Math Review

Fabio Cannizzo

Probability Functions

- The *cumulative distribution function F*(*x*) of a stochastic variable X is defined as

$$F(x) = P(X \leq x)$$

- The probability density function f(x) of X is defined as:

$$f(x) = \frac{dF(x)}{dx} \qquad \Rightarrow \qquad F(x) = \int_{-\infty}^{x} f(x)dx$$

- Let g(X) a function of X, its expected value is:

$$E[g(x)] = \int_{-\infty}^{+\infty} g(x)f(x)dx = \int_{-\infty}^{+\infty} g(x)dF(x)$$

Moments

- $E[X]$ is the expectation of X (first moment)
- $E[X^2]$ is the second moment of X
- $E[X^n]$ is the n-th moment of X

Probability Formulas

$$E\left[a + bX + cY\right] = a + bE\left[X\right] + cE\left[Y\right]$$

$$Var[X] = E[X^2] - E[X]^2$$

$$Var[a] = 0$$

$$Var[aX] = a^2 Var[X]$$

$$Var[aX + bY] = a^2 Var[X] + 2abCov[X, Y] + b^2 Var[Y]$$

$$Cov[X, Y] = E\left[XY\right] - E\left[X\right]E\left[Y\right]$$

$$Cov[X, X] = Var[X]$$

$$Cov[a, X] = 0$$

$$Cov[a + bX, cY] = bcCov[X, Y]$$

$$Corr[X, Y] = \frac{Cov[X, Y]}{\sqrt{Var[X]Var[Y]}}$$

Probability Formula Multi Dimensional

- Given a vector of random variables

  $X=(X_1, X_2, \ldots, X_n)^T$
- Its expectation is also a vector and its covariance is a symmetric matrix semi positive definite

  $E[X]=(E[X_1], E[X_2], \ldots, E[X_n])^T$

$$Cov[X]=\begin{bmatrix} Cov[X_1,X_1] & Cov[X_1,X_2] & \cdots & Cov[X_1,X_n] \\ Cov[X_1,X_2] & Cov[X_2,X_2] & \cdots & Cov[X_2,X_n] \\ \vdots & \vdots & \ddots & \vdots \\ Cov[X_1,X_n] & Cov[X_2,X_n] & \cdots & Cov[X_n,X_n] \end{bmatrix}$$

Sample Estimators

- Unbiased sample estimators for the mean and variance of a stochastic variable X are respectively

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{N} X_i$$

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{N} \left(X_i - \hat{\mu}\right)^2$$

Conditional Probability

- Conditional probability (Bayes):

  $P(A \wedge B) = P(A|B)*P(B) = P(B|A)*P(A)$

- Recursive conditional expectation
  - $E[\, E[A|B] \,] = E[A]$

Uniform Distribution

- X is uniformly distributed in [a,b] if its CDF is

$$P(x < X) = F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le \mathrm{x} \le b \\ 1 & x > b \end{cases}$$

- Its probability density function is

$$f_X(x) = \frac{1}{b-a} I_{a \le x \le b}$$

- and its properties are

$$E[X] = (a+b)/2$$

$$Var[X] = (b-a)^2/12$$

Normal Distribution

No close form for F(x)

$$X \sim N\left(\mu, \sigma^2\right), \qquad f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

Standard normal. Computer algorithm offer approximations for F(z)

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1), \qquad f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \qquad F_Z(z) = \int_{-\infty}^{z} f_Z(u)\,du$$

CDF of normal can be obtained via the CDF of standard normal

$$F_X(x) = P(X < x) = P\left(\frac{X-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) = P\left(Z < \frac{x-\mu}{\sigma}\right) = F_Z\left(\frac{x-\mu}{\sigma}\right)$$

Normal Distribution

- The sum of Gaussian variables is Gaussian

$$X \sim N\left(\mu_X, \sigma_X^2\right) \text{ and } Y \sim N\left(\mu_Y, \sigma_Y^2\right) \;\Rightarrow\; \mathrm{Z} = a + bX + cY \sim N\left(\mu_Z, \sigma_Z^2\right)$$

$$\text{where} \quad \mu_Z = a + b\mu_X + c\mu_Y, \quad \sigma_Z^2 = b^2\sigma_X^2 + c^2\sigma_Y^2 + 2bcCov(X,Y)$$

- Multi variate joint Gaussian density distribution

$$X \sim N(\mu, \Sigma)$$

$$f_X\left(x_1, x_1, \ldots, x_n\right) = \frac{1}{|\Sigma|^{1/2}(2\pi)^{n/2}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}$$

where $|\Sigma|$ is the determinant of Σ

Central Limit Theorem

- Let X_1, X_2, …, X_n be a sample of *n* random variables i.i.d with mean *m* and variance s^2, then the distribution of the average of these variables

  $Z=(X_1+X_2+ \ldots +X_n)/n$ (sample mean estimator)

  for large *n* tend to the normal distribution $N(m,s^2/n)$
- This is true regardless of the initial distribution of X
- Corollary: the distribution of the sum also tend to the normal distribution $N(nm,ns^2)$

LogNormal Distribution

- If X is normally distributed with mean μ and variance σ^2, then $Y=e^X$ is *lognormally distributed*. I.e.

$$f(y)=\frac{d}{dy}P(Y<y)=\frac{d}{dy}P(e^X<y)=\frac{d}{dy}P(X<\ln y)$$

$$=\frac{d}{dy}\int_{-\infty}^{\ln y}p(x)dX=\frac{1}{y\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln y-\mu}{\sigma}\right)^2}$$

Lognormal probability density function

$$E[Y]=e^{E[X]+\frac{1}{2}Var[X]}$$

$$Var[Y]=e^{2E[X]+Var[X]}\left(e^{Var[X]}-1\right)$$

$$E[Y^2]=e^{2E[X]+Var[X]}e^{Var[X]}$$

Expectation, second moment and variance, expressed as a function of expectation and variance of X

Stochastic Process

- A stochastic process X_t is a time series of random variables $X_0, X_1, \ldots, X_N$
- We categorize:
  - discrete time, discrete variable
  - discrete time, continuous variable
  - continuous time, discrete variable
  - continuous time, continuous variable

Markov Process and Martingale

- A **Markov** Process is a stochastic process for which everything that we know about its future is summarized by its current value, i.e. the past is irrelevant

$$P[X_T < x \mid X_t, X_{t-1}, \ldots] = P[X_T < x \mid X_t], \qquad t < T$$

- Recursive expectation:

$$E[\, X_{t2} \mid F_{t0} \,] = E[\, E[\, X_{t2} \mid F_{t1}] \mid F_{t0} \,]$$

- A **Martingale** is a stochastic process for which the current value is an unbiased predictor of all future values, i.e. the process is **driftless**

$$E[X_T \mid F_t] = X_t, \qquad t < T$$

Random Walk

- Consider the discrete time continuous variable **random walk** process

$$W_{t+\Delta t} = W_t + \sqrt{\Delta t}\,\varepsilon_t, \quad \text{with} \quad W_0 = 0$$

$$\text{where} \quad \varepsilon_t \sim i.i.d.\,N(0,1)$$

- If $\Delta t \rightarrow 0$, we obtain a continuous time continuous variable random process names **Brownian motion**

$$dW_t = W_{t+dt} - W_t = \sqrt{dt}\,\varepsilon_t \sim N(0, dt)$$

- *dt* is infinitesimal, therefore we can ignore dt^{α} when α>1

dW_t Properties

- $dW_t \sim N(0, dt)$
- $E[dW_t\, dt] = \mathrm{E}[\varepsilon_t]\, dt^{3/2} = 0$
- $\mathrm{E}[dW_t^2] = \mathrm{E}[\varepsilon_t^2]\, dt = (\mathrm{E}[\varepsilon_t^2] - \mathrm{E}[\varepsilon_t]^2)\, dt = \mathrm{Var}[\varepsilon_t]\, dt = dt$
- δW_t is independent on δW_{t+1}

W_t Properties

- W_t is Markov
- W_t is martingale
- $W_t \sim N(0,t)$
- W_t is continuous
- W_t is differentiable nowhere
- W_t will eventually hit every real number
- Non overlapping W_t increments are independent
- If $t \rightarrow \infty$ then $\mathrm{Var}[W_t] \rightarrow \infty$

Ito's Process

- An Ito's process is defined as:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

$$X_t = X_0 + \int_0^t \mu(u, X_u)du + \int_0^t \sigma(u, X_u)dW_u$$

Standard Riemann integral

Ito's integral

Properties of Ito's Integral

$$E\left[\int_0^t f(u, X_u)\,dW_u\right] = 0$$

$$Var\left[\int_0^t f(u, X_u)\,dW_u\right] = \int_0^t f(u, X_u)^2\,du$$

$$\int_0^t f(u, X_u)\,dW_u$$ is a Martingale

$$\int_0^t f(u)\,dW_u$$ is normal if f() does not depend on X

Property of Ito's Process

- X_t is Markov
- If $\mu(X_t,t)=0$ then X_t is Martingale
- $\mathrm{d}X_t\mathrm{d}t=0$
- $(\mathrm{d}X_t)^2=\sigma(X_t,t)^2\mathrm{d}t$

Ito's Lemma

- Consider the Ito's process

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

- A function $f(X_t,t)$ satisfy the SDE:

$$df(t, X_t) = \frac{\partial}{\partial t} f(t, X_t)dt + \frac{\partial}{\partial X_t} f(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2}{\partial X_t^2} f(t, X_t)(dX_t)^2$$

$$= \left[f_t + f_X \mu + \frac{1}{2} f_{XX} \sigma^2 \right] dt + f_X \sigma \, dW_t$$

We can derive it by taking a Taylor expansion and drop all terms of order higher than 1

Product Rule

- Let $f(X_t)$ and $g(X_t)$ be functions of the stochastic processes X_t and Y_t

$$d\left(f\left(X_t\right)g\left(X_t\right)\right) = f\left(X_t\right)dg\left(X_t\right) + g\left(X_t\right)df\left(X_t\right) + df\left(X_t\right)dg\left(X_t\right)$$

and in this term we
get rid of all terms of
order higher than dt

Arithmetic Brownian Motion

- A process X_t follows an **Arithmetic Brownian motion** if:

$$dX_t = \mu\, dt + \sigma\, dW_t$$

- It has solution

$$\int_t^{t+\Delta t} dX_t = \int_t^{t+\Delta t} \mu\, du + \int_t^{t+\Delta t} \sigma\, dW_u \quad \Rightarrow \quad X_{t+\Delta t} = X_t + \mu\, \Delta t + \sigma\left(W_{t+\Delta t} - W_t\right)$$

- because the solution is the sum of a constant and a Gaussian stochastic term (the Brownian increment), we conclude X is also Gaussian

Arithmetic Brownian Motion

- Its properties are:

$$
\begin{aligned}
&E_t\left[X_{t+\Delta t}\right] = E_t\left[X_t + \mu\,\Delta t + \sigma\left(W_{t+\Delta t} - W_t\right)\right] = X_t + \mu\,\Delta t \\
&Var_t\left[X_t\right] = Var_t\left[X_t + \mu\,\Delta t + \sigma\left(W_{t+\Delta t} - W_t\right)_t\right] = \sigma^2 \Delta t \\
&X_t \sim N\left(X_t + \mu\,\Delta t, \sigma^2 \Delta t\right)
\end{aligned}
$$

- Note that the variance grows linearly in Δt
- The process can go negative

Geometric Brownian Motion

- A process X follows a **Geometric Brownian motion** if:

$$\frac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t$$

- It has solution

$$X_{t+\Delta t} = X_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\, \Delta W_t}$$

- and has properties:

$$X_{t+\Delta t} \sim LogN\left(\ln X_t + \left(\mu - \frac{\sigma^2}{t}\right)\Delta t, \sigma^2 \Delta t\right)$$

$$E_t\left[X_t\right] = Xte^{\mu\,\Delta t}, \quad Var_t\left[X_t\right] = X_t^2 e^{2\mu\,\Delta t}\left(e^{\sigma^2 \Delta t} - 1\right)$$

Geometric Brownian Motion

- The solution can be obtained via a simple application of Ito's lemma

$$\frac{dX_t}{X_t} = \mu\, dt + \sigma\, dW_t$$

$$\text{let} \quad Z_t = f(X_t) = \ln X_t$$

$$f_X(X_t) = \frac{1}{X_t}, \quad f_{XX}(X_t) = \frac{-1}{X_t^2}, \quad f_t(X_t) = 0$$

partial derivatives

$$dZ_t = \frac{1}{X_t} X_t(\mu\, dt + \sigma\, dW_t) + \frac{1}{2}\frac{-1}{X_t^2} X_t^2 (\sigma\, dW_t)^2$$

$$= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t$$

Z_t follows an ABM, for which we know the solution

Geometric Brownian Motion

$$Z_{t+\Delta t} = Z_t + \left(\mu - \frac{\sigma^2}{2} \right) \Delta t + \sigma \, \Delta W_t$$

$$X_{t+\Delta t} = \exp(Z_{t+\Delta t}) = X_t \exp\left[\left(\mu - \frac{\sigma^2}{2} \right) \Delta t + \sigma \, \Delta W_t \right]$$

- Because Z_t follows an ABM, it is normally distributed. Therefore X_t is **lognormally** distributed and its moments can be obtained trivially from the moments of Z_t (which are known), via the formula linking the properties of normal and lognormal distribution

Geometric Brownian Motion

$$Z_{t+\Delta t} = Z_t + \left(\mu - \frac{\sigma^2}{2} \right) \Delta t + \sigma \, \Delta W_t$$

$$E_t\left[Z_{t+\Delta t}\right] = Z_t + \left(\mu - \frac{\sigma^2}{2} \right) \Delta t$$

$$Var_t\left[Z_{t+\Delta t}\right] = \sigma^2 \Delta t$$

$$E_t\left[X_{t+\Delta t}\right] = e^{E_t\left[Z_{t+\Delta t}\right] + \frac{1}{2} Var_t\left[Z_{t+\Delta t}\right]} = e^{Z_t + \left(\mu - \frac{\sigma^2}{2} \right) \Delta t + \frac{1}{2} \sigma^2 \Delta t} = X_t e^{\mu \, \Delta t}$$

$$Var_t\left[X_{t+\Delta t}\right] = e^{2 E_t\left[Z_{t+\Delta t}\right] + Var_t\left[Z_{t+\Delta t}\right]} \left(e^{Var_t\left[Z_{t+\Delta t}\right]} - 1 \right) = X_t^2 e^{2 \mu \, \Delta t} \left(e^{\sigma^2 \Delta t} - 1 \right)$$

- Note that the process cannot become negative (because of the exponential)
- The variance grows exponentially with Δt

Ornstein Uhlenbeck

- A process X follows an **Ornstein Uhlenbeck**, (also known as **Vasicek**, for its application in interest rates) if:

$$dX_t = \kappa\left(\mu - X_t\right)dt + \sigma\, dW_t$$

Mean reverting

- It has solution:

$$X_{t+\Delta t} = \mu + \left(X_t - \mu\right)e^{-\kappa\,\Delta t} + \sigma\, e^{-\kappa\,\Delta t}\int_t^{t+\Delta t} e^{-\kappa\,(t-u)}dW_u$$

- and has properties:

$$X_{t+\Delta t} \sim N\left(\mu + \left(X_t - \mu\right)e^{-\kappa\,\Delta t}, \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa\,\Delta t}\right)\right)$$

Ornstein Uhlenbeck

- Note that a Ornstein Uhlenbeck:
  - Can become negative
  - It is mean reverting around μ
  - Its variance is bounded for $\Delta t \rightarrow \infty$

Ornstein Uhlenbeck

- The solution can be obtained via a simple application of Ito's lemma

$$dX_t = \kappa(\mu - X_t)\,dt + \sigma\,dW_t$$

$$\text{let}\quad Z_t = f(X_t) = X_t e^{\kappa t}$$

$$f_X(X_t) = e^{\kappa t},\quad f_{XX}(X_t) = 0,\quad f_t(X_t) = X_t \kappa\, e^{\kappa t}$$ partial derivatives

$$dZ_t = e^{\kappa t}\left(\kappa(\mu - X_t)\,dt + \sigma\,dW_t\right) + X_t \kappa\, e^{\kappa t} dt$$

$$= e^{\kappa t}\left(\kappa\,\mu\,dt + \sigma\,dW_t\right)$$

Z_t is easy to integrate, as there is no stochastic terms in the drift coefficient

Ornstein Uhlenbeck

$$Z_{t+\Delta t} = Z_t + \kappa\,\mu \int_t^{t+\Delta t} e^{\kappa u} du + \sigma \int_t^{t+\Delta t} e^{\kappa u} dW_u$$

$$= Z_t + \mu\, e^{\kappa t}\left(e^{\kappa \Delta t} - 1\right) + \sigma \int_t^{t+\Delta t} e^{\kappa u} dW_u$$

$$X_{t+\Delta t} = Z_{t+\Delta t} e^{-\kappa(t+\Delta t)} = \mu + \left(X_t - \mu\right) e^{-\kappa \Delta t} + \sigma\, e^{-\kappa \Delta t} \int_t^{t+\Delta t} e^{-\kappa(t-u)} dW_u$$

- The moments of X_t can be computed trivially, as all terms in the solution are deterministic except for the Ito's integral, which is normal. In particular:

$$Var_t\left[X_{t+\Delta t}\right] = \sigma^2 e^{-2\kappa \Delta t}\, Var_t\left[\int_0^{\Delta t} e^{\kappa u} dW_u\right]$$

$$= \sigma^2 e^{-2\kappa \Delta t} \int_0^{\Delta t} e^{2\kappa u} du = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa \Delta t}\right)$$

Square Root

- A process X follows a **Square Root** (also known as **Cox Ingersoll Ross**) if:

$$dX_t = \kappa\left(\mu - X_t\right)dt + \sigma\sqrt{X_t}\,dW_t$$

- A solution in close form is not available:

$$X_{t+\Delta t} = \mu + \left(X_t - \mu\right)e^{-\kappa\,\Delta t} + \sigma\,e^{-\kappa\left(t+\Delta t\right)}\int_t^{t+\Delta t} e^{\kappa\,u}\sqrt{X_u}\,dW_u$$

- and has properties:

$$E\left[X_{t+\Delta t}\right] = \mu + \left(X_t - \mu\right)e^{-\kappa\,\Delta t}$$

$$Var\left[X_{t+\Delta t}\right] = \frac{\mu\sigma^2}{2\kappa}\left(1 - e^{-\kappa\,\Delta t}\right)^2 + \frac{\sigma^2}{\kappa}X_t\left(e^{-\kappa\,\Delta t} - e^{-2\kappa\,\Delta t}\right)$$

- X_t has non central X^2 distribution
- The process is mean reverting and cannot go negative

Square Root

- The properties can be obtained as follows:

$$\text{let } Z_t = X_t e^{\kappa t}$$

$$dZ_t = e^{\kappa t} dX_t + \kappa e^{\kappa t} X_t dt = \kappa \mu e^{\kappa t} dt + \sigma e^{\kappa t} \sqrt{X_t}\, dW_t$$

$$Z_{t+\Delta t} = Z_t + \mu e^{\kappa t}\left(e^{\kappa \Delta t} - 1\right) + \sigma \int_{\mathrm{t}}^{t+\Delta t} e^{\kappa u} \sqrt{X_u}\, dW_u$$

$$X_{t+\Delta t} = \mu + \left(X_t - \mu\right) e^{-\kappa \Delta t} + \sigma e^{-\kappa (t+\Delta t)} \int_{\mathrm{t}}^{t+\Delta t} e^{\kappa u} \sqrt{X_u}\, dW_u$$

- The mean is immediate to compute, because the mean of the Ito's integral is null. For the variance we use Ito for the process X^2 and then compute the expectation
- The process is mean reverting and cannot go negative

Square Root

$$Var_t\left[X_{t+\Delta t}\right]=\sigma^2\, e^{-2\kappa(t+\Delta t)}E_t\left[\int_t^{t+\Delta t}\left(e^{\kappa u}\sqrt{X_u}\right)^2 du\right]$$

$$=\sigma^2\, e^{-2\kappa(t+\Delta t)}\int_t^{t+\Delta t} e^{2\kappa u}E_t\left[X_u\right]du$$

$$=\sigma^2\, e^{-2\kappa(t+\Delta t)}\int_t^{t+\Delta t} e^{2\kappa u}\left(\mu+\left(X_t-\mu\right)e^{-\kappa(u-t)}\right)du$$

$$=\left[\frac{\sigma^2\mu}{2\kappa}\left(1-e^{-\kappa\Delta t}\right)^2+\frac{\sigma^2}{\kappa}X_t\left(e^{-\kappa\Delta t}-e^{-2\kappa\Delta t}\right)\right]$$

- X_t could be constructed as the sum of square independent normal variables, hence it has non central X^2 distribution

Further Readings

- Mood, Graybill, Boes, *Introduction to Statistics*
- Steven Shreve, *Stochastic Calculus for Finance; Volume II: Continuous-Time Models*