Compute the differentials dZ_t and $d(X_tZ_t)$ and then prove that Z_t and X_tZ_t are martingales.

Solution: By the equivalence between differential and integral forms (5.5) (5.6),

$$dQ_t = -\Theta_t dW_t - \frac{1}{2}\Theta_t^2 dt.$$

Then $(dQ_t)^2 = \Theta_t^2 dt$. By Itô formula,

$$dZ_t = e^{Q_t}dQ_t + \frac{1}{2}e^{Q_t}(dQ_t)^2 = Z_t\left(dQ_t + \frac{1}{2}(dQ_t)^2\right) = -Z_t\Theta_t dW_t,$$

which means $Z_t = Z_0 + \int_0^t(-Z_s\Theta_s)dW_s$. Hence Z_t is a martingale. In particular, $\mathbb{E}[Z_t] = Z_0 = 1$.

To calculate $d(X_tZ_t)$, one need to use the product rule (5.31):

$$\begin{aligned} d(X_tZ_t) &= X_t dZ_t + Z_t dX_t + dX_t dZ_t \\ &= -X_tZ_t\Theta_t dW_t + Z_t(dW_t + \Theta_t dt) - Z_t\Theta_t dt \\ &= (-X_tZ_t\Theta_t + Z_t)dW_t. \end{aligned}$$

Hence X_tZ_t is also a martingale.

Remark: : The Girsanov theorem that we will learn in the next section says that if we define a probability measure $\tilde{\mathbb{P}}$ with $\tilde{\mathbb{P}}(A) = \int_A Z_T(\omega)d\mathbb{P}(\omega)$ with Z_T defined by (7.56), then the X_t in (7.57) is a Brownian motion in this new (or imaginary) world with probability measure $\tilde{\mathbb{P}}$.

In particular, by taking $\Theta = \theta$ being a constant, we find a way to construct $\tilde{\mathbb{P}}$ so that the $\tilde{W}$ that we have introduced in (7.14) (which is the X_t in (7.57) with $\Theta =$ constant) is a Brownian motion in this new world with $\tilde{\mathbb{P}}$.

Example 7.5 *Our goal is to find the price of an option, but we indeed obtain a bit more than that. c can represent any financial asset as long as we can relate the value of the financial asset at time t with stock price $S(t)$, i.e., $c = c(t, S(t))$.*

For example,

- *If we require $c(T, x) = 1$, then $c(T, S(T)) = 1$. 1 is the value of a zero-coupon bond with face value = 1 dollar at time T. The solution of (7.11) is $c(t, x) = e^{-r(T-t)}$ (one can easily check that $c(t, x) = e^{-r(T-t)}$ satisfies (7.11)). So, the price of a zero-coupon bond with unit face value at time t is $c(t, S_t) = e^{-r(T-t)}$.*
- *If we require $c(T, x) = x$, then $c(T, S(T)) = S(T)$. S_T is the price of the stock a time T. The solution of (7.11) is $c(t, x) = x$ (one can easily check that $c(t, x) = x$ satisfies (7.11)). So, the price of stock at time t is $c(t, S(t)) = S(t)$.*

The above two solutions plus the case with $c(T, S(T)) = (S(T)-K)^+$ simply say that we can use $\Phi = \Delta S + B$ to replicate bond, stock, and option. See Question 2 of Homework VII for the forward contract.