FE5208: problem set 2

Each group of ≤ 5 people submits a copy on LumiNUS

Due on 3 April

Part 1:

1. Establish Equation (6) on Slide 15 of Lecture 4.

   Answer: To prove equation $\tilde{E}[X|\mathcal{F}_s] = E\left[X\frac{Y(t)}{Y(s)}\middle|\mathcal{F}_s\right]$, it is to verify that:

1) $E\left[X\frac{Y(t)}{Y(s)}\middle|\mathcal{F}_s\right]$ is $\mathcal{F}_s$ — *measurable.*

2) $E\left[X\frac{Y(t)}{Y(s)}\middle|\mathcal{F}_s\right]$ is the conditional expectation of X under $\tilde{P}$:

It is obvious that $E\left[X\frac{Y(t)}{Y(s)}\middle|\mathcal{F}_s\right]$ is $\mathcal{F}_s$ -measurable. To see 2), recall from Slide 15 of Lecture 4 that for $X_s \in \mathcal{F}_s$, we

have

$$\tilde{\mathsf{E}}(X_s) = \mathsf{E}(X_s \cdot Y(s)). \tag{1}$$

Since $E\left[X\frac{Y(t)}{Y(s)}\middle|\mathcal{F}_s\right]$ is $\mathcal{F}_s$ -measurable, it follows that any $A \in \mathcal{F}_s$, we have

$$\begin{aligned}
&\tilde{E}\left[1_A E\left[X\frac{Y(t)}{Y(s)}\middle|\mathcal{F}_s\right]\right]\\
&= E\left[1_A E\left[X\frac{Y(t)}{Y(s)}\middle|\mathcal{F}_s\right]Y(s)\right]\\
&= E\left[1_A E\left[X\frac{Y(t)}{Y(s)}Y(s)\middle|\mathcal{F}_s\right]\right]\\
&= E\left[1_A E[XY(t)|\mathcal{F}_s]\right]\\
&= E[1_A XY(t)]\\
&= \tilde{E}[1_A X]
\end{aligned} \tag{2}$$

where 1_A is indicator function and we use (1) to obtain (2). Thus: $\tilde{E}[X|\mathcal{F}_s] = E\left[X\frac{Y(t)}{Y(s)}\middle|\mathcal{F}_s\right]$ for $0 < s < t < T$.

2. Calculate the Vasicek bond price following the steps on Slide 25 of Lecture 5.

   Answer: Recall that

$$r(u) = e^{-au}r(0) + ab\int_0^u e^{-a(u-s)}ds + \sigma\int_0^u e^{-a(u-s)}d\tilde{W}(s);$$
$$\int_t^T r(u)du = \int_t^T e^{-au}r(0)du + ab\int_t^T\int_0^u e^{-a(u-s)}dsdu + \sigma\int_t^T\int_0^u e^{-a(u-s)}d\tilde{W}(s)du.$$

   Thus,

$$\begin{aligned}\frac{1-e^{-a(T-t)}}{a}r(t) &= \left(\frac{1}{a}e^{-at} - \frac{1}{a}e^{-aT}\right)r(0) + b\left(\int_0^t e^{-a(t-s)}ds - \int_0^t e^{-a(T-s)}ds\right)\\ &\quad + \frac{\sigma}{a}\left(\int_0^t e^{-a(t-s)}d\tilde{W}(s) - \int_0^t e^{-a(T-s)}d\tilde{W}(s)\right) \qquad (3)\end{aligned}$$

   We calculate the two double integrals respectively as follows:

$$\begin{aligned}&\int_t^T\int_0^u e^{-a(u-s)}dsdu\\ &= \int_0^t\int_t^T e^{-a(u-s)}duds + \int_t^T\int_s^T e^{-a(u-s)}duds\\ &= \int_0^t -\frac{1}{a}\left[e^{-a(T-s)} - e^{-a(t-s)}\right]ds + \int_t^T -\frac{1}{a}\left(e^{-a(T-s)} - 1\right)ds\\ &= -\frac{1}{a}\int_0^T e^{-a(T-s)}ds + \frac{1}{a}\int_0^t e^{-a(t-s)}ds + \frac{1}{a}(T-t)\end{aligned}$$

Likewise,

$$\begin{aligned}&\int_t^T\int_0^u e^{-a(u-s)}d\tilde{W}(s)du\\ &= \int_0^t\int_t^T e^{-a(u-s)}dud\tilde{W}(s) + \int_t^T\int_s^T e^{-a(u-s)}dud\tilde{W}(s)\\ &= \int_0^t -\frac{1}{a}\left[e^{-a(T-s)} - e^{-a(t-s)}\right]d\tilde{W}(s) + \int_t^T -\frac{1}{a}(e^{-a(T-s)} - 1)d\tilde{W}(s)\\ &= -\frac{1}{a}\int_0^T e^{-a(T-s)}d\tilde{W}(s) + \frac{1}{a}\int_0^t e^{-a(t-s)}d\tilde{W}(s) + \frac{1}{a}\int_t^T d\tilde{W}(s)\end{aligned}$$

Therefore, it follows from (3) that

$$\begin{aligned}
\int_t^T r(u)du &= \int_t^T e^{-au} r(0)du + ab\int_t^T \int_0^u e^{-a(u-s)}dsdu + \sigma \int_t^T \int_0^u e^{-a(u-s)}d\tilde{W}(s)du. \\
&= \left(\frac{1}{a}e^{-at} - \frac{1}{a}e^{-aT}\right) r(0) - b\int_0^T e^{-a(T-s)}ds + b\int_0^t e^{-a(t-s)}ds + b(T-t) \\
&- \frac{\sigma}{a}\int_0^T e^{-a(T-s)}d\tilde{W}(s) + \frac{\sigma}{a}\int_0^t e^{-a(t-s)}d\tilde{W}(s) + \frac{\sigma}{a}\int_t^T d\tilde{W}(s) \\
&= \frac{1-e^{-a(T-t)}}{a} r(t) - b\int_t^T e^{-a(t-s)}ds + b(T-t) - \frac{\sigma}{a}\int_t^T (e^{-a(T-s)} - 1)d\tilde{W}(s)
\end{aligned}$$

Let $B(t,T) = \frac{1-e^{-a(T-t)}}{a}$. It follows that

$$E_Q\left[\int_t^T r(u)du|\mathcal{F}_t\right] = B(t,T)(r(t)-b) + b(T-t).$$

Moreover,

$$\begin{aligned}
& Var\left[\int_t^T r(u)du|\mathcal{F}_t\right] \\
= & \frac{\sigma^2}{a^2}\int_t^T (e^{-a(T-s)} - 1)^2 ds \\
= & \frac{\sigma^2}{a^2}\left[\frac{1}{2a}(1-e^{-2a(T-t)}) + (T-t) - \frac{2}{a}(1-e^{-a(T-t)})\right] \\
= & \frac{\sigma^2}{a^2}\left[-\frac{a}{2}(B(t,T))^2 - B(t,T) + (T-t)\right].
\end{aligned}$$

$$\frac{\sigma^2}{2a^2}\left[-\frac{a}{2}(B(t,T))^2 - B(t,T) + (T-t)\right] + bB(t,T) - b(T-t)$$

Finally, we obtain that

$$\begin{aligned}
P(t,T) &= \exp\left[-E_Q\left[\int_t^T r(u)du|\mathcal{F}_t\right] + \frac{1}{2}Var_Q\left[\int_t^T r(u)du|\mathcal{F}_t\right]\right] \\
&= \exp(A(t,T) - B(t,T)r(t)) \\
\text{where } B(t,T) &= \frac{1-e^{-a(T-t)}}{a} \\
A(t,T) &= (B(t,T) - (T-t))(b - \frac{\sigma^2}{2a^2}) - \frac{\sigma^2}{4a}(B(t,T))^2.
\end{aligned}$$

3. Verify that the bond price solution to the Ho-Lee model on Slide 30 of Lecture 5 satisfies the bond PDE with the boundary condition.

Answer: Recall the bond PDE

$$\frac{\partial P}{\partial t} + f(t,r)\frac{\partial P}{\partial r} + \frac{1}{2}\rho^2(t,r)\frac{\partial^2 P}{\partial r^2} - R(r)P = 0$$

where $h(t,r) = 0$ is omitted and

$$\begin{aligned} R(r) &= r; \\ dr(u) &= f(u, r(u))du + \rho(u, r(u))d\tilde{W}(u); \end{aligned}$$

Ho-Lee model: $f(t,r) = \theta(t)$ and $\rho(t,r) = \sigma$.

$$\begin{aligned} &B(t,T) = T - t \\ &A(t,T) = \int_t^T \theta(s)(s-T)ds + \frac{1}{6}\sigma^2(T-t)^3 \\ &P(t,T) = \exp\left[A(t,T) - B(t,T)r(t)\right] \\ \text{then } &\frac{\partial P}{\partial t} = \left(-r\frac{\partial B}{\partial t} + \frac{\partial A}{\partial t}\right)P = \left(r - \theta(t)(t-T) - \frac{1}{2}\sigma^2(T-t)^2\right)P \\ &\frac{\partial P}{\partial r} = -B(t,T)P = -(T-t)P \\ &\frac{\partial^2 P}{\partial r^2} = B^2(t,T)P = (T-t)^2 P \end{aligned}$$

Then, in the bond PDE,

$$\begin{aligned} &\frac{\partial P}{\partial t} + f(t,r)\frac{\partial P}{\partial r} + \frac{1}{2}\rho^2(t,r)\frac{\partial^2 P}{\partial r^2} - R(r)P \\ &= \left[r - \theta(t)(t-T) - \frac{1}{2}\sigma^2(T-t)^2\right]P + \theta(t)(-(T-t))P \\ &+ \frac{1}{2}\sigma^2(T-t)^2 P - rP \\ &= rP - \theta(t)(t-T)P - \frac{1}{2}\sigma^2(T-t)^2 P - \theta(t)(T-t)P + \frac{1}{2}\sigma^2(T-t)^2 P - rP \\ &= 0 \end{aligned}$$

4. Verify the bond price solution to the Vasicek model on Slide 31 of Lecture 5 satisfies the bond PDE with the boundary condition.

Answer: Vasicek: $f(t,r) = \alpha\left(\mu - r\left(t\right)\right)$ and $\rho(t,r) = \sigma$.

$$
\begin{aligned}
B(t,T) &= \frac{1 - e^{-\alpha(T-t)}}{\alpha} \\
A(t,T) &= \left(B(t,T) - (T-t)\right)\left(\mu - \frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2}{4\alpha}B(t,T)^2 \\
P &= \exp[A(t,T) - B(t,T)r(t)] \\
\text{then } \frac{\partial P}{\partial r} &= -B(t,T)P = \frac{e^{-\alpha(T-t)} - 1}{\alpha}P \\
\frac{\partial^2 P}{\partial r^2} &= B^2(t,T)P = \frac{(1 - e^{-\alpha(T-t)})^2}{\alpha^2}P \\
\frac{\partial P}{\partial t} &= \left(-r\frac{\partial B}{\partial t} + \frac{\partial A}{\partial t}\right)P \\
&= \left((-r)(-e^{-\alpha(T-t)}) + \left(\mu - \frac{\sigma^2}{2\alpha^2}\right)\left(\frac{\partial B}{\partial t} + 1\right) - \frac{\sigma^2}{4\alpha}\frac{\partial(B(t,T)^2)}{\partial t}\right)P \\
&= \left[re^{-\alpha(T-t)} + \mu - \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha(T-t)})^2 - \mu e^{-\alpha(T-t)}\right]P
\end{aligned}
$$

Then, in the bond PDE, we have

$$
\begin{aligned}
&\frac{\partial P}{\partial t} + f(t,r)\frac{\partial P}{\partial r} + \frac{1}{2}\rho^2(t,r)\frac{\partial^2 P}{\partial r^2} - R(r)P \\
&= \left[re^{-\alpha(T-t)} + \mu - \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha(T-t)})^2 - \mu e^{-\alpha(T-t)}\right]P \\
&- \left(\mu - r\right)\left(1 - e^{-\alpha(T-t)}\right)P + \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha(T-t)})^2 P - rP \\
&= 0
\end{aligned}
$$

5. Establish the two Facts stated on Slide 19 of Lecture 6.

Answer:

(a) Observe that

$$
\begin{aligned}
\int wdF - \int wdH &= wF|_0^\omega - \int Fdw - \left(wH|_0^\omega - \int Hdw\right) \\
&= wF|_0^\omega - wH|_0^\omega - \int (F-H)dw
\end{aligned}
$$

Note that $F(\omega) = G(\omega) = 1, F(0) = G(0) = 0$. Since w is nondecreasing, $w' \geq 0$ and $dw = w'dx \geq 0$. Hence, $\int wdF - \int wdH = -\int (F-H)dw \geq 0$ iff $F(x) \leq H(x)$ for all x.

(b) According to Krishna (2002, Appendix D), affiliation of signals implies that

$$
\frac{g(x,y')}{g(x,y)} \leq \frac{g(x',y')}{g(x',y)},\ (y \leq y',\, x \leq x')
$$

where $g(x,y)$ is the joint density of X_1 and Y_1. Hence, $\frac{g(y|x')}{g(y|x)} \leq \frac{g(y'|x')}{g(y|x)}$ which makes $\frac{g(\cdot|x')}{g(\cdot|x)}$ a nondecreasing function for $x' \geq x$. For all $t > \tilde{t}, t < x$,

$$\frac{g(\tilde{t}|x)}{g(\tilde{t}|t)} \leq \frac{g(t|x)}{g(t|t)} \Rightarrow \frac{g(\tilde{t}|x)}{g(t|x)} \leq \frac{g(\tilde{t}|t)}{g(t|t)}$$

Since this holds for any $t > \tilde{t}$, we obtain

$$\begin{aligned} &\int_0^t \frac{g(\tilde{t}|x)}{g(t|x)} d\tilde{t} \leq \int_0^t \frac{g(\tilde{t}|t)}{g(t|t)} d\tilde{t} \\ \Longrightarrow\ & \frac{G(t|x)}{g(t|x)} \leq \frac{G(t|t)}{g(t|t)} \Longrightarrow \frac{g(t|t)}{G(t|t)} \leq \frac{g(t|x)}{G(t|x)}. \end{aligned}$$

6. Derive the first-order condition for the second-price auction. Verify that the bidding strategy β^{II} defined on Slide 14 of Lecture 6 satisfies the first-order condition.

   Answer: Bidder i's expected payoff with signal x and bid b is

$$\Pi^{\mathrm{II}}(b,x) = \int_0^{(\beta)^{-1}(b)} (u(x,y) - \beta(y))\, g(y|x) dy = \int_0^{(\beta)^{-1}(b)} (u(x,y) - u(y,y))\, g(y|x) dy$$

   Its first-order condition is

$$\begin{aligned} \frac{\partial \Pi^{\mathrm{II}}(b,x)}{\partial b} &= \frac{\partial}{\partial b} \int_0^{(\beta)^{-1}(b)} (u(x,y) - u(y,y))\, g(y|x) dy \\ &= \left(u\left(x, \beta^{-1}(b)\right) - u\left(\beta^{-1}(b), \beta^{-1}(b)\right)\right) \\ &\quad \times g\left(\beta^{-1}(b)|x\right) \frac{1}{\beta'(\beta^{-1}(b))} \\ &= 0 \end{aligned}$$

   Hence, if we set $\beta = \beta^{\mathrm{II}}$ and $b = \beta^{\mathrm{II}}(x)$, the first-order condition becomes:

$$\begin{aligned} \frac{\partial \Pi^{\mathrm{II}}(b,x)}{\partial b} &= \left(u\left(x, (\beta^{\mathrm{II}})^{-1}(b)\right) - u\left((\beta^{\mathrm{II}})^{-1}(b), (\beta^{\mathrm{II}})^{-1}(b)\right)\right) \\ &\quad \times g\left((\beta^{\mathrm{II}})^{-1}(b)|x\right) \frac{1}{\beta^{\mathrm{II}\prime}((\beta^{\mathrm{II}})^{-1}(b))} \\ &= (u(x,x) - u(x,x))\, g(x|x) \frac{1}{\beta^{\mathrm{II}\prime}(x)} = 0 \end{aligned}$$

7. Consider a common-value first-price auction where X_i is uniformly distributed in $[0,1]$ and independently across the bidders. Assume that the common value $V = \frac{1}{n}\sum_{i=1}^n x_i$ where x_i is the realization of X_i. Again each bidder i knows only the realization of his signal X_i but not that of the other bidders' signals. Consider a symmetry equilibrium where $\beta(x) = \alpha x$ with $\alpha > 0$. Solve α. What is the value of α approaching when $n \to \infty$?

Answer: $X_i \sim U[0,1]$ and independent and $V = \frac{1}{n}\sum_{i=1}^{n} x_i$, where x_i is the realisation of X_i. Observe first that

$$\begin{aligned} u(x,y) &= E\left[V|X_i = x_i, Y_i = y\right] \\ &= \frac{1}{n}\left(x + y + (n-2)\frac{y}{2}\right). \end{aligned}$$

This is because $\{X_i\}$ are all independent, $X_i = x$ and $Y_1 = \max_{j\neq i} X_j = y$ (the highest signal realization among other bidders is y, and finally the remaining $n-2$ signals all have expectated value $\frac{y}{2}$). Moreover, since $\{X_i\}$ are all independent,

$$\begin{aligned} G(y|x) &= G(y) = \prod_{i=1}^{n-1} P(Y_i \leq y) = y^{n-1}; \\ g(y|x) &= (n-1)y^{n-2}. \end{aligned}$$

Hence,

$$\frac{g(x|x)}{G(x|x)} = \frac{n-1}{x}$$

therefore,

$$\begin{aligned} L(y|x) &= \exp\left(\int_x^y \frac{g(t|t)}{G(t|t)}dt\right) = \exp\left(\int_x^y \frac{n-1}{t}dt\right) \\ &= \left(\frac{y}{x}\right)^{n-1} \end{aligned}$$

We consider a symmetric equilibrium where $\beta(x) = \alpha x$ in a first-price auction.

$$\begin{aligned} \beta(x) &= \int_0^x V(y,y)dL(y|x) \\ &= \int_0^x \left(\frac{n+2}{2n}y\right)(n-1)\frac{y^{n-2}}{x^{n-1}}dy \\ &= \frac{(n+2)(n-1)}{2x^{n-1}n}\int_0^x y^{n-1}dy \\ &= \frac{(n+2)(n-1)}{n\cdot 2x^{n-1}}\cdot\frac{x^n}{n} \\ &= \frac{(n+2)(n-1)}{2n^2}x \end{aligned}$$

that is, $\alpha = \frac{n^2+n-2}{2n^2}x$, when $n \to \infty$, $\alpha = \frac{1}{2}$.