Basic Topology

M. A. Armstrong

June 11, 2020

Contents

1 Introduction

1.1 Abstract spaces

We ask for a set X and for each point x of X a nonempty collection of subsets of X, called neighbourhoods of x. These neighbourhoods are required to satisfy four axioms

1. x lies in each of its neighbourhoods
2. The intersection of two neighbourhoods of x is itself a neighbourhood of x
3. If N is a neighbourhood of x and if U is a subset of X which contains N, then U is a neighbourhood of x
4. If N is a neighbourhood of x and if $\mathring{N}$ denotes the set $\{z \in N \mid N \text{ is a neighbourhood of } z\}$, then $\mathring{N}$ is a neighbourhood of x. (The set $\mathring{N}$ is called the **interior** of N)

This whole structure is called a **topological space**. The assignment of a collection of neighbourhoods satisfying axioms (1) $\to$ (4) to each point $x \in X$ is called a **topology** on the set X.

Let X and Y be topological spaces. A function $f : X \to Y$ is **continuous** if for each point $x \in X$ and each neighbourhood N of $f(x)$ in Y the set $f^{-1}(N)$ is a neighbourhood of x in X. A function $h : X \to Y$ is called a **homeomorphism** if it is one-one, onto, continuous and has a continuous inverse. When such a function exists, X and Y are called **homeomorphic** spaces

Example 1.1. 1. Let X be a topological space and let Y be a subset of X. We can define a topology on Y as follows. Given a point $y \in Y$ take the collection of its neighbourhoods in the topological space X and intersect each of these neighbourhood with Y. The resulting sets are the neighbourhoods of y in Y. We say that Y has the **subspace topology**.

Definition 1.1. A **surface** is a topological space in which each point has a neighbourhood homeomorphic to the plane, and for which any two distinct points possess disjoint neighbourhoods

2 Continuity

2.1 Open and closed sets

Let X be a topological space and call a subset O of X **open** if it is a neighbourhood of each of its points. The union of any collection of open sets is open by axiom (3) and the intersection of *finite* number of open sets is open by axiom (2).

Suppose we have a set X together with a nonempty collection of subsets of X, which we call open sets, such that any union of open sets is itself open, any finite intersection of open sets is open, and both the whole set and the empty set are open. Given a point x of X, we shall call a subset N of x a **neighbourhood of** x if we can find an open set O s.t. $x \in O \subseteq N$

We claim that this definition of neighbourhood makes X into a topological space.

Verification for axiom (4). Suppose N is a neighbourhood of x and let $\mathring{N}$ denote the set of points z s.t. N is a neighbourhood of z. Choose an open set O s.t. $x \in O \subseteq N$. Now O, being open, is a neighbourhood of each of its points, so O is contained in $\mathring{N}$.

Definition 2.1. A **topology** on a set X is a nonempty collection of subsets of X, called open sets, such that any union of open sets is open, any finite intersection of open sets is open, and both X and the empty set are open. A set together with a topology on it is called a **topological space**

The open sets of the "usual" topology on $\mathbb{R}^n$ are characterized as follows. A set U is open if given $x \in U$ we can always find a positive real number ϵ s.t. the ball with centre x and radius ϵ lies entirely in U.

For **discrete topology** on X, every subset of X is an open set and we call X discrete space.

A subset of a topological space is **closed** if its complement is open.

Consider the set A on $\mathbb{R}^2$ whose points x, y satisfy $x \geq 0$ and $y > 0$. This set is neither closed nor open. Take the space X whose points are those points $(x, y) \in \mathbb{R}^2$ s.t. $x \geq 1$ or $x \leq -1$ and whose topology is induced from $\mathbb{R}^2$. The subsets of X consisting of those points with positive first coordinate is both open and closed.

Let A be a subset of a topological space X and call a point p of X a **limit point** (or accumulation point) of A if every neighbourhood of p contains at least one point of $A - \{p\}$

Example 2.1. 1. Take X to be the real line $\mathbb{R}$, and let A consist of the points $1/n$, $n = 1, 2, \ldots$. Then A has exactly one limit point, namely the origin

2. X the real line, take $A = [0, 1)$. Then each point of A is a limit point of A, and in addition 1 is a limit point
3. Let $X \subseteq \mathbb{R}^3$ and let A consist of those points all of whose coordinates are rational. Then every point of $\mathbb{R}^3$ is a limit point of A
4. Let $A \subseteq \mathbb{R}^3$ be the set of points which have integer coordinates. Then A does not have any limit points
5. Take X to be the set of all real numebers with the so called **finite-complement topology**. Here a set is open if its complement is finite or all of X. If we now take A to be an infinite subset of X (say the set of all integers), then every point of X is a limit point of A. On the other hand a finite subset of X has no limit points in this topology

Theorem 2.2. *A set is closed if and only if it contains all its limit points*

Proof. If A is closed, its complement $X - A$ is open. Since an open set is a neighbourhood of each of its points, no point of $X - A$ can be a limit point of A.

Suppose A contains all its limit point and let $x \in X - A$. Since x is not a limit point of A, we can find a neighbourhood N of x which does not meet A. So N is inside $X - A$, showing $X - A$ to be a neighbourhood of each of its points and consequently open. Therefore A is clsoed. $\square$

The union of A and all its limit points is called the **closure** of A and is written A

Theorem 2.3. *The closure of A is the smallest closed set containing A, in other words the intersection of all closed sets which contain A*

Proof. For if $x \in X - A$, we can find an open neighbourhood U of x which does not contain any points of A. Since an open set is a neighbourhood of each of its points, U cannot contain any of the limit points of A. Therefore we have an open set U s.t. $x \in U \subseteq X - A$. Consequently $X - A$ is a neighbourhood of each of its points and must be open.

Now let B be a closed set which contains A. Then every limit point of A is a limit point of B and therefore must lie in B since B is closed. This gives $A \subseteq B$ $\square$

Corollary 2.4. *A set is closed if and only if it is equal to its closure*

A set whose closure is the whole space is said to be **dense** in the space

The **interior** of a set, usually written $\mathring{A}$, is the union of all open sets contained in A. A point lies in $\mathring{A}$ if and only if it's a neighbourhood of A.

We define the **frontier** of A to be the $A \cap X - A$.

Suppose we have a topology on a set X, and a collection β of open set s.t. every open set is a union of members of β. Then β is called a **base** for the topology and elements of β are called **basic open sets**. An equivalent formulation is to ask that given a point $x \in X$, and a neighbourhood N of x, there is always an element B of β s.t. $x \in B \subseteq N$.

Theorem 2.5. *Let β be a nonempty collection of subsets of a set X. If the intersection of any finite number of members of β is always in β, and if $\bigcup \beta = X$, then β is a base for a topology on X*

2.2 Continuous functions

Theorem 2.6. *A function from X to Y is continuous if and only if the inverse image of each open set of Y is open in X*

A continuous function is often called a **map**

Theorem 2.7. *The composition of two maps is a map*

Theorem 2.8. *Suppose $f : X \to Y$ is continuous, and let $A \subseteq X$ have the subspace topology. Then the restriction $f|A : A \to Y$ is continuous*

Theorem 2.9. *The following are equivalent*

1. *$f : X \to Y$ is a map*
2. *If β is a base for the topology of Y, the inverse image of every member of β is open in X*
3. *$f(A) \subseteq f(A)$ for any subset A of X*
4. *$f^{-1}(B) \subseteq f^{-1}(B)$ for any subset B of Y*
5. *The inverse image of each closed set in Y is closed in X*