Model Theory: An Introduction

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Contents

1 Structures and Theories

1.1 Languages and Structures

Definition 1.1. A language $\mathcal{L}$ is given by specifying the following data

1. A set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$
2. a set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$
3. a set of constant symbols $\mathcal{C}$

Definition 1.2. An $\mathcal{L}$-structure $\mathcal{M}$ is given by the following data

1. a nonempty set M called the **universe**, **domain** or **underlying set** of $\mathcal{M}$
2. a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$
3. a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$
4. an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$

We refer to $f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$ as the **interpretations** of the symbols f, R and c. We often write the structure as $\mathcal{M} = (M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C})$

Definition 1.3. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures with universes M and N respectively. An $\mathcal{L}$-**embedding** $\eta : \mathcal{M} \to \mathcal{N}$ is a one-to-one map $\eta : M \to N$ that

1. $\eta(f^{\mathcal{M}}(a_1, \dots, a_{n_f})) = f^{\mathcal{N}}(\eta(a_1), \dots, \eta(a_{n_f}))$ for all $f \in \mathcal{F}$ and $a_1, \dots, a_{n_f} \in M$
2. $(a_1, \dots, a_{m_R}) \in R^{\mathcal{M}}$ if and only if $(\eta(a_1), \dots, \eta(a_{m_R})) \in R^{\mathcal{N}}$ for all $R \in \mathcal{R}$ and $a_1, \dots, a_{m_R} \in M$
3. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for $c \in \mathcal{C}$

A bijective $\mathcal{L}$-embedding is called an $\mathcal{L}$-**isomorphism**. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-embedding, we say either $\mathcal{M}$ is a **substrcture** of $\mathcal{N}$ or that $\mathcal{N}$ is an **extension** of $\mathcal{M}$

The **cardinality** of $\mathcal{M}$ is $|M|$

Definition 1.4. The set of $\mathcal{L}$-**terms** is the smallest set $\mathcal{T}$ s.t.

1. $c \in \mathcal{T}$ for each constant symbol $c \in \mathcal{C}$
2. each variable symbol $v_i \in \mathcal{T}$ for $i = 1, 2, \dots$
3. if $t_1, \dots, t_{n_f} \in \mathcal{T}$ and $f \in \mathcal{F}$ then $f(t_1, \dots, n_{n_f}) \in \mathcal{T}$

Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and that t is a term built using variables from $\bar{v} = (v_{i_1}, \dots, v_{i_m})$. We want to interpret t as a function $t^{\mathcal{M}} : M^m \to M$. For s a subterm of t and $\bar{a} = (a_{i_1}, \dots, a_{i_m}) \in M$, we inductively define $s^{\mathcal{M}}(\bar{a})$ as follows.

1. If s is a constant symbol c, then $s^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$
2. If s is the variable v_{i_j}, then $s^{\mathcal{M}}(\bar{a}) = a_{i_j}$
3. If s is the term $f(t_1, \ldots, t_{n_f})$, where f is a function symbol of $\mathcal{L}$ and $t_1, \ldots, t_{n_f}$ are terms, then $s^{\mathcal{M}}(\bar{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_f}^{\mathcal{M}}(\bar{a}))$

The function $t^{\mathcal{M}}$ is defined by $\bar{a} \mapsto t^{\mathcal{M}}(\bar{a})$

Definition 1.5. ϕ is an **atomic $\mathcal{L}$-formula** if ϕ is either

1. $t_1 = t_2$ where t_1 and t_2 are terms
2. $R(t_1, \ldots, t_{n_R})$

The set of $\mathcal{L}$-**formulas** is the smallest set $\mathcal{W}$ containing the atomic formulas s.t.

1. if $\phi \in \mathcal{W}$, then $\neg\phi \in \mathcal{W}$
2. if $\phi, \psi \in \mathcal{W}$, then $(\phi \wedge \psi), (\phi \vee \psi) \in \mathcal{W}$
3. if $\phi \in \mathcal{W}$, then $\exists v_i \phi, \forall v_i \phi \in \mathcal{W}$

We say a variable v **occurs freely** in a formula ϕ if it is not inside a $\exists v$ or $\forall v$ quantifier; otherwise we say that it's **bound**. We call a formula a **sentence** if it has no free variables. We often write $\phi(v_1, \ldots, v_n)$ to make explicit the free variables in ϕ

Definition 1.6. Let ϕ be a formula with free variables from $\bar{v} = (v_{i_1, \ldots, v_{i_m}})$ and let $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M^m$. We inductively define $\mathcal{M} \models \phi\bar{a}$ as follows

1. If ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$
2. If ϕ is $R(t_1, \ldots, t_{m_R})$ then $\mathcal{M} \models \phi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{m_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$
3. If ϕ is $\neg\psi$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \not\models \psi(\bar{a})$
4. If ϕ is $(\psi \wedge \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \theta(\bar{a})$
5. If ϕ is $(\psi \vee \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ or $\mathcal{M} \models \theta(\bar{a})$
6. If ϕ is $\exists v_j \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if there is $b \in M$ s.t. $\mathcal{M} \models \psi(\bar{a}, b)$
7. If ϕ is $\forall v_j \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a}, b)$ for all $b \in M$

If $\mathcal{M} \models \phi(\bar{a})$ we say that $\mathcal{M}$ **satisfies** $\phi(\bar{a})$ or $\phi(\bar{a})$ is **true** in $\mathcal{M}$

Proposition 1.7. *Suppose that $\mathcal{M}$ is a substrcture of $\mathcal{N}$, $\bar{a} \in M$ and $\phi(\bar{v})$ is a quantifier-free formula. Then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \psi(\bar{a})$*

Proof. **Claim** If $t(\bar{v})$ is a term and $\bar{b} \in M$ then $t^{\mathcal{M}}(\bar{b}) = t^{\mathcal{N}}(\bar{b})$. $\square$

Definition 1.8. We say that two $\mathcal{L}$-strctures $\mathcal{M}$ and $\mathcal{N}$ are {elementarily equivalent} and write $\mathcal{M} \equiv \mathcal{N}$ if

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{N} \models \phi$$

for all $\mathcal{L}$-sentences ϕ

We let Th($\mathcal{M}$), the **full theory** of $\mathcal{M}$ be the set of $\mathcal{L}$-sentences ϕ s.t. $\mathcal{M} \models \phi$

Theorem 1.9. *Suppose that $j : \mathcal{M} \to \mathcal{N}$ is an isomorphism. Then $\mathcal{M} \equiv \mathcal{N}$*

Proof. Show by induction on formulas that $\mathcal{M} \models \phi(a_1, \ldots, a_n)$ if and only if $\mathcal{N} \models \phi(j(a_1), \ldots, j(a_n))$ for all formulas ϕ $\square$

1.2 Theories

Let $\mathcal{L}$ be a language. An $\mathcal{L}$-**theory** T is a set of $\mathcal{L}$-sentences. We say that $\mathcal{M}$ is a **model** of T and write $\mathcal{M} \models T$ if $\mathcal{M} \models \phi$ for all sentences $\phi \in T$. A theory is **satisfiable** if it has a model.

A class of $\mathcal{L}$-structures $\mathcal{K}$ is an **elementary class** if there is an $\mathcal{L}$-theory T s.t. $\mathcal{K} = \{\mathcal{M} : \mathcal{M} \models T\}$

Example 1.10 (Groups)**.** Let $\mathcal{L} = \{\cdot, e\}$ where $\cdot$ is a binary function symbol and e is a constant symbol. The class of groups is axiomatized by

$$\begin{aligned}
&\forall x\ e \cdot x = x \cdot e = x \\
&\forall x \forall y \forall z\ x \cdot (y \cdot z) = (x \cdot y) \cdot z \\
&\forall x \exists y\ x \cdot y = y \cdot x = e
\end{aligned}$$

Example 1.11 (Rings and Fields)**.** Let $\mathcal{L}_r$ be the language of rings $\{+, -, \cdot, 0, 1\}$, where $+, -$ and $\cdot$ are binary function symbols and 0 and 1 are constants. The axioms for rings are given by

$$\begin{aligned}
&\forall x \forall y \forall z\ (x - y = z \leftrightarrow x = y + z) \\
&\forall x\ x \cdot 0 = 0 \\
&\forall x \forall y \forall z\ x \cdot (y \cdot z) = (x \cdot y) \cdot z \\
&\forall x\ x \cdot 1 = 1 \cdot x = x \\
&\forall x \forall y \forall z\ x \cdot (y + z) = (x \cdot y) + (x \cdot z) \\
&\forall x \forall y \forall z\ (x + y) \cdot z = (x \cdot z) + (y \cdot z)
\end{aligned}$$

We axiomatize the class of fields by adding

$$\begin{aligned}
&\forall x \forall y\ x \cdot y = y \cdot x \\
&\forall x\ (x \neq 0 \to \exists y\ x \cdot y = 1)
\end{aligned}$$

We axiomatize the class of algebraically closed fields by adding to the field axioms the sentences

$$\forall a_0 \ldots \forall a_{n-1} \exists x\ x^n + \sum_{i=1}^{n-1} a_i x^i = 0$$

for $n = 1, 2, \ldots$. Let ACF be the axioms for algebraically closed fields.

Let ψ_p be the $\mathcal{L}_r$-sentence $\forall x \; \underbrace{x + \cdots + x}_{p\text{-times}} = 0$, which asserts that a field has characteristic p. For $p > 0$ a prime, let $\mathrm{ACF}_p = \mathrm{ACF} \cup \{\psi_p\}$ and $\mathrm{ACF}_0 = \mathrm{ACF} \cup \{\neg\psi_p : p > 0\}$ be the theories of algebraically closed fields of characteristic p and zero respectively

Definition 1.12. Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence. We say that ϕ is a **logical consequence** of T and write $T \models \phi$ if $\mathcal{M} \models \phi$ whenever $\mathcal{M} \models T$

Proposition 1.13.
1. *Let $\mathcal{L} = \{+, <, 0\}$ and let T be the theory of ordered abelian groups. Then $\forall x(x \neq 0 \to x + x \neq 0)$ is a logical consequence of T*
2. *Let T be the theory of groups where every element has order 2. Then $T \not\models \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_1 \neq x_3)$*

Proof. 1. $\mathbb{Z}/2\mathbb{Z} \models T \wedge \neg\exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_1 \neq x_3)$

□

1.3 Definable Sets and Interpretability

Definition 1.14. Let $\mathcal{M} = (M, \ldots)$ be an $\mathcal{L}$-structure. We say that $X \subseteq M^n$ is **definable** if and only if there is an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ and $\bar{b} \in M^b$ s.t. $X = \{\bar{a} \in M^n : \mathcal{M} \models \phi(\bar{a}, \bar{b})\}$. We say that $\phi(\bar{v}, \bar{b})$ **defines** X. We say that X is A-**definable** or **definable over** A if there is a formula $\psi(\bar{v}, w_1, \ldots, w_l)$ and $\bar{b} \in A^l$ s.t. $\psi(\bar{v}, \bar{b})$ defines X

A number of examples using $\mathcal{L}_r$, the language of rings

- Let $\mathcal{M} = (R, +, -, \cdot, 0, 1)$ be a ring. Let $p(X) \in R[X]$. Then $Y = \{x \in R : p(x) = 0\}$ is definable. Suppose that $p(X) = \sum_{i=0}^{m} a_i X^i$. Let $\phi(v, w_0, \ldots, w_n)$ be the formula

$$w_n \cdot \underbrace{v \cdots v}_{n\text{-times}} + \cdots + w_1 \cdot v + w_0 = 0$$

  Then $\phi(v, a_0, \ldots, a_n)$ defines Y. Indeed, Y is A-definable for any $A \supseteq \{a_0, \ldots, a_n\}$
- Let $\mathcal{M} = (\mathbb{R}, +, -, \cdot, 0, 1)$ be the field of real numbers. Let $\phi(x, y)$ be the formula

$$\exists z(z \neq 0 \wedge y = x + z^2)$$

  Because $a < b$ if and only if $\mathcal{M} \models \phi(a, b)$, the ordering is $\emptyset$-definable

- Consider the natural numbers $\mathbb{N}$ as an $\mathcal{L} = \{+, \cdot, 0, 1\}$ structure. There is an $\mathcal{L}$-formula $T(e, x, s)$ s.t. $\mathbb{N} \models T(e, x, s)$ if and only if the Turing machine with program coded by e halts on input x in at most s steops. Thus the Turing machine with program e halts on input x if and only if

$\mathbb{N} \models \exists s\, T(e, x, s)$. So the halting computations is definable

Proposition 1.15. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Suppose that D_n is a collection of subsets of M^n for all $n \geq 1$ and $\mathcal{D} = (D_n : n \geq 1)$ is the smallest collection s.t.*

1. *$M^n \in D_n$*
2. *for all n-ary function symbols f of $\mathcal{L}$, the graph of $f^{\mathcal{M}}$ is in D_{n+1}*
3. *for all n-ary relation symbols R of $\mathcal{L}$, $R^{\mathcal{M}} \in D_n$*
4. *for all $i, j \leq n$, $\{(x_1, \ldots, x_n) \in M^n : x_i = x_j\} \in D_n$*
5. *if $X \in D_n$, then $M \times X \in D_{n+1}$*
6. *each D_n is cloed under complement, union and intersection*
7. *if $X \in D_{n+1}$ and $\pi : M^{n+1} \to M^n$ is the projection $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)$, then $\pi(X) \in D_n$*
8. *if $X \in D_{n+m}$ and $b \in M^m$, then $\{a \in M^n : (a, b) \in X\} \in D_n$*

Thus $X \subseteq M^n$ is definable if and only if $X \in D_n$

Proposition 1.16. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. If $X \subset M^n$ is A-definable, then every $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes A pointwise fixes X setwise(that is, if σ is an automorphism of M and $\sigma(a) = a$ for all $a \in A$, then $\sigma(X) = X$)*

Proof.

$$\mathcal{M} \models \psi(\bar{v}, \bar{a}) \leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{v}), \sigma(\bar{a})) \leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{v}), \bar{a})$$

In other words, $\bar{b} \in X$ if and only if $\sigma(\bar{b}) \in X$ □

Definition 1.17. A subset S of a field L is **algebraically independent** over a subfield K if the elements of S do not satisfy any non-trivial polynomial equation with coefficients in K

Corollary 1.18. *The set of real numbers is not definable in the field of complex numbers*

Proof. If $\mathbb{R}$ where definable, then it would be definable over a finite $A \subset \mathbb{C}$. Let $r, s \in \mathbb{C}$ be algebraically independent over A with $r \in \mathbb{R}$ and $s \notin \mathbb{R}$. There is an automorphism σ of $\mathbb{C}$ s.t. $\sigma | A$ is the identity and $\sigma(r) = s$. Thus $\sigma(\mathbb{R}) \neq \mathbb{R}$ and $\mathbb{R}$ is not definable over A □

We say that an $\mathcal{L}_0$-structure $\mathcal{N}$ is **definably interpreted** in an $\mathcal{L}$-structure $\mathcal{M}$ if and only if we can find a definable $X \subseteq M^n$ for some n and we can interpret the symbols of $\mathcal{L}_0$ as definable subsets and functions on X so that the resulting $\mathcal{L}_0$-structure is isomorphic to $\mathcal{M}$

For example, let K be a field and G be $\mathrm{GL}_2(K)$, the group of invertible 2×2 matrices over K. Let $X = \{(a, b, c, d) \in K^4 : ad - bc \neq 0\}$. Let $f : X^2 \to X$ by

$$f((a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2)) = \\ (a_1a_2 + b_1c_2, a_1b_2 + b_1d_2, c_1a_2 + d_1c_2, c_1b_2 + d_1d_2)$$

X and f are definable in $(K, +, \cdot)$, and the set X with operation f is isomorphic to $\mathrm{GL}_2(K)$, where the identity element of X is $(1, 0, 0, 1)$

Clearly, $(\mathrm{GL}_n(K), \cdot, e)$ is definably interpreted in $(K, +, \cdot, 0, 1)$. A **linear algebraic group** over K is a subgroup of $\mathrm{GL}_n(K)$ defined by polynomial equations over K. Any linear algebraic group over K is definably interpreted in K

Let F be an infinite field and let G be the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a, b \in F, a \neq 0$. This group is isomorphic to the group of affine transformations $x \mapsto ax + b$, where $a, b \in F$ and $a \neq 0$

We will show that F is definably interpreted in the group G. Let

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}$$

where $\tau \neq 0$. Let

$$A = \{g \in G : g\alpha = \alpha g\} = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F\} B = \{g \in G : g\beta = \beta g\} = \{\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} : x \neq 0\}$$

Clearly A, B are definable using parameters α and β

B acts on A by conjugation

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{y}{x} \\ 0 & 1 \end{pmatrix}$$

We can define the map $i : A\backslash\{1\} \to B$ by $i(a) = b$ if and only if $b^{-1}ab = \alpha$, that is

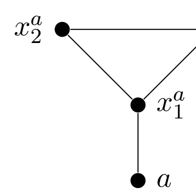
$$i\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

Define an operation $*$ on A by

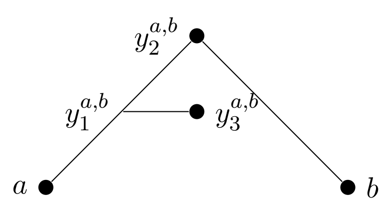
$$a * b = \begin{cases} i(b)a(i(b))^{-1} & \text{if } b \neq I \\ 1 & \text{if } b = I \end{cases}$$

where I is the identity matrix. Now $(F, +, \cdot, 0, 1) \cong (A, \cdot, *, 1, \alpha)$

Very complicated structures can often be interpreted in seemingly simpler ones. For example, any structure in a countable language can be interpreted in a graph. Let $(A, <)$ be a linear order. For each $a \in A$, G_A will have vertices a, x_1^a, x_2^a, x_3^a and contain the subgraph

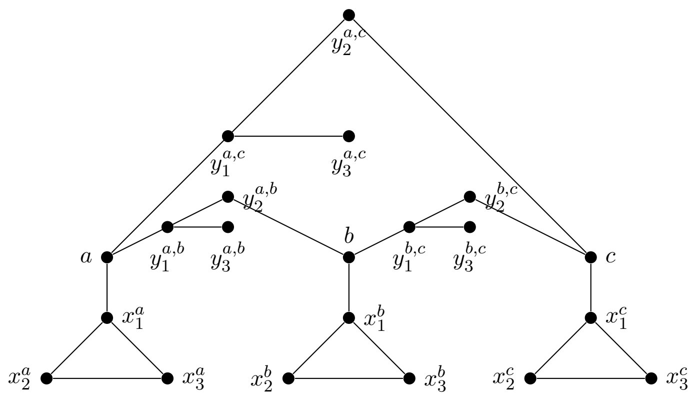


If $a < b$, then G_A will have vertices $y_1^{a,b}, y_2^{a,b}, y_3^{a,b}$ and contain the subgraph



Let $V_A = A \cup \{x_1^a, x_2^a, x_3^a : a \in A\} \cup \{y_1^{a,b}, y_2^{a,b}, y_3^{a,b} : a, b \in A \text{ and } a < b\}$, and let R_A be the smallest symmetric relation containing all edges drawn above.

For example, if A is the three-element linear order $a < b < c$, then G_A is the graph



Let $\mathcal{L} = \{R\}$ where R is a binary relation. Let $\phi(x, u, v, w)$ be the formula asserting that x, u, v, w are distinct, there are edges $(x, u), (u, v), (v, w), (u, w)$ and these are the only edges involving u, v, w. $G_A \models \phi(a, x_1^a, x_2^a, x_3^a)$ for all $a \in A$.

$\psi(x, y, u, v, w)$ asserts that x, y, u, v, w are distinct. $(x, u), (u, v), (u, w), (v, y)$

Define $\theta_i(z)$ as follows:

$$
\begin{aligned}
\theta_0(z) &:= \exists u \exists v \exists w \ \phi(z, u, v, w) \\
\theta_1(z) &:= \exists x \exists v \exists w \ \phi(x, z, v, w) \\
\theta_2(z) &:= \exists u \exists u \exists w \ \phi(x, u, z, w) \\
\theta_3(z) &:= \exists x \exists y \exists v \exists w \ \psi(x, y, z, v, w) \\
\theta_4(z) &:= \exists x \exists y \exists u \exists w \ \psi(x, y, u, z, w) \\
\theta_5(z) &:= \exists x \exists y \exists u \exists v \ \psi(x, y, u, v, z)
\end{aligned}
$$

If $a, b \in A$ and $a < b$, then

$$G_A \models \theta_0(a) \wedge \theta_1(x_1^a) \wedge \theta_2(x_2^a) \wedge \theta_2(x_3^a)$$

and

$$G_A \models \theta_3(y_1^{a,b}) \wedge \theta_4(y_2^{a,b}) \wedge \theta_5(y_3^{a,b})$$

Lemma 1.19. *If $(A, <)$ is a linear order, then for all vertices x in G, there is a unique $i \leq 5$ s.t. $G_A \models \theta_i(x)$*

Let T be the $\mathcal{L}$-theory with the following axioms

1. R is symmetric and irreflexive
2. for all x, exactly one θ_i holds

3. if $\theta_0(x)$ and $\theta_0(y)$ then $\neg R(x,y)$
4. if $\exists u \exists v \exists w \ \psi(x,y,u,v,w)$
   then $\forall u_1 \forall v_1 \forall w_1 \neg\psi(y,x,u_1,v_1,w_1)$
5. if $\exists u \exists v \exists w \ \psi(x,y,u,v,w)$ and $\exists u \exists v \exists w \ \psi(y,z,u,v,w)$ then
   $\exists u \exists v \exists w \ \psi(x,z,u,v,w)$
6. if $\theta_0(x)$ and $\theta_0(y)$, then either $x = y$ or $\exists u \exists v \exists w \ \psi(x,y,u,v,w)$ or
   $\exists u \exists v \exists w \ \psi(y,x,u,v,w)$
7. if $\phi(x,u,v,w) \wedge \phi(x,u',v',w')$, then $u = u', v = v', w = w'$
8. if $\psi(x,y,u,v,w) \wedge \psi(x,y,u',v',w')$, then $u' = u, v = v', w = w'$

If $(A,<)$ is a linear order, then $G_A \models T$

Suppose $G \models T$. Let $X_G = \{x \in G : G \models \theta_0(x)\}$

Lemma 1.20. *If $(A,<)$ is a linear order, then $(X_{G_A}, <_{G_A}) \cong (A,<)$. Moreover, $G_{X_G} \cong G$ for all $G \models T$*

Definition 1.21. An $\mathcal{L}_0$-structure $\mathcal{N}$ is **interpretable** in an $\mathcal{L}$-structure M if there is a definable $X \subseteq M^n$, a definable equivalence relation E on X, and for each symbol of $\mathcal{L}_0$ we can find definable E-invariant sets on X s.t. X/E with the induced structure is isomorphic to $\mathcal{N}$

1.4 Answers to Exercises

Exercise 1.4.1.

1. transform ψ to CNF
2. prenex normal form

Exercise 1.4.2.

1. $\begin{array}{cc} s & rs \\ \bullet & \bullet \\ e & r \\ \bullet & \bullet \end{array}$
2. enumerate $\mathcal{M}$'s functions, relations and constants

Exercise 1.4.3. [1] Note that every $\mathcal{L}$-structure $\mathcal{M}$ of size κ is isomorphic to an $\mathcal{L}$-structure with domain κ. For each relation symbols, we have 2^κ options. If the language has size λ, this is at most $(2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^{\max(\lambda,\kappa)}$

Exercise 1.4.4.

$$\begin{aligned} T \models \phi &\Leftrightarrow \forall \mathcal{M}\ \mathcal{M} \models T \to \mathcal{M} \models \phi \\ &\Leftrightarrow \forall \mathcal{M}\ \mathcal{M} \models T' \to \mathcal{M} \models \phi \\ &\Leftrightarrow T' \models \phi \end{aligned}$$

Exercise 1.4.5. Follow the definition

[1] from StackExchange

Exercise 1.4.6. Since there is no model $\mathcal{M}$ s.t. $\mathcal{M} \models T$. It's true that $T \models \phi$

Exercise 1.4.7.
1. Suppose $\mathcal{M} \models \phi$, then $E^{\mathcal{M}}$ is an equivalent relation and each equivalence class's cardinality is 2
2. follows from number theory
3. [DJMM12]

Exercise 1.4.8. TBD

Exercise 1.4.9.
1. $G(f) = \{(\bar{x}, \bar{y}) \in M^{n+m} \mid \phi(\bar{x}, \bar{y})\}$ and $G(g) = \{(\bar{y}, \bar{z}) \in M^{m+l} \mid \psi(\bar{y}, \bar{z})\}$. Hence $G(g \circ f) = \{(\bar{x}, \bar{z}) \in M^{n+l} \mid \phi(\bar{x}, \bar{y}) \wedge \psi(\bar{y}, \bar{z})\}$
2.

References

[DJMM12] Arnaud Durand, Neil D. Jones, Johann A. Makowsky, and Malika More. Fifty years of the spectrum problem: survey and new results. *Bulletin of Symbolic Logic*, 18(4):505–553, 2012.