Finite Model Theory

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Contents

1 Preliminaries

1.1 Structures

Vocabularies are finite sets that consist of **relation symbols** and **constant symbols**. We denote vocabularies by τ, σ,.... A *vocabulary is *relational if it does not contain constants.

1.2 Graph

Let $\tau = \{E\}$ with a binary relation symbol E. A **graph** (or **undirected graph**) is a τ-structure $\mathcal{G} = (G, E^G)$ satisfying

1. for all $a \in G$: not $E^G aa$
2. for all $a, b \in G$: if $E^G ab$ then $E^G ba$

By GRAPH we denote the class of **finite** graphs. If only (1) is required, we speak of a **digraph**

A subset X of the universe of a graph $\mathcal{G}$ is a **clique**, if $E^G ab$ for all $a, b \in X$, $a \neq b$

Let $\mathcal{G}$ be a digraph. If $n \geq 1$ and

$$E^G a_0 a_1, E^G a_1 a_2, \ldots, E^G a_{n-1} a_n$$

then $a_0, \ldots, a_n$ is a **path** from a_0 to a_n of **length** n. If $a_0 = a_n$, then $a_0, \ldots, a_n$ is a **cycle**. A path $a_0, \ldots, a_n$ is **Hamiltonian** if $G = \{a_0, \ldots, a_n\}$ and $a_i \neq a_j$ for $i \neq j$. If, in addition, $E^G a_n a_0$ we speak of a **Hamiltonian circuit**

Let $\mathcal{G}$ be a graph. Write $a \sim b$ if $a = b$ or if there is a path from a to b. The equivalence class of a is called the **(connected) component** of a. Let CONN be the class of finite connected graphs

Denote by $d(a, b)$ the length of a shortest path from a to b; more precisely, define the **distance function** $d : G \times G \to \mathbb{N} \cup \{\infty\}$ by

$$d(a, b) = \infty \text{ iff } a \not\sim b, \quad d(a, b) = 0 \text{ iff } a = b$$

and otherwise

$$d(a, b) = \min\{n \geq 1 \mid \text{there is a path from } a \text{ to } b \text{ of length } n\}$$

We give the following definitions only for **finite** digraphs. A vertex b is a successor of a vertex a if $E^G ab$. The **in-degree** of a vertex is the number of its predecessors, the **out-degree** the number of its successors.

A **root** of a digraph is a vertex with in-degree 0 and a **leaf** a vertex with out-degree 0.

A **forest** is an acyclic digraph where each vertex has in-degree at most 1. A **tree** is a forest with connected underlying graph. Let TREE be the class of finite trees.

1.3 Syntax and Semantics of First-Order Logic

Denote FO[τ] the set of formulas of first-order logic of vocabulary τ.

When only taking into consideration finite structures, we use the notation $\Phi \models_{\text{fin}} \psi$

The **quantifier rank** $\text{qr}(\varphi)$ of a formula φ is the maximum number of nested quantifiers occurring in it

It can be shown that every first-order formula is logically equivlent to a formula in prenex normal form, that is, to a formula of the form $Q_1 x_1, \ldots, Q_s x_s \psi$ where $Q_1, \ldots, Q_s \in \{\forall, \exists\}$, and where ψ is quantifier-free. Such a formula is called Σ_n if the string of n consecutive blocks, where in each block all quantifiers all of the same type, adjacent blocks contain quantifiers of different type, and the first block is existential. Π_n formulas are defined in the same way but now we require that the first block consists of universal quantifiers. A Δ_n-formula is a formula logically equivalent to both a Σ_n-formula and a Π_n-formula

Given a formula $\varphi(x, \bar{z})$ and $n \geq 1$,

$$\exists^{\geq n} x \varphi(x, \bar{z})$$

is an abbreviation for the formula

$$\exists x_1, \ldots \exists x_n (\bigwedge_{1 \leq i \leq n} \varphi(x_i, \bar{z}) \wedge \bigwedge_{1 \leq i < j \leq n} \neg x_i = x_j)$$

We set

$$\varphi_{\geq n} := \exists^{\geq n} x \; x = x$$

Clearly

$$\mathcal{A} \models \varphi_{\geq n} \quad \text{iff} \quad \|A\| \geq n$$

1.4 Some Classical Results of First-Order Logic

Theorem 1.1. *The set of logically valid sentences of first-order logic is r.e.*

Theorem 1.2 (Compactness Theorem)**.** Φ *is satisfiable iff every finite subset of* Φ *is satisfiable*

Neither Theorem 1.1 nor 1.2 remain valid if one only considers finite structures. A counterexample for the Compactness Theorem is given by the set $\Phi_\infty := \{\varphi_{\geq n} \mid n \geq 1\}$: Each finite subset of Φ_∞ has a finite model, but Φ_∞ has no finite model

The failure of Theorem 1.1 is documented by

Theorem 1.3 (Trahtenbrot's Theorem)**.** *The set of sentences of first-order logic valid in all finite structures is not r.e.*

Lemma 1.4. *Let $\varphi \in \mathrm{FO}[\tau]$ and for $i \in I$, let $\Phi^i \subseteq \mathrm{FO}[\tau]$. Assume that*

$$\models \varphi \leftrightarrow \bigvee_{i \in I} \bigwedge \Phi^i$$

Then there is a finite $I_0 \subseteq I$ and for every $i \in I_0$, a finite $\Phi^i_0 \subseteq \Phi^i$ s.t.

$$\models \varphi \leftrightarrow \bigvee_{i \in I_0} \bigwedge \Phi^i_0$$

Proof. For simplicity we assume that φ is a sentence and that every Φ^i is a set of sentences. By hypothesis, for some $i \in I$, we have $\Phi^i \models \varphi$; hence, by the Compactness Theorem, $\Phi^i_0 \models \varphi$ for some finite $\Phi^i_0 \subseteq \Phi^i$.

If there is not such I_0 with $\models \varphi \to \bigvee_{i \in I_0} \bigwedge \Phi^i_0$, then each finite subset of $\{\varphi\} \cup \{\neg \bigwedge \Phi^i_0 \mid i \in I\}$ has a model. Hence by the Compactness Theorem, there is a contradiction $\square$

Corollary 1.5. *Let Φ be a set of first-order sentences. Assume that any two structures that satisfy the same sentences of Φ are elementarily equivalent. Then any first-order sentence is equivalent to a boolean combination of sentences of Φ*

Proof. For any structure $\mathcal{A}$ set

$$\Phi(\mathcal{A}) := \{\psi \mid \psi \in \Phi, \mathcal{A} \models \psi\} \cup \{\neg\psi \mid \psi \in \Phi, \mathcal{A} \models \neg\psi\}$$

Let φ be any first-order sentence. By the preceding lemma it suffices to show that

$$\models \varphi \leftrightarrow \bigvee_{\mathcal{A} \models \phi} \bigwedge \Phi(\mathcal{A})$$

If $\mathcal{B} \models \varphi$ then $\mathcal{B} \models \bigvee_{\mathcal{A} \models \varphi} \bigwedge \Phi(\mathcal{A})$. Suppose $\mathcal{A} \models \bigvee_{\mathcal{A} \models \varphi} \bigwedge \Phi(\mathcal{A})$. Then for some model $\mathcal{A}$ of φ, $\mathcal{B} \models \Phi(\mathcal{A})$. By the definition of $\Phi(\mathcal{A})$, $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences of Φ $\square$

1.5 Model Classes and Global Relations

Fix a vocabulary τ. For a sentence φ of FO[τ] we denote by Mod(φ) the class of **finite** models of φ.

Mod(φ) is closed under isomorphisms

For $\varphi(x_1, \dots, x_n) \in \mathrm{FO}[\tau]$ and a structure $\mathcal{A}$ let

$$\varphi^{\mathcal{A}}(-) := \{(a_1, \dots, a_n) \mid \mathcal{A} \models \varphi[a_1, \dots, a_n]\}$$

be the set of n-tuples **defined by** φ **in** $\mathcal{A}$

Definition 1.6. Let K be a class of τ-structures. An n-ary **global relation** Γ **on** K is a mapping assigning to each $A \in K$ an n-ary relation $\Gamma(\mathcal{A})$ on $\mathcal{A}$ satisfying

$$\Gamma(\mathcal{A})a_1 \dots a_n \quad \text{iff} \quad \Gamma(\mathcal{B})\pi(a_1) \dots \pi(a_n)$$

for every isomorphism $\pi : \mathcal{A} \cong \mathcal{B}$ and every $a_1, \dots, a_n \in A$. If K is the class of all finite τ-structures, then we just speak of an n-ary **global relation**

Example 1.1.
1. Any formula $\varphi(x_1, \dots, x_n) \in \mathrm{FO}[\tau]$ defines the global relation $\mathcal{A} \mapsto \varphi^{\mathcal{A}}(-)$
2. The "transitive closure relation" TC is the binary global relation on GRAPH with

$$\mathrm{TC}(\mathcal{G}) := \{(a, b) \mid a, b \in G, \text{ there is a path from } a \text{ to } b\}$$

1.6 Relational Databases and Query Languages

2 EhrenfeuchtFraïssé Method

2.1 Elementary Classes

Proposition 2.1. *Every finite structure can be characterized in first-order logic up to isomorphism, i.e., for every finite structure $\mathcal{A}$ there is a sentence $\varphi_{\mathcal{A}}$ of first-order logic s.t. for all structures $\mathcal{B}$ we have*

$$\mathcal{B} \models \varphi_{\mathcal{A}} \quad \textit{iff} \quad \mathcal{A} \cong \mathcal{B}$$

Proof. Suppose $A = \{a_1, \dots, a_n\}$. Set $\bar{a} = a_1 \dots a_n$. Let

$$\Theta_n := \{\psi \mid \psi \text{ has the form } Rx_1 \dots x_k, x = y \text{ or } c = x, \text{ and variables among } v_1, \dots, v_n\}$$

and

$$\varphi_{\mathcal{A}} := \exists v_1 \ldots \exists v_n (\bigwedge\{\psi \mid \psi \in \Theta_n, \mathcal{A} \models \psi[\bar{a}]\} \wedge$$
$$\bigwedge\{\neg\psi \mid \psi \in \Theta_n, \mathcal{A} \models \neg\psi[\bar{a}]\} \wedge \forall v_{n+1}(v_{n+1} = v_n \vee \cdots \vee v_{n+1} = v_n))$$

□

Corollary 2.2. *Let K be a class of finite structures. Then there is a set Φ of first-order sentences s.t.*

$$K = \mathrm{Mod}(\Phi)$$

that is, K is the class of finite models of Φ

Proof. For each n there is only a finite number of pairwise nonisomorphic structures of cardinality n. Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be a maximal subset of K of pairwise nonisomorphic structures of cardinality n. Set

$$\psi_n := (\varphi_{=n} \rightarrow (\varphi_{\mathcal{A}_1} \vee \cdots \vee \varphi_{\mathcal{A}_k}))$$

Then $K = \mathrm{Mod}(\{\psi_n \mid n \geq 1\})$ □

Definition 2.3. Let K be a class of finite structures. K is called **axiomatizable in first-order logic** or **elementary** if there is a setence φ of first-order logic s.t. $K = \mathrm{Mod}(\varphi)$

For structures $\mathcal{A}$ and $\mathcal{B}$ and $m \in \mathbb{N}$ we write $\mathcal{A} \equiv_m \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are m**-equivalent** if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same first-order sentences of quantifier rank $\leq m$

Theorem 2.4. *Let K be a class of finite structures. Suppose that for every m there are fintie structures $\mathcal{A}$ and $\mathcal{B}$ s.t.*

$$\mathcal{A} \in K, \mathcal{B} \notin K, \text{ and } \mathcal{A} \equiv_m \mathcal{B}$$

Then K is not axiomatizable in first-order logic

Proof. Let φ be any first-order sentence. Set $m := \mathrm{qr}(\varphi)$ and $\mathcal{A} \equiv_m \mathcal{B}$ □

2.2 Ehrenfeucht's Theorem

Definition 2.5. Assume $\mathcal{A}$ and $\mathcal{B}$ are structures. Let p be a map with $\mathrm{dom}(p) \subseteq A$ and $\mathrm{im}(p) \subseteq B$. Then p is said to be a **partial isomorphism** from $\mathcal{A}$ to $\mathcal{B}$ if

- p is injective

- for every $c \in \tau$: $c^{A} \in \mathrm{dom}(p)$ and $p(c^{A}) = c^{B}$
- for every n-ary $R \in \tau$ and all $a_1, \ldots, a_n \in \mathrm{dom}(p)$

$$R^{A} a_1 \ldots a_n \quad \text{iff} \quad R^{B} p(a_1) \ldots p(a_n)$$

We write $\mathrm{Part}(\mathcal{A}, \mathcal{B})$ for the set of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$

Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures, $\bar{a} \in A^s$, $\bar{b} \in B^s$, and $m \in \mathbb{N}$. The **Ehrenfeucht game** $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ is played by two players called the **spoiler** and the **duplicator**. Each player has to make m moves in the course of a play. In his i-th move the spoiler first selects a structure, $\mathcal{A}$ or $\mathcal{B}$, and an element in this structure. If the spoiler chooses e_i in $\mathcal{A}$ the nthe duplicator in his i-th move must choose an element f_i in $\mathcal{B}$. If the spoiler chooses f_i in $\mathcal{B}$ then the duplicator must choose an element e_i in $\mathcal{A}$

	$\mathcal{A}, \bar{a}$	$\mathcal{B}, \bar{b}$
first move	e_1	f_1
second move	e_2	f_2
$\vdots$	$\vdots$	$\vdots$
m-th move	e_m	f_m