Abstract and Concrete Categories
The Joy of Cats

Jií Adámek & Horst Herrlich & George E. Strecker

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1 Categories, Functors, and Natural Transformations

1.1 Categories and Functors

1.1.1 Categories

Definition 1.1. A **category** is a quadruple $\mathbf{A} = (\mathcal{O}, \hom, id, \circ)$ consisting

1. a class $\mathcal{O}$, whose members are called **A-objects**
2. for each pair (A, B) of **A**-objects, a set $\hom(A, B)$ whose members are called **A-morphisms from** A **to** B

Example 1.1. 1. The following **constructs**; i.e., categories of structured sets and structure-preserving functions between them

(a) **Alg**(Ω) with objects all Ω**-algebras** and morphisms all Ω**-homomorphisms** between them. Here $\Omega = (n_i)_{i \in I}$ is a family of natural numbers n_i, indexed by a set I. An Ω-algebra is a pair $X, (\omega_i)_{i \in I}$ consisting of a set X and a family of functions $\omega_i : X^{n_i} \to X$, called n_i**-ary operations** on X. An Ω-homomorphism $f : (X, (\omega_i)_{i \in I} \to (\widehat{X}, (\widehat{\omega}_i)_{i \in I})$ is a function $f : X \to \widehat{X}$ for which the diagram

$$\begin{tikzcd} X^{n_i} \arrow[r, "f^{n_i}"] \arrow[d, "\omega_i"'] & \widehat{X}^{n_i} \arrow[d, "\widehat{\omega}_i"] \\ X \arrow[r, "f"'] & \widehat{X} \end{tikzcd}$$

commutes for each $i \in I$.

(b) Σ**-Seq** with objects all (deterministic, sequential) Σ**-acceptor**, where Σ is a finite set of input symbols. Objects are quadruples (Q, δ, q_0, F) where Q is a finite set of states, $\delta : \Sigma \times Q \to Q$ is a transition map, $q_0 \in Q$ is the initial state, and $F \subseteq Q$ is the set of final states.
A morphism $f : (Q, \delta, q_0, F) \to (Q', \delta', q_0', F')$ (called a **simulation**) is a function $f : Q \to Q'$ that preserves

i. transitions, i.e., $\delta'(\sigma, f(q)) = f(\delta(\sigma, q))$
ii. the initial state, i.e., $f(q_0) = q_0'$
iii. the final states, i.e., $f[F] \subseteq F'$

2. The following categories where the objects and morphisms are *not* constructed sets and structure-preserving functions:

(a) Mat with objects all natural numbers, and for which $\hom(m, n)$ is the set of all real $m \times n$ matrices, $id_n : n \to n$ is the unit diagonal matrix, and composition is defined by $A \circ B = BA$

(b) **Aut** with objects all (deterministic, sequential, Moore) **automata**. Objects are sectuples $(Q, \Sigma, Y, \delta, q_0, y)$, where Q is the set of states, Σ and Y are the sets of input symbols and output symbols, respectively, $\delta : \Sigma \times Q \to Q$ is the transition map, $q_0 \in Q$ is the initial state, and $y : Q \to Y$ is the output map. Morphisms from an automaton $(Q, \Sigma, Y, \delta, q_0, y)$ to an automaton $(Q', \Sigma', Y', \delta', q_0', y')$ are triples (f_Q, f_Σ, f_Y) of functions satisfying the following conditions

i. preservation of transitions: $\delta'(f_\Sigma(\sigma), f_Q(q)) = f_Q(\delta(\sigma, q))$
ii. preservation of outputs: $f_Y(y(q)) = y'(f_Q(q))$
iii. preservation of initial state: $f_Q(q_0) = q_0'$

1.2 The Dual Principle

Definition 1.2. For any category $\mathbf{A} = (\mathcal{O}, \hom_{\mathbf{A}}, id, \circ)$ the **dual** (or **opposite**) **category of A** is the category $\mathbf{A}^{\mathrm{op}} = (\mathcal{O}, \hom_{\mathbf{A}^{\mathrm{op}}}, id, \circ^{\mathrm{op}})$, where $\hom_{\mathbf{A}^{\mathrm{op}}}(A, B) = \hom_{\mathbf{A}}(B, A)$ and $f \circ^{\mathrm{op}} g = g \circ f$

Consider the property of objects X in **A**:

$\mathcal{P}_{\mathbf{A}}(X) \equiv$ *For any* $\mathbf{A}$*-object* A *there exists exactly one* $\mathbf{A}$*-morphism* $f : A \to X$

Step1: In $\mathcal{P}_{\mathbf{A}}(X)$ replace all occurrences of $\mathbf{A}$ by $\mathbf{A}^{\mathrm{op}}$, thus obtaining the property

$\mathcal{P}_{\mathbf{A}^{\mathrm{op}}}(X) \equiv$ *For any* $\mathbf{A}^{\mathrm{op}}$*-object* A *there exists exactly one* $\mathbf{A}^{\mathrm{op}}$*-morphism* $f : A \to X$

Step2: Translate $\mathcal{P}_{\mathbf{A}^{\mathrm{op}}}(X)$ into the logically equivalent statement

$\mathcal{P}^{\mathrm{op}}_{\mathbf{A}}(X) \equiv$ *For any* $\mathbf{A}$*-object* A *there exists exactly one* $\mathbf{A}$*-morphism* $f : X \to A$

The **Duality Principle for Categories** states

Whenever a property $\mathcal{P}$ holds for all categories,
then the property $\mathcal{P}^{\mathrm{op}}$ holds for all categories.

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