

Model Theory for Dummies: An Introduction

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June 8, 2020

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1 Structures and Theories

1.1 Languages and Structures

Definition 1.1. A language \mathcal{L} is given by specifying the following data

1. A set of function symbols \mathcal{F} and positive integers n_f for each $f \in \mathcal{F}$
2. a set of relation symbols \mathcal{R} and positive integers n_R for each $R \in \mathcal{R}$
3. a set of constant symbols \mathcal{C}

Definition 1.2. An \mathcal{L} -structure \mathcal{M} is given by the following data

1. a nonempty set M called the **universe**, **domain** or **underlying set** of \mathcal{M}
2. a function $f^{\mathcal{M}} : M^{n_f} \rightarrow M$ for each $f \in \mathcal{F}$
3. a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$
4. an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$

We refer to $f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$ as the **interpretations** of the symbols f, R and c . We often write the structure as $\mathcal{M} = (M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C})$

Definition 1.3. Suppose that \mathcal{M} and \mathcal{N} are \mathcal{L} -structures with universes M and N respectively. An \mathcal{L} -**embedding** $\eta : \mathcal{M} \rightarrow \mathcal{N}$ is a one-to-one map $\eta : M \rightarrow N$ that

1. $\eta(f^{\mathcal{M}}(a_1, \dots, a_{n_f})) = f^{\mathcal{N}}(\eta(a_1), \dots, \eta(a_{n_f}))$ for all $f \in \mathcal{F}$ and $a_1, \dots, a_{n_f} \in M$
2. $(a_1, \dots, a_{m_R}) \in R^{\mathcal{M}}$ if and only if $(\eta(a_1), \dots, \eta(a_{m_R})) \in R^{\mathcal{N}}$ for all $R \in \mathcal{R}$ and $a_1, \dots, a_{m_R} \in M$
3. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for $c \in \mathcal{C}$

A bijective \mathcal{L} -embedding is called an \mathcal{L} -**isomorphism**. If $M \subseteq N$ and the inclusion map is an \mathcal{L} -embedding, we say either \mathcal{M} is a **substructure** of \mathcal{N} or that \mathcal{N} is an **extension** of \mathcal{M}

The **cardinality** of \mathcal{M} is $|M|$, the cardinality of the universe of \mathcal{M}

Definition 1.4. The set of \mathcal{L} -**terms** is the smallest set \mathcal{T} s.t.

1. $c \in \mathcal{T}$ for each constant symbol $c \in \mathcal{C}$
2. each variable symbol $v_i \in \mathcal{T}$ for $i = 1, 2, \dots$
3. if $t_1, \dots, t_{n_f} \in \mathcal{T}$ and $f \in \mathcal{F}$ then $f(t_1, \dots, t_{n_f}) \in \mathcal{T}$

Suppose that \mathcal{M} is an \mathcal{L} -structure and that t is a term built using variables from $\bar{v} = (v_{i_1}, \dots, v_{i_m})$. We want to interpret t as a function $t^{\mathcal{M}} : M^m \rightarrow M$. For s a subterm of t and $\bar{a} = (a_{i_1}, \dots, a_{i_m}) \in M$, we inductively define $s^{\mathcal{M}}(\bar{a})$ as follows.

1. If s is a constant symbol c , then $s^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$
 2. If s is the variable v_{i_j} , then $s^{\mathcal{M}}(\bar{a}) = a_{i_j}$
 3. If s is the term $f(t_1, \dots, t_{n_f})$, where f is a function symbol of \mathcal{L} and t_1, \dots, t_{n_f} are terms, then $s^{\mathcal{M}}(\bar{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \dots, t_{n_f}^{\mathcal{M}}(\bar{a}))$
- The function $t^{\mathcal{M}}$ is defined by $\bar{a} \mapsto t^{\mathcal{M}}(\bar{a})$

Definition 1.5. ϕ is an **atomic \mathcal{L} -formula** if ϕ is either

1. $t_1 = t_2$ where t_1 and t_2 are terms
2. $R(t_1, \dots, t_{n_R})$

The set of **\mathcal{L} -formulas** is the smallest set \mathcal{W} containing the atomic formulas s.t.

1. if $\phi \in \mathcal{W}$, then $\neg\phi \in \mathcal{W}$
2. if $\phi, \psi \in \mathcal{W}$, then $(\phi \wedge \psi), (\phi \vee \psi) \in \mathcal{W}$
3. if $\phi \in \mathcal{W}$, then $\exists v_i \phi, \forall v_i \phi \in \mathcal{W}$

We say a variable v **occurs freely** in a formula ϕ if it is not inside a $\exists v$ or $\forall v$ quantifier; otherwise we say that it's **bound**. We call a formula a **sentence** if it has no free variables. We often write $\phi(v_1, \dots, v_n)$ to make explicit the free variables in ϕ

Definition 1.6. Let ϕ be a formula with free variables from $\bar{v} = (v_{i_1}, \dots, v_{i_m})$ and let $\bar{a} = (a_{i_1}, \dots, a_{i_m}) \in M^m$. We inductively define $\mathcal{M} \models \phi(\bar{a})$ as follows

1. If ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$
2. If ϕ is $R(t_1, \dots, t_{m_R})$ then $\mathcal{M} \models \phi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), \dots, t_{m_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$
3. If ϕ is $\neg\psi$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \not\models \psi(\bar{a})$
4. If ϕ is $(\psi \wedge \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \theta(\bar{a})$
5. If ϕ is $(\psi \vee \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ or $\mathcal{M} \models \theta(\bar{a})$
6. If ϕ is $\exists v_j \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if there is $b \in M$ s.t. $\mathcal{M} \models \psi(\bar{a}, b)$
7. If ϕ is $\forall v_j \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a}, b)$ for all $b \in M$

If $\mathcal{M} \models \phi(\bar{a})$ we say that \mathcal{M} **satisfies** $\phi(\bar{a})$ or $\phi(\bar{a})$ is **true** in \mathcal{M}

Proposition 1.7. Suppose that \mathcal{M} is a substructure of \mathcal{N} , $\bar{a} \in M$ and $\phi(\bar{v})$ is a quantifier-free formula. Then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \phi(\bar{a})$

Proof. **Claim** If $t(\bar{v})$ is a term and $\bar{b} \in M$ then $t^{\mathcal{M}}(\bar{b}) = t^{\mathcal{N}}(\bar{b})$. □

Definition 1.8. We say that two \mathcal{L} -structures \mathcal{M} and \mathcal{N} are **elementarily equivalent** and write $\mathcal{M} \equiv \mathcal{N}$ if

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{N} \models \phi$$

for all \mathcal{L} -sentences ϕ

We let $\text{Th}(\mathcal{M})$, the **full theory** of \mathcal{M} be the set of \mathcal{L} -sentences ϕ s.t. $\mathcal{M} \models \phi$

Theorem 1.9. Suppose that $j : \mathcal{M} \rightarrow \mathcal{N}$ is an isomorphism. Then $\mathcal{M} \equiv \mathcal{N}$

Proof. Show by induction on formulas that $\mathcal{M} \models \phi(a_1, \dots, a_n)$ if and only if $\mathcal{N} \models \phi(j(a_1), \dots, j(a_n))$ for all formulas ϕ \square

1.2 Theories

Let \mathcal{L} be a language. An \mathcal{L} -**theory** T is a set of \mathcal{L} -sentences. We say that \mathcal{M} is a **model** of T and write $\mathcal{M} \models T$ if $\mathcal{M} \models \phi$ for all sentences $\phi \in T$. A theory is **satisfiable** if it has a model.

A class of \mathcal{L} -structures \mathcal{K} is an **elementary class** if there is an \mathcal{L} -theory T s.t. $\mathcal{K} = \{\mathcal{M} : \mathcal{M} \models T\}$

Example 1.1 (Linear Orders). Let $\mathcal{L} = \{<\}$, where $<$ is a binary relation symbol. The class of linear order is axiomatized by the \mathcal{L} -sentences

$$\begin{aligned} \forall x \neg(x < x) \\ \forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z) \\ \forall x \forall y (x < y \vee x = y \vee y < x) \end{aligned}$$

Example 1.2 (Groups). Let $\mathcal{L} = \{\cdot, e\}$ where \cdot is a binary function symbol and e is a constant symbol. The class of groups is axiomatized by

$$\begin{aligned} \forall x \ e \cdot x = x \cdot e = x \\ \forall x \forall y \forall z \ x \cdot (y \cdot z) = (x \cdot y) \cdot z \\ \forall x \exists y \ x \cdot y = y \cdot x = e \end{aligned}$$

Example 1.3 (Left R -modules). Let R be a ring with multiplicative identity 1. Let $\mathcal{L} = \{+, 0\} \cup \{r : r \in R\}$ where $+$ is a binary function symbol, 0 is a constant, and r is a unary function symbol for $r \in R$. In an R -module, we will interpret r as scalar multiplication by R . The axioms for R -modules are

$$\begin{aligned} \forall x \ r(x + y) &= r(x) + r(y) \text{ for each } r \in R \\ \forall x \ (r + s)(x) &= r(x) + s(x) \text{ for each } r, s \in R \\ \forall x \ r(s(x)) &= rs(x) \text{ for } r, s \in R \\ \forall x \ 1(x) &= x \end{aligned}$$

Example 1.4 (Rings and Fields). Let \mathcal{L}_r be the language of rings $\{+, -, \cdot, 0, 1\}$, where $+$, $-$ and \cdot are binary function symbols and 0 and 1 are constants. The axioms for rings are given by

$$\begin{aligned} \forall x \forall y \forall z (x - y = z &\leftrightarrow x = y + z) \\ \forall x x \cdot 0 &= 0 \\ \forall x \forall y \forall z x \cdot (y \cdot z) &= (x \cdot y) \cdot z \\ \forall x x \cdot 1 &= 1 \cdot x = x \\ \forall x \forall y \forall z x \cdot (y + z) &= (x \cdot y) + (x \cdot z) \\ \forall x \forall y \forall z (x + y) \cdot z &= (x \cdot z) + (y \cdot z) \end{aligned}$$

We axiomatize the class of fields by adding

$$\begin{aligned} \forall x \forall y x \cdot y &= y \cdot x \\ \forall x (x \neq 0 &\rightarrow \exists y x \cdot y = 1) \end{aligned}$$

We axiomatize the class of algebraically closed fields by adding to the field axioms the sentences

$$\forall a_0 \dots \forall a_{n-1} \exists x x^n + \sum_{i=1}^{n-1} a_i x^i = 0$$

for $n = 1, 2, \dots$. Let ACF be the axioms for algebraically closed fields.

Let ψ_p be the \mathcal{L}_r -sentence $\forall x \underbrace{x + \dots + x}_{p\text{-times}} = 0$, which asserts that a field has characteristic p . For $p > 0$ a prime, let $\text{ACF}_p = \text{ACF} \cup \{\psi_p\}$ and $\text{ACF}_0 = \text{ACF} \cup \{\neg\psi_p : p > 0\}$ be the theories of algebraically closed fields of characteristic p and zero respectively

Definition 1.10. Let T be an \mathcal{L} -theory and ϕ an \mathcal{L} -sentence. We say that ϕ is a **logical consequence** of T and write $T \models \phi$ if $\mathcal{M} \models \phi$ whenever $\mathcal{M} \models T$

Proposition 1.11. 1. Let $\mathcal{L} = \{+, <, 0\}$ and let T be the theory of ordered abelian groups. Then $\forall x (x \neq 0 \rightarrow x + x \neq 0)$ is a logical consequence of T

2. Let T be the theory of groups where every element has order 2. Then $T \not\models \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_1 \neq x_3)$

Proof. 1. $\mathbb{Z}/2\mathbb{Z} \models T \wedge \neg \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_1 \neq x_3)$

□

1.3 Definable Sets and Interpretability

Definition 1.12. Let $\mathcal{M} = (M, \dots)$ be an \mathcal{L} -structure. We say that $X \subseteq M^n$ is **definable** if and only if there is an \mathcal{L} -formula $\phi(v_1, \dots, v_n, w_1, \dots, w_m)$ and $\bar{b} \in M^m$ s.t. $X = \{\bar{a} \in M^n : \mathcal{M} \models \phi(\bar{a}, \bar{b})\}$. We say that $\phi(\bar{v}, \bar{b})$ **defines** X . We say that X is **A-definable** or **definable over A** if there is a formula $\psi(\bar{v}, w_1, \dots, w_l)$ and $\bar{b} \in A^l$ s.t. $\psi(\bar{v}, \bar{b})$ defines X

A number of examples using \mathcal{L}_r , the language of rings

- Let $\mathcal{M} = (R, +, -, \cdot, 0, 1)$ be a ring. Let $p(X) \in R[X]$. Then $Y = \{x \in R : p(x) = 0\}$ is definable. Suppose that $p(X) = \sum_{i=0}^m a_i X^i$. Let $\phi(v, w_0, \dots, w_m)$ be the formula

$$w_m \cdot \underbrace{v \cdots v}_{n\text{-times}} + \cdots + w_1 \cdot v + w_0 = 0$$

Then $\phi(v, a_0, \dots, a_m)$ defines Y . Indeed, Y is A -definable for any $A \supseteq \{a_0, \dots, a_m\}$

- Let $\mathcal{M} = (\mathbb{R}, +, -, \cdot, 0, 1)$ be the field of real numbers. Let $\phi(x, y)$ be the formula

$$\exists z (z \neq 0 \wedge y = x + z^2)$$

Because $a < b$ if and only if $\mathcal{M} \models \phi(a, b)$, the ordering is \emptyset -definable

- Consider the natural numbers \mathbb{N} as an $\mathcal{L} = \{+, \cdot, 0, 1\}$ structure. There is an \mathcal{L} -formula $T(e, x, s)$ s.t. $\mathbb{N} \models T(e, x, s)$ if and only if the Turing machine with program coded by e halts on input x in at most s steps. Thus the Turing machine with program e halts on input x if and only if $\mathbb{N} \models \exists s T(e, x, s)$. So the halting computations is definable

Proposition 1.13. Let \mathcal{M} be an \mathcal{L} -structure. Suppose that D_n is a collection of subsets of M^n for all $n \geq 1$ and $\mathcal{D} = (D_n : n \geq 1)$ is the smallest collection s.t.

1. $M^n \in D_n$
2. for all n -ary function symbols f of \mathcal{L} , the graph of $f^{\mathcal{M}}$ is in D_{n+1}
3. for all n -ary relation symbols R of \mathcal{L} , $R^{\mathcal{M}} \in D_n$
4. for all $i, j \leq n$, $\{(x_1, \dots, x_n) \in M^n : x_i = x_j\} \in D_n$
5. if $X \in D_n$, then $M \times X \in D_{n+1}$
6. each D_n is closed under complement, union and intersection
7. if $X \in D_{n+1}$ and $\pi : M^{n+1} \rightarrow M^n$ is the projection $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$, then $\pi(X) \in D_n$
8. if $X \in D_{n+m}$ and $b \in M^m$, then $\{a \in M^n : (a, b) \in X\} \in D_n$

Thus $X \subseteq M^n$ is definable if and only if $X \in D_n$

Proposition 1.14. *Let \mathcal{M} be an \mathcal{L} -structure. If $X \subset M^n$ is A -definable, then every \mathcal{L} -automorphism of \mathcal{M} that fixes A pointwise fixes X setwise (that is, if σ is an automorphism of M and $\sigma(a) = a$ for all $a \in A$, then $\sigma(X) = X$)*

Proof.

$$\mathcal{M} \models \psi(\bar{v}, \bar{a}) \leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{v}), \sigma(\bar{a})) \leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{v}), \bar{a})$$

In other words, $\bar{b} \in X$ if and only if $\sigma(\bar{b}) \in X$ □

Definition 1.15. A subset S of a field L is **algebraically independent** over a subfield K if the elements of S do not satisfy any non-trivial polynomial equation with coefficients in K

Corollary 1.16. *The set of real numbers is not definable in the field of complex numbers*

Proof. If \mathbb{R} were definable, then it would be definable over a finite $A \subset \mathbb{C}$. Let $r, s \in \mathbb{C}$ be algebraically independent over A with $r \in \mathbb{R}$ and $s \notin \mathbb{R}$. There is an automorphism σ of \mathbb{C} s.t. $\sigma|_A$ is the identity and $\sigma(r) = s$. Thus $\sigma(\mathbb{R}) \neq \mathbb{R}$ and \mathbb{R} is not definable over A □

We say that an \mathcal{L}_0 -structure \mathcal{N} is **definably interpreted** in an \mathcal{L} -structure \mathcal{M} if and only if we can find a definable $X \subseteq M^n$ for some n and we can interpret the symbols of \mathcal{L}_0 as definable subsets and functions on X so that the resulting \mathcal{L}_0 -structure is isomorphic to \mathcal{M}

For example, let K be a field and G be $\text{GL}_2(K)$, the group of invertible 2×2 matrices over K . Let $X = \{(a, b, c, d) \in K^4 : ad - bc \neq 0\}$. Let $f : X^2 \rightarrow X$ by

$$f((a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2)) = (a_1a_2 + b_1c_2, a_1b_2 + b_1d_2, c_1a_2 + d_1c_2, c_1b_2 + d_1d_2)$$

X and f are definable in $(K, +, \cdot)$, and the set X with operation f is isomorphic to $\text{GL}_2(K)$, where the identity element of X is $(1, 0, 0, 1)$

Clearly, $(\text{GL}_n(K), \cdot, e)$ is definably interpreted in $(K, +, \cdot, 0, 1)$. A **linear algebraic group** over K is a subgroup of $\text{GL}_n(K)$ defined by polynomial equations over K . Any linear algebraic group over K is definably interpreted in K

Let F be an infinite field and let G be the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a, b \in F, a \neq 0$. This group is isomorphic to the group of affine transformations $x \mapsto ax + b$, where $a, b \in F$ and $a \neq 0$

We will show that F is definably interpreted in the group G . Let

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}$$

where $\tau \neq 0$. Let

$$A = \{g \in G : g\alpha = \alpha g\} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F \right\}$$

$$B = \{g \in G : g\beta = \beta g\} = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} : x \neq 0 \right\}$$

Clearly A, B are definable using parameters α and β

B acts on A by conjugation

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{y}{x} \\ 0 & 1 \end{pmatrix}$$

We can define the map $i : A \setminus \{1\} \rightarrow B$ by $i(a) = b$ if and only if $b^{-1}ab = \alpha$, that is

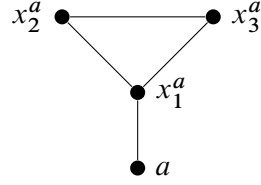
$$i \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

Define an operation $*$ on A by

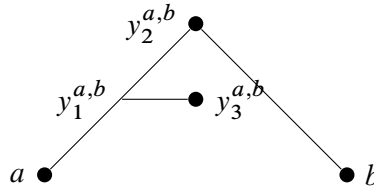
$$a * b = \begin{cases} i(b)a(i(b))^{-1} & \text{if } b \neq I \\ 1 & \text{if } b = I \end{cases}$$

where I is the identity matrix. Now $(F, +, \cdot, 0, 1) \cong (A, \cdot, *, 1, \alpha)$

Very complicated structures can often be interpreted in seemingly simpler ones. For example, any structure in a countable language can be interpreted in a graph. Let $(A, <)$ be a linear order. For each $a \in A$, G_A will have vertices a, x_1^a, x_2^a, x_3^a and contain the subgraph

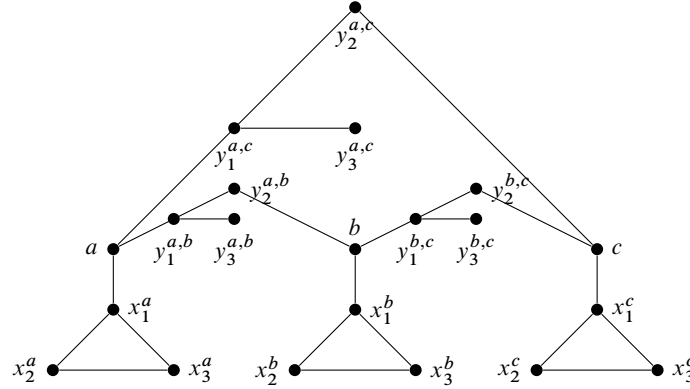


If $a < b$, then G_A will have vertices $y_1^{a,b}, y_2^{a,b}, y_3^{a,b}$ and contain the subgraph



Let $V_A = A \cup \{x_1^a, x_2^a, x_3^a : a \in A\} \cup \{y_1^{a,b}, y_2^{a,b}, y_3^{a,b} : a, b \in A \text{ and } a < b\}$, and let R_A be the smallest symmetric relation containing all edges drawn above.

For example, if A is the three-element linear order $a < b < c$, then G_A is the graph



Let $\mathcal{L} = \{R\}$ where R is a binary relation. Let $\phi(x, u, v, w)$ be the formula asserting that x, u, v, w are distinct, there are edges $(x, u), (u, v), (v, w), (u, w)$ and these are the only edges involving u, v, w . $G_A \models \phi(a, x_1^a, x_2^a, x_3^a)$ for all $a \in A$.

$\psi(x, y, u, v, w)$ asserts that x, y, u, v, w are distinct. $(x, u), (u, v), (u, w), (v, y)$

Define $\theta_i(z)$ as follows:

$$\begin{aligned}\theta_0(z) &:= \exists u \exists v \exists w \phi(z, u, v, w) \\ \theta_1(z) &:= \exists x \exists v \exists w \phi(x, z, v, w) \\ \theta_2(z) &:= \exists u \exists u \exists w \phi(x, u, z, w) \\ \theta_3(z) &:= \exists x \exists y \exists v \exists w \psi(x, y, z, v, w) \\ \theta_4(z) &:= \exists x \exists y \exists u \exists w \psi(x, y, u, z, w) \\ \theta_5(z) &:= \exists x \exists y \exists u \exists v \psi(x, y, u, v, z)\end{aligned}$$

If $a, b \in A$ and $a < b$, then

$$G_A \models \theta_0(a) \wedge \theta_1(x_1^a) \wedge \theta_2(x_2^a) \wedge \theta_2(x_3^a)$$

and

$$G_A \models \theta_3(y_1^{a,b}) \wedge \theta_4(y_2^{a,b}) \wedge \theta_5(y_3^{a,b})$$

Lemma 1.17. *If $(A, <)$ is a linear order, then for all vertices x in G , there is a unique $i \leq 5$ s.t. $G_A \models \theta_i(x)$*

Let T be the \mathcal{L} -theory with the following axioms

1. R is symmetric and irreflexive
2. for all x , exactly one θ_i holds
3. if $\theta_0(x)$ and $\theta_0(y)$ then $\neg R(x, y)$
4. if $\exists u \exists v \exists w \psi(x, y, u, v, w)$
then $\forall u_1 \forall v_1 \forall w_1 \neg \psi(y, x, u_1, v_1, w_1)$
5. if $\exists u \exists v \exists w \psi(x, y, u, v, w)$ and $\exists u \exists v \exists w \psi(y, z, u, v, w)$ then
 $\exists u \exists v \exists w \psi(x, z, u, v, w)$
6. if $\theta_0(x)$ and $\theta_0(y)$, then either $x = y$ or $\exists u \exists v \exists w \psi(x, y, u, v, w)$ or
 $\exists u \exists v \exists w \psi(y, x, u, v, w)$
7. if $\phi(x, u, v, w) \wedge \phi(x, u', v', w')$, then $u = u', v = v', w = w'$
8. if $\psi(x, y, u, v, w) \wedge \psi(x, y, u', v', w')$, then $u' = u, v = v', w = w'$

If $(A, <)$ is a linear order, then $G_A \models T$

Suppose $G \models T$. Let $X_G = \{x \in G : G \models \theta_0(x)\}$

Lemma 1.18. *If $(A, <)$ is a linear order, then $(X_{G_A}, <_{G_A}) \cong (A, <)$. Moreover, $G_{X_G} \cong G$ for all $G \models T$*

Definition 1.19. An \mathcal{L}_0 -structure \mathcal{N} is **interpretable** in an \mathcal{L} -structure M if there is a definable $X \subseteq M^n$, a definable equivalence relation E on X , and for each symbol of \mathcal{L}_0 we can find definable E -invariant sets on X s.t. X/E with the induced structure is isomorphic to \mathcal{N}

1.4 Answers to Exercises

Exercise 1.4.1. 1. transform ψ to CNF
2. prenex normal form



Exercise 1.4.2. 1.
2. enumerate \mathcal{M} 's functions, relations and constants

Exercise 1.4.3. ¹ Note that every \mathcal{L} -structure \mathcal{M} of size κ is isomorphic to an \mathcal{L} -structure with domain κ . For each relation symbols, we have 2^κ options. If the language has size λ , this is at most $(2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^{\max(\lambda, \kappa)}$

Exercise 1.4.4.

$$\begin{aligned} T \models \phi &\Leftrightarrow \forall \mathcal{M} \mathcal{M} \models T \rightarrow \mathcal{M} \models \phi \\ &\Leftrightarrow \forall \mathcal{M} \mathcal{M} \models T' \rightarrow \mathcal{M} \models \phi \\ &\Leftrightarrow T' \models \phi \end{aligned}$$

Exercise 1.4.5. Follow the definition

Exercise 1.4.6. Since there is no model \mathcal{M} s.t. $\mathcal{M} \models T$. It's true that $T \models \phi$

Exercise 1.4.7. 1. Suppose $\mathcal{M} \models \phi$, then $E^{\mathcal{M}}$ is an equivalent relation and each equivalence class's cardinality is 2
2. follows from number theory
3. [DJMM12]

Exercise 1.4.8. TBD

Exercise 1.4.9. $G(f) = \{(\bar{x}, \bar{y}) \in M^{n+m} \mid \phi(\bar{x}, \bar{y})\}$ and $G(g) = \{(\bar{y}, \bar{z}) \in M^{m+l} \mid \psi(\bar{y}, \bar{z})\}$. Hence $G(g \circ f) = \{(\bar{x}, \bar{z}) \in M^{n+l} \mid \phi(\bar{x}, \bar{y}) \wedge \psi(\bar{y}, \bar{z})\}$

Exercise 1.4.10. $\phi(\bar{a}, b)$ really defines a function and since $\phi(\bar{a}, y) \rightarrow y = b$

2 Basic Techniques

2.1 The Compactness Theorem

Some points of proofs

- Proofs are finite
- (Soundness) If $T \vdash \phi$, then $T \models \phi$

¹stackexchange

- If T is a finite set of sentences, then there is an algorithm that, when given a sequence of \mathcal{L} -formulas σ and an \mathcal{L} -sentence ϕ , will decide whether σ is a proof of ϕ from T

A language \mathcal{L} is **recursive** if there is an algorithm that decides whether a sequence of symbols is an \mathcal{L} -formula. An \mathcal{L} -theory T is **recursive** if there is an algorithm that when given an \mathcal{L} -sentence ϕ as input, decides whether $\phi \in T$

Proposition 2.1. *If \mathcal{L} is a recursive language and T is a recursive \mathcal{L} -theory, then $\{\phi : T \vdash \phi\}$ is recursively enumerable; that is, there is an algorithm that when given ϕ as input will halt accepting if $T \vdash \phi$ and not halt if $T \not\vdash \phi$*

Proof. There is $\sigma_0, \sigma_1, \dots$ a computable listing of all finite sequence of \mathcal{L} -formulas. At stage i , we check to see whether σ_i is a proof of ψ from T . If it is, then halt. \square

Theorem 2.2 (Gödel's Completeness Theorem). *Let T be an \mathcal{L} -theory and ϕ an \mathcal{L} -sentence, then $T \models \phi$ if and only if $T \vdash \phi$*

We say that an \mathcal{L} -theory T is **inconsistent** if $T \vdash (\phi \wedge \neg\phi)$ for some sentence ϕ .

Corollary 2.3. *T is consistent if and only if T is satisfiable*

Proof. Suppose that T is not satisfiable, then every model of T is a model of $\phi \wedge \neg\phi$. Thus by the Completeness theorem $T \vdash (\phi \wedge \neg\phi)$ \square

Theorem 2.4 (Compactness Theorem). *T is satisfiable if and only if every finite subset of T is satisfiable*

Proof. If T is not satisfiable, then T is inconsistent. Let σ be a proof of a contradiction from T . Because σ is finite, only finitely many assumptions from T are used in the proof. Thus there is a finite $T_0 \subseteq T$ s.t. σ is a proof of a contradiction from T_0 \square

2.1.1 Henkin Constructions

A theory T is **finitely satisfiable** if every finite subset of T is satisfiable. We will show that every finitely satisfiable theory T is satisfiable.

Definition 2.5. We say that an \mathcal{L} -theory T has the **witness property** if whenever $\phi(v)$ is an \mathcal{L} -formula with one free variable v , then there is a constant symbol $c \in \mathcal{L}$ s.t. $T \vdash (\exists v\phi(v)) \rightarrow \phi(c) \in T$

An \mathcal{L} -theory T is **maximal** if for all ϕ either $\phi \in T$ or $\neg\phi \in T$

Lemma 2.6. Suppose T is a maximal and finitely satisfiable \mathcal{L} -theory. If $\Delta \subseteq T$ is finite and $\Delta \models \psi$, then $\psi \in T$

Proof. If $\psi \notin T$, then $\neg\psi \in T$ but $\Delta \cup \{\psi\}$ is unsatisfiable □

Lemma 2.7. Suppose that T is a maximal and finitely satisfiable \mathcal{L} -theory with the witness property. Then T has a model. In fact, if κ is a cardinal and \mathcal{L} has at most κ constant symbols, then there is $\mathcal{M} \models T$ with $|\mathcal{M}| \leq \kappa$

Proof. Let \mathcal{C} be the set of constant symbols of \mathcal{L} . For $c, d \in \mathcal{C}$, we say $c \sim d$ if $c = d \in T$

Claim 1 \sim is an equivalence relation.

The universe of our model will be $M = \mathcal{C} / \sim$. Clearly $|M| \leq \kappa$. We let c^* denote the equivalence class of c and interpret c as its equivalence class, that is, $c^{\mathcal{M}} = c^*$

Suppose that R is an n -ary relation symbol of \mathcal{L}

Claim 2 Suppose that $c_1, \dots, c_n, d_1, \dots, d_n \in \mathcal{C}$ and $c_i \sim d_i$ for $i = 1, \dots, n$, then $R(\bar{c})$ if and only if $R(\bar{d})$

By Lemma 2.6, if one of $R(\bar{c})$ and $R(\bar{d})$ is in T , then both are in T

$$R^{\mathcal{M}} = \{(c_1^*, \dots, c_n^*) : R(c_1, \dots, c_n) \in T\}$$

Suppose that f is an n -ary function symbol of \mathcal{L} and $c_1, \dots, c_n \in \mathcal{C}$. Because $\emptyset \models \exists v f(c_1, \dots, c_n) = v$, and T has the witness property, then there is $c_{n+1} \in \mathcal{C}$ s.t. $f(c_1, \dots, c_n) = c_{n+1} \in T$. As above, if $d_i \sim c_i$ for $i = 1, \dots, n+1$, then $f(d_1, \dots, d_n) = d_{n+1} \in T$. Thus we get a well-defined function $f^{\mathcal{M}} : M^n \rightarrow M$ by

$$f^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^* \text{ if and only if } f(c_1, \dots, c_n) = d \in T$$

Claim 3 Suppose that t is a term using free variables from v_1, \dots, v_n . If $c_1, \dots, c_n, d \in \mathcal{C}$, then $t(c_1, \dots, c_n) = d \in T$ if and only if $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$

(\Rightarrow) If t is a constant symbol, then $c = d \in T$ and $c^{\mathcal{M}} = c^* = d^*$

If t is the variable v_i , then $c_i = d \in T$ and $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = c_i^* = d^*$

Suppose that the claim is true for t_1, \dots, t_m and t is $f(t_1, \dots, t_m)$. Using the witness property and Lemma 2.6, we can find $d, d_1, \dots, d_m \in \mathcal{C}$ s.t. $t_i(c_1, \dots, c_n) = d_i \in T$ for $i \leq m$ and $f(d_1, \dots, d_m) = d \in T$. By our induction hypothesis, $t_i^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d_i^*$ and $f^{\mathcal{M}}(d_1^*, \dots, d_m^*) = d^*$. Thus $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$

(\Leftarrow) Suppose $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = d^*$. By the witness property, there is a $e \in \mathcal{C}$ s.t. $t(c_1, \dots, c_n) = e \in T$. Using the (\Rightarrow) direction of the proof, $t^{\mathcal{M}}(c_1^*, \dots, c_n^*) = e^*$. Thus $e^* = d^*$ and $e = d \in T$

Claim 4 For all \mathcal{L} -formulas $\phi(v_1, \dots, v_n)$ and $c_1, \dots, c_n \in \mathcal{C}$, $\mathcal{M} \models \phi(\bar{c}^*)$ if and only if $\phi(\bar{c}) \in T$

Suppose that ϕ is $t_1 = t_2$. By Lemma 2.6 and the witness property, we can find d_1 and d_2 s.t. $t_1(\bar{c}) = d_1, t_2(\bar{c}) = d_2 \in T$. By Claim 3, $t_i^{\mathcal{M}}(\bar{c}^*) = d_i^*$. Then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{c}^*) &\Leftrightarrow d_1^* = d_2^* \\ &\Leftrightarrow d_1 = d_2 \in T \\ &\Leftrightarrow t_1(\bar{c}) = t_2(\bar{c}) \in T \end{aligned}$$

Suppose that ϕ is $R(t_1, \dots, t_m)$. There are $d_1, \dots, d_m \in \mathcal{C}$ s.t. $t_i(\bar{c}) = d_i \in T$. Thus

$$\begin{aligned} \mathcal{M} \models \phi(\bar{c}^*) &\Leftrightarrow \bar{d}^* \in R^{\mathcal{M}} \\ &\Leftrightarrow R(\bar{d}) \in T \\ &\Leftrightarrow \phi(\bar{c}) \in T \end{aligned}$$

Suppose that the claim is true for ϕ . If $\mathcal{M} \models \neg\phi(\bar{c}^*)$, then $\mathcal{M} \not\models \phi(\bar{c}^*)$. By the inductive hypothesis, $\phi(\bar{c}) \notin T$. Thus by maximality, $\neg\phi(\bar{c}) \in T$. On the other hand, if $\neg\phi(\bar{c}) \in T$, then because T is finitely satisfiable, $\phi(\bar{c}) \notin T$. Thus, by induction, $\mathcal{M} \models \phi(\bar{c}^*)$. \square

Lemma 2.8. *Let T be a finitely satisfiable \mathcal{L} -theory. There is a language $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ a finitely satisfiable \mathcal{L}^* -theory s.t. any \mathcal{L}^* -theory extending T^* has the witness property. We can choose \mathcal{L}^* s.t. $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$*

Proof. We first show that there is a language $\mathcal{L}_1 \supseteq \mathcal{L}$ and a finitely satisfiable \mathcal{L}_1 -theory $T_1 \supseteq T$ s.t. for any \mathcal{L} -formula $\phi(v)$ there is an \mathcal{L}_1 -constant symbol c s.t. $T_1 \models (\exists v \phi(v)) \rightarrow \phi(c)$. For each \mathcal{L} -formula $\phi(v)$, let c_ϕ be a new constant symbol and let $\mathcal{L}_1 = \mathcal{L} \cup \{c_\phi : \phi(v) \text{ an } \mathcal{L}\text{-formula}\}$. For each \mathcal{L} -formula $\phi(v)$, let Θ_ϕ be the \mathcal{L}_1 -sentence $(\exists v \phi(v)) \rightarrow \phi(c_\phi)$. Let $T_1 = T \cup \{\Theta_\phi : \phi(v) \text{ an } \mathcal{L}\text{-formula}\}$

Claim T_1 is finitely satisfiable

Suppose that Δ is a finite subset of T_1 . Then $\Delta = \Delta_0 \cup \{\Theta_{\phi_1}, \dots, \Theta_{\phi_n}\}$ where Δ_0 is a finite subset of T and there is $\mathcal{M} \models \Delta_0$. We will make \mathcal{M} into an $\mathcal{L} \cup \{c_{\phi_1}, \dots, c_{\phi_n}\}$ -structure \mathcal{M}' . If $\mathcal{M} \models \exists v \phi(v)$, choose a_i some element of M s.t. $\mathcal{M} \models \phi(a_i)$ and let $c_{\phi_i}^{\mathcal{M}'} = a_i$. Otherwise, let $c_{\phi_i}^{\mathcal{M}'}$ be any element of M . Clearly $\mathcal{M}' \models \Theta_{\phi_i}$ for $i \leq n$. Thus T_1 is finitely satisfiable.

We now iterate the construction above to build a sequence of languages $\mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \dots$ and a sequence of finitely satisfiable \mathcal{L}_i -theories $T \subseteq$

$T_1 \subseteq T_2 \subseteq \dots$ s.t. if $\phi(v)$ is an \mathcal{L}_i -formula then there is a constant symbol $c \in \mathcal{L}_{i+1}$ s.t. $T_{i+1} \models (\exists v \phi(v)) \rightarrow \phi(c)$

Let $\mathcal{L}^* = \bigcup \mathcal{L}_i$ and $T^* = \bigcup T_i$. If $|\mathcal{L}_i|$ is the number of relation, function and constant symbols in \mathcal{L}_i , then there are at most $|\mathcal{L}_i| + \aleph_0$ formulas in \mathcal{L}_i . Thus by induction, $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$ \square

Lemma 2.9. Suppose that T is a finitely satisfiable \mathcal{L} -theory and ϕ is an \mathcal{L} -sentence, then either $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely satisfiable

Corollary 2.10. If T is a finitely satisfiable \mathcal{L} -theory, then there is a maximal finitely satisfiable \mathcal{L} -theory $T' \supseteq T$

Proof. Let I be the set of all finitely satisfiable \mathcal{L} -theory containing T . We partially order I by inclusion. If $C \subseteq I$ is a chain, let $T_C = \bigcup \{\Sigma : \Sigma \in C\}$. If Δ is a finite subset of T_C , then there is a $\Sigma \in C$ s.t. $\Delta \subseteq \Sigma$, so T_C is finitely satisfiable and $T_C \supseteq \Sigma$ for all $\Sigma \in C$. Thus every chain in I has an upper bound, and we can apply Zorn's lemma to find a $T' \in I$ maximal w.r.t. the partial order. \square

Theorem 2.11 (strengthening of Compactness Theorem). If T is a finitely satisfiable \mathcal{L} -theory and κ is an infinite cardinal with $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality at most κ

Proof. By Lemma 2.8, we can find $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ a finitely satisfiable \mathcal{L}^* -theory s.t. any \mathcal{L}^* -theory extending T^* has the witness property and the cardinality of \mathcal{L}^* is at most κ . By Corollary 2.10, we can find a maximal finitely satisfiable \mathcal{L}^* -theory $T' \supseteq T^*$. Because T' has the witness property, Lemma 2.7 ensures that there is $\mathcal{M} \models T$ with $|M| \leq \kappa$ \square

Proposition 2.12. Let $\mathcal{L} = \{\cdot, +, <, 0, 1\}$ and let $\text{Th}(\mathbb{N})$ be the full \mathcal{L} -theory of the natural numbers. There is $\mathcal{M} \models \text{Th}(\mathbb{N})$ and $a \in M$ s.t. a is larger than every natural number

Proof. Let $\mathcal{L}^* = \mathcal{L} \cup \{c\}$ where c is a new constant symbol and let

$$T = \text{Th}(\mathbb{N}) \cup \left\{ \underbrace{1 + 1 + \dots + 1}_{n\text{-times}} < c : \text{for } n = 1, 2, \dots \right\}$$

If Δ is a finite subset of T we can make \mathbb{N} a model of Δ by interpreting c as a suitably large natural number. Thus T is finitely satisfiable and there is $\mathcal{M} \models T$. \square

Lemma 2.13. If $T \models \phi$, then $\Delta \models T$ for some finite $\Delta \subseteq T$

Proof. Suppose not. Let $\Delta \subseteq T$ be finite. Because $\Delta \not\models \phi$, $\Delta \cup \{\neg\phi\}$ is satisfiable. Thus $T \cup \{\neg\phi\}$ is finitely satisfiable and by the compactness theorem, $T \not\models \phi$ \square

2.2 Complete Theories

Definition 2.14. An \mathcal{L} -theory T is called **complete** if for any \mathcal{L} -sentence ϕ either $T \models \phi$ or $T \models \neg\phi$

For \mathcal{M} an \mathcal{L} -structure, then the full theory

$$\text{Th}(\mathcal{M}) = \{\phi : \phi \text{ is an } \mathcal{L}\text{-sentence and } \mathcal{M} \models \phi\}$$

is a complete theory.

Proposition 2.15. Let T be an \mathcal{L} -theory with infinite models. If κ is an infinite cardinal and $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality κ

Proof. Let $\mathcal{L}^* = \mathcal{L} \cup \{c_\alpha : \alpha < \kappa\}$, where each c_α is new constant symbol, and let T^* be the \mathcal{L}^* -theory $T \cup \{c_\alpha \neq c_\beta : \alpha, \beta < \kappa, \alpha \neq \beta\}$. Clearly if $\mathcal{M} \models T^*$, then \mathcal{M} is a model of T of cardinality at least κ . Thus by Theorem 2.11, it suffices to show that T^* is finitely satisfiable. But if $\Delta \subseteq T^*$ is finite, then $\Delta \subseteq T \cup \{c_\alpha \neq c_\beta : \alpha \neq \beta, \alpha, \beta \in I\}$, where I is a finite subset of κ . Let \mathcal{M} be an infinite model of T . We can interpret the symbols $\{c_\alpha : \alpha \in I\}$ as $|I|$ distinct elements of M . Because $\mathcal{M} \models \Delta$, T^* is finitely satisfiable. \square

Definition 2.16. Let κ be an infinite cardinal and let T be a theory with models of size κ . We say that T is κ -**categorical** if any two models of T of cardinality κ are isomorphic.

Let $\mathcal{L} = \{+, 0\}$ be the language of additive groups and let T be the \mathcal{L} -theory of torsion-free divisible Abelian groups. The axioms of T are the axioms for Abelian groups together with the axioms

$$\begin{aligned} \forall x (x \neq 0 \rightarrow \underbrace{x + \cdots + x}_{n\text{-times}} \neq 0) \\ \forall y \exists x \underbrace{x + \cdots + x}_{n\text{-times}} = y \end{aligned}$$

for $n = 1, 2, \dots$

Proposition 2.17. The theory of torsion-free divisible Abelian groups is κ -categorical for all $\kappa > \aleph_0$

Proof. We first argue that models of T are essentially vector spaces over the field of rational numbers \mathbb{Q} . If V is any vector space over \mathbb{Q} , then the underlying additive group V is a model of T . Check StackExchange. On the other hand, if $G \models T$, $g \in G$ and $n \in \mathbb{N}$ with $g > 0$, we can find $h \in G$ s.t. $nh = g$. If $nk = g$, then $n(h - k) = 0$. Because G is torsion-free there is a unique $h \in G$ s.t. $nh = g$. We call this element g/n . We can view G as a \mathbb{Q} -vector space under the action $\frac{m}{n}g = m(g/n)$

Two \mathbb{Q} -vector spaces are isomorphic if and only if they have the same dimension. Thus the model of T are determined up to isomorphism by their dimension. If G has dimension λ , then $|G| = \lambda + \aleph_0$. If κ is uncountable and G has cardinality κ , then G has dimension κ . Thus for $\kappa > \aleph_0$ any two models of T of cardinality κ are isomorphic \square

Lemma 2.18. *Field of uncountable cardinality κ has transcendence degree κ ²*

Proof. We prove the theorem for fields with characteristic $p = 0$.

Since each characteristic 0 field contains a copy of \mathbb{Q} as its prime field, we can view F as a field extension over \mathbb{Q} . We will show that F has a subset of cardinality κ which is algebraically independent over \mathbb{Q} .

We build the claimed subset of F by transfinite induction and implicit use of the axiom of choice.

Let $S_0 = \emptyset$

Let S_1 be a singleton containing some element of F which is not algebraic over \mathbb{Q} . This is possible since algebraic numbers are countable

Define $S_{\alpha+1}$ to be S_α together with an element of F which is not a root of any non-trivial polynomial with coefficients in $\mathbb{Q} \cup S_\alpha$ since there are only $|\mathbb{Q} \cup S_\alpha| = \aleph_0 + |\alpha| < \kappa$ polynomials

Define $S_\beta = \bigcup_{\alpha < \beta} S_\alpha$

Let $P(x_1, \dots, x_n)$ be a non-trivial polynomial with coefficients in \mathbb{Q} and elements a_1, \dots, a_n in F . W.L.O.G., it is assumed that a_n was added at an ordinal $\alpha + 1$ later than the other elements. Then $P(a_1, \dots, a_{n-1}, x_n)$ is a polynomial with coefficients in $\mathbb{Q} \cup S_\alpha$. Hence $P(a_1, \dots, a_n) \neq 0$. \square

Proposition 2.19. *ACF_p is κ -categorical for all uncountable cardinals κ*

Proof. Two algebraically closed fields are isomorphic if and only if they have the same characteristic and transcendence degree. See AdvancedModernAlgebra.org. By Lemma 2.18, an algebraically closed field of transcendence degree λ has cardinality $\lambda + \aleph_0$. \square

²proofwiki

Theorem 2.20 (Vaught's Test). *Let T be a satisfiable theory with no finite models that is κ -categorical for some infinite cardinal $\kappa \geq |\mathcal{L}|$. Then T is complete*

Proof. Suppose T is not complete. Then there is a sentence ϕ s.t. $T \not\models \phi$ and $T \not\models \neg\phi$. Because $T \not\models \psi$ if and only if $T \cup \{\neg\psi\}$ is satisfiable, the theories $T_0 = T \cup \{\phi\}$ and $T_1 = T \cup \{\neg\phi\}$ are satisfiable. Because T has no finite models, both T_0 and T_1 have infinite models. By Proposition 2.15 we can find \mathcal{M}_0 and \mathcal{M}_1 of cardinality κ with $\mathcal{M}_i \models T_i$. Because \mathcal{M}_0 and \mathcal{M}_1 disagree about ϕ , they are not elementarily equivalent, and hence by Theorem 1.9, nonisomorphic. \square

Definition 2.21. We say that an \mathcal{L} -theory T is **decidable** if there is an algorithm that when given an \mathcal{L} -sentence ϕ as input decides whether $T \models \phi$

Lemma 2.22. *Let T be a recursive complete satisfiable theory in a recursive language \mathcal{L} . Then T is decidable*

Proof. Because T is satisfiable $A = \{\phi : T \models \phi\}$ and $B = \{\phi : T \models \neg\phi\}$ are disjoint. Because T is consistent $A \cup B$ is the set of all \mathcal{L} -sentences. By the Completeness Theorem, $A = \{\phi : T \vdash \phi\}$ and $B = \{\phi : T \vdash \neg\phi\}$. By Proposition 2.1 A and B are recursively enumerable. But any recursively enumerable set with a recursively enumerable complement is recursive. \square

Corollary 2.23. *For $p = 0$ or p prime, ACF_p is decidable. In particular, $\text{Th}(\mathbb{C})$, the first-order theory of the field of complex numbers, is decidable*

Corollary 2.24. *Let ϕ be a sentence in the language of rings. The following are equivalent*

1. ϕ is true in the complex number
2. ϕ is true in every algebraically closed field of characteristic zero
3. ϕ is true in some algebraically closed field of characteristic zero
4. There are arbitrarily large primes p s.t. ϕ is true in some algebraically closed field of characteristic p
5. There is an m s.t. for all $p > m$, ϕ is true in all algebraically closed fields of characteristic p

Proof. By Proposition 2.19 and Vaught's Test, ACF_p is complete.

(2) \rightarrow (5). Suppose that $ACF_0 \models \phi$. By Lemma 2.13, there is a finite $\Delta \subseteq ACF_0$ s.t. $\Delta \models \phi$. Thus if we choose p large enough, then $ACF_p \models \Delta$.

(4) \rightarrow (2). Suppose $ACF_0 \not\models \phi$. Because ACF_0 is complete, $ACF_0 \models \neg\phi$. \square

2.3 Up and Down

Definition 2.25. If \mathcal{M} and \mathcal{N} are \mathcal{L} -structures, then an \mathcal{L} -embedding $j : \mathcal{M} \rightarrow \mathcal{N}$ is called an **elementary embedding** if

$$\mathcal{M} \models \phi(a_1, \dots, a_n) \leftrightarrow \mathcal{N} \models \phi(j(a_1), \dots, j(a_n))$$

for all \mathcal{L} -formulas $\phi(v_1, \dots, v_n)$ and all $a_1, \dots, a_n \in M$

If \mathcal{M} is a substructure of \mathcal{N} , we say that it is an **elementary substructure** and write $\mathcal{M} \prec \mathcal{N}$ if the inclusion map is elementary. \mathcal{N} is an **elementary extension** of \mathcal{M}

Definition 2.26. \mathcal{M} is an \mathcal{L} -structure. Let \mathcal{L}_M be the language where we add to \mathcal{L} constant symbols m for each element of M . The **atomic diagram** of \mathcal{M} is $\{\phi(m_1, \dots, m_n) : \phi \text{ is either an atomic } \mathcal{L}\text{-formula or the negation of an atomic } \mathcal{L}\text{-formula and } \mathcal{M} \models \phi(m_1, \dots, m_n)\}$. The **elementary diagram** of \mathcal{M} is

$$\{\phi(m_1, \dots, m_n) : \mathcal{M} \models \phi(m_1, \dots, m_n), \phi \text{ an } \mathcal{L}\text{-formula}\}$$

We let $\text{Diag}(\mathcal{M})$ and $\text{Diag}_{\text{el}}(\mathcal{M})$ denote the atomic and elementary diagrams of \mathcal{M}

Lemma 2.27. 1. Suppose that \mathcal{N} is an \mathcal{L}_M -structure and $\mathcal{N} \models \text{Diag}(\mathcal{M})$, then viewing \mathcal{N} as an \mathcal{L} -structure, there is an \mathcal{L} -embedding of \mathcal{M} into \mathcal{N}
2. If $\mathcal{N} \models \text{Diag}_{\text{el}}(\mathcal{M})$, then there is an elementary embedding of \mathcal{M} into \mathcal{N}

Proof. 1. Let $j : M \rightarrow N$ by $j(m) = m^{\mathcal{N}}$. If $m_1 \neq m_2 \in \text{Diag}(\mathcal{M})$; thus $j(m_1) \neq j(m_2)$ so j is an embedding. If f is a function symbols of \mathcal{L} and $f^{\mathcal{M}}(m_1, \dots, m_n) = m_{n+1}$, then $f(m_1, \dots, m_n) = m_{n+1}$ is a formula in $\text{Diag}(\mathcal{M})$ and $f^{\mathcal{N}}(j(m_1), \dots, j(m_n)) = j(m_{n+1})$. If R is a relation symbol and $\bar{m} \in R^{\mathcal{M}}$, then $R(m_1, \dots, m_n) \in \text{Diag}(\mathcal{M})$ and $(j(m_1), \dots, j(m_n)) \in R^{\mathcal{N}}$. Hence j is an \mathcal{L} -embedding
2. j is elementary. □

Theorem 2.28 (Upward LöwenheimSkolem Theorem). Let \mathcal{M} be an infinite \mathcal{L} -structure and κ be an infinite cardinal $\kappa \geq |\mathcal{M}| + |\mathcal{L}|$. Then, there is \mathcal{N} an \mathcal{L} -structure of cardinality κ and $j : \mathcal{M} \rightarrow \mathcal{N}$ is elementary

Proof. Because $\mathcal{M} \models \text{Diag}_{\text{el}}(\mathcal{M})$, $\text{Diag}_{\text{el}}(\mathcal{M})$ is satisfiable. By Theorem 2.11, there is $\mathcal{N} \models \text{Diag}_{\text{el}}(\mathcal{M})$ of cardinality κ . By Lemma 2.27, there is an elementary $j : \mathcal{M} \rightarrow \mathcal{N}$ □

Proposition 2.29 (Tarski-Vaught Test). *Suppose that \mathcal{M} is a substructure of \mathcal{N} . Then \mathcal{M} is an elementary substructure if and only if, for any formula $\phi(v, \bar{w})$ and $\bar{a} \in M$, if there is $b \in N$ s.t. $\mathcal{N} \models \phi(b, \bar{a})$, then there is $c \in M$ s.t. $\mathcal{N} \models \phi(c, \bar{a})$*

Proof. We need to show that for all $\bar{a} \in M$ and all \mathcal{L} -formulas $\psi(\bar{v})$

$$\mathcal{M} \models \psi(\bar{a}) \Leftrightarrow \mathcal{N} \models \psi(\bar{a})$$

In Proposition 1.7, we showed that if $\phi(\bar{v})$ is quantifier free then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\phi(\bar{a})$ \square

We say that an \mathcal{L} -theory T has **built-in Skolem functions** if for all \mathcal{L} -formulas $\phi(v, w_1, \dots, w_n)$ there is a function symbol f s.t. $T \models \forall \bar{w}((\exists v \phi(v, \bar{w})) \rightarrow \phi(f(\bar{w}), \bar{w}))$. In other words, there are enough function symbols in the language to witness all existential statements.

Lemma 2.30. *Let T be an \mathcal{L} -theory. There are $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ an \mathcal{L}^* -theory s.t. T^* has built-in Skolem functions, and if $\mathcal{M} \models T$, then we can expand \mathcal{M} to $\mathcal{M}^* \models T^*$. We can choose \mathcal{L}^* s.t. $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$.*

We call T^ a **skolemization** of T*

Proof. We build a sequence of languages $\mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \dots$ and \mathcal{L}_i -theories T_i s.t. $T = T_0 \subseteq T_1 \subseteq \dots$

Given \mathcal{L}_i , let $\mathcal{L}_{i+1} = \mathcal{L} \cup \{f_\phi : \phi(v, w_1, \dots, w_n) \text{ an } \mathcal{L}_i\text{-formula}, n = 1, 2, \dots\}$, where f_ϕ is an n -ary function symbol. For $\phi(v, \bar{w})$ an \mathcal{L}_i -formula, let Ψ_ϕ be the sentence

$$\forall \bar{w}((\exists v \phi(v, \bar{w})) \rightarrow \phi(f_\phi(\bar{w}), \bar{w}))$$

and let $T_{i+1} = T_i \cup \{\Psi_\phi : \phi \text{ an } \mathcal{L}_i\text{-formula}\}$

Claim If $\mathcal{M} \models T_i$, then we can interpret the function symbols of $\mathcal{L}_{i+1} \setminus \mathcal{L}_i$ so that $\mathcal{M} \models T_{i+1}$

Let c be some fixed element of M . If $\phi(v, w_1, \dots, w_n)$ is an \mathcal{L}_i -formula, we find a function $g : M^n \rightarrow M$ s.t. $\bar{a} \in M^n$ and $X_{\bar{a}} = \{b \in M : \mathcal{M} \models \phi(b, \bar{a})\}$ is nonempty, then $g(\bar{a}) \in X_{\bar{a}}$, and if $X_{\bar{a}} = \emptyset$, then $g(\bar{a}) = c$. Thus if $\mathcal{M} \models \exists v \phi(v, \bar{a})$, then $\mathcal{M} \models \phi(g(\bar{a}), \bar{a})$. If we interpret f_ϕ as g , then $\mathcal{M} \models \Psi_\phi$

Let $\mathcal{L}^* = \bigcup \mathcal{L}_i$ and $T^* = \bigcup T_i$. If $\phi(v, \bar{w})$ is an \mathcal{L}^* -formula, then $\phi \in \mathcal{L}_i$ for some i and $\Psi_\phi \in T_{i+1} \subseteq T^*$, so T^* has built in Skolem functions. By iterating the claim, we see that for any $\mathcal{M} \models T$ we can interpret the symbols of $\mathcal{L}^* \setminus \mathcal{L}$ to make $\mathcal{M} \models T^*$

$$|\mathcal{L}_{i+1}| = |\mathcal{L}_i| + \aleph_0 \quad \square$$

Theorem 2.31 (LöwenheimSkolem Theorem). *Suppose that \mathcal{M} is an \mathcal{L} -structure and $X \subseteq M$, there is an elementary submodel \mathcal{N} of \mathcal{M} s.t. $X \subseteq N$ and $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$*

Proof. By Lemma 2.30, we may assume that $\text{Th}(\mathcal{M})$ has built in Skolem functions (otherwise we may extend \mathcal{L} to some \mathcal{L}^*). Let $X_0 = X$. Given X_i , let $X_{i+1} = X_i \cup \{f^{\mathcal{M}}(\bar{a}) : f \text{ an } n\text{-ary function symbol, } \bar{a} \in X_i^n, n = 1, 2, \dots\}$. Let $N = \bigcup X_i$, then $|N| \leq |X| + |\mathcal{L}| + \aleph_0$. If f is an n -ary function symbol of \mathcal{L} and $\bar{a} \in N^n$, then $\bar{a} \in X_i^n$ for some i and $f^{\mathcal{M}}(\bar{a}) \in X_{i+1} \subseteq N$. Thus $f^{\mathcal{M}}|N : N^n \rightarrow N$. Thus we can interpret f as $f^{\mathcal{N}} = f^{\mathcal{M}}|N^n$. If R is an n -ary relation symbol, let $R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^n$. If c is a constant symbol of \mathcal{L} , there is a Skolem function $f \in \mathcal{L}$ s.t. $f(x) = c^{\mathcal{M}}$ for all $x \in M$ (for example, f is the Skolem function for the formula $v = c$). Thus $c^{\mathcal{N}} \in N$.

If $\phi(v, \bar{w})$ is any \mathcal{L} -formula, $\bar{a}, \bar{b} \in M$ and $\mathcal{M} \models \phi(\bar{b}, \bar{a})$, then $\mathcal{M} \models \phi(f(\bar{a}), \bar{a})$ for some function symbol f of \mathcal{L} . By construction, $f^{\mathcal{M}}(\bar{a}) \in N$. Thus by Proposition 2.29 $\mathcal{N} \prec \mathcal{M}$ \square

Definition 2.32. A **universal sentence** is one of the form $\forall \bar{v} \phi(\bar{v})$, where ϕ is quantifier-free. We say that an \mathcal{L} -theory T has a **universal axiomatization** if there is a set of universal \mathcal{L} -sentences Γ s.t. $\mathcal{M} \models \Gamma$ if and only if $\mathcal{M} \models T$ for all \mathcal{L} -structures \mathcal{M}

Theorem 2.33. *An \mathcal{L} -theory T has a universal axiomatization if and only if whenever $\mathcal{M} \models T$ and \mathcal{N} is a substructure of \mathcal{M} , then $\mathcal{N} \models T$. In other words, a theory is preserved under substructure if and only if it has a universal axiomatization*

Proof. Suppose that $\mathcal{N} \subseteq \mathcal{M}$. By Proposition 1.7, if $\phi(\bar{v})$ is a quantifier-free formula and $\bar{a} \in N$, then $\mathcal{N} \models \phi(\bar{a})$ if and only if $\mathcal{M} \models \phi(\bar{a})$. Thus if $\mathcal{M} \models \forall \bar{v} \phi(\bar{v})$, then so does \mathcal{N} .

Suppose that T is preserved under substructures. Let $\Gamma = \{\phi : \phi \text{ is universal and } T \models \phi\}$. Clearly, if $\mathcal{N} \models T$, then $\mathcal{N} \models \Gamma$. For the other direction, suppose that $\mathcal{N} \models \Gamma$. We claim that $\mathcal{N} \models T$.

Claim $T \cup \text{Diag}(\mathcal{N})$ is satisfiable

Suppose not. Then, by the Compactness Theorem, there is a finite $\Delta \subseteq \text{Diag}(\mathcal{N})$ s.t. $T \cup \Delta$ is not satisfiable. Let $\Delta = \{\psi_1, \dots, \psi_n\}$. Let \bar{c} be the new constant symbols from N used in ψ_1, \dots, ψ_n and say $\psi_i = \phi_i(\bar{c})$, where ϕ_i is a quantifier-free \mathcal{L} -formula. Because the constants in \bar{c} do not occur in T , if there is a model of $T \cup \{\exists \bar{v} \bigwedge \phi_i(\bar{v})\}$, then by interpreting \bar{c} as witness to the existential formula, $T \cup \Delta$ would be satisfiable. Thus $T \models \forall \bar{v} \bigvee \neg \phi_i(\bar{v})$. As the latter formula is universal, $\forall \bar{v} \bigvee \neg \phi_i(\bar{v}) \in \Gamma$, contradicting $\mathcal{N} \models \Gamma$.

By Lemma 2.27, there is $\mathcal{M} \models T$ with $\mathcal{M} \supseteq \mathcal{N}$. Because T is preserved under substructure, $\mathcal{N} \models T$ and Γ is a universal axiomatization \square

Definition 2.34. Suppose that $(I, <)$ is a linear order. Suppose that \mathcal{M}_i is an \mathcal{L} -structure for $i \in I$. We say that $(\mathcal{M}_i : i \in I)$ is a chain of \mathcal{L} -structures if $\mathcal{M}_i \subseteq \mathcal{M}_j$ for $i < j$. If $\mathcal{M}_i \prec \mathcal{M}_j$ for $i < j$, we call $(\mathcal{M}_i : i \in I)$ an **elementary chain**

If $(\mathcal{M}_i : i \in I)$ is a nonempty chain of structures, then we can define $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$, the union of the chain, as follows. $M = \bigcup_{i \in I} M_i$. if c is a constant in the language, then $c^{\mathcal{M}_i} = c^{\mathcal{M}_j}$ for all $i, j \in I$. Let $c^{\mathcal{M}} = c^{\mathcal{M}_i}$.

Suppose that $\bar{a} \in M$. Because I is linearly ordered, we can find $i \in I$ s.t. $\bar{a} \in M_i$. If f is a function symbol of \mathcal{L} and $i < j$, then $f^{\mathcal{M}_i}(\bar{a}) = f^{\mathcal{M}_j}(\bar{a})$. Thus $f^{\mathcal{M}} = \bigcup_{i \in I} f^{\mathcal{M}_i}$ is a well-defined function. Similarly, $R^{\mathcal{M}} = \bigcup_{i \in I} R^{\mathcal{M}_i}$

Proposition 2.35. Suppose that $(I, <)$ is a linear order and $(\mathcal{M}_i : i \in I)$ is an elementary chain. Then $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$ is an elementary extension of each \mathcal{M}_i

Proof. We prove by induction on formulas that

$$\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{M}_i \models \phi(\bar{a})$$

for all $i \in I$, all formulas $\phi(\bar{v})$, and all $\bar{a} \in M_i^n$

Because \mathcal{M}_i is a substructure of \mathcal{M} , by Proposition 1.7 this is true for all atomic ϕ . $\neg\phi$ and $\phi \vee \psi$ is easy.

Suppose that ϕ is $\exists v \psi(v, \bar{w})$ and the chain holds for ψ . If $\mathcal{M}_i \models \psi(b, \bar{a})$, then so does \mathcal{M} . Thus if $\mathcal{M}_i \models \phi(\bar{a})$, then so does \mathcal{M} . On the other hand, if $\mathcal{M} \models \psi(b, \bar{a})$, there is $j \geq i$ s.t. $b \in M_j$. By induction, $\mathcal{M}_j \models \psi(b, \bar{a})$, so $\mathcal{M}_j \models \phi(\bar{a})$. Because $\mathcal{M}_i \prec \mathcal{M}_j$, $\mathcal{M}_i \models \phi(\bar{a})$ \square

2.4 Back and Forth

2.4.1 Dense Linear Orders

Let $\mathcal{L} = \{<\}$ and let DLO be the theory of dense linear orders without endpoints. DLO is axiomatized by the axioms for linear orders plus the axioms

$$\begin{aligned} \forall x \forall y (x < y \rightarrow \exists z x < z < y) \\ \forall x \exists y \exists z y < x < z \end{aligned}$$

Theorem 2.36. The theory DLO is \aleph_0 -categorical and complete

Proof. Let $(A, <)$ and $(B, <)$ be two countable models of DLO. Let a_0, a_1, a_2, \dots and b_0, b_1, b_2, \dots be one-to-one enumerations of A and B . We will build a sequence of partial bijections $f_i : A_i \rightarrow B_i$ where $A_i \subset A$ and $B_i \subset B$ are finite s.t. $f_0 \subseteq f_1 \subseteq \dots$ and if $x, y \in A_i$ and $x < y$, then $f_i(x) < f_i(y)$. We call f_i a **partial embedding**. We will build these sequences s.t. $A = \bigcup A_i$ and $B = \bigcup B_i$. In this case, $f = \bigcup f_i$ is the desired isomorphism from $(A, <)$ to $(B, <)$

At odd stages of the construction we will ensure that $\bigcup A_i = A$, and at even stages we will ensure that $\bigcup B_i = B$

stage 0: Let $A_0 = B_0 = f_0 = \emptyset$

stage $n + 1 = 2m + 1$: We will ensure that $a_m \in A_{n+1}$.

If $a_m \in A_n$, then let $A_{n+1} = A_n$, $B_{n+1} = B_n$ and $f_{n+1} = f_n$. Suppose that $a_m \notin A_n$. To add a_m to the domain of our partial embedding, we must find $b \in B \setminus B_n$ s.t.

$$\alpha < a_m \Leftrightarrow f_n(\alpha) < b$$

for all $\alpha \in A_n$. In other words, we must find $b \in B$, which is the image under f_n of the cut of a_m in A_n . Exactly one of the following holds:

1. a_m is greater than every element of A_n , or
2. a_m is less than every element of A_n , or
3. there are α and $\beta \in A_n$ s.t. $\alpha < \beta$, $\gamma \leq \alpha$ or $\gamma \geq \beta$ for all $\gamma \in A_n$ and $\alpha < a_m < \beta$

In case 1 because B_n is finite and $B \models \text{DLO}$, we can find $b \in B$ greater than every element of B_n . Similar for case 2. In case 3, because f_n is a partial embedding, $f_n(\alpha) < f_n(\beta)$ and we can choose $b \in B_n$ s.t. $f_n(\alpha) < b < f_n(\beta)$. Note that

$$\alpha < a_m \Leftrightarrow f_n(\alpha) < b$$

for all $\alpha \in A_n$

stage $n + 1 = 2m + 2$: We will ensure $b_m \in B_{n+1}$

Again, if b_m is already in B_n , then we make no changes. Otherwise, we must find $a \in A$ s.t. the image of the cut of a in A_n is the cut of b_m in B_n . This is done in odd case.

Clearly, at odd stages we have ensured that $\bigcup A_n = A$ and at even stages we have ensured that $\bigcup B_n = B$. Because each f_n is a partial embedding, $f = \bigcup f_n$ is an isomorphism from A onto B

But there are no finite dense linear orders, Vaught's test implies that DLO is complete \square

2.4.2 The Random Graph

Let $\mathcal{L} = \{R\}$, where R is a binary relation symbol. We will consider an \mathcal{L} -theory containing the graph axioms $\forall x \neg R(x, x)$ and $\forall x \forall y R(x, y) \rightarrow R(y, x)$. Let ψ_n be the “extension axiom”

$$\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n \left(\bigwedge_{i=1}^n \bigwedge_{j=1}^n x_i \neq y_j \rightarrow \exists z \bigwedge_{i=1}^n (R(x_i, z) \wedge \neg R(y_i, z)) \right)$$

We let T be the theory of graphs where we add $\{\exists x \exists y x \neq y\} \cup \{\psi_n : n = 1, 2, \dots\}$ to the graph axioms. A model of T is a graph where for any finite disjoint sets X and Y we can find a vertex with edges going to every vertex in X and no vertex in Y

Theorem 2.37. *T is satisfiable and \aleph_0 -categorical. In particular, T is complete and decidable*

Proof. We first build a countable model of T . Let G_0 be any countable graph

Claim There is a graph $G_1 \supseteq G_0$ s.t. G_1 is countable and if X and Y are disjoint finite subsets of G_0 then there is $z \in G_1$ s.t. $R(x, z)$ for $x \in X$ and $\neg R(y, z)$ for $y \in Y$

Let the vertices of G_1 be the vertices of G_0 plus new vertices z_X for each $X \subseteq G_0$. The edges of G_1 are the edges of G together with new edges between x and z_X whenever $X \subseteq G_0$ is finite and $x \in X$.

We iterate this construction to build a sequence of countable graphs $G_0 \subset G_1 \subset \dots$ s.t. if X and Y are disjoint finite subsets of G_i , then there is $z \in G_{i+1}$ s.t. $R(x, z)$ for $x \in X$ and $\neg R(y, z)$ for $y \in Y$. Thus $G = \bigcup G_n$ is a countable model of T

Next we show that T is \aleph_0 -categorical. Let G_1 and G_2 be countable models of T . Let a_0, a_1, \dots list G_1 , and let b_0, b_1, \dots list G_2 . We will build a sequence of finite partial one-to-one maps $f_0 \subseteq f_1 \subseteq f_2 \subseteq \dots$ s.t. for all x, y in the domain of f_s ,

$$G_1 \models R(x, y) \Leftrightarrow G_2 \models R(f_s(x), f_s(y))$$

Let $f_0 = \emptyset$ stage $s + 1 = 2i + 1$: We make sure that a_i is in the domain

If a_i is in the domain of f_s , let $f_{s+1} = f_s$. If not, let $\alpha_1, \dots, \alpha_m$ list the domain of f_s and let $X = \{j \leq m : R(\alpha_j, a_i)\}$ and let $Y = \{j \leq m : \neg R(\alpha_j, a_i)\}$. Because $G_2 \models T$, we can find $b \in G_2$ s.t. $G_2 \models R(f_s(\alpha_j), b)$ for $j \in X$ and $G_2 \models \neg R(f_s(\alpha_j), b)$ for $j \in Y$. Let $f_{s+1} = f_s \cup \{(a_i, b)\}$.

stage $s + 1 = 2i + 2$: Similar

□

Let \mathcal{G}_N be the set of all graphs with vertices $\{1, 2, \dots, N\}$. We consider a probability measure on \mathcal{G}_N where we make all graphs equally likely. This is the same as constructing a random graph where we independently decide whether there is an edge between i and j with probability $\frac{1}{2}$. For any \mathcal{L} -sentence ϕ ,

$$p_N(\phi) = \frac{|\{G \in \mathcal{G}_N : G \models \phi\}|}{|\mathcal{G}_N|}$$

is the probability that a random element of \mathcal{G}_N satisfies ϕ

Lemma 2.38. $\lim_{N \rightarrow \infty} p_N(\psi_n) = 1$

Proof. Fix n . Let G be a random graph in \mathcal{G}_N where $N > 2n$. Fix $x_1, \dots, x_n, y_1, \dots, y_n, z \in G$ distinct. Let q be the probability that

$$\neg \left(\bigwedge_{i=1}^n (R(x_i, z)) \wedge \neg R(y_i, z) \right)$$

Then $q = 1 - 2^{-2n}$. Because these probabilities are independent, the probability that

$$G \models \neg \exists z \neg \left(\bigwedge_{i=1}^n (R(x_i, z)) \wedge \neg R(y_i, z) \right)$$

is q^{N-2n} . Let M be the number of pairs of disjoint subsets of G of size n . Thus

$$p_N(\neg \psi_n) \leq M q^{N-2n} < N^{2n} q^{N-2n}$$

Because $q < 1$

$$\lim_{N \rightarrow \infty} p_N(\neg \psi_n) = \lim_{N \rightarrow \infty} N^{2n} q^N = 0$$

□

Theorem 2.39 (Zero-One Law for Graphs). *For any \mathcal{L} -sentence ϕ either $\lim_{N \rightarrow \infty} p_N(\phi) = 0$ or $\lim_{N \rightarrow \infty} p_N(\phi) = 1$. Moreover, T axiomatizes $\{\phi : \lim_{N \rightarrow \infty} p_N(\phi) = 1\}$, the **almost sure theory of graphs**. The almost sure theory of graphs is decidable and complete*

Proof. If $T \models \phi$, then there is n s.t. if G is a graph and $G \models \psi_n$, then $G \models \phi$. Thus, $p_N(\phi) \geq p_N(\psi_n)$ and by Lemma 2.38, $\lim_{N \rightarrow \infty} p_N(\phi) = 1$. □

2.4.3 Ehrenfeucht-Fraïssé Games

Let \mathcal{L} be a language and $\mathcal{M} = (M, \dots)$ and $\mathcal{N} = (N, \dots)$ be two \mathcal{L} -structures with $M \cap N = \emptyset$. If $A \subseteq M$, $B \subseteq N$ and $f : A \rightarrow B$, we say that f is a **partial embedding** if $f \cup \{(c^{\mathcal{M}}, c^{\mathcal{N}}) : c \text{ a constant of } \mathcal{L}\}$ is a bijection preserving all relations and functions of \mathcal{L} .

We will define an infinite two-player game $G_\omega(\mathcal{M}, \mathcal{N})$. We will call the two players player I and player II; together they will build a partial embedding f from M to N . A play of the game will consist of ω stages. At the i th-stage, player I moves first and either plays $m_i \in M$, challenging player II to put m_i into the domain of f , or $n_i \in N$, challenging player II to put n_i into the range. If player I plays $m_i \in M$, then player II must play $n_i \in N$, whereas if player I plays $n_i \in N$, then player II must play $m_i \in M$. Player II wins the play of the game if $f = \{(m_i, n_i) : i = 1, 2, \dots\}$ is the graph of a partial embedding.

A **strategy** for player II in $G_\omega(\mathcal{M}, \mathcal{N})$ is a function τ s.t. if player I's first n moves are c_1, \dots, c_n , then player II's n th move will be $\tau(c_1, \dots, c_n)$. We say that player II uses the strategy τ in the play of the game if the play looks like

Player I	Player II
c_1	
	$\tau(c_1)$
c_2	
	$\tau(c_1, c_2)$
c_3	
	$\tau(c_1, c_2, c_3)$
\vdots	\vdots

We say that τ is a **winning strategy** for player II, if for any sequence of plays c_1, \dots player I makes, player II will win by following τ . We define strategies for player I analogously.

For example, suppose that $\mathcal{M}, \mathcal{N} \models \text{DLO}$. Then player II has a winning strategy. Suppose that up to stage n they have built a partial embedding $g : A \rightarrow B$. If player I plays $a \in M$, then player II plays $b \in N$ s.t. the cut b makes in B is the image of the cut of a in A under g . Similar for player I's $b \in N$.

Proposition 2.40. *If \mathcal{M} and \mathcal{N} is countable, then the second player has a winning strategy in G_ω if and only if $\mathcal{M} \cong \mathcal{N}$*

Proof. If $\mathcal{M} \cong \mathcal{N}$, player II can win by playing according to the isomorphism

Suppose that player II has a winning strategy. Let m_0, m_1, \dots list M and n_0, n_1, \dots list N . Consider a play of the game where the second player uses the winning strategy and the first player plays $m_0, n_0, m_1, n_1, m_2, n_2, \dots$. If f is the partial embedding build during this play of the game then the domain of f is M and the range of f is N . Thus f is an isomorphism \square

Fix \mathcal{L} a finite language with no function symbols, and let \mathcal{M} and \mathcal{N} be \mathcal{L} -structures. We define a game $G_n(\mathcal{M}, \mathcal{N})$ for $n = 1, 2, \dots$. The game will have n rounds similar to ω rounds. Player II wins if $\{(a_i, b_i) : i = 1, \dots, n\}$ is the graph of a partial embedding from \mathcal{M} into \mathcal{N} . We call $G_n(\mathcal{M}, \mathcal{N})$ an **Ehrenfeucht-Fraïssé Games**

Theorem 2.41. *Let \mathcal{L} be a finite language without function symbols and let \mathcal{M} and \mathcal{N} be \mathcal{L} -structures. Then $\mathcal{M} \equiv \mathcal{N}$ if and only if the second player has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ for all n*

We need several lemmas.

Lemma 2.42. *One of the players has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$*

Proof. Suppose that player II does not have a winning strategy. Then there is some move player I can make in round one so that player II has no move available to force a win. Player I makes that move. Now, whatever player II does, there is still a move that if made by player I means that player II cannot force a win. \square

We inductively define $\text{depth}(\phi)$, the **quantifier depth** of an \mathcal{L} -formula ϕ , as follows

- $\text{depth}(\phi) = 0$ if and only if ϕ is quantifier-free
- $\text{depth}(\neg\phi) = \text{depth}(\phi)$
- $\text{depth}(\phi \wedge \psi) = \text{depth}(\phi \vee \psi) = \max\{\text{depth}(\phi), \text{depth}(\psi)\}$
- $\text{depth}(\exists v\phi) = \text{depth}(\phi) + 1$

We say that $\mathcal{M} \equiv_n \mathcal{N}$ if $\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$ for all sentences of depth at most n . We will show player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ if and only if $\mathcal{M} \equiv_n \mathcal{N}$

Lemma 2.43. *For each n and l , there is a finite list of formulas ϕ_1, \dots, ϕ_k of depth at most n in free variables x_1, \dots, x_l s.t. every formula of depth at most n in free variables x_1, \dots, x_l is equivalent to some ϕ_i*

Proof. We first prove this for quantifier-free formulas. Because \mathcal{L} is finite and has no function symbols, there are only finitely many atomic \mathcal{L} -formulas in free variables x_1, \dots, x_l . Let $\sigma_1, \dots, \sigma_s$ list all such formulas.

If ϕ is a Boolean combination of formulas τ_1, \dots, τ_s , then there is S a collection of subsets of $\{1, \dots, s\}$ s.t.

$$\models \phi \leftrightarrow \bigvee_{X \in S} \left(\bigwedge_{i \in X} \tau_i \wedge \bigwedge_{i \notin X} \neg \tau_i \right)$$

This gives a list of 2^{2^s} formulas s.t. every Boolean combination of τ_1, \dots, τ_s is equivalent to a formula in this list. In particular, because quantifier free formulas are Boolean combinations of atomic formulas, there is a finite list of depth-zero formulas s.t. every depth-zero formula is equivalent to one in the list.

Because formulas of depth $n + 1$ are Boolean combinations of $\exists v\phi$ and $\forall v\phi$ where ϕ has depth at most n \square

Lemma 2.44. *Let \mathcal{L} be a finite language without function symbols and \mathcal{M} and \mathcal{N} be \mathcal{L} -structures. The second player has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ if and only if $\mathcal{M} \equiv_n \mathcal{N}$*

Proof. Induction on n

Suppose that $\mathcal{M} \equiv_n \mathcal{N}$. Consider a play of the game where in round one player I plays $a \in M$. We claim that there is $b \in N$ s.t. $\mathcal{M} \models \phi(a) \Leftrightarrow \mathcal{N} \models \phi(b)$ whenever $\text{depth}(\phi) < n$. Let $\phi_0(v), \dots, \phi_m(v)$ list, up to equivalence, all formulas of depth less than n . Let $X = \{i \leq m : \mathcal{M} \models \phi_i(a)\}$, and let $\Phi(v)$ be the formula

$$\bigwedge_{i \in X} \phi_i(v) \wedge \bigwedge_{i \notin X} \neg \phi_i(v)$$

Then, $\text{depth}(\exists v \Phi(v)) \leq n$ and $\mathcal{M} \models \Phi(a)$; thus there is $b \in N$ s.t. $\mathcal{N} \models \Phi(b)$. Player II plays b in round one

If $n = 1$, the game has now concluded and $a \mapsto b$ is a partial embedding so player II wins. Suppose that $n > 1$

Let $\mathcal{L}^* = \mathcal{L} \cup \{c\}$, where c is a new constant symbol. View \mathcal{M} and \mathcal{N} as \mathcal{L}^* -structures (\mathcal{M}, a) and (\mathcal{N}, b) where we interpret the new constant as a and b respectively. Because

$$\mathcal{M} \models \phi(a) \Leftrightarrow \mathcal{N} \models \phi(b)$$

for $\phi(v)$ an \mathcal{L} -formula with $\text{depth}(\phi) < n$, $(\mathcal{M}, a) \equiv_{n-1} (\mathcal{N}, b)$. By induction, player II has a winning strategy in $G_{n-1}((\mathcal{M}, a), (\mathcal{N}, b))$. If player's second play is d , player II responds as if d was player I's first play in $G_{n-1}((\mathcal{M}, a), (\mathcal{N}, b))'$ and continues playing using this strategy, that is, in round i player I has plays a, d_2, \dots, d_i , then player II plays $\tau(d_2, \dots, d_i)$, where τ is his winning strategy in $G((\mathcal{M}, a), (\mathcal{N}, b))$. \square

2.5 Exercises

Exercise 2.5.1. We say that an ordered group $(G, +, <)$ is **Archimedean** if for all $x, y \in G$ with $x, y > 0$ there is an integer m s.t. $|x| < m|y|$. Show that there are non-Archimedean fields elementarily equivalent to the field of real numbers

Exercise 2.5.2. Let T be an \mathcal{L} -theory and T_\forall be all of the universal sentences ϕ s.t. $T \models \phi$. Show that $\mathcal{A} \models T_\forall$ if and only if there is $\mathcal{M} \models T$ with $\mathcal{A} \subseteq \mathcal{M}$

Proof. Comes from Quantifier Elimination Tests and Examples

Consider the theory $T' = T \cup \text{Diag}(\mathcal{A})$ in the language $\mathcal{L}_\mathcal{A}$. We will show by contradiction that T' is satisfiable.

Suppose that T' is not satisfiable. Then by the Compactness Theorem, already some finite subset $\Delta \subseteq T'$ is not satisfiable. By forming conjunctions we may assume that the part of Δ coming from $\text{Diag}(\mathcal{A})$ consists only of one formula $\phi(\bar{a})$ for some $\bar{a} \in \mathcal{A}$, where $\phi(\bar{a})$ is a conjunction of atomic formulas and the negation of atomic formulas. Thus we will assume that $T \cup \{\phi(\bar{a})\}$ is not satisfiable.

On the other hand, viewing T as an $\mathcal{L}_{\bar{a}}$ -theory, and because $T \cup \{\phi(\bar{a})\}$ is not satisfiable, we obtain $T \models \neg\phi(\bar{a})$. We will show that this implies $T \models \forall \bar{v} \neg\phi(\bar{v})$: Let \mathcal{C} be an \mathcal{L} -structure with $\mathcal{C} \models T$. Let n be the number of components in \bar{a} and $c_1, \dots, c_n \in C$. Let \mathcal{C}' be the $\mathcal{L}_{\bar{a}}$ -structure which expands \mathcal{C} by the constant symbols that we interpret as c_1, \dots, c_n respectively. Then $\mathcal{C}' \models T$ and hence $\mathcal{C}' \models \neg\phi(\bar{c})$. As this follows for any tuple in C , we get $\mathcal{C} \models \forall \bar{v} \neg\phi(\bar{v})$

Since T_\forall consists exactly of the universal formulas which hold in all models of T , we obtain $T_\forall \models \forall x \neg\phi(x)$. Hence also $\mathcal{A} \models \forall x \neg\phi(x)$, a contradiction

Therefore T' is indeed satisfiable □

3 Algebraic Examples

3.1 Quantifier Elimination

Definition 3.1. We say that a theory T has **quantifier elimination** if for every formula ϕ there is a quantifier-free formula ψ s.t.

$$T \models \phi \leftrightarrow \psi$$

Lemma 3.2. Let $(A, <)$ and $(B, <)$ be countable dense linear orders, $a_1, \dots, a_n \in A$, $b_1, \dots, b_n \in B$, s.t. $a_1 < \dots < a_n$ and $b_1 < \dots < b_n$. Then there is an isomorphism $f : A \rightarrow B$ s.t. $f(a_i) = b_i$ for all $i = 1, \dots, n$

Proof. Modify the proof of Theorem 2.36 starting with $A_0 = \{a_1, \dots, a_n\}$, $B_0 = \{b_1, \dots, b_n\}$, and the partial isomorphism $f_0 : A_0 \rightarrow B_0$, where $f_0(a_i) = b_i$. \square

Theorem 3.3. *DLO has quantifier elimination*

Proof. First, suppose that ϕ is a sentence. If $\mathbb{Q} \models \phi$, then because DLO is complete, $\text{DLO} \models \phi$, and

$$\text{DLO} \models \phi \leftrightarrow x_1 = x_1$$

whereas if $\mathbb{Q} \models \neg\phi$

$$\text{DLO} \models \phi \leftrightarrow x_1 \neq x_1$$

Now suppose that ϕ is a formula with free variables x_1, \dots, x_n where $n \geq 1$. We will show that there is a quantifier-free formula ψ with free variables from among x_1, \dots, x_n s.t.

$$\mathbb{Q} \models \forall \bar{x} (\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$$

Because DLO is complete,

$$\text{DLO} \models \forall \bar{x} (\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$$

so this will suffice.

For $\sigma : \{(i, j) : 1 \leq i < j \leq n\} \rightarrow 3$, let $\chi_\sigma(x_1, \dots, x_n)$ be the formula

$$\bigwedge_{\sigma(i,j)=0} x_i = x_j \wedge \bigwedge_{\sigma(i,j)=1} x_i < x_j \wedge \bigwedge_{\sigma(i,j)=2} x_i > x_j$$

We call χ_σ a **sign condition**.

Let \mathcal{L} be the language of linear orders and ϕ be an \mathcal{L} -formula with $n \geq 1$ free variables. Let Λ_ϕ be the set of sign conditions s.t. there is $\bar{a} \in \mathbb{Q}$ s.t.

$$\mathbb{Q} \models \chi_\sigma(\bar{a}) \wedge \phi(\bar{a})$$

case 1: $\Lambda_\phi = \emptyset$

Then $\mathbb{Q} \models \forall \bar{x} \neg\phi(\bar{x})$ and $\mathbb{Q} \models \phi(\bar{x}) \leftrightarrow x_1 \neq x_1$

case 2: $\Lambda_\phi \neq \emptyset$

Let

$$\psi_\phi(\bar{x}) = \bigwedge_{\sigma \in \Lambda_\phi} \chi_\sigma(\bar{x})$$

By choice of Λ_ϕ ,

$$\mathbb{Q} \models \phi(\bar{x}) \rightarrow \psi_\phi(\bar{x})$$

On the other hand, suppose that $\bar{b} \in \mathbb{Q}$ and $\mathbb{Q} \models \psi_\phi(\bar{b})$. Let $\sigma \in \Lambda_\phi$ s.t. $\mathbb{Q} \models \chi_\sigma(\bar{b})$. There is $\bar{a} \in \mathbb{Q}$ s.t. $\mathbb{Q} \models \phi(\bar{a}) \wedge \chi_\sigma(\bar{a})$. By Theorem 2.36, there is f , an automorphism of $(\mathbb{Q}, <)$ s.t. $f(\bar{a}) = \bar{b}$. By Theorem 1.9, $\mathbb{Q} \models \phi(\bar{b})$. Thus $\phi(\bar{b}) \leftrightarrow \psi_\phi(\bar{b})$ \square

Theorem 3.4. Suppose that \mathcal{L} contains a constant symbol c , T is an \mathcal{L} -theory, and $\phi(\bar{v})$ is an \mathcal{L} -formula. The following are equivalent:

1. There is a quantifier-free \mathcal{L} -formula $\psi(\bar{v})$ s.t. $T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$
2. If \mathcal{M} and \mathcal{N} are models of T , \mathcal{A} is an \mathcal{L} -structure, $\mathcal{A} \subseteq \mathcal{M}$, and $\mathcal{A} \subseteq \mathcal{N}$, then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \phi(\bar{a})$ for all $\bar{a} \in \mathcal{A}$

Proof. (1) \rightarrow (2). Suppose that $T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$, where ψ is quantifier-free. Let $\bar{a} \in \mathcal{A}$, where \mathcal{A} is a common substructure of \mathcal{M} and \mathcal{N} and the latter structures are models of T . In Proposition 1.7, we saw that quantifier-free formulas are preserved under substructure and extension. Thus

$$\begin{aligned} \mathcal{M} \models \phi(\bar{a}) &\Leftrightarrow \mathcal{M} \models \psi(\bar{a}) \\ &\Leftrightarrow \mathcal{A} \models \psi(\bar{a}) \\ &\Leftrightarrow \mathcal{N} \models \psi(\bar{a}) \\ &\Leftrightarrow \mathcal{N} \models \phi(\bar{a}) \end{aligned}$$

(2) \rightarrow (1). First, if $T \models \forall \bar{v}\phi(\bar{v})$, then $T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow c = c)$. Second, if $T \models \forall \bar{v}\neg\phi(\bar{v})$, then $T \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow c \neq c)$.

Thus, we may assume that both $T \cup \{\phi(\bar{v})\}$ and $T \cup \{\neg\phi(\bar{v})\}$ are satisfiable

Let $\Gamma(\bar{v}) = \{\psi(\bar{v}) : \psi \text{ is quantifier free and } T \models \forall \bar{v}(\phi(\bar{v}) \rightarrow \psi(\bar{v}))\}$. Let d_1, \dots, d_m be new constant symbols. We will show that $T \cup \Gamma(\bar{d}) \models \phi(\bar{d})$. Then, by compactness, there are $\psi_1, \dots, \psi_n \in \Gamma$ s.t.

$$T \models \forall \bar{v} \left(\bigwedge_{i=1}^n \psi_i(\bar{v}) \rightarrow \phi(\bar{v}) \right)$$

Thus

$$T \models \forall \bar{v} \left(\bigwedge_{i=1}^n \psi_i(\bar{v}) \leftrightarrow \phi(\bar{v}) \right)$$

and $\bigwedge_{i=1}^n \psi_i(\bar{v})$ is quantifier-free

Claim $T \cup \Gamma(\bar{d}) \models \phi(\bar{d})$

Suppose not. Let $\mathcal{M} \models T \cup \Gamma(\bar{d}) \cup \{\neg\phi(\bar{d})\}$. Let \mathcal{A} be the substructure of \mathcal{M} generated by \bar{d}

Let $\Sigma = T \cup \text{Diag}(\mathcal{A}) \cup \phi(\vec{d})$. If Σ is unsatisfiable, then there are quantifier-free formulas $\psi_1(\vec{d}), \dots, \psi_n(\vec{d}) \in \text{Diag}(\mathcal{A})$ s.t.

$$T \models \forall \vec{v} \left(\bigwedge_{i=1}^n \psi_i(\vec{v}) \rightarrow \neg \phi(\vec{v}) \right)$$

But then

$$T \models \forall \vec{v} \left(\phi(\vec{v}) \rightarrow \bigvee_{i=1}^n \neg \psi_i(\vec{v}) \right)$$

so $\bigvee_{i=1}^n \neg \psi_i(\vec{v}) \in \Gamma$ and $\mathcal{A} \models \bigvee_{i=1}^n \neg \psi_i(\vec{d})$, a contradiction. Thus, Σ is satisfiable

Let $\mathcal{N} \models \Sigma$. Then $\mathcal{N} \models \phi(\vec{d})$. Because $\Sigma \supseteq \text{Diag}(\mathcal{A})$, $\mathcal{A} \subseteq \mathcal{N}$, by Lemma 2.27. But $\mathcal{M} \models \neg \phi(\vec{d})$; thus $\mathcal{N} \models \neg \phi(\vec{d})$, a contradiction \square

Lemma 3.5. *Let T be an \mathcal{L} -theory. Suppose that for every quantifier-free \mathcal{L} -formula $\theta(\vec{v}, w)$ there is a quantifier-free formula $\psi(\vec{v})$ s.t. $T \models \exists w \theta(\vec{v}, w) \leftrightarrow \psi(\vec{v})$. Then T has quantifier elimination*

Proof. Let $\phi(\vec{v})$ be an \mathcal{L} -formula. We wish to show to show that $T \models \forall \vec{v} (\phi(\vec{v}) \leftrightarrow \psi(\vec{v}))$ for some quantifier-free formula $\psi(\vec{v})$

If ϕ is quantifier-free, there is nothing to prove. Suppose that for $i = 0, 1$, $T \models \forall \vec{v} (\theta_i(\vec{v}) \leftrightarrow \psi_i(\vec{v}))$, where ψ_i is quantifier-free.

If $\phi(\vec{v}) = \neg \theta_0(\vec{v})$, then $T \models \forall \vec{v} (\phi(\vec{v}) \leftrightarrow \neg \psi_0(\vec{v}))$

Suppose that $T \models \forall \vec{v} (\theta(\vec{v}, w) \leftrightarrow \psi_0(\vec{v}, w))$, where ψ_0 is quantifier-free and $\phi(\vec{v}) = \exists w \theta(\vec{v}, w)$. Then $T \models \forall \vec{v} (\phi(\vec{v}) \leftrightarrow \exists \psi_0(\vec{v}, w))$. By our assumptions, there is a quantifier-free $\psi(\vec{v})$ s.t. $T \models \forall \vec{v} (\exists w \psi_0(\vec{v}, w) \leftrightarrow \psi(\vec{v}))$. But then $T \models \forall \vec{v} (\phi(\vec{v}) \leftrightarrow \psi(\vec{v}))$ \square

Combining Theorem 3.4 and Lemma 3.5 gives us the following test for quantifier elimination

Corollary 3.6 (NEED TO RECONSIDER). *Let T be an \mathcal{L} -theory. Suppose that for all quantifier-free formulas $\phi(\vec{v}, w)$, if $\mathcal{M}, \mathcal{N} \models T$, \mathcal{A} is a common substructure of \mathcal{M} and \mathcal{N} , $\vec{a} \in \mathcal{A}$, and there is $b \in \mathcal{M}$ s.t. $\mathcal{M} \models \phi(\vec{a}, b)$, then there is $c \in \mathcal{N}$ s.t. $\mathcal{N} \models \phi(\vec{a}, c)$. Then T has a quantifier elimination*

Proof. Check this notes Quantifier Elimination Tests and Examples

We need to show that $T \models \forall \vec{v} (\exists w \phi(\vec{v}, w) \leftrightarrow \psi(\vec{v}))$. Suppose that $\mathcal{M} \models \exists w \phi(\vec{v}, w)$, then $\mathcal{N} \models \exists w \phi(\vec{v}, w)$. Note that \mathcal{M} and \mathcal{N} are interchangeable. \square

3.1.1 Divisible Abelian Groups

Work with the language $\mathcal{L} = \{+, -, 0\}$ because its convenient

Let DAG be the \mathcal{L} -theory of nontrivial torsion-free divisible Abelian groups

Lemma 3.7. *Suppose G and H are nontrivial torsion free divisible Abelian groups, $G \subseteq H$, $\psi(\bar{v}, w)$ is quantifier-free, $\bar{a} \in G$, $b \in H$, and $H \models \phi(\bar{a}, b)$. Then there is $c \in G$ s.t. $G \models \phi(\bar{a}, c)$*

Proof. We first note that ψ can be put in disjunctive normal form.

Because $H \models \psi(\bar{a}, b)$, $H \models \bigwedge_{j=1}^m \theta_{i,j}(\bar{a}, b)$ for some i . Thus, without loss of generality, we may assume that ψ is a conjunction of atomic and negated atomic formulas. If $\theta(v_1, \dots, v_m, w)$ is an atomic formula, then for some integers n_1, \dots, n_m, m , $\theta(\bar{v}, w)$ is $\sum n_i v_i + mw = 0$

Thus we may assume that

$$\psi(\bar{a}, w) = \bigwedge_{i=1}^s \sum_{j=1}^m n_{i,j} a_j + m_i w = 0 \wedge \bigwedge_{i=1}^s \sum_{j=1}^m n'_{i,j} a_j + m'_i w \neq 0$$

Let $g_i = \sum n_{i,j} a_j$ and $h_i = \sum n'_{i,j} a'_j$. Then $g_i, h_i \in G$ and

$$\psi(\bar{a}, w) \leftrightarrow \bigwedge g_i + m_i w = 0 \wedge \bigwedge h_i + m'_i w \neq 0$$

If any $m_i \neq 0$, then $b = -g_i/m_i \in G$ and $G \models \theta(\bar{a}, b)$, so suppose that $\psi(\bar{a}, w) = \bigwedge h_i + m'_i w \neq 0$. Thus $\psi(\bar{a}, w)$ is satisfied by any element of H that is not equal to any one of $\frac{-h_1}{m'_1}, \dots, \frac{-h_s}{m'_s}$. Because G is infinite, there is an element of G satisfying $\psi(\bar{a}, w)$ \square

Lemma 3.8. *Suppose that G is a torsion-free Abelian group. Then there is a torsion-free divisible Abelian group H , called the **divisible hull** of G , and an embedding $i : G \rightarrow H$ s.t. if $j : G \rightarrow H'$ is an embedding of G into a torsion-free divisible Abelian group, then there is $h : H \rightarrow H'$ s.t. $j = h \circ i$*

Proof. If G is the trivial group, then we take $H = \mathbb{Q}$ since every torsion free divisible Abelian group can be viewed as a vector space over \mathbb{Q} . So suppose that G is non-trivial

Let $X = \{(g, n) : g \in G, n \in \mathbb{N}, n > 0\}$. We think of (g, n) as g/n

We define an equivalence relation \sim on X by $(g, n) \sim (h, m)$ if and only if $mg = nh$. Let $H = X / \sim$. For $(g, n) \in X$, let $[(g, n)]$ denote the \sim -class of (g, n) . We define $+$ on H by $[(g, n)] + [(h, m)] = [(mg + nh, mn)]$. We must show that $+$ is well defined

Suppose that $(g_0, n_0) \sim (g, n)$. We claim that $(mg_0 + n_0h, mn_0) \sim (mg + nh, mn)$.

Similarly we can define $-$ by $[(g, n)] - [(h, m)] = [(mg - nh, mn)]$. It is easy to show that $(H, +)$ is an Abelian group

If $[(g, m)] \in H$ and $n > 0$, then $n[(g, m)] = [(ng, m)]$. If $(ng, m) \sim (0, k)$, then $kng = 0$. Because $k, n > 0$ and G is torsion free, $g = 0$. Then $[(g, m)] = [(0, 1)]$. Thus H is torsion free.

Suppose that $[(g, m)] \in H$ and $n > 0$, then $n[(g, mn)] = [(g, m)]$. Thus H is divisible.

We can embed G into H by the map $i(g) = [(g, 1)]$

Suppose that H' is a divisible torsion-free Abelian group and $j : G \rightarrow H'$ is an embedding. Let $h : H \rightarrow H'$ by $h([(g, n)]) = j(g)/n$ \square

Theorem 3.9. *DAG has quantifier elimination*

Proof. Suppose that G_0 and G_1 are torsion-free divisible Abelian groups, G is a common subgroup of G_0 and G_1 , $\bar{g} \in G$, $h \in G_0$ and $G_0 \models \phi(\bar{g}, h)$, where ϕ is quantifier-free. Let H be the divisible hull of G . Because we can embed H into G_0 , by Lemma 3.7, $H \models \exists w \phi(\bar{g}, w)$. Because we can embed H into G_1 , there is $h' \in G_1$ s.t. $G_1 \models \phi(\bar{g}, h')$. By Corollary 3.6, DAG has quantifier elimination \square

Quantifier elimination gives us a good picture of the definable sets in a model of DAG. Suppose that $\phi(v_1, \dots, v_n, w_1, \dots, w_m)$ is an atomic formula. Then there are integers k_1, \dots, k_n and l_1, \dots, l_m s.t. $\phi(\bar{v}, \bar{w}) \leftrightarrow \sum k_i x_i + \sum l_i y_i = 0$. If $G \models \text{DAG}$ and $a_1, \dots, a_m \in G$, $\phi(\bar{v}, \bar{a})$ defines $\{\bar{g} \in G^n : \sum k_i g_i + \sum l_i a_i = 0\}$, a hyperplane in G^n . Because any \mathcal{L} -formula $\phi(\bar{v}, \bar{w})$ is equivalent in DAG to a Boolean combination of atomic \mathcal{L} -formulas, every definable subset of G^n is a Boolean combination of hyperplanes

In particular, suppose that $\bar{a} \in G^m$ and $\phi(v, \bar{a})$ defines a subset of G . The “hyperplanes” in G are just single points. Thus, $\{g \in G : G \models \phi(g, \bar{a})\}$ is either finite or cofinite. Thus every definable subset of G was definable already in the language of equality

Definition 3.10. We say that an \mathcal{L} -theory T is **strongly minimal** if for any $\mathcal{M} \models T$ every definable subset of M is either finite or cofinite

Corollary 3.11. *DAG is strongly minimal*

If T is a theory then T_\forall is the set of all universal consequences of T . In Exercise 2.5.2 we saw that $\mathcal{A} \models T_\forall$ if and only if there is $\mathcal{M} \models T$ with $\mathcal{A} \subseteq \mathcal{M}$. One consequence of Lemma 3.8 is that every torsion-free Abelian

group is a substructure of a nontrivial divisible Abelian group. Because the axioms for torsion-free Abelian groups are universal, DAG_V is *exactly the theory of torsion-free Abelian groups*.

We say that a theory T has **algebraically prime models** if for any $\mathcal{A} \models T_V$ there is $\mathcal{M} \models T$ and an embedding $i : \mathcal{A} \rightarrow \mathcal{M}$ s.t. for all $\mathcal{N} \models T$ and embeddings $j : \mathcal{A} \rightarrow \mathcal{N}$ there is $h : \mathcal{M} \rightarrow \mathcal{N}$ s.t. $j = h \circ i$.

$$\begin{array}{ccc} \mathcal{A} \models T_V & \xrightarrow{i} & \mathcal{M} \models T \\ & \searrow j & \downarrow h \\ & & \mathcal{N} \models T \end{array}$$

If $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{M} \subseteq \mathcal{N}$, we say that \mathcal{M} is **simply closed** in \mathcal{N} and write $\mathcal{M} \prec_s \mathcal{N}$ if for any quantifier free formula $\phi(\bar{v}, w)$ and any $\bar{a} \in M$, if $\mathcal{N} \models \exists \phi(\bar{a}, w)$ then so does \mathcal{M} . Lemma 3.7 says that if G and H are models of DAG and $G \subseteq H$, then $G \prec_s H$.

Corollary 3.12. *Suppose that T is an \mathcal{L} -theory s.t.*

1. *T has algebraically prime models and*
 2. *$\mathcal{M} \prec_s \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ are models of T*
- Then T has quantifier elimination*

Proof. Suppose $\mathcal{A} \models T_V$, then □

Definition 3.13. An \mathcal{L} -theory T is **model-complete** $\mathcal{M} \prec \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M}, \mathcal{N} \models T$

Proposition 3.14. *If T has quantifier elimination, then T is model-complete*

Proof. Suppose that $\mathcal{M} \subseteq \mathcal{N}$ are models of T . Let $\phi(\bar{v})$ be an \mathcal{L} -formula, and let $\bar{a} \in M$. There is a quantifier-free formula $\psi(\bar{v})$ s.t. $\mathcal{M} \models \forall \bar{v}(\phi(\bar{v}) \leftrightarrow \psi(\bar{v}))$. Because quantifier-free formulas are preserved under substructures and extensions, $\mathcal{M} \models \psi(\bar{a})$ if and only if $\mathcal{N} \models \psi(\bar{a})$. Thus $\mathcal{M} \prec \mathcal{N}$ □

Proposition 3.15. *Let T be a model-complete theory. Suppose that there is $\mathcal{M}_0 \models T$ s.t. \mathcal{M}_0 embeds into every model of T . Then T is complete*

Proof. If $\mathcal{M} \models T$, then $\mathcal{M}_0 \prec \mathcal{M}$. In particular $\mathcal{M}_0 \equiv \mathcal{M}$. □

Because $(\mathbb{Q}, +, 0)$ embeds in every model of DAG , this gives another proof of the completeness of DAG

3.1.2 Ordered Divisible Abelian Groups

Let $\mathcal{L} = \{+, 0, <, 0\}$ and let ODAG be the theory of nontrivial divisible ordered Abelian groups.

Lemma 3.16. *Let G be an ordered Abelian group and H be the divisible hull of G . We can order H s.t. $i : G \rightarrow H$ is order-preserving, $(H, +, <) \models \text{ODAG}$ and if $H' \models \text{ODAG}$ and $j : G \rightarrow H'$ is an embedding, then there is an embedding $h : H \rightarrow H'$ s.t. $j = h \circ i$*

4 Set Theory

4.1 Cardinal Arithmetic

Corollary 4.1. 1. If $|I| = \kappa$ and $|A_i| \leq \kappa$ for all $i \in I$, then $|\bigcup A_i| \leq \kappa$
 2. If κ is regular, $|I| < \kappa$ and $|A_i| < \kappa$ for all $i \in I$, then $|\bigcup A_i| < \kappa$
 3. Let κ be an infinite cardinal. Let X be a set and \mathcal{F} a set of functions $f : X^{n_f} \rightarrow X$. Suppose that $|\mathcal{F}| \leq \kappa$ and $A \subseteq X$ with $|A| \leq \kappa$. Let $\text{CL}(A)$ be the smallest subset of X containing A closed under the functions in \mathcal{F} . Then $|\text{CL}(A)| \leq \kappa$

5 Algebra

Comes from [Lan05]

5.1 Real Fields

5.1.1 Ordered Fields

Let K be a field. An **ordering** of K is a subset P of K having the following properties

ORD 1. Given $x \in K$, we have either $x \in P$, or $x = 0$ or $-x \in P$, and these three possibilities are mutually exclusive

ORD 2. If $x, y \in P$, then $x + y, xy \in P$

K is **ordered by P** , and we call P the set of **positive elements**

Suppose K is ordered by P . Since $1 \neq 0$ and $1 = 1^2 = (-1)^2$, we see that $1 \in P$. By **ORD 2**, it follows that $1 + \dots + 1 \in P$, whence K has characteristic

0. If $x \in P$ and $x \neq 0$, then $xx^{-1} = 1 \in P$ implies that $x^{-1} \in P$

Let E be a field. Then a product of sums of squares in E is a sum of squares.

If $a, b \in E$ are sum of squares and $b \neq 0$, then a/b is a sum of squares

Consider complex number:)

Let $x, y \in K$. We define $x < y$ to mean that $y - x \in P$. If $x < 0$ we say that x is **negative**.

If K is ordered and $x \in K, x \neq 0$, then x^2 is positive

If E has characteristic $\neq 2$, and -1 is a sum of squares in E , then every element $a \in E$ is a sum of squares, because $4a = (1+a)^2 - (1-a)^2$

If K is a field with an ordering P , and F is a subfield, then obviously, $P \cap F$ defines an ordering of F , which is called the **induced** ordering

Let K be an ordered field and let F be a subfield with the induced ordering. We put $|x| = x$ if $x > 0$ and $|x| = -x$ if $x < 0$. An element $\alpha \in K$ is **infinitely large** over F if $|\alpha| \geq x$ for all $x \in F$. It is **infinitely small** over F if $0 \leq |\alpha| \leq |x|$ for all $x \in F, x \neq 0$. α is infinitely large if and only if α^{-1} is infinitely small. K is **archimedean** over F if K has no elements which are infinitely large over F . An intermediate field $F_1, K \supset F_1 \supset F$ is **maximal archimedean over F** in K if it is archimedean over F and no other intermediate field containing F_1 is archimedean over F . We say that F is **maximal archimedean in K** if it is maximal archimedean over itself in K

Let K be an ordered field and F a subfield. Let \mathfrak{o} be the set of elements of K which are not infinitely large over F . Then \mathfrak{o} is a ring and that for any $\alpha \in K$, we have α or $\alpha^{-1} \in \mathfrak{o}$. Hence \mathfrak{o} is what is called a valuation ring, containing F . Let \mathfrak{m} be the ideal of all $\alpha \in K$ which are infinitely small over F . Then \mathfrak{m} is the unique maximal ideal of \mathfrak{o} , because any element in \mathfrak{o} which is not in \mathfrak{m} has an inverse in \mathfrak{o} . We call \mathfrak{o} the **valuation ring determined by the ordering of K/F**

Proposition 5.1. *Let K be an ordered field and F a subfield. Let \mathfrak{o} be the valuation ring determined by the ordering of K/F , and let \mathfrak{m} be its maximal ideal. Then $\mathfrak{o}/\mathfrak{m}$ is a real field.*

Proof. Otherwise, we could write

$$-1 = \sum \alpha_i^2 + a$$

with $\alpha_i \in \mathfrak{o}$ and $a \in \mathfrak{m}$. Since $\sum \alpha_i^2$ is positive and a is infinitely small, such a relation is clearly impossible \square

6 Reference

References

- [DJMM12] Arnaud Durand, Neil D. Jones, Johann A. Makowsky, and Malika More. Fifty years of the spectrum problem: survey and new results. *Bulletin of Symbolic Logic*, 18(4):505–553, 2012.
- [Lan05] S. Lang. *Algebra*. Graduate Texts in Mathematics. Springer New York, 2005.

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