FE5222 Solutions to Homework 3

November 10, 2019

1. (35 Points) In Lecture 5, we derived the pricing formula for a Quanto option in two ways by using a different currency as the domestic market (see p. 54 and p 60. in the lecture notes). Prove that the two pricing formulae are equivalent.

   Hint: Compare the SDEs for stock price and foreign exchange rate in domestic market and foreign market after change of numeraire and use the fact that these SDEs can be written as

   $$\frac{dS(t)}{S(t)} = rdt + \sigma_1 d\widetilde{W}_1(t)$$

   and

   $$\frac{dX(t)}{X(t)} = (r - r')dt + \sigma_2 d\widetilde{W}_3(t)$$

   where

   $$d\widetilde{W}_1(t)d\widetilde{W}_3(t) = \rho dt$$

   Solution

   In this solution, we use notions different from lecture notes. In particular,

   (a) We use subscript u and s to denote quantities for USD market and SGD market respectively;

   (b) The notation $W_{u,i}$ and $W_{s,i}$, $i = 1, 2$ are used for the Brownian motion under risk neutral measure for USD and SGD market, as opposed to $\widetilde{W}_{u,i}$ and $\widetilde{W}_{s,i}$ used in the lecture notes.

   Let $S_u(t)$ be the stock price in USD, $X(t)$ be the exchange rate for SGD measured as units of USD per SGD, $Y(t) = \frac{1}{X(t)}$ be the exchange for USD in the unit of SGD and r_u and r_s be the risk-free interest rate for USD and SGD respectively. Let $W_{u,1}(t)$ and $W_{u,2}(t)$ be two independent Brownian motions under the risk neutral measure when USD is used as the domestic market.

   $$\frac{dS_u(t)}{S_u(t)} = r_u dt + \sigma_{u,1} dW_{u,1}(t) \tag{1}$$

   and

   $$\frac{dX(t)}{X(t)} = (r_u - r_s)\, dt + \sigma_{u,2} \left(\rho_u dW_{u,1}(t) + \sqrt{1-\rho_u^2} dW_{u,2}(t)\right) \tag{2}$$

   Let $S_s(t) = \frac{S_u(t)}{X(t)}$ which is the stock price in the unit of SGD. Applying the change of numeraire formulae on p. 49 Lecture Notes 5, we have

   $$\frac{dY(t)}{Y(t)} = (r_s - r_u)\, dt - \sigma_{u,2} \left(\rho_u dW_1^f(t) + \sqrt{1-\rho_u^2} dW_2^f(t)\right) \tag{3}$$

and

$$\frac{dS_s(t)}{S_s(t)} = r_s dt + (\sigma_{u,1} - \sigma_{u,2}\rho_u)\, dW_1^f(t) - \sigma_{u,2}\sqrt{1-\rho_u^2}dW_2^f(t) \tag{4}$$

From Equation (3) and (4), we have

$$\frac{dY(t)}{Y(t)}\frac{dY(t)}{Y(t)} = \sigma_{u,2}^2 dt \tag{5}$$

$$\frac{dS_s(t)}{S_s(t)}\frac{dS_s(t)}{S_s(t)} = \left((\sigma_{u,1} - \sigma_{u,2}\rho_u)^2 + \sigma_{u,2}^2\left(1-\rho_u^2\right)\right) dt = \left(\sigma_{u,1}^2 - 2\sigma_{u,1}\sigma_{u,2}\rho_u + \sigma_{u,2}^2\right) dt \tag{6}$$

and

$$\frac{dY(t)}{Y(t)}\frac{dS_s(t)}{S_s(t)} = \left(-\sigma_{u,2}\rho_u\left(\sigma_{u,1} - \sigma_{u,2}\rho_u\right) + \sigma_{u,2}^2\left(1-\rho_u^2\right)\right) dt = \left(\sigma_{u,2}^2 - \sigma_{u,1}\sigma_{u,2}\rho_u\right) dt \tag{7}$$

Now if instead we star from SGD market and use it as the domestic market, we have

$$\frac{dS_s(t)}{S_s(t)} = r_s dt + \sigma_{s,1} dW_{s,1}(t)$$

and

$$\frac{dY(t)}{Y(t)} = (r_s - r_u)\, dt + \sigma_{s,2}\left(\rho_s dW_{s,1}(t) + \sqrt{1-\rho_s^2}dW_{s,2}(t)\right)$$

where $W_{s,2}(t)$ and $W_{s,2}(t)$ are two independent Brownian motion under the risk neutral measure for SGD.

It is easy to see that

$$\frac{dY(t)}{Y(t)}\frac{dY(t)}{Y(t)} = \sigma_{s,2}^2 dt \tag{8}$$

,

$$\frac{dS_s(t)}{S_s(t)}\frac{dS_s(t)}{S_s(t)} = \sigma_{s,1}^2 dt \tag{9}$$

and

$$\frac{dY(t)}{Y(t)}\frac{dS_s(t)}{S_s(t)} = \sigma_{s,1}\sigma_{s,2}\rho_s \tag{10}$$

Comparing Equations (5, (6) and (7) with Equations (8), (9) and (10), we must have

$$\sigma_{u,2} = \sigma_{s,2} \tag{11}$$

$$\sigma_{u,1}^2 - 2\sigma_{u,1}\sigma_{u,2}\rho_u + \sigma_{u,2}^2 = \sigma_{s,1}^2 \tag{12}$$

and

$$\sigma_{u,2}^2 - \sigma_{u,1}\sigma_{u,2}\rho_u = \sigma_{s,1}\sigma_{s,2}\rho_s \tag{13}$$

Using symmetry (or solving from the above Equations (11), (12) and (13)), we have

$$\sigma_{s,2} = \sigma_{u,2} \tag{14}$$

$$\sigma_{s,1}^2 - 2\sigma_{s,1}\sigma_{s,2}\rho_s + \sigma_{s,2}^2 = \sigma_{u,1}^2 \tag{15}$$

and

$$\sigma_{s,2}^2 - \sigma_{s,1}\sigma_{s,2}\rho_s = \sigma_{u,1}\sigma_{u,2}\rho_u \tag{16}$$

Using USD as domestic market, the quanto option price is

$$V = \bar{Q}\left(e^{(r_u - r_s + \rho_u\sigma_{u,1}\sigma_{u,2})T} S_u(0)\Phi(d_{u,1}) - e^{-r_s T} K\Phi(d_{u,2})\right) \tag{17}$$

where

$$d_{u,1/2} = \frac{\ln\left(\frac{S_u(0)}{K}\right) + \left(r_u + \rho_u\sigma_{u,1}\sigma_{u,2} \pm \frac{1}{2}\sigma_{u,1}^2\right)T}{\sigma_{u,1}\sqrt{T}}$$

From Equation (16), we have

$$V = \bar{Q}\left(e^{\left(r_u - r_s + \sigma_{s,2}^2 - \sigma_{s,1}\sigma_{s,2}\rho_s\right)T} S_u(0)\Phi(d_{u,1}) - e^{-r_s T} K\Phi(d_{u,2})\right) \tag{18}$$

From Equation (16) and (15), we have

$$d_{u,1/2} = \frac{\ln\left(\frac{S_u(0)}{K}\right) + \left(r_u + \sigma_{s,2}^2 - \sigma_{s,1}\sigma_{s,2}\rho_s \pm \frac{1}{2}\sigma_{u,1}^2\right)T}{\sigma_{u,1}\sqrt{T}} \tag{19}$$

and

$$\sigma_{u,1}^2 = \sigma_{s,1}^2 - 2\sigma_{s,1}\sigma_{s,2}\rho_s + \sigma_{s,2}^2 \tag{20}$$

Comparing Equation (18), (19) and (20) with the pricing formula for Quanto option using SGD as domestic market (see p.59), we can see immediately that they coincide. *Q.E.D.*

2. (25 Points) (**Garman-Kohlhagen Formula**) Consider the cross-currency market model. Let X be the foreign exchange rate measured as the number of units of domestic currency per unit of the foreign currency. A call option on a unit of foreign currency pays $(X(T) - K)^+$ in domestic currency. Price this option (in the unit of domestic currency).

   Solution

   See Shreve, Section 9.3.6.

3. (25 Points) The price of an American put option satisfies the smooth pasting condition. In this exercise we will provide a heuristic proof.

   (a) Argue that we can't have $\frac{\partial V}{\partial S}(t, S^*) < -1$ by using the Figure (1).

   (b) Show that if $\frac{\partial V}{\partial S}(t, S^*) > -1$ we can then increase the price V near the exercise boundary S^* by using a smaller exercise boundary (see Figure (2)).

   Solution

   (a) We first prove that $\frac{\partial V}{\partial S}(t, S^*) < -1$ is impossible.
   If $\frac{\partial V}{\partial S}(t, S^*) < -1$, then there exists $S' > S^*$ such that

   $$V(t, S') < V(t, S^*) - (S' - S^*)$$

   Since $V(t, S^*) = K - S^*$, it follows that

   $$V(t, S') < K - S'$$

   which implies that when the stock price is S', we shall exercise the put option, contradicting the fact that S^* is the on the exercise boundary.

Figure 1: $\frac{\partial V}{\partial S}\left(t, S^*\right) < -1$

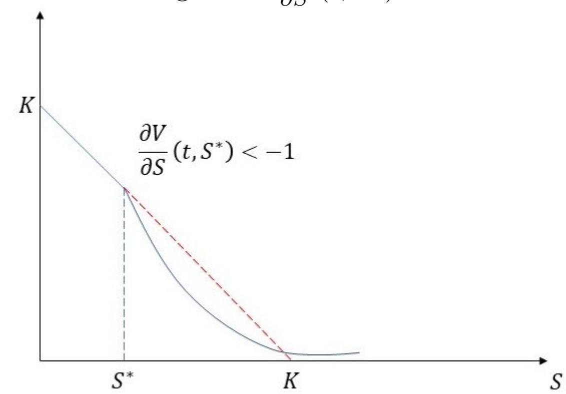


Figure 2: $\frac{\partial V}{\partial S}\left(t, S^*\right) > -1$

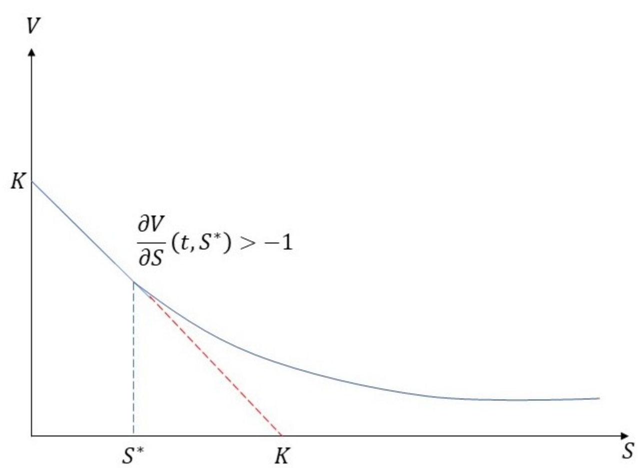
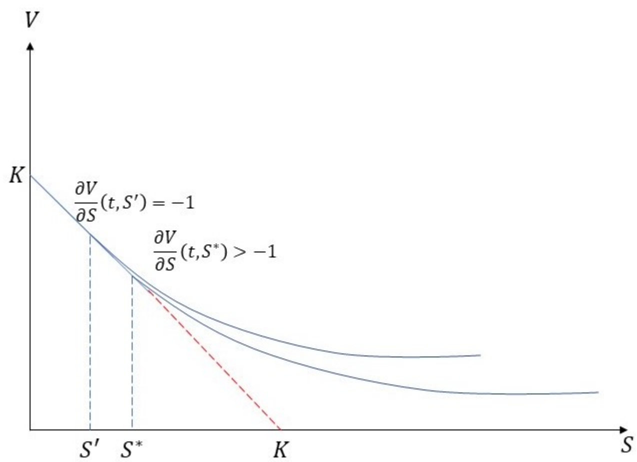


Figure 3: $\frac{\partial V}{\partial S}(t, S') = -1$



(b) Now suppose $\frac{\partial V}{\partial S}(t, S^*) > -1$.
We can find $S' < S^*$ such that $\frac{\partial V}{\partial S}(t, S^*) = -1$. Now look at the solution for American option with boundary condition

$$V(t, S') = K - S'$$

It is obvious that the new solution has a higher value between S' and S^*.
Another argument is based on arbitrage principle. Consider a portfolio that longs one put option and long a stock. When the stock price moves from S^* to $S^* - \Delta S$ for $\Delta S > 0$, the value of put option moves from $K - S^*$ to $K - S^* + \Delta S$. The value of the long position on stock moves from S^* to $S^* - \Delta S$. The change in the value of portfolio is zero in this case. When ΔS is small enough and the stock price moves from S^* to $S^* + \Delta S$. The option price moves approximately from V to $V + \frac{\partial V}{\partial S}(t, S^*)\Delta S$. The change in the value of portfolio is

$$\frac{\partial V}{\partial S}(t, S^*)\Delta S + \Delta S > 0$$

Hence there is an arbitrage for the instantaneous time between t to $t + dt$.

4. (15 Points) A function f is convex if for any $0 \leq \lambda \leq 1$

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$$

Let X be a random variable. Jensen's inequality says

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$$

Let $X = x_i, i = 1, \ldots, n$ and $\mathbb{P}(X = x_i) = p_i$ where $\sum p_i = 1$. Prove Jensen's inequality for this special case from first principles (i.e. without using Jensen's inequality). Explain the geometric meaning of this inequality (you are not required to submit this).

Solution

This can be proved using mathematical induction.