Model Theory for Dummies: An Introduction

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Contents

1 Structures and Theories

1.1 Languages and Structures

Definition 1.1. A language $\mathcal{L}$ is given by specifying the following data

1. A set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$
2. a set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$
3. a set of constant symbols $\mathcal{C}$

Definition 1.2. An $\mathcal{L}$-structure $\mathcal{M}$ is given by the following data

1. a nonempty set M called the **universe**, **domain** or **underlying set** of $\mathcal{M}$
2. a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$
3. a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$
4. an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$

We refer to $f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$ as the **interpretations** of the symbols f, R and c. We often write the structure as $\mathcal{M} = (M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C})$

Definition 1.3. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures with universes M and N respectively. An **$\mathcal{L}$-embedding** $\eta : \mathcal{M} \to \mathcal{N}$ is a one-to-one map $\eta : M \to N$ that

1. $\eta(f^{\mathcal{M}}(a_1, \ldots, a_{n_f})) = f^{\mathcal{N}}(\eta(a_1), \ldots, \eta(a_{n_f}))$ for all $f \in \mathcal{F}$ and $a_1, \ldots, a_{n_f} \in M$
2. $(a_1, \ldots, a_{m_R}) \in R^{\mathcal{M}}$ if and only if $(\eta(a_1), \ldots, \eta(a_{m_R})) \in R^{\mathcal{N}}$ for all $R \in \mathcal{R}$ and $a_1, \ldots, a_{m_R} \in M$
3. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for $c \in \mathcal{C}$

A bijective $\mathcal{L}$-embedding is called an **$\mathcal{L}$-isomorphism**. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-embedding, we say either $\mathcal{M}$ is a **substrcture** of $\mathcal{N}$ or that $\mathcal{N}$ is an **extension** of $\mathcal{M}$

The **cardinality** of $\mathcal{M}$ is $|M|$, the cardinality of the universe of $\mathcal{M}$

Definition 1.4. The set of **$\mathcal{L}$-terms** is the smallest set $\mathcal{T}$ s.t.

1. $c \in \mathcal{T}$ for each constant symbol $c \in \mathcal{C}$
2. each variable symbol $v_i \in \mathcal{T}$ for $i = 1, 2, \ldots$
3. if $t_1, \ldots, t_{n_f} \in \mathcal{T}$ and $f \in \mathcal{F}$ then $f(t_1, \ldots, n_{n_f}) \in \mathcal{T}$

Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and that t is a term built using variables from $\bar{v} = (v_{i_1}, \ldots, v_{i_m})$. We want to interpret t as a function $t^{\mathcal{M}} : M^m \to M$. For s a subterm of t and $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M$, we inductively define $s^{\mathcal{M}}(\bar{a})$ as follows.

1. If s is a constant symbol c, then $s^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$
2. If s is the variable v_{i_j}, then $s^{\mathcal{M}}(\bar{a}) = a_{i_j}$
3. If s is the term $f(t_1, \ldots, t_{n_f})$, where f is a function symbol of $\mathcal{L}$ and $t_1, \ldots, t_{n_f}$ are terms, then $s^{\mathcal{M}}(\bar{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_f}^{\mathcal{M}}(\bar{a}))$

The function $t^{\mathcal{M}}$ is defined by $\bar{a} \mapsto t^{\mathcal{M}}(\bar{a})$

Definition 1.5. ϕ is an **atomic $\mathcal{L}$-formula** if ϕ is either
1. $t_1 = t_2$ where t_1 and t_2 are terms
2. $R(t_1, \ldots, t_{n_R})$

The set of **$\mathcal{L}$-formulas** is the smallest set $\mathcal{W}$ containing the atomic formulas s.t.
1. if $\phi \in \mathcal{W}$, then $\neg\phi \in \mathcal{W}$
2. if $\phi, \psi \in \mathcal{W}$, then $(\phi \wedge \psi), (\phi \vee \psi) \in \mathcal{W}$
3. if $\phi \in \mathcal{W}$, then $\exists v_i \phi, \forall v_i \phi \in \mathcal{W}$

We say a variable v **occurs freely** in a formula ϕ if it is not inside a $\exists v$ or $\forall v$ quantifier; otherwise we say that it's **bound**. We call a formula a **sentence** if it has no free variables. We often write $\phi(v_1, \ldots, v_n)$ to make explicit the free variables in ϕ

Definition 1.6. Let ϕ be a formula with free variables from $\bar{v} = (v_{i_1, \ldots, v_{i_m}})$ and let $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M^m$. We inductively define $\mathcal{M} \models \phi\bar{a}$ as follows
1. If ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$
2. If ϕ is $R(t_1, \ldots, t_{m_R})$ then $\mathcal{M} \models \phi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{m_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$
3. If ϕ is $\neg\psi$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \not\models \psi(\bar{a})$
4. If ϕ is $(\psi \wedge \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \theta(\bar{a})$
5. If ϕ is $(\psi \vee \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ or $\mathcal{M} \models \theta(\bar{a})$
6. If ϕ is $\exists v_j \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if there is $b \in M$ s.t. $\mathcal{M} \models \psi(\bar{a}, b)$
7. If ϕ is $\forall v_j \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a}, b)$ for all $b \in M$

If $\mathcal{M} \models \phi(\bar{a})$ we say that $\mathcal{M}$ **satisfies** $\phi(\bar{a})$ or $\phi(\bar{a})$ is **true** in $\mathcal{M}$

Proposition 1.7. *Suppose that $\mathcal{M}$ is a substrcture of $\mathcal{N}$, $\bar{a} \in M$ and $\phi(\bar{v})$ is a quantifier-free formula. Then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \psi(\bar{a})$*

Proof. **Claim** If $t(\bar{v})$ is a term and $\bar{b} \in M$ then $t^{\mathcal{M}}(\bar{b}) = t^{\mathcal{N}}(\bar{b})$. □

Definition 1.8. We say that two $\mathcal{L}$-strctures $\mathcal{M}$ and $\mathcal{N}$ are **elementarily equivalent** and write $\mathcal{M} \equiv \mathcal{N}$ if

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{N} \models \phi$$

for all $\mathcal{L}$-sentences ϕ

We let $\mathrm{Th}(\mathcal{M})$, the **full theory** of $\mathcal{M}$ be the set of $\mathcal{L}$-sentences ϕ s.t. $\mathcal{M} \models \phi$

Theorem 1.9. *Suppose that $j : \mathcal{M} \to \mathcal{N}$ is an isomorphism. Then $\mathcal{M} \equiv \mathcal{N}$*

Proof. Show by induction on formulas that $\mathcal{M} \models \phi(a_1, \ldots, a_n)$ if and only if $\mathcal{N} \models \phi(j(a_1), \ldots, j(a_n))$ for all formulas ϕ $\square$

1.2 Theories

Let $\mathcal{L}$ be a language. An **$\mathcal{L}$-theory** T is a set of $\mathcal{L}$-sentences. We say that $\mathcal{M}$ is a **model** of T and write $\mathcal{M} \models T$ if $\mathcal{M} \models \phi$ for all sentences $\phi \in T$. A theory is **satisfiable** if it has a model.

A class of $\mathcal{L}$-structures $\mathcal{K}$ is an **elementary class** if there is an $\mathcal{L}$-theory T s.t. $\mathcal{K} = \{\mathcal{M} : \mathcal{M} \models T\}$

Example 1.1 (Linear Orders)**.** Let $\mathcal{L} = \{<\}$, where $<$ is a binary relation symbol. The class of linear order is axiomatized by the $\mathcal{L}$-sentences

$$\begin{aligned}
&\forall x\ \neg(x < x)\\
&\forall x \forall y \forall z\ ((x < y \wedge y < z) \to x < z)\\
&\forall x \forall y\ (x < y \vee x = y \vee y < x)
\end{aligned}$$

Example 1.2 (Groups)**.** Let $\mathcal{L} = \{\cdot, e\}$ where $\cdot$ is a binary function symbol and e is a constant symbol. The class of groups is axiomatized by

$$\begin{aligned}
&\forall x\ e \cdot x = x \cdot e = x\\
&\forall x \forall y \forall z\ x \cdot (y \cdot z) = (x \cdot y) \cdot z\\
&\forall x \exists y\ x \cdot y = y \cdot x = e
\end{aligned}$$

Example 1.3 (Left R-modules)**.** Let R be a ring with multiplicative identity 1. Let $\mathcal{L} = \{+, 0\} \cup \{r : r \in R\}$ where $+$ is a binary function symbol, 0 is a constant, and r is a unary function symbol for $r \in R$. In an R-module, we will interpret r as scalar multiplication by R. The axioms for R-modules are

$$\begin{aligned}
&\forall x\ \ r(x + y) = r(x) + r(y) \text{ for each } r \in R\\
&\forall x\ \ (r + s)(x) = r(x) + s(x) \text{ for each } r, s \in R\\
&\forall x\ \ r(s(x)) = rs(x) \text{ for } r, s \in R\\
&\forall x\ \ 1(x) = x
\end{aligned}$$

Example 1.4 (Rings and Fields). Let $\mathcal{L}_r$ be the language of rings $\{+, -, \cdot, 0, 1\}$, where $+, -$ and $\cdot$ are binary function symbols and 0 and 1 are constants. The axioms for rings are given by

$$\begin{aligned}
&\forall x \forall y \forall z\ (x - y = z \leftrightarrow x = y + z) \\
&\forall x\ x \cdot 0 = 0 \\
&\forall x \forall y \forall z\ x \cdot (y \cdot z) = (x \cdot y) \cdot z \\
&\forall x\ x \cdot 1 = 1 \cdot x = x \\
&\forall x \forall y \forall z\ x \cdot (y + z) = (x \cdot y) + (x \cdot z) \\
&\forall x \forall y \forall z\ (x + y) \cdot z = (x \cdot z) + (y \cdot z)
\end{aligned}$$

We axiomatize the class of fields by adding

$$\begin{aligned}
&\forall x \forall y\ x \cdot y = y \cdot x \\
&\forall x\ (x \neq 0 \to \exists y\ x \cdot y = 1)
\end{aligned}$$

We axiomatize the class of algebraically closed fields by adding to the field axioms the sentences

$$\forall a_0 \ldots \forall a_{n-1} \exists x\ x^n + \sum_{i=1}^{n-1} a_i x^i = 0$$

for $n = 1, 2, \ldots$. Let ACF be the axioms for algebraically closed fields.

Let ψ_p be the $\mathcal{L}_r$-sentence $\forall x\ \underbrace{x + \cdots + x}_{p\text{-times}} = 0$, which asserts that a field has characteristic p. For $p > 0$ a prime, let $\mathrm{ACF}_p = \mathrm{ACF} \cup \{\psi_p\}$ and $\mathrm{ACF}_0 = \mathrm{ACF} \cup \{\neg\psi_p : p > 0\}$ be the theories of algebraically closed fields of characteristic p and zero respectively

Definition 1.10. Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence. We say that ϕ is a **logical consequence** of T and write $T \models \phi$ if $\mathcal{M} \models \phi$ whenever $\mathcal{M} \models T$

Proposition 1.11. *1. Let $\mathcal{L} = \{+, <, 0\}$ and let T be the theory of ordered abelian groups. Then $\forall x(x \neq 0 \to x + x \neq 0)$ is a logical consequence of T*

2. Let T be the theory of groups where every element has order 2. Then $T \not\models \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_1 \neq x_3)$

Proof. 1. $\mathbb{Z}/2\mathbb{Z} \models T \wedge \neg\exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_1 \neq x_3)$

□

1.3 Definable Sets and Interpretability

Definition 1.12. Let $\mathcal{M} = (M, \dots)$ be an $\mathcal{L}$-structure. We say that $X \subseteq M^n$ is **definable** if and only if there is an $\mathcal{L}$-formula $\phi(v_1, \dots, v_n, w_1, \dots, w_m)$ and $\bar{b} \in M^b$ s.t. $X = \{\bar{a} \in M^n : \mathcal{M} \models \phi(\bar{a}, \bar{b})\}$. We say that $\phi(\bar{v}, \bar{b})$ **defines** X. We say that X is A**-definable** or **definable over** A if there is a formula $\psi(\bar{v}, w_1, \dots, w_l)$ and $\bar{b} \in A^l$ s.t. $\psi(\bar{v}, \bar{b})$ defines X

A number of examples using $\mathcal{L}_r$, the language of rings

- Let $\mathcal{M} = (R, +, -, \cdot, 0, 1)$ be a ring. Let $p(X) \in R[X]$. Then $Y = \{x \in R : p(x) = 0\}$ is definable. Suppose that $p(X) = \sum_{i=0}^{m} a_i X^i$. Let $\phi(v, w_0, \dots, w_n)$ be the formula

$$w_n \cdot \underbrace{v \cdots v}_{n\text{-times}} + \cdots + w_1 \cdot v + w_0 = 0$$

  Then $\phi(v, a_0, \dots, a_n)$ defines Y. Indeed, Y is A-definable for any $A \supseteq \{a_0, \dots, a_n\}$
- Let $\mathcal{M} = (\mathbb{R}, +, -, \cdot, 0, 1)$ be the field of real numbers. Let $\phi(x, y)$ be the formula

$$\exists z (z \neq 0 \wedge y = x + z^2)$$

  Because $a < b$ if and only if $\mathcal{M} \models \phi(a, b)$, the ordering is $\emptyset$-definable
- Consider the natural numbers $\mathbb{N}$ as an $\mathcal{L} = \{+, \cdot, 0, 1\}$ structure. There is an $\mathcal{L}$-formula $T(e, x, s)$ s.t. $\mathbb{N} \models T(e, x, s)$ if and only if the Turing machine with program coded by e halts on input x in at most s steops. Thus the Turing machine with program e halts on input x if and only if

$\mathbb{N} \models \exists s\, T(e, x, s)$. So the halting computations is definable

Proposition 1.13. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Suppose that D_n is a collection of subsets of M^n for all $n \geq 1$ and $\mathcal{D} = (D_n : n \geq 1)$ is the smallest collection s.t.*

1. *$M^n \in D_n$*
2. *for all n-ary function symbols f of $\mathcal{L}$, the graph of $f^{\mathcal{M}}$ is in D_{n+1}*
3. *for all n-ary relation symbols R of $\mathcal{L}$, $R^{\mathcal{M}} \in D_n$*
4. *for all $i, j \leq n$, $\{(x_1, \dots, x_n) \in M^n : x_i = x_j\} \in D_n$*
5. *if $X \in D_n$, then $M \times X \in D_{n+1}$*
6. *each D_n is cloed under complement, union and intersection*
7. *if $X \in D_{n+1}$ and $\pi : M^{n+1} \to M^n$ is the projection $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$, then $\pi(X) \in D_n$*
8. *if $X \in D_{n+m}$ and $b \in M^m$, then $\{a \in M^n : (a, b) \in X\} \in D_n$*

Thus $X \subseteq M^n$ is definable if and only if $X \in D_n$

Proposition 1.14. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. If $X \subset M^n$ is A-definable, then every $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes A pointwise fixes X setwise(that is, if σ is an automorphism of M and $\sigma(a) = a$ for all $a \in A$, then $\sigma(X) = X$)*

Proof.

$$\mathcal{M} \models \psi(\bar{v}, \bar{a}) \leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{v}), \sigma(\bar{a})) \leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{v}), \bar{a})$$

In other words, $\bar{b} \in X$ if and only if $\sigma(\bar{b}) \in X$ □

Definition 1.15. A subset S of a field L is **algebraically independent** over a subfield K if the elements of S do not satisfy any non-trivial polynomial equation with coefficients in K

Corollary 1.16. *The set of real numbers is not definable in the field of complex numbers*

Proof. If $\mathbb{R}$ where definable, then it would be definable over a finite $A \subset \mathbb{C}$. Let $r, s \in \mathbb{C}$ be algebraically independent over A with $r \in \mathbb{R}$ and $s \notin \mathbb{R}$. There is an automorphism σ of $\mathbb{C}$ s.t. $\sigma|A$ is the identity and $\sigma(r) = s$. Thus $\sigma(\mathbb{R}) \neq \mathbb{R}$ and $\mathbb{R}$ is not definable over A □

We say that an $\mathcal{L}_0$-structure $\mathcal{N}$ is **definably interpreted** in an $\mathcal{L}$-structure $\mathcal{M}$ if and only if we can find a definable $X \subseteq M^n$ for some n and we can interpret the symbols of $\mathcal{L}_0$ as definable subsets and functions on X so that the resulting $\mathcal{L}_0$-structure is isomorphic to $\mathcal{M}$

For example, let K be a field and G be $\mathrm{GL}_2(K)$, the group of invertible 2×2 matrices over K. Let $X = \{(a, b, c, d) \in K^4 : ad - bc \neq 0\}$. Let $f : X^2 \to X$ by

$$\begin{aligned} &f((a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2)) = \\ &\qquad (a_1a_2 + b_1c_2, a_1b_2 + b_1d_2, c_1a_2 + d_1c_2, c_1b_2 + d_1d_2) \end{aligned}$$

X and f are definable in $(K, +, \cdot)$, and the set X with operation f is isomorphic to $\mathrm{GL}_2(K)$, where the identity element of X is $(1, 0, 0, 1)$

Clearly, $(\mathrm{GL}_n(K), \cdot, e)$ is definably interpreted in $(K, +, \cdot, 0, 1)$. A **linear algebraic group** over K is a subgroup of $\mathrm{GL}_n(K)$ defined by polynomial equations over K. Any linear algebraic group over K is definably interpreted in K

Let F be an infinite field and let G be the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a, b \in F, a \neq 0$. This group is isomorphic to the group of affine transformations $x \mapsto ax + b$, where $a, b \in F$ and $a \neq 0$

We will show that F is definably interpreted in the group G. Let

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}$$

where $\tau \neq 0$. Let

$$A = \{g \in G : g\alpha = \alpha g\} = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F\}$$

$$B = \{g \in G : g\beta = \beta g\} = \{\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} : x \neq 0\}$$

Clearly A, B are definable using parameters α and β

B acts on A by conjugation

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{y}{x} \\ 0 & 1 \end{pmatrix}$$

We can define the map $i : A\backslash\{1\} \rightarrow B$ by $i(a) = b$ if and only if $b^{-1}ab = \alpha$, that is

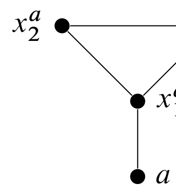
$$i \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

Define an operation $*$ on A by

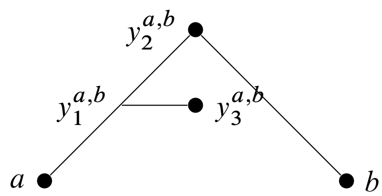
$$a * b = \begin{cases} i(b)a(i(b))^{-1} & \text{if } b \neq I \\ 1 & \text{if } b = I \end{cases}$$

where I is the identity matrix. Now $(F, +, \cdot, 0, 1) \cong (A, \cdot, *, 1, \alpha)$

Very complicated structures can often be interpreted in seemingly simpler ones. For example, any structure in a countable language can be interpreted in a graph. Let $(A, <)$ be a linear order. For each $a \in A$, G_A will have vertices a, x_1^a, x_2^a, x_3^a and contain the subgraph

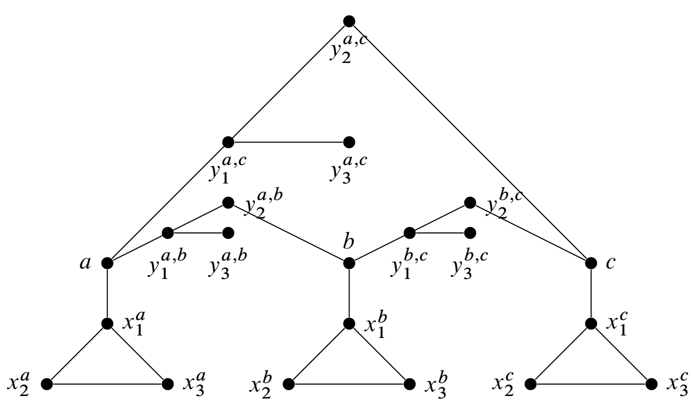


If $a < b$, then G_A will have vertices $y_1^{a,b}, y_2^{a,b}, y_3^{a,b}$ and contain the subgraph



Let $V_A = A \cup \{x_1^a, x_2^a, x_3^a : a \in A\} \cup \{y_1^{a,b}, y_2^{a,b}, y_3^{a,b} : a, b \in A \text{ and } a < b\}$, and let R_A be the smallest symmetric relation containing all edges drawn above.

For example, if A is the three-element linear order $a < b < c$, then G_A is the graph



Let $\mathcal{L} = \{R\}$ where R is a binary relation. Let $\phi(x, u, v, w)$ be the formula asserting that x, u, v, w are distinct, there are edges (x, u), (u, v), (v, w), (u, w) and these are the only edges involving u, v, w. $G_A \models \phi(a, x_1^a, x_2^a, x_3^a)$ for all $a \in A$.

$\psi(x, y, u, v, w)$ asserts that x, y, u, v, w are distinct. (x, u), (u, v), (u, w), (v, y)

Define $\theta_i(z)$ as follows:

$$\begin{aligned}
\theta_0(z) &:= \exists u \exists v \exists w\ \phi(z,u,v,w)\\
\theta_1(z) &:= \exists x \exists v \exists w\ \phi(x,z,v,w)\\
\theta_2(z) &:= \exists u \exists u \exists w\ \phi(x,u,z,w)\\
\theta_3(z) &:= \exists x \exists y \exists v \exists w\ \psi(x,y,z,v,w)\\
\theta_4(z) &:= \exists x \exists y \exists u \exists w\ \psi(x,y,u,z,w)\\
\theta_5(z) &:= \exists x \exists y \exists u \exists v\ \psi(x,y,u,v,z)
\end{aligned}$$

If $a, b \in A$ and $a < b$, then

$$G_A \models \theta_0(a) \wedge \theta_1(x_1^a) \wedge \theta_2(x_2^a) \wedge \theta_2(x_3^a)$$

and

$$G_A \models \theta_3(y_1^{a,b}) \wedge \theta_4(y_2^{a,b}) \wedge \theta_5(y_3^{a,b})$$

Lemma 1.17. *If $(A, <)$ is a linear order, then for all vertices x in G, there is a unique $i \leq 5$ s.t. $G_A \models \theta_i(x)$*

Let T be the $\mathcal{L}$-theory with the following axioms

1. R is symmetric and irreflexive
2. for all x, exactly one θ_i holds
3. if $\theta_0(x)$ and $\theta_0(y)$ then $\neg R(x,y)$
4. if $\exists u \exists v \exists w\ \psi(x,y,u,v,w)$ then $\forall u_1 \forall v_1 \forall w_1 \neg\psi(y,x,u_1,v_1,w_1)$
5. if $\exists u \exists v \exists w\ \psi(x,y,u,v,w)$ and $\exists u \exists v \exists w\ \psi(y,z,u,v,w)$ then $\exists u \exists v \exists w\ \psi(x,z,u,v,w)$
6. if $\theta_0(x)$ and $\theta_0(y)$, then either $x = y$ or $\exists u \exists v \exists w\ \psi(x,y,u,v,w)$ or $\exists u \exists v \exists w\ \psi(y,x,u,v,w)$
7. if $\phi(x,u,v,w) \wedge \phi(x,u',v',w')$, then $u = u', v = v', w = w'$
8. if $\psi(x,y,u,v,w) \wedge \psi(x,y,u',v',w')$, then $u' = u, v = v', w = w'$

If $(A, <)$ is a linear order, then $G_A \models T$

Suppose $G \models T$. Let $X_G = \{x \in G : G \models \theta_0(x)\}$

Lemma 1.18. *If $(A, <)$ is a linear order, then $(X_{G_A}, <_{G_A}) \cong (A, <)$. Moreover, $G_{X_G} \cong G$ for all $G \models T$*

Definition 1.19. An $\mathcal{L}_0$-structure $\mathcal{N}$ is **interpretable** in an $\mathcal{L}$-structure M if there is a definable $X \subseteq M^n$, a definable equivalence relation E on X, and for each symbol of $\mathcal{L}_0$ we can find definable E-invariant sets on X s.t. X/E with the induced structure is isomorphic to $\mathcal{N}$

1.4 Answers to Exercises

Exercise 1.4.1.

1. transform ψ to CNF
2. prenex normal form

Exercise 1.4.2.

1. $\begin{array}{cc} s & rs \\ \bullet & \bullet \\ e & r \\ \bullet & \bullet \end{array}$
2. enumerate $\mathcal{M}$'s functions, relations and constants

Exercise 1.4.3. [1] Note that every $\mathcal{L}$-structure $\mathcal{M}$ of size κ is isomorphic to an $\mathcal{L}$-structure with domain κ. For each relation symbols, we have 2^κ options. If the language has size λ, this is at most $(2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^{\max(\lambda,\kappa)}$

Exercise 1.4.4.

$$\begin{aligned} T \models \phi &\Leftrightarrow \forall \mathcal{M}\ \mathcal{M} \models T \to \mathcal{M} \models \phi \\ &\Leftrightarrow \forall \mathcal{M}\ \mathcal{M} \models T' \to \mathcal{M} \models \phi \\ &\Leftrightarrow T' \models \phi \end{aligned}$$

Exercise 1.4.5. Follow the definition

Exercise 1.4.6. Since there is no model $\mathcal{M}$ s.t. $\mathcal{M} \models T$. It's true that $T \models \phi$

Exercise 1.4.7.

1. Suppose $\mathcal{M} \models \phi$, then $E^{\mathcal{M}}$ is an equivalent relation and each equivalence class's cardinality is 2
2. follows from number theory
3. [DJMM12]

Exercise 1.4.8. TBD

Exercise 1.4.9. $G(f) = \{(\bar{x},\bar{y}) \in M^{n+m} \mid \phi(\bar{x},\bar{y})\}$ and $G(g) = \{(\bar{y},\bar{z}) \in M^{m+l} \mid \psi(\bar{y},\bar{z})\}$. Hence $G(g \circ f) = \{(\bar{x},\bar{z}) \in M^{n+l} \mid \phi(\bar{x},\bar{y}) \wedge \psi(\bar{y},\bar{z})\}$

Exercise 1.4.10. $\phi(\bar{a},b)$ really defines a function and since $\phi(\bar{a},y) \to y = b$

2 Basic Techniques

2.1 The Compactness Theorem

Some points of proofs

- Proofs are finite
- (Soundness) If $T \vdash \phi$, then $T \models \phi$

[1]stackexchange

- If T is a finite set of sentences, then there is an algorithm that, when given a sequence of $\mathcal{L}$-formulas σ and an $\mathcal{L}$-sentence ϕ, will decide whether σ is a proof of ϕ from T

A language $\mathcal{L}$ is **recursive** if there is an algorithm that decides whether a sequence of symbols is an $\mathcal{L}$-formula. An $\mathcal{L}$-theory T is **recursive** if there is an algorithm that when given an $\mathcal{L}$-sentence ϕ as input, decides whether $\phi \in T$

Proposition 2.1. *If $\mathcal{L}$ is a recursive language and T is a recursive $\mathcal{L}$-theory, then $\{\phi : T \vdash \phi\}$ is recursively enumerable; that is, there is an algorithm that when given ϕ as input will halt accepting if $T \vdash \phi$ and not halt if $T \nvdash \phi$*

Proof. There is $\sigma_0, \sigma_1, \ldots$ a computable listing of all finite sequence of $\mathcal{L}$-formulas. At stage i, we check to see whether σ_i is a proof of ψ from T. If it is, then halt. $\square$

Theorem 2.2 (Gödel's Completeness Theorem)**.** *Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence, then $T \models \phi$ if and only if $T \vdash \phi$*

We say that an $\mathcal{L}$-theory T is **inconsistent** if $T \vdash (\phi \wedge \neg\phi)$ for some sentence ϕ.

Corollary 2.3. *T is consistent if and only if T is satisfiable*

Proof. Supose that T is not satisfiable, then every model of T is a model of $\phi \wedge \neg\phi$. Thus by the Completeness theorem $T \vdash (\phi \wedge \neg\phi)$ $\square$

Theorem 2.4 (Compactness Theorem)**.** *T is satisfiable if and only if every finite subset of T is satisfiable*

Proof. If T is not satisfiable, then T is inconsistent. Let σ be a proof of a contradiction from T. Because σ is finite, only finitely many assumptions from T are used in the proof. Thus there is a finite $T_0 \subseteq T$ s.t. σ is a proof of a contradiction from T_0 $\square$

2.1.1 Henkin Constructions

A theory T is **finitely satisfiable** if every finite subset of T is satisfiable. We will show that every finitely satisfiable theory T is satisfiable.

Definition 2.5. We say that an $\mathcal{L}$-theory T has the **witness property** if whenever $\phi(v)$ is an $\mathcal{L}$-formula with one free variable v, then there is a constant symbol $c \in \mathcal{L}$ s.t. $T \vdash (\exists v \phi(v)) \to \phi(c) \in T$

An $\mathcal{L}$-theory T is **maximal** if for all ϕ either $\phi \in T$ or $\neg\phi \in T$

Lemma 2.6. *Suppose T is a maximal and finitely satisfiable $\mathcal{L}$-theory. If $\Delta \subseteq T$ is finite and $\Delta \models \psi$, then $\psi \in T$*

Proof. If $\psi \notin T$, then $\neg\psi \in T$ but $\Delta \cup \{\psi\}$ is unsatisfiable $\square$

Lemma 2.7. *Suppose that T is a maximal and finitely satisfiable $\mathcal{L}$-theory with the witness property. Then T has a model. In fact, if κ is a cardinal and $\mathcal{L}$ has at most κ constant symbols, then there is $\mathcal{M} \models T$ with $|\mathcal{M}| \leq \kappa$*

Proof. Let $\mathcal{C}$ be the set of constant symbols of $\mathcal{L}$. For $c, d \in \mathcal{C}$, we say $c \sim d$ if $c = d \in T$

Claim 1 $\sim$ is an equivalence relation.

The universe of our model will be $M = \mathcal{C}/\sim$. Clearly $|M| \leq \kappa$. We let c^* denote the equivalence class of c and interprete c as its equivalence class, that is, $c^{\mathcal{M}} = c^*$

Suppose that R is an n-ary relation symbol of $\mathcal{L}$

Claim 2 Suppose that $c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathcal{C}$ and $c_i \sim d_i$ for $i = 1, \ldots, n$, then $R(\bar{c})$ if and only if $R(\bar{d})$

By Lemma 2.6, if one of $R(\bar{c})$ and $R(\bar{d})$ is in T, then both are in T

$$R^{\mathcal{M}} = \{(c_1^*, \ldots, c_n^*) : R(c_1, \ldots, c_n) \in T\}$$

Suppose that f is an n-ary function symbol of $\mathcal{L}$ and $c_1, \ldots, c_n \in \mathcal{C}$. Because $\underline{\emptyset \models \exists v f(c_1, \ldots, c_n) = v}$, and T has the witness property, then there is $c_{n+1} \in \mathcal{C}$ s.t. $f(c_1, \ldots, c_n) = c_{n+1} \in T$. As above, if $d_i \sim c_i$ for $i = 1, \ldots, n+1$, then $f(d_1, \ldots, d_n) = d_{n+1} \in T$. Thus we get a well-defined function $f^{\mathcal{M}} : M^n \to M$ by

$$f^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^* \text{ if and only if } f(c_1, \ldots, c_n) = d \in T$$

Claim 3 Suppose that t is a term using free variables from $v_1, \ldots, v_n$. If $c_1, \ldots, c_n, d \in \mathcal{C}$, then $t(c_1, \ldots, c_n) = d \in T$ if and only if $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$

($\Rightarrow$) If t is a constant symbol, then $c = d \in T$ and $c^{\mathcal{M}} = c^* = d^*$

If t is the variable v_i, then $c_i = d \in T$ and $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = c_i^* = d^*$

Suppose that the claim is true for $t_1, \ldots, t_m$ and t is $f(t_1, \ldots, t_m)$. Using the witness property and Lemma 2.6, we can find $d, d_1, \ldots, d_n \in \mathcal{C}$ s.t. $t_i(c_1, \ldots, c_n) = d_i \in T$ for $i \leq m$ and $f(d_1, \ldots, d_m) = d \in T$. By our induction hypothesis, $t_i^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d_i^*$ and $f^{\mathcal{M}}(d_1^*, \ldots, d_m^*) = d^*$. Thus $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$

($\Leftarrow$) Suppose $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$. By the witness property, there is a $e \in \mathcal{C}$ s.t. $t(c_1, \ldots, c_n) = e \in T$. Using the ($\Rightarrow$) direction of the proof, $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = e^*$. Thus $e^* = d^*$ and $e = d \in T$

Claim 4 For all $\mathcal{L}$-formulas $\phi(v_1, \ldots, v_n)$ and $c_1, \ldots, c_n \in \mathcal{C}$, $\mathcal{M} \models \phi(\bar{c}^*)$ if and only if $\phi(\bar{c}) \in T$

Suppose that ϕ is $t_1 = t_2$. By Lemma 2.6 and the witness property, we can find d_1 and d_2 s.t. $t_1(\bar{c}) = d_1, t_2(\bar{c}) = d_2 \in T$. By Claim 3, $t_i^{\mathcal{M}}(\bar{c}^*) = d_i^*$. Then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{c}^*) &\Leftrightarrow d_1^* = d_2^* \\ &\Leftrightarrow d_1 = d_2 \in T \\ &\Leftrightarrow t_1(\bar{c}) = t_2(\bar{c}) \in T \end{aligned}$$

Suppose that ϕ is $R(t_1, \ldots, t_m)$. There are $d_1, \ldots, d_m \in \mathcal{C}$ s.t. $t_i(\bar{c}) = d_i \in T$. Thus

$$\begin{aligned} \mathcal{M} \models \phi(\bar{c}^*) &\Leftrightarrow \bar{d}^* \in R^{\mathcal{M}} \\ &\Leftrightarrow R(\bar{d}) \in T \\ &\Leftrightarrow \phi(\bar{c}) \in T \end{aligned}$$

Suppose that the claim is true for ϕ. If $\mathcal{M} \models \neg\phi(\bar{c}^*)$, then $\mathcal{M} \not\models \phi(\bar{c}^*)$. By the inductive hypothesis, $\phi(\bar{c}) \notin T$. Thus by maximality, $\neg\phi(\bar{c}) \in T$. On the other hand, if $\neg\phi(\bar{c}) \in T$, then because T is finitely satisfiable, $\phi(\bar{c}) \notin T$. Thus, by induction, $\mathcal{M} \not\models \phi(\bar{c}^*)$. $\square$

Lemma 2.8. *Let T be a finitely satisfiable $\mathcal{L}$-theory. There is a language $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ a finitely satisfiable $\mathcal{L}^*$-theory s.t. any $\mathcal{L}^*$-theory extending T^* has the witness property. We can choose $\mathcal{L}^*$ s.t. $\left|\mathcal{L}^*\right| = |\mathcal{L}| + \aleph_0$*

Proof. We first show that there is a language $\mathcal{L}_1 \supseteq \mathcal{L}$ and a finitely satisfiable $\mathcal{L}_1$-theory $\mathcal{L}_1 \supseteq T$ s.t. for any $\mathcal{L}$-formula $\phi(v)$ there is an $\mathcal{L}_1$-constant symbol c s.t. $T_1 \models (\exists v \phi(v)) \to \phi(c)$. For each $\mathcal{L}$-formula $\phi(v)$, let c_ϕ be a new constant symbol and let $\mathcal{L}_1 = \mathcal{L} \cup \{c_\phi : \phi(v) \text{ an } \mathcal{L}\text{-formula}\}$. For each $\mathcal{L}$-formula $\phi(v)$, let Θ_ϕ be the $\mathcal{L}_1$-sentence $(\exists v \phi(v)) \to \phi(c_\phi)$. Let $T_1 = T \cup \{\Theta_\phi : \phi(v) \text{ an } \mathcal{L}\text{-formula}\}$

Claim T_1 is finitely satisfiable

Suppose that Δ is a finite subset of T_1. Then $\Delta = \Delta_0 \cup \{\Theta_{\phi_1}, \ldots, \Theta_{\phi_n}\}$ where Δ_0 is a finite subset of T and there is $\mathcal{M} \models \Delta_0$. We will make $\mathcal{M}$ into an $\mathcal{L} \cup \{c_{\phi_1}, \ldots, c_{\phi_n}\}$-structure $\mathcal{M}'$. If $\mathcal{M} \models \exists v \phi(v)$, choose a_i some element of M s.t. $\mathcal{M} \models \phi(a_i)$ and let $c_{\phi_i}^{\mathcal{M}'} = a_i$. Otherwise, let $c_{\phi_i}^{\mathcal{M}'}$ be any element of $\mathcal{M}$. Clearly $\mathcal{M}' \models \Theta_{\phi_i}$ for $i \leq n$. Thus T_1 is finitely satisfiable.

We now iterate the construction above to build a sequence of languages $\mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \ldots$ and a sequence of finitely satisfiable $\mathcal{L}_i$-theories $T \subseteq$

$T_1 \subseteq T_2 \subseteq \ldots$ s.t. if $\phi(v)$ is an $\mathcal{L}_i$-formula then there is a constant symbol $c \in \mathcal{L}_{i+1}$ s.t. $T_{i+1} \models (\exists v \phi(v)) \to \phi(c)$

Let $\mathcal{L}^* = \bigcup \mathcal{L}_i$ and $T^* = \bigcup T_i$. If $|\mathcal{L}_i|$ is the number of relation, function and constant symbols in $\mathcal{L}_i$, then there are at most $|\mathcal{L}_i| + \aleph_0$ formulas in $\mathcal{L}_i$. Thus by induction, $\left|\mathcal{L}^*\right| = |\mathcal{L}| + \aleph_0$ □

Lemma 2.9. *Suppose that T is a finitely satisfiable $\mathcal{L}$-theory and ϕ is an $\mathcal{L}$-sentence, then either $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely satisfiable*

Corollary 2.10. *If T is a finitely satisfiable $\mathcal{L}$-theory, then there is a maximal finitely satisfiable $\mathcal{L}$-theory $T' \supseteq T$*

Proof. Let I be the set of all finitely satisfiable $\mathcal{L}$-theory containing T. We partially order I by inclusion. If $C \subseteq I$ is a chain, let $T_C = \bigcup\{\Sigma : \Sigma \in C\}$. If Δ is a finite subset of T_C, then there is a $\Sigma \in C$ s.t. $\Delta \subseteq \Sigma$, so T_C is finitely satisfiable and $T_C \supseteq \Sigma$ for all $\Sigma \in C$. Thus every chain in I has an upper bound, and we can apply Zorn's lemma to find a $T' \in I$ maximal w.r.t. the partial order. □

Theorem 2.11 (strengthening of Compactness Theorem)**.** *If T is a finitely satisfiable $\mathcal{L}$-theory and κ is an infinite cardinal with $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality at most κ*

Proof. By Lemma 2.8, we can find $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ a finitely satisfiable $\mathcal{L}^*$-theory s.t. any $\mathcal{L}^*$-theory extending T^* has the witness property and the cardinality of $\mathcal{L}^*$ is at most κ. By Corollary 2.10, we can find a maximal finitely satisfiable $\mathcal{L}^*$-theory $T' \supseteq T^*$. Because T' has the witness property, Lemma 2.7 ensures that there is $\mathcal{M} \models T$ with $|M| \leq \kappa$ □

Proposition 2.12. *Let $\mathcal{L} = \{\cdot, +, <, 0, 1\}$ and let* Th$(\mathbb{N})$ *be the full $\mathcal{L}$-theory of the natural numbers. There is $\mathcal{M} \models$* Th$(\mathbb{N})$ *and $a \in M$ s.t. a is larger than every natural number*

Proof. Let $\mathcal{L}^* = \mathcal{L} \cup \{c\}$ where c is a new constant symbol and let

$$T = \mathrm{Th}(\mathbb{N}) \cup \{\underbrace{1 + 1 + \cdots + 1}_{n\text{-times}} < c : \text{for } n = 1, 2, \ldots\}$$

If Δ is a finite subset of T we can make $\mathbb{N}$ a model of Δ by interpreting c as a suitably large natural number. Thus T is finitely satisfiable and there is $\mathcal{M} \models T$. □

Lemma 2.13. *If $T \models \phi$, then $\Delta \models T$ for some finite $\Delta \subseteq T$*

Proof. Suppose not. Let $\Delta \subseteq T$ be finite. Because $\Delta \not\models \phi$, $\Delta \cup \{\neg\phi\}$ is satisfiable. Thus $T \cup \{\neg\phi\}$ is finitely satisfiable and by the compactness theorem, $T \not\models \phi$ $\square$

2.2 Complete Theories

Definition 2.14. An $\mathcal{L}$-theory T is called **complete** if for any $\mathcal{L}$-sentence ϕ either $T \models \phi$ or $T \models \neg\phi$

For $\mathcal{M}$ an $\mathcal{L}$-structure, then the full theory

$$\text{Th}(\mathcal{M}) = \{\phi : \phi \text{ is an } \mathcal{L}\text{-sentence and } \mathcal{M} \models \phi\}$$

is a complete theory.

Proposition 2.15. *Let T be an $\mathcal{L}$-theory with infinite models. If κ is an infinite cardinal and $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality κ*

Proof. Let $\mathcal{L}^* = \mathcal{L} \cup \{c_\alpha : \alpha < \kappa\}$, where each c_α is new constant symbol, and let T^* be the $\mathcal{L}^*$-theory $T \cup \{c_\alpha \neq c_\beta : \alpha, \beta < \kappa, \alpha \neq \beta\}$. Clearly if $\mathcal{M} \models T^*$, then $\mathcal{M}$ is a model of T of cardinality at least κ. Thus by Theorem 2.11, it suffices to show that T^* is finitely satisfiable. But if $\Delta \subseteq T^*$ is finite, then $\Delta \subseteq T \cup \{c_\alpha \neq c_\beta : \alpha \neq \beta, \alpha, \beta \in I\}$, where I is a finite subset of κ. Let $\mathcal{M}$ be an infinite model of T. We can interpret the symbols $\{c_\alpha : \alpha \in I\}$ as $|I|$ distinct elements of M. Because $\mathcal{M} \models \Delta$, T^* is finitely satisfiable. $\square$

Definition 2.16. Let κ be an infinite cardinal and let T be a theory with models of size κ. We say that T is **κ-categorical** if any two models of T of cardinality κ are isomorphic.

Let $\mathcal{L} = \{+, 0\}$ be the language of additive groups and let T be the $\mathcal{L}$-theory of torsion-free divisible Abelian groups. The axioms of T are the axioms for Abelian groups together with the axioms

$$\forall x (x \neq 0 \rightarrow \underbrace{x + \cdots + x}_{n\text{-times}} \neq 0)$$

$$\forall y \exists x \underbrace{x + \cdots + x}_{n\text{-times}} = y$$

for $n = 1, 2, \ldots$

Proposition 2.17. *The theory of torsion-free divisible Abelian groups is κ-categorical for all $\kappa > \aleph_0$*

Proof. We first argue that models of T are essentially vector spaces over the field of rational numbers $\mathbb{Q}$. If V is any vector space over $\mathbb{Q}$, then the underlying additive group V is a model of T. Check StackExchange. On the other hand, if $G \models T$, $g \in G$ and $n \in \mathbb{N}$ with $g > 0$, we can find $h \in G$ s.t. $nh = g$. If $nk = g$, then $n(h - k) = 0$. Because G is torsion-free there is a unique $h \in G$ s.t. $nh = g$. We call this element g/n. We can view G as a $\mathbb{Q}$-vector space under the action $\frac{m}{n}g = m(g/n)$

Two $\mathbb{Q}$-vector spaces are isomorphic if and only if they have the same dimension. Thus the model of T are determined up to isomorphism by their dimension. If G has dimension λ, then $|G| = \lambda + \aleph_0$. If κ is uncountable and G has cardinality κ, then G has dimension κ. Thus for $\kappa > \aleph_0$ any two models of T of cardinality κ are isomorphic □

Lemma 2.18. *Field of uncountable cardinality κ has transcendence degree κ* [2]

Proof. We prove the theorem for fields with characteristic $p = 0$.

Since each characteristic 0 field contains a copy of $\mathbb{Q}$ as its prime field, we can view F as a field extension over $\mathbb{Q}$. We will show that F has a subset of cardinality κ which is algebraically independent over $\mathbb{Q}$.

We build the claimed subset of F by transfinite induction and implicit use of the axiom of choice.

Let $S_0 = \emptyset$

Let S_1 be a singleton containing some element of F which is not algebraic over $\mathbb{Q}$. This is possible since algebraic numbers are countable

Define $S_{\alpha+1}$ to be S_α together with an element of F which is not a root of any non-trivial polynomial with coefficients in $\mathbb{Q} \cup S_\alpha$ since there are only $|\mathbb{Q} \cup S_\alpha| = \aleph_0 + |\alpha| < \kappa$ polynomials

Define $S_\beta = \bigcup_{\alpha<\beta} S_\alpha$

Let $P(x_1, \ldots, x_n)$ be a non-trivial polynomial with coefficients in $\mathbb{Q}$ and elements $a_1, \ldots, a_n$ in F. W.L.O.G., it is assumed that a_n was added at an ordinal $\alpha + 1$ later than the other elements. Then $P(a_1, \ldots, a_{n-1}, x_n)$ is a polynomial with coefficients in $\mathbb{Q} \cup S_\alpha$. Hence $P(a_1, \ldots, a_n) \neq 0$. □

Proposition 2.19. *ACF_p is κ-categorical for all uncountable cardinals κ*

Proof. Two algebraically closed fields are isomorphic if and only if they have the same characteristic and transcendence degree. See `AdvancedModernAlgebra.org`. By Lemma 2.18, an algebraically closed field of transcendence degree λ has cardinality $\lambda + \aleph_0$. □

[2]proofwiki

Theorem 2.20 (Vaught's Test)**.** *Let T be a satisfiable theory with no finite models that is κ-categorical for some infinite cardinal $\kappa \geq |\mathcal{L}|$. Then T is complete*

Proof. Suppose T is not complete. Then there is a sentence ϕ s.t. $T \not\models \phi$ and $T \not\models \neg\phi$. Because $T \not\models \psi$ if and only if $T \cup \{\neg\psi\}$ is satisfiable, the theories $T_0 = T \cup \{\phi\}$ and $T_1 = T \cup \{\neg\phi\}$ are satisfiable. Because T has no finite models, both T_0 and T_1 have infinite models. By Proposition 2.15 we can find $\mathcal{M}_0$ and $\mathcal{M}_1$ of cardinality κ with $\mathcal{M}_i \models T_i$. Because $\mathcal{M}_0$ and $\mathcal{M}_1$ disagree about ϕ, they are not elementarily equivalent, and hence by Theorem 1.9, nonisomorphic. $\square$

Definition 2.21. We say that an $\mathcal{L}$-theory T is **decidable** if there is an algorithm that when given an $\mathcal{L}$-sentence ϕ as input decides whether $T \models \phi$

Lemma 2.22. *Let T be a recursive complete satisfiable theory in a recursive language $\mathcal{L}$. Then T is decidable*

Proof. Because T is satisfiable $A = \{\phi : T \models \phi\}$ and $B = \{\phi : T \models \neg\phi\}$ are disjoint. Because T is consistent $A \cup B$ is the set of all $\mathcal{L}$-sentences. By the Completeness Theorem, $A = \{\phi : T \vdash \phi\}$ and $B = \{\phi : T \vdash \neg\phi\}$. By Proposition 2.1 A and B are recursively enumerable. But any recursively enumerable set with a recursively enumerable complement is recursive. $\square$

Corollary 2.23. *For $p = 0$ or p prime, ACF_p is decidable. In particular,* $\mathrm{Th}(\mathbb{C})$*, the first-order theory of the field of complex numbers, is decidable*

Corollary 2.24. *Let ϕ be a sentence in the language of rings. The following are equivalent*

1. *ϕ is true in the complex number*
2. *ϕ is true in every algebraically closed field of characteristic zero*
3. *ϕ is true in some algebraically closed field of characteristic zero*
4. *There are arbitrarily large primes p s.t. ϕ is true in some algebraically closed field of characteristic p*
5. *There is an m s.t. for all $p > m$, ϕ is true in all algebraically closed fields of characteristic p*

Proof. By Proposition 2.19 and Vaught's Test, ACF_p is complete.

(2) $\to$ (5). Suppose that $\mathrm{ACF}_0 \models \phi$. By Lemma 2.13, there is a finite $\Delta \subseteq \mathrm{ACF}_0$ s.t. $\Delta \models \phi$. Thus if we choose p large enough, then $\mathrm{ACF}_p \models \Delta$.

(4) $\to$ (2). Suppose $\mathrm{ACF}_0 \not\models \phi$. Because ACF_0 is complete, $\mathrm{ACF}_0 \models \neg\phi$. $\square$

2.3 Up and Down

Definition 2.25. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures, then an $\mathcal{L}$-embedding $j : \mathcal{M} \to \mathcal{N}$ is called an **elementary embedding** if

$$\mathcal{M} \models \phi(a_1, \ldots, a_n) \leftrightarrow \mathcal{N} \models \phi(j(a_1), \ldots, j(a_n))$$

for all $\mathcal{L}$-formulas $\phi(v_1, \ldots, v_n)$ and all $a_1, \ldots, a_n \in M$

If $\mathcal{M}$ is a substructure of $\mathcal{N}$, we say that it is an **elementary substructure** and write $\mathcal{M} \prec \mathcal{N}$ if the inclusion map is elementary. $\mathcal{N}$ is an **elementary extension** of $\mathcal{M}$

Definition 2.26. $\mathcal{M}$ is an $\mathcal{L}$-structure. Let $\mathcal{L}_M$ be the language where we add to $\mathcal{L}$ constant symbols m for each element of M. The **atomic diagram** of $\mathcal{M}$ is $\{\phi(m_1, \ldots, m_n) : \phi$ is either an atomic $\mathcal{L}$-formula or the negation of an atomic $\mathcal{L}$-formula and $\mathcal{M} \models \phi(m_1, \ldots .m_n)\}$. The **elementary diagram** of $\mathcal{M}$ is

$$\{\phi(m_1, \ldots, m_n) : \mathcal{M} \models \phi(m_1, \ldots, m_n), \phi \text{ an } \mathcal{L}\text{-formula}\}$$

We let $\mathrm{Diag}(\mathcal{M})$ and $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ denote the atomic and elementary diagrams of $\mathcal{M}$

Lemma 2.27.
1. *Suppose that $\mathcal{N}$ is an $\mathcal{L}_M$-structure and $\mathcal{N} \models \mathrm{Diag}(\mathcal{M})$, then viewing $\mathcal{N}$ as an $\mathcal{L}$-structure, there is an $\mathcal{L}$-embedding of $\mathcal{M}$ into $\mathcal{N}$*
2. *If $\mathcal{N} \models \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$, then there is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$*

Proof.
1. Let $j : M \to N$ by $j(m) = m^{\mathcal{N}}$. If $m_1 \neq m_2 \in \mathrm{Diag}(\mathcal{M})$; thus $j(m_1) \neq j(m_2)$ so j is an embedding. If f is a function symbols of $\mathcal{L}$ and $f^{\mathcal{M}}(m_1, \ldots, m_n) = m_{n+1}$, then $f(m_1, \ldots, m_n) = m_{n+1}$ is a formula in $\mathrm{Diag}(\mathcal{M})$ and $f^{\mathcal{N}}(j(m_1), \ldots, j(m_n)) = j(m_{n+1})$. If R is a relation symbol and $\bar{m} \in R^{\mathcal{M}}$, then $R(m_1, \ldots, m_n) \in \mathrm{Diag}(\mathcal{M})$ and $(j(m_1), \ldots, j(m_n)) \in R^{\mathcal{N}}$. Hence j is an $\mathcal{L}$-embedding
2. j is elementary.

$\square$

Theorem 2.28 (Upward LöwenheimSkolem Theorem)**.** *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure and κ be an infinite cardinal $\kappa \geq |\mathcal{M}| + |\mathcal{L}|$. Then, there is $\mathcal{N}$ an $\mathcal{L}$-structure of cardinality κ and $j : \mathcal{M} \to \mathcal{N}$ is elementary*

Proof. Because $\mathcal{M} \models \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$, $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ is satisfiable. By Theorem 2.11, there is $\mathcal{N} \models \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ of cardinality κ. By Lemma 2.27, there is an elementary $j : \mathcal{M} \to \mathcal{N}$ $\square$

Proposition 2.29 (Tarski-Vaught Test)**.** *Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then $\mathcal{M}$ is an elementary substructure if and only if, for any formula $\phi(v, \bar{w})$ and $\bar{a} \in M$, if there is $b \in N$ s.t. $\mathcal{N} \models \phi(b, \bar{a})$, then there is $c \in M$ s.t. $\mathcal{N} \models \phi(c, \bar{a})$*

Proof. We need to show that for all $\bar{a} \in M$ and all $\mathcal{L}$-formulas $\psi(\bar{v})$

$$\mathcal{M} \models \psi(\bar{a}) \Leftrightarrow \mathcal{N} \models \psi(\bar{a})$$

In Proposition 1.7, we showed that if $\phi(\bar{v})$ is quantifier free then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\phi(\bar{a})$ □

We say that an $\mathcal{L}$-theory T has **built-in Skolem functions** if for all $\mathcal{L}$-formulas $\phi(v, w_1, \ldots, w_n)$ there is a function symbol f s.t.
$T \models \forall \bar{w}((\exists v \phi(v, \bar{w})) \to \phi(f(\bar{w}), \bar{w}))$. In other words, there are enough function symbols in the language to witness all existential statements.

Lemma 2.30. *Let T be an $\mathcal{L}$-theory. There are $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ an $\mathcal{L}^*$-theory s.t. T^* has built-in Skolem functions, and if $\mathcal{M} \models T$, then we can expand $\mathcal{M}$ to $\mathcal{M}^* \models T^*$. We can choose $\mathcal{L}^*$ s.t. $\left|\mathcal{L}^*\right| = |\mathcal{L}| + \aleph_0$.*

*We call T^* a **skolemization** of T*

Proof. We build a sequence of languages $\mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \ldots$ and $\mathcal{L}_i$-theories T_i s.t. $T = T_0 \subseteq T_1 \subseteq \ldots$

Given $\mathcal{L}_i$, let $\mathcal{L}_{i+1} = \mathcal{L} \cup \{f_\phi : \phi(v, w_1, \ldots, w_n) \text{ an } \mathcal{L}_i\text{-formula}, n = 1, 2, \ldots\}$, where f_ϕ is an n-ary function symbol. For $\phi(v, \bar{w})$ an $\mathcal{L}_i$-formula, let Ψ_ϕ be the sentence

$$\forall \bar{w}((\exists v \phi(v, \bar{w})) \to \phi(f_\phi(\bar{w}), \bar{w}))$$

and let $T_{i+1} = T_i \cup \{\Psi_\phi : \phi \text{ an } \mathcal{L}_i\text{-formula}\}$

Claim If $\mathcal{M} \models T_i$, then we can interpret the function symbols of $\mathcal{L}_{i+1} \backslash \mathcal{L}_i$ so that $\mathcal{M} \models T_{i+1}$

Let c be some fixed element of M. If $\phi(v, w_1, \ldots, w_n)$ is an $\mathcal{L}_i$-formula, we find a function $g : M^n \to M$ s.t. $\bar{a} \in M^n$ and $X_{\bar{a}} = \{b \in M : \mathcal{M} \models \phi(b, \bar{a})\}$ is nonempty, then $g(\bar{a}) \in X_{\bar{a}}$, and if $X_{\bar{a}} = \emptyset$, then $g(\bar{a}) = c$. Thus if $\mathcal{M} \models \exists v \phi(v, \bar{a})$, then $\mathcal{M} \models \phi(g(\bar{a}), \bar{a})$. If we interpret f_ϕ as g, then $\mathcal{M} \models \Psi_\phi$

Let $\mathcal{L}^* = \bigcup \mathcal{L}_i$ and $T^* = \bigcup T_i$. If $\phi(v, \bar{w})$ is an $\mathcal{L}^*$-formula, then $\phi \in \mathcal{L}_i$ for some i and $\Psi_\phi \in T_{i+1} \subseteq T^*$, so T^* has built in Skolem functions. By iterating the claim, we see that for any $\mathcal{M} \models T$ we can interpret the symbols of $\mathcal{L}^* \backslash \mathcal{L}$ to make $\mathcal{M} \models T^*$

$|\mathcal{L}_{i+1}| = |\mathcal{L}_i| + \aleph_0$ □

Theorem 2.31 (LöwenheimSkolem Theorem)**.** *Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $X \subseteq M$, there is an elementary submodel $\mathcal{N}$ of $\mathcal{M}$ s.t. $X \subseteq N$ and $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$*

Proof. By Lemma 2.30, we may assume that $\mathrm{Th}(\mathcal{M})$ has built in Skolem functions (otherwise we may extend $\mathcal{L}$ to some $\mathcal{L}^*$). Let $X_0 = X$. Given X_i, let $X_{i+1} = X_i \cup \{f^{\mathcal{M}}(\bar{a}) : f$ an n-ary function symbol, $\bar{a} \in X_i^n, n = 1, 2, \dots\}$. Let $N = \bigcup X_i$, then $|N| \leq |X| + |\mathcal{L}| + \aleph_0$ If f is an n-ary function symbol of $\mathcal{L}$ and $\bar{a} \in N^n$, then $\bar{a} \in X_i^n$ for some i and $f^{\mathcal{M}}(\bar{a}) \in X_{i+1} \subseteq N$. Thus $f^{\mathcal{M}}|N : N^n \to N$. Thus we can interpret f as $f^{\mathcal{N}} = f^{\mathcal{M}}|N^n$. If R is an n-ary relation symbol, let $R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^n$. If c is a constant symbol of $\mathcal{L}$, there is a Skolem function $f \in \mathcal{L}$ s.t. $f(x) = c^{\mathcal{M}}$ for all $x \in M$ (for example, f is the Skolem function for the formula $v = c$) . Thus $c^{\mathcal{N}} \in N$

If $\phi(v, \bar{w})$ is any $\mathcal{L}$-formula, $\bar{a}, b \in M$ and $\mathcal{M} \models \phi(b, \bar{a})$, then $\mathcal{M} \models \phi(f(\bar{a}), \bar{a})$ for some function symbol f of $\mathcal{L}$. By construction, $f^{\mathcal{M}}(\bar{a}) \in N$. Thus by Proposition 2.29 $\mathcal{N} \prec \mathcal{M}$ □

Definition 2.32. A **universal sentence** is one of the form $\forall \bar{v} \phi(\bar{v})$, where ϕ is quantifier-free. We say that an $\mathcal{L}$-theory T has a **universal axiomatization** if there is a set of universal $\mathcal{L}$-sentences Γ s.t. $\mathcal{M} \models \Gamma$ if and only if $\mathcal{M} \models T$ for all $\mathcal{L}$-structures $\mathcal{M}$

Theorem 2.33. *An $\mathcal{L}$-theory T has a universal axiomatization if and only if whenever $\mathcal{M} \models T$ and $\mathcal{N}$ is a substructure of $\mathcal{M}$, then $\mathcal{N} \models T$. In other words, a theory is preserved under substructure if and only if it has a universal axiomatization*

Proof. Suppose that $\mathcal{N} \subseteq \mathcal{M}$. By Proposition 1.7, if $\phi(\bar{v})$ is a quantifier-free formula and $\bar{a} \in N$, then $\mathcal{N} \models \phi(\bar{a})$ if and only if $\phi(\bar{a})$. Thus if $\mathcal{M} \models \forall \bar{v} \phi(\bar{v})$, then so does $\mathcal{N}$

Suppose that T is preserved under substructures. Let $\Gamma = \{\phi : \phi$ is universal and $T \models \phi\}$. Clearly, if $\mathcal{N} \models T$, then $\mathcal{N} \models \Gamma$. For the other direction, suppose that $\mathcal{N} \models \Gamma$. We claim that $\mathcal{N} \models T$

Claim $T \cup \mathrm{Diag}(\mathcal{N})$ is satisfiable

Suppose not. Then, by the Compactness Theorem, there is a finite $\Delta \subseteq \mathrm{Diag}(\mathcal{N})$ s.t. $T \cup \Delta$ is not satisfiable. Let $\Delta = \{\psi_1, \dots, \psi_n\}$. Let $\bar{c}$ be the new constant symbols from N used in $\psi_1, \dots, \psi_n$ and say $\psi_i = \phi_i(\bar{c})$, where ϕ_i is a quantifier-free $\mathcal{L}$-formula. Because the constants in $\bar{c}$ do not occur in T, if there is a model of $T \cup \{\exists \bar{v} \bigwedge \phi_i(\bar{v})\}$, then by interpreting $\bar{c}$ as witness to the existential formula, $T \cup \Delta$ would be satisfiable. Thus $T \models \forall \bar{v} \bigvee \neg \phi_i(\bar{v})$. As the latter formula is universal, $\forall \bar{v} \bigvee \neg \phi_i(\bar{v}) \in \Gamma$, contradicting $\mathcal{N} \models \Gamma$.

By Lemma 2.27, there is $\mathcal{M} \models T$ with $\mathcal{M} \supseteq \mathcal{N}$. Because T is preserved under substructure, $\mathcal{N} \models T$ and Γ is a universal axiomatization $\square$

Definition 2.34. Suppose that $(I, <)$ is a linear order. Suppose that $\mathcal{M}_i$ is an $\mathcal{L}$-structure for $i \in I$. We say that $(\mathcal{M}_i : i \in I)$ is a chain of $\mathcal{L}$-strctures if $\mathcal{M}_i \subseteq \mathcal{M}_j$ for $i < j$. If $\mathcal{M}_i \prec \mathcal{M}_j$ for $i < j$, we call $(\mathcal{M}_i : i \in I)$ an **elementary chain**

If $(\mathcal{M}_i : i \in I)$ is a nonempty chain of structures, then we can define $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$, the union of the chain, as follows. $M = \bigcup_{i \in I} M_i$. if c is a constant in the language, then $c^{\mathcal{M}_i} = c^{\mathcal{M}_j}$ for all $i, j \in I$. Let $c^{\mathcal{M}} = c^{\mathcal{M}_i}$.

Suppose that $\bar{a} \in M$. Because I is linearly ordered, we can find $i \in I$ s.t. $\bar{a} \in M_i$. If f is a function symbol of $\mathcal{L}$ and $i < j$, then $f^{\mathcal{M}_i}(\bar{a}) = f^{\mathcal{M}_j}(\bar{a})$. Thus $f^{\mathcal{M}} = \bigcup_{i \in I} f^{\mathcal{M}_i}$ is a well-defined function. Similarly, $R^{\mathcal{M}} = \bigcup_{i \in I} R^{\mathcal{M}_i}$

Proposition 2.35. *Suppose that $(I, <)$ is a linear order and $(\mathcal{M}_i : i \in I)$ is an elementary chain. Then $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$ is an elementary extension of each $\mathcal{M}_i$*

Proof. We prove by induction on formulas that

$$\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{M}_i \models \phi(\bar{a})$$

for all $i \in I$, all formulas $\phi(\bar{v})$, and all $\bar{a} \in M_i^n$

Because $\mathcal{M}_i$ is a substructure of $\mathcal{M}$, by Proposition 1.7 this is true for all atomic ϕ. $\neg\phi$ and $\phi \vee \psi$ is easy.

Suppose that ϕ is $\exists v \psi(v, \bar{w})$ and the chain holds for ψ. If $\mathcal{M}_i \models \psi(b, \bar{a})$, then so does $\mathcal{M}$. Thus if $\mathcal{M}_i \models \phi(\bar{a})$, then so does $\mathcal{M}$. On the other hand, if $\mathcal{M} \models \psi(b, \bar{a})$, there is $j \geq i$ s.t. $b \in M_j$. By induction, $\mathcal{M}_j \models \psi(b, \bar{a})$, so $\mathcal{M}_j \models \phi(\bar{a})$. Because $\mathcal{M}_i \prec \mathcal{M}_j$, $\mathcal{M}_i \models \phi(\bar{a})$ $\square$

2.4 Back and Forth

2.4.1 Dense Linear Orders

Let $\mathcal{L} = \{<\}$ and let DLO be the theory of dense linear orders without endpoints. DLO is axiomatized by the axioms for linear orders plus the axioms

$$\forall x \forall y \ (x < y \rightarrow \exists z \ x < z < y)$$
$$\forall x \exists y \exists z \ y < x < z$$

Theorem 2.36. *The theory DLO is $\aleph_0$-categorical and complete*

Proof. Let $(A,<)$ and $(B,<)$ be two countable models of DLO. Let $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ be one-to-one enumerations of A and B. We will build a sequence of partial bijections $f_i : A_i \to B_i$ where $A_i \subset A$ and $B_i \subset B$ are finite s.t. $f_0 \subseteq f_1 \subseteq \ldots$ and if $x, y \in A_i$ and $x < y$, then $f_i(x) < f_i(y)$. We call f_i a **partial embedding**. We will build these sequences s.t. $A = \bigcup A_i$ and $B = \bigcup B_i$. In this case, $f = \bigcup f_i$ is the desired isomorphism from $(A,<)$ to $(B,<)$

At odd stages of the construction we will ensure that $\bigcup A_i = A$, and at even stages we will ensure that $\bigcup B_i = B$

<u>stage 0</u>: Let $A_0 = B_0 = f_0 = \emptyset$

<u>stage $n+1 = 2m+1$</u>: We will ensure that $a_m \in A_{n+1}$.

If $a_m \in A_n$, then let $A_{n+1} = A_n$, $B_{n+1} = B_n$ and $f_{n+1} = f_n$. Suppose that $a_m \notin A_n$. To add a_m to the domain of our partial embedding, we must find $b \in B \setminus B_n$ s.t.

$$\alpha < a_m \Leftrightarrow f_n(\alpha) < b$$

for all $\alpha \in A_n$. In other words, we must find $b \in B$, which is the image under f_n of the cut of a_m in A_n. Exactly one of the following holds:

1. a_m is greater than every element of A_n, or
2. a_m is than than every element of A_n, or
3. there are α and $\beta \in A_n$ s.t. $\alpha < \beta$,$\gamma \leq \alpha$ or $\gamma \geq \beta$ for all $\gamma \in A_n$ and $\alpha < a_m < \beta$

In case 1 because B_n is finite and $B \models$ DLO,we can find $b \in B$ greater than every element of B_n. Similar for case 2. In case 3, because f_n is a partial embedding, $f_n(\alpha) < f_n(\beta)$ and we can choose $b \in B_n$ s.t. $f_n(\alpha) < b < f_n(\beta)$. Note that

$$\alpha < a_m \Leftrightarrow f_n(\alpha) < b$$

for all $\alpha \in A_n$

<u>stage $n+1 = 2m+2$</u>: We will ensure $b_m \in B_{n+1}$

Again, if b_m is already in B_n, then we make no changes. Otherwise, we must find $a \in A$ s.t. the image of the cut of a in A_n is the cut of b_m in B_n. This is done in odd case.

Clearly, at odd stages we have ensured that $\bigcup A_n = A$ and at even stages we have ensured that $\bigcup B_n = B$. Because each f_n is a partial embedding, $f = \bigcup f_n$ is an isomorphism from A onto B

But there are no finite dense linear orders, Vaught's test implies that DLO is complete ☐

2.4.2 The Random Graph

Let $\mathcal{L} = \{R\}$, where R is a binary relation symbol. We will consider an $\mathcal{L}$-theory containing the graph axioms $\forall x\ \neg R(x,x)$ and $\forall x \forall y\ R(x,y) \to R(y,x)$. Let ψ_n be the "extension axiom"

$$\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n \left(\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} x_1 \neq y_j \to \exists z \bigwedge_{i=1}^{n} (R(x_i, z) \wedge \neg R(y_i, z)) \right)$$

We let T be the theory of graphs where we add $\{\exists x \exists y\ x \neq y\} \cup \{\psi_n : n = 1, 2, \dots\}$ to the graph axioms. A model of T is a graph where for any finite disjoint sets X and Y we can find a vertex with edges going to every vertex in X and no vertex in Y

Theorem 2.37. *T is satisfiable and $\aleph_0$-categorical. In particular, T is complete and decidable*

Proof. We first build a countable model of T. Let G_0 be any countable graph

Claim There is a graph $G_1 \supseteq G_0$ s.t. G_1 is countable and if X and $Y$are disjoint finite subsets of G_0 then there is $z \in G_1$ s.t. $R(x,z)$ for $x \in X$ and $\neg R(y,z)$ for $y \in Y$

Let the vertices of G_1 be the vertices of G_0 plus new vertices z_X for each $X \subseteq G_0$. The edges of G_1 are the edges of G together with new edges between x and z_X whenever $X \subseteq G_0$ is finite and $x \in X$.

We iterate this construction to build a sequence of countable graphs $G_0 \subset G_1 \subset \dots$ s.t. if X and Y are disjoint finite subsets of G_i, then there is $z \in G_{i+1}$ s.t. $R(x,z)$ for $x \in X$ and $\neg R(y,z)$ for $y \in Y$. Thus $G = \bigcup G_n$ is a countable model of T

Next we show that T is $\aleph_0$-categorical. Let G_1 and G_2 be countable models of T. Let $a_0, a_1, \dots$ list G_1, and let $b_0, b_1, \dots$ list G_2. We will build a sequence of finite partial one-to-one maps $f_0 \subseteq f_1 \subseteq f_2 \subseteq \dots$ s.t. for all x, y in the doamin of f_s,

$$G_1 \models R(x,y) \Leftrightarrow G_2 \models R(f_s(x), f_s(y))$$

Let $f_0 = \emptyset$ <u>stage $s+1 = 2i+1$</u>: We make sure that a_i is in the domain

If a_i is in the domain of f_s, let $f_{s+1} = f_s$. If not, let $\alpha_1, \dots, \alpha_m$ list the domain of f_s and let $X = \{j \leq m : R(\alpha_j, a_i)\}$ and let $Y = \{j \leq m : \neg R(\alpha_j, a_i)\}$. Because $G_2 \models T$, we can find $b \in G_2$ s.t. $G_2 \models R(f_s(\alpha_j), b)$ for $j \in X$ and $G_2 \models \neg R(f_s(\alpha_j), b)$ for $j \in Y$. Let $f_{s+1} = f_s \cup \{(a_i, b)\}$.

<u>stage $s+1 = 2i+2$</u>: Similar $\square$

Let $\mathcal{G}_N$ be the set of all graphs with vertices $\{1, 2, \ldots, N\}$. We consider a probability measure on $\mathcal{G}_N$ where we make all graphs equally likely. This is the same as constructing a random graph where we independently decide whether there is an edge between i and j with probability $\frac{1}{2}$. For any $\mathcal{L}$-sentence ϕ,

$$p_N(\phi) = \frac{\left|\{G \in \mathcal{G}_N : G \models \phi\}\right|}{|\mathcal{G}_N|}$$

is the probability that a random element of $\mathcal{G}_N$ satisfies ϕ

Lemma 2.38. $\lim_{N\to\infty} p_N(\psi_n) = 1$

Proof. Fix n. Let G be a random graph in $\mathcal{G}_N$ where $N > 2n$. Fix $x_1, \ldots, x_n, y_1, \ldots, y_n, z \in G$ distinct. Let q be the probability that

$$\neg\left(\bigwedge_{i=1}^{n} (R(x_i, z)) \wedge \neg R(y_i, z)\right)$$

Then $q = 1 - 2^{-2n}$. Because these probabilities are independent, the probability that

$$G \models \neg\exists z \neg\left(\bigwedge_{i=1}^{n} (R(x_i, z)) \wedge \neg R(y_i, z)\right)$$

is q^{N-2n}. Let M be the number of pairs of disjoint subsets of G of size n. Thus

$$p_N(\neg\psi_n) \leq Mq^{N-2n} < N^{2n}q^{N-2n}$$

Because $q < 1$

$$\lim_{N\to\infty} p_N(\neg\psi_n) = \lim_{N\to\infty} N^{2n}q^N = 0$$

$\square$

Theorem 2.39 (Zero-One Law for Graphs)**.** *For any $\mathcal{L}$-sentence ϕ either $\lim_{N\to\infty} p_N(\phi) = 0$ or $\lim_{N\to\infty} p_N(\phi) = 1$. Moreover, T axiomatizes $\{\phi : \lim_{N\to\infty} p_N(\phi) = 1\}$, the **almost sure theory graphs**. The almost sure theory of graphs is decidable and complete*

Proof. If $T \models \phi$, then there is n s.t. if G is a graph and $G \models \psi_n$, then $G \models \phi$. Thus, $p_N(\phi) \geq \phi_N(\psi_n)$ and by Lemma 2.38, $\lim_{N\to\infty} p_N(\phi) = 1$. $\square$

2.4.3 Ehrenfeucht-Fraïssé Games

Let $\mathcal{L}$ be a language and $\mathcal{M} = (M, \dots)$ and $\mathcal{N} = (N, \dots)$ be two $\mathcal{L}$-structures with $M \cap N = \emptyset$. If $A \subseteq M$, $B \subseteq N$ and $f : A \to B$, we wsay that f is a **partial embedding** if $f \cup \{(c^{\mathcal{M}}, c^{\mathcal{N}}) : c \text{ a constant of } \mathcal{L}\}$is a bijection preserving all relations and functions of $\mathcal{L}$

We will define an infinite two-player game $G_\omega(\mathcal{M}, \mathcal{N})$. We will call the two players player I and player II; together they will build a partial embedding f from M to N. A play of the game will consist of ω stages. At the ith-stage, player I moves first and either plays $m_i \in M$, challenging player II to put m_i into the domain of f, or $n_i \in N$, challenging player II to put n_i into the range. If player I plays $m_i \in M$, then player II must play $n_i \in N$, whereas if player I plays $n_i \in M$, then player II must play $m_i \in M$. Player II wins the play of the game if $f = \{(m_i, n_i) : i = 1, 2, \dots\}$ is the graph of a partial embedding.

A **strategy** for player II in $G_\omega(\mathcal{M}, \mathcal{N})$ is a function τ s.t. if player I's first n moves are $c_1, \dots, c_n$, then player II's nth move will be $\tau(c_1, \dots, c_n)$. We say that player II uses the strategy τ in the play of the game if the play looks like

Player I	Player II
c_1	
	$\tau(c_1)$
c_2	
	$\tau(c_1, c_2)$
c_3	
	$\tau(c_1, c_2, c_3)$
$\vdots$	$\vdots$

We say that τ is a **winning strategy** for player II, if for any sequence of plays $c_1, \dots$ player I makes, player II will win by following τ. We define strategies for player I analogously

For example, suppose that $\mathcal{M}, \mathcal{N} \models \text{DLO}$. Then player II has a winning strategy. Suppose that up to stage n they have built a partial embedding $g : A \to B$. If player I plays $a \in M$, then player II plays $b \in N$ s.t. the cub b makes in B is the image of the cut of a in A under g. Similar for player I's $b \in N$

Proposition 2.40. *If $\mathcal{M}$ and $\mathcal{N}$ is countable, then the second player has a wining strategy in G_ω if and only if $\mathcal{M} \cong \mathcal{N}$*

Proof. If $\mathcal{M} \cong \mathcal{N}$, player II can win by playing according to the isomorphism

Suppose that player II has a winning strategy. Let $m_0, m_1, \ldots$ list M and $n_0, n_1, \ldots$ list N. Consider a play of the game where the second player uses the winning strategy and the first player plays $m_0, n_0, m_1, n_1, m_2, n_2, \ldots$. If f is the partial embedding build during this play of the game then the domain of f is M and the range of f is N. Thus f is an isomorphism $\square$

Fix $\mathcal{L}$ a finite language with no function symbols, and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. We define a game $G_n(\mathcal{M}, \mathcal{N})$ for $n = 1, 2, \ldots$. The game will have n rounds similar to ω rounds . Player II wins if $\{(a_i, b_i) : i = 1, \ldots, n\}$ is the graph of a partial embedding from $\mathcal{M}$ into $\mathcal{N}$. We call $G_n(\mathcal{M}, \mathcal{N})$ an **Ehrenfeucht-Fraïssé Games**

Theorem 2.41. *Let $\mathcal{L}$ be a finite language without function symbols and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. Then $\mathcal{M} \equiv \mathcal{N}$ if and only if the second player has a wining strategy in $G_n(\mathcal{M}, \mathcal{N})$ for all n*

We need several lemmas.

Lemma 2.42. *One of the players has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$*

Proof. Suppose that player II does not have a winning strategy. Then there is some move player I can make in round one so that player II has no move available to force a win. Player I makes that move. Now, whatever player II does, there is still a move that if made by player I means that player II cannot force a win. $\square$

We inductively define depth(ϕ), the **quantifier depth** of an $\mathcal{L}$-formula ϕ, as follows

depth(ϕ) $= 0$ if and only if ϕ is quantifier-free

depth($\neg\phi$) $=$ depth(ϕ)

depth($\phi \wedge \psi$) $=$ depth($\phi \vee \psi$) $= \max\{$depth(ϕ), depth(ψ)$\}$

depth($\exists v \phi$) $=$ depth(ϕ) $+ 1$

We say that $\mathcal{M} \equiv_n \mathcal{N}$ if $\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$ for all sentences of depth at most n. We will show player II has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ if and only if $\mathcal{M} \equiv_n \mathcal{N}$

Lemma 2.43. *For each n and l, there is a finite list of formulas $\phi_1, \ldots, \phi_k$ of depth at most n in free variables $x_1, \ldots, x_l$ s.t. every formula of depth at most n in free variables $x_1, \ldots, x_l$ is equivalent to some ϕ_i*

Proof. We first prove this for quantifier-free formulas. Because $\mathcal{L}$ is finite and has no function symbols, there are only finitely many atomic $\mathcal{L}$-formulas in free variables $x_1, \ldots, x_l$. Let $\sigma_1, \ldots, \sigma_s$ list all such formulas.

If ϕ is a Boolean combination of formulas $\tau_1, \ldots, \tau_s$, then there is S a collection of subsets of $\{1, \ldots, s\}$ s.t.

$$\models \phi \leftrightarrow \bigvee_{X \in S} \left(\bigwedge_{i \in X} \tau_i \wedge \bigwedge_{i \notin X} \neg \tau_i \right)$$

This gives a list of 2^{2^s} formulas s.t. every Boolean combination of $\tau_1, \ldots, \tau_s$ is equivalent to a formula in this list. In particular, because quantifier free formulas are Boolean combinations of atomic formulas, there is a finite list of depth-zero formulas s.t. every depth-zero formula is equivalent to one in the list.

Because formulas of depth $n + 1$ are Boolean combinations of $\exists v \phi$ and $\forall v \phi$ where ϕ has depth at most n $\square$

Lemma 2.44. *Let $\mathcal{L}$ be a finite language without function symbols and $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. The second player has a winning strategy in $G_n(\mathcal{M}, \mathcal{N})$ if and only if $\mathcal{M} \equiv_n \mathcal{N}$*

Proof. Induction on n

Suppose that $\mathcal{M} \equiv_n \mathcal{N}$. Consider a play of the game where in round one player I plays $a \in M$. We claim that there is $b \in \mathcal{N}$ s.t. $\mathcal{M} \models \phi(a) \Leftrightarrow \mathcal{N} \models \phi(b)$ whenever depth(ϕ) $< n$. Let $\phi_0(v), \ldots, \phi_m(v)$ list, up to equivalance, all formulas of depth less than n. Let $X = \{i \leq m : \mathcal{M} \models \phi_i(a)\}$, and let $\Phi(v)$ be the formula

$$\bigwedge_{i \in X} \phi_i(v) \wedge \bigwedge_{i \notin X} \neg \phi_i(v)$$

Then, depth($\exists v \Phi(v)$) $\leq n$ and $\mathcal{M} \models \Phi(a)$; thus there is $b \in N$ s.t. $\mathcal{N} \models \Phi(b)$. Player II plays b in round one

If $n = 1$, the game has now concluded and $a \mapsto b$ is a partial embedding so player II wins. Suppose that $n > 1$

Let $\mathcal{L}^* = \mathcal{L} \cup \{c\}$, where c is a new constant symbol. View $\mathcal{M}$ and $\mathcal{N}$ as $\mathcal{L}^*$-structures $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ where we interpret the new constant as a and b respectively. Because

$$\mathcal{M} \models \phi(a) \Leftrightarrow \mathcal{N} \models \phi(b)$$

for $\phi(v)$ an $\mathcal{L}$-formula with depth(ϕ) $< n$, $(\mathcal{M}, a) \equiv_{n-1} (\mathcal{N}, b)$. By induction, player II has a winning strategy in $G_{n-1}((\mathcal{M}, a), (\mathcal{N}, b))$. If player's second play is d, player II responds as if d was player I's first play in $G_{n-1}((\mathcal{M}, a), (\mathcal{N}, b))'$ and continues playing using this strategy, that is, in round i player I has plays $a, d_2, \ldots, d_i$, then player II plays $\tau(d_2, \ldots, d_i)$, where τ is his winning strategy in $G((\mathcal{M}, a), (\mathcal{N}, b))$. $\square$

3 Algebraic Examples

3.1 Quantifier Elimination

Definition 3.1. We say that a theory T has **quantifier elimination** if for every formula ϕ there is a quantifier-free formula ψ s.t.

$$T \models \phi \leftrightarrow \psi$$

Lemma 3.2. *Let $(A, <)$ and $(B, <)$ be countable dense linear orders, $a_1, \dots, a_n \in A$, $b_1, \dots, b_n \in B$, s.t. $a_1 < \cdots < a_n$ and $b_1, \cdots < b_n$. Then there is an isomorphism $f : A \to B$ s.t. $f(a_i) = b_i$ for all $i = 1, \dots, n$*

Proof. Modify the proof of Theorem 2.36 starting with $A_0 = \{a_1, \dots, a_n\}$, $B_0 = \{b_1, \dots, b_n\}$, and the partial isomorphism $f_0 : A_0 \to B_0$, where $f_0(a_i) = b_i$. □

4 Reference

References

[DJMM12] Arnaud Durand, Neil D. Jones, Johann A. Makowsky, and Malika More. Fifty years of the spectrum problem: survey and new results. *Bulletin of Symbolic Logic*, 18(4):505–553, 2012.

5 Index

A Set Theory

A.1 Cardinal Arithmetic

Corollary A.1. *1. If $|I| = \kappa$ and $|A_i| \leq \kappa$ for all $i \in I$, then $\left|\bigcup A_i\right| \leq \kappa$*

2. If κ is regular, $|I| < \kappa$ and $|A_i| < \kappa$ for all $i \in I$, then $\left|\bigcup A_i\right| < \kappa$

3. Let κ be an infinite cardinal. Let X be a set and $\mathcal{F}$ a set of functions $f : X^{n_f} \to X$. Suppose that $|\mathcal{F}| \leq \kappa$ and $A \subseteq X$ with $|A| \leq \kappa$. Let $\boldsymbol{CL}(A)$ be the smallest subset of X containing A closed under the functions in $\mathcal{F}$. Then $|\boldsymbol{CL}(A)| \leq \kappa$