Time Series: Theory and Methods

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April 23, 2020

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1.1 Stochastic Processes

Definition 1.1. A **stochastic process** is a family of random variables $\{X_t, t \in T\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$

A **probability space** or a **probability triple** $(\Omega, \mathcal{F}, P)$ consists of three elements

1. The sample space Ω - an arbitrary non-empty set
2. The σ-algebra $\mathcal{F} \in 2^\Omega$ - called events, s.t.
    - $\mathcal{F}$ contains the sample space: $\Omega \in \mathcal{F}$
    - $\mathcal{F}$ is closed under complements
    - $\mathcal{F}$ is closed under countable unions
3. The probability measure $P : \mathcal{F} \to [0, 1]$ - a function on $\mathcal{F}$ s.t.
    - P is countably additive: if $\{A_i\}_{i=1}^\infty \subseteq \mathcal{F}$ is a countable collection of pairwise disjoint sets, then $P(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty P(A_i)$
    - the measure of entire sample space is equal to one

A **random variable** is a measurable function $X : \Omega \to E$ from a set of possible outcomes Ω to a measurable space E. The probability that X takes on a value in a measurable set $S \subseteq E$ is written as

$$P(X \in S) = P(\omega \in \Omega \mid X(\omega) \in S)$$

Remark. In time series analysis, the index set T is a set of time points, very often $\{0, \pm 1, \pm 2, \dots\}$,$\{1, 2, 3, \dots\}$,$[0, \infty)$ or $(-\infty, \infty)$

Definition 1.2 (Realizations of a Stochastic Process)**.** The functions $\{X(\omega), \omega \in \Omega\}$ on T are known as the **realizations** or **sample-paths** of the process $\{X_t, t \in T\}$

Example 1.1 (Sinusoid with Random Phase and Amplitude)**.** Let A and Θ be independent random variable with $A \geq 0$ and Θ distributed uniformly on $[0, 2\pi)$. A stochastic process $\{X(t), t \in \mathbb{R}\}$ can then be defined in terms of A and Θ for any given $\nu \geq 0$ and $r > 0$ by

$$X_t = r^{-1} A \cos(\nu t + \Theta)$$

or more explicitly

$$X_t(\omega) = r^{-1} A(\omega) \cos(\nu t + \Theta(\omega))$$