FE5222 Advanced Derivative Pricing

Wu Lei

Risk Management Institute, National University of Singapore

rmiwul@nus.edu.sg

August 28, 2019

Overview

Introduction

Two approaches

- Partial Differential Equation (P.D.E.) Approach
- Risk Neutral Approach

P.D.E. Approach in Black-Scholes-Merton Model

Assume the stock price evolves (in the real world) according to the following process

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dW(t)$$

where α and σ are constant.

P.D.E. Approach in Black-Scholes-Merton Model

Assume the stock price evolves (in the real world) according to the following process

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dW(t)$$

where α and σ are constant.
The quadratic variation of $S(t)$ (in differential form) is

$$dS(t)dS(t) = \sigma^2 S^2(t)dt$$

P.D.E. Approach in Black-Scholes-Merton Model

Let $V(t, S(t))$ be the value of a financial derivative (call/put option etc.) at time t.

P.D.E. Approach in Black-Scholes-Merton Model

Let $V(t, S(t))$ be the value of a financial derivative (call/put option etc.) at time t.
By Ito's Lemma, the change of $V(t, S(t))$ from t to $t + dt$ is

$$dV(t, S(t)) = V_t dt + V_S dS(t) + \frac{1}{2} V_{SS} dS(t) dS(t)$$

P.D.E. Approach in Black-Scholes-Merton Model

Let $V(t, S(t))$ be the value of a financial derivative (call/put option etc.) at time t.
By Ito's Lemma, the change of $V(t, S(t))$ from t to $t + dt$ is

$$\begin{aligned} dV(t, S(t)) &= V_t dt + V_S dS(t) + \tfrac{1}{2} V_{SS} dS(t) dS(t) \\ &= V_t dt + V_S dS(t) + \tfrac{1}{2} \sigma^2 S^2 V_{SS} dt \end{aligned}$$

P.D.E. Approach in Black-Scholes-Merton Model

If we simultaneously hold $-V_S$ shares of stock at t, the value of our portfolio $\pi(t)$ at time t is

$$\pi(t) = V(t, S(t)) - V_S S(t)$$

P.D.E. Approach in Black-Scholes-Merton Model

The change of portfolio value between t to $t + dt$ is

$$\begin{aligned} d\pi(t) &= dV(t, S(t)) - V_s dS(t) \\ &= V_t dt + \frac{1}{2}\sigma^2 S^2(t) V_{SS} dt \end{aligned}$$

P.D.E. Approach in Black-Scholes-Merton Model

The change of portfolio value between t to $t + dt$ is

$$\begin{aligned} d\pi(t) &= dV(t, S(t)) - V_s dS(t) \\ &= V_t dt + \frac{1}{2}\sigma^2 S^2(t) V_{SS} dt \end{aligned}$$

The change of portfolio value is independent of price change!

P.D.E. Approach in Black-Scholes-Merton Model

In other words, this portfolio is not subject to any price risk in the infinitesimal time interval $[t, t + dt]$.

P.D.E. Approach in Black-Scholes-Merton Model

In other words, this portfolio is not subject to any price risk in the infinitesimal time interval $[t, t + dt]$.

$\Rightarrow$ The portfolio is as safe as holding a riskless asset.

P.D.E. Approach in Black-Scholes-Merton Model

In other words, this portfolio is not subject to any price risk in the infinitesimal time interval $[t, t + dt]$.

$\Rightarrow$ The portfolio is as safe as holding a riskless asset.

$\Rightarrow$ Its value shall grow at the same rate as a riskless asset (no arbitrage principle).

P.D.E. Approach in Black-Scholes-Merton Model

In other words, this portfolio is not subject to any price risk in the infinitesimal time interval $[t, t + dt]$.

$\Rightarrow$ The portfolio is as safe as holding a riskless asset.

$\Rightarrow$ Its value shall grow at the same rate as a riskless asset (no arbitrage principle).

$\Rightarrow$

$$d\pi(t) = r\pi(t)dt$$

P.D.E. Approach in Black-Scholes-Merton Model

In other words, this portfolio is not subject to any price risk in the infinitesimal time interval $[t, t + dt]$.

$\Rightarrow$ The portfolio is as safe as holding a riskless asset.

$\Rightarrow$ Its value shall grow at the same rate as a riskless asset (no arbitrage principle).

$\Rightarrow$

$$d\pi(t) = r\pi(t)dt$$

$\Rightarrow$

$$V_t dt + \frac{1}{2}\sigma^2 S^2(t) V_{SS} dt = r\left(V(t, S(t)) - V_S S(t)\right) dt$$

P.D.E. Approach in Black-Scholes-Merton Model

Canceling dt and rearranging it, we get Black-Scholes P.D.E.

$$V_t + rV_s + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV(t, S) = 0$$

P.D.E. Approach in Black-Scholes-Merton Model

From replicating perspective, at time t if we hold a portfolio $X(t)$ of

- V_S shares of stock
- $\frac{1}{r}\left(V_t + \frac{1}{2}\sigma^2 S^2(t) V_{SS}\right)$ cash

P.D.E. Approach in Black-Scholes-Merton Model

From replicating perspective, at time t if we hold a portfolio $X(t)$ of

- V_S shares of stock
- $\frac{1}{r}\left(V_t + \frac{1}{2}\sigma^2 S^2(t) V_{SS}\right)$ cash

The change of $X(t)$ from t to $t + dt$ is

$$V_S dS(t) + \left(V_t + \frac{1}{2}\sigma^2 S^2(t) V_{SS}\right) dt$$

P.D.E. Approach in Black-Scholes-Merton Model

From replicating perspective, at time t if we hold a portfolio $X(t)$ of

- V_S shares of stock
- $\frac{1}{r}\left(V_t + \frac{1}{2}\sigma^2 S^2(t) V_{SS}\right)$ cash

The change of $X(t)$ from t to $t + dt$ is

$$V_S dS(t) + \left(V_t + \frac{1}{2}\sigma^2 S^2(t) V_{SS}\right) dt$$

The is the same as holding the derivative V!.

P.D.E. Approach in Black-Scholes-Merton Model

Example

We can argue that $V(0) = X(0)$. Otherwise there is arbitrage opportunity. Suppose $V(0) > X(0)$.

P.D.E. Approach in Black-Scholes-Merton Model

Example

We can argue that $V(0) = X(0)$. Otherwise there is arbitrage opportunity. Suppose $V(0) > X(0)$.

1. At $t = 0$,
   - Short V
   - Long X
   - Deposit the cash gain $V(0) - X(0)$ at a bank account

Example

We can argue that $V(0) = X(0)$. Otherwise there is arbitrage opportunity. Suppose $V(0) > X(0)$.

1. At $t = 0$,
   - Short V
   - Long X
   - Deposit the cash gain $V(0) - X(0)$ at a bank account
2. At expiry T, the value of our positions is:
   - -$V(T)$
   - $X(T)$
   - $V(0) - X(0) +$ interest

   Since $X(T) - X(0) = V(T) - V(0)$, the net value is the amount of interest.

P.D.E. Approach in Black-Scholes-Merton Model

Example

We can argue that $V(0) = X(0)$. Otherwise there is arbitrage opportunity. Suppose $V(0) > X(0)$.

1. At $t = 0$,
   - Short V
   - Long X
   - Deposit the cash gain $V(0) - X(0)$ at a bank account
2. At expiry T, the value of our positions is:
   - -$V(T)$
   - $X(T)$
   - $V(0) - X(0) +$ interest

   Since $X(T) - X(0) = V(T) - V(0)$, the net value is the amount of interest.

$\Rightarrow$ Lock in riskless gain!

Risk Neutral Approach

Idea:

- Replicate the payoff of a derivative V with a portfolio X consisting of stocks and cash.

Risk Neutral Approach

Idea:

- Replicate the payoff of a derivative V with a portfolio X consisting of stocks and cash.
- Since the discounted stock prices are martingale under risk-neural measure, the discounted value of X is also a martingale.

Risk Neutral Approach

Idea:

- Replicate the payoff of a derivative V with a portfolio X consisting of stocks and cash.
- Since the discounted stock prices are martingale under risk-neural measure, the discounted value of X is also a martingale.
- The discounted value $\widetilde{V}$ of V is also a martingale under risk neutral measure. Hence

$$\widetilde{V}(t) = \widetilde{\mathbb{E}}[\widetilde{V}(T)|\mathcal{F}_t]$$

Girsanov's Theorem

Definition

Let Z be a positive random variable such that $\mathbb{E}[Z] = 1$. The Radon-Nikodym derivative process $Z(t)$ is defined as

$$Z(t) = \mathbb{E}[Z|\mathcal{F}_t]$$

Girsanov's Theorem

- $Z(t) > 0$

Girsanov's Theorem

- $Z(t) > 0$
- $Z(t)$ is a martingale.

Girsanov's Theorem

- $Z(t) > 0$
- $Z(t)$ is a martingale.
- $\mathbb{E}[Z(t)] = 1$.

Girsanov's Theorem

Lemma

Let Z be a positive random variable and $\mathbb{E}[Z] = 1$, $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z$. Let Y be an integrable random variable. Assume that Y is $\mathcal{F}_t$ measurable. Then

$$\widetilde{\mathbb{E}}[Y] = \mathbb{E}[YZ(t)]$$

where $\widetilde{\mathbb{E}}$ is the expectation w.r.t. the probability measure $\widetilde{\mathbb{P}}$.

Girsanov's Theorem

Note that

$$\widetilde{\mathbb{E}}[Y] = \mathbb{E}[YZ]$$

always holds. $Z(t)$ is the estimate of Z given the information $\mathcal{F}_t$. When Y is known at time t, we can refine the expectation on the RHS with available information to use $Z(t)$.

Girsanov's Theorem

Proof.

$$\widetilde{\mathbb{E}}[Y] = \mathbb{E}[YZ]$$

Girsanov's Theorem

Proof.

$$\begin{aligned} \widetilde{\mathbb{E}}[Y] &= \mathbb{E}[YZ] \\ &= \mathbb{E}\left[\mathbb{E}[YZ|\mathcal{F}_t]\right] \end{aligned}$$

Girsanov's Theorem

Proof.

$$\begin{aligned}\widetilde{\mathbb{E}}[Y] &= \mathbb{E}[YZ] \\ &= \mathbb{E}\left[\mathbb{E}[YZ|\mathcal{F}_t]\right] \\ &= \mathbb{E}\left[Y\mathbb{E}[Z|\mathcal{F}_t]\right] \\ &= \mathbb{E}[YZ(t)]\end{aligned}$$

$\square$

Girsanov's Theorem

Lemma

Let $s < t$ and Y be an $\mathcal{F}_t$ measurable random variable. Then

$$\widetilde{\mathbb{E}}[Y|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]$$

Girsanov's Theorem

- This is the condition expectation version of the previous lemma

Girsanov's Theorem

- This is the condition expectation version of the previous lemma
- In change of measure, the 'scaling factor' Z needs to be normalized (i.e., $\mathbb{E}[Z] = 1$). However the conditional expectation $\mathbb{E}[Z|\mathcal{F}_s] = Z(t) \neq 1$. Hence we need to rescale it by a factor of $\frac{1}{Z(s)}$ such that $\mathbb{E}[\frac{Z(t)}{Z(s)}|\mathcal{F}_s] = 1$.

Girsanov's Theorem

Proof.

We shall prove that $\frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]$ is the conditional expectation w.r.t. $\widetilde{\mathbb{P}}$ of Y given $\mathcal{F}_s$. To do this, we need to verify

- $\frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]$ is $\mathcal{F}_s$-measurable.

Girsanov's Theorem

Proof.

We shall prove that $\frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]$ is the conditional expectation w.r.t. $\widetilde{\mathbb{P}}$ of Y given $\mathcal{F}_s$. To do this, we need to verify

- $\frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]$ is $\mathcal{F}_s$-measurable.
- For any $A \in \mathcal{F}_s$,

$$\widetilde{\mathbb{E}}\left[1_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\right] = \widetilde{\mathbb{E}}\left[1_A Y\right]$$

$\square$

Proof.

Since both $\frac{1}{Z(s)}$ and $\mathbb{E}[YZ(t)|\mathcal{F}_s]$ are $\mathcal{F}_s$-measurable, $\frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]$ is $\mathcal{F}_s$-measurable. $\square$

Proof.

$$\widetilde{\mathbb{E}}\left[1_A \frac{1}{Z(s)} \mathbb{E}[YZ(t)|\mathcal{F}_s]\right] = \mathbb{E}\left[1_A \frac{1}{Z(s)} \mathbb{E}[YZ(t)|\mathcal{F}_s] Z(s)\right]$$

Proof.

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[1_A \frac{1}{Z(s)} \mathbb{E}[YZ(t)|\mathcal{F}_s]\right] &= \mathbb{E}\left[1_A \frac{1}{Z(s)} \mathbb{E}[YZ(t)|\mathcal{F}_s] Z(s)\right] \\
&= \mathbb{E}\left[1_A \mathbb{E}\left[YZ(t)|\mathcal{F}_s\right]\right]
\end{aligned}$$

Proof.

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[1_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\right] &= \mathbb{E}\left[1_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]Z(s)\right] \\
&= \mathbb{E}\left[1_A \mathbb{E}\left[YZ(t)|\mathcal{F}_s\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[1_A YZ(t)|\mathcal{F}_s\right]\right]
\end{aligned}$$

Proof.

$$
\begin{aligned}
\widetilde{\mathbb{E}}\left[1_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\right] &= \mathbb{E}\left[1_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]Z(s)\right] \\
&= \mathbb{E}\left[1_A \mathbb{E}\left[YZ(t)|\mathcal{F}_s\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[1_A YZ(t)|\mathcal{F}_s\right]\right] \\
&= \mathbb{E}\left[1_A YZ(t)\right]
\end{aligned}
$$

Proof.

$$\begin{aligned}\widetilde{\mathbb{E}}\left[1_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]\right] &= \mathbb{E}\left[1_A \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s]Z(s)\right] \\ &= \mathbb{E}\left[1_A \mathbb{E}\left[YZ(t)|\mathcal{F}_s\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[1_A YZ(t)|\mathcal{F}_s\right]\right] \\ &= \mathbb{E}\left[1_A YZ(t)\right] \\ &= \widetilde{\mathbb{E}}\left[1_A Y\right]\end{aligned}$$

$\square$

Girsanov's Theorem

Theorem

Let $W(t), 0 \leq t \leq T$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathcal{F}_t\}_{t \geq 0}$ be a filtration for the Brownian motion, $\Theta(t)$ is an adapted process. Suppose $\Theta(t)$ satisfies Novikov's condition

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T \Theta^2(s)ds}\right] < \infty$$

Define

$$Z(t) = e^{-\int_0^t \Theta(s)dW(s) - \frac{1}{2}\int_0^t \Theta^2(s)ds}$$

then $Z(t)$ is a martingale and $\mathbb{E}Z(t) = 1, \forall 0 \leq t \leq T$.

Girsanov's Theorem

Theorem (Cont'd)

Furthermore, if we let $Z = Z(T)$,

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z$$

and

$$\widetilde{W}(t) = \int_0^t \Theta(s)ds + W(t)$$

Then $\widetilde{W}(t)$ *is a Brownian motion under the measure* $\widetilde{\mathbb{P}}$.

Girsanov's Theorem

Theorem (Cont'd)

Furthermore, if we let $Z = Z(T)$*,*

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = Z$$

and

$$\widetilde{W}(t) = \int_0^t \Theta(s)ds + W(t)$$

Then $\widetilde{W}(t)$ *is a Brownian motion under the measure* $\widetilde{\mathbb{P}}$*.*

Note that we often use differential form

$$d\widetilde{W}(t) = \Theta(t)dt + dW(t)$$

Girsanov's Theorem

Outline of the proof

1. Prove $Z(t)$ is a martingale by showing that $dZ(t)$ has zero drift term
2. Show that $\widetilde{W}$
   - is a martingale (under the probability measure $\widetilde{\mathbb{P}}$)
   - has continuous sample paths; and
   - unit quadratic variation per unit time

     $$[\widetilde{W}, \widetilde{W}](t) = t$$

   $\Longrightarrow$ By Levy's Theorem $\widetilde{W}$ is a Brownian motion under the probability measure $\widetilde{\mathbb{P}}$

Girsanov's Theorem

Proof.

We first prove $Z(t)$ is a martingale.

Girsanov's Theorem

Proof.

We first prove $Z(t)$ is a martingale.
Let

$$X(t) = -\int_0^t \Theta(s)dW(s) - \frac{1}{2}\int_0^t \Theta^2(s)ds$$

which written in differential form becomes

$$dX(t) = -\Theta(t)dW(t) - \frac{1}{2}\Theta^2(t)dt$$

Girsanov's Theorem

Proof.

We first prove $Z(t)$ is a martingale.
Let

$$X(t) = -\int_0^t \Theta(s)dW(s) - \frac{1}{2}\int_0^t \Theta^2(s)ds$$

which written in differential form becomes

$$dX(t) = -\Theta(t)dW(t) - \frac{1}{2}\Theta^2(t)dt$$

Hence

$$dX(t)dX(t) = \Theta^2(t)dt$$

$\square$

Girsanov's Theorem

Proof.

Since $Z(t) = e^{X(t)}$, we can apply Ito's Lemma to the function $f(t, x) = e^x$ and get

$$dZ(t) = f_x dX(t) + \frac{1}{2} f_{xx} dX(t) dX(t)$$

Girsanov's Theorem

Proof.

Since $Z(t) = e^{X(t)}$, we can apply Ito's Lemma to the function $f(t,x) = e^x$ and get

$$dZ(t) = f_x dX(t) + \frac{1}{2} f_{xx} dX(t) dX(t)$$

Note that $f_x = f_{xx} = e^x$, we have

$$\begin{aligned} dZ(t) &= Z(t)\left(-\Theta(t)dW(t) - \tfrac{1}{2}\Theta^2(t)dt\right) + \tfrac{1}{2}Z(t)\Theta^2(t)dt \\ &= -\Theta(t)Z(t)dW(t) \end{aligned}$$

Girsanov's Theorem

Proof.

Since $Z(t) = e^{X(t)}$, we can apply Ito's Lemma to the function $f(t,x) = e^x$ and get

$$dZ(t) = f_x dX(t) + \frac{1}{2} f_{xx} dX(t) dX(t)$$

Note that $f_x = f_{xx} = e^x$, we have

$$\begin{aligned} dZ(t) &= Z(t)\left(-\Theta(t)dW(t) - \tfrac{1}{2}\Theta^2(t)dt\right) + \tfrac{1}{2}Z(t)\Theta^2(t)dt \\ &= -\Theta(t)Z(t)dW(t) \end{aligned}$$

Hence

$$Z(t) = Z(0) - \int_0^t \Theta(s)Z(s)dW(s)$$

is a martingale. $\square$

Girsanov's Theorem

Proof.

Now we show $\widetilde{W}(t)$ is a Brownian motion.

Girsanov's Theorem

Proof.

Now we show $\widetilde{W}(t)$ is a Brownian motion.

- It is trivial that $\widetilde{W}(t)$ has a continuous sample path.

Girsanov's Theorem

Proof.

Now we show $\widetilde{W}(t)$ is a Brownian motion.

- It is trivial that $\widetilde{W}(t)$ has a continuous sample path.
- $$\begin{aligned} d\widetilde{W}(t)d\widetilde{W}(t) &= (\Theta(t)dt + dW(t))^2 \\ &= \Theta^2(t)dtdt + 2\Theta(t)dW(t)dt + dW(t)dW \\ &= dt \end{aligned}$$

  Hence $\widetilde{W}(t)$ has unit quadratic variation per unit time

Girsanov's Theorem

Proof.

Now we show $\widetilde{W}(t)$ is a Brownian motion.

- It is trivial that $\widetilde{W}(t)$ has a continuous sample path.
- $$\begin{aligned} d\widetilde{W}(t)d\widetilde{W}(t) &= (\Theta(t)dt + dW(t))^2 \\ &= \Theta^2(t)dtdt + 2\Theta(t)dW(t)dt + dW(t)dW \\ &= dt \end{aligned}$$

  Hence $\widetilde{W}(t)$ has unit quadratic variation per unit time
- It's left to show that $\widetilde{W}(t)$ is a martingale under $\widetilde{\mathbb{P}}$

□

Girsanov's Theorem

Proof.

Fix $s < t$, we need to show

$$\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}_s] = \widetilde{W}(s)$$

where $\widetilde{\mathbb{E}}$ is the expectation w.r.t. $\widetilde{\mathbb{P}}$.

Girsanov's Theorem

Proof.

Fix $s < t$, we need to show

$$\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}_s] = \widetilde{W}(s)$$

where $\widetilde{\mathbb{E}}$ is the expectation w.r.t. $\widetilde{\mathbb{P}}$.
We notice that $Z(t)$ is a martingale, hence

$$Z(t) = \mathbb{E}[Z(T)|\mathcal{F}_t] = \mathbb{E}[Z|\mathcal{F}_t]$$

$Z(t)$ is a Radom-Nikodym process.

Girsanov's Theorem

Proof.

Fix $s < t$, we need to show

$$\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}_s] = \widetilde{W}(s)$$

where $\widetilde{\mathbb{E}}$ is the expectation w.r.t. $\widetilde{\mathbb{P}}$.
We notice that $Z(t)$ is a martingale, hence

$$Z(t) = \mathbb{E}[Z(T)|\mathcal{F}_t] = \mathbb{E}[Z|\mathcal{F}_t]$$

$Z(t)$ is a Radom-Nikodym process.
We can use the change of measure formula for conditional expectation and get

$$\widetilde{\mathbb{E}}[\widetilde{W}(t)|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[\widetilde{W}(t)Z(t)|\mathcal{F}_s]$$

Girsanov's Theorem

Proof.

It is sufficient to show

$$\frac{1}{Z(s)}\mathbb{E}[\widetilde{W}(t)Z(t)|\mathcal{F}_s] = \widetilde{W}(s)$$

Girsanov's Theorem

Proof.

It is sufficient to show

$$
\begin{aligned}
& \frac{1}{Z(s)}\mathbb{E}[\widetilde{W}(t)Z(t)|\mathcal{F}_s] = \widetilde{W}(s) \\
\iff \quad & \mathbb{E}[\widetilde{W}(t)Z(t)|\mathcal{F}_s] = \widetilde{W}(s)Z(s)
\end{aligned}
$$

Girsanov's Theorem

Proof.

It is sufficient to show

$$
\begin{aligned}
& \frac{1}{Z(s)}\mathbb{E}[\widetilde{W}(t)Z(t)|\mathcal{F}_s] = \widetilde{W}(s) \\
\iff \quad & \mathbb{E}[\widetilde{W}(t)Z(t)|\mathcal{F}_s] = \widetilde{W}(s)Z(s) \\
\iff \quad & \widetilde{W}(t)Z(t) \text{ is a martingale under } \mathbb{P}
\end{aligned}
$$

$\square$

Girsanov's Theorem

Proof.

We now prove $\widetilde{W}(t)Z(t)$ is a martingale under $\mathbb{P}$.

Girsanov's Theorem

Proof.

We now prove $\widetilde{W}(t)Z(t)$ is a martingale under $\mathbb{P}$.

$$d\left(\widetilde{W}(t)Z(t)\right) = Z(t)d\widetilde{W}(t) + \widetilde{W}(t)dZ(t) + d\widetilde{W}(t)dZ(t)$$

Girsanov's Theorem

Proof.

We now prove $\widetilde{W}(t)Z(t)$ is a martingale under $\mathbb{P}$.

$$
\begin{aligned}
d\left(\widetilde{W}(t)Z(t)\right) &= Z(t)d\widetilde{W}(t) + \widetilde{W}(t)dZ(t) + d\widetilde{W}(t)dZ(t) \\
&= Z(t)\left(\Theta dt + dW(t)\right) - \widetilde{W}(t)\Theta Z(t)dW(t) \\
&\quad -\left(\Theta dt + dW(t)\right)\Theta Z(t)dW(t)
\end{aligned}
$$

Girsanov's Theorem

Proof.

We now prove $\widetilde{W}(t)Z(t)$ is a martingale under $\mathbb{P}$.

$$
\begin{aligned}
d\left(\widetilde{W}(t)Z(t)\right) &= Z(t)d\widetilde{W}(t) + \widetilde{W}(t)dZ(t) + d\widetilde{W}(t)dZ(t) \\
&= Z(t)\left(\Theta dt + dW(t)\right) - \widetilde{W}(t)\Theta Z(t)dW(t) \\
&\quad - \left(\Theta dt + dW(t)\right)\Theta Z(t)dW(t) \\
&= Z(t)\Theta dt + Z(t)dW(t) - \widetilde{W}(t)\Theta Z(t)dW(t) \\
&\quad -\Theta^2 Z(t)dW(t)dt - \Theta Z(t)dW(t)dW(t)
\end{aligned}
$$

Girsanov's Theorem

Proof.

We now prove $\widetilde{W}(t)Z(t)$ is a martingale under $\mathbb{P}$.

$$
\begin{aligned}
d\left(\widetilde{W}(t)Z(t)\right) &= Z(t)d\widetilde{W}(t) + \widetilde{W}(t)dZ(t) + d\widetilde{W}(t)dZ(t) \\
&= Z(t)\left(\Theta dt + dW(t)\right) - \widetilde{W}(t)\Theta Z(t)dW(t) \\
&\quad -\left(\Theta dt + dW(t)\right)\Theta Z(t)dW(t) \\
&= Z(t)\Theta dt + Z(t)dW(t) - \widetilde{W}(t)\Theta Z(t)dW(t) \\
&\quad -\Theta^2 Z(t)dW(t)dt - \Theta Z(t)dW(t)dW(t) \\
&= Z(t)\left(1 - \widetilde{W}(t)\Theta\right)dW(t)
\end{aligned}
$$

Since $d\left(\widetilde{W}(t)Z(t)\right)$ has no drift term, it is a martingale under $\mathbb{P}$. This completes the proof. $\square$

Risk Neutral Measure

Model for stock market in the real world measure $\mathbb{P}$

- Stock price process

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \forall 0 \leq t \leq T$$

where $\alpha(t)$ and $\sigma(t)$ are two adapted processes, and $\sigma(t) > 0$.

Risk Neutral Measure

Model for stock market in the real world measure $\mathbb{P}$

- Stock price process

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \forall 0 \leq t \leq T$$

  where $\alpha(t)$ and $\sigma(t)$ are two adapted processes, and $\sigma(t) > 0$.

- Interest rate process $R(t)$, $R(t)$ is adapted

Risk Neutral Measure

Model for stock market in the real world measure $\mathbb{P}$

- Stock price process

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), \forall 0 \leq t \leq T$$

where $\alpha(t)$ and $\sigma(t)$ are two adapted processes, and $\sigma(t) > 0$.

- Interest rate process $R(t)$, $R(t)$ is adapted
- Discount process

$$D(t) = e^{-\int_0^t R(s)ds}$$

Note that

$$dD(t) = -R(t)dt$$

Risk Neutral Measure

The discounted price $D(t)S(t)$ follows

$$d(D(t)S(t)) = S(t)dD(t) + D(t)dS(t) + dD(t)dS(t)$$

Risk Neutral Measure

The discounted price $D(t)S(t)$ follows

$$
\begin{aligned}
d(D(t)S(t)) &= S(t)dD(t) + D(t)dS(t) + dD(t)dS(t) \\
&= -R(t)D(t)S(t)dt + \alpha(t)D(t)S(t)dt \\
&\quad +\sigma(t)D(t)S(t)dW(t)
\end{aligned}
$$

Risk Neutral Measure

The discounted price $D(t)S(t)$ follows

$$\begin{aligned} d(D(t)S(t)) &= S(t)dD(t) + D(t)dS(t) + dD(t)dS(t) \\ &= -R(t)D(t)S(t)dt + \alpha(t)D(t)S(t)dt \\ &\quad +\sigma(t)D(t)S(t)dW(t) \\ &= \sigma(t)D(t)S(t)\left(\frac{\alpha(t)-R(t)}{\sigma(t)} + dW(t)\right) \end{aligned}$$

Risk Neutral Measure

Let

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}$$

Then

$$\begin{aligned} d(D(t)S(t)) &= \sigma(t)D(t)S(t)\left(\Theta(t) + dW(t)\right) \\ &= \sigma(t)D(t)S(t)d\widetilde{W}(t) \end{aligned}$$

where

$$\widetilde{W}(t) = \int_0^t \Theta(s)ds + W(t)$$

is a Brownian motion under the probability measure $\widetilde{\mathbb{P}}$ defined as

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = e^{-\frac{1}{2}\int_0^t \Theta^2(s)ds - \int_0^t \Theta(s)dW(s)}$$

Risk Neutral Measure

- $\Theta(t)$ is called the *market price of risk*

Risk Neutral Measure

- $\Theta(t)$ is called the *market price of risk*
- $\widetilde{\mathbb{P}}$ is the risk neutral measure

Risk Neutral Measure

- $\Theta(t)$ is called the *market price of risk*
- $\widetilde{\mathbb{P}}$ is the risk neutral measure
- $$D(t)S(t) = S(0) + \int_0^t \sigma(s)D(s)S(s)d\widetilde{W}(s)$$

  is a martingale under the risk neutral measure $\widetilde{\mathbb{P}}$

Risk Neutral Measure

- Substituting $dW(t) = -\Theta(t)dt + d\widetilde{W}(t)$ to the stock price $dS(t)$, we get

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t)$$

i.e.,

$$\frac{dS(t)}{S(t)} = R(t)dt + \sigma(t)d\widetilde{W}(t)$$

Risk Neutral Measure

- Substituting $dW(t) = -\Theta(t)dt + d\widetilde{W}(t)$ to the stock price $dS(t)$, we get

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t)$$

i.e.,

$$\frac{dS(t)}{S(t)} = R(t)dt + \sigma(t)d\widetilde{W}(t)$$

- The mean rate of return for $S(t)$ changes from $\alpha(t)$ to $R(t)$ from real world measure to risk neutral measure

Risk Neutral Measure

- Substituting $dW(t) = -\Theta(t)dt + d\widetilde{W}(t)$ to the stock price $dS(t)$, we get

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t)$$

i.e.,

$$\frac{dS(t)}{S(t)} = R(t)dt + \sigma(t)d\widetilde{W}(t)$$

- The mean rate of return for $S(t)$ changes from $\alpha(t)$ to $R(t)$ from real world measure to risk neutral measure
- The instantaneous volatility $\sigma(t)$ does not change. However if $\sigma(t)$ is random, its distribution has changed from real world measure to risk neutral measure.

Value of Portfolio under Risk Neutral Measure

Consider a portfolio X of stocks and a money market account

- Initial capital $X(0)$
- At time t, hold $\Delta(t)$ shares of stock and invest the rest $X(t) - \Delta(t)S(t)$ in a money market account

Value of Portfolio under Risk Neutral Measure

From t to $t + dt$, the change of the value of portfolio is

$$dX(t) = \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt$$

Value of Portfolio under Risk Neutral Measure

From t to $t + dt$, the change of the value of portfolio is

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt \\ &= \Delta(t)\left(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\right) \\ &\quad + R(t)(X(t) - \Delta(t)S(t))dt \end{aligned}$$

Value of Portfolio under Risk Neutral Measure

From t to $t + dt$, the change of the value of portfolio is

$$
\begin{aligned}
dX(t) &= \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt \\
&= \Delta(t)\left(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\right) \\
&\quad + R(t)(X(t) - \Delta(t)S(t))dt \\
&= R(t)X(t)dt + \sigma(t)\Delta(t)S(t)\left(\Theta(t)dt + dW(t)\right)
\end{aligned}
$$

Value of Portfolio under Risk Neutral Measure

From t to $t + dt$, the change of the value of portfolio is

$$
\begin{aligned}
dX(t) &= \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt \\
&= \Delta(t)\left(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)\right) \\
&\quad + R(t)(X(t) - \Delta(t)S(t))dt \\
&= R(t)X(t)dt + \sigma(t)\Delta(t)S(t)\left(\Theta(t)dt + dW(t)\right) \\
&= R(t)X(t)dt + \sigma(t)\Delta(t)S(t)d\widetilde{W}(t)
\end{aligned}
$$

Value of Portfolio under Risk Neutral Measure

The change of the discounted value of portfolio is

$$d\left(D(t)X(t)\right) = D(t)dX(t) + X(t)dD(t) + dX(t)dD(t)$$

Value of Portfolio under Risk Neutral Measure

The change of the discounted value of portfolio is

$$\begin{aligned} d\left(D(t)X(t)\right) &= D(t)dX(t) + X(t)dD(t) + dX(t)dD(t) \\ &= D(t)dX(t) + X(t)dD(t) \end{aligned}$$

Value of Portfolio under Risk Neutral Measure

The change of the discounted value of portfolio is

$$\begin{aligned} d\left(D(t)X(t)\right) &= D(t)dX(t) + X(t)dD(t) + dX(t)dD(t) \\ &= D(t)dX(t) + X(t)dD(t) \\ &= D(t)dX(t) - R(t)D(t)X(t)dt \end{aligned}$$

Value of Portfolio under Risk Neutral Measure

The change of the discounted value of portfolio is

$$\begin{aligned} d\left(D(t)X(t)\right) &= D(t)dX(t) + X(t)dD(t) + dX(t)dD(t) \\ &= D(t)dX(t) + X(t)dD(t) \\ &= D(t)dX(t) - R(t)D(t)X(t)dt \\ &= D(t)\left(R(t)X(t)dt + \sigma(t)\Delta(t)S(t)d\widetilde{W}(t)\right) \\ &\quad -R(t)D(t)X(t)dt \end{aligned}$$

Value of Portfolio under Risk Neutral Measure

The change of the discounted value of portfolio is

$$\begin{aligned}
d\left(D(t)X(t)\right) &= D(t)dX(t) + X(t)dD(t) + dX(t)dD(t) \\
&= D(t)dX(t) + X(t)dD(t) \\
&= D(t)dX(t) - R(t)D(t)X(t)dt \\
&= D(t)\left(R(t)X(t)dt + \sigma(t)\Delta(t)S(t)d\widetilde{W}(t)\right) \\
&\quad - R(t)D(t)X(t)dt \\
&= \sigma(t)D(t)\Delta(t)S(t)d\widetilde{W}(t)
\end{aligned}$$

Value of Portfolio under Risk Neutral Measure

The discounted value of portfolio is a martingale under the risk neutral measure.

Value of Portfolio under Risk Neutral Measure

The discounted value of portfolio is a martingale under the risk neutral measure.

$$\Longrightarrow D(t)X(t) = \widetilde{\mathbb{E}}\left[X(T)D(T)|\mathcal{F}_t\right]$$

Value of Portfolio under Risk Neutral Measure

The discounted value of portfolio is a martingale under the risk neutral measure.

$$\Longrightarrow D(t)X(t) = \widetilde{\mathbb{E}}\left[X(T)D(T)|\mathcal{F}_t\right]$$

$$\Longrightarrow X(t) = \frac{1}{D(t)}\widetilde{\mathbb{E}}\left[X(T)D(T)|\mathcal{F}_t\right]$$

Risk Neutral Pricing

Let $V(T)$ be the payoff of a derivative and $V(T)$ is $\mathcal{F}_T$-measurable. Suppose we can choose a portfolio $X(t)$ of stocks and a money market account with an initial capital $X(0)$ such that $X(T) = V(T)$.

Risk Neutral Pricing

From

$$X(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}}\left[X(T)D(T)|\mathcal{F}_t\right]$$

we have

$$X(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}}\left[V(T)D(T)|\mathcal{F}_t\right]$$

Risk Neutral Pricing

From

$$X(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}} \left[X(T) D(T) | \mathcal{F}_t \right]$$

we have

$$X(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}} \left[V(T) D(T) | \mathcal{F}_t \right]$$

By non-arbitrage argument, we must have

$$X(t) = V(t) \ \forall t$$

Risk Neutral Pricing

From

$$X(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}} [X(T)D(T)|\mathcal{F}_t]$$

we have

$$X(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}} [V(T)D(T)|\mathcal{F}_t]$$

By non-arbitrage argument, we must have

$$X(t) = V(t) \; \forall t$$

This implies

$$V(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}} [V(T)D(T)|\mathcal{F}_t]$$

The discounted value $D(t)V(t)$ is a martingale under risk neutral measure.

Risk Neutral Pricing

To summarize, under the assumptions that $V(T)$ is $\mathcal{F}_T$-measurable and there is a replicating portfolio of stocks and a money market account with initial capital $X(0)$ we must have

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-\int_t^T R(s)ds} V(T) | \mathcal{F}_t\right]$$

Risk Neutral Pricing

- The condition $V(T)$ is $\mathcal{F}_T$-measurable means the payoff of the derivative must be based on the information available up to time T, including path dependent derivative.

Risk Neutral Pricing

- The condition $V(T)$ is $\mathcal{F}_T$-measurable means the payoff of the derivative must be based on the information available up to time T, including path dependent derivative.
- The actual amount of initial capital $X(0)$ does not really matter.

Risk Neutral Pricing

- The condition $V(T)$ is $\mathcal{F}_T$-measurable means the payoff of the derivative must be based on the information available up to time T, including path dependent derivative.
- The actual amount of initial capital $X(0)$ does not really matter.
- The existence of a replicating portfolio will be justified later.

Risk Neutral Pricing

Example

We consider the call option price in the Black-Scholes-Merton model (we assume constant interest rate and volatility). Using risk neutral pricing approach, we have

$$c(0, S(t)) = \widetilde{\mathbb{E}}[(S(T) - K)^{+} | \mathcal{F}_t]$$

where $\widetilde{\mathbb{E}}$ is the expectation under the risk neutral measure $\widetilde{\mathbb{P}}$.

Risk Neutral Pricing

Example

Under risk neutral measure, stock price follows

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t)$$

Risk Neutral Pricing

Example

Under risk neutral measure, stock price follows

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t)$$

Solving it, we have

$$S(T) = S(t)e^{(r-\frac{1}{2})(T-t)+\sigma(W(T)-W(t))}$$

Risk Neutral Pricing

Example

Under risk neutral measure, stock price follows

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{W}(t)$$

Solving it, we have

$$S(T) = S(t)e^{(r-\frac{1}{2})(T-t)+\sigma(W(T)-W(t))}$$

Substituting it into the pricing formula we have

$$c(0, S(t)) = \widetilde{\mathbb{E}}[(S(t)e^{(r-\frac{1}{2})(T-t)+\sigma(W(T)-W(t))} - K)^+|\mathcal{F}_t]$$

which can be easily solved.

Risk Neutral Pricing

Two outstanding issues

- Does there always exit a replicating portfolio?

Risk Neutral Pricing

Two outstanding issues

- Does there always exit a replicating portfolio?
- If it exits, how do we find it (in theory)?

Martingale Representation Theorem

Theorem

Let $W(t), 0 \leq t \leq T$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t\}_{t \geq 0}$ be the filtration generated by $W(t)$. Let $M(t)$ be a martingale w.r.t. $\{\mathcal{F}_t\}$. Then there exists an adapted process $\Gamma(t)$ such that

$$M(t) = M(0) + \int_0^t \Gamma(s) dW(u)$$

Martingale Representation Theorem

- The filtration needs to be generated by $W(t)$. That is the only source of uncertainty comes from the Brownian motion.

Martingale Representation Theorem

- The filtration needs to be generated by $W(t)$. That is the only source of uncertainty comes from the Brownian motion.
- From hedging perspective, we shall be able to hedge uncertainty with stock which is driven by the same Brownian motion.

Replicating Portfolio

Our goal is to find the process $\Delta(t)$ of shares of stock to replicate (with correct initial capital) the payoff $V(T)$.

- Define $V(t)$ as

$$V(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}}[V(T)D(T)|\mathcal{F}_t]$$

$D(t)V(t)$ is a martingale.

Replicating Portfolio

Our goal is to find the process $\Delta(t)$ of shares of stock to replicate (with correct initial capital) the payoff $V(T)$.

- Define $V(t)$ as

$$V(t) = \frac{1}{D(t)}\widetilde{\mathbb{E}}[V(T)D(T)|\mathcal{F}_t]$$

  $D(t)V(t)$ is a martingale.

- By Martingale Representation Theorem,

$$D(t)V(t) = V(0) + \int_0^t \Gamma(u)d\widetilde{W}(u)$$

Replicating Portfolio

Our goal is to find the process $\Delta(t)$ of shares of stock to replicate (with correct initial capital) the payoff $V(T)$.

- Define $V(t)$ as

$$V(t) = \frac{1}{D(t)} \widetilde{\mathbb{E}}[V(T)D(T)|\mathcal{F}_t]$$

  $D(t)V(t)$ is a martingale.

- By Martingale Representation Theorem,

$$D(t)V(t) = V(0) + \int_0^t \Gamma(u)d\widetilde{W}(u)$$

- Suppose we have found $\Delta(t)$. Under risk neutral measure the portfolio $X(t)$ is

$$D(t)X(t) = X(0) + \int_0^t \Delta(u)\sigma(u)D(u)S(u)d\widetilde{W}(u)$$

Replicating Portfolio

Comparing the two equations, we must have

$$V(0) = X(0)$$

and

$$\Delta(t) = \frac{\Gamma(t)}{\sigma(t)D(t)S(t)}$$

Replicating Portfolio

Two important assumptions

- $\sigma(t)$ is positive
- $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the Brownian motion.

Under these two assumptions, every $\mathcal{F}_T$-measurable derivatives can be hedged. Such as model is said to be complete.

Thank you!