A Course In Universal Algebra

Stanley Burris & H. P. Sankappanavar

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Contents

1 Lattices

1.1 Definitions of Lattices

Definition 1.1. A nonempty set L together with two binary operations $\vee$ and $\wedge$ (read "join" and "meet" respectively) on L is called a **lattice** if it satisfies the following identities

L1: (a) $x \vee y \approx y \vee x$
(b) $x \wedge y \approx y \wedge x$ (commutative laws)
L2: (a) $x \vee (y \vee z) \approx (x \vee y) \vee z$
(b) $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$ (associate laws)
L3: (a) $x \vee x \approx x$
(b) $x \wedge x \approx x$ (idempotent laws)
L4: (a) $x \approx x \vee (x \wedge y)$
(b) $x \approx x \wedge (x \vee y)$ (absorption laws)

Definition 1.2. Let A be a subset of a poset P. An element p in P is an **upper bound** for A if $a \leq p$ for every a in A. An element p in P is the **least upper bound** of A (l.u.b. of A) or **supremum** of A ($\sup A$.

For a, b in P we say b **covers** a, or a is **covered by** b if $a < b$ and whenever $a \leq c \leq b$ it follows that $a = c$ or $c = b$. We use the notation $a \prec b$ to denote a is covered by b.

Definition 1.3. A poset L is a lattice iff for every a, b in L both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist

1. If L is a lattice by the first definition, then define $\leq$ on L by $a \leq b$ iff $a = a \wedge b$
2. If L is a lattice by the second definition, then define $\vee$ and $\wedge$ by $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$

1.2 Isomorphism Lattices, and Sublattices

Definition 1.4. Two lattices L_1 and L_2 are **isomorphic** if there is a bijection α from L_1 to L_2 s.t. for every a, b in L_1 the following two equation hold: $\alpha(a \vee b) = \alpha(a) \vee \alpha(b)$ and $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$. Such an α is called an **isomorphism**

Definition 1.5. If P_1 and P_2 are two posets and α is a map from P_1 to P_2, then we say α is **order-preserving** if $\alpha(a) \leq \alpha(b)$ holds in P_2 whenever $a \leq b$ holds in P_1

Theorem 1.6. *Two lattices L_1 and L_2 are isomorphic iff there is a bijection α from L_1 to L_2 s.t. both α and α^{-1} are order-preserving*

Definition 1.7. If L is a lattice and $L' \neq \emptyset$ is a subset of L s.t. for every pair of elements a, b in L' both $a \vee b$ and $a \wedge b$ are in L', where $\wedge, \vee$ are the lattice operations of L, then we say that L' with the same operations is a **sublattice** of L

Definition 1.8. A lattice L_1 can be **embedded** into a lattice L_2 if there is a sublattice of L_2 isomorphic to L_1; in this case we also say that L_2 **contains a copy of L_1 as a sublattice**

1.3 Distributive and Modular Lattices

Definition 1.9. A **distributive lattice** is a lattice which satisfies either of the distributive laws,
D1: $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$
D2: $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$

Theorem 1.10. *A lattice L satisfies D1 iff it satisfies D2*

$$\begin{aligned}
x \vee (y \wedge z) &\approx (x \vee (x \wedge z)) \vee (y \wedge z) && \text{(by L4(a))} \\
&\approx x \vee ((x \wedge z) \vee (y \wedge z)) \\
&\approx x \vee ((z \wedge x) \vee (z \wedge y)) \\
&\approx x \vee (z \wedge (x \vee y)) \\
&\approx x \vee ((x \vee y) \wedge z) \\
&\approx (x \wedge (x \vee y)) \vee (x \vee y \wedge z) \\
&\approx ((x \vee y) \wedge x) \vee ((x \vee y) \wedge) \\
&\approx (x \vee y) \wedge (x \vee z)
\end{aligned}$$

Actually every lattice satisfies both of the inequalities $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$ and $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$.

Definition 1.11. A **modular lattice** is any lattice which satisfies the **modular** law
M: $x \leq y \to x \vee (y \wedge z) \approx y \wedge (x \vee z)$

Equivalent to the identity

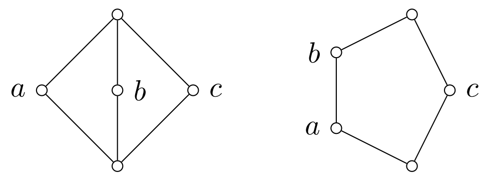
$$(x \wedge y) \vee (y \wedge z) \approx y \wedge ((x \wedge y) \vee z)$$

Every lattice satisfies

$$x \leq y \to x \vee (y \wedge z) \leq y \wedge (x \vee z)$$

Theorem 1.12. *Every distributive lattice is a modular lattice*

Neither M_5 nor N_5 is a distributive lattice in Figure 1



M_5 N_5

Figure 1

Theorem 1.13 (Dedekind). *L is a nonmodular lattice iff N_5 can be embedded into L*

Proof. If L doesn't satisfy the modular law. Then for some a, b, c in L we have $a \leq b$ but $a \vee (b \wedge c) < b \wedge (a \vee c)$. Let $a_1 = a \vee (b \wedge c)$ and $b_1 = b \wedge (a \vee c)$. Then

$$c \wedge b_1 = c \wedge (b \wedge (a \vee c)) = (c \wedge (a \vee c)) \wedge b = c \wedge b$$

and

$$c \vee a_1 = c \vee a$$

Now as $c \wedge b \leq a_1 \leq b_1$, we have $c \wedge b \leq c \wedge a_1 \leq c \wedge b_1 = c \wedge b$, hence $c \wedge a_1 = c \wedge b$. Likewise $c \vee a = c \vee b_1$

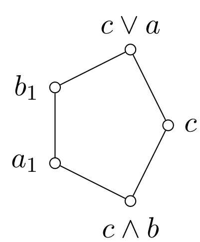


Figure 2

□

Theorem 1.14 (Birkhoff). *L is a nondistributive lattice iff M_5, or N_5 can be embedded into L*

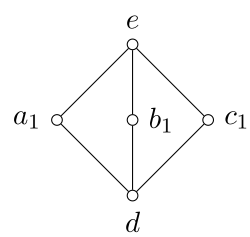


Figure 3

Proof. Let suppose that L is a nondistributive lattice and that L does not contain a copy of N_5 as a sublattice. Thus L is modular by Theorem 1.13. Since the distributive laws do not hold in L, there must be elements a, b, c from L s.t. $(a \wedge b) \vee (a \wedge c) < a \wedge (b \vee c)$. Let us define

$$\begin{aligned} d &= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \\ e &= (a \vee b) \wedge (a \vee c) \wedge (b \vee c) \\ a_1 &= (a \wedge e) \vee d \\ b_1 &= (b \wedge e) \vee d \\ c_1 &= (c \wedge e) \vee d \end{aligned}$$

Then $d \leq a_1, b_1, c_1 \leq e$. Now from

$$a \wedge e = a \wedge (b \vee c)$$

and

$$\begin{aligned} a \wedge d &= \underline{a} \wedge (\underline{(a \wedge b) \vee (a \wedge c)} \vee (b \wedge c)) \\ &= ((a \wedge b) \vee (a \wedge c)) \vee (a \wedge (b \wedge c)) && \text{by M} \\ &= (a \wedge b) \vee (a \wedge c) \end{aligned}$$

it follows that $d < e$

We now show that diagram in Figure 3 is a copy of M_5 in L. To do this it suffices to show that $a_1 \wedge b_1 = a_1 \wedge c_1 = b_1 \wedge c_1 = d$ and $a_1 \vee b_1 = a_1 \vee c_1 = b_1 \vee c_1 = e$.

$$
\begin{aligned}
a_1 \wedge b_1 &= ((a \wedge e) \vee \underline{d}) \wedge (\underline{(b \wedge e) \vee d}) \\
&= ((a \wedge e) \wedge ((b \wedge \underline{e}) \vee d)) \vee d && \text{(by M)} \\
y \wedge z &= ((b \wedge e) \vee d) \wedge d = d \\
&= ((a \wedge e) \wedge ((b \vee d) \wedge e)) \vee d && \text{(by M)} \\
&= ((a \wedge e) \wedge e \wedge (b \vee d)) \vee d \\
&= ((a \wedge e) \wedge (b \vee d)) \vee d \\
&= (a \wedge \underline{(b \vee c)} \wedge (\underline{b} \vee (a \wedge c))) \vee d \\
&= (a \wedge (b \vee ((b \vee c) \wedge (a \vee c)))) \vee d && \text{(by M)} \\
&= (\underline{a} \wedge (b \vee \underline{(a \wedge c)})) \vee d && a \wedge c \leq b \vee c \\
&= (a \wedge c) \vee (b \wedge a) \vee d && \text{(by M)} \\
&= d
\end{aligned}
$$

□

1.4 Complete Lattices, Equivalence Relations, and Algebraic Lattices

Definition 1.15. A poset P is **complete** if for every subset A of P both $\sup A$ and $\inf A$ exists in P. The elements $\sup A$ an $\inf A$ will be denoted by $\bigvee A$ and $\bigwedge A$.

Theorem 1.16. *Let P be a poset s.t. $\bigvee A$ exists for every subset A, or s.t. $\bigwedge A$ exists for every subset A. Then P is a complete lattice*

Proof. Suppose $\bigwedge A$ exists for every $A \subseteq P$. Then letting A^u be the set of upper bounds of A in P, it is routine to verify that $\wedge A^u$ is indeed $\vee A$. □

In the above theorem, the existence of $\bigwedge \emptyset$ guarantees a largest element in P, and likewise the existence of $\bigvee \emptyset$ guarantees a smallest element in P. (Every element is larger than $\emptyset$).

Definition 1.17. A sublattice L' of a complete lattice L is called a **complete sublattice** of L if for every subset A of L' the elements $\bigvee A$ and $\bigwedge A$, as defined in L, are actually in L'