Numerical Analysis

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1 Chap1 Mathematical Preliminaries

1.1 1.2 Roundoff Errors and Computer Arithmetic

Truncation Error : the error involved in using a truncated, or finite, summation to approximate the sum of an infinite series

Roundoff Error: the error produced when performing real number calculations. It occurs because the arithmetic performed in a machine involves numbers with only a finite number of digits.

Suppose $y = 0.d_1 d_2 \dots d_k d_{k+1} d_{k+2} \dots \times 10^n$, then

$$fl(y) = \begin{cases} 0.d_1 d_2 \dots d_k \times 10^n & \text{chopping} \\ chop(y + 5 \times 10^{n-(k+1)}) = 0.\delta_1 \delta_2 \dots \delta_k \times 10^n & \text{Rounding} \end{cases}$$

Definition 1.1. *If $p*$ is an approximation to p, the absolute error is $|p - p*|$, and the relative error is $\frac{|p-p*|}{|p|}$, provided that $p \neq 0$*

Definition 1.2. *The number $p*$ is said to approximate p to t significant digits if t is the largest nonnegative integer for which* $\frac{|p-p*|}{|p|} < 5 \times 10^{-t}$

chopping $|\frac{y-fl(y)}{y}| = |\frac{0.d_1d_2...d_kd_{k+1}\cdots\times 10^n - 0.d_1d_2...d_k\times 10^n}{0.d_1d_2...d_kd_{k+1}\times 10^n}| = |\frac{0.d_{k+1}\cdots}{0.d_1d_2\dots}|\times 10^{-k} \leqslant \frac{1}{0.1}\times 10^{-k} = 10^{-k+1}$

rounding $|\frac{y-fl(y)}{y}| \leqslant \frac{0.5}{0.1}\times 10^{-k} = 0.5\times 10^{-k+1}$

Finite digit arithmetic

- $x \oplus y = fl(fl(x) + fl(y))$
- $x \otimes y = fl(fl(x) \times fl(y))$
- $x \ominus y = fl(fl(x) - fl(y))$
- $x \oplus y = fl(fl(x) \div fl(y))$

1.2 1.3 ALgorithms and Convergence

An algorithm that satisfies that small changes in the initial data produce correspondingly small changes in the final results is called **stable**; otherwise it is **unstable**. An algorithm is called **conditionally stable** if it is stable only for certain choices of initial data.

Suppose that E >0 denotes an initial error and En represents the magnitude of an error after n subsequent operations. If $E_n \approx CnE_0$, where C is a constant independent of n, then the growth of error is said to be **linear**. If $E_n \approx C^nE_0$, for some C >1, then the growth of error is called **exponential**

Suppose $\{\beta_n\}_{n=1}^{\infty}$, $\lim\limits_{n\to\infty}\beta_n = 0$, $\{\alpha_n\}_{n=1}^{\infty}$, $\lim\limits_{n\to\infty}\alpha_n = \alpha$. If a positive constant K exists with $|\alpha_n - \alpha| \leqslant K|\beta_n|$ for large n, then $\{\alpha_n\}_{n=1}^{\infty}$ converges to with **rate, or order, of convergence** $O(\beta_n)$

Suppose $\lim\limits_{h\to 0} G(h) = 0$, $\lim\limits_{h\to 0} F(h) = L$ and $|F(h) - L| \leqslant K|G(h)|$ for sufficiently small h, then we write $F(h) = L + O(G(h))$

2 Chap2 Solutions of equations in one variable

2.1 2.1 Bisection method

Theorem 2.1. *Intermediate Value Theorem If $f \in C[a,b]$, $K \in (f(a), f(b))$, then there exists a number $p \in (a,b)$ for which $f(p) = K$*

Theorem 2.2. *Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The bisection method generates a sequence $\{p_n\}, n = 0, 1, \dots$ approximating a zero p of f with*

$$|p_n - p| \leqslant \frac{b-a}{2^n}, \quad \textit{when } n \geqslant 1$$

2.2 2.2 Fixed-Point Iteration

$f(x) = 0 \xleftrightarrow{\text{equivalent}} x = f(x) + x = g(x)$

Theorem 2.3. *Fixed-Point Theorem Let $g \in C[a,b]$ be s.t. $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose that g' exists on (a,b) and that a constant $0 < k < 1$ exists with $|g'(x)| \leqslant k$ for all $x \in (a,b)$ (hence g' can't converge to 1). Then for any number p_0 in $[a,b]$, the sequence defined by $p_n = g(p_{n-1}), n \geqslant 1$ converges to the unique point p in $[a,b]$*

Corollary 2.1. $|p_n - p| \leqslant \frac{1}{1-k}|p_{n+1} - p_n|$ *and* $|p_n - p| \leqslant \frac{k^n}{1-k}|p_1 - p_0|$

2.3 2.3 Newton's method

Linearize a nonlinear function using **Taylor's expansion**

Let $p_0 \in [a,b]$ be an approximation to p s.t. $f'(p_0) \neq 0$, hence $f(x) = f(p_0) + f'(p_0)(x - p_0) + \frac{f''(\xi_x)}{2!}(x-p_0)^2$, then $0 = f(p) \approx f(p_0) + f'(p_0)(p - p0) \to p \approx p_0 - \frac{f(p_0)}{f'(p_0)}$ $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for} n \geqslant 1$

Theorem 2.4. *Let $f \in C^2[a,b]$. If $p \in [a,b]$ is s.t. $f(p) = 0, f'(p) \neq 0$, then there exists a $\delta > 0$ s.t. Newton's method generates a sequence $\{p_n\}, n \in \mathbb{N}\backslash\{0\}$ converging to p for any initial approximation $p \in [p - \delta, p + \delta]$.*

2.4 2.4 Error analysis for iterative methods

Definition 2.1. *Suppose $\{p_n\}(n = 0, 1, \dots)$ is a sequence that converges to p with $p_n \neq p$ for all n. If positive constants α and λ exist with*

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$$

then $\{p_n\}(n = 0, 1, \dots)$ converges to p of order α, with asymptotic error constant λ

Theorem 2.5. *Let p be a fixed point of $g(x)$. If there exists some constant $\alpha \geqslant 2$ s.t. $g \in C^\alpha[p - \delta, p + \delta]$, $g'(p) = \cdots = g^{\alpha-1}(p) = 0$ and $g^\alpha(p) \neq 0$. Then the iterations with $p_n = g(p_{n-1})$, $n \geqslant 1$ is of order α*

$$p_{n+1} = g(p_n) = g(p) + g'(p)(p_n - p) + \cdots + \frac{g^{\alpha}(\xi_n)}{\alpha!}(p_n - p)^{\alpha}$$

Theorem 2.6. *Let $g \in C[a,b]$ be s.t. $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose in addition that g' is continuous on (a,b) and a positive constant $k < 1$ exists with*

$$|g'(x)| \leqslant k, \quad \text{for all } x \in (a,b)$$

If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a,b]$, the sequence

$$p_n = g(p_{n-1}), \quad \text{for } n \geqslant 1$$

converges only linearly to the unique fixed point in $[a,b]$

Proof.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} &= \lim_{n\to\infty} \frac{|g(p_n) - p|}{|p_n - p|} \\
&= \lim_{n\to\infty} \frac{|g'(\xi)(p_n - p)|}{|p_n - p|} \\
&= |g'(p)|
\end{aligned}$$

□

Theorem 2.7. *Let p be a solution of the equation $x = g(x)$. Suppose that $g'(p) = 0$ and g" is continuous with $|g''(x)| < M$ on an open interval I containing p. Then there exists a $\delta > 0$ s.t. for $p_0 \in [p-\delta, p+\delta]$, the sequence defined by $p_n = g(p_{n-1})$, when $n \geqslant 1$ converges at least quadratically to p. Moreover, for sufficiently large values of n,*

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2$$

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