Notes on Set Theory

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Contents

1 Ordinal

1.1 Linear and partial ordering

Definition 1.1. A binary relation $<$ on a set P is a **partial ordering** of P if:

1. $p \not< p$ for any $p \in P$
2. if $p < q$ and $q < r$ then $p < r$
   $(P, <)$ is called a **partial ordered set**. A partial ordering $<$ of P is a **linear ordering** if moreover
3. $p < q$ or $q < p$ or $p = q$ for all $p, q \in P$

If $(P, <)$ and $(Q, <)$ are poset and $f : P \to Q$, then f is **order-preserving** if $x < y$ implies $f(x) < f(y)$. If P and Q are linearly ordered, then f is also called **increasing**

1.2 Well-Ordering

Definition 1.2. A linear ordering $<$ of a set P is a **well-ordering** if every nonempty subset of P has a least element

Lemma 1.3. *If $(W, <)$ is a well-ordering set and $f : W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$*

Proof. Assume that the set $X = \{x \in W \mid f(x) < x\}$ is nonempty and let z be the least element of X. Hence $f(f(x)) < f(x)$ and $f(x) \in X$, a contradiction. □

Corollary 1.4. *The only automorphism of a well-ordered set is the identity*

Corollary 1.5. *If two well-ordered sets W_1, W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique*

If W is a well-ordered set and $u \in W$, then $\{x \in W : x < u\}$ is an **initial segment** of W

Lemma 1.6. *No well-ordered set is isomorphic to an initial segment of itself*

Proof. If $\operatorname{ran}(f) = \{x : x < u\}$, then $f(u) < u$, contrary to lemma 1.3 □

Theorem 1.7. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds:*

1. $W_1 \cong W_2$
2. *W_1 is isomorphic to an initial segment of W_2*
3. *W_2 is isomorphic to an initial segment of W_1*

Proof. For $u \in W_i, (i = 1, 2)$, let $W_i(u)$ denote the initial segment of W_i given by u. Let

$$f = \{(x, y) \in W_1 \times W_2 \mid W_1(x) \cong W_2(y)\}$$

If $W_1(x) \cong W_w(y)$ and $W_1(x) \cong W_2(y')$, then $W_2(y) \cong W_1(y')$. According to lemma 1.6, $y = y'$. Hence it's easy to see that f is a one-to-one function.

If h is an isomorphism between $W_1(x)$ and $W_2(y)$ and $x' < x$, then $W_1(x') \cong W_2(h(x'))$. It follows that f is order-preserving.

If $\mathrm{dom}(f) = W_1$ and $\mathrm{ran}(f) = W_2$, then case 1 holds.

If $y_1 < y_2$ and $y_2 \in \mathrm{ran}(f)$, then $y_1 \in \mathrm{ran}(f)$. If there is some $y < y_2$ and $y \notin \mathrm{ran}(f)$. Consider the least element y' of $\{y \in W_2 \mid y < y_2 \wedge y \notin \mathrm{ran}(f)\}$. Let $x' = \sup\{x \in W_1 \mid \exists y \in W_2(W_1(x) \cong W_2(y) \wedge y < y')\}$, then $W_1(x') \cong W_2(y')$, a contradiction.

If $\mathrm{ran}(f) \neg W_2$ and y_0 is the least element of $W_2 - \mathrm{ran}(f)$. We have $\mathrm{ran}(f) = W_2(x_0)$. Necessarily, $\mathrm{dom}(f) = W_1$, for otherwise we could have $(x_0, y_0) \in f$ where x_0 =least element of $W_1 - \mathrm{dom}(f)$. Thus case 2 holds.

Similarly, case 3 holds. □

If $W_1 \cong W_2$, we say that they have the same **order-type**

1.3 Ordinal Numbers

The idea is to define ordinal numbers so that

$$\alpha < \beta \Leftrightarrow \alpha \in \beta \wedge \alpha = \{\beta : \beta < \alpha\}$$

Definition 1.8. A set T is **transitive** if every element of T is a subset of T

Definition 1.9. A set is an **ordinal number** (an **ordinal**) if it's transitive and well-ordered by $\in$

The class of all ordinals is denoted by Ord

We define

$$\alpha < \beta \Leftrightarrow \alpha \in \beta$$

Lemma 1.10.
1. *$0 = \emptyset$ is an ordinal*
2. *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal*
3. *If $\alpha \neq \beta$ are ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$*
4. *If α,β are ordinals, then either $\alpha \subset \beta$ or $\beta \subset \alpha$*

Proof.
1. definition
2. definition

3. If $\alpha \subset \beta$, let γ be the least element of the set $\beta - \alpha$. Since α is transitive, it follows that α is the initial segment of β given by γ. Thus $\alpha = \{\xi \in \beta \mid \xi < \gamma\} = \gamma \in \beta$
4. Clearly $\alpha \cap \beta$ is an ordinal γ. We have $\gamma = \alpha$ or $\gamma = \beta$, for otherwise $\gamma \in \alpha$ and $\gamma \in \beta$ by 3. Then $\gamma \in \gamma$ which contradicts the definition of an ordinal

$\square$

Using lemma 1.10 one gets the following facts about ordinal numbers

1. $<$ is a linear ordering of the class Ord
2. For each α, $\alpha = \{\beta : \beta < \alpha\}$
3. If C is a nonempty class of ordinals, then $\bigcap C$ is an ordinal, $\bigcap C \in C$ and $\bigcap C = \inf C$
4. If X is a nonempty set of ordinals, then $\bigcup X$ is an ordinal and $\bigcup X = \sup X$
5. For every α, $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = \inf\{\beta : \beta > \alpha\}$

We thus define $\alpha + 1 = \alpha \cup \{\alpha\}$(the **succesor** of α)

Theorem 1.11. *Every well-ordered set is isomorphic to a unique ordinal number*

Proof. The uniqueness follows from lemma 1.6. Given a well-ordered set W, we find an isomorphic ordinal as follows: Define $F(x) = \alpha$ if α is isomorphic to the initial segment of W given by x. If such an α exists, then it's unique. By the replacement axiom, $F(W)$ is a set. For each $x \in W$, such an α exists. Otherwise consider the least x such that α doesn't exist. Let $\alpha = \sup\{F(x') \mid x' \in W \wedge x' < x\}$ and $F(x) = \alpha$. If γ is the least $\gamma \notin F(W)$, then $F(W) = \gamma$ and we have an isomorphism of W onto γ $\square$

If $\alpha = \beta + 1$, then α is a **succesor ordinal**. If α is not a succesor ordinal then $\alpha = \sup\{\beta : \beta < \alpha\} = \bigcup \alpha$ is called a **limit ordinal**. We also consider 0 a limit ordinal and define $\sup \emptyset = 0$.

1.4 Induction and Recursion

Theorem 1.12 (Transfinite Induction)**.** *Let C be a class of ordinals and assume*

1. $0 \in C$
2. *if $\alpha \in C$, then $\alpha + 1 \in C$*
3. *if α is a nonzero limit ordinal and $\beta \in C$ for all $\beta < \alpha$, then $\alpha \in C$*

Then C is the class of all ordinals

Proof. Otherwise let α be the least ordinal $\alpha \notin C$ and apply 1, 2 or 3 $\square$

A function whose domain is the set $\mathbb{N}$ is called an **infinite** sequence} (A **sequence** in X is a function $f : \mathbb{N} \to X$). The standard notation for a sequence is

$$\langle a_n : n < \omega \rangle$$

A **finite sequence** is a function s s.t. $\mathrm{dom}(s) = \{i : i < n\}$ for some $n \in \mathbb{N}$; then s is a **sequence of length** n

A **transfinite sequence** is a function whose domain is an ordinal

$$\langle a_\xi : \xi < \alpha \rangle$$

It is also called an α-**sequence** or a **sequence of length** α. We also say that a sequence $\langle a_\xi : \xi < \alpha \rangle$ is an **enumeration** of its range $\{a_\xi : \xi < \alpha\}$. If s is a sequence of length α, then $s^\frown x$ or simply sx denotes the sequence of length $\alpha + 1$ that extends s and whose αth term is x:

$$s^\frown x = sx = s \cap \{(\alpha, x)\}$$

Theorem 1.13 (Transfinite Recursion)**.** *Let G be a function, then 1 below defines a unique function F on Ord s.t.*

$$F(\alpha) = G(F{\restriction}\alpha)$$

for each α

In other words, if we let $a_\alpha = F(\alpha)$, then for each α

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$$

Corollary 1.14. *Let X be a set and θ be an ordinal number. For every function G on the set of all transfinite sequences in X of length $< \theta$ s.t. $\mathrm{ran}(G) \subset X$ there exists a unique θ-sequence in X s.t. $a_\alpha = G(\langle a_\xi : \xi < \theta)$ for every $\alpha < \theta$*

Proof. Let

$$\begin{aligned} F(\alpha) = x \leftrightarrow \text{there is a sequence } \langle a_\xi : \xi < \alpha \rangle \text{ such that} \\ 1.\ (\forall \xi < \alpha) a_\xi = G(\langle a_n \eta : \eta < \xi \rangle) \\ 2.\ x = G(\langle a_\xi : \xi < \alpha \rangle) \end{aligned} \tag{1}$$

For every α, if there is an α-sequence that satisfying 1, then such a sequence is unique. Thus $F(\alpha)$ is determined uniquely by 2 and therefore F is a function. $\square$

Definition 1.15. Let $\alpha > 0$ be a limit ordinal and let $\langle \gamma_\xi : \xi < \alpha \rangle$ be a **nondecreasing** sequence of ordinals (i.e., $\xi < \eta$ implies $\gamma_\xi \leq \gamma_e ta$). We define the **limit** of the sequence by

$$\lim_{\xi \to \alpha} \gamma_\xi = \sup\{\gamma_\xi : \xi < \alpha\}$$

A sequence of ordinals $\langle \gamma_\alpha : \alpha \in Ord \rangle$ is **normal** if it's increasing and **continuous**, i.e., for every limit α, $\gamma_\alpha = \lim_{\xi \to \alpha} \gamma_\xi$

1.5 Ordinal Arithmetic

Definition 1.16 (Addition)**.** For all ordinal numbers α

1. $\alpha + 0 = \alpha$
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, for all β
3. $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$ for all limit $\beta > 0$

Definition 1.17. For all ordinal numbers α

1. $\alpha \cdot 0 = 0$
2. $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$, for all β
3. $\alpha \cdot \beta = \lim_{\xi \to \beta}(\alpha \cdot \xi)$ for all limit $\beta > 0$

Definition 1.18 (Exponentiation)**.** For all ordinal numbers α

1. $\alpha^0 = 1$
2. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$, for all β
3. $\alpha^\beta = \lim_{\xi \to \beta} \alpha^\xi$ for all limit $\beta > 0$

Lemma 1.19. *For all ordinals α, β and γ*

1. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
2. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

Neither $+$ nor $\cdot$ are commutative

$$1 + \omega = \omega \neq \omega + 1, \;\; 2 \cdot \omega = \omega \neq \omega \cdot 2$$

Definition 1.20. Let $(A, <_A)$ and $(B, <_B)$ be disjoint linearly ordered sets. The **sum** of these linear orders is the set $A \cup B$ with the ordering defined as follows: $x < y$ if and only if

1. $x, y \in A$ and $x <_A y$
2. $x, y \in B$ and $x <_B y$
3. $x \in A$ and $y \in B$

Definition 1.21. Let $(A, <)$ and $(B, <)$ be linearly ordered sets. The **product** of these linear orders is the set $A \times B$ with the ordering defined by

$$(a_1, b_1) < (a_2, b_2) \Leftrightarrow b_1 < b_2 \text{ or } (b_1 = b_2 \wedge a_1 < a_2)$$

Lemma 1.22. *For all ordinals α and β, $\alpha + \beta$ and $\alpha \cdot \beta$ are respectively isomorphic to the sum and to the product of α and β*

Proof. Suppose $(A, <_A) \cong \alpha$ and $(B, <_B) \cong \beta$.

1. if $\beta = 0$, then $B = \emptyset$, $A \cup B = A$
2. if $(A \cup B, <_{A \cup B}) \cong \alpha + \beta$, let $B' = B \cup \{c\}$ s.t. $\{c\} \cap A = \{c\} \cap B = \emptyset$ all for all $b \in B$, $b < c$. Hence

$$\alpha + (\beta + 1) = (\alpha + \beta) + 1 \cong (A \cup B) \cup \{c\} = A \cup B'$$

3. if β is a limit ordinal and for all $\xi < \beta$ and $B_\xi \cong \xi$, $(A \cup B_\xi, <_{A \cup B_\xi}) \cong \alpha + \xi$,

$$A \cup B = A \cup \sup B_\xi = \sup(A \cup B_\xi) \cong \sup(\alpha + \xi) = \alpha + \beta$$

□

Lemma 1.23.
1. *If $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$*
2. *If $\alpha < \beta$ then there exists a unique δ s.t. $\alpha + \delta = \beta$*
3. *If $\beta < \gamma$ and $\alpha > 0$, then $\alpha \cdot \beta < \alpha \cdot \gamma$*
4. *If $\alpha > 0$ and γ is arbitrary, then there exist a unique β and a unique $\rho < \alpha$ s.t. $\gamma = \alpha \cdot \beta + \rho$*
5. *If $\beta < \gamma$ and $\alpha > 1$, then $\alpha^\beta < \alpha^\gamma$*

Proof.
2. Let δ be the order-type of the set $\{\xi : \alpha \leq \xi < \beta\}$
4. Let β be the greatest ordinal s.t. $\alpha \cdot \beta \leq \gamma$

□

Theorem 1.24 (Cantor's Normal Form Theorem)**.** *Every ordinal $\alpha > 0$ can be represented uniquely in the form*

$$\alpha = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_n} \cdot k_n$$

where $n \geq 1$, $\alpha \geq \beta_1 > \cdots > \beta_n$ and $k_1, \ldots, k_n$ are nonzero natural numbers.

Proof. By induction on α. For $\alpha = 1$ we have $1 = \omega^0 + 1$; for arbitrary $\alpha > 0$, let β be the greatest ordinal s.t. $\omega^\beta \leq \alpha$. The uniqueness of the normal form is proved by induction □

1.6 Well-Founded Relations

A binary relation E on a set P is **well-founded** if every nonempty $X \subset P$ has an E-**minimal** element.

Given a well-founded relation E on a set P, we can define the **height** of E and assign to each $x \in P$ and ordinal number, the **rank** of x in E

Theorem 1.25. *If E is a well-founded relation on P, then there exists a unique function ρ from P into the ordinals s.t. for all $x \in P$*

$$\rho(x) = \sup\{\rho(y) + 1 : yEx\}$$

The range of ρ is an initial segment of the ordinals, thus an ordinal number. This ordinal is called the **height** of E

Proof. By induction, let

$$\begin{aligned}
&P_0 = \emptyset \\
&P_{\alpha+1} = \{x \in P : \forall y(yEx \to y \in P_\alpha)\} \cup P_\alpha \\
&P_\alpha = \bigcup_{\xi<\alpha} P_\xi \text{ if } \alpha \text{ is a limit ordinal}
\end{aligned}$$

Let θ be the least ordinal s.t. $P_{\theta+1} = P_\theta$. We claim that $P_\theta = P$ □

1.7 Exercise

Exercise 1.7.1. Every normal sequence $\langle \gamma_\alpha : \alpha \in Ord \rangle$ has arbitrarily large **fixed points**, i.e., α s.t. $\gamma_\alpha = \alpha$

Proof. From StackExchange. □

A limit ordinal $\gamma > 0$ is called **indecomposable** if there exist no $\alpha < \gamma$ and $\beta < \gamma$ s.t. $\alpha + \beta = \gamma$

Exercise 1.7.2. A limit ordinal $\gamma > 0$ is indecomposable if and only if $\alpha+\gamma = \gamma$ for all $\alpha < \gamma$ if and only if $\gamma = \omega^\alpha$ for some α

Proof.
1. (3)→(1). Assume $\gamma_1, \gamma_2 < \gamma = \omega^\alpha$. By Cantor's normal form theorem, there exist α' and k s.t. $\gamma_1, \gamma_2 < \omega^{\alpha'} \cdot k$
2. (2)→(3). Assume that γ can't be written as ω^α. Then by Cantor's theorem, $\gamma = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_n} \cdot k_n$. But then $\omega^{\beta_1} < \gamma$ and $\omega^{\beta_1} + \gamma > \gamma$

□

Exercise 1.7.3. (Without the Axiom of Infinity). Let $\omega =$ least limit $\alpha \neq 0$ if it exists, $\omega =$ Ord otherwise. Prove that the following statements are equivalent

1. There exists an inductive set
2. There exists an infinite set
3. ω is a set

2 Cardinal Numbers

2.1 Cardinality

Two sets X, Y have the same *cardinality*

$$|X| = |Y| \tag{2}$$

if there exists a one-to-one mapping of X onto Y.

The relation 2 is an equivalence relation. We assume that we can assign to each set X its *cardinal number* $|X|$ so that two sets are assigned the same cardinal just in case they satisfy condition 2. *Cardinal numbers can be defined either using the Axiom of Regularity (via equivalence classes) or using the Axiom of Choice*

$$|X| \leq |Y|$$

if there exists a one-to-one mapping of X into Y.

Theorem 2.1 (Cantor)**.** *For every set* $X, |X| < \left|P(X)\right|$

Proof. Let f be a function from X into $P(X)$. The set

$$Y = \{x \in X : x \notin f(x)\}$$

is not in the range of f. Thus f is not a function of X onto $P(X)$ □

Theorem 2.2 (Cantor-Bernstein)**.** *If* $|A| \leq |B|$ *and* $|B| \leq |A|$, *then* $|A| = |B|$

Proof. If $f_1 : A \to B$ and $f_2 : B \to A$ are one-to-one, then if we let $B' = f_2(B)$ and $A_1 = f_2(f_1(A))$, we have $A_1 \subset B' \subset A$ and $|A_1| = |A|$. Thus we may assume that $A_1 \subset B \subset A$ and that f is a one-to-one function of A onto A_1; we will show that $|A| = |B|$

We define for all $n \in \mathbb{N}$

$$A_0 = A, \quad A_{n+1} = f(A_n)$$
$$B_0 = B, \quad B_{n+1} = f(B_n)$$

Let g be the function on A defined as follows

$$g(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n \\ x & \text{otherwise} \end{cases}$$

Then g is a one-to-one mapping of A onto B
StackExchange □

The arithmetic operations on cardinals are defined as follows:

$$\begin{aligned} \kappa + \lambda &= |A \cup B| && \text{where } |A| = \kappa, |B| = \lambda, A, B \text{ are disjoint} \\ \kappa \cdot \lambda &= |A \times B| && \text{where } |A| = \kappa, |B| = \lambda \\ \kappa^\lambda &= \left|A^B\right| && \text{where } |A| = \kappa, |B| = \lambda \end{aligned}$$

Lemma 2.3. *If* $|A| = \kappa$*, then* $\left|P(A)\right| = 2^\kappa$

Proof. For every $X \subset A$, let χ_X be the function

$$\chi_X(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \in A - X \end{cases}$$

The mapping $f : X \to \chi_X$ is a one-to-one correspondence between $P(A)$ and $\{0,1\}^A$ □

Facts about cardinal arithmetic

1. $+$ and $\cdot$ are associative, commutative and distributive
2. $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$
3. $(\kappa^\lambda)^\mu == \kappa^{\lambda \cdot \mu}$
4. $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$
5. If $\kappa \leq \lambda$, then $\kappa^\mu \leq \lambda^\mu$
6. If $0 < \lambda \leq \mu$, then $\kappa^\lambda \leq \kappa^\mu$
7. $\kappa^0 = 1; 1^\kappa = 1; 0^\kappa = 0$ if $\kappa > 0$

2.2 Alephs

An ordinal α is called *cardinal number* (a cardinal) if $|\alpha| \neq |\beta|$ for all $\beta < \alpha$

If W is a well-ordered set, then there exists an ordinal α s.t. $|W| = |\alpha|$. Thus we let

$$|W| = \text{the least ordinal s.t. } |W| = |\alpha|$$

All infinite cardinals are limit ordinals. The infinite ordinal numbers that are cardinals are called *alephs*

Lemma 2.4. *1. For every α there is a cardinal number greater than α*

2. If X is a set of cardinals, then $\sup X$ is a cardinal

For every α, let α^+ be the least cardinal number greater than α, the cardinal successor *of α*

Proof. 1. For any set X, let

$$h(X) = \text{the least } \alpha \text{ s.t. there is no one-to-one function of } \alpha \to X$$

There is only a set of possible well-orderings of subsets of X. Hence there is only a set of ordinals for which a one-to-one function of α into X exists. Thus $h(X)$ exists.

If α is an ordinal, then $|\alpha| < \left|h(\alpha)\right|$

2. Let $\alpha = \sup X$. If f is a one-to-one mapping of α onto some $\beta < \alpha$, let $\kappa \in X$ s.t. $\beta < \kappa \leq \alpha$. Then $|\kappa| = \left|\{f(\xi) : \xi < \kappa\}\right| \leq \beta$, a contradiction

□

Using Lemma 2.4 we define the increasing enumeration of all alephs.

$$\aleph_0 = \omega_0 = \omega, \quad \aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+$$
$$\aleph_\alpha = \omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal}$$

Theorem 2.5. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$

2.3 The Canonical Well-Ordering of $\alpha \times \alpha$

We define

$$\begin{aligned}(\alpha,\beta) < (\gamma,\delta) \leftrightarrow\ & \text{either } \max\{\alpha,\beta\} < \max\{\gamma,\delta\},\\ & \text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\} \text{ and } \alpha < \gamma,\\ & \text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\}, \alpha = \gamma \text{ and } \beta < \delta\end{aligned}$$

This relation is a linear ordering of the class Ord $\times$ Ord. Moreover if $X \subset$ Ord $\times$ Ord is nonempty, then X has a least element. Also, for each α,$\alpha \times \alpha$ is the initial segment given by $(0, \alpha)$. If we let

$$\Gamma(\alpha,\beta) = \text{the order-type of the set } \{(\xi,\eta) : (\xi,\eta) < (\alpha,\beta)\}$$

then Γ is a one-to-one mapping of Ord^2 onto Ord and

$$(\alpha,\beta) < (\gamma,\delta) \quad \text{if and only if} \quad \Gamma(\alpha,\beta) < \Gamma(\gamma,\delta)$$

Note that $\Gamma(\omega, \omega) = \omega$ and since $\gamma(\alpha) = \Gamma(\alpha, \alpha)$ is an increasing function of α, we have $\gamma(\alpha) \geq \alpha$. However, $\gamma(\alpha)$ is also continuous and so $\Gamma(\alpha, \alpha) = \alpha$ for arbitrarily large α

Proof of Theorem 2.5. We shall show that $\gamma(\omega_\alpha) = \omega_\alpha$. This is true for $\alpha = 0$. Thus let α be the least ordinal s.t. $\gamma(\omega_\alpha) \neq \omega_\alpha$. Let $\beta, \gamma < \omega_\alpha$ be s.t. $\Gamma(\beta, \gamma) = \omega_\alpha$. Pick $\delta < \omega_\alpha$ s.t. $\delta > \beta$ and $\delta > \gamma$. Since $\delta \times \delta$ is an initial segment of Ord $\times$ Ord in the canonical well-ordering and contains (β, γ), we have $\Gamma(\delta, \delta) \supset \omega_\alpha$ and so $|\delta \times \delta| \geq \aleph_\alpha$. However $|\delta \times \delta| = |\delta| \cdot |\delta|$, and by the minimality of α, $|\delta| \cdot |\delta| = |\delta| < \aleph_\alpha$. A contradiction

As a corollary we have

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max\{\aleph_\alpha, \aleph_\beta\}$$

2.4 Cofinality

Let $\alpha > 0$ be a limit ordinal. We say that an increasing β-sequence $\langle \alpha_\xi : \xi < \beta \rangle$, β a limit ordinal, is *cofinal* in α if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. $A \subset \alpha$ is *cofinal* in α if $\sup A = \alpha$. If α is an infinite limit ordinal, the *cofinality* of α is

$$\begin{aligned} \operatorname{cf} \alpha = &\text{the least limit ordinal } \beta \text{ s.t. there is an increasing} \\ &\beta\text{-sequence } \langle \alpha_\xi : \xi < \beta \rangle \text{ with } \lim_{\xi \to \beta} \alpha_\xi = \alpha \end{aligned}$$

Obviously $\operatorname{cf} \alpha$ is a limit ordinal and $\operatorname{cf} \alpha \leq \alpha$. Examples: $\operatorname{cf}(\omega + \omega) = \operatorname{cf} \aleph_\omega = \omega$

Lemma 2.6. $\operatorname{cf}(\operatorname{cf} \alpha) = \operatorname{cf} \alpha$

Proof. If $\langle \alpha_\xi : \xi < \beta \rangle$ is cofinal in α and $\langle \xi(\nu) : \nu < \gamma \rangle$ is cofinal in β, then $\langle \alpha_{\xi(\nu)} : \nu < \gamma \rangle$ is cofinal in α □

Lemma 2.7. *Let $\alpha > 0$ be a limit ordinal*

1. *If $A \subset \alpha$ and $\sup A = \alpha$, then the order-type of A is at least $\operatorname{cf} \alpha$*
2. *If $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_\xi \leq \ldots, \xi < \gamma$, is a nondecreasing γ-sequence of ordinals in α and $\lim_{\xi \to \gamma} \beta_\xi = \alpha$, then $\operatorname{cf} \gamma = \operatorname{cf} \alpha$*

Proof.

1. The order-type of A is the length of the increasing enumeration of A which is an increasing sequence with limit α
2. If $\gamma = \lim_{\nu \to \operatorname{cf} \gamma} \xi(\nu)$, then $\alpha = \lim_{\nu \to \operatorname{cf} \gamma} \beta_{\xi(\nu)}$, and the nondecreasing sequence $\langle \beta_{\xi(\nu) : \nu < \operatorname{cf} \gamma} \rangle$ has an increasing subsequence of length $\leq \operatorname{cf} \gamma$ with the same limit. Thus $\operatorname{cf} \alpha \leq \operatorname{cf} \gamma$

Let $\alpha = \lim_{\nu \to \mathrm{cf}\,\alpha} \alpha_\nu$. For each $\nu < \mathrm{cf}\,\alpha$, let $\xi(\nu)$ be the least ξ greater than all $\xi(\iota), \iota < \nu$ s.t. $\beta_\xi > \alpha_\nu$. Since $\lim_{\nu \to \mathrm{cf}\,\alpha} \beta_{\xi(\nu)} = \alpha$ it follows that $\lim_{\nu \to \mathrm{cf}\,\alpha} \xi(\nu) = \gamma$ and so $\mathrm{cf}\,\gamma \le \mathrm{cf}\,\alpha$

□

An infinite cardinal $\aleph_\alpha$ is *regular* if $\mathrm{cf}\,\omega_\alpha = \omega_\alpha$. It's *singular* if $\mathrm{cf}\,\omega_\alpha < \omega_\alpha$

Lemma 2.8. *For every limit ordinal α,$\mathrm{cf}\,\alpha$ is a regular cardinal*

Proof. If α is not a cardinal, then using a mapping of $|\alpha|$ onto α, one can construct a cofinal sequence in α of length $\le |\alpha|$. Therefore $\mathrm{cf}\,\alpha < \alpha$

Since $\mathrm{cf}(\mathrm{cf}\,\alpha) = \mathrm{cf}(\alpha)$, it follows that $\mathrm{cf}\,\alpha$ is a cardinal and is regular □

Let κ be a limit ordinal. A subset $X \subset \kappa$ is *bounded* if $\sup X < \kappa$ and *unbounded* if $\sup X = \kappa$

Lemma 2.9. *Let κ be an aleph*

1. *If $X \subset \kappa$ and$|X| < \mathrm{cf}\,\kappa$ then X is bounded*
2. *If $\lambda < \mathrm{cf}\,\kappa$ and $f : \lambda \to \kappa$ then the range of f is bounded*

It follows from 1 that every unbounded subset of a regular cardinal has cardinality κ

Proof. 1. from Lemma 2.7

2. If $X = \mathrm{ran}(f)$ then$|X| \le \lambda$

□

Lemma 2.10. *An infinite cardinal κ is singular if and only if there exists a cardinal $\lambda < \kappa$ and a family $\{S_\xi : \xi < \lambda\}$ of subsets of κ s.t. $|S_\xi| < \kappa$ for each $\xi < \lambda$ and $\kappa = \bigcup_{\xi<\lambda} S_\xi$. The least cardinal λ that satisfies the condition is $\mathrm{cf}\,\kappa$*

Proof. If κ is singular, then there is an increasing sequence $\langle \alpha_\xi : \xi < \mathrm{cf}\,\kappa \rangle$ with $\lim_\xi \alpha_\xi = \kappa$. Let $\lambda = \mathrm{cf}\,\kappa$ and $S_\xi = \alpha_\xi$ for all $\xi < \lambda$

If the condition holds, let $\lambda < \kappa$ be the least cardinal for which there is a family $\{S_\xi : \xi < \lambda\}$ s.t. $\kappa = \bigcup_{\xi<\lambda} S_\xi$ and $|S_\xi| < \kappa$ for each $\xi < \lambda$. For every $\xi < \lambda$, let β_ξ be the order-type of $\bigcup_{\nu<\xi} S_\nu$. The sequence $\langle \beta_\xi : \xi < \lambda \rangle$ is nondecreasing and by the minimality of λ, $\beta_\xi < \kappa$ ofr all $\xi < \lambda$. We shall show that $\lim_\xi \beta_\xi = \kappa$, thus proving that $\mathrm{cf}\,\kappa \le \lambda$

Let $\beta = \lim_{\xi \to \lambda} \beta_\xi$. There is a one-to-one mapping f of $\kappa = \bigcup_{\xi<\lambda} S_\xi$ into $\lambda \times \beta$: if $\alpha \in \kappa$, let $f(\alpha) = (\xi, \gamma)$ where ξ is the least ξ s.t. $\alpha \in S_\xi$ and γ is the order-type of $S_\xi \cap \alpha$. Since $\lambda < \kappa$ and $|\kappa \times \beta| = \lambda \cdot |\beta|$, it follows that $\beta = \kappa$ □

The only cardinal inequality we have proved so far is Cantor's Theorem $\kappa < 2^\kappa$. It follows that $\kappa < \lambda^\kappa$ for every $\lambda > 1$

Theorem 2.11. *If κ is an infinite cardinal, then $\kappa < \kappa^{\operatorname{cf}\kappa}$*

Proof. Let F be a collection of κ functions from $\operatorname{cf}\kappa$ to κ:$F = \{f_\alpha : \alpha < \kappa\}$. It's enough to find $f : \operatorname{cf}\kappa \to \kappa$ that is different from all the f_α. Let $\kappa = \lim_{\xi\to\operatorname{cf}\kappa} \alpha_\xi$. For $\xi < \operatorname{cf}\alpha$, let

$$f(\xi) = \text{least } \gamma \text{ s.t. } \gamma \neq f_\alpha(\xi) \text{ for all } \alpha < \alpha_\xi$$

Such γ exists since $\left|\{f_\alpha(\xi) : \alpha < \alpha_\xi\}\right| \leq \left|\alpha_\xi\right| < \kappa$. Obviously $f \neq f_\alpha$ for all $\alpha < \kappa$ ☐

An uncountable cardinal κ is *weakly inaccessible* if it's a limit cardinal and is regular. The existence of (weakly) inaccessible cardinals is not provable in ZFC.

Note that if $\aleph_\alpha > \aleph_0$ is limit and regular, then $\aleph_\alpha = \operatorname{cf}\aleph_\alpha = \operatorname{cf}\alpha \leq \alpha$, and so $\aleph_\alpha = \alpha$

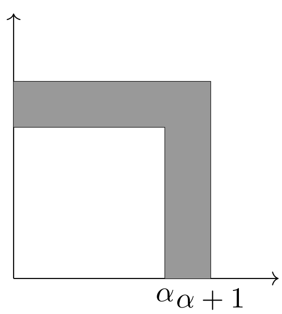
The least fixed point $\aleph_\alpha = \alpha$ has cofinality ω:

$$\kappa = \lim\langle\omega, \omega_\omega, \omega_{\omega_\omega}, \ldots\rangle = \lim_{n\to\omega} \kappa_n$$
$$\text{where } \kappa_0 = \omega, \kappa_{n+1} = \omega_{\kappa_n}$$

2.5 Exercises

Exercise 2.5.1. Show that $\Gamma(\alpha \times \alpha) \leq \omega^\alpha$

Proof. Note that $\gamma_{\alpha+1} = \gamma_\alpha + \alpha + \alpha + 1 = \gamma_\alpha + \alpha \cdot 2 + 1$.



If $\gamma_\alpha \leq \omega^\alpha$, then

$$\gamma_{\alpha+1} \leq \omega^\alpha + \alpha \cdot 2 + 1$$

Since $\alpha \cdot \omega \leq \omega^\alpha$,

$$\gamma_{\alpha+1} \leq \omega^\alpha + \alpha \cdot 2 + 1 \leq \omega^\alpha + \alpha \cdot \omega \leq \omega^\alpha \cdot 2 \leq \omega^{\alpha+1}$$

□

Exercise 2.5.2. There is a well-ordering of the class of all finite sequences of ordinals s.t. for each α, the set of all finite sequences in ω_α is an initial segment and its order-type is ω_α

Proof. We need to show that $\Gamma(\omega_\alpha^\omega) = \omega_\alpha$, which is to show $\Gamma(\omega_\alpha^n) = \omega_\alpha$ for any $n \in \omega$. □

We say that a set B is a *projection* of a set A if there is a mapping of A onto B. Note that B is a projection of A if and only if there is a partition P of A such that $|P| = |B|$. If $|A| \geq |B| > 0$, then B is a projection of A. Conversely using the Axiom of Choice, one shows that if B is a projection of A, then $|A| \geq |B|$. This cannot be proved without the Axiom of Choice

Exercise 2.5.3. The set of all finite subsets of ω_α has cardinality $\aleph_\alpha$

Exercise 2.5.4. $\omega_{\alpha+1}$ is a projection of $P(\omega_\alpha)$

Proof. Consider $f : P(\omega_\alpha \times \omega_\alpha) \to \omega_{\alpha+1}$. If $R \subset \omega_\alpha \times \omega_\alpha$ is a well-ordering, let $f(R) = \mathrm{type}(R)$ and $f(\omega_\alpha \times \omega_\alpha) = \omega_\alpha$ □

Exercise 2.5.5. $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$

Proof. $\aleph_{\alpha+1}$ is a projection of $P(\aleph_\alpha)$. Hence

$$\aleph_{\alpha+1} < 2^{\aleph_{\alpha+1}} \leq 2^{2^{\aleph_\alpha}}$$

□

Exercise 2.5.6 (**ZF**). Show that ω_2 is not a countable union of countable sets.

Proof. Assume that □

3 Real Numbers

The set of all real numbers $\mathbb{R}$ (the *real line* or the *continuum*) is the unique ordered field in which every nonempty bounded set has a least upper bound.

Theorem 3.1 (Cantor)**.** *The set of all real numbers is uncountable*

Proof. Let us assume that the set $\mathbb{R}$ of all reals is countable, and let $c_0, \ldots, c_n, n \in \mathbb{N}$ be an enumeration of $\mathbb{R}$.

Let $a_0 = c_0$ and $b_0 = c_{k_0}$ where k_0 is the least k s.t. $a_0 < c_k$. For each n, let $a_{n+1} = c_{i_n}$, where i_n is the least i s.t. $a_n < c_i < b_n$, and $b_{n+1} = c_{k_n}$ where k_n is the least k s.t. $a_{n+1} < c_k < b_n$. If we let $a = \sup\{a_n : n \in \mathbb{N}\}$, then $a \neq c_k$ for all k $\square$

3.1 The Cardinality of the Continuum

Let $\mathfrak{c}$ denote the cardinality of $\mathbb{R}$. As the set $\mathbb{Q}$ of all rational numbers is dense in $\mathbb{R}$, every real number r is equal to $\sup\{q \in \mathbb{Q} : q < r\}$ and because $\mathbb{Q}$ is countable, it follows that $\mathfrak{c} \leq \left|P(\mathbb{Q})\right| = 2^{\aleph_0}$

Let **C** (the *Cantor Set*) be the set of all reals of the form $\sum_{n=1}^{\infty} a_n/3^n$ where each $a_n = 0$ or 2. **C** is obtained by removing from the closed interval $[0, 1]$. **C** is in a one-to-one correspondence with the set all ω-sequences of 0's and 2's and so $|\mathbf{C}| = 2^{\aleph_0}$

Therefore $\mathfrak{c} \geq 2^{\aleph_0}$ and so by the Cantor-Berstein Theorem we have

$$\mathfrak{c} = 2^{\aleph_0}$$

In ZFC every infinite cardinal is an aleph and so $2^{\aleph_0} \geq \aleph_1$. Cantor's conjecture then becomes the statement

$$2^{\aleph_0} = \aleph_1$$

known as the *Continuum Hypothesis* (CH).

3.2 The Ordering of $\mathbb{R}$

A linear ordering $(P, <)$ is *complete* if every nonempty boudned subset of P has a least upper bound.

Definition 3.2. A lienar ordering $(P, <)$ is *dense* if for all $a < b$ there exists a c s.t. $a < c < b$

A set $D \subset P$ is a *dense subset* if for all $a < b$ in P there exists a $d \in D$ s.t. $a < d < b$.

An ordered set is *unbounded* if it has neither a least nor a greatest element

Theorem 3.3 (Cantor). *1. Any two countable unbounded dense linearly ordered sets are isomorphic*

2. $(\mathbb{R}, <)$ is the unique complete linear ordering that has a countable dense subset isomorphic to $(\mathbb{Q}, <)$

Proof. 1. Let $P_1 = \{a_n : n \in \mathbb{N}\}$ and $P_2 = \{b_n : n \in \mathbb{N}\}$ be two such linearly ordered sets. We construct an isomorphism $f : P_1 \to P_2$ in the following way: we first define $f(a_0)$, then $f^{-1}(b_0)$, then $f(a_1)$, then $f^{-1}(b_1)$, etc., so as to keep f order-preserving. For example, to define $f(a_n)$, if it's not yet defined, we let $f(a_n) = b_k$ where k is the least index s.t. f remains order-preserving (such a k always exists because f has been defined for only finitely many $a \in P_1$)

2. To prove the uniqueness of $\mathbb{R}$, let C and C' be two complete dense unbounded linearly ordered sets, let P and P' be dense in C and C', respectively, and let f be an isomorphism of P onto P'. Then f can be extended to an isomorphism f^* of C and C': For $x \in C$ let $f^*(C) = \sup\{f(p) : p \in P \text{ and } p \leq x\}$

□

The existence of $(\mathbb{R}, <)$ is proved by means of **Dedekind cuts** in $(\mathbb{Q}, <)$. The following theorem is a general version of this construction:

Theorem 3.4. *Let $(P, <)$ be a dense unbounded linearly ordered set. Then there is a complete unbounded linearly ordered set $(C, \prec)$ s.t.*

1. *$P \subset C$ and $<$ and $\prec$ agree on P*
2. *P is dense in C*

Proof. A **Dedekind cut** in P is a pair (A, B) of disjoint nonempty subsets of P s.t.

1. $A \cup B = P$
2. $a < b$ for any $a \in A$ and $b \in B$
3. A does not have a greatest element

Let C be the set of all Dedekind cuts in P and let $(A_1, B_1) \preceq (A_2, B_2)$ if $A_1 \subset A_2$. The set C is complete: If $\{(A_i, B_i) : i \in I\}$ is a nonempty bounded subset of C, then $(\bigcup_{i \in I} A_i, \bigcap_{i \in I} B_i)$ is its supremum.

For $p \in P$, let

$$A_p = \{x \in P : x < p\}, \quad B_p = \{x \in P : x \geq P\}$$

Then $P' = \{(A_p, B_p) : p \in P\}$ is isomorphic to P and is dense in C □

3.3 Suslin's Problem

The real line is, up to isomorphism, the unique linearly ordered set that is dense, unbounded, complete and contains a countable dense subset.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, every nonempty open interval of **R** contains a rational number. Hence if S is a disjoint collection of open intervals, S is at most countable. ($f : S \to \mathbb{Q}$ is injective)

Let P be a dense linearly ordered set. If every disjoint collection of open intervals in P is at most countable, then we say that P satisfies the **countable chain condition**

Suslin's Problem*. Let P be a complete dense unbounded linearly ordered set that satisfies the countable chain condition. Is P isomorphic to the real line?*

This question cannot be decided in ZFC

3.4 The Topology of the Real Line

The real line is a metric space with the metric $d(a, b) = |a - b|$. Its metric topology coincides with the order topology of $(\mathbb{R}, <)$. Since $\mathbb{Q}$ is a dense set in $\mathbb{R}$ and since every Cauchy sequence of real numbers converges, $\mathbb{R}$ is a separable complete metric space. (A metric space is **separable** if it has a countable dense set; it is **complete** if every Cauchy sequence converges)

Every open sets is the union of open intervals with rational endpoints.[1]

Every open interval has cardinality $\mathfrak{c}$

A nonempty closed set is **perfect** if it has no isolated points.

Theorem 3.5. *Every perfect set has cardinality* $\mathfrak{c}$

Proof. Given a perfect set P, we want to find a one-to-one function $F : \{0, 1\}^\omega \hookrightarrow P$. Let S be the set of all fintie sequences of 0's and 1's. By induction on the length of $s \in S$ one can fined closed intervals I_s s.t. for each n and all $s \in S$ of length of n,

1. $I_s \cap P$
2. the diameter of I_s is $\leq 1/n$
3. $I_{s^\frown 0} \subset I_s, I_{s^\frown 1} \subset I_s$ and $I_{s^\frown 0} \cap I_{s^\frown 1} = \emptyset$

For each $f \in \{0, 1\}^\omega$, the set $P \cap \bigcap_{n=0}^{\infty} I_{f\restriction n}$ has exactly one element □

Theorem 3.6 (Cantor-Bendixson)**.** *If F is an uncountable closed set then $F = P \cup S$ where P is perfect and S is at most countable*

Corollary 3.7. *If F is a closed set, then either $|F| \leq \aleph_0$ or $|F| = 2^{\aleph_0}$*

[1]Check StackExchange

Proof. For every $A \subset \mathbb{R}$, let

$$A' = \text{the set of all limit points of } A$$

A' is closed, and if A is closed then $A' \subset A$. Thus we let

$$F_0 = F, \quad F_{\alpha+1} = F'_\alpha$$
$$F_\alpha = \bigcap_{\gamma<\alpha} F_\gamma \text{ if } \alpha > 0 \text{ is a limit ordinal}$$

Since $F_0 \supset F_1 \supset \cdots \supset F_\alpha \supset \ldots$, there exists an ordinal θ s.t. $F_\alpha = F_\theta$ for all $\alpha \geq \theta$. We let $P = F_\theta$

If P is nonempty, then $P' = P$ and so it's perfect. Thus the proof is completed by showing that $F - P$ is at most countable.

Let $\langle J_k : k \in \mathbb{N}\rangle$ be an enumeration of rational intervals. We have $F - P = \bigcup_{\alpha<\theta}(F_\alpha - F'_\alpha)$; hence if $a \in F - P$, then there is a unique α s.t. a is an isolated point of F_α. We let $k(a)$ be the least k s.t. a is the only opoint of F_α in the interval J_k. Note that if $\alpha \leq \beta$, $b \neq a$ and $b \in F_\beta - F'_\beta$, then $b \notin J_{k(a)}$. Thus the correspondence $a \mapsto k(a)$ is one-to-one ☐

A set of reals is called **nowhere dense** if its closure has empty interior.

Theorem 3.8 (The Baire Category Theorem)**.** *If $D_0, D_1, \ldots, D_n, \ldots, n \in \mathbb{N}$ are dense open sets of reals, then the intersection $D = \bigcap_{n=0}^\infty D_n$ is dense in $\mathbb{R}$*

3.5 Borel Sets

Definition 3.9. An **algebra of sets** is a collection of $\mathcal{S}$ of subsets of a given set S s.t.

1. $S \in \mathcal{S}$
2. if $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ then $X \cup Y \in \mathcal{S}$
3. if $X \in \mathcal{S}$ then $S - X \in \mathcal{S}$

A σ**-algebra** is additionally closed under countable unions (and intersections)

4. if $X_n \in \mathcal{S}$ for all n, then $\bigcup_{n=0}^\infty X_n \in \mathcal{S}$

Definition 3.10. A set of reals B is **Borel** if it belongs to the smallest σ-algebra $\mathcal{B}$ of sets of reals that contains all open sets.

4 The Axiom of Choice and Cardinal Arithmetic

Theorem 4.1 (Zermelo's Well-Ordering Theorem)**.** *Every set can be well-ordered*

Proof. Let A be a set. To well order A, it suffices to construct a transfinite one-to-one sequence $\langle a_\alpha : \alpha < \theta \rangle$ that enumerates A. That we can do by induction, using a choice function f for the family S of all nonempty subsets of A. We let for every α

$$a_\alpha = f(A - \{a_\xi : \xi < \alpha\})$$

if $A - \{a_\xi : \xi < \alpha\}$ is nonempty. Let θ be the least ordinal s.t. $A = \{a_\xi : \xi < \theta\}$. Clearly $\langle a_\alpha : \alpha < \theta \rangle$ enumerates A □

In fact, Zermelo's Theorem 4.1 is equivalent to the Axiom of Choice. For any family S of nonempty sets, well-order $\bigcup S$ and let $f(X)$ be the least element of X for every $X \in S$

Of particular importance is the fact that the set of all real numbers can be well-ordered. It follows that $2^{\aleph_0}$ is a cardinal and $2^{\aleph_0} \geq \aleph_1$

If every set can be well-ordered, then every infinite set has a countable subset: Well-order the set and take the first ω elements.

Dealing with cardinalities of sets is much easier when we have **AC**. In the first place, any two sets have comparable cardinals. Another consequence is

$$iff \text{ maps } A \text{ onto } B \text{ then } |B| \leq |A|$$

This is done by choosing one element from $f_{-1}(\{b\})$ for each $b \in B$

Another consequence of the **AC** is

The union of a countable family of countable sets is countable

Let A_n be a countable set for each $n \in \mathbb{N}$. For each n, let us *choose* an enumeration $\langle a_{n,k} : k \in \mathbb{N} \rangle$ of A_n. This gives us a projection of $\mathbb{N} \times \mathbb{N}$ onto $\bigcup_{n=0}^{\infty} A_n$

Lemma 4.2. $\left|\bigcup S\right| \leq |S| \cdot \sup\{|X| : X \in S\}$

Proof. Let $\kappa = |S|$ and $\lambda = \sup\{|X| : X \in S\}$. We have $S = \{X_\alpha : \alpha < \kappa\}$ and for each $\alpha < \kappa$ we choose an enumeration $X_\alpha = \{a_{\alpha,\beta} : \beta < \lambda_\alpha\}$ where $\lambda_\alpha < \lambda$. Again we have a projection

$$(\alpha, \beta) \mapsto a_{\alpha,\beta}$$

of $\kappa \times \lambda$ onto $\bigcup S$ □

Corollary 4.3. *Every $\aleph_{\alpha+1}$ is a regular cardinal*

Proof. This is because otherwise $\omega_{\alpha+1}$ would be the union of at most $\aleph_\alpha$ sets of cardinality at most $\aleph_\alpha$.

Suppose $\omega_{\alpha+1}$ is not regular and hence $\mathrm{cf}(\omega_{\alpha+1}) \leq \omega_\alpha$. Consider a cofinal increasing ω_α-sequence $\langle \beta_\xi : \xi < \omega_\alpha \rangle$ s.t. $\lim_{\xi \to \omega_\alpha} \beta_\xi = \omega_{\alpha+1}$.

Note that $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ □

4.1 Using the Axiom of Choice in Mathematics

Theorem 4.4 (Zorn's Lemma)**.** *If $(P, <)$ is a nonempty partially ordered set s.t. every chain in P has an upper bound, then P has a maximal element*

Proof. We let by induction

$$a_\alpha = \text{ an element of } P \text{ s.t. } a_\alpha > a_\xi \text{ for every } \xi < \alpha \text{ if there is one}$$

If $\alpha > 0$ is a limit ordinal, then $C_\alpha = \{a_\xi : \xi < \alpha\}$ is a chain in P and a_α exists by the assumption. Eventually, there is a θ s.t. there is no $a_{\theta+1} \in P, a_{\theta+1} > a_\theta$. Thus a_θ is a maximal element of P □

4.2 The Countable Axiom of Choice

The Countable Axiom of Choice. *Every countable family of nonempty sets has a choice function*

Proposition 4.5. *Countable **AC** implies that the union of countably many countable sets is countable*

Proof. Well order the countable sets. □

Proposition 4.6. *Countable **AC** implies $\aleph_1$ is regular*

The Principle of Dependent Choices (DC). *If E is a binary relation on a nonempty set A, and if for every $a \in A$ there exists $b \in A$ s.t. bEa, thenthere is a sequence $a_0, \dots, a_n, \dots$ in A s.t.*

$$a_{n+1} E a_n \quad \text{for all } n \in \mathbb{N}$$

Lemma 4.7. 1. *A linear ordering $<$ of a set P is a well-ordering of P if and only if there is no infinite descending sequence*

$$a_0 > a_1 > \cdots > a_n > \dots$$

in A

2. *A relation E on P is well-founded if and only if there is no infinite sequence $\langle a_n : n \in \mathbb{N}\rangle$ in P s.t.*

$$a_{n+1} E a_n \quad \textit{for all } n \in \mathbb{N}$$

Proof. 1 is a special case of 2. □

4.3 Cardinal Arithmetic

In the presence of the **AC**, every set can be well-ordered and so every infinite set has the cardinality of some $\aleph_\alpha$.

Lemma 4.8. *If $2 \le \kappa \le \lambda$ and λ is infinite, then $\kappa^\lambda = 2^\lambda$*

Proof.

$$2^\lambda \le \kappa^\lambda \le (2^\kappa)^\lambda = 2^{\lambda \cdot \kappa} = 2^\lambda$$

□

If κ and λ are infinite and $\lambda < \kappa$, first if $2^\lambda > \kappa$, then we have $\kappa^\lambda = 2^\lambda$ but if $2^\lambda < \kappa$ (because $\kappa^\lambda \le \kappa^\kappa = 2^\kappa$) we can only conclude

$$\kappa \le \kappa^\lambda \le 2^\kappa$$

If λ is a cardinal and $|A| \ge \lambda$ let

$$[A]^\lambda = \{X \subseteq A : |X| = \lambda\}$$

Lemma 4.9. *If $|A| = \kappa \ge \lambda$, then the set $[A]^\lambda$ has cardinality κ^λ*

Proof. On the one hand, every $f : \lambda \to A$ is a subset of $\lambda \times A$, and $|f| = \lambda$. Thus $\kappa^\lambda \le \left|[\lambda \times A]^\lambda\right| = \left|[A]^\lambda\right|$. □

4.4 Infinite Sums and Products

Let $\kappa_i : i \in I$ be an indexed set of cardinal numbers. We define

$$\sum_{i \in I} \kappa_i = \left|\bigcup_{i \in I} X_i\right|$$

where $\{X_i : i \in I\}$ is a disjoint family of sets s.t. $|X_i| = \kappa_i$ for each $i \in I$

Note that if κ and λ are cardinals and $\kappa_i = \kappa$ for each $i < \lambda$, then

$$\sum_{i < \lambda} \kappa_i = \lambda \cdot \kappa$$

Lemma 4.10. *If λ is an infinite cardinal and $\kappa_i > 0$ for each $i < \lambda$, then*

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i$$

Proof. Let $\kappa = \sup_{i<\lambda} \kappa_i$ and $\sigma = \sum_{i<\lambda} \kappa_i$. On the one hand, since $\kappa_i \leq \kappa$ for all i, we have $\sum_{i<\lambda} \kappa \leq \lambda \cdot \kappa$. On the other hand, since $\kappa_i \geq 1$ for all i, we have $\lambda = \sum_{i<\lambda} 1 \leq \sigma$ and since $\sigma \geq \kappa_i$ for all i, we have $\sigma \geq \sup_{i<\lambda} \kappa_i = \kappa$. Therefore $\sigma \geq \lambda \cdot \kappa$ □

In particular, if $\lambda \leq \sup_{i<\lambda} \kappa_i$, we have

$$\sum_{i<\lambda} \kappa_i = \sup_{i<\lambda} \kappa_i$$

Thus we can characterize singular cardinals as follows: An infinite cardinal κ is singular just in case

$$\kappa = \sum_{i<\lambda} \kappa_i$$

where $\lambda < \kappa$ and for each i, $\kappa_i < \kappa$

An infinite product of cardinals is defined using infinite products of sets. If $\{X_i : i \in I\}$ is a family of sets, then the **product** is defined as

$$\prod_{i \in I} X_i = \{f : f \text{ is a function on } I \text{ and } f(i) \in X_i \text{ for each } i \in I\}$$