Introduction To Commutative Algebra

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1 Rings and Ideals

A **unit** is an element u with a **reciprocal** $1/u$ or the **multiplicative inverse**. The units form a multiplicative group, denoted $R^\times$

A ring **homomorphism**, or simply a **ring map**, $\varphi : R \to R'$ is a map preserving sum, products and 1

If there is an unspecified isomorphism between rings R and R', then we write $R = R'$ when it is **canonical**; that is, it does not depend on any artificial choices.

A subset $R'' \subset R$ is a **subring** if R'' is a ring and the inclusion $R'' \hookrightarrow R$ is a ring map. In this case, we call R a **(ring) extension**.

An R**-algebra** is a ring R' that comes equipped with a ring map $\varphi : R \to R'$, called the **structure map**, denoted by R'/R. For example, every ring is canonically a $\mathbb{Z}$-algebra. An R**-algebra homomorphism**, or R**-map**, $R' \to R''$ is a ring map between R-algebras.

A group G is said to **act** on R if there is a homomorphism given from G into the group of automorphism of R. The **ring of invariants** R^G is the subring defined by

$$R^G := \{x \in R \mid gx = g \text{ for all } g \in G\}$$

Similarly a group G is said to **act** on R'/R if G acts on R' and each $g \in G$ is an R-map. Note that R'^G is an R-subalgebra

Boolean rings

The simplest nonzero ring has two elements, 0 and 1. It's denoted $\mathbb{F}_2$

Given any ring R and any set X, let R^X denote the set of functions $f : X \to R$. Then R^X is a ring.

For example, take $R := \mathbb{F}_2$. Given $f : X \to R$, put $S := f^{-1}\{1\}$. Then $f(x) = 1$ if $x \in S$. In other words, f is the **characteristic function** χ_S. Thus *the characteristic functions form a ring, namely,* $\mathbb{F}_2^X$

Given $T \subset X$, clearly $\chi_S \cdot \chi_T = \chi_{S \cap T}$. $\chi_S + \chi_T = \chi_{S \triangle T}$, where $S \triangle T$ is the **symmetric difference**:

$$S \triangle T := (S \cup T) - (S \cap T)$$

Thus *the subsets of X form a ring: sum is symmetric difference, and product is intersection. This ring is canonically isomorphic to* $\mathbb{F}_2^X$

A ring B is called **Boolean** if $f^2 = f$ for all $f \in B$. If so, then $2f = 0$ as $2f = (f + f)^2 = f^2 + 2f + f^2 = 4f$

Suppose X is a topological space, and give $\mathbb{F}_2$ the **discrete** topology; that is, every subset is both open and closed. Consider the continuous functions $f : X \to \mathbb{F}_2$. Clearly, they are just the χ_S where S is both open and closed.

Polynomial rings

Let R be a ring, $P := R[X_1, \dots, X_n]$. P has this **Universal Mapping Property** (UMP): *given a ring map $\varphi : R \to R'$ and given an element x_i of R' for each i, there is a unique ring map $\pi : P \to R'$ with $\pi|R = \varphi$ and $\pi(X_i) = x_i$.* In fact, since π is a ring map, necessarily π is given by the formula:

$$\pi(\sum a_{(i_1,\dots,i_n)} X_1^{i_1} \dots X_n^{i_n}) = \sum \varphi(a_{(i_1,\dots,i_n)}) x_1^{i_1} \dots x_n^{i_n} \tag{1.0.1}$$

In other words, P is universal among R-algebras equipped with a list of n elements

Similarly let $\mathcal{X} := \{X_\lambda\}_{\lambda \in \Lambda}$ be any set of variables. Set $P' := R[\mathcal{X}]$; the elements of P' are the polynomials in any finitely many of the X_λ. P' has essentially the same UMP as P

Ideals

Let R be a ring. A subset $\mathfrak{a}$ is called an **ideal** if

1. $0 \in \mathfrak{a}$
2. whenever $a, b \in \mathfrak{a}$, also $a + b \in \mathfrak{a}$
3. whenever $x \in R$ and $a \in \mathfrak{a}$ also $xa \in \mathfrak{a}$

Given a subset $\mathfrak{a} \subset R$, by the ideal $\langle \mathfrak{a} \rangle$ that $\mathfrak{a}$ **generates**, we mean the smallest ideal containing $\mathfrak{a}$

All ideal containing all the a_λ contains any (finite) **linear combination** $\sum x_\lambda a_\lambda$ with $x_\lambda \in R$ and almost all 0.

Given a single element a, we say that the ideal $\langle a \rangle$ is **principal**

Given a number of ideals $\mathfrak{a}_\lambda$, by their **sum** $\sum \mathfrak{a}_\lambda$ we mean the set of all finite linear combinations $\sum x_\lambda a_\lambda$ with $x_\lambda \in R$ and $a_\lambda \in \mathfrak{a}_\lambda$

Given two ideals $\mathfrak{a}$ and $\mathfrak{b}$, by the **transporter** of $\mathfrak{b}$ into $\mathfrak{a}$ we mean the set

$$(\mathfrak{a} : \mathfrak{b}) := \{x \in R \mid x\mathfrak{b} \subset \mathfrak{a}\}$$

$(\mathfrak{a} : \mathfrak{b})$ is an ideal. Plainly,

$$\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}, \quad \mathfrak{a}, \mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}, \quad \mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$$

Further, for any ideal $\mathfrak{c}$, the distributive law holds: $\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}$

Given an ideal fa, notice $\mathfrak{a} = R$ *if and only if* $1 \in \mathfrak{a}$. It follows that $\mathfrak{a} = R$ iff $\mathfrak{a}$ contains a unit.

Given a ring map $\varphi : R \to R'$, denote by $\mathfrak{a}R'$ or $\mathfrak{a}^e$ the ideal of R' generated by the set $\varphi(\mathfrak{a})$. We call it the **extension** of $\mathfrak{a}$

Given an ideal $\mathfrak{a}'$ of R', its preimage $\varphi^{-1}(\mathfrak{a}')$ is an ideal of R. We call $\varphi^{-1}(\mathfrak{a}')$ the **contraction** of $\mathfrak{a}'$ and sometimes denote it by $\mathfrak{a}'^c$

Residue rings

kernel $\ker(\varphi)$ is defined to be the ideal $\varphi^{-1}(0)$ of R

Let $\mathfrak{a}$ be an ideal of R. Form the set of cosets of $\mathfrak{a}$

$$R/\mathfrak{a} := \{x + \mathfrak{a} \mid x \in R\}$$

$R/\mathfrak{a}$ is called the **residue ring** or **quotient ring** or **factor ring** of R **modulo** $\mathfrak{a}$. From the **quotient map**

$$\kappa : R \to R/\mathfrak{a} \quad \text{by } \kappa x := x + \mathfrak{a}$$

The element $\kappa x \in R/\mathfrak{a}$ is called the **residure** of x.

If $\ker(\varphi) \supset \mathfrak{a}$, *then there is a ring map* $\psi : R/\mathfrak{a} \to R'$ *with* $\psi\kappa = \varphi$; that is, the following diagram is commutative

$$\begin{tikzcd} R \arrow[r, "\kappa"] \arrow[dr, "\varphi"'] & R/\mathfrak{a} \arrow[d, "\psi"] \\ & R' \end{tikzcd}$$

by $\psi(x\mathfrak{a}) = \varphi(x)$. Then we only need to verify that ψ is a map

Conversely, *if* ψ *exists, then* $\ker(\varphi) \supset \mathfrak{a}$, *or* $\varphi\mathfrak{a} = 0$, *or* $\mathfrak{a}R' = 0$, since $\kappa\mathfrak{a} = 0$

Further, *if* ψ *exists, then* ψ *is unique* as κ is surjective

Finally, as κ is surjective, *if* ψ *exists, then* ψ *is surjective iff* ψ *is so*. In addition, ψ *is injective iff* $\mathfrak{a} = \ker(\varphi)$. Hence ψ *is an isomorphism iff* φ *is surjective and* $\mathfrak{a} = \ker(\varphi)$. Therefore,

$$R/\ker(\varphi) \xrightarrow{\sim} \operatorname{im}(\varphi)$$

$R/\mathfrak{a}$ has UMP: $\kappa(\mathfrak{a}) = 0$, and given $\varphi : R \to R'$ s.t. $\varphi : R \to R'$ s.t. $\varphi(\mathfrak{a}) = 0$, there is a unique ring map $\psi : R/\mathfrak{a} \to R'$ s.t. $\psi\kappa = \varphi$. In other words, $R/\mathfrak{a}$ is universal among R-algebras R' s.t. $\mathfrak{a}R' = 0$

If $\mathfrak{a}$ is the ideal generated by elements a_λ,then the UMP can be usefully rephrased as follows: $\kappa(a_\lambda) = 0$ for all λ, and given $\varphi : R \to R'$ s.t. $\varphi(a_\lambda) = 0$ for all λ, there is a unique ring map $\psi : R/\mathfrak{a} \to R'$ s.t. $\psi\kappa = \varphi$

The UMP serves to determine $R/\mathfrak{a}$ up to unique isomorphism. Say R', equipped with $\varphi : R \to R'$ has the UMP too. $\kappa(\mathfrak{a}) = 0$ so there is a unique $\psi' : R' \to R/\mathfrak{a}$ with $\psi'\varphi = \kappa$. Then $\psi'\psi\kappa = \kappa$. Hence $\psi'\psi = 1$ by uniqueness. Thus ψ and ψ' are inverse isomorphism

$$\begin{tikzcd}
 & & R/\mathfrak{a} \arrow[dl, "\psi"] \arrow[dd, "1"] \\
R \arrow[urr, "\kappa"] \arrow[r, "\varphi"] \arrow[drr, "\kappa"'] & R' \arrow[dr, "\psi'"] & \\
 & & R/\mathfrak{a}
\end{tikzcd}$$

Proposition 1.1. *Let R be a ring, $P := R[X]$, $a \in R$ and $\pi : P \to R$ the R-algebra map defined by $\pi(X) := a$. Then*

1. $\ker(\pi) = \{F(X) \in P \mid F(a) = 0\} = \langle X - a \rangle$
2. $R/\langle X - a \rangle \simeq R$

Proof. Set $G := X - a$. Given $F \in P$, let's show $F = GH + r$ with $H \in P$ and $r \in R$. By linearity, we may assume $F := X^n$. If $n \geq 1$, then $F = (G + a)X^{n-1}$, so $F = GH + aX^{n-1}$ with $H := X^{n-1}$.

Then $\pi(F) = \pi(G)\pi(H) + \pi(r) = r$. Hence $F \in \ker(\pi)$ iff $F = GH$. But $\pi(F) = F(a)$ by 1.0.1 □

Degree of a polynomial

Let R be a ring, P the polynomial ring in any number of variables. If F is a monomial $\mathbf{M}$, then its degree $\deg(\mathbf{M})$ is the sum of its exponents; in general, $\deg(F)$ is the largest $\deg(\mathbf{M})$ of all monomials $\mathbf{M}$ in F

Given any $G \in P$ with FG nonzero, notice that

$$\deg(FG) \leq \deg(F) + \deg(G)$$

Order of a polynomial

Let R be a ring, P the polynomial ring in variable X_λ for $\lambda \in \Lambda$, and $(x_\lambda) \in R^\Lambda$ a vector. Let $\varphi_{(x_\lambda)} : P \to P$ denote the R-algebra map defined by $\varphi_{(x_\lambda)} X_\mu := X_\mu + x_\mu$ for all $\mu \in \Lambda$. Fix a nonzero $F \in P$

The **order** of F at the zero vector (0), denoted $\operatorname{ord}_{(0)} F$, is defined as the smallest deg(**M**) of all the monomials **M** in F. In general, the **order** of F at the vector (x_λ), denoted $\operatorname{ord}_{(x_\lambda)} F$ is defined by the formula: $\operatorname{ord}_{(x_\lambda)} F := \operatorname{ord}_{(0)}(\varphi_{(x_\lambda)} F)$

Notice that $\operatorname{ord}_{(x_\lambda)} F = 0$ iff $F(x_\lambda) \neq 0$ as $(\varphi_{x_\lambda} F)(0) = F(x_\lambda)$

Given μ and $x \in R$, form $F_{\mu,x}$ by substituting x for X_μ in F. If $F_{\mu,x_\mu} \neq 0$, then

$$\operatorname{ord}_{(x_\lambda)} F \leq \operatorname{ord}_{(x_\lambda)} F_{\mu,x_\mu}$$

If $x_\mu = 0$, then F_{μ,x_μ} is the sum of the terms without x_μ in F. Hence if $(x_\lambda) = (0)$, then 1 holds. But substituting 0 for X_μ in $\varphi_{(x_\lambda)} F$ is the same as substituting x_μ for X_μ in F and then applying $\varphi_{(x_\lambda)}$ to the result; that is, $(\varphi_{(x_\mu)} F)_{\mu,0} = \varphi_{(x_\lambda)} F_{\mu,x_\mu}$

Given any $G \in P$ with FG nonzero,

$$\operatorname{ord}_{(x_\lambda)} FG \geq \operatorname{ord}_{(x_\lambda)} F +_{(x_\lambda)} G$$

Nested ideals

Let R be a ring, $\mathfrak{a}$ an ideal, and $\kappa : R \to R/\mathfrak{a}$ the quotient map. Given an ideal $\mathfrak{b} \supset \mathfrak{a}$, form the corresponding set of cosets of $\mathfrak{a}$

$$\mathfrak{b}/\mathfrak{a} := \{b + \mathfrak{a} \mid b \in \mathfrak{b}\} = \kappa(\mathfrak{b})$$

Clearly, $\mathfrak{b}/\mathfrak{a}$ is an ideal of $R/\mathfrak{a}$. Also $\mathfrak{b}/\mathfrak{a} = \mathfrak{b}(R/\mathfrak{a})$

Given an ideal $\mathfrak{b} \supset \mathfrak{a}$, form the composition of the quotient maps

$$\varphi : R \to R/\mathfrak{a} \to (R/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a})$$

φ is surjective and $\ker(\varphi) = \mathfrak{b}$. Hence φ factors

$$\begin{array}{ccc} R & \longrightarrow & R/\mathfrak{b} \\ \downarrow & & \simeq\downarrow\psi \\ R/\mathfrak{a} & \longrightarrow & (R/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a}) \end{array}$$

Idempotents

Let R be a ring. Let $e \in R$ be an **idempotent**; that is, $e^2 = e$. Then Re is a ring with e as 1.

Exercise

Exercise 1.0.1. Let $\varphi : R \to R'$ be a map of rings, $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ ideals of R, $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3$ ideals of R'. Prove

1. $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = \mathfrak{a}_1^e + \mathfrak{a}_2^e$