A Course In Universal Algebra

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Contents

1 Lattices

1.1 Definitions of Lattices

Definition 1.1. A nonempty set L together with two binary operations $\vee$ and $\wedge$ (read "join" and "meet" respectively) on L is called a **lattice** if it satisfies the following identities

L1: (a) $x \vee y \approx y \vee x$
(b) $x \wedge y \approx y \wedge x$ (commutative laws)

L2: (a) $x \vee (y \vee z) \approx (x \vee y) \vee z$
(b) $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$ (associate laws)

L3: (a) $x \vee x \approx x$
(b) $x \wedge x \approx x$ (idempotent laws)

L4: (a) $x \approx x \vee (x \wedge y)$
(b) $x \approx x \wedge (x \vee y)$ (absorption laws)

Definition 1.2. Let A be a subset of a poset P. An element p in P is an **upper bound** for A if $a \leq p$ for every a in A. An element p in P is the **least upper bound** of A (l.u.b. of A) or **supremum** of A ($\sup A$.

For a, b in P we say b **covers** a, or a is **covered by** b if $a < b$ and whenever $a \leq c \leq b$ it follows that $a = c$ or $c = b$. We use the notation $a \prec b$ to denote a is covered by b.

Definition 1.3. A poset L is a lattice iff for every a, b in L both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist

1. If L is a lattice by the first definition, then define $\leq$ on L by $a \leq b$ iff $a = a \wedge b$
2. If L is a lattice by the second definition, then define $\vee$ and $\wedge$ by $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$

1.2 Isomorphism Lattices, and Sublattices

Definition 1.4. Two lattices L_1 and L_2 are **isomorphic** if there is a bijection α from L_1 to L_2 s.t. for every a, b in L_1 the following two equation hold: $\alpha(a \vee b) = \alpha(a) \vee \alpha(b)$ and $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$. Such an α is called an **isomorphism**

Definition 1.5. If P_1 and P_2 are two posets and α is a map from P_1 to P_2, then we say α is **order-preserving** if $\alpha(a) \leq \alpha(b)$ holds in P_2 whenever $a \leq b$ holds in P_1

Theorem 1.6. *Two lattices L_1 and L_2 are isomorphic iff there is a bijection α from L_1 to L_2 s.t. both α and α^{-1} are order-preserving*

Definition 1.7. If L is a lattice and $L' \neq \emptyset$ is a subset of L s.t. for every pair of elements a, b in L' both $a \vee b$ and $a \wedge b$ are in L', where $\wedge, \vee$ are the lattice operations of L, then we say that L' with the same operations is a **sublattice** of L

Definition 1.8. A lattice L_1 can be **embedded** into a lattice L_2 if there is a sublattice of L_2 isomorphic to L_1; in this case we also say that L_2 **contains a copy of L_1 as a sublattice**

1.3 Distributive and Modular Lattices

Definition 1.9. A **distributive lattice** is a lattice which satisfies either of the distributive laws,

D1: $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$

D2: $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$

Theorem 1.10. *A lattice L satisfies D1 iff it satisfies D2*

$$\begin{aligned} x \vee (y \wedge z) &\approx (x \vee (x \wedge z)) \vee (y \wedge z) && \text{(by L4(a))} \\ &\approx x \vee ((x \wedge z) \vee (y \wedge z)) \\ &\approx x \vee ((z \wedge x) \vee (z \wedge y)) \\ &\approx x \vee (z \wedge (x \vee y)) \\ &\approx x \vee ((x \vee y) \wedge z) \\ &\approx (x \wedge (x \vee y)) \vee (x \vee y \wedge z) \\ &\approx ((x \vee y) \wedge x) \vee ((x \vee y) \wedge) \\ &\approx (x \vee y) \wedge (x \vee z) \end{aligned}$$

Actually every lattice satisfies both of the inequalities $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$ and $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$.

Definition 1.11. A **modular lattice** is any lattice which satisfies the **modular** law

M: $x \leq y \rightarrow x \vee (y \wedge z) \approx y \wedge (x \vee z)$

Equivalent to the identity

$$(x \wedge y) \vee (y \wedge z) \approx y \wedge ((x \wedge y) \vee z)$$

Every lattice satisfies

$$x \leq y \rightarrow x \vee (y \wedge z) \leq y \wedge (x \vee z)$$

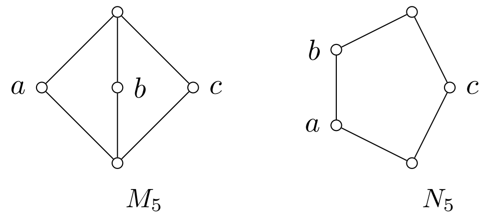

Figure 1

Theorem 1.12. *Every distributive lattice is a modular lattice*

Neither M_5 nor N_5 is a distributive lattice in Figure 1

Theorem 1.13 (Dedekind)**.** *L is a nonmodular lattice iff N_5 can be embedded into L*

Proof. If L doesn't satisfy the modular law. Then for some a, b, c in L we have $a \leq b$ but $a \vee (b \wedge c) < b \wedge (a \vee c)$. Let $a_1 = a \vee (b \wedge c)$ and $b_1 = b \wedge (a \vee c)$. Then

$$c \wedge b_1 = c \wedge (b \wedge (a \vee c)) = (c \wedge (a \vee c)) \wedge b = c \wedge b$$

and

$$c \vee a_1 = c \vee a$$

Now as $c \wedge b \leq a_1 \leq b_1$, we have $c \wedge b \leq c \wedge a_1 \leq c \wedge b_1 = c \wedge b$, hence $c \wedge a_1 = c \wedge b$. Likewise $c \vee a = c \vee b_1$

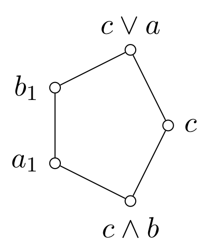

Figure 2

□

Theorem 1.14 (Birkhoff)**.** *L is a nondistributive lattice iff M_5, or N_5 can be embedded into L*

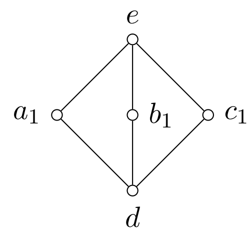

Figure 3

Proof. Let suppose that L is a nondistributive lattice and that L does not contain a copy of N_5 as a sublattice. Thus L is modular by Theorem 1.13. Since the distributive laws do not hold in L, there must be elements a, b, c from L s.t. $(a \wedge b) \vee (a \wedge c) < a \wedge (b \vee c)$. Let us define

$$\begin{aligned} d &= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \\ e &= (a \vee b) \wedge (a \vee c) \wedge (b \vee c) \\ a_1 &= (a \wedge e) \vee d \\ b_1 &= (b \wedge e) \vee d \\ c_1 &= (c \wedge e) \vee d \end{aligned}$$

Then $d \leq a_1, b_1, c_1 \leq e$. Now from

$$a \wedge e = a \wedge (b \vee c)$$

and

$$\begin{aligned} a \wedge d &= \underline{a} \wedge (\underline{(a \wedge b) \vee (a \wedge c)} \vee (b \wedge c)) \\ &= ((a \wedge b) \vee (a \wedge c)) \vee (a \wedge (b \wedge c)) && \text{by M} \\ &= (a \wedge b) \vee (a \wedge c) \end{aligned}$$

it follows that $d < e$

We now show that diagram in Figure 3 is a copy of M_5 in L. To do this it suffices to show that $a_1 \wedge b_1 = a_1 \wedge c_1 = b_1 \wedge c_1 = d$ and $a_1 \vee b_1 = a_1 \vee c_1 = b_1 \vee c_1 = e$.

$$
\begin{aligned}
a_1 \wedge b_1 &= ((a \wedge e) \vee \underline{d}) \wedge (\underline{(b \wedge e) \vee d}) \\
&= ((a \wedge e) \wedge ((b \wedge \underline{e}) \vee d)) \vee d && \text{(by M)} \\
y \wedge z &= ((b \wedge e) \vee d) \wedge d = d \\
&= ((a \wedge e) \wedge ((b \vee d) \wedge e)) \vee d && \text{(by M)} \\
&= ((a \wedge e) \wedge e \wedge (b \vee d)) \vee d \\
&= ((a \wedge e) \wedge (b \vee d)) \vee d \\
&= (a \wedge \underline{(b \vee c)} \wedge (\underline{b} \vee (a \wedge c))) \vee d \\
&= (a \wedge (b \vee ((b \vee c) \wedge (a \vee c)))) \vee d && \text{(by M)} \\
&= (\underline{a} \wedge (b \vee \underline{(a \wedge c)})) \vee d && a \wedge c \leq b \vee c \\
&= (a \wedge c) \vee (b \wedge a) \vee d && \text{(by M)} \\
&= d
\end{aligned}
$$

□

1.4 Complete Lattices, Equivalence Relations, and Algebraic Lattices

Definition 1.15. A poset P is **complete** if for every subset A of P both $\sup A$ and $\inf A$ exists in P. The elements $\sup A$ an $\inf A$ will be denoted by $\bigvee A$ and $\bigwedge A$.

Theorem 1.16. *Let P be a poset s.t. $\bigvee A$ exists for every subset A, or s.t. $\bigwedge A$ exists for every subset A. Then P is a complete lattice*

Proof. Suppose $\bigwedge A$ exists for every $A \subseteq P$. Then letting A^u be the set of upper bounds of A in P, it is routine to verify that $\wedge A^u$ is indeed $\vee A$. □

In the above theorem, the existence of $\bigwedge \emptyset$ guarantees a largest element in P, and likewise the existence of $\bigvee \emptyset$ guarantees a smallest element in P. (Every element is larger than $\emptyset$).

Definition 1.17. A sublattice L' of a complete lattice L is called a **complete sublattice** of L if for every subset A of L' the elements $\bigvee A$ and $\bigwedge A$, as defined in L, are actually in L'

Definition 1.18. The **diagonal relation** Δ_A and the **all relation** A^2 is denoted by ∇_A. $r_1 \circ r_2$ iff there is a $c \in A$ s.t. $\langle a, c \rangle \in r_1$ and $\langle c, b \rangle \in r_2$

$\mathrm{Eq}(A)$ is the set of all equivalence relations on A.

Theorem 1.19. *The poset* $\mathrm{Eq}(A)$ *with* $\subseteq$ *as the partial ordering, is a complete lattice.*

Theorem 1.20. *If* θ_1 *and* θ_2 *are two equivalence relations on* A *then*

$$\theta_1 \vee \theta_2 = \theta_1 \cup (\theta_1 \circ \theta_2) \cup (\theta_1 \circ \theta_2 \circ \theta_1) \cup (\theta_1 \circ \theta_2 \circ \theta_1 \circ \theta_2) \cup \dots$$

or equivalently, $\langle a, b \rangle \in \theta_1 \vee \theta_2$ *iff there is a sequence of elements* $c_1, c_2, \dots, c_n$ *from* A *s.t.*

$$\langle c_i, c_{i+1} \rangle \in \theta_1 \quad \textit{or} \quad \langle c_i, c_{i+1} \rangle \in \theta_2$$

for $i = 1, \dots, n-1$ *and* $a = c, b = c_n$

Definition 1.21. Let θ be a member of $\mathrm{Eq}(A)$. For $a \in A$, the **equivalence class** (or **coset**) **of a modulo** θ is the set $a/\theta = \{b \in A : \langle b, a \rangle \in \theta\}$. The set $\{a/\theta : a \in A\}$ is denoted by A/θ

Theorem 1.22. *p For* $\theta \in \mathrm{Eq}(A)$ *and* $a, b \in A$ *we have*

1. $A = \bigcup_{a \in A} a/\theta$
2. $a/\theta \neq b/\theta$ *implies* $a/\theta \cap b/\theta = \emptyset$

Definition 1.23. A partition π of a set A is a family of nonempty pairwise disjoint subsets of A s.t. $A = \bigcup \pi$. The sets in π are called the **blocks** of π. The set of all partitions of A is denoted by $\Pi(A)$

Theorem 1.24. $\Pi(A)$ *is a complete lattice and it's isomorphic to the lattice* $\mathrm{Eq}(A)$.

Definition 1.25. The lattice $\Pi(A)$ is called the **lattice of partitions** of A

Definition 1.26. Let L be a lattice. An element a in L is **compact** iff whenever $\bigvee A$ exists and $a \leq \bigvee A$ for $A \subseteq L$, then $a \leq \bigvee B$ for some finite $B \subseteq A$. L is **compactly generated** iff every element in L is a sup of compact elements. A lattice is **algebraic** if it is complete and compactly generated.

1.5 Closure Operator

Definition 1.27. If we are given a set A, a mapping $C : \mathcal{P}(A) \to \mathcal{P}(A)$ is called a closure **operator** on A if, for $X, Y \subseteq A$ it satisfies

C1: $X \subseteq C(X)$ (extensive)
C2: $C^2(X) = C(X)$ (idempotent)
C3: $X \subseteq Y$ implies $C(X) \subseteq C(Y)$

A subset X of A is called a **closed subset** if $C(X) = X$. The poset of closed subsets of A with set inclusion is denoted by L_C

Theorem 1.28. *Let C be a closure operator on a set A. Then L_C is a complete lattice with*

$$\bigwedge_{i \in I} C(A_i) = \bigcap_{i \in I} C(A_i)$$

and

$$\bigvee_{i \in I} C(A_i) = C(\bigcup_{i \in I} A_i)$$

Theorem 1.29. *Every complete lattice is isomorphic to the lattice of closed subsets of some set A with a closure operator C*

Proof. Let L be a complete lattice. For $X \subseteq L$ define

$$C(X) = \{a \in L : a \leq \sup X\}$$

Then C is a closure operator on L and the mapping $a \mapsto \{b \in L : b \leq a\}$ gives the desired isomorphism □

Definition 1.30. A closure operator C on the set A is an **algebraic closure operator** if for every $X \subseteq A$
C4: $C(X) = \bigcup\{C(Y) : Y \subseteq X$ and Y is finite$\}$

Theorem 1.31. *If C is an algebraic closure operator on a set A then L_C is an algebraic lattice, and the compact elements of L_C are precisely the closed sets $C(X)$, where X is a finite subset of A*

Proof. First we will show that $C(X)$ is compact if X is finite. Supppose $X = \{a_1, \ldots, a_k\}$ and

$$C(X) \subseteq \bigvee_{i \in I} C(A_i) = C(\bigcup_{i \in I} A_i)$$

For each $a_j \in X$ we have by (C4) a finite $X_j \subseteq \bigcup_{i \in I} A_i$ with $a_j \in C(X_j)$. Since there are finitely many A_i's, say $A_{j1}, \ldots, A_{jn}$, s.t.

$$X_j \subseteq A_{j1} \cup \cdots \cup A_{jn}$$

then

$$a_j \in C(A_{j1} \cup \cdots \cup A_{jn})$$

but then

$$X \subseteq \bigcup_{1 \le j \le k} C(A_{j1} \cup \cdots \cup C_{jn})$$

so

$$X \subseteq C(\bigcup_{\substack{1 \le j \le k \\ 1 \le i \le n}} A_{ji})$$

and hence

$$C(X) \subseteq \bigvee_{\substack{1 \le j \le k \\ 1 \le i \le n}} C(A_{ji})$$

So $C(X)$ is compact

Now suppose $C(Y)$ is not equal to (C) for any fintie X, it's not compact.

$\square$

Definition 1.32. If C is a closure operator on A and Y is closed subset of A, then we say a set X is a **generating set** for Y if $C(X) = Y$. The set Y is **finitely generated** if there is a finite generating set for Y. The set X is **minimal** generating set for Y if X generates Y and no proper subset of X generates Y

Corollary 1.33. *Let C be an algebraic closure operator on A. Then the finitely generated subsets of A are precisely the compact elements of L_C*

Theorem 1.34. *Every algebraic lattice is isomorphic to the lattice of closed subsets of some set A with an algebraic closure operator C*

2 Index