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Notes on Set Theory

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Contents

1 Ordinal

1.1 Linear and partial ordering

Definition 1.1 *A binary relation $<$ on a set P is a* ***partial ordering*** *of P if:*

1. $p \not< p$ *for any* $p \in P$
2. *if* $p < q$ *and* $q < r$ *then* $p < r$

$(P, <)$ is called a ***partial ordered set****. A partial ordering $<$ of P is a* ***linear ordering*** *if moreover*

3. $p < q$ *or* $q < p$ *or* $p = q$ *for all* $p, q \in P$

If $(P, <)$ and $(Q, <)$ are poset and $f : P \to Q$, then f is **order-**

preserving if $x < y$ implies $f(x) < f(y)$. If P and Q are linearly ordered, then f is also called **increasing**

1.2 Well-Ordering

Definition 1.2 () *A linear ordering $<$ of a set P is a **well-ordering** if every nonempty subset of P has a least element*

Lemma 1.3 () *If $(W, <)$ is a well-ordering set and $f : W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$*

Assume that the set $X = \{x \in W \mid f(x) < x\}$ is nonempty and let z be the least element of X. Hence $f(f(x)) < f(x)$ and $f(x) \in X$, a contradiction.

Corollary 1.4 () *The only automorphism of a well-ordered set is the iden-*

tity

Corollary 1.5 () *If two well-ordered sets W_1, W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique*

If W is a well-ordered set and $u \in W$, then $\{x \in W : x < u\}$ is an **initial segment** of W

Lemma 1.6 () *No well-ordered set is isomorphic to an initial segment of itself*

If $\mathrm{ran}(f) = \{x : x < u\}$, then $f(u) < u$, contrary to lemma **??**

Theorem 1.7 () *If W_1 and W_2 are well-ordered sets, then exactly one of* *the following three cases holds:*

1. $W_1 \cong W_2$

2. W_1 *is isomorphic to an initial segment of* W_2

3. W_2 *is isomorphic to an initial segment of* W_1

For $u \in W_i, (i = 1, 2)$, let $W_i(u)$ denote the initial segment of W_i given by u. Let

$$f = \{(x, y) \in W_1 \times W_2 \mid W_1(x) \cong W_2(y)\}$$

If $W_1(x) \cong W_w(y)$ and $W_1(x) \cong W_2(y')$, then $W_2(y) \cong W_1(y')$. According to lemma **??**, $y = y'$. Hence it's easy to see that f is a one-to-one function.

If h is an isomorphism between $W_1(x)$ and $W_2(y)$ and $x' < x$, then $W_1(x') \cong W_2(h(x'))$. It follows that f is order-preserving.

If $\mathrm{dom}(f) = W_1$ and $\mathrm{ran}(f) = W_2$, then case 1 holds.

If $y_1 < y_2$ and $y_2 \in \mathrm{ran}(f)$, then $y_1 \in \mathrm{ran}(f)$. If there is some $y < y_2$ and $y \notin \mathrm{ran}(f)$. Consider the least element y' of $\{y \in W_2 \mid y < y_2 \wedge y \notin \mathrm{ran}(f)\}$. Let $x' = \sup\{x \in W_1 \mid \exists y \in W_2(W_1(x) \cong W_2(y) \wedge y < y')\}$, then $W_1(x') \cong W_2(y')$, a contradiction.

If $\mathrm{ran}(f) \neg W_2$ and y_0 is the least element of $W_2 - \mathrm{ran}(f)$. We have $\mathrm{ran}(f) = W_2(x_0)$. Necessarily, $\mathrm{dom}(f) = W_1$, for otherwise we could have

$(x_0, y_0) \in f$ where x_0 =least element of $W_1 - \mathrm{dom}(f)$. Thus case 2 holds.

Similarly, case 3 holds.

If $W_1 \cong W_2$, we say that they have the same **order-type**

1.3 Ordinal Numbers

The idea is to define ordinal numbers so that

$$\alpha < \beta \Leftrightarrow \alpha \in \beta \wedge \alpha = \{\beta : \beta < \alpha\}$$

Definition 1.8 () *A set T is **transitive** if every element of T is a subset of T*

Definition 1.9 () *A set is an **ordinal number** (an **ordinal**) if it's tran-*

sitive and well-ordered by $\in$

The class of all ordinals is denoted by Ord

We define

$$\alpha < \beta \Leftrightarrow \alpha \in \beta$$

Lemma 1.10 () *1.* $0 = \emptyset$ *is an ordinal*

2. If α *is an ordinal and* $\beta \in \alpha$*, then* β *is an ordinal*

3. If $\alpha \neq \beta$ *are ordinals and* $\alpha \subset \beta$*, then* $\alpha \in \beta$

4. If α*,*β *are ordinals, then either* $\alpha \subset \beta$ *or* $\beta \subset \alpha$

1. definition

2. definition

3. If $\alpha \subset \beta$, let γ be the least element of the set $\beta - \alpha$. Since α is transitive, it follows that α is the initial segment of β given by γ. Thus $\alpha = \{\xi \in \beta \mid \xi < \gamma\} = \gamma \in \beta$

4. Clearly $\alpha \cap \beta$ is an ordinal γ. We have $\gamma = \alpha$ or $\gamma = \beta$, for otherwise $\gamma \in \alpha$ and $\gamma \in \beta$ by 3. Then $\gamma \in \gamma$ which contradicts the definition of an ordinal

Using lemma **??** one gets the following facts about ordinal numbers

1. $<$ is a linear ordering of the class Ord

2. For each α, $\alpha = \{\beta : \beta < \alpha\}$

3. If C is a nonempty class of ordinals, then $\bigcap C$ is an ordinal, $\bigcap C \in C$ and $\bigcap C = \inf C$

4. If X is a nonempty set of ordinals, then $\bigcup X$ is an ordinal and $\bigcup X = \sup X$

5. For every α, $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = \inf\{\beta : \beta > \alpha\}$

We thus define $\alpha + 1 = \alpha \cup \{\alpha\}$(the **succesor** of α)

Theorem 1.11 () *Every well-ordered set is isomorphic to a unique ordinal* *number*

The uniqueness follows from lemma **??**. Given a well-ordered set W, we find an isomorphic ordinal as follows: Define $F(x) = \alpha$ if α is isomorphic to the initial segment of W given by x. If such an α exists, then it's unique. By the replacement axiom, $F(W)$ is a set. For each $x \in W$, such an α exists. Otherwise consider the least x such that α doesn't exist. Let $\alpha = \sup\{F(x') \mid x' \in W \wedge x' < x\}$ and $F(x) = \alpha$. If γ is the least $\gamma \notin F(W)$, then $F(W) = \gamma$ and we have an isomorphism of W onto γ

If $\alpha = \beta + 1$, then α is a **succesor ordinal**. If α is not a succesor ordinal then $\alpha = \sup\{\beta : \beta < \alpha\} = \bigcup \alpha$ is called a **limit ordinal**. We also consider

0 a limit ordinal and define $\sup \emptyset = 0$.

1.4 Induction and Recursion

Theorem 1.12 (Transfinite Induction) *Let C be a class of ordinals and assume*

1. $0 \in C$
2. *if* $\alpha \in C$, *then* $\alpha + 1 \in C$
3. *if* α *is a nonzero limit ordinal and* $\beta \in C$ *for all* $\beta < \alpha$, *then* $\alpha \in C$

Then C is the class of all ordinals

Otherwise let α be the least ordinal $\alpha \notin C$ and apply 1, 2 or 3

A function whose domain is the set $\mathbb{N}$ is called an **{**(infinite) sequence}

(A **sequence** in X is a function $f : \mathbb{N} \to X$). The standard notation for a sequence is

$$\langle a_n : n < \omega \rangle$$

A **finite sequence** is a function s s.t. $\mathrm{dom}(s) = \{i : i < n\}$ for some $n \in \mathbb{N}$; then s is a **sequence of length** n

A **transfinite sequence** is a function whose domain is an ordinal

$$\langle a_\xi : \xi < \alpha \rangle$$

It is also called an α-**sequence** or a **sequence of length** α. We also say that a sequence $\langle a_\xi : \xi < alpha \rangle$ is an **enumeration** of its range $\{a_\xi : \xi < \alpha\}$. If

s is a sequence of length α, then $s^\frown x$ or simply sx denotes the sequence of length $\alpha + 1$ that extends s and whose αth term is x:

$$s^\frown x = sx = s \cap \{(\alpha, x)\}$$

Theorem 1.13 (Transfinite Recursion) *Let G be a function, then* **??** *below defines a unique function F on Ord s.t.*

$$F(\alpha) = G(F\restriction\alpha)$$

for each α

In other words, if we let $a_\alpha = F(\alpha)$, then for each α

$$a_\alpha = G(\langle a_\xi : \xi < \alpha\rangle)$$

Corollary 1.14 () *Let X be a set and θ be an ordinal number. For every function G on the set of all transfinite sequences in X of length $< \theta$ s.t. $\mathrm{ran}(G) \subset X$ there exists a unique θ-sequence in X s.t. $a_\alpha = G(\langle a_\xi : \xi < \theta)$ for every $\alpha < \theta$*

Let

$$F(\alpha) = x \leftrightarrow \text{there is a sequence } \langle a_\xi : \xi < \alpha\rangle \text{ such that} \tag{1}$$

1. $(\forall \xi < \alpha) a_\xi = G(\langle a_n \eta : \eta < \xi\rangle)$

2. $x = G(\langle a_\xi : \xi < \alpha\rangle)$

For every α, if there is an α-sequence that satisfying 1, then such a sequence is unique. Thus $F(\alpha)$ is determined uniquely by 2 and therefore F is a function.

Definition 1.15 () *Let $\alpha > 0$ be a limit ordinal and let $\langle \gamma_\xi : \xi < \alpha \rangle$ be a* ***nondecreasing*** *sequence of ordinals (i.e., $\xi < \eta$ implies $\gamma_\xi \leq \gamma_e ta$). We define the* ***limit*** *of the sequence by*

$$\lim_{\xi \to \alpha} \gamma_\xi = \sup\{\gamma_\xi : \xi < \alpha\}$$

A sequence of ordinals $\langle \gamma_\alpha : \alpha \in Ord \rangle$ is ***normal*** *if it's increasing and* ***continuous****, i.e., for every limit α, $\gamma_\alpha = \lim_{\xi \to \alpha} \gamma_\xi$*

1.5 Ordinal Arithmetic

Definition 1.16 (Addition) *For all ordinal numbers α*

1. $\alpha + 0 = \alpha$

2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, *for all* β

3. $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$ *for all limit* $\beta > 0$

Definition 1.17 *For all ordinal numbers* α

1. $\alpha \cdot 0 = 0$

2. $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$, *for all* β

3. $\alpha \cdot \beta = \lim_{\xi \to \beta}(\alpha \cdot \xi)$ *for all limit* $\beta > 0$

Definition 1.18 (Exponentiation) *For all ordinal numbers* α

1. $\alpha^0 = 1$

2. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$, *for all* β

3. $\alpha^\beta = \lim_{\xi\to\beta} \alpha^\xi$ *for all limit* $\beta > 0$

Lemma 1.19 () *For all ordinals* α, β *and* γ

1. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

2. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

Neither $+$ nor $\cdot$ are commutative

$$1 + \omega = \omega \neq \omega + 1,\;\; 2 \cdot \omega = \omega \neq \omega \cdot 2$$

Definition 1.20 () *Let* $(A, <_A)$ *and* $(B, <_B)$ *be disjoint linearly ordered sets. The* ***sum*** *of these linear orders is the set* $A \cup B$ *with the ordering* *defined as follows:* $x < y$ *if and only if*

1. $x, y \in A$ *and* $x <_A y$

2. $x, y \in B$ *and* $x <_B y$

3. $x \in A$ *and* $y \in B$

Definition 1.21 () *Let* $(A, <)$ *and* $(B, <)$ *be linearly ordered sets. The* ***product*** *of these linear orders is the set* $A \times B$ *with the ordering defined by*

$$(a_1, b_1) < (a_2, b_2) \Leftrightarrow b_1 < b_2 \text{ or } (b_1 = b_2 \wedge a_1 < a_2)$$

Lemma 1.22 () *For all ordinals* α *and* β, $\alpha + \beta$ *and* $\alpha \cdot \beta$ *are respectively isomorphic to the sum and to the product of* α *and* β

Suppose $(A, <_A) \cong \alpha$ and $(B, <_B) \cong \beta$.

1. if $\beta = 0$, then $B = \emptyset, A \cup B = A$

2. if $(A \cup B, <_{A \cup B}) \cong \alpha + \beta$, let $B' = B \cup \{c\}$ s.t. $\{c\} \cap A = \{c\} \cap B = \emptyset$

   all for all $b \in B$, $b < c$. Hence

$$\alpha + (\beta + 1) = (\alpha + \beta) + 1 \cong (A \cup B) \cup \{c\} = A \cup B'$$

3. if β is a limit ordinal and for all $\xi < \beta$ and $B_\xi \cong \xi$,

   $(A \cup B_\xi, <_{A \cup B_\xi}) \cong \alpha + \xi$,

$$A \cup B = A \cup \sup B_\xi = \sup(A \cup B_\xi) \cong \sup(\alpha + \xi) = \alpha + \beta$$

Lemma 1.23 () *1. If $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$*

2. *If* $\alpha < \beta$ *then there exists a unique* δ *s.t.* $\alpha + \delta = \beta$

3. *If* $\beta < \gamma$ *and* $\alpha > 0$, *then* $\alpha \cdot \beta < \alpha \cdot \gamma$

4. *If* $\alpha > 0$ *and* γ *is arbitrary, then there exist a unique* β *and a unique*

   $\rho < \alpha$ *s.t.* $\gamma = \alpha \cdot \beta + \rho$

5. *If* $\beta < \gamma$ *and* $\alpha > 1$, *then* $\alpha^{\beta} < \alpha^{\gamma}$

2. Let δ be the order-type of the set $\{\xi : \alpha \leq \xi < \beta\}$

4. Let β be the greatest ordinal s.t. $\alpha \cdot \beta \leq \gamma$

Theorem 1.24 (Cantor's Normal Form Theorem) *Every ordinal* $\alpha >$

0 can be represented uniquely in the form

$$\alpha = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_n} \cdot k_n$$

where $n \geq 1$, $\alpha \geq \beta_1 > \cdots > \beta_n$ and $k_1, \ldots, k_n$ are nonzero natural numbers.

By induction on α. For $\alpha = 1$ we have $1 = \omega^0 + 1$; for arbitrary $\alpha > 0$, let β be the greatest ordinal s.t. $\omega^\beta \leq \alpha$. The uniqueness of the normal form is proved by induction

1.6 Well-Founded Relations

A binary relation E on a set P is **well-founded** if every nonempty $X \subset P$ has an E-**minimal** element.

Given a well-founded relation E on a set P, we can define the **height** of

E and assign to each $x \in P$ and ordinal number, the **rank** of x in E

Theorem 1.25 () *If E is a well-founded relation on P, then there exists a*

unique function ρ from P into the ordinals s.t. for all $x \in P$

$$\rho(x) = \sup\{\rho(y) + 1 : yEx\}$$

The range of ρ is an initial segment of the ordinals, thus an ordinal number.

This ordinal is called the **height** of E

By induction, let

$$P_0 = \emptyset$$

$$P_{\alpha+1} = \{x \in P : \forall y(yEx \to y \in P_\alpha)\} \cup P_\alpha$$

$$P_\alpha = \bigcup_{\xi<\alpha} P_\xi \text{ if } \alpha \text{ is a limit ordinal}$$

Let θ be the least ordinal s.t. $P_{\theta+1} = P_\theta$. We claim that $P_\theta = P$

1.7 Exercise

Exercise 1.7.1 *Every normal sequence $\langle \gamma_\alpha : \alpha \in Ord \rangle$ has arbitrarily large*

***fixed points**, i.e., α s.t. $\gamma_\alpha = \alpha$*

From StackExchange.

A limit ordinal $\gamma > 0$ is called **indecomposable** if there exist no $\alpha < \gamma$

and $\beta < \gamma$ s.t. $\alpha + \beta = \gamma$

Exercise 1.7.2 *A limit ordinal $\gamma > 0$ is indecomposable if and only if $\alpha + \gamma = \gamma$ for all $\alpha < \gamma$ if and only if $\gamma = \omega^\alpha$ for some α*

1. (3)$\to$(1). Assume $\gamma_1, \gamma_2 < \gamma = \omega^\alpha$. By Cantor's normal form theorem, there exist α' and k s.t. $\gamma_1, \gamma_2 < \omega^{\alpha'} \cdot k$

2. (2)$\to$(3). Assume that γ can't be written as ω^α. Then by Cantor's theorem, $\gamma = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_n} \cdot k_n$. But then $\omega^{\beta_1} < \gamma$ and $\omega^{\beta_1} + \gamma > \gamma$

Exercise 1.7.3 *(Without the Axiom of Infinity). Let ω = least limit $\alpha \neq 0$ if it exists, ω = Ord otherwise. Prove that the following statements are* *equivalent*

1. *There exists an inductive set*

2. *There exists an infinite set*

3. *ω is a set*

2 Cardinal Numbers

2.1 Cardinality

Two sets X, Y have the same *cardinality*

$$X = Y \tag{2}$$

if there exists a one-to-one mapping of X onto Y.

The relation **??** is an equivalence relation. We assume that we can assign to each set X its *cardinal number* X so that two sets are assigned the same cardinal just in case they satisfy condition **??**. *Cardinal numbers can be defined either using the Axiom of Regularity (via equivalence classes) or using the Axiom of Choice*

$$X \leq Y$$

if there exists a one-to-one mapping of X into Y.

Theorem 2.1 (Cantor) *For every set X, $X < P(X)$*

Let f be a function from X into $P(X)$. The set

$$Y = \{x \in X : x \notin f(x)\}$$

is not in the range of f. Thus f is not a function of X onto $P(X)$

Theorem 2.2 (Cantor-Bernstein) *If $A \leq B$ and $B \leq A$, then $A = B$*

If $f_1 : A \to B$ and $f_2 : B \to A$ are one-to-one, then if we let $B' = f_2(B)$ and $A_1 = f_2(f_1(A))$, we have $A_1 \subset B' \subset A$ and $A_1 = A$. Thus we may assume that $A_1 \subset B \subset A$ and that f is a one-to-one function of A onto A_1; we will show that $A = B$

We define for all $n \in \mathbb{N}$

$$A_0 = A, \quad A_{n+1} = f(A_n)$$

$$B_0 = B, \quad B_{n+1} = f(B_n)$$

Let g be the function on A defined as follows

$$g(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n \\ x & \text{otherwise} \end{cases}$$

Then g is a one-to-one mapping of A onto B

StackExchange

The arithmetic operations on cardinals are defined as follows:

$$\kappa + \lambda = A \cup B \quad \text{where } A = \kappa, B = \lambda, A, B \text{ are disjoint}$$

$$\kappa \cdot \lambda = A \times B \quad \text{where } A = \kappa, B = \lambda$$

$$\kappa^{\lambda} = A^{B} \quad \text{where } A = \kappa, B = \lambda$$

Lemma 2.3 () *If $A = \kappa$, then $P(A) = 2^{\kappa}$*

For every $X \subset A$, let χ_X be the function

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \in A - X \end{cases}$$

The mapping $f : X \to \chi_X$ is a one-to-one correspondence between $P(A)$

and $\{0,1\}^A$

Facts about cardinal arithmetic

1. $+$ and $\cdot$ are associative, commutative and distributive

2. $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$

3. $(\kappa^\lambda)^\mu == \kappa^{\lambda \cdot \mu}$

4. $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$

5. If $\kappa \leq \lambda$, then $\kappa^\mu \leq \lambda^\mu$

6. If $0 < \lambda \leq \mu$, then $\kappa^\lambda \leq \kappa^\mu$

7. $\kappa^0 = 1; 1^\kappa = 1; 0^\kappa = 0$ if $\kappa > 0$

2.2 Alephs

An ordinal α is called *cardinal number* (a cardinal) if $\alpha \neq \beta$ for all $\beta < \alpha$

If W is a well-ordered set, then there exists an ordinal α s.t. $W = \alpha$.

Thus we let

$$W = \text{the least ordinal s.t. } W = \alpha$$

All infinite cardinals are limit ordinals. The infinite ordinal numbers that are cardinals are called *alephs*

Lemma 2.4 () *1. For every α there is a cardinal number greater than α*

2. If X is a set of cardinals, then $\sup X$ *is a cardinal*

For every α, let α^+ be the least cardinal number greater than α, the cardinal successor *of α*

1. For any set X, let

$$h(X) = \text{the least } \alpha \text{ s.t. there is no one-to-one function of } \alpha \to X$$

There is only a set of possible well-orderings of subsets of X. Hence there is only a set of ordinals for which a one-to-one function of α into X exists. Thus $h(X)$ exists.

If α is an ordinal, then $\alpha < h(\alpha)$

2. Let $\alpha = \sup X$. If f is a one-to-one mapping of α onto some $\beta < \alpha$, let

   $\kappa \in X$ s.t. $\beta < \kappa \leq \alpha$. Then $\kappa = \{f(\xi) : \xi < \kappa\} \leq \beta$, a contradiction

Using Lemma **??** we define the increasing enumeration of all alephs.

$$\aleph_0 = \omega_0 = \omega, \quad \aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+$$

$$\aleph_\alpha = \omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal}$$

Theorem 2.5 () $\aleph_{\alpha \cdot \aleph_\alpha} = \aleph_\alpha$

2.3 The Canonical Well-Ordering of $\alpha \times \alpha$

We define

$$(\alpha,\beta) < (\gamma,\delta) \leftrightarrow \text{ either } \max\{\alpha,\beta\} < \max\{\gamma,\delta\},$$

$$\text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\} \text{ and } \alpha < \gamma,$$

$$\text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\}, \alpha = \gamma \text{ and } \beta < \delta$$

This relation is a linear ordering of the class Ord $\times$ Ord. Moreover if $X \subset$ Ord $\times$ Ord is nonempty, then X has a least element. Also, for each α,$\alpha \times \alpha$ is the initial segment given by $(0,\alpha)$. If we let

$$\Gamma(\alpha,\beta) = \text{the order-type of the set } \{(\xi,\eta) : (\xi,\eta) < (\alpha,\beta)\}$$

then Γ is a one-to-one mapping of Ord^2 onto Ord and

$$(\alpha, \beta) < (\gamma, \delta) \quad \text{if and only if} \quad \Gamma(\alpha, \beta) < \Gamma(\gamma, \delta)$$

Note that $\Gamma(\omega, \omega) = \omega$ and since $\gamma(\alpha) = \Gamma(\alpha, \alpha)$ is an increasing function of α, we have $\gamma(\alpha) \geq \alpha$. However, $\gamma(\alpha)$ is also continuous and so $\Gamma(\alpha, \alpha) = \alpha$ for arbitrarily large α

Proof of Theorem **??**. We shall show that $\gamma(\omega_\alpha) = \omega_\alpha$. This is true for $\alpha = 0$. Thus let α be the least ordinal s.t. $\gamma(\omega_\alpha) \neq \omega_\alpha$. Let $\beta, \gamma < \omega_\alpha$ be s.t. $\Gamma(\beta, \gamma) = \omega_\alpha$. Pick $\delta < \omega_\alpha$ s.t. $\delta > \beta$ and $\delta > \gamma$. Since $\delta \times \delta$ is an initial segment of $\mathrm{Ord} \times \mathrm{Ord}$ in the canonical well-ordering and contains (β, γ), we have $\Gamma(\delta, \delta) \supset \omega_\alpha$ and so $\delta \times \delta \geq \aleph_\alpha$. However $\delta \times \delta = \delta \cdot \delta$, and by the minimality of α, $\delta \cdot \delta = \delta < \aleph_\alpha$. A contradiction

As a corollary we have

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max\{\aleph_\alpha, \aleph_\beta\}$$

2.4 Cofinality

Let $\alpha > 0$ be a limit ordinal. We say that an increasing β-sequence $\langle \alpha_\xi : \xi < \beta \rangle$, β a limit ordinal, is *cofinal* in α if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. $A \subset \alpha$ is *cofinal* in α if $\sup A = \alpha$. If α is an infinite limit ordinal, the *cofinality* of α is

$$\text{cf}\,\alpha = \text{the least limit ordinal } \beta \text{ s.t. there is an increasing}$$

$$\beta\text{-sequence } \langle \alpha_\xi : \xi < \beta \rangle \text{ with } \lim_{\xi \to \beta} \alpha_\xi = \alpha$$

Obviously cf α is a limit ordinal and cf $\alpha \leq \alpha$. Examples: cf $(\omega + \omega) =$

cf $\aleph_\omega = \omega$

Lemma 2.6 () *cf (cf α) = cf α*

cf

If $\langle \alpha_\xi : \xi < \beta \rangle$ is cofinal in α and $\langle \xi(\nu) : \nu < \gamma \rangle$ is cofinal in β, then

$\langle \alpha_{\xi(\nu)} : \nu < \gamma \rangle$ is cofinal in α

Lemma 2.7 () *Let $\alpha > 0$ be a limit ordinal*

1. *If $A \subset \alpha$ and $\sup A = \alpha$, then the order-type of A is at least cf α*

2. *If $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_\xi \leq \ldots, \xi < \gamma$, is a nondecreasing γ-sequence of*

ordinals in α *and* $\lim_{\xi \to \gamma} \beta_\xi = \alpha$, *then cf* $\gamma =$ *cf* α

cf