Notes on Set Theory

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1 Ordinal

1.1 Linear and partial ordering

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Definition 1.1. A binary relation $<$ on a set P is a **partial ordering** of P if:

1. $p \not< p$ for any $p \in P$
2. if $p < q$ and $q < r$ then $p < r$
   $(P, <)$ is called a **partial ordered set**. A partial ordering $<$ of P is a **linear ordering** if moreover
3. $p < q$ or $q < p$ or $p = q$ for all $p, q \in P$

If $(P, <)$ and $(Q, <)$ are poset and $f : P \to Q$, then f is **order-preserving** if $x < y$ implies $f(x) < f(y)$. If P and Q are linearly ordered, then f is also called **increasing**

1.2 Well-Ordering

Definition 1.2. A linear ordering $<$ of a set P is a **well-ordering** if every nonempty subset of P has a least element

Lemma 1.3. *If $(W, <)$ is a well-ordering set and $f : W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$*

Proof. Assume that the set $X = \{x \in W \mid f(x) < x\}$ is nonempty and let z be the least element of X. Hence $f(f(x)) < f(x)$ and $f(x) \in X$, a contradiction. □

Corollary 1.4. *The only automorphism of a well-ordered set is the identity*

Corollary 1.5. *If two well-ordered sets W_1, W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique*

If W is a well-ordered set and $u \in W$, then $\{x \in W : x < u\}$ is an **initial segment** of W

Lemma 1.6. *No well-ordered set is isomorphic to an initial segment of itself*

Proof. If $\mathrm{ran}(f) = \{x : x < u\}$, then $f(u) < u$, contrary to lemma 1.3 □

Theorem 1.7. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds:*

1. $W_1 \cong W_2$

2. *W_1 is isomorphic to an initial segment of W_2*
3. *W_2 is isomorphic to an initial segment of W_1*

Proof. For $u \in W_i, (i = 1, 2)$, let $W_i(u)$ denote the initial segment of W_i given by u. Let

$$f = \{(x, y) \in W_1 \times W_2 \mid W_1(x) \cong W_2(y)\}$$

If $W_1(x) \cong W_w(y)$ and $W_1(x) \cong W_2(y')$, then $W_2(y) \cong W_1(y')$. According to lemma 1.6, $y = y'$. Hence it's easy to see that f is a one-to-one function.

If h is an isomorphism between $W_1(x)$ and $W_2(y)$ and $x' < x$, then $W_1(x') \cong W_2(h(x'))$. It follows that f is order-preserving.

If $\mathrm{dom}(f) = W_1$ and $\mathrm{ran}(f) = W_2$, then case 1 holds.

If $y_1 < y_2$ and $y_2 \in \mathrm{ran}(f)$, then $y_1 \in \mathrm{ran}(f)$. If there is some $y < y_2$ and $y \notin \mathrm{ran}(f)$. Consider the least element y' of $\{y \in W_2 \mid y < y_2 \wedge y \notin \mathrm{ran}(f)\}$. Let $x' = \sup\{x \in W_1 \mid \exists y \in W_2(W_1(x) \cong W_2(y) \wedge y < y')\}$, then $W_1(x') \cong W_2(y')$, a contradiction.

If $\mathrm{ran}(f)\neg W_2$ and y_0 is the least element of $W_2 - \mathrm{ran}(f)$. We have $\mathrm{ran}(f) = W_2(x_0)$. Necessarily, $\mathrm{dom}(f) = W_1$, for otherwise we could have $(x_0, y_0) \in f$ where x_0 =least element of $W_1 - \mathrm{dom}(f)$. Thus case 2 holds.

Similarly, case 3 holds. □

If $W_1 \cong W_2$, we say that they have the same **order-type**

1.3 Ordinal Numbers

The idea is to define ordinal numbers so that

$$\alpha < \beta \Leftrightarrow \alpha \in \beta \wedge \alpha = \{\beta : \beta < \alpha\}$$

Definition 1.8. A set T is **transitive** if every element of T is a subset of T

Definition 1.9. A set is an **ordinal number** (an **ordinal**) if it's transitive and well-ordered by $\in$

The class of all ordinals is denoted by Ord

We define

$$\alpha < \beta \Leftrightarrow \alpha \in \beta$$

Lemma 1.10. 1. *$0 = \emptyset$ is an ordinal*

2. *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal*
3. *If $\alpha \neq \beta$ are ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$*
4. *If α,β are ordinals, then either $\alpha \subset \beta$ or $\beta \subset \alpha$*

Proof.
1. definition
2. definition
3. If $\alpha \subset \beta$, let γ be the least element of the set $\beta - \alpha$. Since α is transitive, it follows that α is the initial segment of β given by γ. Thus $\alpha = \{\xi \in \beta \mid \xi < \gamma\} = \gamma \in \beta$
4. Clearly $\alpha \cap \beta$ is an ordinal γ. We have $\gamma = \alpha$ or $\gamma = \beta$, for otherwise $\gamma \in \alpha$ and $\gamma \in \beta$ by 3. Then $\gamma \in \gamma$ which contradicts the definition of an ordinal

$\square$

Using lemma 1.10 one gets the following facts about ordinal numbers

1. $<$ is a linear ordering of the class Ord
2. For each α, $\alpha = \{\beta : \beta < \alpha\}$
3. If C is a nonempty class of ordinals, then $\bigcap C$ is an ordinal, $\bigcap C \in C$ and $\bigcap C = \inf C$
4. If X is a nonempty set of ordinals, then $\bigcup X$ is an ordinal and $\bigcup X = \sup X$
5. For every α, $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = \inf\{\beta : \beta > \alpha\}$

We thus define $\alpha + 1 = \alpha \cup \{\alpha\}$(the **succesor** of α)

Theorem 1.11. *Every well-ordered set is isomorphic to a unique ordinal number*

Proof. The uniqueness follows from lemma 1.6. Given a well-ordered set W, we find an isomorphic ordinal as follows: Define $F(x) = \alpha$ if α is isomorphic to the initial segment of W given by x. If such an α exists, then it's unique. By the replacement axiom, $F(W)$ is a set. For each $x \in W$, such an α exists. Otherwise consider the least x such that α doesn't exist. Let $\alpha = \sup\{F(x') \mid x' \in W \wedge x' < x\}$ and $F(x) = \alpha$. If γ is the least $\gamma \notin F(W)$, then $F(W) = \gamma$ and we have an isomorphism of W onto γ $\square$

If $\alpha = \beta + 1$, then α is a **succesor ordinal**. If α is not a succesor ordinal then $\alpha = \sup\{\beta : \beta < \alpha\} = \bigcup \alpha$ is called a **limit ordinal**. We also consider 0 a limit ordinal and define $\sup \emptyset = 0$.

1.4 Induction and Recursion

Theorem 1.12 (Transfinite Induction)**.** *Let C be a class of ordinals and assume*

1. $0 \in C$
2. *if $\alpha \in C$, then $\alpha + 1 \in C$*
3. *if α is a nonzero limit ordinal and $\beta \in C$ for all $\beta < \alpha$, then $\alpha \in C$*

Then C is the class of all ordinals

Proof. Otherwise let α be the least ordinal $\alpha \notin C$ and apply 1, 2 or 3 $\square$

A function whose domain is the set $\mathbb{N}$ is called an **{(infinite) sequence}** (A **sequence** in X is a function $f : \mathbb{N} \to X$). The standard notation for a sequence is

$$\langle a_n : n < \omega \rangle$$

A **finite sequence** is a function s s.t. $\operatorname{dom}(s) = \{i : i < n\}$ for some $n \in \mathbb{N}$; then s is a **sequence of length** n

A **transfinite sequence** is a function whose domain is an ordinal

$$\langle a_\xi : \xi < \alpha \rangle$$

It is also called an α-**sequence** or a **sequence of length** α. We also say that a sequence $\langle a_\xi : \xi < alpha \rangle$ is an **enumeration** of its range $\{a_\xi : \xi < \alpha\}$. If s is a sequence of length α, then $s^\frown x$ or simply sx denotes the sequence of length $\alpha + 1$ that extends s and whose αth term is x:

$$s^\frown x = sx = s \cap \{(\alpha, x)\}$$

Theorem 1.13 (Transfinite Recursion)**.** *Let G be a function, then 1 below defines a unique function F on Ord s.t.*

$$F(\alpha) = G(F \restriction \alpha)$$

for each α

In other words, if we let $a_\alpha = F(\alpha)$, then for each α

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$$

Corollary 1.14. *Let X be a set and θ be an ordinal number. For every function G on the set of all transfinite sequences in X of length $< \theta$ s.t. $\operatorname{ran}(G) \subset X$ there exists a unique θ-sequence in X s.t. $a_\alpha = G(\langle a_\xi : \xi < \theta)$ for every $\alpha < \theta$*

Proof. Let

$$\begin{aligned} F(\alpha) = x \leftrightarrow \text{there is a sequence } & \langle a_\xi : \xi < \alpha \rangle \text{ such that} \\ & 1.\ (\forall \xi < \alpha) a_\xi = G(\langle a_n \eta : \eta < \xi \rangle) \\ & 2.\ x = G(\langle a_\xi : \xi < \alpha \rangle) \end{aligned} \tag{1}$$

For every α, if there is an α-sequence that satisfying 1, then such a sequence is unique. Thus $F(\alpha)$ is determined uniquely by 2 and therefore F is a function. $\square$

Definition 1.15. Let $\alpha > 0$ be a limit ordinal and let $\langle \gamma_\xi : \xi < \alpha \rangle$ be a **nondecreasing** sequence of ordinals (i.e., $\xi < \eta$ implies $\gamma_\xi \leq \gamma_e ta$). We define the **limit** of the sequence by

$$\lim_{\xi \to \alpha} \gamma_\xi = \sup\{\gamma_\xi : \xi < \alpha\}$$

A sequence of ordinals $\langle \gamma_\alpha : \alpha \in Ord \rangle$ is **normal** if it's increasing and **continuous**, i.e., for every limit α, $\gamma_\alpha = \lim_{\xi \to \alpha} \gamma_\xi$

1.5 Ordinal Arithmetic

Definition 1.16 (Addition)**.** For all ordinal numbers α

1. $\alpha + 0 = \alpha$
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, for all β
3. $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$ for all limit $\beta > 0$

Definition 1.17 (Multiplication)**.** For all ordinal numbers α

1. $\alpha \cdot 0 = 0$
2. $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$, for all β
3. $\alpha \cdot \beta = \lim_{\xi \to \beta}(\alpha \cdot \xi)$ for all limit $\beta > 0$

Definition 1.18 (Exponentiation)**.** For all ordinal numbers α

1. $\alpha^0 = 1$
2. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$, for all β
3. $\alpha^\beta = \lim_{\xi \to \beta} \alpha^\xi$ for all limit $\beta > 0$

Lemma 1.19. *For all ordinals α, β and γ*

1. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
2. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

Neither $+$ nor $\cdot$ are commutative

$$1 + \omega = \omega \neq \omega + 1, \;\; 2 \cdot \omega = \omega \neq \omega \cdot 2$$

Definition 1.20. Let $(A, <_A)$ and $(B, <_B)$ be disjoint linearly ordered sets. The **sum** of these linear orders is the set $A \cup B$ with the ordering defined as follows: $x < y$ if and only if

1. $x, y \in A$ and $x <_A y$
2. $x, y \in B$ and $x <_B y$
3. $x \in A$ and $y \in B$

Definition 1.21. Let $(A,<)$ and $(B,<)$ be linearly ordered sets. The **product** of these linear orders is the set $A \times B$ with the ordering defined by

$$(a_1,b_1) < (a_2,b_2) \Leftrightarrow b_1 < b_2 \text{ or } (b_1 = b_2 \wedge a_1 < a_2)$$

Lemma 1.22. *For all ordinals α and β, $\alpha+\beta$ and $\alpha\cdot\beta$ are respectively isomorphic to the sum and to the product of α and β*

Proof. Suppose $(A,<_A) \cong \alpha$ and $(B,<_B) \cong \beta$.

1. if $\beta = 0$, then $B = \emptyset$, $A \cup B = A$
2. if $(A\cup B, <_{A\cup B}) \cong \alpha+\beta$, let $B' = B \cup \{c\}$ s.t. $\{c\} \cap A = \{c\} \cap B = \emptyset$ all for all $b \in B$, $b < c$. Hence

$$\alpha + (\beta+1) = (\alpha+\beta)+1 \cong (A\cup B)\cup\{c\} = A \cup B'$$

3. if β is a limit ordinal and for all $\xi < \beta$ and $B_\xi \cong \xi$, $(A \cup B_\xi, <_{A\cup B_\xi}) \cong \alpha+\xi$,

$$A \cup B = A \cup \sup B_\xi = \sup(A \cup B_\xi) \cong \sup(\alpha+\xi) = \alpha+\beta$$

□

Lemma 1.23. *1. If $\beta < \gamma$ then $\alpha+\beta < \alpha+\gamma$*

2. *If $\alpha < \beta$ then there exists a unique δ s.t. $\alpha+\delta = \beta$*
3. *If $\beta < \gamma$ and $\alpha > 0$, then $\alpha\cdot\beta < \alpha\cdot\gamma$*
4. *If $\alpha > 0$ and γ is arbitrary, then there exist a unique β and a unique $\rho < \alpha$ s.t. $\gamma = \alpha\cdot\beta+\rho$*
5. *If $\beta < \gamma$ and $\alpha > 1$, then $\alpha^\beta < \alpha^\gamma$*

Proof. 2. Let δ be the order-type of the set $\{\xi : \alpha \leq \xi < \beta\}$

4. Let β be the greatest ordinal s.t. $\alpha\cdot\beta \leq \gamma$

□

Theorem 1.24 (Cantor's Normal Form Theorem)**.** *Every ordinal $\alpha > 0$ can be represented uniquely in the form*

$$\alpha = \omega^{\beta_1}\cdot k_1 + \cdots + \omega^{\beta_n}\cdot k_n$$

where $n \geq 1$, $\alpha \geq \beta_1 > \cdots > \beta_n$ and $k_1,\dots,k_n$ are nonzero natural numbers.

Proof. By induction on α. For $\alpha = 1$ we have $1 = \omega^0+1$; for arbitrary $\alpha > 0$, let β be the greatest ordinal s.t. $\omega^\beta \leq \alpha$. The uniqueness of the normal form is proved by induction □

1.6 Well-Founded Relations

A binary relation E on a set P is **well-founded** if every nonempty $X \subset P$ has an E-**minimal** element.

Given a well-founded relation E on a set P, we can define the **height** of E and assign to each $x \in P$ and ordinal number, the **rank** of x in E

Theorem 1.25. *If E is a well-founded relation on P, then there exists a unique function ρ from P into the ordinals s.t. for all $x \in P$*

$$\rho(x) = \sup\{\rho(y) + 1 : yEx\}$$

The range of ρ is an initial segment of the ordinals, thus an ordinal number. This ordinal is called the **height** of E

Proof. By induction, let

$$\begin{aligned} &P_0 = \emptyset \\ &P_{\alpha+1} = \{x \in P : \forall y(yEx \to y \in P_\alpha)\} \cup P_\alpha \\ &P_\alpha = \bigcup_{\xi<\alpha} P_\xi \text{ if } \alpha \text{ is a limit ordinal} \end{aligned}$$

Let θ be the least ordinal s.t. $P_{\theta+1} = P_\theta$. We claim that $P_\theta = P$ □

1.7 Exercise

1. Every normal sequence $\langle \gamma_\alpha : \alpha \in Ord\rangle$ has arbitrarily large **fixed points**, i.e., α s.t. $\gamma_\alpha = \alpha$

   Proof. From StackExchange. □

A limit ordinal $\gamma > 0$ is called **indecomposable** if there exist no $\alpha < \gamma$ and $beta < \gamma$ s.t. $\alpha + \beta = \gamma$

2. A limit ordinal $\gamma > 0$ is indecomposable if and only if $\alpha + \gamma = \gamma$ for all $\alpha < \gamma$ if and only if $\gamma = \omega^\alpha$ for some α

   Proof. (a) $(3) \to (1)$. Assume $\gamma_1, \gamma_2 < \gamma = \omega^\alpha$. By Cantor's normal form theorem, there exist α' and k s.t. $\gamma_1, \gamma_2 < \omega^{\alpha'} \cdot k$

   (b) $(2) \to (3)$. Assume that γ can't be written as ω^α. Then by Cantor's theorem, $\gamma = \omega^{\beta_1} \cdot k_1 + \cdots + \omega^{\beta_n} \cdot k_n$. But then $\omega^{\beta_1} < \gamma$ and $\omega^{\beta_1} + \gamma > \gamma$

   □

2 Models of Set - Sertraline

2.1 Some mathematical logic

Theorem 2.1 (Gödel's second incompleteness theorem)**.** *If a consistent recursive axiom set T contains* ***ZFC*** *, then*

$$T \nvdash Con(t)$$

especially, ***ZFC*** $\nvdash$ *Con(****ZFC*** *)*

Definition 2.2. Suppose (M, E_M) and (N, E_N) are two models of set theory, then

1. if for any formula σ, $M \models \sigma$ if and only if $N \models \sigma$, then M and N are **elementary equivalent**, denoted by $M \equiv N$
2. If bijection $f : M \to N$ satisfies: for any $a, b \in M$, $aE_M b$ iff $f(a)E_N f(b)$, then $f : M \cong N$ is an **isomorphism**
3. If $M \subseteq N$ and $E_M = E_N \restriction M$, then M is N's submodel
4. If M is isomorphic to a submodel of N by injection f, and for any formula $\varphi(x_1, \ldots, x_n)$, for any $a_1, \ldots, a_n \in M$, $M \models \varphi[a_1, \ldots, a_n]$ iff $N \models \varphi[f(a_1), \ldots, f(a_n)]$, then f is called an **elementary embedding** from M to N, written as $f : M \prec N$
5. If $M \subseteq N$ and $M \prec N$, then M is a **elementary submodel** of N

Lemma 2.3. *Suppose $N \models$* ***ZFC*** *, $M \subseteq N$, then $M \prec N$ iff $\forall \varphi(x, x_1, \ldots, x_n)$, $\forall(a_1, \ldots, a_n) \in M$, if $\exists a \in N$ s.t. $N \models \varphi[a, a_1, \ldots, a_n]$, then $\exists a \in M$ s.t. $M \models \varphi[a, a_1, \ldots, a_n]$*

Definition 2.4. Suppose $(M, E) \models$ **ZFC**

1. $h_\varphi : M^n \to M$ is φ's **Skolem function** if $\forall a_1, \ldots, a_n \in M$, if $\exists a \in M$ s.t. $M \models \varphi[a, a_1, \ldots, a_n]$, then $M \models \varphi[h_\varphi(a_1, \ldots, a_n), a_1, \ldots, a_n]$. requires **AC**
2. Let $\mathcal{H} = \{h_\varphi \mid \varphi \text{is a formula on set theory}\}$. For any $S \subseteq M$, **Skolem hull** $\mathcal{H}(S)$ is the smallest set consisting of S and closed under $\mathcal{H}$

Lemma 2.5. *$N \models$* ***ZFC*** *, $S \subseteq N$, if $M = \mathcal{H}(S)$, then $M \prec N$*

Theorem 2.6 (Löwenheim-Skolem theorem)**.** *Suppose $N \models$* ***ZFC*** *and is infinite, then there is a model M s.t. $|M| = \omega$ and $M \prec N$*

2.2 Cumulative Hierarchy

This section works in $\mathbf{ZF}^-$(a.k.a. **ZF** $-$ axiom of foundation)

Definition 2.7. For any α, define sequence V_α

1. $V_0 = \emptyset$
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
3. For any limit ordinal λ, $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$

And $\mathbf{WF} = \bigcup_{\alpha \in \mathbf{On}} V_\alpha$

Lemma 2.8. *For any ordinal α*

1. *V_α is transitive*
2. *if $\xi \leq \alpha$, then $V_\xi \subseteq V_\alpha$*
3. *if κ is inaccessible cardinal, then $|V_\kappa| = \kappa$*

Proof. 1. Obviously $\kappa \leq V_\kappa$. Since κ is inaccessible, then for any $\alpha < \kappa$, $|V_\alpha| < \kappa$.

□

Definition 2.9. For any set $x \in \mathbf{WF}$,

$$\mathrm{rank}(x) = \min\{\beta \mid x \in V_{\beta+1}\}$$

Lemma 2.10.

1. *$V_\alpha = \{x \in \mathbf{WF} \mid \mathit{rank}(x) < \alpha\}$*
2. ***WF** is transitive*
3. *For any $x, y \in \mathbf{WF}$, if $x \in y$, then $\mathit{rank}(x) < \mathit{rank}(y)$*
4. *for any $y \in \mathbf{WF}$, $\mathit{rank}(y) = \sup\{\mathit{rank}(x) + 1 \mid x \in y\}$*

Lemma 2.11. *Supoose α is an ordinal*

1. *$\alpha \in \mathbf{WF}$ and $\mathit{rank}(\alpha) = \alpha$*
2. *$V_\alpha \cap \mathbf{On} = \alpha$*

Lemma 2.12.

1. *If $x \in \mathbf{WF}$, then $\bigcup x, \mathcal{P}(x), \{x\} \in \mathbf{WF}$, and their ranks are all less than $\mathit{rank}(x) + \omega$*
2. *If $x, y \in \mathbf{WF}$, then $x \times y, x \cup y, x \cap y, \{x, y\}, (x, y), x^y \in \mathbf{WF}$, and their ranks are all less than $\mathit{rank}(x) + \mathit{rank}(y) + \omega$*
3. *$\mathbb{Z}, \mathbb{Q}, \mathbb{R} \in V_{\omega+\omega}$*
4. *for any set x, $x \in \mathbf{WF}$ iff $x \subset \mathbf{WF}$*

Lemma 2.13. *Suppose **AC***

1. *for any group G, there exists group $G' \cong G$ in **WF***
2. *for any topological space T, there exists $T' \cong T$ in **WF***

Definition 2.14. Binary relation $<$ on set A is **well-founded** if for any nonempty $X \subseteq A$, X has minimal element under $<$

Theorem 2.15. *If $A \in$ **WF** , then $\in$ is a well-founded relation on A*

Lemma 2.16. *If set A is transitive and $\in$ is well-founded on A, then $A \in$ **WF***

Lemma 2.17. *For any set x, there is a smallest transitive set $\operatorname{trcl}(x)$ s.t. $x \subseteq \operatorname{trcl}(x)$*

Proof.

$$\begin{aligned} x_0 &= x \\ x_{n+1} &= \bigcup x_n \\ \operatorname{trcl}(x) &= \bigcup_{n<\omega} x_n \end{aligned}$$

□

$\operatorname{trcl}(x)$ is called **transitive closure** of x

Lemma 2.18. *Without axiom of power set*

1. *if x is transitive, then $\operatorname{trcl}(x) = x$*
2. *if $y \in x$, then $\operatorname{trcl}(y) \subseteq \operatorname{trcl}(x)$*
3. $\operatorname{trcl}(x) = x \cup \bigcup\{\operatorname{trcl}(y) \mid y \in x\}$

Theorem 2.19. *For any set X, the following are equivalent*

1. $X \in$ ***WF***
2. $\operatorname{trcl}(X) \in$ ***WF***
3. *$\in$ is a well-founded relation on $\operatorname{trcl}(X)$*

Theorem 2.20. *The following propositions are equivalent*

1. *Axiom of foundation*
2. *For any set X, $\in$ is a well-founded relation on X*
3. ***V*** $=$ ***WF***

2.3 Relativization

Definition 2.21. Let **M** be a class φ a formula, the **relativization** of φ to **M** is $\varphi^{\mathbf{M}}$ defined inductively

$$\begin{aligned} (x \in y)^{\mathbf{M}} &\leftrightarrow x = y \\ (x \in y)^{\mathbf{M}} &\leftrightarrow x \in y \\ (\varphi \to \psi)^{\mathbf{M}} &\leftrightarrow \varphi^{\mathbf{M}} \to \psi^{\mathbf{M}} \\ (\neg\varphi)^{\mathbf{M}} &\leftrightarrow \neg\varphi^{\mathbf{M}} \\ (\forall x\varphi)^{\mathbf{M}} &\leftrightarrow (\forall x \in \mathbf{M})\varphi^{\mathbf{M}} \end{aligned}$$

Note $\varphi^{\mathbf{V}} = \varphi$ and

$$f^{\mathbf{M}} = \{(x_1, \dots, x_n, x_{n+1}) \in \mathbf{M} \mid \varphi^{\mathbf{M}}(x_1, \dots, x_n, x_{n+1})\}$$

Definition 2.22. For any theory T, any class $\mathbf{M}$, $\mathbf{M} \models T$ iff for any axiom φ of T, $\varphi^{\mathbf{M}}$ holds

Theorem 2.23 (ZF$^-$). ***WF*** $\models$ ***ZF***

Proof.

- **Axiom of existence**
  $(\exists x(x = x))^{\mathbf{M}} \leftrightarrow \exists x \in \mathbf{M}\,(x = x)$, which is equivalent to $\mathbf{M}$ being nonempty
- **Axiom of extensionality**

$$\forall X \forall Y \forall u((u \in X \leftrightarrow u \in Y) \to X = Y)^{\mathbf{M}} \Leftrightarrow$$
$$\forall X \in \mathbf{M}\, \forall Y \in \mathbf{M}\, \forall u \in \mathbf{M}\, ((u \in X \leftrightarrow u \in Y) \to X = Y)$$

Lemma 2.24. *If* $\mathbf{M}$ *is transitive, then axiom of extensionality holds in* $\mathbf{M}$

- **Axiom schema of specification**

$$\forall X \in \mathbf{M}\, \exists Y \in \mathbf{M}\, \forall u \in \mathbf{M}\, (u \in Y \leftrightarrow u \in X \wedge \varphi^{\mathbf{M}}(u))$$

  Since for any $X \in \mathbf{WF}$, $\mathcal{P}(X) \subseteq \mathbf{WF}$
- **Axiom of paring**
- **Axiom of union**
- **Axiom of power set**

$$\forall X \in \mathbf{M}\, \exists Y \in \mathbf{M}\, \forall u \in \mathbf{M}\, (u \in Y \leftrightarrow (u \subseteq X)^{\mathbf{M}})$$

  and

$$(u \subseteq X)^{\mathbf{M}} \leftrightarrow \forall x \in \mathbf{M}\, (x \in u \to x \in X) \leftrightarrow u \cap \mathbf{M} \subseteq X$$

- **Axiom of foundation**
- **Axiom schema of replacement**

□

2.4 Absoluteness

Definition 2.25. For any formula $\psi(x_1, \dots, x_n)$ and any class $\mathbf{M}$,$\mathbf{N}$, $\mathbf{M} \subseteq \mathbf{N}$, if

$$\forall x_1 \dots \forall x_n \in \mathbf{M}\, (\psi^{\mathbf{M}}(x_1, \dots, x_n) \leftrightarrow \psi^{\mathbf{N}}(x_1, \dots, x_n))$$

then $\psi(x_1, \dots, x_n)$ is **absolute** for $\mathbf{M}$,cn. If $\mathbf{N} = \mathbf{V}$, then ψ is **absolute** for $\mathbf{M}$

Lemma 2.26. *Suppose $\boldsymbol{M} \subseteq \boldsymbol{N}$ and φ,ψ are formulas, then*

1. *if φ,ψ are absolute for $\boldsymbol{M}$,cn, then so are $\neg\varphi, \varphi \to \psi$*
2. *if φ doesn't contain any quantifiers, then φ is absolute for any $\boldsymbol{M}$*
3. *if $\boldsymbol{M}$,$\boldsymbol{N}$ are transitive and φ is absolute for them, then so are $\forall x \in y\varphi$*

Definition 2.27. Δ_0 formula

1. $x = y, x \in y$ are Δ_0 formulas
2. if φ,ψ are Δ_0, then so are $\neg\varphi, \varphi \to \psi$
3. if φ is Δ_0, y is any set, then $(\forall x \in y)\varphi$ is Δ_0

If φ is Δ_0, then $\exists x_1 \dots \exists x_n\varphi$ is Σ_1 formula, $\forall x_1 \dots \forall x_n\varphi$ is Π_1

Lemma 2.28. *$\boldsymbol{M} \subseteq \boldsymbol{N}$ are both transitive, $\psi(x_0, \dots, x_n)$ is a formula, then*

1. *if ψ is Δ_0, then it's absolute for $\boldsymbol{M}$,cn*
2. *if ψ is Σ_1, then*

$$\forall x_1 \dots x_n(\psi^{\boldsymbol{M}}(x_1, \dots, x_n) \to \psi^{\boldsymbol{N}}(x_1, \dots, x_n))$$

3. *if ψ is Π_1, then*

$$\forall x_1 \dots x_n(\psi^{\boldsymbol{N}}(x_1, \dots, x_n) \to \psi^{\boldsymbol{M}}(x_1, \dots, x_n))$$

Lemma 2.29. *If $\boldsymbol{M} \subseteq \boldsymbol{N}$, $\boldsymbol{M} \models \Sigma, \boldsymbol{N} \models \Sigma$ and*

$$\Sigma \vdash \forall x_1 \dots \forall x_n(\varphi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n))$$

then φ is absolute for $\boldsymbol{M}$,$\boldsymbol{N}$ if and only if ψ is absolute for $\boldsymbol{M}$,$\boldsymbol{N}$

Definition 2.30. Suppose **M** $\subseteq$ **N** , $f(x_1, \dots, x_n)$ is a function. f is **absolute** for **M** and **N** if and only if $\varphi(x_1, \dots, x_n, x_{n+1})$ defining f is absolute.

Theorem 2.31. *Following relations and functions can be defined in **ZF**$^-$ $-$ Pow $-$ Inf and are equivalent to some Δ_0 formulas. So they are absolute for any transitive model $\boldsymbol{M}$ on **ZF**$^-$ $-$ Pow $-$ Inf*

1. $x \in y$
2. $x = y$
3. $x \subset y$
4. $\{x, y\}$
5. $\{x\}$
6. (x, y)
7. $\emptyset$
8. $x \cup y$
9. $x - y$
10. $x \cap y$

11. x^+
12. x is a transitive set
13. $\bigcup x$
14. $\bigcap x$ ($\bigcap \emptyset = \emptyset$)

Lemma 2.32. *Absoluteness is closed under operation composition*

Theorem 2.33. *Following relations and functions are absolute for any transitive model $\boldsymbol{M}$ on $\boldsymbol{ZF}^- - Pow - Inf$*

1. *z is an ordered pair*
2. $A \times B$
3. *R is a relation*
4. $\mathrm{dom}(R)$
5. $\mathrm{ran}(R)$
6. *f is a function*
7. $f(x)$
8. *f is injective*

2.5 Relative consistence of the axiom of foundation

Lemma 2.34. *Suppose transitive class $\boldsymbol{M} \models \boldsymbol{ZF}^- - Pow - inf$ and $\omega \in \boldsymbol{M}$, then the axiom of infinity is true in $\boldsymbol{M}$. Hence the axiom of infinity is true in $\boldsymbol{WF}$*

Theorem 2.35. *Let T be a theory of set theory language and Σ a set of sentences. Suppose $\boldsymbol{M}$ is a class and $T \vdash \boldsymbol{M} \neq \emptyset$, then if $\boldsymbol{M} \models_T \Sigma$, then*

1. *for any sentences φ, if $\Sigma \vdash \varphi$, then $T \vdash \varphi^{\boldsymbol{M}}$*
2. *if T is consistent, then so is $Cn(\Sigma)$*

Theorem 2.36. *The axiom of foundation is consistent with $\boldsymbol{ZF}^-$.*

Proof. By 2.35, let T be $\mathbf{ZF}^-$, Σ be **ZF** and **M** be **WF** □

Lemma 2.37 ($\mathbf{ZF}^-$). *Suppose transitive model $\boldsymbol{M} \models \boldsymbol{ZF}^- - Pow - Inf$. If $X, R \in \boldsymbol{M}$ and R is a well-order on X, then*

$$(R \textit{ is a well-order on } X)^{\boldsymbol{M}}$$

Theorem 2.38 ($\mathbf{ZF}^-$). $V_\omega \models \boldsymbol{ZFC} - Inf + \neg Inf$

Proof. For any $X \in V_\omega$, X is finite hence there is a well-ordering on X □

Corollary 2.39. $Con\boldsymbol{ZF}^- \to Con\boldsymbol{ZFC} - Inf + \neg Inf$

2.6 Induction and recursion based on well-order relation

Definition 2.40. $\boldsymbol{R}$ is a well-founded relation on $\boldsymbol{X}$ if and only if

$$\forall U \subset \boldsymbol{X}(U \neq \emptyset \to \exists y \in U(\neg \exists z \in U(z\boldsymbol{R}y)))$$

Definition 2.41. Relation $\boldsymbol{R}$ is **set-like** on $\boldsymbol{X}$ iff for any $x \in \boldsymbol{X}$, $\{y \in \boldsymbol{X} \mid y\boldsymbol{R}x\}$ is a set

Definition 2.42. If $\boldsymbol{R}$ is a set-like relation on $\boldsymbol{X}$ and $x \in \boldsymbol{X}$, define

$$\begin{aligned}
\mathrm{pred}^0(\boldsymbol{X}, x, \boldsymbol{R}) &= \{y \in \boldsymbol{X} \mid y\boldsymbol{R}x\} \\
\mathrm{pred}^{n+1}(\boldsymbol{X}, x, bR) &= \bigcup\{\mathrm{pred}(\boldsymbol{X}, y, \boldsymbol{R}) \mid y \in \mathrm{pred}^n(\boldsymbol{X}, x, \boldsymbol{R})\} \\
\mathrm{cl}(\boldsymbol{X}, x, \boldsymbol{R}) &= \bigcup_{n\in\omega} \mathrm{pred}^n(\boldsymbol{X}, x, \boldsymbol{R})
\end{aligned}$$

Lemma 2.43. *If $\boldsymbol{R}$ is a set-like relation on $\boldsymbol{X}$, then for any $y \in$ cl$(\boldsymbol{X}, x, \boldsymbol{R})$, pred$(\boldsymbol{X}, y, \boldsymbol{R}) \subseteq$ cl$(\boldsymbol{X}, x, \boldsymbol{R})$*

Theorem 2.44 (Induction on well-founded set-like relation)**.** *If $\boldsymbol{R}$ is a well-founded set-like relation on $\boldsymbol{X}$, then every nonempty $\boldsymbol{Y} \subseteq \boldsymbol{X}$ has minimal element under $\boldsymbol{R}$*

Theorem 2.45. *Suppose $\boldsymbol{R}$ is a well-founded set-like relation on $\boldsymbol{X}$. If $\boldsymbol{F} : \boldsymbol{X} \times \boldsymbol{V} \to \boldsymbol{V}$, then there is a unique $\boldsymbol{G} : \boldsymbol{X} \to \boldsymbol{V}$ s.t.*

$$\forall x \in \boldsymbol{X}(\boldsymbol{G}(x) = \boldsymbol{F}(x, \boldsymbol{G}\restriction \mathit{pred}(\boldsymbol{X}, x, \boldsymbol{R})))$$

Definition 2.46. If $\boldsymbol{R}$ is a set-like well-founded relation on $\boldsymbol{X}$, define

$$\mathrm{rank}(x, \boldsymbol{X}, \boldsymbol{R}) = \sup\{\mathrm{rank}(y, \boldsymbol{X}, \boldsymbol{R}) + 1 \mid y\boldsymbol{R}x \wedge y \in \boldsymbol{X}\}$$

Note that

$$\boldsymbol{F}(x, h) = \sup\{\alpha + 1 \mid \alpha \in \mathrm{ran}(h)\}$$

Lemma 2.47 (**ZF**$^-$)**.** *If $\boldsymbol{X}$ is transitive and $\in$ is well-founded on $\boldsymbol{X}$, then $\boldsymbol{X} \subseteq$ **WF** and for any $x \in \boldsymbol{X}$, rank$(x, \boldsymbol{X}, \in) =$ rank(x)*

Definition 2.48. $\boldsymbol{R}$ is a set-like well-founded relation on $\boldsymbol{X}$, **Mostowski function $\boldsymbol{G}$** on $(\boldsymbol{X}, \boldsymbol{R})$ is

$$\boldsymbol{G}(x) = \{\boldsymbol{G}(y) \mid y \in \boldsymbol{X} \wedge y\boldsymbol{R}x\}$$

$\mathbf{M} = \mathrm{ran}(\boldsymbol{G})$ is called the **Mostowski collapse** of $(\boldsymbol{X}, \boldsymbol{R})$

Lemma 2.49. *1.* $\forall x, y \in \boldsymbol{X}(x\boldsymbol{R}y \to \boldsymbol{G}(x) \in \boldsymbol{G}(y))$

2. *$\boldsymbol{M}$ is transitive*
3. *If the axiom of power set holds, $\boldsymbol{M} \subseteq \boldsymbol{WF}$*
4. *if the axiom of power set holds and $x \in \boldsymbol{X}$, then $rank(x, \boldsymbol{X}, \boldsymbol{R}) = rank(\boldsymbol{G}(x))$*

Definition 2.50. $\boldsymbol{R}$ is extensional on $\boldsymbol{X}$ iff

$$\forall x, y \in \boldsymbol{X}(\forall z \in \boldsymbol{X}(z\boldsymbol{R}x \leftrightarrow z\boldsymbol{R}y) \to x = y)$$

Lemma 2.51. *If $\boldsymbol{X}$ is transitive then $\in$ is extensional on $\boldsymbol{X}$*

Lemma 2.52. *Let $\boldsymbol{R}$ be a set-like well-founded relation on $\boldsymbol{X}$, $\boldsymbol{G}$ is a Mostowski function on it. If $\boldsymbol{R}$ is extensional, then $\boldsymbol{G}$ is an isomorphism*

Theorem 2.53 (Mostowski collapse theorem)**.** *Suppose $\boldsymbol{R}$ is set-like well-founded extensional on $\boldsymbol{X}$, then there are unique transitive class $\boldsymbol{M}$ and bijection $\boldsymbol{G} : \boldsymbol{X} \to \boldsymbol{M}$ s.t. $\boldsymbol{G} : (\boldsymbol{X}, \boldsymbol{R}) \cong (\boldsymbol{M}, \in)$*

2.7 Absoluteness under the axiom of foundation

Theorem 2.54. *The following relations and functions can be defined by formulas in $\boldsymbol{ZF} - Pow$ and are equivalent to some Δ_0 formulas*

1. *x is an ordinal*
2. *x is a limit ordinal*
3. *x is a successor ordinal*
4. *ω*
5. *x is a finite ordinal*
6. *$0, 1, 2, \dots, 20, \dots$*

Theorem 2.55. *If transitive model $\boldsymbol{M} \models \boldsymbol{ZF} - Pow$, then every finite subset of $\boldsymbol{M}$ belongs to $\boldsymbol{M}$*

Proof. prove

$$\forall x \subset \mathbf{M} (|x| = n \to x \in \mathbf{M})$$

□

Theorem 2.56. *The following concepts are absolute for any transitive model of $\boldsymbol{ZF} - Pow$*

1. *x is finite*
2. *X^n*
3. *$X^{<\omega}$*

4. R *is a well-ordering on* X
5. *type*(X, R)
6. $\alpha + 1$
7. $\alpha - 1$
8. $\alpha + \beta$
9. $\alpha \cdot \beta$

Class $\boldsymbol{X}$ is in fact a formula $\boldsymbol{X}(x)$. It's absolute for $\mathbf{M}$ if and only if $\forall x \in \mathbf{M}\,(\boldsymbol{X}^{\mathbf{M}}(x) \leftrightarrow \boldsymbol{X}(x))$, which is equivalent to $\{x \in \mathbf{M} \mid \boldsymbol{X}(x)\} = \{x \in \mathbf{M} \mid \boldsymbol{X}^{\mathbf{M}}(x)\}$. Hence $\boldsymbol{X}$ is absolute for $\mathbf{M}$ if and only if $\boldsymbol{X}^{\mathbf{M}} = \mathbf{M} \cap \boldsymbol{X}$

Theorem 2.57. *Suppose* $\boldsymbol{R}$ *is a well-founded set-like relation on* $\boldsymbol{X}$, $\boldsymbol{F} : \boldsymbol{X} \times \boldsymbol{V} \to \boldsymbol{V}$,

$$\forall x \in \boldsymbol{X}(\boldsymbol{G}(x) = \boldsymbol{F}(x, \boldsymbol{G}\restriction(\boldsymbol{X}, x, \boldsymbol{R})))$$

transitive model $\boldsymbol{M} \models \boldsymbol{ZF} - Pow$ *and*

1. $\boldsymbol{F}$ *is absolute for* $\boldsymbol{M}$
2. $\boldsymbol{X}, \boldsymbol{R}$ *are absolute for* $\boldsymbol{M}$, $(\boldsymbol{R}$ *is set-like on* $\boldsymbol{X})^{\boldsymbol{M}}$ *and*

$$\forall x \in \boldsymbol{M}\,(pred(\boldsymbol{X}, x, \boldsymbol{R}) \subseteq \boldsymbol{M})$$

then $\boldsymbol{G}$ *is absolute for* $\boldsymbol{M}$

Theorem 2.58. *The following concept is absolute for any transitive model of* $\boldsymbol{ZF} - Pow$

1. α^{β}
2. *rank*(x)
3. $\mathrm{trcl}(x)$

Lemma 2.59. *transitive* $\boldsymbol{M} \models \boldsymbol{ZF}$

1. *if* $x \in \boldsymbol{M}$, *then* $\mathcal{P}^{\boldsymbol{M}}(x) = \mathcal{P}(x) \cap \boldsymbol{M}$
2. *if* $\alpha \in \boldsymbol{M}$, *then* $V_{\alpha}^{\boldsymbol{M}} = V_{\alpha} \cap \boldsymbol{M}$

2.8 Unaccessible cardinal and models of ZFC

$\boldsymbol{Z} = \mathbf{ZF} - \mathrm{Rep}, \mathbf{ZF}^{-} = \mathbf{ZFC} - \mathrm{Rep}$

Theorem 2.60. *If* $\gamma > \omega$ *is a limit ordinal, then* $V_{\gamma} \models_{\mathbf{ZF}} \boldsymbol{Z}$ *and* $V_{\gamma} \models_{\mathbf{ZFC}} \boldsymbol{ZC}$

Corollary 2.61. $V_{\omega+\omega}$ *doesn't satisfies the axiom of replacement*

Proof. □

Theorem 2.62. $\boldsymbol{ZC} \not\vdash \exists x(x = V_{\omega}), \boldsymbol{ZC} \not\vdash \forall x \exists y(\mathrm{trcl}(x) = y)$

Theorem 2.63. *If κ is an inaccessible cardinal, then $V_\kappa \models_{\boldsymbol{ZF}^-}$ **ZF**,*
*$V_\kappa \models_{\boldsymbol{ZFC}^-}$ **ZFC***

Proof. Since κ is inaccessible, $|V_\kappa| = \kappa$. For any $A \in V_\kappa$, $|A| < \kappa$. Since κ is regular, any $f : A \to V_\kappa$ is bounded. Hence there exists $\alpha < \kappa$ s.t. $\mathrm{ran}(f) \subseteq V_\alpha$ □

Corollary 2.64. *We cannot prove "there is some inaccessible cardinals" in **ZFC***

Proof. Suppose we could. Then we have $V_\kappa \models$ **ZFC** , which contradicts Gödel's second incompleteness theorem □

Lemma 2.65. *Suppose κ is inaccessible. The following concepts are absolute for V_κ*

1. *x is a cardinal*
2. *x is a regular cardinal*
3. *x is an inaccessible cardinal*

Lemma 2.66. *Con(**ZFC**) $\to$ Con(**ZFC** + "there is no inaccessible cardinal")*

Proof. If κ is the smallest inaccessible cardinal, then
$V_\kappa \models$ **ZFC** + "there is no inaccessible cardinal". Define

$$\mathbf{M} = \bigcap\{V_\kappa \mid \kappa \text{ is inaccessible}\}$$

□

If there are, then $\mathbf{M} = V_\kappa$

Corollary 2.67. *Con(**ZFC**)$\neg\to$Con(**ZFC**+"there are some inaccessible cardinals")*

Definition 2.68. For any infinite cardinal κ, $H_\kappa = \{x \mid |\mathrm{trcl}(x)| < \kappa\}$ is the collection of sets which **hereditarily have size less than** κ. Element of H_ω is called **hereditarily finite set**. Element of H_{ω_1} is called **hereditarily countable set**

Lemma 2.69. *For any infinite cardinal κ, $H_\kappa \subseteq V_\kappa$*

Lemma 2.70. *If κ is regular, then $H_\kappa = V_\kappa$ if and only if κ is inaccessible*

Proof. which implies $|V_\kappa| = \kappa$ □

Lemma 2.71. *For any infinite cardinal κ*

1. *H_κ is transitive*
2. *$H_\kappa \cap$ **On** $= \kappa$*

3. *If $x \in H_\kappa$, then $\bigcup x \in H_\kappa$*
4. *If $x, y \in H_\kappa$, then $\{x, y\} \in H_\kappa$*
5. *If $x \in H_\kappa, y \subseteq x$, then $y \in H_\kappa$*
6. *if κ is regular, then $\forall x(x \in H_\kappa \leftrightarrow x \subset H_\kappa \wedge |x| < \kappa)$*

Theorem 2.72. *If κ is uncountable regular cardinal, then $H_\kappa \models_{\mathbf{ZFC}}$ **ZFC** $-$ Pow*

Theorem 2.73. *If κ is uncountable regular cardianl, then the following propositions are equivalent*

1. *$H_\kappa \models$ **ZFC***
2. *$H_\kappa = V_\kappa$*
3. *κ is inaccessible*

Corollary 2.74. *Con(**ZFC**) $\rightarrow$ Con(**ZFC** $-$ pow $+ \forall x(x$ is countable$))$*

2.9 Reflection theorem

Lemma 2.75. *$\boldsymbol{M} \subseteq \boldsymbol{N}$ are classes. $\varphi_1, \ldots, \varphi_n$ is a sequence closed under subformula, then the following propositions are equivalent*

1. *$\varphi_1, \ldots, \varphi_n$ are absolute for **M** and **N***
2. *if $\varphi_i = \exists \varphi_j(x, y_1, \ldots, y_m)$, then*

$$\forall y_1, \ldots, y_m \in \boldsymbol{M}\,(\exists x \in \boldsymbol{N}\,\varphi_j^{\boldsymbol{N}}\,(x, y_1, \ldots, y_m) \rightarrow \exists x \in \boldsymbol{M}\,\varphi_j^{\boldsymbol{M}}\,(x, y_1, \ldots, y_m))$$

Theorem 2.76 (reflection theorem(**ZF**))**.** *For any finite formula set $F = \{\varphi_1, \ldots, \varphi_n\}$, for any V_α, there exists V_β s.t. $V_\alpha \subseteq V_\beta$ and $\varphi_1, \ldots, \varphi_n$ are absolute for V_β*

Corollary 2.77 (ZF). *$F = \{\sigma_1, \ldots, \sigma_n\}$ are finite subsets of **ZF**, then*

$$\forall \alpha \exists \beta > \alpha(\sigma_1^{V_\beta} \wedge \cdots \wedge \sigma_n^{V_\beta})$$

Corollary 2.78. *$F = \{\sigma_1, \ldots, \sigma_n\}$ is a finite subset of **ZF**. Unless **ZF** is unconsistent, F cannot prove all axioms of **ZF***

Theorem 2.79 (ZFC). *For any finite formula set $F = \{\varphi_1, \ldots, \varphi_n\}$, for any set N, there exists set M s.t.*

1. *$N \subseteq M$*
2. *$\varphi_1, \ldots, \varphi_n$ are absolute for $(M, \in)$*
3. *$|M| \leq |N| \cdot \omega$*

Corollary 2.80 (ZFC). *For any finite formula set $F = \{\varphi_1, \ldots, \varphi_n\}$, for any set N, there exists set M s.t.*

1. *$N \subseteq M$*
2. *$\varphi_1, \ldots, \varphi_n$ are absolute for $(M, \in)$*
3. *$|M| \leq |N| \cdot \omega$*
4. *M is transitive*

3 Constructable Set - Venlafaxine

3.1 Definablity and Gödel operation

Definition 3.1. M is a set, $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a formula, $X \subseteq M^n$ is **definable in** M **from parameters from** ψ if and only if there are $y_1, \ldots, y_m \in M$ s.t.

$$X = \{(x_1, \ldots, x_n) \mid (\psi^M(x_1, \ldots, x_n, y_1, \ldots, y_m))\}$$

$$\mathrm{Def}(M) = \{X \subseteq M \mid \exists \psi, X \text{ is definable in } M \text{ from } \psi\}$$

Definition 3.2. Gödel operation

1. $G_1(X, Y) = \{X, Y\}$
2. $G_2(X, Y) = X \times Y$
3. $G_3(X, Y) = \in \restriction X \times Y$
4. $G_4(X, Y) = X - Y$
5. $G_5(X, Y) = X \cap Y$
6. $G_6(X, Y) = \bigcap X$
7. $G_7(X, Y) = \mathrm{dom}(X)$
8. $G_8(X, Y) = \{(x, y) \mid (y, x) \in X\}$
9. $G_9(X, Y) = \{(x, y, z) \mid (x, z, y) \in X\}$
10. $G_{10}(X, Y) = \{(x, y, z) \mid (y, z, x) \in X\}$

Class C is closed under Gödel operation if for any X, Y, X,Y∈ C$ implies $G_i(X, Y) \in C$. For any set M, $\mathrm{cl}_G(M)$ is the **closure under Gödel operation**

Definition 3.3. ψ is a **normal form** if

1. only $\neg, \wedge, \exists$ are logical symbol
2. = doesn't appear
3. if $x_i \in x_j$ then $i \neq j$
4. $\exists$ only shown as: $\exists x_{m+1} \in x_i \varphi(x_1, \ldots, x_{m+1})$, $1 \leq i \leq m$

Lemma 3.4. *Any Δ_0 formula can be transformed into normal form*

Theorem 3.5. *For any Δ_0 formula $\psi(x_1, \ldots, x_n)$, there is Gödel operations' composition G s.t. for any $X_1, \ldots, X_n$*

$$\begin{aligned} G(X_1, \ldots, X_n) = &\{(x_1, \ldots, x_n) \mid \\ &\qquad x_1 \in X_1 \wedge \cdots \wedge x_n \in X_n \wedge \psi(x_1, \ldots, x_n)\} \end{aligned}$$

Corollary 3.6. *If M is transitive and $M = cl_G(M)$, then for any Δ_0 formula $\psi(x, y_1, \ldots, y_m)$, any set $X \in M$, any $y_1, \ldots, y_m \in M$ if*

$$Y = \{x \in X \mid \psi(x, y_1, \ldots, y_m)\}$$

then $Y \in M$. Hence Δ_0 schema of specification holds in M

Lemma 3.7. *If $G(X_1, \ldots, X_n)$ is Gödel operations' composition, then $Z = G(X_1, \ldots, X_n)$ is equivalent to a Δ_0 formula*

Theorem 3.8. *For any transitive set M, Def$(M) =$ cl$_G(M \cup \{M\}) \cap \mathcal{P}(M)$*

Lemma 3.9. *If transitive $\boldsymbol{M} \models$ **ZF**, then for any transitive set $M \in \boldsymbol{M}$, Def(M) is absolute for $\boldsymbol{M}$*

Lemma 3.10. *For any transitive set M*

1. *Def$(M) \subseteq \mathcal{P}(M)$*
2. *$M \subseteq$ Def(M)*
3. *for any $X \subseteq M$, if X is finite, then $X \in$ Def(M)*
4. *assume **AC** and $|M| \geq \omega$, then $|\text{Def}(M)| = |M|$*

3.2 Gödel's L

Definition 3.11. for any α

1. $L_0 = \emptyset$
2. $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$
3. For any limit α, $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$

$\mathbf{L} = \bigcup\limits_{\alpha \in \mathbf{On}} L_\alpha$. Element of GLis called constructible set

Lemma 3.12. *For any ordinal α*

1. *L_α is transitive*
2. *If $\alpha < \beta$, then $L_\alpha \subseteq L_\beta$*
3. *$L_\alpha \subseteq V_\alpha$*

Definition 3.13. $x \in \mathbf{L}$

$$\mathrm{rank}_{\mathbf{L}}(x) = \min\{\beta \mid x \in \mathbf{L}_{\beta+1}\}$$

Lemma 3.14. *For any α*

$$L_\alpha = \{x \in \boldsymbol{L} \mid \mathit{rank}_{\boldsymbol{L}}(x) < \alpha\}$$

Lemma 3.15. *For any ordinal α*

1. *$L_\alpha \cap \boldsymbol{On} = \alpha$*
2. *$\alpha \in \boldsymbol{L} \cap \mathit{rank}_{\boldsymbol{L}}(\alpha) = \alpha$*

Proof. since "α is a cardinal" is absolute for any transitive set.

$$\begin{aligned}\alpha = L_\alpha \cap \mathbf{On} &= \{\eta \in L_\alpha \mid \eta \text{ is a ordinal}\} \\ &= \{\eta \in L_\alpha \mid (\eta \text{ is an ordinal}^{L_\alpha})\} \in \mathrm{Def}(L_\alpha)\end{aligned}$$

□

Lemma 3.16. *for any ordinal* α

1. $L_\alpha \in L_{\alpha+1}$
2. *any finite subset of* L_α *belongs to* $L_{\alpha+1}$

Lemma 3.17.

1. $\forall n \in \omega (L_n = V_n)$
2. $L_\omega = V_\omega$

Lemma 3.18. *If AC, then for any* $\alpha \geq \omega, |L_\alpha| = |\alpha|$

Theorem 3.19. $\boldsymbol{L} \models \boldsymbol{ZF}$

3.3 Axiom of constructibility and relativization

Theorem 3.20 (Axiom of constructibility)**.** $\boldsymbol{V} = \boldsymbol{L}$

Lemma 3.21. *function* $\alpha \mapsto L_\alpha$ *is absolute for any transitive model of* ***ZF***

Theorem 3.22. $\boldsymbol{L} \models \boldsymbol{ZF} + \boldsymbol{V} = \boldsymbol{L}$

Proof. $(\mathbf{V} = \mathbf{L})^{\mathbf{L}}$ is $\forall x \in \mathbf{L} \exists \alpha \in \mathbf{L} (x \in L_\alpha)^{\mathbf{L}}$. By 3.21, $(x \in L_\alpha)^{\mathbf{L}} \Leftrightarrow x \in L_\alpha$. Hence $\mathbf{L} \models \mathbf{V} = \mathbf{L}$ □

Hence

Theorem 3.23. $Con(\boldsymbol{ZF}) \to Con(\boldsymbol{ZF} + \boldsymbol{V} = \boldsymbol{L})$

Theorem 3.24. *Suppose transitive proper class* $\boldsymbol{M} \models \boldsymbol{ZF} - Pow$, *then* $\boldsymbol{L} = \boldsymbol{L}^{\boldsymbol{M}} \subseteq \boldsymbol{M}$

Proof. For any ordinal α, since $\mathbf{M}$ is proper, $\mathbf{M} \not\subseteq V_\alpha$. Hence there is $x \in \mathbf{M}$ s.t. $\text{rank}(x) \geq \alpha$. Since rank is absolute, $\text{rank}(x) \in \mathbf{M}$. And $\mathbf{M}$ is transitive, hence $\alpha \in \mathbf{M}$. By 3.21, $L_\alpha \in \mathbf{M}$

$$\begin{aligned}
\mathbf{L}^{\mathbf{M}} &= \{x \in \mathbf{M} \mid (\exists \alpha \in \mathbf{On}\ (x \in L_\alpha))^{\mathbf{M}}\} \\
&= \{x \mid \exists \alpha \in \mathbf{On} \cap \mathbf{M} (x \in L_\alpha \cap \mathbf{M})\} \\
&= \{x \mid \exists \alpha \in \mathbf{On}\ (x \in L_\alpha)\} \\
&= \mathbf{L}
\end{aligned}$$

□

Definition 3.25. If transitive model $\mathbf{M} \models \mathbf{ZF}$ contains all ordinals, then it's an **inner model**

Lemma 3.26. *there is a finite set of axioms* $\{\psi_1, \dots, \psi_n\}$ *of* $\boldsymbol{ZF} - Pow$ *s.t. ordinals,rank and* L_α *are absolute for any model of* $\{\psi_1, \dots, \psi_n\}$

Lemma 3.27. *If set M is transitive, then $M \cap \boldsymbol{On}$ is a ordinall and is the least that doesn't belong to M, denoted by α^M*

Theorem 3.28. *There is a finite subset $\{\psi_1, \dots, \psi_n\}$ of axioms of $\boldsymbol{ZF} - Pow$ satisfying*

$$\forall M(M \text{ is transitive } \wedge \psi_1^M \wedge \cdots \wedge \psi_n^m \rightarrow (L_{\alpha^M} = \boldsymbol{L}^M \subseteq M))$$

Theorem 3.29. *The is a finite subset $\{\psi_1, \dots, \psi_{n+1}\}$ of axioms of $\boldsymbol{ZF} - Pow + \boldsymbol{V} = \boldsymbol{L}$ satisfying*

1. *If $\boldsymbol{M}$ is a transitive proper class and $\psi_1^{\boldsymbol{M}} \wedge \cdots \wedge \psi_{n+1}^{\boldsymbol{M}}$, then $\boldsymbol{M} = \boldsymbol{L}$*
2. *$\forall M(M \text{ is transitive } \wedge \psi_1^M \wedge \cdots \wedge \psi_n^m \rightarrow (L_{\alpha^M} = M))$*

Theorem 3.30. *There is a well-ordering on $\boldsymbol{L}$. Hence $\boldsymbol{V} = \boldsymbol{L} \rightarrow \boldsymbol{AC}$*

If $\mathbf{V} = \mathbf{L}$, hence $\aleph_\alpha \subseteq L_{\aleph_{\alpha+1}}$. Because $|L_{\alpha_{\alpha+1}}| = \aleph_{\alpha+1}$, $2^{\aleph_\alpha} \leq \aleph_{\alpha+1}$

Theorem 3.31. *If $\boldsymbol{V} = \boldsymbol{L}$, then for any infinite ordinal α, $\mathcal{P}(L_\alpha) \subseteq L_{|\alpha|^+}$*

Corollary 3.32 (ZF). $(\boldsymbol{AC} + \boldsymbol{GCH})^{\boldsymbol{L}}$

Theorem 3.33 (ZF). *$Con(\boldsymbol{ZF}) \rightarrow Con(\boldsymbol{ZFC} + \boldsymbol{GCH})$*

Theorem 3.34 (ZF). *Suppose $S_0 = \{\psi_1, \dots, \psi_n\} \subseteq \boldsymbol{ZF} + \boldsymbol{V} = \boldsymbol{L}$, then*

$$\boldsymbol{ZF} \vdash \exists M(|M| = \omega \wedge M \text{ is transitive} \wedge (\psi_1^M \wedge \cdots \wedge \psi_n^M))$$

Lemma 3.35. *Suppose $\boldsymbol{V} = \boldsymbol{L}$. For any uncountable regular cardinal κ, $L_\kappa = H_\kappa$*

Corollary 3.36. *If κ is a uncountable regular cardinal, then $_\kappa \models \boldsymbol{ZF} - Pow + \boldsymbol{V} = \boldsymbol{L}$. If κ is inaccessible, then $L_\kappa \models \boldsymbol{ZF} + \boldsymbol{V} = \boldsymbol{L}$*

4 The end

Learn and forget