

Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets

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1 Recursive Functions

1.1 Formal Definitions of Computable Functions

1.1.1 Primitive Recursive Functions

Definition 1.1. The class of primitive recursive functions is the smallest class \mathcal{C} of functions closed under the following schema

1. the **successor function**, $\lambda x[x + 1] \in \mathcal{C}$
2. the **constant functions**, $\lambda x_1 \dots x_n[m] \in \mathcal{C}, 0 \leq n, m$
3. the **identity function**, $\lambda x_1 \dots x_n[x_i] \in \mathcal{C}, 1 \leq i \leq n$
4. (Composition) If $g_1, \dots, g_m, h \in \mathcal{C}$, then

$$f(x_1, \dots, x_n) = h(g_1(x_1, \dots, x_n), \dots, g_m(x_1, \dots, x_n))$$

is in \mathcal{C} where g_1, \dots, g_m are functions of n variables and h is a function of m variables

5. (Primitive Recursion) If $g, h \in \mathcal{C}$ and $n \geq 1$ then $f \in \mathcal{C}$ where

$$\begin{aligned} f(0, x_2, \dots, x_n) &= g(x_2, \dots, x_n) \\ f(x_1 + 1, x_2, \dots, x_n) &= h(x_1, f(x_1, \dots, x_n), x_2, \dots, x_n) \end{aligned}$$

Hence a function is primitive recursive if there is a **derivation**, namely a sequence $f_1, \dots, f_k = f$ s.t. for each $f_i, i \leq k$ is either an initial function or obtained from 4 or 5.

A predicate (relation) is **primitive recursive** if its characteristic function is.

1.1.2 Diagonalization and Partial Recursive Functions

Although the primitive recursive functions include all the usual functions from elementary number theory they fail to include **all** computable functions. Each derivation of a primitive recursive function is a finite string of symbols from a fixed finite alphabet, and thus all derivations can be effectively listed. Let f_n be the function corresponding to the n th derivation in this listing. Then the function $g(x) = f_x(x) + 1$ cannot be primitive recursive.

The same argument applies to any effective set of schemata which produces only **total** functions. *Thus to obtain all computable functions we are forced to consider computable **partial** functions.*

Definition 1.2 (Kleene). The class of **partial recursive** (p.r.) functions is the least class obtained by closing under schemata 1 through 5 for the primitive recursive functions and the following schemata 6. A **total recursive** function (abbreviated **recursive** function) is a partial recursive function which is total.

6. (Unbounded Search) If $\theta(x_1, \dots, x_n, y)$ is a partial recursive function of $n + 1$ variables, and

$$\psi(x_1, \dots, x_n) = \mu y [\theta(x_1, \dots, x_n, y) \downarrow = 0 \wedge (\forall z \leq y) [\theta(x_1, \dots, x_n, z) \downarrow]]$$

Definition 1.3. A relation $R \subseteq \omega^n, n \geq 1$ is **recursive** (**primitive recursive**, has property P) if its characteristic function χ_R is recursive (primitive recursive) where $\chi_R(x_1, \dots, x_n) = 1$ if and only if $(x_1, \dots, x_n) \in R$.

1.1.3 Turing Computable Functions

A **Turing machine** M includes a two-way infinite **tape** divided into **cells**, a **reading head** which scans one cell of the tape at a time, and a finite set of internal **states** $Q = \{q_0, \dots, q_n\}, n \geq 1$. Each cell is either blank (B) or has written on it the symbol 1. In a single step the machine may simultaneously

1. change from one state to another
2. change the scanned symbol s to another symbol $s' \in S = \{1, B\}$
3. move the reading head one cell to the right (R) or left (L)

The operation of M is controlled by a partial map $\delta : Q \times S \rightarrow Q \times S \times \{R, L\}$

The map δ viewed as a finite set of quintuples is called a **Turing program**. The **input** integer x is represented by a string of $x + 1$ consecutive 1's.

1.1.4 Exercises

Exercise 1.1.1 (Definition by cases). If $g_1(x), \dots, g_n(x)$ are primitive recursive functions and $R_1(x), \dots, R_n(x)$ are primitive recursive relations which are mutually exclusive and exhaustive show that f is primitive where $f(x) = g_1(x)$ if $R_1(x), \dots, f(x) = g_n(x)$ if $R_n(x)$

Proof. $f(x) = \sum_{i=1}^n \chi_{R_i}(x) \times g_i(x)$ □

1.2 The Basic Results

Church's Thesis asserts that these functions coincide with the intuitively computable functions. We shall accept Church's Thesis and from now on

shall use the terms “partial recursive” “Turing computable” and “computable” interchangeably

Definition 1.4. Let P_e be the Turing program with code number (Gödel number) e (also called **index** e) in this listing and let $\varphi_e^{(n)}$ be the partial functions of n variables computed by P_e , where φ_e abbreviates $\varphi_e^{(1)}$

Lemma 1.5 (Padding Lemma). *Each partial recursive function φ_x has \aleph_0 indices, and furthermore for each x we can effectively find an infinite set A_x of indices for the same partial function*

Proof. For any program P_x mentioning internal states $\{q_0, \dots, q_n\}$ add extraneous instructions $q_{n+1} B q_{n+1} B R, q_{n+2} B q_{n+2}, B R, \dots$ to get new programs for the same functions \square

Theorem 1.6 (Normal Form Theorem (Kleene)). *There exist a predicate $T(e, x, y)$ (called the **Kleene T-predicate**) and a function $U(y)$ which are recursive (indeed primitive recursive) s.t.*

$$\varphi_e(x) = U(\mu y T(e, x, y))$$

Proof. Informally, the predicate $T(e, x, y)$ asserts that y is the code number of some Turing computation according to program P_e with input x . To see whether $T(e, x, y)$ holds we first effectively recover from e the Program P_e ; then recover from y the computation c_0, c_1, \dots, c_n if y codes such a computation. Now check whether c_0, \dots, c_n is a computation according to P_e with x as the input in c_0 . If so $U(y)$ simply outputs the number of 1's in the final configuration c_n . \square

It follows from the Normal Form Theorem that every Turing computable partial function is partial recursive. To prove the converse one constructs Turing machines corresponding to the schemata (1) \rightarrow (6).

Note by Theorem ?? it follows that every partial recursive function can be obtained from two primitive recursive functions by **one** application of the μ -operator

Theorem 1.7 (Enumeration Theorem). *There is a p.r. function of 2 variables $\varphi_z^{(2)}(e, x)$ s.t. $\varphi_z^{(2)}(e, x) = \varphi_e(x)$. Indeed the Enumeration Theorem holds for p.r. functions of n variables*

Proof. Let $\varphi_z^{(2)}(e, x) = U(\mu y T(e, x, y))$. For $\varphi_z^{(n)}(e, x_1, \dots, x_{n-1})$, by s - m - n theorem,

$$\varphi_z^{(n)}(e, \bar{x}) = \varphi_{s_{n-1}^2(z, e)}^{(n-1)}(\bar{x})$$

Thus we only need to make sure that $s_{n-1}^2(z, e) \in A_e$, which can be effectively found. \square

Theorem 1.8 (Parameter Theorem (*s-m-n Theorem*)). *For every $m, n \geq 1$ there exists a 1:1 recursive function s_n^m of $m + 1$ variables s.t. for all x, y_1, y_2, \dots, y_m*

$$\varphi_{s_n^m(x, y_1, \dots, y_m)}^{(n)} = \lambda z_1, \dots, z_n (\varphi_x^{(m+n)}(y_1, \dots, y_m, z_1, \dots, z_n))$$

Proof. (informal). For simplicity consider the case $m = n = 1$. $\varphi_{s_1^1(x, y)}^{(1)} = \lambda z (\varphi_x^{(2)}(y, z))$ The program $P_{s_1^1(x, y)}$ on input z first obtains P_x and then applies P_x to input (y, z) . Now $s = s_1^1$ is a recursive function by Church's Thesis since this is an effective procedure in x and y . If s is not already 1:1 it may be replaced by a 1:1 recursive function s' s.t. $\varphi_{s(x, y)} = \varphi_{s'(x, y)}$ by using the padding lemma, and by defining $s'(x, y)$ in increasing order of $\langle x, y \rangle$, where $\langle x, y \rangle$ is the image of (x, y) under the pairing function \square

Remark. Here is an interesting question in StackExchange

The *s-m-n* theorem asserts that y may be treated as a fixed parameter in the program $P_{s(x, y)}$ which operate on z and furthermore that the index $s(x, y)$ of this program is effective in x and y . A simple application of the *s-m-n* theorem is the existence of a recursive function $f(x)$ s.t. $\varphi_{f(x)} = 2\varphi_x$. Let $\psi(x, y) = 2\varphi_x(y)$. By Church's Thesis $\psi(x, y) = \varphi_e^{(2)}(x, y)$ for some e . Let $f(x) = s_1^1(e, x)$

We let $\langle x, y \rangle$ denote the image of (x, y) under the standard pairing function $\frac{1}{2}(x^2 + 2xy + y^2 + 3x + y)$ which is a bijective recursive function from $\omega^2 \rightarrow \omega$. Let π_1 and π_2 denote the inverse functions $\pi_1(\langle x, y \rangle) = x$

For a relation $R \subseteq \omega^n, n > 1$, we say that R has some property P iff the set $\{\langle x_1, \dots, x_n \rangle : R(x_1, \dots, x_n)\}$ has property P

Definition 1.9. We write $\varphi_{e, s}(x) = y$ if $x, y, e < s$ and y is the output $\varphi_e(x)$ in $< s$ steps of the Turing machine P_e . If such a s exists we say $\varphi_{e, s}(x)$ **converges**, which we write as $\varphi_{e, s}(x) \downarrow$, and **diverges** ($\varphi_{e, s}(x) \uparrow$). Similarly, we write $\varphi_e(x) \downarrow$ if $\varphi_{e, s}(x) \downarrow$ for some s

Theorem 1.10. 1. The set $\{\langle e, x, s \rangle : \varphi_{e, s}(x) \downarrow\}$ is recursive
2. The set $\{\langle e, x, y, s \rangle : \varphi_{e, s}(x) = y\}$ is recursive

Proof. From Church's Thesis since they are all computable \square

1.2.1 Exercises

Exercise 1.2.1. Prove the following alternative definition of $\varphi_{e,s}(x) = y$ also satisfies Theorem ?? as well as the convenient properties:

$$\varphi_{e,s}(x) = y \implies e, x, y < s$$

and

$$(\forall s)(\exists \text{ at most one } \langle e, x, y \rangle)[\varphi_{e,s}(x) = y \ \& \ \varphi_{e,s-1}(x) \uparrow]$$

and hence

$$(\forall s)(\exists \text{ at most one } \langle e, x \rangle)[x \in W_{e,s+1} - W_{e,s}]$$

Define $\varphi_{e,s}(x) = y$ by recursion on s on follows. Let $\varphi_{e,0}(x) \uparrow$ for all x . Let $\varphi_{e,s+1}(x) = y$ iff $\varphi_{e,s}(x) = y$, or $s = \langle e, x, y, t \rangle$ for some $t > 0$ and y is the output of $\varphi_e(x)$ in $\leq t$ steps of the Turing program P_e

1.3 Recursively Enumerable Sets and Unsolvable Problems

Definition 1.11. 1. A set A is **recursively enumerable** (r.e.) if A is the domain of some p.r. function
2. let the e th r.e. set be denoted by

$$W_e = \text{dom}(\varphi_e) = \{x : \varphi_e(x) \downarrow\} = \{x : (\exists y)T(e, x, y)\}$$

3. $W_{e,s} = \text{dom}(\varphi_{e,s})$

Note that $\varphi_e(x) = x$ iff $(\exists s)[\varphi_{e,s} = y]$ and $x \in W_e$ iff $(\exists s)(x \in W_{e,s})$

Definition 1.12. Let $K = \{x : \varphi_x(x) \text{ converges} \} = \{x : x \in W_x\}$

Proposition 1.13. K is r.e.

Proof. K is the domain of the following p.r. function

$$\psi(x) = \begin{cases} x & \text{if } \varphi_x(x) \text{ converges,} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Now ψ is p.r. by Church's Thesis since $\psi(x)$ can be computed by applying program P_x to input x and giving output x only if $\varphi(x)$ converges. Alternatively and more formally, $K = \text{dom}(\theta)$ where $\theta(x) = \varphi_z^{(2)}(x, x)$ for $\varphi_z^{(2)}$ the p.r. function defined in the Enumeration Theorem ?? \square

Corollary 1.14. K is not recursive

Proof. If K had a recursive characteristic function χ_K then the following function would be recursive

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K \end{cases}$$

However f cannot be recursive since $f \neq \varphi_x$ for any x □

Definition 1.15. $K_0 = \{\langle x, y \rangle : x \in W_y\}$

K_0 is p.r. $K_0 = \text{dom } \theta_0$, where $\theta(\langle x, y \rangle) = \varphi_z^{(2)}(y, x)$

Corollary 1.16. K_0 is not recursive

Proof. $x \in K$ iff $\langle x, x \rangle \in K_0$ □

The **halting problem** is to decide for arbitrary x and y whether $\varphi_x(y) \downarrow$. Corollary ?? asserts the unsolvability of the halting problem.

Definition 1.17. 1. A is a **many-one reducible** (**m -reducible**) to B (written $A \leq_m B$) if there is a recursive function f s.t. $f(A) \subseteq B$ and $f(\bar{A}) \subseteq \bar{B}$, i.e. $x \in A$ iff $f(x) \in B$
 2. A is **one-one reducible** (**1-reducible**) to B ($A \leq_1 B$) if $A \leq_m B$ by a 1:1 recursive function

The proof of corollary ?? established that $K \leq_1 K_0$ via the function $f(x) = \langle x, x \rangle$

Definition 1.18. 1. $A \equiv_m B$ if $A \leq_m B$ and $B \leq_m A$
 2. $A \equiv_1 B$ if $A \leq_1 B$ and $B \leq_1 A$
 3. $\text{deg}_m(A) = \{B : A \equiv_m B\}$
 4. $\text{deg}_1(A) = \{B : A \equiv_1 B\}$

The equivalence classes under \equiv_m and \equiv_1 are called the **m-degrees** and **1-degrees** respectively

Proposition 1.19. If $A \leq_m B$ and B is recursive then A is recursive

Proof. $\chi_A(x) = \chi_B(f(x))$ □

Theorem 1.20. $K \leq_1 \text{Tot} := \{x : \varphi_x \text{ is a total function}\}$

Proof. Define the function

$$\psi(x, y) = \begin{cases} 1 & \text{if } x \in K \\ \text{undefined} & \text{otherwise} \end{cases}$$

By s - m - n theorem, there is a 1:1 recursive function f s.t. $\varphi_{f(x)}(y) = \psi(x, y)$. Choose e s.t. $\varphi_e(x, y) = \psi(x, y)$ since ψ is p.r. and define $f(x) = s_1^1(e, x)$. Note that

$$\begin{aligned} x \in K &\implies \varphi_{f(x)} = \lambda y[1] \implies \varphi_{f(x)} \text{ total} \implies f(x) \in \text{Tot} \\ x \notin K &\implies \varphi_{f(x)} = \lambda y[\text{undefined}] \implies \varphi_{f(x)} \text{ not total} \implies f(x) \notin \text{Tot} \end{aligned}$$

□

Definition 1.21. A set $A \subseteq \omega$ is an **index set** if for all x and y

$$(x \in A \wedge \varphi_x = \varphi_y) \implies y \in A$$

Theorem 1.22. If A is a nontrivial index set, i.e., $A \neq \emptyset, \omega$, then either $K \leq_1 A$ or $K \leq_1 \bar{A}$

Proof. Choose e_0 s.t. $\varphi_{e_0}(y)$ is undefined for all y . If $e_0 \in \bar{A}$, then $K \leq_1 A$ as follows. Since $A \neq \emptyset$ we can choose $e_1 \in A$. Now $\varphi_{e_1} \neq \varphi_{e_0}$ because A is an index set. By s - m - n theorem define a 1:1 recursive function f s.t.

$$\varphi_{f(x)}(y) = \begin{cases} \varphi_{e_1}(y) & x \in K \\ \text{undefined} & x \notin K \end{cases}$$

Now

$$\begin{aligned} x \in K &\implies \varphi_{f(x)} = \varphi_{e_1} \implies f(x) \in A \\ x \notin K &\implies \varphi_{f(x)} = \varphi_{e_0} \implies f(x) \in \bar{A} \end{aligned}$$

□

It's possible that both $K \leq_1 A$ and $K \leq_1 \bar{A}$ for an index set A , for example if $A = \text{Tot}$

Corollary 1.23 (Rice's Theorem). Let \mathcal{C} be any class of partial recursive functions. Then $\{n : \varphi_n \in \mathcal{C}\}$ is recursive iff $\mathcal{C} = \emptyset$ or \mathcal{C} is the set of all partial recursive functions

Proof. \mathcal{C} is an index set and hence is trivial.

□

Definition 1.24.

$$\begin{aligned}
K_1 &= \{x : W_x \neq \emptyset\} \\
\text{Fin} &= \{x : W_x \text{ is finite}\} \\
\text{Inf} &= \omega - \text{Fin} = \{x : W_x \text{ is infinite}\} \\
\text{Tot} &= \{x : \varphi_x \text{ is total}\} = \{x : W_x = \omega\} \\
\text{Con} &= \{x : \varphi_x \text{ is total and constant}\} \\
\text{Cof} &= \{x : W_x \text{ is cofinite}\} \\
\text{Rec} &= \{x : W_x \text{ is recursive}\} \\
\text{Ext} &= \{x : \varphi_x \text{ is extendible to a total recursive function}\}
\end{aligned}$$

Definition 1.25. An r.e. set A is **1-complete** if $W_e \leq_1 A$ for every r.e. set W_e

K_0 is 1-complete because $x \in W_e$ iff $\langle x, e \rangle \in K_0$

Definition 1.26. Let A join B written $A \oplus B$ be

$$\{2x : x \in A\} \cup \{2x + 1 : x \in B\}$$

1.3.1 Exercises

- Exercise 1.3.1.* 1. $A \leq_m A \oplus B$ and $B \leq_m A \oplus B$
2. if $A \leq_m C$ and $B \leq_m C$ then $A \oplus B \leq_m C$

Proof. 1.
2. Easy

□

Exercise 1.3.2. $K \equiv_1 K_0 \equiv_1 K_1$

Proof. $K \leq_1 A$ for $A = K_1$, con or Inf.

$K_0 \leq K$ for the same reason.

For $K \leq K_1$

$$\varphi_{f(x)}(y) = \begin{cases} x & x \in K \\ \text{undefined} & x \notin K \end{cases}$$

For $K_0 \leq_1 K$, the same (find a x s.t. $x \in W_x$)

Also note that K and K_1 are 1-complete

□

Exercise 1.3.3. Prove directly (without Rice's theorem) that $K \leq_1 \text{Fin}$

Proof. Let

$$\varphi_{f(x)}(s) = \begin{cases} 0 & x \notin K_s \\ \text{undefined} & x \in K_s \end{cases}$$

where $K_s = W_{e,s}$ for some e s.t. $K = W_e$. If $x \in K$, then $\text{dom}(\varphi_{f(x)})$ is finite \square

Exercise 1.3.4. For any x show that $\overline{K} \leq_1 \{y : \varphi_x = \varphi_y\}$ and $\overline{K} \leq_1 \{y : W_x = W_y\}$

Proof. Use the method of exercise ?? . If $x \notin W_x$, then $\text{dom}(\varphi_{f(x)}) = \omega$. \square

Exercise 1.3.5. $\text{Ext} \neq \omega$

Proof. Use K . If $\psi(x)$ can be extended to a recursive function, then K would be recursive. \square

Exercise 1.3.6. 1. Disjoint sets A and B are **recursively inseparable** if there is no recursive set C s.t. $A \subseteq C$ and $C \cap B = \emptyset$. Show that there exists disjoint r.e. sets which are recursively inseparable.

2. Give an alternative proof that $\text{Ext} \neq \omega$
3. For A and B as in part 1, prove that $K \equiv_1 A$ and $K \equiv_1 B$

Proof. 1. Consider $A = \{x : \varphi_x(0) = 0\}$ and $B = \{x : \varphi_x(0) = 1\}$.
 2. corollary from 1.
 3. \square

Exercise 1.3.7. A set A is **cylinder** if $(\forall B)[B \leq_m A \implies B \leq_1 A]$

1. Show that any index set is a cylinder
2. Show that any set of the form $A \times \omega$ is a cylinder
3. Show that A is a cylinder iff $A \equiv_1 B \times \omega$ for some set B

Proof. 1. If different $x, y \in B$ and $f(x) = f(y)$, we could just add redundant computation and $\varphi_{f(x)} = \varphi_{f(y)}$
 2. to make sure images are different by ω
 3. \square

Exercise 1.3.8. Show that the partial recursive functions are not closed under μ , i.e., there is a p.r. function ψ s.t. $\lambda x[\mu y[\psi(x, y) = 0]]$ is not p.r.

Proof. $\psi(x, y) = 0$ if $y = 1$ or $y = 0$ and $\varphi_x(x) \downarrow$. \square

Exercise 1.3.9. If A is recursive and B, \overline{B} are each $\neq \emptyset$, then $A \leq_m B$

Proof. choose elements $b \in B$ and $b' \in \overline{B}$. Then

$$\psi_{f(x)}(s) = \begin{cases} b & x \in A \\ b' & x \notin A \end{cases}$$

□

Exercise 1.3.10. Prove that $\text{Inf} \equiv_1 \text{Tot} \equiv_1 \text{Con}$

Proof. $\text{Tot} \equiv_1 \text{Con}$ is obvious. For $\text{Inf} \leq_1 \text{Con}$, define

$$\psi(e, x) = \begin{cases} 0 & \text{if } (\exists y > x)[\varphi_e(y) \downarrow] \\ \uparrow & \text{otherwise} \end{cases}$$

□

Exercise 1.3.11. $\text{Fin} \leq_1 \text{Cof}$

Proof.

$$\varphi_{f(e)}(s) = \begin{cases} \uparrow & \text{if } W_{e,s+1} - W_{e,s} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

□

1.4 Recursive Permutation and Myhill's Isomorphism Theorem

Definition 1.27. 1. A **recursive permutation** is a 1:1, recursive function from ω to ω
 2. A property of set is **recursively invariant** if it's invariant under all recursive permutation

Examples:

1. A is r.e. ($A \leq_1 \text{im}(A)$)
2. A has cardinality n
3. A is recursive

Properties that not recursively invariant:

1. $2 \in A$
2. A contains the even integers
3. A is an index set

Definition 1.28. A is **recursively isomorphic** to B (written $A \equiv B$) if there is a recursive permutation p s.t. $p(A) = B$

Definition 1.29. The equivalence classes under \equiv are called **recursive isomorphism types**

Theorem 1.30 (Myhill Isomorphism Theorem). $A \equiv B \iff A \equiv_1 B$

Proof. (\implies) trivial.

(\impliedby) Let $A \leq_1 B$ via f and $B \leq_1 A$ via g . We define a recursive permutation h by stages so that $h(A) = B$. We let $h = \bigcup_s h_s$, where $h_0 = \emptyset$ and h_s is that portion of h defined by the end of stage s . Assume h_s is given so that in particular we can effectively check for membership in $\text{dom } h_s$ and $\text{ran}(h_s)$ which we both assume finite

Stage $s + 1 = 2x + 1$. Assume that h_s is $1 : 1$, $\text{dom } h_s$ is finite and $y \in A$ iff $h_s(y) \in B$ for all $y \in \text{dom } h_s$. If $h_s(x)$ is defined, do nothing. Otherwise enumerate the set $\{f(x), f(h_s^{-1}f(x)), \dots, f(h_s^{-1}f)^n(x), \dots\}$ until the first element y not yet in $\text{ran}(h_s)$. Define $h_{s+1}(x) = y$. y must exist since f and h_s are $1 : 1$ and $x \notin \text{dom } h_s$

Stage $s + 1 = 2x + 2$. Define $h^{-1}(x)$ similarly with f, h_s, dom and ran replaced by $g, h_s^{-1}, \text{ran}, \text{dom}$ respectively \square

Definition 1.31. A function f **dominates** a function g if $f(x) \geq g(x)$ for almost every (all but finitely many) $x \in \omega$

1.4.1 Exercises

Exercise 1.4.1 (\times). Prove that the primitive recursive permutations do not form a group under composition

Proof. Define $g(x) = \mu y T(e, x, y)$. g dominates all primitive recursive functions since $y \geq U(y)$ for all y . Suppose f is a primitive recursive permutation and $f(g(x)) = x$ if x is even. Note that given y we can primitively recursively compute whether there is an x s.t. $g(x) = y$ \square

Exercise 1.4.2. Let $\omega = \bigcup_n A_n = \bigcup_n B_n$ where the sequences $\{A_n\}_{n \in \omega}$ and $\{B_n\}_{n \in \omega}$ are each pairwise disjoint. Let f and g be $1:1$ recursive functions s.t. $f(A_n) \subseteq B_n$ and $g(B_n) \subseteq A_n$ for all n . Show that the construction of Theorem ?? produces a recursive permutation h s.t. $h(A_n) = B_n$ for all n

Proof. *stage $s + 1 = 2x + 1$:* assume h_s is $1:1$, $\text{dom } h_s$ is finite. Hence there is $a \in \omega$ not in $\text{dom } h_s$. Then by... \square

Exercise 1.4.3 (Rogers). Let \mathcal{P} be the class of partial recursive functions of one variable. A **numbering** of the p.r. function is a map π from ω onto \mathcal{P} . The numbering $\{\varphi_e\}_{e \in \omega}$ is called the **standard numbering**. Let $\hat{\pi}$ be another numbering and let ψ_e denote $\hat{\pi}(e)$. Then $\hat{\pi}$ is an **acceptable** numbering if there are recursive functions f and g s.t.

1. $\varphi_{f(x)} = \psi_x$
2. $\psi_{g(x)} = \varphi_x$

Show that for any acceptable numbering $\hat{\pi}$, there is a recursive permutation p of ω s.t. $\varphi_x = \psi_{p(x)}$ for all x

Proof. Define $e_1 \sim e_2$ if φ_{e_1} and φ_{e_2} computes the same p.r. function. Then we get an enumeration $([e_i])_{i \in \omega} = A / \sim$. Define $A_i = [e_i]$. Obviously $f(A_i) \subseteq B_i$ and vice versa

By exercise ?? with appropriate definitions of A_n and B_n it suffices to convert f and g to a 1:1 recursive functions f_1 and g_1 satisfying (1) and (2).

To define f_1 from f use the Padding Lemma ?. To define $g_1(x)$ we must be able (uniformly in x) to effectively generate an infinite set S_x of indices s.t. for each $y \in S_x$ $\psi_y = \psi_{g(x)}$. Take any two recursively inseparable r.e. sets A and B , such as those of Exercise ??, and define

$$\varphi_{k(x,y)}(z) = \begin{cases} \varphi_x(z) & y \in A \\ 0 & y \in B \\ \text{undefined} & \text{otherwise} \end{cases}$$

and similarly $\varphi_{l(x,y)}$ with 1 in place of 0. Let $C_x = \{k(x, y) : y \in A\}$ and $D_x = \{l(x, y) : y \in A\}$. If $\varphi_x \neq \lambda z[0]$, then $g(C_x)$ cannot be finite or else A and B are recursively separable. Hence $S_x = g(C_x) \cup g(D_x)$ is infinite. Note we do not have to know this in order to see that S_x is infinite \square

2 Fundamentals of Recursively Enumerable Sets and the Recursion Theorem

2.1 Equivalent Definitions of Recursively Enumerable Sets

- Definition 2.1.**
1. A set A is a **projection** of some relation $R \subseteq \omega \times \omega$ if $A = \{x : (\exists y) R(x, y)\}$
 2. A set A is in Σ_1 -**form** (abbreviated “ A is Σ_1 ”) if A is the projection of some recursive relation $R \subseteq \omega \times \omega$.

Theorem 2.2 (Normal Form Theorem for r.e. sets). *A set A is r.e. iff A is Σ_1*

Proof. If A is r.e., then $A = W_e$ for some e . Hence

$$x \in W_e \Leftrightarrow (\exists s)[x \in W_{e,s}] \Leftrightarrow (\exists s)T(e, x, s)$$

and $T(e, x, s)$ is primitive recursive

Let $A = \{x : (\exists y)R(x, y)\}$, where R is recursive. Then $A = \text{dom } \psi$, where $\psi(x) = (\mu y)R(x, y)$ \square

Theorem 2.3 (Quantifier Contraction Theorem). *If there is a recursive relation*

$$R \subseteq \omega^{n+1}$$

and

$$A = \{x : (\exists y_1) \dots (\exists y_n)R(x, y_1, \dots, y_n)\}$$

then A is Σ_1

Proof. Define the recursive relation $S \subseteq \omega^2$ by

$$S(x, z) \Leftrightarrow R(x, (z)_1, \dots, (z)_n)$$

where $z = p_1^{(z)_1} \dots p_k^{(z)_k}$ \square

Corollary 2.4. *The projection of an r.e. relation is r.e.*

Definition 2.5. The **graph** of a (partial) function ψ is the relation

$$(x, y) \in \text{graph } \psi \Leftrightarrow \psi(x) = y$$

Using Theorem ?? the following sets and relations are r.e.:

1. $K = \{e : e \in W_e\} = \{e : (\exists s, y)[\varphi_{e,s}(e) = y]\}$
2. $K_0 = \{\langle x, e \rangle : x \in W_e\} = \{\langle x, e \rangle : (\exists s, y)[\varphi_{e,s}(x) = y]\}$
3. $K_1 = \{e : W_e \neq \emptyset\} = \{e : (\exists s, x)[x \in W_{e,s}]\}$
4. $\text{im } \varphi_e = \{y : (\exists s, x)[\varphi_{e,s}(x) = y]\}$
5. $\text{graph } \varphi_e = \{(x, y) : (\exists s)[\varphi_{e,s}(x) = y]\}$

Theorem 2.6 (Uniformization Theorem). *If $R \subseteq \omega^2$ is an r.e. relation, then there is a p.r. function ψ (called a **selector function** for R) s.t.*

$$\psi(x) \downarrow \Leftrightarrow (\exists y)R(x, y)$$

and in this case $(x, \psi(x)) \in R$

Proof. Since R is r.e. and hence Σ_1 , there is a recursive relation S s.t. $R(x, y)$ holds iff $(\exists z)S(x, y, z)$. Define the p.r. function

$$\theta(x) = (\mu u)S(x, (u)_1, (u)_2)$$

and set $\psi(x) = (\theta(x))_1$ □

Theorem 2.7 (Graph Theorem). *A partial function ψ is partial recursive iff its graph is r.e.*

Proof. If the graph of ψ is r.e., then ψ is its own selector function.

If ψ is p.r., there is e s.t. $\varphi_e = \psi$ □

Theorem 2.8 (Listing Theorem). *A set A is r.e. iff $A = \emptyset$ or A is the range of a total recursive function.. Furthermore, f can be found uniformly in an index for A as explained in Exercise ??*

Proof. Let $A = W_e \neq \emptyset$. Find the least integer $\langle a, t \rangle$ s.t. $a \in W_{e,t}$. Define the recursive function f by

$$f(\langle s, t \rangle) = \begin{cases} x & x \in W_{e,s+1} - W_{e,s} \\ a & \text{otherwise} \end{cases}$$

Clearly $A = \text{im } f$.

If A is the range of a total recursive function, A is Σ_1 □

Theorem 2.9 (Union Theorem). *The r.e. sets are closed under union and intersection uniformly effectively, namely there are recursive functions f and g s.t. $W_{f(x,y)} = W_x \cup W_y$, and $W_{g(x,y)} = W_x \cap W_y$*

Proof. Using the s - m - n Theorem define $f(x, y)$ by enumerating $z \in W_{f(x,y)}$ if $(\exists s)[z \in W_{x,s} \cup W_{y,s}]$ □

Corollary 2.10 (Reduction Principle for r.e. sets). *Given any two r.e. sets A and B , there exist r.e. sets $A_1 \subseteq A$ and $B_1 \subseteq B$ s.t. $A_1 \cap B_1 = \emptyset$ and $A_1 \cup B_1 = A \cup B$*

Proof. Define the relation $R := A \times \{0\} \cup B \times \{1\}$ which is r.e. by Theorem ??. By the Uniformization Theorem ??, let ψ be the p.r. selector function for R . Let $A_1 = x : \psi(x) = 0$ and $B_1 = x : \psi(x) = 1$ □

Definition 2.11. A set A is in Δ_1 -form (abbreviated “ A is Δ_1 ”) if both A and \bar{A} is Σ_1 .

Theorem 2.12 (Complementation Theorem). *A set A is recursive iff both A and \bar{A} are r.e. (i.e., iff $A \in \Delta_1$)*

Proof. Let $A = W_e$, $\bar{A} = W_i$. Define the recursive function

$$f(x) = (\mu s)[x \in W_{e,s} \vee x \in W_{i,s}]$$

Then $x \in A$ iff $x \in W_{e,f(x)}$, so A is recursive □

Corollary 2.13. \bar{K} is not r.e.

- Definition 2.14.** 1. A **lattice** $\mathcal{L} = (L; \leq, \vee, \wedge)$ is a partially ordered set (poset) in which any two elements have a least upper bound and greatest lower bound. If a and b are elements of a lattice \mathcal{L} , $a \vee b$ denote the least upper bound (lub) of a and b , $a \wedge b$ the greatest lower bound (glb). If \mathcal{L} contains a least element and greatest element these are called the **zero** element and **unit** element 1. In such a lattice a is the **complement** of b if $a \vee b = 1$
2. A lattice is **distributive** if all its elements satisfy the distributive laws $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ and $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$
 3. A lattice is **complemented** if every element has a complement
 4. A poset closed under suprema but not necessarily under infima is an **upper semi-lattice**
 5. $\mathcal{M} = (M; \leq, \vee, \wedge)$ is a **sublattice** of \mathcal{L} if $M \subseteq L$ and M is closed under the operations \vee and \wedge in \mathcal{L}
 6. A nonempty subset $I \subseteq L$ forms an **ideal** $\mathcal{I} = (I, \leq, \wedge, \vee)$ of \mathcal{L} if I satisfies the conditions
 - (a) $[a \in L \ \& \ a \leq b \in I] \implies a \in I$
 - (b) $[a \in I \ \& \ b \in I] \implies a \vee b \in I$
 7. A subset $D \subseteq L$ forms a **filter** $\mathcal{D} = (D; \leq, \wedge, \vee)$ of \mathcal{L} if it satisfies the dual conditions
 - (a) $[a \in L \ \& \ a \geq b \in D] \implies a \in D$
 - (b) $[a \in D \ \& \ b \in D] \implies a \wedge b \in D$
 8. Let \mathcal{L} be an upper semi-lattice. The definitions of ideal and filter are the same except that we require (2) only when $a \wedge b$ exists. Furthermore, we say \mathcal{D} is a **strong filter** in \mathcal{L} if \mathcal{D} satisfies (1) and also:
 - (a) $[a \in \mathcal{D} \ \& \ b \in \mathcal{D}] \Leftrightarrow (\exists c \in \mathcal{D})[c \leq a \ \& \ c \leq b]$

The collection of all subsets of ω forms a Boolean algebra, $\mathcal{N} = (2^\omega; \subseteq, \cup, \cap)$ with \emptyset as least element and ω as the greatest element. The finite sets form an ideal \mathcal{F} of \mathcal{N} and the cofinite sets form a filter \mathcal{C} in \mathcal{N}

Definition 2.15. 1. By Theorem ?? the r.e. sets form a distributive lattice \mathcal{E} under inclusion with greatest element ω and least element \emptyset
 2. By Theorem ?? an r.e. set $A \in \mathcal{E}$ is recursive iff $\bar{A} \in \mathcal{E}$. Hence the recursive sets form a Boolean algebra $\mathcal{R} \subseteq \mathcal{E}$.

2.1.1 exercise

Exercise 2.1.1. 1. Prove that $A \leq_m B$ and B r.e. imply A r.e.
 2. Show that Fin and Tot are not r.e.
 3. Show that Cof is not r.e.

Proof. 1. Let $f : A \rightarrow B$, then $A = \{a : (\exists b)((a, b) \in \text{graph } f)\}$?

□

Exercise 2.1.2. Prove that if A is r.e. and ψ is p.r. then $\psi(A)$ is r.e. and $\psi^{-1}(A)$ is r.e.

Proof. Let $\psi = \varphi_e$ and $\psi(A) = \{y : (\exists s, x)\varphi_{e,s}(x) = y\}$

□

Exercise 2.1.3. Prove that if f is recursive, then $\text{graph } f$ is recursive

Exercise 2.1.4. A function f is **increasing** if $f(x) < f(x + 1)$ for all x . Show that an infinite set A is recursive iff A is the range of an increasing recursive function

Proof.

$$\chi_A(x) = \begin{cases} 1 & (\exists y < x) f(y) = x \\ 0 & \end{cases}$$

□

Exercise 2.1.5. Prove that any infinite r.e. set is the range of a 1:1 recursive function

Exercise 2.1.6. Prove that every infinite r.e. set contains an infinite recursive subset

Exercise 2.1.7. A set A is **co-r.e.** (or equivalently Π_1) if \bar{A} is r.e. Use Exercise ?? to prove that the reduction principle fails for Π_1 sets

Exercise 2.1.8. The **separation principle** holds for a class \mathcal{C} of sets if for every $A, B \in \mathcal{C}$ s.t. $A \cap B = \emptyset$ there exists C s.t. $C, \bar{C} \in \mathcal{C}$, $A \subseteq C$ and $B \subseteq \bar{C}$. By Exercise ?? the separation fails for r.e. sets. Use Corollary ?? to show that the separation principle holds for co-r.e. sets

Exercise 2.1.9. Prove that if $A \leq_1 B$ and A and B are r.e. and A is infinite then $A \leq_1 B$ via some f s.t. $f(A) = B$

Exercise 2.1.10. Show that the proof of Theorem ?? is uniform in e in the sense that there is a p.r. function $\psi(e, y)$ s.t. if $W_e \neq \emptyset$ then $\lambda y \psi(e, y)$ is total and $W_e = \{\psi(e, y) : y \in \omega\}$.

2.2 Uniformity and Indices for Recursive and Finite Sets

A theorem will be said to hold **uniformly** if such an effective procedure exists.

Definition 2.16. 1. We say that e is Σ_1 -**index** (r.e. index) for a set A if $A = W_e = \{x : (\exists y)T(e, x, y)\}$
 2. $\langle e, i \rangle$ is a Δ_1 -**index** for a recursive set A if $A = W_e$ and $\bar{A} = W_i$
 3. e is a Δ_0 -**index** (**characteristic index**) for A if φ_e is the characteristic function for A

Theorem 2.17. *There is no p.r. function ψ s.t. if $W_x = A$ and A is recursive then $\psi(x)$ converges and $W_{\psi(x)} = \bar{A}$. (There is no uniformly effective way to pass from Σ_1 -indices to Δ_0 -indices for recursive sets)*

Proof. Define the recursive function f by

$$W_{f(x)} = \begin{cases} \omega & x \in K \\ \emptyset & x \notin K \end{cases}$$

Now

$$\begin{aligned} x \in K &\implies W_{f(x)} = \omega \implies W_{\psi f(x)} = \emptyset \\ x \notin K &\implies W_{f(x)} = \emptyset \implies W_{\psi f(x)} = \omega \end{aligned}$$

Hence

$$x \in \bar{K} \iff W_{\psi f(x)} \neq \emptyset \iff (\exists y, s)[y \in W_{\psi f(x), s}]$$

so \bar{K} is Σ_1 and hence r.e., contradicting Corollary ?? □

Corollary 2.18. *The recursive sets are closed under \cup, \cap and complementation. The closure under \cup and \cap is uniformly effective w.r.t. both Σ_1 and Δ_1 -indices. The closure under complementation is uniformly effective w.r.t. Δ_1 -indices*

A finite set, being recursive, has both a Σ_1 -index and Δ_0 -index.

- Definition 2.19.** 1. Given a finite set $A = \{x_1, \dots, x_k\}$, where $x_1 < x_2 < \dots < x_k$, the number $y = 2^{x_1} + \dots + 2^{x_k}$ is the **canonical index** of A . Let D_y denote finite set with canonical index y and D_0 denote \emptyset .
2. A sequence $\{D_{f(x)}\}_{x \in \omega}$ for some recursive function f is called a **recursive sequence** or a **strong array** of finite sets.

There is no p.r. function ψ s.t. if φ_x is the characteristic function of D_y , then $\psi(x)$ converges and $\psi(x) = |D_y|$. (If ψ exists, define $\varphi_{f(x)}(s) = 1$ if $x \in K_{s+1} - K_s$ and $\varphi_{f(x)}(s) = 0$ otherwise. Thus $\psi \circ f$ is actually the characteristic function of K)

- Definition 2.20.** 1. A sequence $\{V_n\}_{n \in \omega}$ of r.e. sets is **uniformly r.e. (u.r.e.)**, also called **simultaneously r.e. (s.r.e.)** if there is a recursive function f s.t. $V_n = W_{f(n)}$ for all n .
2. A sequence $\{V_n\}_{n \in \omega}$ of recursive sets is **uniformly recursive** if there is a recursive function $g(x, n)$ s.t. $\lambda x [g(x, n)]$ is the characteristic function of V_n for all n .

From now on we assume that we have define $\varphi_{e,s}$ and $W_{e,s}$ using Exercise ??

Definition 2.21. A **recursive enumeration** (usually called simply an **enumeration**) of an r.e. set A consists of a strong array $\{A_s\}_{s \in \omega}$ (of finite sets) s.t. $A_s \subseteq A_{s+1}$ and $A = \bigcup_s A_s$.

For example, $\{W_{e,s}\}_{s \in \omega}$ is an enumeration of W_e .

- Definition 2.22.** 1. A **simultaneous (recursive) enumeration** of a u.r.e. sequence $\{V_n\}_{n \in \omega}$ of r.e. sets is a strong array $\{V_{n,s}\}_{n,s \in \omega}$ s.t. for all $s, n \in \omega$
- (a) $V_{n,s} \subseteq V_{n,s+1}$
 - (b) $|V_{n,s+1} - V_{n,s}| \leq 1$
 - (c) $V_n = \bigcup_{s \in \omega} V_{n,s}$
2. A **standard enumeration** (of the r.e. sets) is a simultaneous enumeration of $\{V_n\}_{n \in \omega}$ where $\{V_n\}_{n \in \omega}$ is some acceptable numbering of the r.e. sets as defined in Exercise ??

For example, an easy way to give a simultaneous enumeration of any u.r.e. sequence $\{V_n\}_{n \in \omega}$ is to choose a 1:1 recursive function f with range $\{\langle x, n \rangle : x \in V_n\}$ and to define

$$V_{n,s} = \{x : (\exists t < s)[f(t) = \langle x, n \rangle]\}$$

Definition 2.23. Let $\{X_s\}_{s \in \omega}$ and $\{Y_s\}_{s \in \omega}$ be recursive enumeration of r.e. sets X and Y

1. Define $X \setminus Y = \{z : (\exists s)[z \in X_s - Y_s]\}$, the elements enumerated in X before being enumerated in Y
2. Define $X \searrow Y = (X \setminus Y) \cap Y$, the elements enumerated in X and later in Y

2.2.1 Exercises

- Exercise 2.2.1.* 1. Given recursive enumeration $\{X_s\}_{s \in \omega}$ and $\{Y_s\}_{s \in \omega}$ of r.e. sets X and Y prove that both $X \setminus Y$ and $X \searrow Y$ are r.e. sets
2. Prove that $X \setminus Y = (X - Y) \cup (X \searrow Y)$
 3. Prove that if $X - Y$ is nonrecursive then $X \searrow Y$ is infinite
 4. Give an alternative proof of Corollary ?? by letting $A_1 = W_x \setminus W_y$ and $B_1 = W_y \setminus W_x$ where $W_x = A$ and $W_y = B$
 5. Let f be a 1:1 recursive function from ω onto K_0 . Define

$$W_{e,s} = \{x : (\exists t \leq s)[f(t) = \langle x, e \rangle]\}$$

Show that $\{W_{e,s} : e, s \in \omega\}$ satisfies condition

$$(\forall s)(\exists \text{ at most one } \langle e, x \rangle)[x \in W_{e,s+1} - W_{e,s}]$$

- Proof.* 1. Prove (x, z) is recursive
- 3.

□