FE5222 Homework 3: Due by Thursday, Oct 10

September 19, 2019

1. (35 Points) In Lecture 5, we derived the pricing formula for a Quanto option in two ways by using a different currency as the domestic market (see p. 54 and p 60. in the lecture notes). Prove that the two pricing formulae are equivalent.

   Hint: Compare the SDEs for stock price and foreign exchange rate in domestic market and foreign market after change of numeraire and use the fact that these SDEs can be written as

   $$\frac{dS(t)}{S(t)} = rdt + \sigma_1 d\widetilde{W}_1(t)$$

   and

   $$\frac{dX(t)}{X(t)} = (r - r')dt + \sigma_2 d\widetilde{W}_3(t)$$

   where

   $$d\widetilde{W}_1(t)d\widetilde{W}_3(t) = \rho dt$$

2. (25 Points) (**Garman-Kohlhagen Formula**) Consider the cross-currency market model. Let X be the foreign exchange rate measured as the number of units of domestic currency per unit of the foreign currency. A call option on a unit of foreign currency pays $(X(T) - K)^+$ in domestic currency. Price this option (in the unit of domestic currency).

3. (25 Points) The price of an American put option satisfies the smooth pasting condition. In this exercise we will provide a heuristic proof.

   (a) Argue that we can't have $\frac{\partial V}{\partial S}(t, S^*) < -1$ by using the Figure (1).

   (b) Show that if $\frac{\partial V}{\partial S}(t, S^*) > -1$ we can then increase the price V near the exercise boundary S^* by using a smaller exercise boundary (see Figure (2)).

4. (15 Points) A function f is convex if for any $0 \leq \lambda \leq 1$

   $$\lambda f(x) + (1 - \lambda)f(y) \geq f(\lambda x + (1 - \lambda)y)$$

   Let X be a random variable. Jensen's inequality says

   $$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$$

   Let $X = x_i, i = 1, \ldots, n$ and $\mathbb{P}(X = x_i) = p_i$ where $\sum p_i = 1$. Prove Jensen's inequality for this special case from first principles (i.e. without using Jensen's inequality). Explain the geometric meaning of this inequality (you are not required to submit this).

Figure 1: $\frac{\partial V}{\partial S}(t, S^*) < -1$

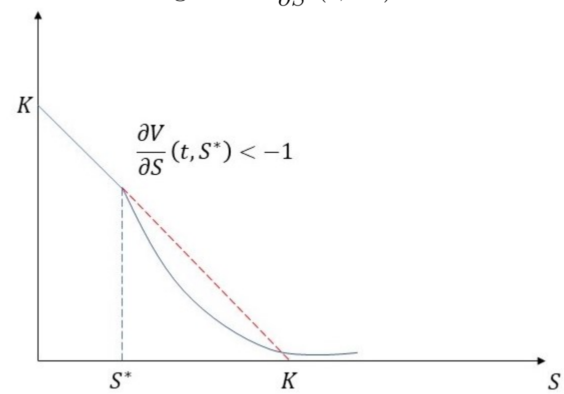

Figure 2: $\frac{\partial V}{\partial S}(t, S^*) > -1$