FE5208: problem set 1 - solution

Each group of ≤ 5 people submits a copy on LumiNUS

Due on 17 February

1. Calculate the interest rates on slide 17 of Lecture 1.

   Answer: For:

T	1	2	3	4	5
$F(0, T-1, T)$	0.0420	0.0500	0.0550	0.0560	0.0530

   and

$$P(0,T) = exp\left[-\sum_{t=1}^{T} F(0,t-1,t)\right]$$
$$R(0,T) = -\frac{\log P(0,T)}{T}$$
$$\rho(0,T) = \frac{1 - P(t,T)}{\sum_{S=1}^{T} P(0,S)}$$

   we can get:

$$P(0,1) = exp(-0.0420) = 0.95887$$

$$P(0,2) = exp(-0.0420 - 0.0500) = 0.91211$$

$$P(0,3) = exp(-0.0420 - 0.0500 - 0.0550) = 0.86329$$

$$P(0,4) = exp(-0.0420 - 0.0500 - 0.0550 - 0.056) = 0.81628$$

$$P(0,5) = exp(-0.0420 - 0.0500 - 0.0550 - 0.056 - 0.053) = 0.77414$$

$$R(0,1) = -\frac{ln(0.95887)}{1.00} = 0.042 \qquad R(0,2) = -\frac{ln(0.91211)}{2.00} = 0.0460$$
$$R(0,3) = -\frac{ln(0.86329)}{3.00} = 0.0490 \qquad R(0,4) = -\frac{ln(0.81628)}{4.00} = 0.05075$$
$$R(0,5) = -\frac{ln(0.77414)}{5.00} = 0.0512$$

$$\rho(0,1) = \frac{1-0.95887}{0.95887} = 0.0429$$

$$\rho(0,2) = \frac{1-0.91211}{0.95887+0.91211} = 0.0470$$

$$\rho(0,3) = \frac{1-0.86329}{0.95887+0.91211+0.86329} = 0.0500$$

$$\rho(0,4) = \frac{1-0.81628}{0.95887+0.91211+0.86329+0.81628} = 0.0517$$

$$\rho(0,5) = \frac{1-0.77414}{0.95887+0.91211+0.86329+0.81628+0.77414} = 0.0522$$

2. Derive $F(t+1, T-1, T, D(t+1))$ on slide 17 of Lecture 2 for the case with $D(t+1) = D(t) + 1$ ("down" at $t+1$).

   Answer: See p. 36 of the textbook Cairns (2004).

3. Compute $P(t, 4, x)$ for $t = 2, 3$ and $x = 0, 1, 2, 3$ on slide 21 of Lecture 2.

   Answer:

   For $t = 2$, the value of x can be $0, 1, 2$. When

$$x = 0: P(2,4,0) = u_3 \cdot \frac{P(1,4,0)}{P(1,2,0)} = u_3 \cdot \frac{u_4 \cdot \frac{P(0,4,0)}{P(0,1,0)}}{u_2 \cdot \frac{P(0,2,0)}{P(0,1,0)}} = 0.96258$$

$$x = 1: P(2,4,1) = u_3 \cdot \frac{P(1,4,1)}{P(1,2,1)} = u_3 \cdot \frac{d_4 \cdot \frac{P(0,4,0)}{P(0,1,0)}}{d_2 \cdot \frac{P(0,2,0)}{P(0,1,0)}} = 0.91335$$

$$x = 2: P(2,4,2) = d_3 \cdot \frac{P(1,4,1)}{P(1,2,1)} = d_3 \cdot \frac{d_4 \cdot \frac{P(0,4,0)}{P(0,1,0)}}{d_2 \cdot \frac{P(0,2,0)}{P(0,1,0)}} = 0.86664$$

For $t=3$, the value of x can be $0,1,2,3$. When

$$x=0: P(3,4,0)=u_2\cdot\frac{P(2,4,0)}{P(2,3,0)}=u_2\cdot\frac{P(2,4,0)}{u_2\cdot\frac{P(1,3,0)}{P(1,2,0)}}=u_2\cdot\frac{P(2,4,0)}{u_2\cdot\frac{u_3\cdot\frac{P(0,3,0)}{P(0,1,0)}}{u_2\cdot\frac{P(0,2,0)}{P(0,1,0)}}}$$

$$=\frac{u_2P(2,4,0)P(0,2,0)}{u_3P(0,3,0)}=0.98812$$

$$x=1: P(3,4,1)=d_2\cdot\frac{P(2,4,0)}{P(2,3,0)}=d_2\cdot\frac{P(3,4,0)}{u_2}=0.96252$$

$$x=2: P(3,4,2)=d_2\cdot\frac{P(2,4,1)}{P(2,3,1)}=d_2\cdot\frac{d_3\cdot\frac{P(1,4,0)}{P(1,2,0)}}{d_2\cdot\frac{P(1,3,0)}{P(1,2,0)}}=d_3\cdot\frac{P(1,4,0)}{P(1,3,0)}$$

$$=d_3\cdot\frac{u_4\cdot\frac{P(0,4,0)}{P(0,1,0)}}{u_3\cdot\frac{P(0,3,0)}{P(0,1,0)}}=0.93759$$

$$x=3: P(3,4,3)=d_2\cdot\frac{P(2,4,2)}{P(2,3,2)}=d_2\cdot\frac{d_3\cdot\frac{P(1,4,1)}{P(1,2,1)}}{d_2\cdot\frac{P(1,3,1)}{P(1,2,1)}}=d_3\cdot\frac{P(1,4,1)}{P(1,3,1)}$$

$$=d_3\cdot\frac{d_4\cdot\frac{P(0,4,0)}{P(0,1,0)}}{d_3\cdot\frac{P(0,3,0)}{P(0,1,0)}}=d_4\cdot\frac{P(0,4,0)}{P(0,3,0)}=0.913299$$

4. Find a portfolio which replicates the derivative $f(P(T,S))$ on slide 13 of lecture 3 the derivative and prove that it is self-financing.

Answer: Fix T and S as given by the derivative contract. Since $D(t)$ defined on slide 13 is a Q-martingale and we have known $Z(t,T)$ is a Q-martingale (in proving the Fundamental Theorem), we can also apply the Binomial Representation Theorem to write

$$D(t)=D(0)+\sum_{u=1}^{t}\phi(u)\,Z(u,S)$$

where $\phi(u)$ is a previsible process. We now construct the portfolio: At time $t-1$, buy $\phi(t)$ units of $P(t-1,S)$ and $\psi(t)$ unit of $B(t-1)$ where

$$\psi(t)=D(t-1)-\phi(t)\,Z(t-1,S) \tag{1}$$

is also a previsible process.

- The portfolio replicates: The value of the portfolio at time T is

$$\begin{aligned}V(T) &= \phi(T+1)\times P(T,S)+\psi(T+1)\,B(t)\\ &= \phi(T+1)\times P(T,S)+\left[D(T)-\phi(T+1)\,Z(T,S)\right]B(T)\ \text{(by (1))}\\ &= B(T)\,D(T)\\ &= f(P(T,S)).\end{aligned}$$

- The portfolio is self-financing: This is similar to our proof of the Fundamental Theorem. Value of the portfolio at time t after rebalancing is

$$
\begin{aligned}
& \phi(t+1) P(t, S)+\psi(t+1) B(t) \\
= & B(t)[\phi(t+1) Z(t, S)+\psi(t+1)] \text{ (def of } Z) \\
= & B(t) D(t) \text{ (def of } \psi) \\
= & B(t)[D(t-1)+\phi(t) \Delta Z(t, S)] \text{ (representation)} \\
= & B(t)\left[\begin{array}{c}\psi(t)+\phi(t) Z(t-1, S) \\ +\phi(t) \Delta Z(t, S)\end{array}\right] \text{ (def of } \psi) \\
= & B(t)[\psi(t)+\phi(t) Z(t, S)] \text{ (def of } \Delta Z) \\
= & B(t) \psi(t)+\phi(t) P(t, S) \text{ (def of } Z)
\end{aligned}
$$

which is the value of the portfolio at time t before rebalancing.

5. Derive the two tables on slides 19 and 22 of Lecture 3. Show your work.

Answer:

$$
\begin{aligned}
\text{At } t=3:\quad P(3,4,1) &= e^{-r(3,1)}[qP(4,4,2)+(1-q)P(4,4,1)] \\
&= e^{-0.05}[0.5\times 100+0.5\times 100] = 95.1229 \\
P(3,4,0) &= e^{-r(3,0)}[qP(4,4,1)+(1-q)P(4,4,0)] \\
&= e^{-0.03}[0.5\times 100+0.5\times 100] = 97.0446 \\
\text{At } t=2:\quad P(2,4,1) &= e^{-r(2,1)}[qP(3,4,2)+(1-q)P(3,4,1)] \\
&= e^{-0.06}[0.5\times 93.2394+0.5\times 95.1229] = 88.6965 \\
P(2,4,0) &= e^{-r(2,0)}[qP(3,4,1)+(1-q)P(3,4,0)] \\
&= e^{-0.04}[0.5\times 95.1229+0.5\times 97.0446] = 92.3163 \\
\text{At } t=1:\quad P(1,4,1) &= e^{-r(1,1)}[qP(2,4,2)+(1-q)P(2,4,1)] \\
&= e^{-0.07}[0.5\times 85.2186+0.5\times 88.6965] = 81.0787 \\
P(1,4,0) &= e^{-r(1,0)}[qP(2,4,1)+(1-q)P(2,4,0)] \\
&= e^{-0.05}[0.5\times 88.6965+0.5\times 92.3163] = 86.0923
\end{aligned}
$$

At $t = 0$: $P(0, 4, 0) = e^{-r(0,0)}[qP(1, 4, 1) + (1 - q)P(1, 4, 0)]$

$= e^{-0.06}[0.5 \times 81.078 + 0.5 \times 86.0923]$

$= 78.7179$

At $t = 3$: $V(3, 2) = min\{100e^{-0.055}, e^{-r(3,2)}[qV(4, 3) + (1 - q)V(4, 2)]\}$

$= min\{100e^{-0.055}, e^{-0.07}[0.5 \times 100 + 0.5 \times 100]\}$

$= min\{94.6485, 93.2394\} = 93.2394$

$V(3, 1) = min\{100e^{-0.055}, e^{-r(3,1)}[qV(4, 2) + (1 - q)V(4, 1)]\}$

$= min\{94.6485, 95.1229\} = 94.6485$

At $t = 2$: $V(2, 2) = min\{100e^{-0.055\times2}, e^{-r(2,2)}[qV(3, 3) + (1 - q)V(3, 2)]\}$

$= min\{89.5834, e^{-0.08}[0.5 \times 91.3931 + 0.5 \times 93.2394]\}$

$= min\{89.5834, 85.2186\} = 85.2186$

$V(2, 1) = min\{100e^{-0.11}, e^{-r(2,1)}[qV(3, 2) + (1 - q)V(3, 1)]\}$

$= min\{89.5834, 88.4731\} = 88.4731$

$V(2, 0) = min\{100e^{-0.11}, e^{-r(2,0)}[qV(3, 1) + (1 - q)V(3, 0)]\}$

$= min\{89.5834, 90.9373\} = 89.5834$

At $t = 1$: $V(1, 1) = \min\{100e^{-0.055\times3}, e^{-r(1,1)}[qV(2, 2) + (1 - q)V(2, 1)]\}$

$= min\{84.7894, e^{-0.07}[0.5 \times 85.2186 + 0.5 \times 88.4731]\}$

$= min\{84.7894, 80.9745\} = 80.9745$

$V(1, 0) = min\{100e^{-0.055\times3}, e^{-r(1,0)}[qV(2, 1) + (1 - q)V(2, 0)]\}$

$= min\{84.7894, 84.6863\} = 84.6863$

At $t = 0$: $V(0, 0) = min\{100e^{-0.055\times4}, e^{-r(0,0)}[qV(1, 1) + (1 - q)V(1, 0)]\}$

$= min\{80.2519, 78.0067\} = 78.0067$

6. Show that $\frac{B(0)}{B(2)}$ and $P(2, 3)$ on slide 32 of Lecture 3 are positively correlated.

Answer: Set $X = \frac{B(0)}{B(2)}, Y = P(2,3)$. Under Q,

$$\begin{aligned}
E(Y) &= E_Q[P(2,3)|\mathcal{F}_0] = f(0,2,3) = 0.947523 \\
E(X) &= E_Q\left[\frac{B(0)}{B(2)}|\mathcal{F}_0\right] \overset{B(0)=1}{=} E_Q\left[exp\left(-\sum_{s=0}^{1} r(s)\right)|\mathcal{F}_0\right] \\
&= 0.6 \times e^{-0.05-0.06} + 0.4 \times e^{-0.05-0.04} = 0.903073 \\
E(XY) &= E_Q\left[\frac{B(0)}{B(2)}P(2,3)|\mathcal{F}_0\right] = E_Q\left[\frac{P(2,3)}{B(2)}|\mathcal{F}_0\right] \\
&\overset{\text{def of } D(t,T)=Z(t,T)}{=} E_Q\left[\frac{1}{B(3)}|\mathcal{F}_0\right] \\
&= 0.6^2 \times e^{-0.05-0.06-0.07} + 0.6 \times 0.4 \times (e^{-0.05-0.06-0.05} + e^{-0.05-0.04-0.05}) \\
&+ 0.4^2 \times e^{-0.05-0.04-0.03} \\
&= 0.855765 \\
\text{Hence, } Cov(X,Y) &= E(XY) - E(X)E(Y) \\
&= 0.855765 - 0.903073 \times 0.947523 = 0.000083 > 0.
\end{aligned}$$

That is, $\frac{B(0)}{B(2)}$ and $P(2,3)$ are positively correlated.

7. Following the argument of binomial representation theorem (see the file binomial representation.pdf), show that

$$D(t,T) = D(0,T) + \sum_{s=1}^{t} \phi(s,T)\,\Delta Z(s,s+1) \tag{2}$$

where D and Z are defined as in Lecture 3. Is it possible to employ a similar argument to express Z in terms of D to prove $Z(t,T)$ is a martingale directly? Explain why.

Answer: First, we show that

$$D(t+1,T) - D(t,T) = \phi(t+1,T)\,\Delta Z(t+1,t+2) \text{ for each } t \tag{3}$$

for some previsible process $\phi(\cdot,T)$. Recall that $\Delta Z(t+1,t+2) = Z(t+1,t+2) - Z(t,t+2)$. At time t, given $r(t)$, $D(t,T)$, and $Z(t,t+2)$ (all are $\mathcal{F}_t$-measurable), let $D(t+1,t+2,u)$ and $Z(t+1,t+2,u)$ be the values of D and Z when $r(t+1)$ goes up from $r(t)$ and $D(t+1,t+2,d)$ and $Z(t+1,t+2,d)$ be the values of D and Z when $r(t+1)$ goes down from $r(t)$. There is a unique linear equation which is satisfied by the values of D and Z in both "u" and "d", i.e.,

$$
\begin{aligned}
& D(t+1,T,u) - D(t,T) \\
= \; & \phi(t+1,T)\left(Z(t+1,t+2,u) - Z(t,t+2)\right) + k(t+1)\,; \qquad (4) \\
& D(t+1,T,d) - D(t,T) \\
= \; & \phi(t+1,T)\left(Z(t+1,t+2,d) - Z(t,t+2)\right) + k(t+1)\,. \qquad (5)
\end{aligned}
$$

Solving the equations, we have

$$\phi(t+1,T) = \frac{D(t+1,T,u) - D(t+1,T,d)}{Z(t+1,t+2,u) - Z(t+1,t+2,d)}.$$

Observe that the value of $\phi(t+1,T)$ is known when $D(t,T)$, and $Z(t,t+2)$ are known (at time t). Hence, $\phi(\cdot,T)$ is a previsible process. Hence, $k(t+1)$ is also $\mathcal{F}_t$-measurable. Therefore,

$$
\begin{aligned}
k(t+1) &= E_Q\left[k(t+1)\,|\mathcal{F}_t\right] \\
&= E_Q\left[\left(D(t+1,T) - D(t,T) - \phi(t+1,T)\left(Z(t+1,t+2) - Z(t,t+2)\right)\right)|\mathcal{F}_t\right] \\
&= E_Q\left[\left(D(t+1,T) - D(t,T)\right)|\mathcal{F}_t\right] \\
&\quad -\phi(t+1,T)\,E_Q\left[\left(Z(t+1,t+2) - Z(t,t+2)\right)|\mathcal{F}_t\right] \\
&= 0 \qquad (6)
\end{aligned}
$$

where the first equality follows because $k(t+1)$ is also $\mathcal{F}_t$-measurable; the second is because (4) and (5); the third because $\phi(t+1,T)$ is also $\mathcal{F}_t$-measurable; and the fourth is because $D(t,T)$ and $Z(s,t+2)$ $(s = t, t+1)$ are both Q-martingale (where the latter is shown at the beginning of our proof of the Fundamental Theorem).

Finally, it is not possible to express Z in terms of D by following a similar argument as above. The reason can be seen from the last step (6) where we use the fact that $D(t,T)$ is a Q-martingale. In contrast, we only know $Z(s,t+2)$ $(s = t, t+1)$ is a Q-martingale instead of $Z(s,T)$ being a Q-martingale. To sum up, we can't reverse of the role of D and Z in the argument above.