Algebra

Serge Lang

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1 Real Fields

1.1 Ordered Fields

Let K be a field. An **ordering** of K is a subset P of K having the following properties

ORD 1. Given $x \in K$, we have either $x \in P$,or $x = 0$ or $-x \in P$, and these three possibilities are mutually exclusive

ORD 2. If $x, y \in P$, then $x + y, xy \in P$

K is **ordered by** P, and we call P the set of **positive elements**

Suppose K is ordered by P. Since $1 \neq 0$ and $1 = 1^2 = (-1)^2$, we see that $1 \in P$. By **ORD 2**, it follows that $1 + \cdots + 1 \in P$, whence K has characteristic 0. If $x \in P$ and $x \neq 0$, then $xx^{-1} = 1 \in P$ implies that $x^{-1} \in P$

Let E be a field. Then a product of sums of squares in E is a sum of squares. If $a, b \in E$ are sum of squares and $b \neq 0$, then a/b is a sum of squares

Consider complex number:)

Let $x, y \in K$. We define $x < y$ to mean that $y - x \in P$. If $x < 0$ we say that x is **negative**.

If K is ordered and $x \in K$, $x \neq 0$, then x^2 is positive

If E has characteristic $\neq 2$, and -1 is a sum of squares in E, then every element $a \in E$ is a sum of squares, because $4a = (1 + a)^2 - (1 - a)^2$

If K is a field with an ordering P, and F is a subfield, then obviously, $P \cap F$ defines an ordering of F, which is called the **induced** ordering

Let K be an ordered field and let F be a subfield with the induced ordering. We put $|x| = x$ if $x > 0$ and $|x| = -x$ if $x < 0$. An element $\alpha \in K$ is **infinitely large** over F if $|\alpha| \geq x$ for all $x \in F$. It is **infinitely small** over F if $0 \leq |\alpha| \leq |x|$ for all $x \in F$, $x \neq 0$. α is infinitely large if and only if α^{-1} is infinitely small. K is **archimedean** over F if K has no elements which are infinitely large over F. An intermediate field F_1, $K \supset F_1 \supset F$ is **maximal archimedean over** F in K if it is archimedean over F and no other intermediate field containing F_1 is archimedean over F. We say that F is **maximal archimedean in** K if it is maximal archimedean over itself in K

Let K be an ordered field and F a subfield. Let K be an ordered field and F a subfield. Let $\mathfrak{o}$ be the set of elements of K which are not infinitely large over F. Then $\mathfrak{o}$ is a ring and that for any $\alpha \in K$, we have α or $\alpha^{-1} \in \mathfrak{o}$. Hence $\mathfrak{o}$ is what is called a valuation ring, containing F. Let $\mathfrak{m}$ be the ideal of all $\alpha \in K$ which are infinitely small over F. Then $\mathfrak{m}$ is the unique maximal ideal of $\mathfrak{o}$, because any element in $\mathfrak{o}$ which is not in $\mathfrak{m}$ has an inverse in $\mathfrak{o}$. We call $\mathfrak{o}$ the **valuation ring determined by the ordering of** K/F

Proposition 1.1. *Let K be an ordered field and F a subfield. Let $\mathfrak{o}$ be the valuation ring determined by the ordering of K/F, and let $\mathfrak{m}$ be its maximal ideal. Then $\mathfrak{o}/\mathfrak{m}$ is a real field.*

Proof. Otherwise, we could write

$$-1 = \sum \alpha_i^2 + a$$

with $\alpha_i \in \mathfrak{o}$ and $a \in \mathfrak{m}$. Since $\sum \alpha_i^2$ is positive and a is infinitely small, such a relation is clearly impossible $\square$