Lecture 3 - Discrete-Time Binomial Models II

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Time-Homogeneous Model

- The drift is time-dependent in the first binomial model in Lecture 2 and hence the current $r(t)$ does not fully pin down future term structure.

- We now consider a time-homogeneous Markov model of short rate.

- Denote the set of possible interest rates by

$$A = \{..., r_{-2}, r_{-1}, r_0, r_1, r_2, ...\}$$

- Suppose that

$r(t+1) = r_{i-1}$ or r_{i+1} with (*real world*) probability one given $r(t)$

Time-Homogeneous Model

- Recall when the interest rate process is deterministic, we have

$$P(t, T) \equiv \exp\left[-\sum_{s=t}^{T-1} F(0, s, s+1)\right].$$

- Now it is stochastic yet Markov (depending only on r_t). Hence, if $r(t) = r_i$, we may set

$$\begin{aligned} P(t, t+1, r(t)) &= e^{-r_i}; \\ P(t, t+2, r(t)) &= e^{-r_i}\left[q_i e^{-r_{i+1}} + (1-q_i)\, e^{-r_{i-1}}\right] \\ &= e^{-r(t)} E_Q\left[P(t+1, t+2)\,|r(t)\right] \\ &\dots \end{aligned}$$

for some $q_i \in (0, 1)$.

Fundamental Theorem

- The bond price:

$$
\begin{aligned}
P(t,T) &= E_Q\left[\exp\left(-\sum_{s=t}^{T-1} r(s)\right)\middle| r(t)\right] \\
&= P(t,t+1)\,E_Q\left[P(t+1,T)\,|r(t)\right]
\end{aligned}
$$

  where the (risk neutral) transition probability is

$$
\begin{aligned}
\Pr{}_Q\left[r(t+1)=r_{i+1}|r(t)=r_i\right] &= q_i; \\
\Pr{}_Q\left[r(t+1)=r_{i-1}|r(t)=r_i\right] &= 1-q_i.
\end{aligned}
$$

- The value of $P(t,T)$ given the value of $r(t)$ will be denoted as $P(t,T,r(t))$.

Random Walk

- An important special case where $q_i = q$ for each i is the probability of "up" with a "tick/step" size $\delta > 0$; moreover, each "up" or "down" is drawn independently of the history.

- We have

$$\begin{aligned} P(0,2) &= P(0,1)\, E_Q\left[P(1,2)\,|r(0)\right] \\ &= e^{-r(0)}\left[q e^{-(r(0)+\delta)} + (1-q)\, e^{-(r(0)-\delta)}\right]. \end{aligned}$$

- Hence,

$$q = \frac{e^{-(r(0)-\delta)} - P(0,2)\, e^{r(0)}}{e^{-(r(0)-\delta)} - P(0,2)\, e^{-(r(0)+\delta)}}.$$

Fundamental Theorem

- We aim to show

$$P(t,T) = E_Q\left[\exp\left(-\sum_{s=t}^{T-1} r(s)\right)|r(t)\right] = E_Q\left[\frac{B(t)}{B(T)}|r(t)\right].$$

- This is equivalent to showing $Z(t,T) = D(t,T)$ where

$$\begin{aligned} Z(t,T) &= \frac{P(t,T)}{B(t)}; \\ D(t,T) &= E_Q\left[\frac{1}{B(T)}|r(t)\right]. \end{aligned}$$

- $D(t,T)$ is a martingale under Q.

Fundamental Theorem

- $Z(s, t+1)$ is also a martingale under Q from t to $t+1$:

$$\begin{aligned}
Z(t, t+2) &= \frac{P(t, t+2)}{B(t)} \\
&= \exp\left(-\sum_{s=0}^{t-1} r(s)\right) P(t, t+2) \text{ (def of } B(t)\text{)} \\
&= \exp\left(-\sum_{s=0}^{t} r(s)\right) E_Q\left[P(t+1, t+2) \,|\, r(t)\right] \\
&= E_Q\left[\frac{P(t+1, t+2)}{B(t+1)} | r(t)\right] \text{ (def of } B(t+1)\text{)} \\
&= E_Q\left[Z(t+1, t+2) \,|\, r(t)\right] \text{ (def of } Z\text{)}
\end{aligned}$$

where in the third equality, we use

$$P(t, t+2, r(t)) = e^{-r(t)} E_Q\left[P(t+1, t+2) \,|\, r(t)\right]$$

Replicating ZCB

- Martingale/Binomial representation theorem:

$$D(t, T) = D(0, T) + \sum_{s=1}^{t} \phi(s, T) \Delta Z(s, s+1)$$

where $\phi(s, T)$ is predictable process and
$\Delta Z(s, s+1) \equiv Z(s, s+1) - Z(s-1, s+1)$.

- Buy at time $t-1$
  - $\phi(t, T)$ units of $P(t-1, t+1)$ and
  - $\psi(t, T)$ (also predictable) units of $B(t-1)$ such that

$$\psi(t, T) = D(t-1, T) - \phi(t, T) Z(t-1, t+1).$$

  - Note: ψ is equivalent to what I wrote in Lecture 2.
- The portfolio is replicating the ZCB $P(t, T)$:
$B(T) D(T, T) = P(T, T) = 1$.

Self-Financing

- Value of the portfolio at time t after rebalancing

$$\begin{aligned}
& \phi(t+1,T)\,P(t,t+2)+\psi(t+1,T)\,B(t) \\
= \;& B(t)\left[\phi(t+1,T)\,Z(t,t+2)+\psi(t+1,T)\right] \;\text{(def of } Z) \\
= \;& B(t)\,D(t,T) \;\text{(def of } \psi) \\
= \;& B(t)\left[D(t-1,T)+\phi(t,T)\,\Delta Z(t,t+1)\right] \;\text{(representation)} \\
= \;& B(t)\left[\begin{array}{c} \psi(t,T)+\phi(t,T)\,Z(t-1,t+1) \\ +\phi(t,T)\,\Delta Z(t,t+1) \end{array}\right] \;\text{(def of } \psi) \\
= \;& B(t)\left[\psi(t,T)+\phi(t,T)\,Z(t,t+1)\right] \;\text{(def of } \Delta Z) \\
= \;& B(t)\,\psi(t,T)+\phi(t,T)\,P(t,t+1) \;\text{(def of } Z)
\end{aligned}$$

which is the value of the portfolio at time t before rebalancing.

No Arbitrage

- Law of one price:

$$\begin{aligned} P(t,T) &= B(t)\, D(t,T) \\ &= B(t)\, E_Q\left[\frac{1}{B(T)} | r(t)\right] \\ &= E_Q\left[\exp\left(-\sum_{s=t}^{T-1} r(s)\right) | r(t)\right]. \end{aligned}$$

- This also means that $Z(t,T) = D(t,T)$ and hence

$$E\left[\frac{P(t+1,T)}{B(t+1)} | r(t)\right] = \frac{P(t,T)}{B(t)} \Leftrightarrow$$
$$e^{-r(t)} E_Q\left[P(t+1,T) | r(t)\right] = P(t,T).$$

Example 3.9

- Suppose that $r(0) = 0.05$ and

$$r(t+1) = \begin{cases} r(t) + 0.01, & \text{if “up” at } t+1; \\ r(t) - 0.01, & \text{if “down” at } t+1. \end{cases}$$

- Suppose that $P(0,2) = P(0,2,0) = 0.909407$. Now calculate

$$q = \frac{e^{-0.04} - 0.909407 \times e^{0.05}}{e^{-0.04} - e^{-0.06}} = 0.25.$$

- For $T = 1$:

$$P(0,1) = P(0,1,0) = e^{-0.05} = 0.951229.$$

- For $T = 2$:

$$\begin{aligned} P(2,2,x) &= 1 \text{ for } x = 0,1,2, \\ P(1,2,1) &= e^{-0.06} = 0.941765, \\ P(1,2,0) &= e^{-0.04} = 0.960789, \\ P(0,2,0) &= 0.909407. \end{aligned}$$

Example 3.9

For $T = 3:$

$P(3, 3, u) = 1$ for $u = 0, 1, 2, 3,$

$$P(2,3,2) = e^{-0.07} = 0.932394,$$
$$P(2,3,1) = e^{-0.05} = 0.951228,$$
$$P(2,3,0) = e^{-0.03} = 0.970446,$$

$$P(1,3,1) = P(1,2,1)[qP(2,3,2) + (1-q)P(2,3,1)] = 0.891400$$
$$P(1,3,0) = P(1,2,0)[qP(2,3,1) + (1-q)P(2,3,0)] = 0.927778,$$
$$P(0,3,0) = P(0,1,0)[qP(1,3,1) + (1-q)P(1,3,0)] = 0.873878.$$

Derivative Prices

- The price of derivatives with payoffs which are contingent on bond prices at a given point in time can be calculated similarly.
- Suppose that a derivative pays off Y at time T where $Y = f(P(T,S))$.
- Denote by $V(t,x)$ the derivative price at t when there are x up-steps in the risk-free rate up to time t. Then, by backward induction,

$$\begin{aligned} V(T,x) &= f(P(T,S)); \\ V(t-1,x) &= P(t-1,t,x)\left[qV(t,x+1) + (1-q)V(t,x)\right] \end{aligned}$$

Derivative Prices

- The unique no-arbitrage price at time t for this contract is

$$V(t) = E_Q\left[\exp\left(-\int_t^T r(s)ds\right) f(P(T,S))|\mathcal{F}_t\right]$$
$$= E_Q\left[\exp\left(-\sum_{s=t}^{T-1} r(s)\right) f(P(T,S))|\mathcal{F}_t\right].$$

- Recall $Z(t,S) = P(t,S)/B(t)$ is a martingale under Q. Define

$$D(t) = E_Q\left[\frac{f(P(T,S))}{B(T)}|\mathcal{F}_t\right].$$

- As before, we could find a portfolio which is replicating $(D(T)B(T) = f(P(T,S)))$ the derivative and self-financing (**Exercise**).

Call Option

- Suppose that we have a call option on $P(t,3)$ which matures at $t=2$ with a strike price of 0.95; that is,

$$f(p) = \max\{p - 0.95, 0\}.$$

- Recall from Example 3.9 that

$$\begin{aligned}
P(2,3,2) &= 0.932394 \Rightarrow V(2,2) = 0. \\
P(2,3,1) &= 0.951229 \Rightarrow V(2,1) = 0.001229 \\
P(2,3,0) &= 0.970446 \Rightarrow V(2,0) = 0.020446
\end{aligned}$$

- Thus,

$$\begin{aligned}
V(1,1) &= P(1,2,1)[qV(2,2) + (1-q)V(2,1)] = 0.000868; \\
V(1,0) &= P(1,2,0)[qV(2,1) + (1-q)V(2,0)] = 0.015028; \\
V(0,0) &= P(0,1,0)[qV(1,1) + (1-q)V(1,0)] = 0.010928.
\end{aligned}$$

Callable Bonds

- Suppose that $r(0) = 0.06$ and $q = 0.5$.

- Consider a zero-coupon, callable bond with a nominal value of 100 and a maximum term of four years. At each of times $t = 1, 2, 3$, the bond may be redeemed early at the option of the issuer. The early redemption price at time t is

$$100 \times \exp[-0.055(4 - t)].$$

- At time 4 the bond will be redeemed at par (100) if this has not already happened.

- Calculate the price for this bond at time 0 and for the equivalent zero-coupon bond with no early redemption option.

Callable Bonds

- The recombining binomial tree for the risk-free rate of interest is given in the table below, where $r(t, x)$ represents the risk-free rate of interest from t to $t+1$ given x.

	t				
x	0	1	2	3	4
4	—	—	—	—	0.10
3	—	—	—	0.09	0.08
2	—	—	0.08	0.07	0.06
1	—	0.07	0.06	0.05	0.04
0	0.06	0.05	0.04	0.03	0.02

- We calculate the prices $P(t, 4, x)$ of the conventional zero-coupon bond, where x is the number of steps up by time 4.

Callable Bonds

- We start with $P(4,4,x) = 100$ for $x = 0,1,2,3,4$. For all t and for all $0 \leq x \leq t$ we have

  $$P(t,4,x) = e^{-r(t,x)}[qP(t+1,4,x+1) + (1-q)P(t+1,4,x)].$$

- Sample calculations:

$$\begin{aligned}
P(3,4,3) &= e^{-r(3,3)}[qP(4,4,4) + (1-q)P(4,4,3)] \\
&= e^{-0.09}[0.5 \times 100 + 0.5 \times 100] \\
&= 91.3931, \\
P(3,4,2) &= e^{-r(3,2)}[qP(4,4,3) + (1-q)P(4,4,2)] \\
&= e^{-0.07}[0.5 \times 100 + 0.5 \times 100] \\
&= 93.2394, \\
P(2,4,2) &= e^{-r(2,2)}[qP(3,4,3) + (1-q)P(3,4,2)] \\
&= e^{-0.08}[0.5 \times 91.3931 + 0.5 \times 93.2394] \\
&= 85.2186.
\end{aligned}$$

Callable Bonds

The complete set of prices corresponding to the above table for $r(t)$ is given below (**Exercise**: derive the table):

$P(t, 4, x)$					
			t		
x	0	1	2	3	4
4	—	—	—	—	100.0000
3	—	—	—	91.3931	100.0000
2	—	—	85.2186	93.2394	100.0000
1	—	81.0787	88.6965	95.1229	100.0000
0	78.7197	86.0923	92.3163	97.0446	100.0000

Callable Bonds

- Thus, the price process $V(t,x)$ evolves according to the following recursive scheme:

$$V(4,x) = 100 \text{ for } x = 0,1,2,3,4.$$

- For each $t = 3,2,1$ and $0 \leq x \leq t$:

$$V(t,x) =$$
$$\min\{100e^{-0.055(4-t)}, e^{-r(t,x)}(qV(t+1,x+1) + (1-q)V(t+1,x)$$

Callable Bonds

$$
\begin{aligned}
V(3,3) &= \min\{100e^{-0.055}, e^{-r(3,3)}(qV(4,4) + (1-q)V(4,3))\}. \\
&= \min\{100e^{-0.055}, e^{-0.09}(\frac{1}{2} \times 100 + \frac{1}{2} \times 100)\}. \\
&= \min\{94.6485, 91.3931\} \\
&= 91.3931, \\
V(3,0) &= \min\{100e^{-0.055}, e^{-r(3,0)}(qV(4,1) + (1-q)V(4,0))\}. \\
&= \min\{100e^{-0.055}, e^{-0.03}(\frac{1}{2} \times 100 + \frac{1}{2} \times 100)\}. \\
&= \min\{94.6485, 97.0446\} \\
&= 94.6485.
\end{aligned}
$$

Callable Bonds

- The complete set of prices corresponding to the above table for $r(t)$ is given below: (**Exercise**: derive the table)

			$V(t,x)$		
			t		
x	0	1	2	3	4
4	—	—	—	—	100.0000
3	—	—	—	91.3931	100.0000
2	—	—	85.2186	93.2394	100.0000
1	—	80.9745	88.4731	**94.6485**	100.0000
0	78.0067	84.6863	**89.5834**	**94.6485**	100.0000

- Those cells which have been typeset in bold indicate that early exercise is optimal; that is, $(t,x) = (2,0), (3,0)$ and $(3,1)$.

- The prices, $V(t,x)$, are generally lower than $P(t,4,x)$ because the option characteristic favours the issuer rather than the holder of the bond.

Futures Contracts

- Let $f(t, S, T)$ be the futures price at time t for delivery at time S of the zero-coupon bond which matures at time T, where $S < T$. Clearly, $f(S, S, T) = P(S, T)$
- In models for the equity market with a constant risk-free rate of interest we know that the forward and futures prices for an equity contract are equal.
- When the risk-free rate of interest is stochastic, forward and futures prices are not equal.

Futures Contracts

- The futures price varies over time in such a way that immediately after the adjustment at time t the contract has value 0.
- Since the futures price varies over time, the futures exchange requires regular margin payments to pay for the adjustments. The mechanism employed by the exchange usually proceeds as follows.
- Consider an investor who has purchased one futures contract at time 0. At time $t = 0$, the net cash flow is 0 (no cost to set up the contract).
- At time $t = 1, 2, \dots, S$, the net cash flow to the investor is $f(t, S, T) - f(t-1, S, T)$ (called the margin payment at t).

Futures Contracts

- For all $t = 0, 1, \ldots, S-1$ we must set $f(t, S, T)$ in order that

$$E_Q\left[\sum_{n=t+1}^{S} \frac{B(t)}{B(n)}(f(n,S,T) - f(n-1,S,T))|\mathcal{F}_t\right] = 0.$$

- This is consistent with our derivative pricing formula since

$$\frac{B(t)}{B(n)} = \exp\left(-\sum_{s=t}^{n-1} r(s)\right).$$

- The problem is solved by backward induction.

Futures Contracts

- First, set $f(S, S, T) = P(S, T)$.
- Suppose that $f(m, S, T)$ is known for $m = t + 1, \ldots, S$. Thus, for each $n = t + 1, \ldots, S$, we already know that

$$E_Q\left[\sum_{m=n+1}^{S} \frac{B(n)}{B(m)}(f(m, S, T) - f(m-1, S, T))|\mathcal{F}_n\right] = 0.$$

- Now set $f(t, S, T)$ such that

$$E_Q\left[\sum_{n=t+1}^{S} \frac{B(t)}{B(n)}(f(n, S, T) - f(n-1, S, T))|\mathcal{F}_t\right] = 0. \quad (1)$$

Futures Contracts

$$
\begin{aligned}
E_Q & \left[\sum_{n=t+1}^{S} \frac{B(t)}{B(n)} (f(n,S,T) - f(n-1,S,T)) | \mathcal{F}_t \right] \\
&= E_Q \left[\frac{B(t)}{B(t+1)} (f(t+1,S,T) - f(t,S,T)) | \mathcal{F}_t \right] \\
&+ \left[\sum_{n=t+2}^{S} \frac{B(t)}{B(n)} (f(n,S,T) - f(n-1,S,T)) | \mathcal{F}_t \right]
\end{aligned}
$$

$$
\begin{aligned}
=& P(t,t+1) E_Q [f(t+1,S,T) - f(t,S,T) | \mathcal{F}_t] \\
&+ E_Q \left[E_Q \left[\sum_{n=t+2}^{S} \frac{B(t)}{B(n)} (f(n,S,T) - f(n-1,S,T)) | \mathcal{F}_{t+1} \right] | \mathcal{F}_t \right]
\end{aligned}
$$

by Tower Property.

Futures Contracts

$$
\begin{aligned}
&= P(t, t+1) E_Q[f(t+1, S, T) - f(t, S, T) | \mathcal{F}_t] \\
&+ E_Q\left[\frac{B(t)}{B(t+1)} \times 0 | \mathcal{F}_t\right] \text{ (by equation (1))}
\end{aligned}
$$

$$
\begin{aligned}
=& P(t, t+1) E_Q[f(t+1, S, T) - f(t, S, T) | \mathcal{F}_t] \\
=& 0
\end{aligned}
$$

- Hence,

$$f(t, S, T) = E_Q[f(t+1, S, T) | \mathcal{F}_t].$$

Futures Contracts

- The result is true for $t = S$ since $f(S, S, T) = P(S, T)$ by definition.

- Suppose the result is true for $t + 1, \ldots, S$. By induction, we have

$$f(t, S, T) = E_Q[E_Q(P(S, T)|\mathcal{F}_{t+1})|\mathcal{F}_t] = E_Q[P(S, T)|\mathcal{F}_t].$$

- This is in contrast to the forward contract under which, denoting the exercise price by K,

$$E_Q\left[\frac{B(t)}{B(S)}(P(S, T) - K)|\mathcal{F}_t\right] = 0 \implies K = \frac{P(t, T)}{P(t, S)}.$$

- The futures and forward prices are not equal because $P(S, T)$ and $B(t)/B(S)$ may be correlated conditional on $\mathcal{F}_t$, i.e.,

$$E_Q\left[\frac{B(t)}{B(S)}|\mathcal{F}_t\right] E_Q\left[P(S, T)|\mathcal{F}_t\right] \neq E_Q\left[\frac{B(t)}{B(S)}(P(S, T)|\mathcal{F}_t\right].$$

Example 3.14

- Suppose that $r(0) = 0.05$ and again,

$$r(t+1) = \begin{cases} r(t) + 0.01, & \text{if "up" at } t+1; \\ r(t) - 0.01, & \text{if "down" at } t+1. \end{cases}$$

- Consider next the futures contract which delivers at time $S = 2$ the zero-coupon bond which matures at time $T = 3$. We will write $f(t, S, T, r)$ meaning $f(t, S, T)$ when $r(t) = r$ and, likewise, $P(t, T, r)$.

r	$P(2,3,r)$	$f(2,2,3,r)$
0.07	0.932394	0.932394
0.05	0.951229	0.951229
0.03	0.970446	0.970446

Example 3.14

- Now consider $f(1,2,3,r)$. First take $r(1) = 0.06$. We require

$$\begin{aligned} 0 &= E_Q[f(2,2,3,r(2)) - f(1,2,3,r(1))|\mathcal{F}_1] \\ &= (0.6 \times 0.932394 + 0.4 \times 0.951229) - f(1,2,3,0.06) \\ &= 0.939928 - f(1,2,3,0.06) \\ &\implies f(1,2,3,0.06) = 0.939928. \end{aligned}$$

- Similarly,

$$\begin{aligned} f(1,2,3,0.04) &= 0.6 \times 0.951229 + 0.4 \times 0.970446 = 0.958916; \\ f(0,2,3,0.05) &= 0.6 \times f(1,2,3,0.06) + 0.4 \times f(1,2,3,0.04) \\ &= 0.947523 \end{aligned}$$

Example 3.14

- As a check we can calculate $f(0,2,3,0.05)$ directly using the relation

$$f(t,S,T) = E_Q[P(S,T)|\mathcal{F}_t]$$

$$\begin{aligned} \implies f(0,2,3,0.05) &= 0.6^2 \times 0.932394 + 2 \times 0.6 \times 0.4 \times 0.951229 \\ &\quad + 0.4^2 \times 0.970446 \\ &= 0.947523. \end{aligned}$$

- At time 0, we have $P(0,2) = 0.903073$ and $P(0,3) = 0.855765$.

- It follows that the forward price at time 0 for delivery of $P(2,3)$ at time 2 is

$$K = \frac{P(0,3)}{P(0,2)} = 0.947614.$$

This is slightly higher than the futures price because $\frac{B(0)}{B(2)}$ and $P(2,3)$ are positively correlated (**Exercise**).