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Notes on Set Theory

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1 Foreword

Notes for the entrance examination

2 Models of Set - Sertraline

2.1 Some mathematical logic

Theorem 2.1 (Gödels second incompleteness theorem) *If a consistent recursive axiom set T contains* ***ZFC****, then*

$$T \nvdash \mathrm{Con}(t)$$

especially, $\boldsymbol{ZFC} \nvdash \mathrm{Con}(\boldsymbol{ZFC})$

Definition 2.2 () *Suppose (M, E_M) and (N, E_N) are two models of set theory, then*

1. *if for any formula σ, $M \models \sigma$ if and only if $N \models \sigma$, then M and N are* ***elementary equivalent****, denoted by $M \equiv N$*

2. *If bijection $f : M \to N$ satisfies: for any $a, b \in M$, $aE_M b$ iff $f(a)E_N f(b)$, then $f : M \cong N$ is an* ***isomorphism***

3. *If $M \subseteq N$ and $E_M = E_N \restriction M$, then M is N's submodel*

4. *If M is isomorphic to a submodel of N by injection f, and for any formula $\varphi(x_1, \ldots, x_n)$, for any $a_1, \ldots, a_n \in M$, $M \models \varphi[a_1, \ldots, a_n]$ iff $N \models \varphi[f(a_1), \ldots, f(a_n)]$, then f is called an* ***elementary embedding*** *from M to N, written as $f : M \prec N$*

5. *If $M \subseteq N$ and $M \prec N$, then M is a* ***elementary submodel*** *of N*

Lemma 2.3 () *Suppose $N \models$ **ZFC**$, M \subseteq N$, then $M \prec N$ iff $\forall \varphi(x, x_1, \ldots, x_n)$, $\forall (a_1, \ldots, a_n) \in M$, if $\exists a \in N$ s.t. $N \models \varphi[a, a_1, \ldots, a_n]$, then $\exists a \in M$ s.t. $M \models \varphi[a, a_1, \ldots, a_n]$*

Definition 2.4 () *Suppose $(M, E) \models$* ***ZFC***

1. *$h_\varphi : M^n \to M$ is φ's* ***Skolem function*** *if $\forall a_1, \ldots, a_n \in M$, if $\exists a \in M$ s.t. $M \models \varphi[a, a_1, \ldots, a_n]$, then $M \models \varphi[h_\varphi(a_1, \ldots, a_n), a_1, \ldots, a_n]$. requires* ***AC***

2. *Let* $\mathcal{H} = \{h_\varphi \mid \varphi$ *is a formula on set theory*$\}$*. For any* $S \subseteq M$*,* ***Skolem hull*** $\mathcal{H}(S)$ *is the smallest set consisting of* S *and closed under* $\mathcal{H}$

Lemma 2.5 () $N \models$ ***ZFC***$, S \subseteq N$*, if* $M = \mathcal{H}(S)$*, then* $M \prec N$

Theorem 2.6 (Löwenheim-Skolem theorem) *Suppose* $N \models$ ***ZFC*** *and is infinite, then there is a model* M *s.t.* $M = \omega$ *and* $M \prec N$

2.2 Cumulative Hierarchy

This section works in **ZF**(a.k.a. **ZF** $-$ axiom of foundation)

Definition 2.7 () *For any* α*, define sequence* V_α

1. $V_0 = \emptyset$
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
3. *For any limit ordinal* λ*,* $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$

And ***WF*** $= \displaystyle\bigcup_{\alpha \in \boldsymbol{On}} V_\alpha$

Lemma 2.8 () *For any ordinal* α

1. V_α *is transitive*
2. *if* $\xi \leq \alpha$*, then* $V_\xi \subseteq V_\alpha$
3. *if* κ *is inaccessible cardinal, then* $V_\kappa = \kappa$

1. Obviously $\kappa \leq V_\kappa$. Since κ is inaccessible, then for any $\alpha < \kappa$, $V_\alpha < \kappa$.

Definition 2.9 () *For any set* $x \in$ ***WF*** *,*

$$\operatorname{rank}(x) = \min\{\beta \mid x \in V_{\beta+1}\}$$

Lemma 2.10 () 1. $V_\alpha = \{x \in$ ***WF*** $\mid \operatorname{rank}(x) < \alpha\}$

2. ***WF*** *is transitive*
3. *For any* $x, y \in$ ***WF*** *, if* $x \in y$*, then* $\operatorname{rank}(x) < \operatorname{rank}(y)$
4. *for any* $y \in$ ***WF*** *,* $\operatorname{rank}(y) = \sup\{\operatorname{rank}(x) + 1 \mid x \in y\}$

Lemma 2.11 () *Supoose α is an ordinal*

1. $\alpha \in \boldsymbol{WF}$ *and* $\mathrm{rank}(\alpha) = \alpha$
2. $V_\alpha \cap \boldsymbol{On} = \alpha$

Lemma 2.12 () 1. *If $x \in \boldsymbol{WF}$, then $\bigcup x, \mathcal{P}(x), \{x\} \in \boldsymbol{WF}$, and their ranks are all less than* $\mathrm{rank}(x) + \omega$

2. *If $x, y \in \boldsymbol{WF}$, then $x \times y, x \cup y, x \cap y, \{x, y\}, (x, y), x^y \in \boldsymbol{WF}$, and their ranks are all less than* $\mathrm{rank}(x) + \mathrm{rank}(y) + \omega$
3. $\mathbb{Z}, \mathbb{Q}, \mathbb{R} \in V_{\omega+\omega}$
4. *for any set x, $x \in \boldsymbol{WF}$ iff $x \subset \boldsymbol{WF}$*

Lemma 2.13 () *Suppose* $\boldsymbol{AC}$

1. *for any group G, there exists group $G' \cong G$ in $\boldsymbol{WF}$*
2. *for any topological space T, there exists $T' \cong T$ in $\boldsymbol{WF}$*

Definition 2.14 () *Binary relation $<$ on set A is* ***well-founded*** *if for any nonempty $X \subseteq A$, X has minimal element under $<$*

Theorem 2.15 () *If $A \in \boldsymbol{WF}$, then $\in$ is a well-founded relation on A*

Lemma 2.16 () *If set A is transitive and $\in$ is well-founded on A, then $A \in \boldsymbol{WF}$*

Lemma 2.17 () *For any set x, there is a smallest transitive set* $\mathrm{trcl}(x)$ *s.t.* $x \subseteq \mathrm{trcl}(x)$

$$\begin{aligned} x_0 &= x \\ x_{n+1} &= \bigcup x_n \\ \mathrm{trcl}(x) &= \bigcup_{n<\omega} x_n \end{aligned}$$

$\mathrm{trcl}(x)$ is called **transitive closure** of x

Lemma 2.18 () *Without axiom of power set*

1. *if x is transitive, then* $\operatorname{trcl}(x) = x$

2. *if $y \in x$, then* $\operatorname{trcl}(y) \subseteq \operatorname{trcl}(x)$

3. $\operatorname{trcl}(x) = x \cup \bigcup\{\operatorname{trcl}(y) \mid y \in x\}$

Theorem 2.19 () *For any set X, the following are equivalent*

1. $X \in \boldsymbol{WF}$

2. $\operatorname{trcl}(X) \in \boldsymbol{WF}$

3. *$\in$ is a well-founded relation on* $\operatorname{trcl}(X)$

Theorem 2.20 () *The following propositions are equivalent*

1. *Axiom of foundation*

2. *For any set X, $\in$ is a well-founded relation on X*

3. $\boldsymbol{V} = \boldsymbol{WF}$

2.3 Relativization

Definition 2.21 () *Let $\boldsymbol{M}$ be a class φ a formula, the **relativization** of φ to $\boldsymbol{M}$ is $\varphi^{\boldsymbol{M}}$ defined inductively*

$$\begin{aligned}
(x \in y)^{\boldsymbol{M}} &\leftrightarrow x = y \\
(x \in y)^{\boldsymbol{M}} &\leftrightarrow x \in y \\
(\varphi \to \psi)^{\boldsymbol{M}} &\leftrightarrow \varphi^{\boldsymbol{M}} \to \psi^{\boldsymbol{M}} \\
(\neg\varphi)^{\boldsymbol{M}} &\leftrightarrow \neg\varphi^{\boldsymbol{M}} \\
(\forall x \varphi)^{\boldsymbol{M}} &\leftrightarrow (\forall x \in \boldsymbol{M})\varphi^{\boldsymbol{M}}
\end{aligned}$$

Note $\varphi^{\mathbf{V}} = \varphi$ and

$$f^{\mathbf{M}} = \{(x_1, \dots, x_n, x_{n+1}) \in \mathbf{M} \mid \varphi^{\mathbf{M}}(x_1, \dots, x_n, x_{n+1})\}$$

Definition 2.22 () *For any theory T, any class $\boldsymbol{M}$, $\boldsymbol{M} \models T$ iff for any axiom φ of T, $\varphi^{\boldsymbol{M}}$ holds*

Theorem 2.23 (ZF) $\boldsymbol{WF} \models \boldsymbol{ZF}$

- **Axiom of existence**

  $(\exists x(x = x))^{\mathbf{M}} \leftrightarrow \exists x \in \mathbf{M}\ (x = x)$, which is equivalent to $\mathbf{M}$ being nonempty

- **Axiom of extensionality**

$$\forall X \forall Y \forall u((u \in X \leftrightarrow u \in Y) \to X = Y)^{\mathbf{M}} \Leftrightarrow$$
$$\forall X \in \mathbf{M}\ \forall Y \in \mathbf{M}\ \forall u \in \mathbf{M}\ ((u \in X \leftrightarrow u \in Y) \to X = Y)$$

  Lemma 2.24 *If $\boldsymbol{M}$ is transitive, then axiom of extensionality holds in $\boldsymbol{M}$*

- **Axiom schema of specification**

$$\forall X \in \mathbf{M}\ \exists Y \in \mathbf{M}\ \forall u \in \mathbf{M}\ (u \in Y \leftrightarrow u \in X \wedge \varphi^{\mathbf{M}}(u))$$

  Since for any $X \in \mathbf{WF}$, $\mathcal{P}(X) \subseteq \mathbf{WF}$

- **Axiom of paring**
- **Axiom of union**
- **Axiom of power set**

$$\forall X \in \mathbf{M}\ \exists Y \in \mathbf{M}\ \forall u \in \mathbf{M}\ (u \in Y \leftrightarrow (u \subseteq X)^{\mathbf{M}})$$

  and

$$(u \subseteq X)^{\mathbf{M}} \leftrightarrow \forall x \in \mathbf{M}\ (x \in u \to x \in X) \leftrightarrow u \cap \mathbf{M} \subseteq X$$

- **Axiom of foundation**
- **Axiom schema of replacement**

2.4 Absoluteness

Definition 2.25 () *For any formula $\psi(x_1, \dots, x_n)$ and any class $\boldsymbol{M}$,$\boldsymbol{N}$, $\boldsymbol{M} \subseteq \boldsymbol{N}$, if*

$$\forall x_1 \dots \forall x_n \in \boldsymbol{M} \ (\psi^{\boldsymbol{M}} (x_1, \dots, x_n) \leftrightarrow \psi^{\boldsymbol{N}} (x_1, \dots, x_n))$$

*then $\psi(x_1, \dots, x_n)$ is **absolute** for $\boldsymbol{M}$,cn. If $\boldsymbol{N} = \boldsymbol{V}$, then ψ is **absolute** for $\boldsymbol{M}$*

Lemma 2.26 () *Suppose $\boldsymbol{M} \subseteq \boldsymbol{N}$ and φ,ψ are formulas, then*

1. *if φ,ψ are absolute for $\boldsymbol{M}$,cn, then so are $\neg\varphi, \varphi \to \psi$*
2. *if φ doesn't contain any quantifiers, then φ is absolute for any $\boldsymbol{M}$*
3. *if $\boldsymbol{M}$,$\boldsymbol{N}$ are transitive and φ is absolute for them, then so are $\forall x \in y \varphi$*

Definition 2.27 () *Δ_0 formula*

1. *$x = y, x \in y$ are Δ_0 formulas*
2. *if φ,ψ are Δ_0, then so are $\neg\varphi, \varphi \to \psi$*
3. *if φ is Δ_0, y is any set, then $(\forall x \in y)\varphi$ is Δ_0*

If φ is Δ_0, then $\exists x_1 \dots \exists x_n \varphi$ is Σ_1 formula, $\forall x_1 \dots \forall x_n \varphi$ is Π_1

Lemma 2.28 () *$\boldsymbol{M} \subseteq \boldsymbol{N}$ are both transitive, $\psi(x_0, \dots, x_n)$ is a formula, then*

1. *if ψ is Δ_0, then it's absolute for $\boldsymbol{M}$,cn*
2. *if ψ is Σ_1, then*

$$\forall x_1 \dots x_n(\psi^{\boldsymbol{M}} (x_1, \dots, x_n) \to \psi^{\boldsymbol{N}} (x_1, \dots, x_n))$$

3. *if ψ is Π_1, then*

$$\forall x_1 \dots x_n(\psi^{\boldsymbol{N}} (x_1, \dots, x_n) \to \psi^{\boldsymbol{M}} (x_1, \dots, x_n))$$

Lemma 2.29 () *If $\boldsymbol{M} \subseteq \boldsymbol{N}$, $\boldsymbol{M} \models \Sigma, \boldsymbol{N} \models \Sigma$ and*

$$\Sigma \vdash \forall x_1 \dots \forall x_n(\varphi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n))$$

then φ is absolute for $\boldsymbol{M}$,$\boldsymbol{N}$ if and only if ψ is absolute for $\boldsymbol{M}$,$\boldsymbol{N}$

Definition 2.30 () *Suppose* $\boldsymbol{M} \subseteq \boldsymbol{N}$ *,* $f(x_1, \ldots, x_n)$ *is a function.* f *is **absolute** for* $\boldsymbol{M}$ *and* $\boldsymbol{N}$ *if and only if* $\varphi(x_1, \ldots, x_n, x_{n+1})$ *defining* f *is absolute.*

Theorem 2.31 () *Following relations and functions can be defined in* $\boldsymbol{ZF}^- - Pow - Inf$ *and are equivalent to some* Δ_0 *formulas. So they are absolute for any transitive model* $\boldsymbol{M}$ *on* $\boldsymbol{ZF}^- - Pow - Inf$

1. $x \in y$
2. $x = y$
3. $x \subset y$
4. $\{x, y\}$
5. $\{x\}$
6. (x, y)
7. $\emptyset$
8. $x \cup y$
9. $x - y$
10. $x \cap y$
11. x^+
12. x *is a transitive set*
13. $\bigcup x$
14. $\bigcap x$ $(\bigcap \emptyset = \emptyset)$

Lemma 2.32 () *Absoluteness is closed under operation composition*

Theorem 2.33 () *Following relations and functions are absolute for any transitive model* $\boldsymbol{M}$ *on* $\boldsymbol{ZF}^- - Pow - Inf$

1. z *is an ordered pair*
2. $A \times B$
3. R *is a relation*

4. dom(R)

5. ran(R)

6. *f is a function*

7. $f(x)$

8. *f is injective*

2.5 Relative consistence of the axiom of foundation

Lemma 2.34 () *Suppose transitive class $\boldsymbol{M} \models \boldsymbol{ZF}^{-} - Pow - inf$ and $\omega \in \boldsymbol{M}$, then the axiom of infinity is true in $\boldsymbol{M}$. Hence the axiom of infinity is true in $\boldsymbol{WF}$*

Theorem 2.35 () *Let T be a theory of set theory language and Σ a set of sentences. Suppose $\boldsymbol{M}$ is a class and $T \vdash \boldsymbol{M} \neq \emptyset$, then if $\boldsymbol{M} \models_T \Sigma$, then*

1. *for any sentences φ, if $\Sigma \vdash \varphi$, then $T \vdash \varphi^{\boldsymbol{M}}$*

2. *if T is consistent, then so is $Cn(\Sigma)$*

Theorem 2.36 () *The axiom of foundation is consistent with $\boldsymbol{ZF}$.*

By **??**, let T be **ZF**, Σ be **ZF** and **M** be **WF**

Lemma 2.37 (ZF^{-}) *Suppose transitive model $\boldsymbol{M} \models \boldsymbol{ZF}^{-} - Pow - Inf$. If $X, R \in \boldsymbol{M}$ and R is a well-order on X, then*

$$(R \textit{ is a well-order on } X)^{\boldsymbol{M}}$$