VII confirms that (7.21) satisfies (7.11) with proper terminal condition), we must have

$$p_{BS} + S_0 = c_{BS} + Ke^{-rT}.$$

   As we have seen in Chapter 1, in the absence of arbitrage opportunities, put-call parity also holds for the market prices, so that

$$p_{mkt} + S_0 = c_{mkt} + Ke^{-rT}.$$

   Subtracting these two equations, we get

$$p_{BS} - p_{mkt} = c_{BS} - c_{mkt}. \tag{7.54}$$

   Suppose that the implied volatility of the put option is, say, 22%. This means that $p_{BS} = p_{mkt}$ when $\sigma = 22\%$ is used in (7.2): $dS_t = \alpha S_t dt + \sigma S_t dW_t$. The same σ and the same (7.2) is used by Black-Scholes to derive the c_{BS}. Because of (7.54), $p_{BS} = p_{mkt}$ implies $c_{BS} = c_{mkt}$. This means that when $\sigma = 22\%$ is used in (7.2), we get $c_{BS} = c_{mkt}$. Hence $\sigma = 22\%$ is also the implied volatility of the call option. This finishes the proof.

10. Let $\{W_t, t \geq 0\}$ be a standard Brownian motion and Θ an arbitrary process. Define

$$Q_t = -\int_0^t \Theta_s dW_s - \frac{1}{2}\int_0^t \Theta_s^2 ds, \tag{7.55}$$

$$Z_t = e^{Q_t}, \tag{7.56}$$

$$X_t = W_t + \int_0^t \Theta_s ds. \tag{7.57}$$

Compute the differentials dZ_t and $d(X_tZ_t)$ and then prove that Z_t and X_tZ_t are martingales.

Solution: By the equivalence between differential and integral forms (5.5) (5.6),

$$dQ_t = -\Theta_t dW_t - \frac{1}{2}\Theta_t^2 dt.$$

Then $(dQ_t)^2 = \Theta_t^2 dt$. By Itô formula,

$$dZ_t = e^{Q_t}dQ_t + \frac{1}{2}e^{Q_t}(dQ_t)^2 = Z_t\left(dQ_t + \frac{1}{2}(dQ_t)^2\right) = -Z_t\Theta_t dW_t,$$

which means $Z_t = Z_0 + \int_0^t(-Z_s\Theta_s)dW_s$. Hence Z_t is a martingale. In particular, $\mathbb{E}[Z_t] = Z_0 = 1$.

To calculate $d(X_tZ_t)$, one need to use the product rule (5.31):

$$\begin{aligned} d(X_tZ_t) &= X_t dZ_t + Z_t dX_t + dX_t dZ_t \\ &= -X_tZ_t\Theta_t dW_t + Z_t(dW_t + \Theta_t dt) - Z_t\Theta_t dt \\ &= (-X_tZ_t\Theta_t + Z_t)dW_t. \end{aligned}$$

Hence X_tZ_t is also a martingale.

Remark: : The Girsanov theorem that we will learn in the next section says that if we define a probability measure $\tilde{\mathbb{P}}$ with $\tilde{\mathbb{P}}(A) = \int_A Z_T(\omega)d\mathbb{P}(\omega)$ with Z_T defined by (7.56), then the X_t in (7.57) is a Brownian motion in this new (or imaginary) world with probability measure $\tilde{\mathbb{P}}$.

In particular, by taking $\Theta = \theta$ being a constant, we find a way to construct $\tilde{\mathbb{P}}$ so that the $\tilde{W}$ that we have introduced in (7.14) (which is the X_t in (7.57) with $\Theta =$ constant) is a Brownian motion in this new world with $\tilde{\mathbb{P}}$.