# Model Theory: An Introduction

David Marker

December 18, 2019

## Contents

# 1 Structures and Theories

## 1.1 Languages and Structures

**Definition 1.1.** A language $\mathcal{L}$ is given by specifying the following data

1. A set of function symbols $\mathcal{F}$ and positive integers $n_f$ for each $f \in \mathcal{F}$
2. a set of relation symbols $\mathcal{R}$ and positive integers $n_R$ for each $R \in \mathcal{R}$
3. a set of constant symbols $\mathcal{C}$

**Definition 1.2.** An $\mathcal{L}$-structure $\mathcal{M}$ is given by the following data

1. a nonempty set $M$ called the **universe**, **domain** or **underlying set** of $\mathcal{M}$
2. a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$
3. a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$
4. an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$

We refer to $f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$ as the **interpretations** of the symbols $f, R$ and $c$. We often write the structure as $\mathcal{M} = (M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C})$

**Definition 1.3.** Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures with universes $M$ and $N$ respectively. An $\mathcal{L}$-**embedding** $\eta : \mathcal{M} \to \mathcal{N}$ is a one-to-one map $\eta : M \to N$ that

1. $\eta(f^{\mathcal{M}}(a_1, \ldots, a_{n_f})) = f^{\mathcal{N}}(\eta(a_1), \ldots, \eta(a_{n_f}))$ for all $f \in \mathcal{F}$ and $a_1, \ldots, a_{n_f} \in M$
2. $(a_1, \ldots, a_{m_R}) \in R^{\mathcal{M}}$ if and only if $(\eta(a_1), \ldots, \eta(a_{m_R})) \in R^{\mathcal{N}}$ for all $R \in \mathcal{R}$ and $a_1, \ldots, a_{m_R} \in M$
3. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for $c \in \mathcal{C}$

A bijective $\mathcal{L}$-embedding is called an $\mathcal{L}$-**isomorphism**. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-embedding, we say either $\mathcal{M}$ is a **substrcture** of $\mathcal{N}$ or that $\mathcal{N}$ is an **extension** of $\mathcal{M}$

The **cardinality** of $\mathcal{M}$ is $|M|$

**Definition 1.4.** The set of $\mathcal{L}$-**terms** is the smallest set $\mathcal{T}$ s.t.

1. $c \in \mathcal{T}$ for each constant symbol $c \in \mathcal{C}$
2. each variable symbol $v_i \in \mathcal{T}$ for $i = 1, 2, \ldots$
3. if $t_1, \ldots, t_{n_f} \in \mathcal{T}$ and $f \in \mathcal{F}$ then $f(t_1, \ldots, n_{n_f}) \in \mathcal{T}$

Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and that $t$ is a term built using variables from $\bar{v} = (v_{i_1}, \ldots, v_{i_m})$. We want to interpret $t$ as a function $t^{\mathcal{M}} : M^m \to M$. For $s$ a subterm of $t$ and $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M$, we inductively define $s^{\mathcal{M}}(\bar{a})$ as follows.

1. If $s$ is a constant symbol $c$, then $s^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$
2. If $s$ is the variable $v_{i_j}$, then $s^{\mathcal{M}}(\bar{a}) = a_{i_j}$
3. If $s$ is the term $f(t_1, \ldots, t_{n_f})$, where $f$ is a function symbol of $\mathcal{L}$ and $t_1, \ldots, t_{n_f}$ are terms, then $s^{\mathcal{M}}(\bar{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_f}^{\mathcal{M}}(\bar{a}))$

The function $t^{\mathcal{M}}$ is defined by $\bar{a} \mapsto t^{\mathcal{M}}(\bar{a})$

**Definition 1.5.** $\phi$ is an **atomic $\mathcal{L}$-formula** if $\phi$ is either
1. $t_1 = t_2$ where $t_1$ and $t_2$ are terms
2. $R(t_1, \ldots, t_{n_R})$

The set of $\mathcal{L}$-**formulas** is the smallest set $\mathcal{W}$ containing the atomic formulas s.t.
1. if $\phi \in \mathcal{W}$, then $\neg\phi \in \mathcal{W}$
2. if $\phi, \psi \in \mathcal{W}$, then $(\phi \wedge \psi), (\phi \vee \psi) \in \mathcal{W}$
3. if $\phi \in \mathcal{W}$, then $\exists v_i \phi, \forall v_i \phi \in \mathcal{W}$

We say a variable $v$ **occurs freely** in a formula $\phi$ if it is not inside a $\exists v$ or $\forall v$ quantifier; otherwise we say that it's **bound**. We call a formula a **sentence** if it has no free variables. We often write $\phi(v_1, \ldots, v_n)$ to make explicit the free variables in $\phi$

**Definition 1.6.** Let $\phi$ be a formula with free variables from $\bar{v} = (v_{i_1, \ldots, v_{i_m}})$ and let $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M^m$. We inductively define $\mathcal{M} \models \phi\bar{a}$ as follows
1. If $\phi$ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$
2. If $\phi$ is $R(t_1, \ldots, t_{m_R})$ then $\mathcal{M} \models \phi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{m_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$
3. If $\phi$ is $\neg\psi$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \not\models \psi(\bar{a})$
4. If $\phi$ is $(\psi \wedge \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \theta(\bar{a})$
5. If $\phi$ is $(\psi \vee \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ or $\mathcal{M} \models \theta(\bar{a})$
6. If $\phi$ is $\exists v_j \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if there is $b \in M$ s.t. $\mathcal{M} \models \psi(\bar{a}, b)$
7. If $\phi$ is $\forall v_j \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a}, b)$ for all $b \in M$

If $\mathcal{M} \models \phi(\bar{a})$ we say that $\mathcal{M}$ **satisfies** $\phi(\bar{a})$ or $\phi(\bar{a})$ is **true** in $\mathcal{M}$

**Proposition 1.7.** *Suppose that $\mathcal{M}$ is a substrcture of $\mathcal{N}$, $\bar{a} \in M$ and $\phi(\bar{v})$ is a quantifier-free formula. Then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \psi(\bar{a})$*

*Proof.* **Claim** If $t(\bar{v})$ is a term and $\bar{b} \in M$ then $t^{\mathcal{M}}(\bar{b}) = t^{\mathcal{N}}(\bar{b})$. □

**Definition 1.8.** We say that two $\mathcal{L}$-strctures $\mathcal{M}$ and $\mathcal{N}$ are {elementarily equivalent} and write $\mathcal{M} \equiv \mathcal{N}$ if

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{N} \models \phi$$

for all $\mathcal{L}$-sentences $\phi$

We let Th($\mathcal{M}$), the **full theory** of $\mathcal{M}$ be the set of $\mathcal{L}$-sentences $\phi$ s.t. $\mathcal{M} \models \phi$

**Theorem 1.9.** *Suppose that $j : \mathcal{M} \to \mathcal{N}$ is an isomorphism. Then $\mathcal{M} \equiv \mathcal{N}$*

*Proof.* Show by induction on formulas that $\mathcal{M} \models \phi(a_1, \dots, a_n)$ if and only if $\mathcal{N} \models \phi(j(a_1), \dots, j(a_n))$ for all formulas $\phi$ $\square$

## 1.2 Theories

Let $\mathcal{L}$ be a language. An $\mathcal{L}$-**theory** $T$ is a set of $\mathcal{L}$-sentences. We say that $\mathcal{M}$ is a **model** of $T$ and write $\mathcal{M} \models T$ if $\mathcal{M} \models \phi$ for all sentences $\phi \in T$. A theory is **satisfiable** if it has a model.

A class of $\mathcal{L}$-structures $\mathcal{K}$ is an **elementary class** if there is an $\mathcal{L}$-theory $T$ s.t. $\mathcal{K} = \{\mathcal{M} : \mathcal{M} \models T\}$

**Example 1.10** (Groups)**.** Let $\mathcal{L} = \{\cdot, e\}$ where $\cdot$ is a binary function symbol and $e$ is a constant symbol. The class of groups is axiomatized by

$$\forall x\ e \cdot x = x \cdot e = x$$
$$\forall x \forall y \forall z\ x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$\forall x \exists y\ x \cdot y = y \cdot x = e$$

**Example 1.11** (Rings and Fields)**.** Let $\mathcal{L}_r$ be the language of rings $\{+, -, \cdot, 0, 1\}$, where $+, -$ and $\cdot$ are binary function symbols and $0$ and $1$ are constants. The axioms for rings are given by

$$\forall x \forall y \forall z\ (x - y = z \leftrightarrow x = y + z)$$
$$\forall x\ x \cdot 0 = 0$$
$$\forall x \forall y \forall z\ x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$\forall x\ x \cdot 1 = 1 \cdot x = x$$
$$\forall x \forall y \forall z\ x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$
$$\forall x \forall y \forall z\ (x + y) \cdot z = (x \cdot z) + (y \cdot z)$$

We axiomatize the class of fields by adding

$$\forall x \forall y\ x \cdot y = y \cdot x$$
$$\forall x\ (x \neq 0 \to \exists y\ x \cdot y = 1)$$

We axiomatize the class of algebraically closed fields by adding to the field axioms the sentences

$$\forall a_0 \dots \forall a_{n-1} \exists x\ x^n + \sum_{i=1}^{n-1} a_i x^i = 0$$

for $n = 1, 2, \ldots$. Let ACF be the axioms for algebraically closed fields.

Let $\psi_p$ be the $\mathcal{L}_r$-sentence $\forall x \underbrace{x + \cdots + x}_{p\text{-times}} = 0$, which asserts that a field has characteristic $p$. For $p > 0$ a prime, let $\mathrm{ACF}_p = \mathrm{ACF} \cup \{\psi_p\}$ and $\mathrm{ACF}_0 = \mathrm{ACF} \cup \{\neg\psi_p : p > 0\}$ be the theories of algebraically closed fields of characteristic $p$ and zero respectively

**Definition 1.12.** Let $T$ be an $\mathcal{L}$-theory and $\phi$ an $\mathcal{L}$-sentence. We say that $\phi$ is a **logical consequence** of $T$ and write $T \models \phi$ if $\mathcal{M} \models \phi$ whenever $\mathcal{M} \models T$

**Proposition 1.13.**
1. *Let $\mathcal{L} = \{+, <, 0\}$ and let $T$ be the theory of ordered abelian groups. Then $\forall x(x \neq 0 \to x + x \neq 0)$ is a logical consequence of $T$*
2. *Let $T$ be the theory of groups where every element has order 2. Then $T \not\models \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_1 \neq x_3)$*

*Proof.* 1. $\mathbb{Z}/2\mathbb{Z} \models T \wedge \neg\exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_1 \neq x_3)$

□

## 1.3 Definable Sets and Interpretability

**Definition 1.14.** Let $\mathcal{M} = (M, \ldots)$ be an $\mathcal{L}$-structure. We say that $X \subseteq M^n$ is **definable** if and only if there is an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n, w_1, \ldots, w_m)$ and $\bar{b} \in M^b$ s.t. $X = \{\bar{a} \in M^n : \mathcal{M} \models \phi(\bar{a}, \bar{b})\}$. We say that $\phi(\bar{v}, \bar{b})$ **defines** $X$. We say that $X$ is $A$-**definable** or **definable over** $A$ if there is a formula $\psi(\bar{v}, w_1, \ldots, w_l)$ and $\bar{b} \in A^l$ s.t. $\psi(\bar{v}, \bar{b})$ defines $X$

A number of examples using $\mathcal{L}_r$, the language of rings

- Let $\mathcal{M} = (R, +, -, \cdot, 0, 1)$ be a ring. Let $p(X) \in R[X]$. Then $Y = \{x \in R : p(x) = 0\}$ is definable. Suppose that $p(X) = \sum_{i=0}^{m} a_i X^i$. Let $\phi(v, w_0, \ldots, w_n)$ be the formula

$$w_n \cdot \underbrace{v \cdots v}_{n\text{-times}} + \cdots + w_1 \cdot v + w_0 = 0$$

  Then $\phi(v, a_0, \ldots, a_n)$ defines $Y$. Indeed, $Y$ is $A$-definable for any $A \supseteq \{a_0, \ldots, a_n\}$
- Let $\mathcal{M} = (\mathbb{R}, +, -, \cdot, 0, 1)$ be the field of real numbers. Let $\phi(x, y)$ be the formula

$$\exists z(z \neq 0 \wedge y = x + z^2)$$

  Because $a < b$ if and only if $\mathcal{M} \models \phi(a, b)$, the ordering is $\emptyset$-definable

- Consider the natural numbers $\mathbb{N}$ as an $\mathcal{L} = \{+, \cdot, 0, 1\}$ structure. There is an $\mathcal{L}$-formula $T(e, x, s)$ s.t. $\mathbb{N} \models T(e, x, s)$ if and only if the Turing machine with program coded by $e$ halts on input $x$ in at most $s$ steops. Thus the Turing machine with program $e$ halts on input $x$ if and only if

$\mathbb{N} \models \exists s\, T(e, x, s)$. So the halting computations is definable

**Proposition 1.15.** *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Suppose that $D_n$ is a collection of subsets of $M^n$ for all $n \geq 1$ and $\mathcal{D} = (D_n : n \geq 1)$ is the smallest collection s.t.*

1. *$M^n \in D_n$*
2. *for all $n$-ary function symbols $f$ of $\mathcal{L}$, the graph of $f^{\mathcal{M}}$ is in $D_{n+1}$*
3. *for all $n$-ary relation symbols $R$ of $\mathcal{L}$, $R^{\mathcal{M}} \in D_n$*
4. *for all $i, j \leq n$, $\{(x_1, \ldots, x_n) \in M^n : x_i = x_j\} \in D_n$*
5. *if $X \in D_n$, then $M \times X \in D_{n+1}$*
6. *each $D_n$ is cloed under complement, union and intersection*
7. *if $X \in D_{n+1}$ and $\pi : M^{n+1} \to M^n$ is the projection $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)$, then $\pi(X) \in D_n$*
8. *if $X \in D_{n+m}$ and $b \in M^m$, then $\{a \in M^n : (a, b) \in X\} \in D_n$*

*Thus $X \subseteq M^n$ is definable if and only if $X \in D_n$*

**Proposition 1.16.** *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. If $X \subset M^n$ is $A$-definable, then every $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes $A$ pointwise fixes $X$ setwise(that is, if $\sigma$ is an automorphism of $M$ and $\sigma(a) = a$ for all $a \in A$, then $\sigma(X) = X$)*

*Proof.*

$$\mathcal{M} \models \psi(\bar{v}, \bar{a}) \leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{v}), \sigma(\bar{a})) \leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{v}), \bar{a})$$

In other words, $\bar{b} \in X$ if and only if $\sigma(\bar{b}) \in X$ □

**Definition 1.17.** A subset $S$ of a field $L$ is **algebraically independent** over a subfield $K$ if the elements of $S$ do not satisfy any non-trivial polynomial equation with coefficients in $K$

**Corollary 1.18.** *The set of real numbers is not definable in the field of complex numbers*

*Proof.* If $\mathbb{R}$ where definable, then it would be definable over a finite $A \subset \mathbb{C}$. Let $r, s \in \mathbb{C}$ be algebraically independent over $A$ with $r \in \mathbb{R}$ and $s \notin \mathbb{R}$. There is an automorphism $\sigma$ of $\mathbb{C}$ s.t. $\sigma|A$ is the identity and $\sigma(r) = s$. Thus $\sigma(\mathbb{R}) \neq \mathbb{R}$ and $\mathbb{R}$ is not definable over $A$ □

We say that an $\mathcal{L}_0$-structure $\mathcal{N}$ is **definably interpreted** in an $\mathcal{L}$-structure $\mathcal{M}$ if and only if we can find a definable $X \subseteq M^n$ for some $n$ and we can interpret the symbols of $\mathcal{L}_0$ as definable subsets and functions on $X$ so that the resulting $\mathcal{L}_0$-structure is isomorphic to $\mathcal{M}$