FE5222 Advanced Derivative Pricing

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October 16, 2019

Overview

Reflection Principle

Notations:

- Probability space: $(\Omega, \mathcal{F}, \mathbb{P})$
- Brownian motion: $\{W_t\}_{t\geq 0}$
- Given any process $\{X_t\}_{t\geq 0}$

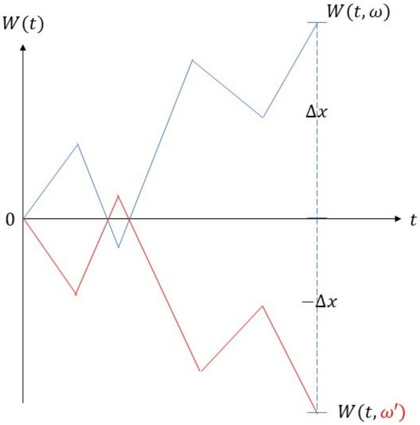
$$X_t^* = \max_{0\leq s\leq t} X_s$$

and

$$X_t^\# = \min_{0\leq s\leq t} X_s$$

X_t^* and $X_t^\#$ are the maximum and minimum of the process $\{X_t, t \geq 0\}$ up to time t respectively.

Reflection Principle

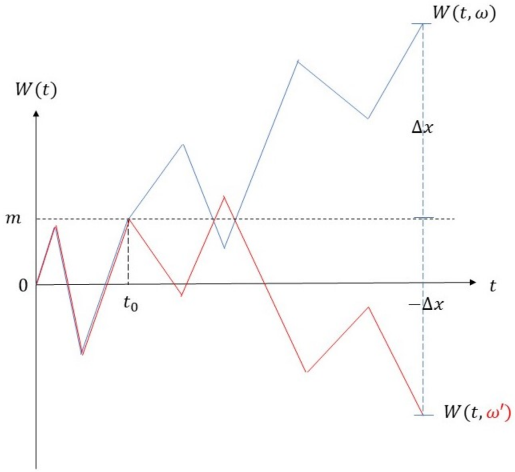


Reflection Principle

If we are not picky, we may write this relationship as

$$\mathbb{P}\left(W(t)=\Delta x\right)=\mathbb{P}\left(W(t)=-\Delta x\right)$$

Reflection Principle



Reflection Principle

Similarly we have

$$
\begin{aligned}
& \mathbb{P}\left(W(t)=m+\Delta x \mid W(t_0)=m\right) \\
= \ & \mathbb{P}\left(W(t)=m-\Delta x \mid W(t_0)=m\right)
\end{aligned}
$$

Reflection Principle

We can extend it

$$
\begin{aligned}
& \mathbb{P}\left(W(t) = m + \Delta x \middle| W(t_0) = m \text{ for some } 0 \leq t_0 \leq t\right) \\
= \ & \mathbb{P}\left(W(t) = m - \Delta x \middle| W(t_0) = m \text{ for some } 0 \leq t_0 \leq t\right)
\end{aligned}
$$

Reflection Principle

Making it more meaningful

$$\begin{aligned} & \mathbb{P}\left(W(t) \geq m + \Delta x | W(t_0) = m \text{ for some } 0 \leq t_0 \leq t\right) \\ = \; & \mathbb{P}\left(W(t) \leq m - \Delta x | W(t_0) = m \text{ for some } 0 \leq t_0 \leq t\right) \end{aligned}$$

Reflection Principle

For $m > 0$

$$W(t_0) = m \text{ for some } 0 \le t_0 \le t$$

$$\Longleftrightarrow$$

$$W_t^* \ge m$$

Reflection Principle

Hence

$$
\begin{aligned}
& \mathbb{P}\left(W(t) \geq m + \Delta x | W_t^* \geq m\right) \\
= \; & \mathbb{P}\left(W(t) \leq m - \Delta x | W_t^* \geq m\right)
\end{aligned}
$$

Reflection Principle

From

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

we have

$$\begin{aligned} & \mathbb{P}(W(t) \geq m + \Delta x, W_t^* \geq m) \\ = \; & \mathbb{P}(W(t) \leq m - \Delta x, W_t^* \geq m) \end{aligned}$$

Reflection Principle

When $\Delta x \geq 0$, $W(t) \geq m + \Delta$ implies $W_t^* \geq m$.

$\Longrightarrow$

$$\mathbb{P}\left(W(t) \geq m + \Delta x, W_t^* \geq m\right) = \mathbb{P}\left(W(t) \geq m + \Delta x\right)$$

$\Longrightarrow$

$$\mathbb{P}\left(W(t) \geq m + \Delta x\right) = \mathbb{P}\left(W(t) \leq m - \Delta x, W_t^* \geq m\right)$$

$\Longrightarrow$ Reflection Principle

Reflection Principle

Theorem (Reflection Principle)

For $m \geq 0, w \leq m$

$$\mathbb{P}\left(W_t^* \geq m, W_t \leq w\right) = \mathbb{P}\left(W_t \geq 2m - w\right) \tag{1}$$

Reflection Principle

Note that if we let

$$\tau_m = \inf\{s : W(s) = m\}$$

then

$$[\tau_m \leq t] = [W_t^* \geq m]$$

Reflection Principle

$$\mathbb{P}(\tau_m \leq t, W_t \leq w) = \mathbb{P}(W_t \geq 2m - w)$$

Distribution of W_t^* and $W_t^\#$

Theorem

The p.d.f. of W_t^* is

$$p^*(m) = \begin{cases} \frac{2}{\sqrt{2\pi t}} e^{-\frac{m^2}{2t}} & m \geq 0 \\ 0 & m < 0 \end{cases}$$

Distribution of W_t^* and $W_t^{\#}$

Theorem

The p.d.f. of $W_t^{\#}$ is

$$p^{\#}(m) = \begin{cases} \frac{2}{\sqrt{2\pi t}} e^{-\frac{m^2}{2t}} & m \leq 0 \\ 0 & m > 0 \end{cases}$$

Distribution of W_t^* and $W_t^\#$

Proof.

Taking $m = w$ in equation (1), we have

$$\mathbb{P}\left(W_t^* \geq m, W_t \leq m\right) = \mathbb{P}\left(W_t \geq m\right)$$

Note that

$$\mathbb{P}\left(W_t^* \geq m, W_t > m\right) = \mathbb{P}\left(W_t > m\right) = \mathbb{P}\left(W_t \geq m\right)$$

Hence

$$\begin{aligned} & \mathbb{P}\left(W_t^* \geq m\right) \\ = \ & \mathbb{P}\left(W_t^* \geq m, W(t) \leq m\right) + \mathbb{P}\left(W_t^* \geq m, W(t) > m\right) \\ = \ & 2\mathbb{P}\left(W_t \geq m\right) \end{aligned}$$

$\square$

Distribution of W_t^* and $W_t^\#$

Proof.

Hence

$$\mathbb{P}(W_t^* \geq m) = \frac{2}{\sqrt{2\pi t}} \int_m^\infty e^{-\frac{x^2}{2t}} dx$$

It follows that

$$\mathbb{P}(W_t^* \leq m) = 1 - \frac{2}{\sqrt{2\pi t}} \int_m^\infty e^{-\frac{x^2}{2t}} dx$$

$\square$

Distribution of W_t^* and $W_t^{\#}$

Proof.

Differentiating the above equation, we have

$$p^*(m) = \begin{cases} \frac{2}{\sqrt{2\pi t}} e^{-\frac{m^2}{2t}} & m > 0 \\ 0 & m \leq 0 \end{cases}$$

$\square$

Distribution of W_t^* and $W_t^{\#}$

Proof.

By symmetry, for $m < 0$, we have

$$\mathbb{P}\left(W_t^{\#} \leq m\right) = \frac{2}{\sqrt{2\pi t}} \int_{-m}^{\infty} e^{-\frac{x^2}{2t}} dx,$$

and hence

$$p^{\#}(m) = \begin{cases} \frac{2}{\sqrt{2\pi t}} e^{-\frac{m^2}{2t}} & m < 0 \\ 0 & m \geq 0 \end{cases}$$

$\square$

Joint Distribution of (W_t^*, W_t) and $(W_t^\#, W_t)$

To price barrier options, we need the joint distribution of the random vector W_t^* (or $W_t^\#$) and W_t.

Joint Distribution of (W_t^*, W_t) and $(W_t^\#, W_t)$

Theorem

The p.d.f. of (W_t^*, W_t) is

$$f^*(m,w) = \begin{cases} \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}} & m \geq 0, w \leq m \\ 0 & otherwise \end{cases}$$

Joint Distribution of (W_t^*, W_t) and $(W_t^{\#}, W_t)$

Theorem

The p.d.f. of $(W_t^{\#}, W_t)$ is

$$f^{\#}(m,w) = \begin{cases} \frac{2(-2m+w)}{t\sqrt{2\pi t}} e^{-\frac{(-2m+w)^2}{2t}} & m \le 0, w \ge m \\ 0 & otherwise \end{cases}$$

Joint Distribution of (W_t^*, W_t) and $(W_t^\#, W_t)$

Proof.

It is trivial that $f^*(m, w) = 0$ for $m < 0$ or $w > m$.

We focus on $f^*(m, w)$ on the region $m \geq 0, w \leq m$.

By the definition of p.d.f, we have

$$\mathbb{P}(W_t^* \geq m, W_t \leq w) = \int_m^\infty \int_{-\infty}^w f^*(x, y) dy dx.$$

By reflection principle, we have

$$\mathbb{P}\left(W_t^* \geq m, W_t \leq w\right) = \mathbb{P}\left(W_t \geq 2m - w\right).$$

□

Joint Distribution of (W_t^*, W_t) and $(W_t^\#, W_t)$

Proof.

Hence,

$$\int_m^\infty \int_{-\infty}^w f^*(x,y)dydx = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^\infty e^{-\frac{s^2}{2t}} ds.$$

Differentiating the above equality w.r.t. m and w respectively, we get

$$f^*(m,w) = \begin{cases} \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}} & m \geq 0, w \leq m \\ 0 & \text{otherwise} \end{cases}$$

$\square$

Brownian Motion with Drift

Definition

Let $\{W_t, t \geq 0\}$ be a Brownian motion, α be a real number, the process $\widetilde{W}_t = \alpha t + W_t$ is called a *Brownian motion with drift*.

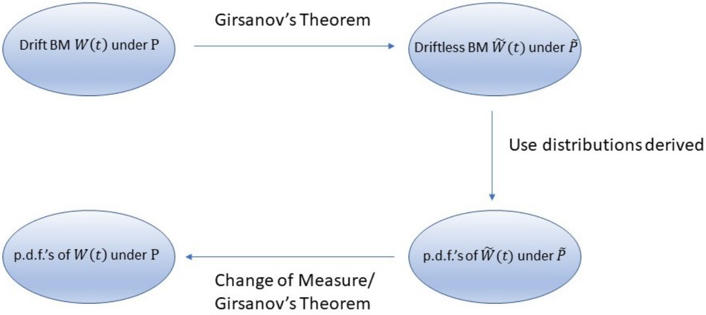
This is not a standard term. In fact, $\widetilde{W}_t$ is not even a Brownian motion in the original probability space.

Brownian Motion with Drift

In pricing barrier options, we usually have to consider Brownian motion with drift.

$\Rightarrow$ The probability distributions we have derived so far apply only to (driftless) Brownian motion.

Brownian Motion with Drift



Brownian Motion with Drift

From Girsanov's theorem, $\widetilde{W}_t$ is a (driftless) Brownian motion on the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$, where

$$\begin{aligned} \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} &= e^{-\frac{1}{2}\int_0^t \alpha^2 dt - \int_0^t \alpha dW_t} \\ &= e^{-\frac{1}{2}\alpha^2 t - \alpha W_t} \end{aligned}$$

Hence the p.d.f.'s we have derived still hold for $\widetilde{W}_t$ on the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$.

Brownian Motion with Drift

Also recall that

$$\widetilde{\mathbb{E}}\left[X\right] = \mathbb{E}\left[X\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right]$$

and

$$\widetilde{\mathbb{E}}\left[Y\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}}\right] = \mathbb{E}\left[Y\right]$$

where

$$\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}} = 1/\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}$$

Distribution of $\left(\widetilde{W}_t^*, \widetilde{W}_t\right)$ and $\left(\widetilde{W}_t^\#, \widetilde{W}_t\right)$

Theorem

The p.d.f. of $(\widetilde{W}_t^*, \widetilde{W}_t)$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is*

$$\widetilde{f}^*(m, w) = \begin{cases} \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}+\alpha w-\frac{1}{2}\alpha^2 t} & m \geq 0, w \leq m \\ 0 & otherwise \end{cases}$$

Distribution of $\left(\widetilde{W}_t^*, \widetilde{W}_t\right)$ and $\left(\widetilde{W}_t^{\#}, \widetilde{W}_t\right)$

Theorem

The p.d.f. of $(\widetilde{W}_t^{\#}, \widetilde{W}_t)$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is*

$$\widetilde{f}^{\#}(m, w) = \begin{cases} \frac{2(-2m+w)}{t\sqrt{2\pi t}} e^{-\frac{(-2m+w)^2}{2t} + \alpha w - \frac{1}{2}\alpha^2 t} & m \le 0, w \ge m \\ 0 & otherwise \end{cases}$$

Distribution of $\left(\widetilde{W}_t^*, \widetilde{W}_t\right)$ and $\left(\widetilde{W}_t^{\#}, \widetilde{W}_t\right)$

Proof.

It is trivial that $\widetilde{f}^*(m, w) = 0$ when $m < 0$ or $w > m$. $\square$

Distribution of $\left(\widetilde{W}_t^*, \widetilde{W}_t\right)$ and $\left(\widetilde{W}_t^\#, \widetilde{W}_t\right)$

Proof.

For $m \geq 0, w \leq m$, by definition, we have

$$\mathbb{P}\left(\widetilde{W}_t^* \leq m, \widetilde{W}_t \leq w\right) = \int_{-\infty}^{m} \int_{-\infty}^{w} \widetilde{f}^*(x, y) dy dx. \qquad (2)$$

□

Distribution of $\left(\widetilde{W}_t^*, \widetilde{W}_t\right)$ and $\left(\widetilde{W}_t^\#, \widetilde{W}_t\right)$

Proof.

On the other hand, we have

$$
\begin{aligned}
& \mathbb{P}\left(\widetilde{W}_t^* \leq m, \widetilde{W}_t \leq w\right) \\
= & \mathbb{E}\left[1_{[\widetilde{W}_t^* \leq m, \widetilde{W}_t \leq w]}\right] \\
= & \widetilde{\mathbb{E}}\left[1_{[\widetilde{W}_t^* \leq m, \widetilde{W}_t \leq w]} \frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}}\right] \\
= & \widetilde{\mathbb{E}}\left[1_{[\widetilde{W}_t^* \leq m, \widetilde{W}_t \leq w]} e^{\frac{1}{2}\alpha^2 t + \alpha W_t}\right] \\
= & \widetilde{\mathbb{E}}\left[1_{[\widetilde{W}_t^* \leq m, \widetilde{W}_t \leq w]} e^{-\frac{1}{2}\alpha^2 t + \alpha \widetilde{W}_t}\right]
\end{aligned}
$$

□

Distribution of $\left(\widetilde{W}_t^*, \widetilde{W}_t\right)$ and $\left(\widetilde{W}_t^\#, \widetilde{W}_t\right)$

Proof.

Since the p.d.f. of $(\widetilde{W}_t^*, \widetilde{W}_t)$ on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$ has been derived before, we have

$$\begin{aligned} & \widetilde{\mathbb{E}}\left[1_{[\widetilde{W}_t^* \le m, \widetilde{W}_t \le w]} e^{-\frac{1}{2}\alpha^2 t + \alpha \widetilde{W}_t}\right] \\ = \; & \int_{-\infty}^{m} \int_{-\infty}^{w} e^{-\frac{1}{2}\alpha^2 t + \alpha y} f^*(x, y) dy dx. \end{aligned}$$

$\square$

Distribution of $\left(\widetilde{W}_t^*, \widetilde{W}_t\right)$ and $\left(\widetilde{W}_t^\#, \widetilde{W}_t\right)$

Proof.

Comparing it with equation (2), we have

$$\widetilde{f}^*(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t} + \alpha w - \frac{1}{2}\alpha^2 t}$$

for all $m \geq 0, w \leq m$. $\square$

Distribution of $\left(\widetilde{W}_t^*, \widetilde{W}_t\right)$ and $\left(\widetilde{W}_t^\#, \widetilde{W}_t\right)$

Proof.

The joint p.d.f. of $\left(\widetilde{W}_t^\#, \widetilde{W}_t\right)$ can be proved in a similar way. $\square$

Down-and-in Put Option

Let H be the barrier level, K be the strike, and S_0 be the spot price. We assume $S_0 > H$ and $K > H$.

The payoff of a down-and-in put option at expiry T is

$$V_T = (K - S_T)^+ 1_{[S_T^{\#} \leq H]}$$

Down-and-in Put Option

Using risk neutral pricing formula, we have

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}\left[(K - S_T)^+ 1_{[S_T^\# \leq H]}\right] \\ &= e^{-rT}\mathbb{E}\left[(K - S_T) 1_{[S_T^\# \leq H, S_T \leq K]}\right] \end{aligned}$$

where r is the risk-free interest rate and the expectation is taken w.r.t. the risk neutral measure $(\Omega, \mathcal{F}, \mathbb{P})$.

Down-and-in Put Option

We assume BSM model

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t$$

Down-and-in Put Option

Solving S_t, we have

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}$$

Equivalently

$$S_t = S_0 e^{\sigma\left(\frac{(r-\frac{1}{2}\sigma^2)}{\sigma}t+W_t\right)}$$

Down-and-in Put Option

Let $\alpha = \frac{(r - \frac{1}{2}\sigma^2)}{\sigma}$ and $\widetilde{W}_t = \alpha t + W_t$. Then

$$S_t = S_0 e^{\sigma \widetilde{W}_t}$$

and

$$S_t^{\#} = S_0 e^{\sigma \widetilde{W}_t^{\#}}$$

Down-and-in Put Option

Since

$$S_T^{\#} \leq H \Longleftrightarrow \widetilde{W}_T^{\#} \leq \frac{1}{\sigma} \ln\left(\frac{H}{S_0}\right)$$

and

$$S_T \leq K \Longleftrightarrow \widetilde{W}_T \leq \frac{1}{\sigma} \ln\left(\frac{K}{S_0}\right)$$

we have

$$V_0 = e^{-rT}\mathbb{E}\left[(K - S_0 e^{\sigma \widetilde{W}_T})1_{[\widetilde{W}_T^{\#} \leq m, \widetilde{W}_T \leq w]}\right]$$

where $m = \frac{1}{\sigma} ln(\frac{H}{S_0}), w = \frac{1}{\sigma} ln(\frac{K}{S_0})$.

Down-and-in Put Option

Note that

- $H < S_0$ and $K < S_0 \Longrightarrow m < 0, m \geq w$.
- Since the p.d.f. of $(\widetilde{W}_T^{\#}, \widetilde{W}_T)$ is known, V_0 can be calculated.

Computation of V_0

Let

$$I_0 = \mathbb{E}\left[(K - S_0 e^{\sigma \widetilde{W}_T}) 1_{[\widetilde{W}_T^{\#} \leq m, \widetilde{W}_T \leq w]}\right]$$

Then

$$V_0 = e^{-rT} I_0$$

Computation of V_0

$$
\begin{aligned}
I_0 &= \int_{-\infty}^{m}\left[\int_{x}^{w}(K - S_0 e^{\sigma y})\widetilde{f}^{\#}(x,y)dy\right]dx \\
&= \int_{m}^{w}\left[\int_{-\infty}^{m}(K - S_0 e^{\sigma y})\widetilde{f}^{\#}(x,y)dx\right]dy \\
&\quad + \int_{-\infty}^{m}\left[\int_{-\infty}^{y}(K - S_0 e^{\sigma y})\widetilde{f}^{\#}(x,y)dx\right]dy
\end{aligned}
$$

Computation of V_0

Let

$$I_1 = \int_m^w \left[\int_{-\infty}^m (K - S_0 e^{\sigma y}) \widetilde{f}^{\#}(x, y) dx \right] dy$$

and

$$I_2 = \int_{-\infty}^m \left[\int_{-\infty}^y (K - S_0 e^{\sigma y}) \widetilde{f}^{\#}(x, y) dx \right] dy$$

Then $I_0 = I_1 + I_2$. We can compute I_1 and I_2.

The rest of slides will not be discussed. It's better for you to work out the details on your own and compare the solution with these slides.

Computation of V_0

First, we note that for any $A < 0$,

$$
\begin{aligned}
& \int_{-\infty}^{A} (K - S_0 e^{\sigma y}) \widetilde{f}^{\#}(x, y) dx \\
= \quad & \int_{-\infty}^{A} (K - S_0 e^{\sigma y}) \frac{2(-2x+y)}{T\sqrt{2\pi T}} e^{-\frac{(-2x+y)^2}{2T} + \alpha y - \frac{1}{2}\alpha^2 T} dx \\
= \quad & \frac{1}{\sqrt{2\pi T}} (K - S_0 e^{\sigma y}) e^{\alpha y - \frac{1}{2}\alpha^2 T} \int_{-\infty}^{A} \frac{2(-2x+y)}{T} e^{-\frac{(-2x+y)^2}{2T}} dx \\
= \quad & \frac{1}{\sqrt{2\pi T}} (K - S_0 e^{\sigma y}) e^{\alpha y - \frac{1}{2}\alpha^2 T} \int_{-\infty}^{-\frac{(-2A+y)^2}{2T}} e^s ds \\
= \quad & \frac{1}{\sqrt{2\pi T}} (K - S_0 e^{\sigma y}) e^{\alpha y - \frac{1}{2}\alpha^2 T} e^{-\frac{(-2A+y)^2}{2T}}
\end{aligned}
$$

Computation of V_0

Hence

$$
\begin{aligned}
& I_1 \\
= \; & \int_m^w \frac{1}{\sqrt{2\pi T}} (K - S_0 e^{\sigma y}) e^{\alpha y - \frac{1}{2}\alpha^2 T} e^{-\frac{(-2m+y)^2}{2T}} dy \\
= \; & K e^{-\frac{1}{2}\alpha^2 T} \int_m^w \frac{1}{\sqrt{2\pi T}} e^{\alpha y} e^{-\frac{(-2m+y)^2}{2T}} dy \\
& - S_0 e^{-\frac{1}{2}\alpha^2 T} \int_m^w \frac{1}{\sqrt{2\pi T}} e^{(\alpha+\sigma) y} e^{-\frac{(-2m+y)^2}{2T}} dy \\
= \; & K e^{-\frac{1}{2}\alpha^2 T} e^{2m\alpha + \frac{1}{2}\alpha^2 T} \left\{ \Phi\left(\frac{w-2m-\alpha T}{\sqrt{T}}\right) - \Phi\left(\frac{-m-\alpha T}{\sqrt{T}}\right) \right\} \\
& - S_0 e^{-\frac{1}{2}\alpha^2 T} e^{2m(\sigma+\alpha) + \frac{1}{2}(\sigma+\alpha)^2 T} \left\{ \Phi\left(\frac{w-2m-(\sigma+\alpha)T}{\sqrt{T}}\right) \right. \\
& \left. - \Phi\left(\frac{-m-(\sigma+\alpha)T}{\sqrt{T}}\right) \right\}
\end{aligned}
$$

Computation of V_0

Hence

$$
\begin{aligned}
&= \ldots \\
&= Ke^{2m\alpha}\left\{\Phi\left(\frac{w-2m-\alpha T}{\sqrt{T}}\right)-\Phi\left(\frac{-m-\alpha T}{\sqrt{T}}\right)\right\} \\
&\quad -S_0 e^{2\frac{r+1/2\sigma^2}{\sigma^2}ln(\frac{H}{S_0})+rT}\left\{\Phi(\frac{w-2m-(\sigma+\alpha)T}{\sqrt{T}}) \right. \\
&\quad \left. -\Phi(\frac{-m-(\sigma+\alpha)T}{\sqrt{T}})\right\} \\
&= K(\frac{H}{S_0})^{2\frac{r-1/2\sigma^2}{\sigma^2}}\left\{\Phi\left(\frac{m+\alpha T}{\sqrt{T}}\right)-\Phi\left(\frac{2m+\alpha T-w}{\sqrt{T}}\right)\right\} \\
&\quad -S_0 e^{2\frac{r+1/2\sigma^2}{\sigma^2}ln(\frac{H}{S_0})+rT}\left\{\Phi\left(\frac{m+(\sigma+\alpha)T}{\sqrt{T}}\right)-\Phi\left(\frac{2m+(\sigma+\alpha)T-w}{\sqrt{T}}\right)\right\} \\
&= K(\frac{H}{S_0})^{2\frac{r-1/2\sigma^2}{\sigma^2}}\left\{\Phi\left(\frac{m+\alpha T}{\sqrt{T}}\right)-\Phi\left(\frac{2m+\alpha T-w}{\sqrt{T}}\right)\right\} \\
&\quad -S_0 e^{rT}(\frac{H}{S_0})^{2\frac{r+1/2\sigma^2}{\sigma^2}}\left\{\Phi\left(\frac{m+(\sigma+\alpha)T}{\sqrt{T}}\right)-\Phi\left(\frac{2m+(\sigma+\alpha)T-w}{\sqrt{T}}\right)\right\}
\end{aligned}
$$

Computation of V_0

$$
\begin{aligned}
& I_2 \\
= \;& \int_{-\infty}^{m} \int_{-\infty}^{y} (K - S_0 e^{\sigma y}) \widetilde{f}^{\#}(x, y) dx dy \\
= \;& \int_{-\infty}^{m} \frac{1}{\sqrt{2\pi T}} (K - S_0 e^{\sigma y}) e^{\alpha y - \frac{1}{2}\alpha^2 T} exp\{-\frac{y^2}{2T}\} dy \\
= \;& e^{-\frac{1}{2}\alpha^2 T} \int_{-\infty}^{m} \frac{1}{\sqrt{2\pi T}} (K - S_0 e^{\sigma y}) e^{-\frac{y^2}{2T} + \alpha y} dy \\
= \;& e^{-\frac{1}{2}\alpha^2 T} \Bigg\{ K \int_{-\infty}^{m} \frac{1}{\sqrt{2\pi T}} e^{\alpha y - e^{\frac{y^2}{2T}}} dy \\
& -S_0 \int_{-\infty}^{m} \frac{1}{\sqrt{2\pi T}} e^{(\alpha+\sigma) y - \frac{y^2}{2T}} dy \Bigg\} \\
= \;& e^{-\frac{1}{2}\alpha^2 T} \left\{ K e^{\frac{1}{2}\alpha^2 T} \Phi\left(\frac{m - \alpha T}{\sqrt{T}}\right) - S_0 e^{\frac{1}{2}(\alpha+\sigma)^2 T} \Phi\left(\frac{m - (\alpha+\sigma) T}{\sqrt{T}}\right) \right\} \\
= \;& K \Phi\left(\frac{m - \alpha T}{\sqrt{T}}\right) - S_0 e^{rT} \Phi\left(\frac{m - (\alpha+\sigma) T}{\sqrt{T}}\right)
\end{aligned}
$$

Computation of V_0

Substituting I_1 and I_2 to I_0, we have

$$
\begin{aligned}
& I_0 \\
= \quad & -S_0 e^{rT}\Phi\left(\frac{ln(H/S_0)-(r+1/2\sigma^2)T}{\sigma\sqrt{T}}\right) + K\Phi\left(\frac{ln(H/S_0)-(r-1/2\sigma^2)T}{\sigma\sqrt{T}}\right) \\
& S_0 e^{rT}\left(\frac{H}{S_0}\right)^{2\frac{r+1/2\sigma^2}{\sigma^2}}\left\{\Phi\left(\frac{ln(H^2/(S_0K))+(r+1/2\sigma^2)T}{\sigma\sqrt{T}}\right)\right. \\
& \left.-\Phi\left(\frac{ln(H/S_0)+(r+1/2\sigma^2)T}{\sigma\sqrt{T}}\right)\right\} \\
& -K\left(\frac{H}{S_0}\right)^{2\frac{r-1/2\sigma^2}{\sigma^2}}\left\{\Phi\left(\frac{ln(H^2/(S_0K))+(r-1/2\sigma^2)T}{\sigma\sqrt{T}}\right)\right. \\
& \left.-\Phi\left(\frac{ln(H/S_0)+(r-1/2\sigma^2)T}{\sigma\sqrt{T}}\right)\right\}
\end{aligned}
$$

Computation of V_0

Let $\lambda = \frac{r+1/2\sigma^2}{\sigma^2}$, $x = \frac{ln(H/S_0)-\lambda\sigma^2 T}{\sigma\sqrt{T}}$, $y = \frac{ln(H^2/(S_0 K))+\lambda\sigma^2 T}{\sigma\sqrt{T}}$ and $z = \frac{ln(H/S_0)+\lambda\sigma^2 T}{\sigma\sqrt{T}}$, Then

$$\begin{aligned} I_0 \quad = \quad & -S_0 e^{rT}\Phi(x) + K\Phi(x+\sigma\sqrt{T}) \\ & +S_0 e^{rT}(\tfrac{H}{S_0})^{2\lambda}\{\Phi(y)-\Phi(z)\} \\ & -K(\tfrac{H}{S_0})^{2\lambda-2}\left\{\Phi(y-\sigma\sqrt{T})-\Phi(z-\sigma\sqrt{T})\right\} \end{aligned}$$

Computation of V_0

The value of down-and-in put option is

$$\begin{aligned} V_0 &= e^{-rT} I_0 \\ &= -S_0\Phi(x) + Ke^{-rT}\Phi(x+\sigma\sqrt{T}) \\ &\quad +S_0\left(\frac{H}{S_0}\right)^{2\lambda}\{\Phi(y)-\Phi(z)\} \\ &\quad -Ke^{-rT}\left(\frac{H}{S_0}\right)^{2\lambda-2}\left\{\Phi(y-\sigma\sqrt{T})-\Phi(z-\sigma\sqrt{T})\right\} \end{aligned}$$

Thank you!