Rough Set Theory: A True Landmark in Data Analysis

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Contents

1 Rough Sets on Fuzzy Approximation Spaces and Intuitionistic Fuzzy Approximation Spaces

1.1 Introduction

1.1.1 Fuzzy Sets

1.1.2 Intuitionistic Fuzzy Sets

the membership and nonmembership values of an element with respect to a collection of elements from a universe may not add up to 1 in all possible cases

Definition 1.1. An *intuitionistic fuzzy set* A on a universe U is defined by two functions: *membership function* μ_A and *non-membership function* ν s.t.

$$\mu_A, \nu_A : U \to [0, 1]$$

where $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in U$.

The *hesitation function* Π_A for an intuitionistic fuzzy set is given by

$$\Pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$$

1.1.3 Rough set

A knowledge base is also called an **approximation space**

1.1.4 Motivation

1.1.5 Fuzzy proximity relation

Definition 1.2. Let U be a universal set and $X \subseteq U$. Then a *fuzzy relation* on X is defined as any fuzzy set defined on $X \times X$

Definition 1.3. A fuzzy relation R is said to be *fuzzy reflexive* on $X \subseteq U$ if it satisfies

$$\mu_R(x, x) = 1 \quad \text{for all } x$$

Definition 1.4. A fuzzy relation R is said to be *fuzzy symmetric* on $X \subseteq U$ if it satisfies

$$\mu_R(x, y) = \mu_R(y, x) \quad \text{for all } x, y \in X$$

Definition 1.5. A fuzzy relation on $X \subseteq U$ is said to be a *fuzzy proximity relation* if it is fuzzy reflexive and fuzzy symmetric.

Definition 1.6. Let $X, Y \subseteq U$. A fuzzy relation from X to Y is a fuzzy set defined on $X \times Y$ characterized by the membership function $\mu_R : X \times Y \to [0, 1]$

Definition 1.7. For any $\alpha \in [0, 1]$, the *α-cut* of R, denoted by R_α is a subset of $X \times Y$ given by $R_\alpha = \big\{(x, y) : \mu_R(x, y) \geq \alpha\big\}$

Let R be a fuzzy proximity relation on U. Then for any $\alpha \in [0, 1]$ the elements of R_α are said to be *α-similar* to each other. $xR_\alpha y$.

Two elements x and y in U are said to be *α-identical* w.r.t. R $(xR(\alpha)y)$ if either x and y are α-similar or x and y are *transitively α-similar*, that is, there exists a sequence of elements $u_1, u_2, \ldots, u_n$ in U s.t. $xR_\alpha u_1, u_1 R_\alpha u_2, \ldots, u_n R_\alpha y$

1.1.6 Intuitionistic fuzzy proximity relation

Definition 1.8. An *intuitionistic fuzzy relation* on a universal set U is an intuitionistic fuzzy set defined on $U \times U$

Definition 1.9. An intuitionistic fuzzy relation R on a universal set U is said to be *intuitionstic fuzzy reflexive* if

$$\mu_R(x, x) = 1 \text{ and } \nu_R(x, x) = 0 \quad \text{for all } x \in X$$

Definition 1.10. An intuitionistic fuzzy relation R on a universal set U is said to be *intuitionistic fuzzy symmetric* if

$$\mu_R(x, y) = \mu_R(y, x) \text{ and } \nu_R(x, y) = \nu_R(y, x) \quad \text{for all } x, y \in X$$

Definition 1.11. *intuitionistic fuzzy proximity*

Define

$$J = \big\{(m, n) \mid m, n \in [0, 1] \text{ and } 0 \leq m + n \leq 1\big\}$$

Definition 1.12. Le R be an IF-proximity relation on U. Then for any $(\alpha, \beta) \in J$ the *(α, β)-cut* of R, denoted by $\mathrm{R}_{\alpha,\beta}$ is

$$R_{\alpha,\beta} = \big\{(x, y) \mid \mu_R(x, y) \geq \alpha \text{ and } \nu_R(x, y) \leq \beta\big\}$$

The relation $R(\alpha, \beta)$ is an equivalence relation.

1.2 Rough Sets on Fuzzy Approximation

1.2.1 Preliminaries

Definition 1.13. For any set of fuzzy proximity relation $K = (U, \mathfrak{R})$ is called a *fuzzy approximation space*

For any fixed $\alpha \in [0, 1]$, $\mathfrak{R}$ generates a set of equivalence relation $\mathfrak{R}(\alpha)$ and we call the associated space $\boldsymbol{K}(\alpha) = (U, \mathfrak{R}(\alpha))$ as the *generated approximation space* corresponding to $\boldsymbol{K}$ and α

1.2.2 Properties

1.2.3 Reduction of Knowledge in Fuzzy Approximation Spaces

Definition 1.14. Let $\mathfrak{R}$ be a family of fuzzy proximity relations on U and $\alpha \in [0, 1]$. For any $R \in \mathfrak{R}$, we say that R is *α-dispensable* or *α-superfluous* in $\mathfrak{R}$ if and only if $IND(\mathfrak{R}(\alpha)) = IND(\mathfrak{R}(\alpha) - R(\alpha))$

Consider $U = \{x_1, \ldots, x_n\}$. Define the fuzzy proximity relations P, Q, R and S over U corresponding to the attributes a, b, c and d respectively.

Table 1: Fuzzy proximity relation for attribute R

P	x_1	x_2	x_3	x_4	x_5
x_1	1	0.3	0.6	0.8	0.5
x_2	0.3	1	0.7	0.4	0.4
x_3	0.6	0.7	1	0.2	0.8
x_4	0.8	0.4	0.2	1	0.5
x_5	0.5	0.4	0.8	0.5	1

Table 2: Fuzzy proximity relation for attribute Q

P	x_1	x_2	x_3	x_4	x_5
x_1	1	0.3	0.4	0.2	0.5
x_2	0.3	1	0.8	0.6	0.6
x_3	0.4	0.8	1	0.3	0.9
x_4	0.2	0.6	0.3	1	0.7
0.5	0.2	0.2	0.9	0.7	1

Table 3: Fuzzy proximity relation for attribute R

R	x_1	x_2	x_3	x_4	x_5
x_1	1	0.3	0.2	0.8	0.7
x_2	0.3	1	0.5	0.3	0.5
x_3	0.2	0.5	1	0.6	0.4
x_4	0.8	0.3	0.6	1	0.9
x_5	0.7	0.5	0.4	0.9	1

Table 4: Fuzzy proximity relation for attribute S

S	x_1	x_2	x_3	x_4	x_5
x_1	1	0.3	0.2	0.2	0.5
x_2	0.3	1	0.5	0.3	0.2
x_3	0.2	0.5	1	0.2	0.4
x_4	0.2	0.3	0.2	1	0.5
x_5	0.5	0.4	0.4	0.5	1

Table 5: Fuzzy proximity relation for $IND(\mathfrak{R}(\alpha))$

$IND(\mathfrak{R}(\alpha))$	x_1	x_2	x_3	x_4	x_5
x_1	1	0.3	0.2	0.2	0.5
x_2	0.3	1	0.3	0.3	0.2
x_3	0.2	0.3	1	0.2	0.4
x_4	0.2	0.3	0.2	1	0.4
x_5	0.5	0.2	0.4	0.4	1

Suppose $\alpha = 0.6$, then we get

$$U/P(\alpha) = \{\{x_1, x_2, x_3, x_4, x_5\}\}$$
$$U/Q(\alpha) = \{\{x_1\}, \{x_2, x_3, x_4, x_5\}\}$$
$$U/R(\alpha) = \{\{x_1, x_3, x_4, x_5\}, \{x_2\}\}$$
$$U/S(\alpha) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\}$$

1.2.4 Relative reducts and relative core of knowledge in fuzzy approximation spaces

Definition 1.15. Let P and Q be two fuzzy proximity relations over the universe U. For every fixed $\alpha \in [0, 1]$, the *α-positive region* of P w.r.t. Q can be defined as

$$\alpha\text{-}POS_P Q = \bigcup_{X_\alpha \in U/Q} \underline{P} X_\alpha$$

Definition 1.16. Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be two families of fuzzy proximity relations on U. For every fixed $\alpha \in [0, 1]$ and $R \in \boldsymbol{P}$,R is $(\boldsymbol{Q}, \alpha)$-*dispensable* in $\boldsymbol{P}$ if

$$\alpha\text{-}POS_{IND}(\boldsymbol{Q}) = \alpha\text{-}POS_{IND(\boldsymbol{P}-\{R\})} IND(\boldsymbol{Q})$$

If every $R \in \boldsymbol{P}$ is $(\boldsymbol{Q}, \alpha)$-indispensable, then $\boldsymbol{P}$ is $(\boldsymbol{Q}, \alpha)$-*indepedent*

Definition 1.17. For every fixed $\alpha \in [0, 1]$, the family $\boldsymbol{S} \subseteq \boldsymbol{P}$ is a $(\boldsymbol{Q}, \alpha)$-reduct of $\boldsymbol{P}$ if and only if

$$\boldsymbol{S} \text{ is } (\boldsymbol{Q}, \alpha)\text{-indepedent}$$
$$\alpha\text{-}POS_{\boldsymbol{S}} \boldsymbol{Q} = \alpha\text{-}POS_{\boldsymbol{P}} \boldsymbol{Q}$$

Consider another attribute T, let's find the relative reduct and the relative core, that is (T, α)-reduct and (T, α)-core of the family of fuzzy proximity relations $\mathcal{R} = \{P, Q, R, S\}$

$$U/T(\alpha) = \{\{x_1, x_4, x_5\}, \{x_2, x_3\}\}$$
$$U/IND\mathcal{R}(\alpha) = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\}$$
$$\alpha\text{-}POS_{\mathcal{R}(\alpha)} T(\alpha) = \bigcup_{X \in U/T(\alpha)} \mathcal{R} X_\alpha = U$$

Table 6: Fuzzy proximity relation for attribute T

T	x_1	x_2	x_3	x_4	x_5
x_1	1	0.4	0.5	0.6	0.8
x_2	0.4	1	0.9	0.3	0.5
x_3	0.5	0.9	1	0.2	0.4
x_4	0.6	0.3	0.2	1	0.5
x_5	0.8	0.5	0.4	0.5	1

1.2.5 Dependency of knowledge in fuzzy approximation spaces

Definition 1.18. We say knowledge $\boldsymbol{Q}$ is α-*derivable* from knowledge $\boldsymbol{P}$ if the elementary α-categories of $\boldsymbol{Q}$ can be defined in terms of some elementary α-categories of knowledge P

Definition 1.19. If $\boldsymbol{Q}$ is α-derivable from $\boldsymbol{P}$, we say that knowledge $\boldsymbol{Q}$ α-*depends* on knowledge $\boldsymbol{P}$ and we denote it by $\boldsymbol{P} \overset{\alpha}{\Rightarrow} \boldsymbol{Q}$. So, $\boldsymbol{P} \overset{\alpha}{\Rightarrow} \boldsymbol{Q}$ if and only if $IND(\boldsymbol{P}(\alpha)) \subseteq IND(\boldsymbol{Q}(\alpha))$

Definition 1.20. Knowledge $\boldsymbol{P}$ and $\boldsymbol{Q}$ are α-*equivalent*, denoted by $\boldsymbol{P} \overset{\alpha}{\equiv} \boldsymbol{Q}$ iff $IND(\boldsymbol{P}(\alpha)) = IND(\boldsymbol{Q}(\alpha))$

1.2.6 partial dependency of knowledge in fuzzy approximation spaces

It may happen that the α-derivation of one knowledge P from another knowledge Q can be partial. That is only a part of knowledge P can be α-derivation from P

Suppose $\mathcal{K}(\alpha) = (U, \mathcal{R}(\alpha))$ is a fuzzy approximation space and $P, Q \subseteq \mathcal{R}$. Then Q *α-depends in a degree* $k(\alpha)$, $0 \leq k(\alpha) \leq 1$ denoted by $P \overset{\alpha}{\Rightarrow}_k Q$ if

$$k(\alpha) = \gamma_{P(\alpha)}(Q(\alpha)) = \frac{\left|POS_{P(\alpha)}(Q(\alpha))\right|}{|U|}$$

$k(\alpha) = 1$ then *totally*, $0 < k(\alpha) < 1$ *partially(roughly)*

1.3 Rough Sets in Intuitionistic Fuzzy Approximation Spaces

1.3.1 knowledge reduction in IF-Approximation spaces

2 Granular structures and approximations in rough sets and knowledge spaces

2.1 Introduction

Granular computing is an emerging field of study focusing on structured thinking, structured problem solving and structured information processing with multiple levels of granularity

A primitive notion of granular computing is that of granules. Granules may be considered as parts of a whole. A granule may be understood as a unit that we use for describing and representing a problem or a focal point of our attention at a specific point of time. Granules can be organized based on their inherent properties and interrelationships. The results are a multilevel granular structure. Each level is populated by granules of the similar size or the similar nature. Depending on a particular context, levels of granularity may be interpreted as levels of abstraction, levels of details, levels of processing, levels of understanding, levels of interpretation, levels of control, and many more. An ordering of levels based on granularity provides a hierarchical granular structure.

2.2 Granular spaces

2.2.1 A Set-theoretic interpretation of granules

Categorization or classification is one of the fundamental tasks of human intelligence.

The process of categorization covers two important issues of granulation, namely, the construction of granules and the naming of granules. Objects in the same categories must be more similar to each other, and objects in different granules are more dissimilar to each other.

2.2.2 A formulation of granules as concept

The classical view of concepts defines a concept jointly by a set of objects, called the extension of the concept, and a set of intrinsic properties common to the set of objects, called the intension of the concept. Typically, the name of a concept reflects the intension of a concept. The extension of a concept is the set of objects which are concrete examples of a concept. One may introduce a logic language so that the intension of a concept is represented by a formula and the extension is represented by the set of objects satisfying the formula.

For an individual $x \in U$, if it satisfies an atomic formula p, we write $x \models p$. An individual satisfies a formula if the individual has the properties as specified by the formula.

If ϕ is a formula, the set $m(\phi)$ defined by

$$m(\phi) = \left\{x \in U \mid x \models \phi\right\}$$

is called the *meaning* of the formula ϕ. In other words, ϕ can be viewed as the description of the set of object $m(\phi)$. As a result, a concept can be expressed by a pair $(\phi, m(\phi))$ where $\phi \in \mathcal{L}$. ϕ is the intension of a concept while $m(\phi)$ is the extension of a concept.

2.2.3 Granular spaces and granular structures

Each atomic formula in $\mathcal{A}$ is associated with a subset of U. This subset may be viewed as an elementary granule in U. Each formula is obtained by taking logic operations on atomic formulas. The meaning set of the formula can be obtained from the elementary granules through set-theoretic operations

A subset or a granule $X \subseteq U$ is *definable* if and only if there exists a formula ϕ in the language $\mathcal{L}$ s.t.

$$X = m(\phi)$$

Family of all definable granules is given by

$$Def(\mathcal{L}(\mathcal{A}, \{\neg, \wedge, \vee\}, U)) = \left\{m(\phi) \mid \phi \in \mathcal{L}(\mathcal{A}, \{\neg, \wedge, \vee\}, U)\right\}$$

which is a subsystem of the power set 2^U closed under set complement, intersection and union. A *granular space*

$$(U, \mathcal{S}_0, \mathcal{S})$$

where U is the universe, $\mathcal{S}_0 \subseteq 2^U$ is a family of elementary granules, i.e., $\mathcal{S}_0 = \left\{m(p) \mid p \in \mathcal{A}\right\}$, $\mathcal{S} \subseteq 2^U$ is a family of definbale granules

$\mathcal{S}$ is an σ-algebra

2.3 Rough Set Analysis

2.3.1 Granular spaces and granular structures

Information table

$$M = (U, At, \{V_a \mid a \in At\}, \{I_a \mid a \in At\})$$

where $I_a : U \to V_a$ is an information function.

For a set of attributes $P \subseteq At$, we can define an equivalence relation on the set of objects

$$xE_p y \iff \forall a \in P(I_a(x) = I_a(y))$$

By taking the union of a family of equivalence classes, we can obtain a composite granule. The family of all such granules contains the entire set U and the empty set $\emptyset$, and is closed under set complement, intersection and union. More specifically, the family is an σ-algebra, denoted by $\sigma(U/E)$, with the basis U/E

For an attribute-value pair (a, v) ,where $a \in At, v \in V_a$, we have an atomic formula $a = v$. The meaning of $a = v$ is

$$m(a = v) = \{x \in U \mid I_a(x) = v\}$$

Hence $[x] = \bigvee_{a \in At} a = I_a(x)$ and is a definable granule.

2.3.2 Rough Set Approximation

$$\underline{apr}(A) = \bigcup \{X \in \sigma(U/E) \mid X \subseteq A\}$$
$$\overline{apr}(A) = \bigcap \{X \in \sigma(U/E) \mid A \subseteq X\}$$

Hence $(\underline{apr}(A), \overline{apr}(A))$ is the tightest approximation.

2.4 Knowledge space theory

In knowledge spaces, we consider a pair $(Q, \mathcal{K})$ where Q is a finite set of questions and $\mathcal{K} \subseteq 2^Q$. Each $K \subseteq \mathcal{K}$ is called a *knowledge state* and \$K\$is the set of all possible knowledge state.

Intuitively, the knowledge state of an individual is represented by the set of questions that he is capable of answering. Each knowledge state can be considered as a granule. The collection of all the knowledge states together with the empty set and the whole set Q is called a knowledge structure, and may be viewed as a granular knowledge structure in the terminology of granular computing

2.4.1 Granular spaces associated to surmise relations

A surmise relation on the set Q of questions is a reflexive and transitive relation S. By aSb, we can surmise the mastery of a if a student can correctly question b. For example, aSb means that if a knowledge state contains b, it must also contain a.

Formally, for a surmise relation S on the finite set Q of questions, the associated knowledge structure $\mathcal{K}$ is defined by

$$\mathcal{K} = \left\{ K \mid \forall q, q' \in Q((qSq', q' \in K) \Rightarrow q \in K) \right\}$$

For each question q in Q, under a surmise relation, we can find a unique prerequistie question set $R_p(q) = \left\{ q' \mid q'Sq \right\}$. The family of the prerequistie question sets for all the questions is denoted by $\mathcal{B}$. By taking the union of prerequisite sets for a family of questions, we can obtain a knowledge structure $\mathcal{K}$ associated to the surmise relation S. It defines a granular space $(Q, \mathcal{B}, \mathcal{K})$. All knowledge states are called granules in $(Q, \mathcal{B}, \mathcal{K})$.

3 On Approximation of classifications, Rough Equalities and Rough Equivalences

4 A generic scheme for generating prediction rules using rough sets

4.1 Rough set prediction model

1. *pre-processing phase*
2. *analysis and rule generating phase*
3. *classification and prediction phase*

4.1.1 Pre-processing phase

1. Data completion