FE 5101:
FIXED INCOME AND DERIVATIVES
SUPPLEMENTARY READING

Overview of Interest Rate Maths

Welcome to *FE 5101: Fixed Income and Derivatives*. I look forward to teaching the first half (six sessions) of this course, which will focus on Bonds, Interest Rate Derivatives, and the mathematics of interest rates. We have 5 class sessions and a midterm exam in the 6^{th} session. Much of the learning will be done in the context of rate mathematics, and to this end, I have prepared this primer on basic practical rate maths.

Some, perhaps many, of you may be quite new to interest rate maths. In such case, I strongly recommend a thorough review and understanding of this primer. In class, we will move quite quickly through arithmetic calculations having to do with securities valuation, pricing, analysis, and curve-derivation.

Several of you will already be quite expert in rate maths based on jobs you have been doing or otherwise. As such you may not find much need for the contents of this reading. However, I would recommend you review it briefly to acquaint yourself with the symbology and terminology to be used in class.

I encourage all to download and review all corresponding lecture materials before coming to class, this because better questions and dialog will make this topic far more accessible than it may otherwise be. And I look forward to teaching and getting to know you.

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CASHFLOWS: TIME VALUE

All future cashflows are defined by three parametres:

1. Amount – actually a currency and amount $1,000,000
2. Date – when is the cashflow is promised
3. Direction – whether you pay or receive the cashflow, with the amount labelled mathematically respectively by a "-" or "+" sign

Two cashflows being of equal size, the one occurring sooner is worth more. This is intuitive: everyone wants to be paid sooner and to pay their obligations later. But the time-related difference in value is quantifiable through the mathematics of interest rates.

A dollar you receive now is worth more than a dollar you would receive in the future because, having it now, you could place the dollar on deposit and earn interest until the date of the future payment, by which time it would have accreted interest. The dollar coming in the future is not earning interest between now and the payment date. So on a future value basis, the money now is worth more than the same amount on a future date, the difference measurable by the interest accrued until that time.

This brings up the concept of cashflows being comparable only if they occur on the same date. And if they don't, at least one of them must be adjusted in order to make them comparable. Generally cashflows are adjusted to their present value in order, among other processes, to make them properly comparable.

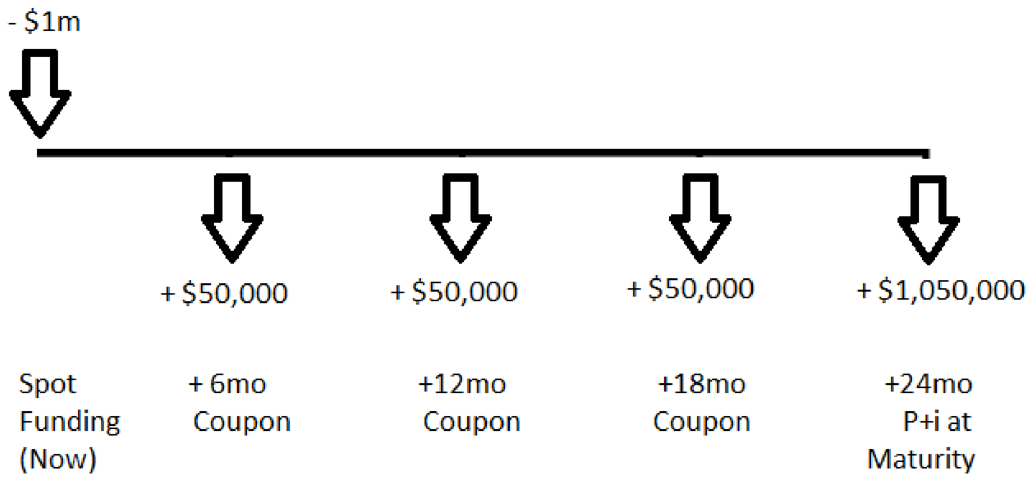
And while it is not sound to add or subtract cashflows in their nominal amounts if they occur on different days, it is possible to add or subtract them once converted into their present values, PVs.

CASHFLOW CONCEPTUALISATION

It is often preferable to depict an investment's cashflows with a diagram.

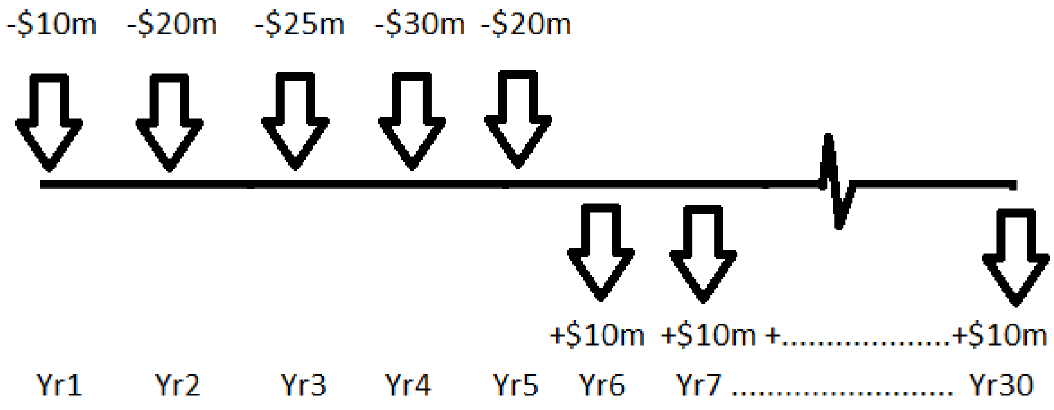
Example 1: Cashflow Diagram of a bond

We are going to buy a bond for $1,000,000 now, and then receive 4 annual payments of $50,000 at the end of each year. At the end of the 4th year, we'll receive the bond's principal (par) value along with the final interest payment. Notice that the cashflows are identified as outflows (-) or inflows (+):



Example 2: Cashflow Diagram of an Infrastructure Project

A government will invest in building a road, expected to last 30 years, and generating toll revenues from year 5 onward. Before that, the road's construction will cause outflows for the first 5 years, and then be followed by cash inflows.


-$10m -$20m -$25m -$30m -$20m
+$10m +$10m +..................+$10m
Yr1 Yr2 Yr3 Yr4 Yr5 Yr6 Yr7 .... Yr30

INTEREST ACCRUAL IN PRACTICE

Interest accrues according to rules regarding compounding and day-count. If you place S$1m on deposit in SGD for 30 days, by convention your interest accruals will be calculated as follows

$$\text{Interest}_{\text{Singapore Dollars}} = \text{Deposit} \times \text{Rate} \times \frac{\text{Days}}{365}$$

$$\text{Interest}_{\text{SGD}} = \$1\text{m} \times .05 \times \frac{30}{365} = \$4{,}109.59$$

If our deposit is in US Dollars or Euros or Japanese Yen, conventionally we will earn:

$$\text{Interest}_{\text{USD}} = \text{Deposit} \times \text{Rate} \times \frac{\text{Days}}{360}$$

$$\text{Interest}_{\text{USD}} = \$1\text{m} \times .05 \times \frac{30}{360} = \$4{,}166.67$$

The actual/360 and actual/365 day-counts are both used in money markets and banking, but generally are country- or currency-specific. USD, JPY, CHF, and EUR are the main users of Actual/360. GBP, AUS, SGD, INR are among those using Actual/365. Always check day-count convention when calculating interest or valuing securities and derivatives.

Counting Accrual Days

The term "actual" in Actual/365 or Actual/360 means the total number of days as follows:

From and including the initial (e.g. loan funding) date
To but excluding the final (e.g. first interest payment) date

Keep in mind that due to weekends and holidays actual days can vary somewhat from the stated instrument accrual period. For example a 6-month deposit funded on 1st Nov and repaying on 1st May is 181 actual days long; but one funding on 1st May and repaying on 1st Nov is 184 actual days, so will accrue 3 more days of interest.

Periods may be longer due to weekends and holidays. If 1st May is a Thursday, then that means 1st Nov is a Saturday, so under *Modified Following* rules, the maturity of the 6 month deposit would take place on Monday 3rd Nov, and accrue a total 186 days of interest. Modified Following is the most common of the rules for dealing with non-banking payment dates..

See Appendix 2 for various date-roll conventions to deal with having an expected payment date occur on a weekend or holiday.

Future Value

An easy way to think of future value is to ask the size of the future cashflow. For example, if you place a deposit for a 3mo (90 day) period at 5%, you will receive principal + interest (P+i), a future value, which can be expressed as:

$$\text{Principal} + \text{Interest} = \text{Principal} \times \left(1 + \text{rate} \times \frac{\text{Days}}{\text{Day - ct}} \right)$$

Which is the same in this case as the more general formula for simple-interest future values

$$\text{Future Value} = \text{PresentValue} \times \left(1 + \text{Rate} \times \frac{\text{Days}}{365} \right)$$

$$\text{P} + \text{i} = \text{FV} = \$1\text{m} \times \left(1 + .05 \times \frac{90}{365} \right) = \$1{,}012{,}328.77$$

The future value of a cashflow will always be larger than the present value, as long as the contractual interest rate is greater than zero.

Any time a cash flow happens further into the future than what we commonly call the spot value date, that cash flow is a future value. It may consist of interest or principal or both (e.g. a mortgage payment), or may not be financial in nature, but form part of a future sale or purchase.

Present Value, PV

The present value of a cash flow is its value adjusted as if it were occurring now (or "today" or "spot") but in any case, simply or conventionally at the present. The principal of a loan funded today is its PV.

To calculate a PV for a known FV cash flow, we can algebraically manipulate the FV equation

$$\text{Future Value} = \text{PresentValue} \times \left(1 + \text{Rate} \times \frac{\text{Days}}{365}\right)$$

To this one:

$$\text{PresentValue} = \frac{\text{Future Value}}{\left(1 + \text{Rate} \times \frac{\text{Days}}{365}\right)}$$

Using this equation, you could figure out how much to pay now for a larger cash inflow in the future. Let's say someone offered you \$10,000 90 days in the future, and you can normally earn a return on depositing or lending 3-month funds at 5% p.a. Actual/365.

As long as you were not taking on additional risk of any sort, you would be willing to pay as much as the following PV. Doing so, you'd earn a 5% per annum return on your funds for the 90 day period.

$$\text{Present Value} = \frac{\$10{,}000}{\left(1 + 0.05 \times \frac{90}{365}\right)} = \$9{,}878.21$$

The above equations are for dealing with simple loans and money market instruments, often referred to as *zero-coupon* investments, because they all share the following attributes

- Funding Spot of the Principal (PV)
- Redeeming at maturity the P+i (FV)
- No cashflows intervening between funding and maturity

But of course, there are more complex use cases for discounted cash flow methodology.

Compounding

Most interest-bearing instruments are not simple-interest instruments like fixed deposits. They involve multiple interest payments, and sometimes involve compounding. With compounding the earned interest is re-invested along with principal, and thereby total future value accumulates faster over time as interest is now being earned on reinvested interest along with the principal.

The frequency of compounding is generally benchmarked to a year, so a $1m deposit rolling over P+i every 3months at 5% will grow to $1,050,945.34 by the end of the year. The total yield here for the year being 5.0945%, even though the contractual rate is 5%.

The longer the time period the higher the interest components will be when compounded using a single rate for all periods and the interest is reinvested.

Now we can use a more complex formula to calculate FVs etc for instruments involving compounding:

$$FV = PV \times \left(1 + [\text{rate}/\text{freq}]\right)^n$$

$$\begin{aligned} FV &= \text{Future Value (or } P + i) \\ PV &= \text{Present Value (or Principal)} \\ r &= \text{Interest Rate per annum, and} \\ \text{freq} &= \text{number of compound periods in a year} \\ n &= \text{Number of compound periods } (\text{Term} \times \text{Frequency}) \end{aligned}$$

And in similar fashion, where we start with a known FV and want to find the PV, we can manipulate the above equation algebraically:

$$PV = \frac{FV}{\left(1 + \dfrac{Rate_{p.a.}}{Freq}\right)^n}$$

And if we know the PV and FV as well as the frequency of compounding, we can calculate the rate, by manipulating the formula to isolate and solve for the rate:

$$(\sqrt[n]{FV \div PV} - 1) \times \text{freq} = \text{rate}$$

Discount Factors

A discount factor is the PV of 1 unit of currency. When working with spreadsheets and programs for which many of the same type of calculations need be done, discount factors (dfs) are robust, reusable, and scalable, so can save work for the programmer:

$$PV = \frac{FV}{(1 + r/\text{freq})^n} = FV \times \frac{1}{(1 + r/\text{freq})^n}$$

That is to say that if I know the dollar-size of the future value cashflow and I know the PV of $1, I can multiply those to get the PV of the cashflow’s FV.

$$df = \frac{1}{(1 + r/\text{freq})^n}$$

Or for simple-interest instruments:

$$df = \frac{1}{(1 + (r \times \text{days}/\text{day-count}))}$$

$$PV = FV \times df$$

By the way:

$$df = PV \div FV$$

Finally, while day-count is important to take into account in real-world applications of interest rate maths, for the remainder of this primer, we’ll assume equal fractions of a year to simplify calculations and not draw focus away from key learning points.

NOMINAL AND EFFECTIVE INTEREST RATES

As we can see, for a given cashflow and nominal interest rate, the more frequent the compounding, the more interest is earned, or the deeper the discount of PV to FV.

The ***nominal rate*** or the contractual rate is the number that is plugged into the formula for calculating interest (or taking the PV). The actual return that we receive over a year, after all compounding has been factored in, is referred to as the ***effective rate.***

The nominal rate is also referred to as the "contractual," the "stated" interest rate, or also as the "quoted" interest rate or sometimes the "annual percentage rate [APR]" or even sometimes the "coupon" or "coupon rate".

The effective rate is sometimes called the "annualised" or "equivalent annual rate [EAR]."

The more frequent the compounding, the more nominal and effective rates diverge. As an example $100,000 plus one year's interest at 9% p.a. compounded annually annum equals $109,000, whereas $100,000 at 9% p.a. compounded quarterly will result in a total accumulated value of $109,308.22. Thus 9% compounded *quarterly is equivalent* to 9.30822% *compounded annually*.

Here we see that the 9% nominal rate has various effective rates depending on the compounding frequency:

Compounding Frequency	**Effective rate**
Annual (freq = 1)	9%
Semi-Annual (freq = 2)	9.2025%
Quarterly(freq = 4)	9.3083%
Monthly(freq = 12)	9.3806%
Daily(freq = 365)	9.4162%
Continuous(freq = ∞)	$= e^{rt} - 1 = 9.4174\%$

The greater the frequency of compounding, the greater will be the effective rate, given a nominal interest rate.

Of course, with lower rates (e.g. 3%), the impact of compounding is reduced.

Converting Between Nominal and Effective

Given a nominal rate of r% p.a., the effective rate can be calculated using the formula:

$$\text{Effective Rate} = \left(1 + \frac{r}{\text{Freq}}\right)^{\text{Freq}} - 1$$

For example, a nominal rate of 9% compounded quarterly is equivalent to:

$$\text{Effective Rate} = \left(1 + \frac{9\%}{4}\right)^{4} - 1$$
$$= 9.30833\%$$

The formula can also be manipulated to *de-compound* an effective rate into a nominal rate:

$$\text{Nominal Rate} = \left(\sqrt[\text{Freq}]{1 + \text{EffectiveRate}} - 1\right) \times \text{Freq}$$

For example, given an effective rate of 9.3806% (with monthly compounding):

$$\text{Nominal Rate} = \left(\sqrt[12]{1 + 0.093806} - 1\right) \times 12$$
$$= 9.0\% \text{ decompounded annual rate}$$

VALUING A SERIES OF CASHFLOWS

If we have a series of cashflows occurring repeatedly throughout a contractual maturity, we take each one's PV, and thereby value-adjust them for how far in the future they occur.

If you take the PV of each cashflow in the series, and then add them together, you can determine a quantum of value, PV, or the profitability of a venture (investment, project, loan, etc) using Net Present Value, NPV.

$$PV_{\text{Series of CFs}} = \frac{CF_1}{(1+r_1)^1} + \frac{CF_2}{(1+r_2)^2} + \ldots + \frac{CF_n}{(1+r_n)^n}$$

Be aware the above example assumes annual cashflows so there is no need to divide the rate by frequency. In practice frequency is often greater than once per year and the rates in the denominator are accrdingly divided.

In the above equation, the series can contain cash in-flows (positive), and out-flows (negative).

ANNUITIES: A special case of a series of cashflows being valued through discounting, is an annuity, which is a series of equally sized, equally timed cashflows, all inflows or all outflows. These usually arise in some sort of loan repayment or instalment purchase (e.g. 12 monthly payments of $50 to the gym you belong to, or the series of interest coupons on a bond).

$$PV_{\text{Annuity}} = \sum_{n=1}^{N} \frac{CF_n}{(1+r_n)^N}$$

NET PRESENT VALUE (NPV)

Net Present Value **(NPV)** is a related use-case of PV. It is applied to determine the profitability of an investment or venture. In a simple example, you might calculate the PV of an annuity as the total value you'll receive on all its future cash in-flows, and then subtract out the cost of buying it (also a PV, but of negative value), to determine if its NPV > 0 (profitable) or NPV < 0 (money-losing).

Example: The NPV of a Car Loan

Assume you work for an investment company that buys originated car loans from car dealers. Your cost of funds is 5% per annum and given an opportunity to buy a newly-originated car-loan of 36 monthly cash flows (car payments from the borrower) of $1000. The loan-originator is offering you this loan for $28,000 now, and in exchange will receive 36 monthly payments of $1000 starting one month from now. You want to determine profitability, so you calculate the loan's NPV.

$$\mathrm{NPV}_{\mathrm{Annuity}} = \sum_{n=1}^{n} \frac{CF_{\mathrm{n}}}{\left(1+\mathrm{r}_{\mathrm{n}}\right)^{\mathrm{n}}} - \mathrm{PV}(\text{Costs of aquiring annuity})$$

$$\mathrm{NPV}_{\mathrm{Carloan}} = \sum_{n=1}^{36} \frac{\$1{,}000}{\left(1+.05/12\right)^{\mathrm{n}}} - \mathrm{PV}(\$28{,}000\text{ now})$$

$$\mathrm{NPV} = \$33{,}366 - \$28{,}000 = \$5{,}366$$

In this case, the NPV is > 0, so the loan is profitable and you would buy it.

NPV use-cases are numerous, including but not limited to project finance, derivatives valuation, and commercial real estate.

INTERNAL RATE OF RETURN (IRR)

For a given series of cash flows, the internal rate of return (IRR) is the single interest rate that makes the series's NPV = 0. Otherwise stated the IRR rate makes

PV of cash in-flows = PV of cash outflows

Finding the IRR might be a case of simple algebra for a single period instrument such as a 3-month fixed term deposit in which you place $100 principal (PV) on deposit and receive principal + interest of $101 at maturity (FV).

Finding the IRR for this is not difficult:

$$PV = \frac{FV}{\left(1 + \frac{Rate_{p.a.}}{Freq}\right)}$$

So

$$IRR = (FV \div PV) - 1) \times Freq$$

$$= (101 \div 100) - 1) \times 4 = 4\% \text{ quarterly}$$

However, most investments involve a series of cashflows over time, and therefore we don't have the convenience of algebraic manipulation, but instead must iterate through the valuation formula to find the IRR.

For example, say you could buy a 2yr 4% semi-annual coupon bond for $97.60, and it would pay you $2 every six months for two years and on the two year date would pay you the principal as well.

Your cash flows would be:

Now, "T_0" (n=0):	- 97.60
In 6 mo (n=1):	+2.00
In 12 mo (n=2):	+2.00
In 18 mo (n=3):	+2.00
In 2 yr (n=4):	+102.00

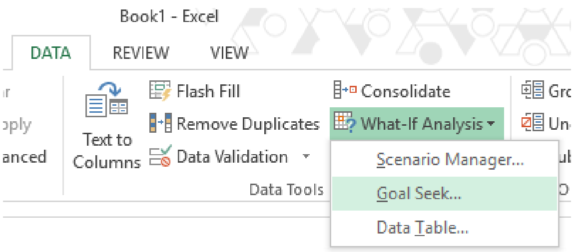
The valuation model for this bond is the PV of its $2 interest payments added to the PV of its $100 par (principal) repayment.

$$PV_{Bond} = Price_{mkt} = \sum_{n=1}^{4} \frac{\$2}{(1+r/2)^{n}} + \frac{\$100}{(1+r/2)^{4}}$$

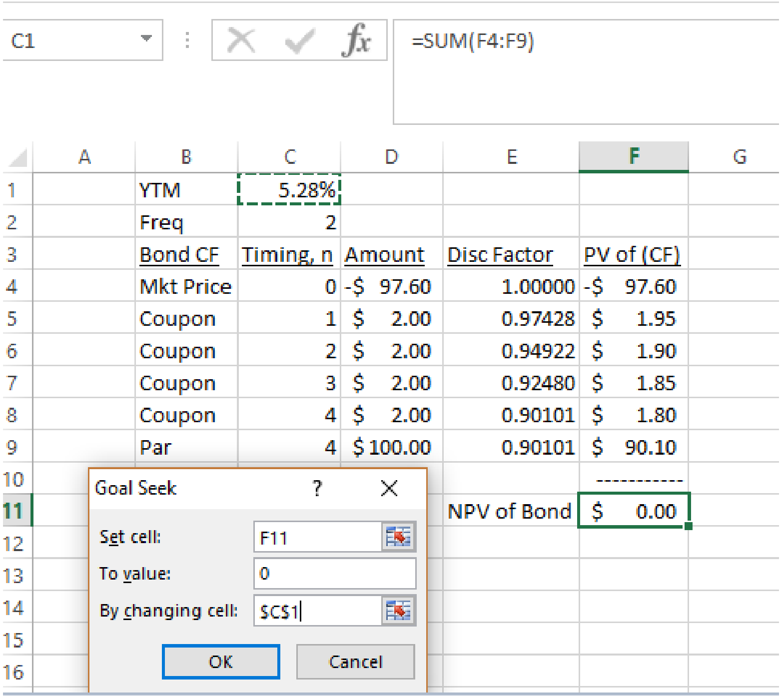
But as you can see, we already know the PV of the bond, which is its market price and are left to calculate for r, which can be done with the help of a spreadsheet like excel (goal-seek function):

$$\$97.60 = \sum_{n=1}^{4} \frac{\$2}{(1+r/2)^{n}} + \frac{\$100}{(1+r/2)^{4}}$$

In case you're wondering, it's 5.28% per annum, semi-annual. The way to get it is to use the Data/Goal-seek function in Excel:



Here's what to do in excel to map out a bond's cashflows, including it's price today, which is a **negative cashflow**, and thereby figure it's single rate (called "YTM") using goal-seek by iterating:



C1 =SUM(F4:F9)

	A	B	C	D	E	F	G
1		YTM	5.28%				
2		Freq	2				
3		Bond CF	Timing, n	Amount	Disc Factor	PV of (CF)	
4		Mkt Price	0	-$ 97.60	1.00000	-$ 97.60	
5		Coupon	1	$ 2.00	0.97428	$ 1.95	
6		Coupon	2	$ 2.00	0.94922	$ 1.90	
7		Coupon	3	$ 2.00	0.92480	$ 1.85	
8		Coupon	4	$ 2.00	0.90101	$ 1.80	
9		Par	4	$ 100.00	0.90101	$ 90.10	
10						----------	
11					NPV of Bond	$ 0.00	

Valuing Bonds to Discount Rates

With bonds, the IRR is usually referred to as the **YTM (yield-to-maturity)**. There are two ways to come to a bond's PV. One involves using the YTM, assuming you already know it. In such case you'd use this formula:

$$\text{Price} = \sum_{n=1}^{N} \frac{Coupon}{(1 + \text{YTM/Freq})^{\text{n}}} + \frac{Par}{(1 + YTM\,/\,Freq)^{N}}$$

A more complex valuation methodology involves using not the YTM, but a series of rates corresponding to the timing of each cash flow in a bond's series. In such case you'd use this formula:

$$\text{Price} = \sum_{n=1}^{N} \frac{Coupon}{(1 + \text{r}_\text{n}/\text{Freq})^{\text{n}}} + \frac{Par}{(1 + r_N\,/\,Freq)^{N}}$$

Valuing a bond to a series of rates is a sounder methodology in many cases than using YTM. But to do so you need a series of rates.

Attaining the series of r_n to use this valuation methodology is called "boot-strapping," and is what we'll be doing in class during the second and third lectures.

APPENDIX 1: Other Day-count Conventions

There are about a dozen or so day-count conventions. Most of what we'll do in class is using Actual/360 and Actual/365. But at the end of this reading are a few examples you find commonly in bond markets. None is particularly hard to understand, but it is crucial to remember that there are several possible day-counts, and to always use the correct day-count for accruals, pricing, and valuation.

30/360

The 30/360 convention is found in most Eurobonds. The calculation assumes every month is 30 days long, and that a year has 360 days. Therefore, each 6 month period will equate to exactly half of the interest rate (assuming semi-annual interest).

$$\text{Interest}_{30/360} = \text{Par} \times \left(\text{Interest Rate} \times \frac{\text{Months} \times 30}{360} \right)$$

There are a number of variants (only minor differences) of the 30/360 day rule, which come into play in determining accruals on bonds being sold in the secondary markets.

Actual/Actual (Government Bonds)

Actual/Actual (Act/Act, A/A or Act/UST) convention uses the number of actual days in the calculation period divided by the number of days in the calculation period multiplied by the frequency.

$$\text{Interest}_{\text{Act/Act}} = \text{Par} \times \left(\text{Interest Rate} \times \frac{\text{Days}_{\text{In Period}}}{\text{Days}_{\text{In Period}} \times \text{Freq}} \right)$$

Act/Act will produce a coupon the same size as under 30/360. There is a small difference in handling accrued interest between these two.

There are variants of the Act/Act rule, some of which take the actual number of days in the year into account.

It is not recommended to memorise the various day-counts' rules. More importantly it is imperative to ascertain the correct day-count to for whatever financial instrument is at hand, and apply it accordingly.

APPENDIX 2: Date Roll Conventions

Date roll conventions determine the rescheduling of payments when they would otherwise occur on days banks are not open to pay and receive payments, such as weekends and public holidays (non-business days).

In certain cases (e.g. money market instruments) a date-roll might lengthen or shorten an interest accrual period, and thereby the interest amount, as well as the payment. In other cases, the payment might stay the same and be paid on an adjusted date.

Following
If the payment date falls on a non-business day (e.g. Sun 15Apr), the date will change to the next business day (Mon 16Apr).

Preceding
If the payment date falls on a non-business day (e.g. Sun 15Apr), then the date changes to the preceding business day (Fri 13Apr).

Modified Following
If the date falls on a non-business day (e.g. Sun 30Apr), the date changes to the next business day, but not if that date is in the next calendar month. In such case, the date changes to the last business day of the previous month (Fri 28Apr). *Modified following is the most common rule.*

Modified Preceding
If the date falls on a non-business day, then the date changes to the preceding business day. However, if that date were in the previous month, the date changes to the following business day.

End-Month
The date always falls on the last business day of a month.

End-Month Following
The date falls on the last day of a month, unless that day is not a business day. If that day is not a business day, the date rolls to the first business day in the next month.

IMM Dates
Dates are set to the IMM (CME) futures' monthly contract dates:the third Wednesday of each month in question.

IMM Quarter
Dates are set to the IMM (CME 3-mo LIBOR) futures' quarterly contract dates (i.e. the third Wednesday of each of Mar-Jun-Sep-Dec)

FRN Convention

The date stays the same from month to month, rolling forward permanently whenever the date falls on a non-business day.