Rough Sets: Theoretical aspects of reasoning about data

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Contents

1 Knowledge

1.1 Knowledge base

Given a finite set $U \neq \emptyset$ (the universe). Any subset $X \subset U$ of the universe is called a **concept** or a **category** in U. And any family of concepts in U will be referred to as **abstract knowledge** about U.

partition or **classification** of a certain universe U is a family $C = \{X_1, X_2, \ldots, X_n\}$ s.t. $X_i \subset U, X_i \neq \emptyset, X_i \cap X_j = \emptyset$ and $\bigcup X_i = U$

A family of classifications is called a **knowledge base** over U

R an equivalence relation over U, U/R family of all equivalence classes of R, referred to be **categories** or **concepts** of R, and $[x]_R$ denotes a category in R containing an element $x \in U$

By a **knowledge base** we can understand a relational system $K = (U, \boldsymbol{R})$, $\boldsymbol{R}$ is a family of equivalence relations over U

If $\boldsymbol{P} \subset \boldsymbol{R}$ and $\boldsymbol{P} \neq \emptyset$, then $\bigcap \boldsymbol{P}$ is also an equivalence relation, and will be denoted by $IND(\boldsymbol{P})$, called an **indiscernibility relation** over $\boldsymbol{P}$

$$[x]_{IND(\boldsymbol{P})} = \bigcap_{R \in \boldsymbol{P}} [x]_R$$

$U/IND(\boldsymbol{P})$ called **$\boldsymbol{P}$-basic knowledge about** U in K. For simplicity, $U/\boldsymbol{P} = U/IND(\boldsymbol{P})$ and $\boldsymbol{P}$ will be also called **$\boldsymbol{P}$-basic knowledge** . Equivalence classes of $IND(\boldsymbol{P})$ are called **basic categories** of knowledge $\boldsymbol{P}$. If $Q \in \boldsymbol{R}$, then Q is a **Q-elementary knowledge** and equivalence classes of Q are referred to as **Q-elementary concepts** of knowledge $\boldsymbol{R}$

The family of all $\boldsymbol{P}$-basic categories for all $\neq \boldsymbol{P} \subset \boldsymbol{R}$ will be called the **family of basic categories** in knowledge base $K = (U, \boldsymbol{R})$

Let $K = (U, \boldsymbol{R})$ be a knowledge base. By $IND(K)$ we denote the family of all equivalence relations defined in K as $IND(K) = \{IND(\boldsymbol{P}) : \emptyset \neq \boldsymbol{P} \subseteq \boldsymbol{R}\}$.

Thus $IND(K)$ is the minimal set of equivalence relations.

Every union of $\boldsymbol{P}$-basic categories will be **$\boldsymbol{P}$-category**

The family of all categories in the knowledge base $K = (U, \boldsymbol{R})$ will be referred to as **K-categories**

1.2 Equivalence, generalization and specialization of knowledge

Let $K = (U, \boldsymbol{P}), K' = (U, \boldsymbol{Q})$. K and K' are **equivalent** $K \simeq K', (\boldsymbol{P} \simeq \boldsymbol{Q})$ if $IND(\boldsymbol{P}) = IND(\boldsymbol{Q})$. Hence $K \simeq K'$ if both K and K' have the same set of elememtary categories. *This means that knowledge in knowledge bases K and K' enables us to express exactly the same facts about the universe.*

If $IND(\boldsymbol{P}) \subset IND(\boldsymbol{Q})$ then knowledge $\boldsymbol{P}$ is **finer** than knowledge $\boldsymbol{Q}$ (**coarser**). $\boldsymbol{P}$ is **specialization** of $\boldsymbol{Q}$ and $\boldsymbol{Q}$ is **generalization** of $\boldsymbol{P}$

2 Imprecise categories, approximations and rough sets

2.1 Rough sets

Let $X \subseteq U$. X is R-**definable** or R-**exact** if X is the union of some R-basic categories. otherwise R-**undefinable**, R-**rough**, R-**inexact** .

2.2 Approximations of set

Given $K = (U, \boldsymbol{R}), R \in IND(K)$

$$\underline{R}X = \bigcup \{Y \in U/R : Y \subseteq X\}$$
$$\overline{R}X = \bigcup \{Y \in U/R : Y \cap X \neq \emptyset\}$$

called the R-**lower** and R-**upper approximation** of X

$BN_R(X) = \overline{R}X - \underline{R}X$ is R-**boundary** of X. $BN_R(X)$ is the set of elements which cannot be classified either to X or to $-X$ having knowledge R

$$POS_R(X) = \underline{R}X, R\text{-positive region of } X$$
$$NEG_R(X) = U - \overline{R}X, R\text{-negative region of } X$$
$$BN_R(X) - R\text{-borderline region of } X$$

If $x \in POS(X)$, then x will be called an R-**positive example of** X

Proposition 2.1. *1. X is R-definable if and only if $\underline{R}X = \overline{R}X$*

2. X is rought w.r.t. R if and only if $\underline{R}X \neq \overline{R}X$

2.3 Properties of approximations

Proposition 2.2 (2.2)**.** *1.* $\underline{R}X \subseteq X \subseteq \overline{R}X$

2. $\underline{R}\emptyset = \underline{R}\emptyset = \emptyset$; $\underline{R}U = \overline{R}U = U$

3. $\overline{R}(X \cup Y) = \overline{R}X \cup \overline{R}Y$

4. $\underline{R}(X \cap Y) = \underline{R}X \cap \underline{R}Y$

5. $X \subseteq Y$ *implies* $\underline{R}X \subseteq \underline{R}Y$

6. $X \subseteq Y$ *implies* $\overline{R}X \subseteq \overline{R}Y$

7. $\underline{R}(X \cup Y) \subseteq \underline{R}X \cup \underline{R}Y$

8. $\underline{R}(-X) = -\overline{R}X$

9. $\overline{R}(-X) = -\underline{R}X$

10. $\overline{R}(-X) = -\underline{R}X$

11. $\underline{R}\underline{R}X = \overline{R}\underline{R}X = \underline{R}X$

12. $\overline{R}\overline{R}X = \underline{R}\overline{R}X = \overline{R}X$

The equivalence relation R over U uniquely defines a topological space $T = (U, DIS(R))$ where $DIS(R)$ is the familty of all open and closed set in T and U/R is a base for T. The R-lower and R-upper approximation of X in A are **interior** and **closure** operations in the topological space T

2.4 Approximations and membership relation

$$x \underline{\in}_R X \text{if and only if } x \in \underline{R}X$$
$$x \overline{\in}_R X \text{if and only if } x \in \overline{R}X$$

where $\underline{\in}_R$ read "x **surely belongs** to X w.r.t. R" and $\overline{\in}_R$ - "x **possibly belongs** to X w.r.t. R". The **lower** and **upper** membership.

Proposition 2.3. *1.* $x \underline{\in} X$ *implies* $x \in X$ *implies* $x \overline{\in} X$

2. $X \subset Y$ *implies (*$x \underline{\in} X$ *implies* $x \underline{\in} Y$ *and* $x \overline{\in} X$ *implies* $x \overline{\in} Y$*)*

3. $x \overline{\in} (X \cup Y)$ *if and only if* $x \overline{\in} X$ *or* $x \overline{\in} Y$

4. $x \underline{\in} (X \cap Y)$ *if and only if* $x \underline{\in} X$ *and* $x \underline{\in} Y$

5. $x \underline{\in} X$ *or* $x \underline{\in} Y$ *implies* $x \underline{\in} (X \cup Y)$

6. $x \overline{\in} X \cap Y$ *implies* $x \overline{\in} X$ *and* $x \overline{\in} Y$

7. $x \underline{\in} (-X)$ *if and only if non* $x \overline{\in} X$

8. $x \overline{\in} (-X)$ *if and only if non* $x \underline{\in} X$

2.5 Numerical characterization of imprecision

accuracy measure

$$\alpha_R(X) = \frac{card\ \underline{R}}{card\ \overline{R}}$$

2.6 Topological characterization of imprecision

Definition 2.1. 1. *If $\underline{R}X \neq \emptyset$ and $\overline{R}X \neq U$, then we say that X is **roughly R-definable**. We can decide whether some elements belong to X or $-X$*

2. *If $\underline{R}X = \emptyset$ and $\overline{R}X \neq U$, then we say that X is **internally R-undefinable**. We can decide whether some elemnts belong to $-X$*

3. *If $\underline{R}X \neq \emptyset$ and $\overline{R}X = U$, then we say that X is **externally R-undefinable**. We can decide whether some elements belong to X*

4. *If $\underline{R}X = \emptyset$ and $\overline{R}X = U$, then we say that X is **totally R-undefinable**. unable to decide*

Proposition 2.4 (2.4)**.** 1. *Set X is R-definable(roughly R-definable, totally R-undefinable) if and only if so is $-X$*

2. *Set X is externally R-undefinable if and only if $-X$ is internally R-undefinable*

Proof. 1.

$$\begin{aligned} R\text{-definable} &\Leftrightarrow \underline{R}X = \overline{R}X, \underline{R} \neq \emptyset, \overline{R} \neq U \\ &\Leftrightarrow -\underline{R}X = -\overline{R}X \\ &\Leftrightarrow \overline{R}(-X) = \underline{R}(-X) \end{aligned}$$

$$
\begin{aligned}
X \text{ is roughly } R\text{-definable} &\Leftrightarrow \underline{R}X \neq \emptyset \wedge \overline{R}X \neq U \\
&\Leftrightarrow -\underline{R}X \neq U \wedge -\overline{R}X \neq \emptyset \\
&\Leftrightarrow \overline{R}(-X) \neq U \wedge \underline{R}(-X) \neq \emptyset
\end{aligned}
$$

□

2.7 Approximation of classifications

If $F = \{X_1, \ldots, X_n\}$ is a family of non empty sets, then $\underline{R}F = \{\underline{R}X_1, \ldots, \underline{R}X_n\}$ and $\overline{R}F = \left\{\overline{R}X_1, \ldots, \overline{R}X_n\right\}$, called the R**-lower approximation** and the R**-upper approximation** of the family F

The **accuracy of approximation** of F by R is

$$\alpha_R(F) = \frac{\sum card\ \underline{R}X_i}{\sum card\ \overline{R}X_i}$$

quality of approximation of F by R

$$\gamma_R(F) = \frac{\sum card\ \underline{R}X_i}{card\ U}$$

Proposition 2.5 (2.5)**.** *Let $F = \{X_1, \ldots, X_n\}$ where $n > 1$ be a classification of U and let R be an equivalence relation. If there exists $i \in \{1, 2, \ldots, n\}$ s.t. $\underline{R}X_i \neq \emptyset$, then for each $j \neq i$ and $j \in \{1, \ldots, n\}$, $\overline{R}X_j \neq U$*

Proof. If $\underline{R}X_i \neq \emptyset$ then there exists $x \in X$ s.t. $[x]_R \subseteq X$, which implies $[x]_R \cap X_j = \emptyset$ for each $j \neq i$. This yields $\overline{R}X_j \cap [x]_R = \emptyset$. □

Proposition 2.6 (2.6)**.** *Let $F = \{X_1, \ldots, X_n\}, n > 1$ be a classification of U and let R be an equivalence relation. If there exists $i \in \{1, \ldots, n\}$ s.t. $\overline{R}X_i = U$, then for each $j \neq i$ and $j \in \{1, \ldots, n\}$ $\underline{R}X_j = \emptyset$*

Proposition 2.7 (2.7)**.** *Let $F = \{X_1, \ldots, X_n\}, n > 1$ be a classification of U and let R be an equivalence relation. If for each $i \in \{1, 2, \ldots, n\}$ $\underline{R}X_i \neq \emptyset$ holds, then $\overline{R}X_i \neq U$ for each $i \in \{1, \ldots, n,\}$*

Proposition 2.8. *Let $F = \{X_1, \ldots, X_n\}, n > 1$ be a classification of U and let R be an equivalence relation. If for each $i \in \{1, 2, \ldots, n\}$ $\overline{R}X_i = U$ holds, then $\underline{R}X_i = \emptyset$ for each $i \in \{1, \ldots, n\}$*

2.8 Rough equality of sets

Definition 2.2. *Let $K = (U, \boldsymbol{R})$ be a knowledge base, $X, Y \subseteq U$ and $R \in IND(K)$, then*

1. *Sets X and Y are* ***bottom*** *R-**equal*** $(X \eqsim_R Y)$ *if* $\underline{R}X = \underline{R}Y$
2. *Sets X and Y are* ***top*** $R -$ ***equal*** $(X \simeq_R Y)$ *if* $\overline{R}X = \overline{R}Y$
3. *Sets X and Y are R-**equal*** $(X \approx_R Y)$ *if* $X \simeq_R Y$ *and* $X \eqsim_R Y$

Proposition 2.9 (2.9). 1. $X \eqsim Y$ *iff* $X \cap Y \eqsim X$ *and* $X \cap Y \eqsim Y$

2. $X \simeq Y$ *iff* $X \cup Y \simeq X$ *and* $X \cup Y \simeq Y$
3. *If* $X \simeq X'$ *and* $Y \simeq Y'$ *then* $X \cup Y \simeq X' \cup Y'$
4. *If* $X \eqsim X'$ *and* $Y \eqsim Y'$ *then* $X \cap Y \eqsim X' \cap Y'$
5. *If* $X \simeq Y$*, then* $X \cup -Y \simeq U$
6. *If* $X \eqsim Y$*, then* $X \cap -Y \eqsim \emptyset$
7. *If* $X \subseteq Y$ *and* $Y \simeq \emptyset$*, then* $X \simeq \emptyset$
8. *If* $X \subseteq Y$ *and* $X \subseteq U$ *then* $Y \subseteq U$
9. $X \simeq Y$ *iff* $-X \eqsim -Y$
10. *If* $X \eqsim \emptyset$ *or* $Y \eqsim \emptyset$*, then* $X \cap Y \eqsim \emptyset$
11. *If* $X \simeq U$ *or* $Y \simeq U$*, then* $X \cup Y \simeq U$

Proposition 2.10 (2.10)**.** *For any equivalence relation R*

1. $\underline{R}X$ *is the intersection of all* $Y \subseteq U$ *s.t.* $X \eqsim_R Y$
2. $\overline{R}$ *is the union of all* $Y \subseteq U$ *s.t.* $X \simeq_R Y$

2.9 Rough inclusion of sets

Definition 2.3. *Let $K = (U, \boldsymbol{R})$ be a knowledge base, $X, Y \subseteq U$ and $R \in IND(K)$.*

1. *Set X is* ***bottom*** *R-**included*** *in Y* $(X \subsetsim_R Y)$ *iff* $\underline{R}X \subseteq \underline{R}Y$
2. *Set X is* ***top*** *R-**included*** *in Y* $(X \tilde{\subset}_R Y)$ *iff* $\overline{R}X \subseteq \overline{R}Y$

3. *Set X is R-**included** in Y ($X \underset{\sim}{\tilde{\subset}}_R Y$) iff $X \tilde{\subset}_R Y$ and $X \underset{\sim}{\subset}_R Y$*

Proposition 2.11 (2.11)**.** *1. If $X \subseteq Y$, then $X \underset{\sim}{\subset} Y, X \tilde{\subset} Y$ and $X \underset{\sim}{\tilde{\subset}} Y$*

2. *If $X \underset{\sim}{\subset} Y$ and $Y \underset{\sim}{\subset} X$, then $X \eqsim Y$*

3. *If $X \tilde{\subset} Y$ and $Y \tilde{\subset} X$, then $X \simeq Y$*

4. *If $X \underset{\sim}{\tilde{\subset}} Y$ and $Y \underset{\sim}{\tilde{\subset}} X$ then $X \approx Y$*

5. *$X \tilde{\subset} Y$ iff $X \cup Y \simeq Y$*

6. *$X \underset{\sim}{\subset} Y$ iff $X \cap Y \eqsim X$*

7. *If $X \subseteq Y, X \eqsim X', Y \eqsim Y'$, then $X' \underset{\sim}{\subset} Y'$*

8. *If $X \subseteq Y, X \simeq X', Y \simeq Y'$, then $X' \tilde{\subset} Y'$*

9. *If $X \subseteq Y, X \approx X', Y \approx Y'$, then $X' \underset{\sim}{\tilde{\subset}} Y'$*

10. *If $X' \tilde{\subset} X$ and $Y' \tilde{\subset} Y$, then $X' \cup Y' \tilde{\subset} X \cup Y$*

11. *If $X' \underset{\sim}{\subset} X, Y' \underset{\sim}{\subset}$ then $X' \cap Y' \underset{\sim}{\subset} X \cap Y$*

12. *$X \cap Y \underset{\sim}{\subset} X \tilde{\subset} X \cup Y$*

13. *If $X \underset{\sim}{\subset} Y$ and $X \eqsim Z$ then $Z \underset{\sim}{\subset} Y$*

14. *If $X \tilde{\subset} Y$ and $X \simeq Z$ then $Z \tilde{\subset} Y$*

15. *If $X \underset{\sim}{\tilde{\subset}} Y$ and $X \approx$ then $Z \underset{\sim}{\tilde{\subset}} Y$*

3 Reduction of knowledge

3.1 Reduct and Core of Knowledge

Let $\boldsymbol{R}$ be a family of equivalence relations and let $P \in \boldsymbol{R}$. R is **dispensable** in $\boldsymbol{R}$ if $IND(\boldsymbol{R}) = IND(\boldsymbol{R} - \{R\})$. Otherwise R is **indispensable** in $\boldsymbol{R}$. The family of $\boldsymbol{R}$ is **independent** if each $R \in \boldsymbol{R}$ is indispensable in $\boldsymbol{R}$. Otherwise $\boldsymbol{R}$ is **dependent**

Proposition 3.1 (3.1)**.** *If $\boldsymbol{R}$ is independent and $\boldsymbol{P} \subseteq \boldsymbol{R}$, then $\boldsymbol{P}$ is also independent*

Proof. $IND(\boldsymbol{R}) = IND(\boldsymbol{P} \cup (\boldsymbol{R} - \boldsymbol{P})) = IND(\boldsymbol{P}) \cap IND(\boldsymbol{R} - \boldsymbol{P})$ □

$\boldsymbol{Q} \subseteq \boldsymbol{R}$ is a **reduct** of $\boldsymbol{P}$ if $\boldsymbol{Q}$ is independent and $IND(\boldsymbol{Q}) = IND(\boldsymbol{P})$

The set of all indispensable relations in $\boldsymbol{P}$ is called the **core** of $\boldsymbol{P}$ denoted by $CORE(\boldsymbol{P})$

Proposition 3.2 (3.2)**.**

$$CORE(\boldsymbol{P}) = \bigcap RED(\boldsymbol{P})$$

where $RED(\boldsymbol{P})$ is the family of all reducts of $\boldsymbol{P}$

Proof. If $\boldsymbol{Q}$ is a reduct of $\boldsymbol{P}$ and $R \in \boldsymbol{P} - \boldsymbol{Q}$, then $IND(\boldsymbol{P}) = IND(\boldsymbol{Q})$. If $\boldsymbol{Q} \subseteq \boldsymbol{R} \subseteq \boldsymbol{P}$ then $IND(\boldsymbol{Q}) = IND(\boldsymbol{R})$. Assuming $\boldsymbol{R} = \boldsymbol{P} - \{R\}$ then $R \notin CORE(\boldsymbol{P})$

If $R \notin CORE(\boldsymbol{P})$. This means $IND(\boldsymbol{P}) = IND(\boldsymbol{P} - \{R\})$ which implies that there exists an independent subset $\boldsymbol{S} \subseteq \boldsymbol{P} - \{R\}$ s.t. $IND(\boldsymbol{S}) = IND(\boldsymbol{P})$. Hence $R \notin \bigcap RED(\boldsymbol{P})$ □

3.2 Relative reduct and relative core of knowledge

Let P and Q be equivalence relations over U

P**-positive**

$$POS_P(Q) = \bigcup_{X \in U/Q} \underline{P}X$$

The P-positive region of Q is the set of all objects of the universe U which can be properly classified to classes of U/Q employing knowledge expressed by the classification U/P

Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be families of equivalence relations over U

$R \in \boldsymbol{P}$ is $\boldsymbol{Q}$**-dispensable** in $\boldsymbol{P}$ if

$$POS_{IND(\boldsymbol{P})}(IND(\boldsymbol{Q})) = POS_{IND(\boldsymbol{P}-\{R\})}(IND(\boldsymbol{Q}))$$

otherwise R is $\boldsymbol{Q}$-indispensable in $\boldsymbol{P}$

If every R in $\boldsymbol{P}$ is $\boldsymbol{Q}$-indispensable, we will say that $\boldsymbol{P}$ is $\boldsymbol{Q}$**-independent** or $\boldsymbol{P}$ is **independent w.r.t.** $\boldsymbol{Q}$

The family $\boldsymbol{S} \subseteq \boldsymbol{P}$ will be called a $\boldsymbol{Q}$**-reduct** of $\boldsymbol{P}$ if and only if $\boldsymbol{S}$ is the $\boldsymbol{Q}$-independent subfamily of $\boldsymbol{P}$ and $POS_{\boldsymbol{S}}(\boldsymbol{Q}) = POS_{\boldsymbol{P}}(\boldsymbol{Q})$

The set of all $\boldsymbol{Q}$-indispensable elmentary relations in $\boldsymbol{P}$ will be called the $\boldsymbol{Q}$**-core** of $\boldsymbol{P}$ and will be denoted as $CORE_{\boldsymbol{Q}}(\boldsymbol{P})$

Proposition 3.3 (3.3)**.**

$$CORE_{\boldsymbol{Q}}(\boldsymbol{P}) = \bigcap RED_{\boldsymbol{Q}}(\boldsymbol{P})$$

where $RED_{\boldsymbol{Q}}(\boldsymbol{P})$ is the family of all $\boldsymbol{Q}$-reducts of $\boldsymbol{P}$

3.3 Reduction of categories

Basic categories are pieces of knowledge, which can be considered as "building blocks" of concepts. Every concept in the knowledge base can be only expressed (exactly or approximately) in terms of basic categories. On the other hand, every basic category is "built up" (is an intersection) of some elementary categories. Thus the question arises whether all the elementary categories are necessary to define the basic categories in question.

Let $F = \{X_1, \ldots, X_n\}$ be a family of sets s.t. $X_i \subseteq U$.

X_i is **dispensable** in F if $\bigcap(F - \{X_i\}) = \bigcap F$, otherwise the set X_i is **indispensable** in F

The family F is **independent** if all of its components are indispensable in F. Otherwise F is **dependent**

The family $H \subseteq F$ is a **reduct** of F if H is independent and $\bigcap H = \bigcap F$

The family of all indispensable sets in F will be called the **core** of F, denoted $CORE(F)$

Proposition 3.4 (3.4)**.**

$$CORE(F) = \bigcap RED(F)$$

3.4 Relative reduct and core of categories

$F = \{X_1, \ldots, X_n\}, X_i \subseteq U$ and a subset $Y \subseteq U$ s.t. $\bigcap F \subseteq Y$

X_i is **Y-dispensable** in $\bigcap F$ if $\bigcap(F - \{X_i\}) \subseteq Y$. Otherwise X_i is **Y-indispensable**

The family F is **Y-independent** in $\bigcap F$ if all of its components are **Y-indispensable** in $\bigcap F$

The family $H \subseteq F$ is a **Y-reduct** of $\bigcap F$ if H is Y-independent in $\bigcap F$ and $\bigcap H \subseteq Y$

The family of all Y-indispensable sets in $\bigcap F$ will be called the Y**core** of F and will be denoted by $CORE_Y(F)$

Proposition 3.5 (3.5)**.**

$$CORE_Y(F) = \bigcap RED_Y(F)$$

4 Dependencies in knowledge base

4.1 Dependency of knowledge

Knowledge $\boldsymbol{Q}$ is **derivable** from knowledge $\boldsymbol{P}$ if all elementary categories of $\boldsymbol{Q}$ can be defined in terms of some elementary categories of knowledge $\boldsymbol{P}$. If

$\boldsymbol{Q}$ is derivable from $\boldsymbol{P}$ we will also say that $\boldsymbol{Q}$ **depends** on $\boldsymbol{P}$ and can be written $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$

Let $K = (U, \boldsymbol{R})$ be a knowledge base and let $\boldsymbol{P}, \boldsymbol{Q} \subseteq \boldsymbol{R}$

1. Knowledge $\boldsymbol{Q}$ **depends on knowledge** $\boldsymbol{P}$ iff $IND(\boldsymbol{P}) \subseteq IND(\boldsymbol{Q})$ note that $IND(\boldsymbol{P})$ is a set of pair

2. Knowledge $\boldsymbol{P}$ and $\boldsymbol{Q}$ are **equivalent** denoted as $\boldsymbol{P} \equiv \boldsymbol{Q}$ iff $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$ and $\boldsymbol{Q} \Rightarrow \boldsymbol{P}$

3. Knowledge $\boldsymbol{P}$ and $\boldsymbol{Q}$ are **independent** denoted as $\boldsymbol{P} \not\equiv \boldsymbol{Q}$ iff neither $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$ nor $\boldsymbol{Q} \Rightarrow \boldsymbol{P}$

Obiviously $\boldsymbol{P} \equiv \boldsymbol{Q}$ if and only if $IND(\boldsymbol{P}) = IND(\boldsymbol{Q})$

Proposition 4.1 (4.1)**.** *The following conditions are equivalent*

1. $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$

2. $IND(\boldsymbol{P} \cup \boldsymbol{Q}) = IND(\boldsymbol{P})$

3. $POS_{\boldsymbol{P}}(\boldsymbol{Q}) = POS_{IND(\boldsymbol{P})}(\boldsymbol{Q}) = U$

4. $\underline{\boldsymbol{P}}X = X$ *for all* $X \in U/Q$

where $\underline{\boldsymbol{P}}X$ *denotes* $\underline{IND(\boldsymbol{P})}X$

Proposition 4.2 (4.2)**.** *If* $\boldsymbol{P}$ *is a reduct of* $\boldsymbol{Q}$ *then* $\boldsymbol{P} \Rightarrow \boldsymbol{Q} - \boldsymbol{P}$ *and* $IND(\boldsymbol{P}) = IND(\boldsymbol{Q})$

Proof.

1. (1) $\rightarrow$ (2)

   $IND(\boldsymbol{P}) \subseteq IND(\boldsymbol{P} \cup \boldsymbol{Q}) \subseteq IND(\boldsymbol{P})$

2. (2) $\rightarrow$ (3)

$$
\begin{aligned}
POS_{IND(\boldsymbol{P})}(\boldsymbol{Q}) &= \bigcup_{X \in U/\boldsymbol{Q}} \underline{IND(\boldsymbol{P})}X \\
&= \bigcup_{X \in U/\boldsymbol{Q}} \underline{IND(\boldsymbol{P} \cup \boldsymbol{Q})}X
\end{aligned}
$$

   Since $\boldsymbol{Q} \subseteq \boldsymbol{P} \cup \boldsymbol{Q}$, $IND(\boldsymbol{P} \cup \boldsymbol{Q}) \subseteq IND(\boldsymbol{Q})$ and for each $x \in U$, $[x]_{IND(\boldsymbol{P} \cup \boldsymbol{Q})} \subseteq [x]_{IND(\boldsymbol{Q})}$, which means for any $Y \in U/\boldsymbol{P} \cup \boldsymbol{Q}$, there exists some $X \in U/\boldsymbol{Q}$ s.t. $Y \subseteq X$. Hence $POS_{\boldsymbol{P}}(\boldsymbol{Q}) = U$

3. $(3) \rightarrow (4)$

$$
\begin{aligned}
POS_{\boldsymbol{P}}(\boldsymbol{Q}) &= \bigcup_{X \in U/\boldsymbol{Q}} \underline{IND(\boldsymbol{P})}X \\
&= \bigcup_{X \in U/bQ} \underline{\boldsymbol{P}}X = U
\end{aligned}
$$

And $\underline{\boldsymbol{P}}X \subseteq X$

4. $(4) \rightarrow (1)$

$$
\begin{aligned}
\boldsymbol{P} \Rightarrow \boldsymbol{Q} &\Leftrightarrow IND(\boldsymbol{P}) \subseteq IND(\boldsymbol{Q}) \\
&\Leftrightarrow \forall x \in U, [x]_{IND(\boldsymbol{P})} \subseteq [x]_{IND(\boldsymbol{Q})}
\end{aligned}
$$

□

Proof. $\boldsymbol{P} \Rightarrow \boldsymbol{Q} - \boldsymbol{P} \Leftrightarrow IND(\boldsymbol{P} \cup \boldsymbol{Q} - \boldsymbol{P}) = IND(\boldsymbol{P})$ □

Proposition 4.3 (4.3)**.** *1. If $\boldsymbol{P}$ is dependent, then there exists a subset $\boldsymbol{Q} \subset \boldsymbol{P}$ s.t. $\boldsymbol{Q}$ is a reduct of $\boldsymbol{P}$*

2. If $\boldsymbol{P} \subseteq \boldsymbol{Q}$ and $\boldsymbol{P}$ is dependent, then all basic relations in $\boldsymbol{P}$ are pairwise independent

3. If $\boldsymbol{P} \subseteq \boldsymbol{Q}$ and $\boldsymbol{P}$ is independent, then every subset $\boldsymbol{R}$ of $\boldsymbol{P}$ is independent

Proposition 4.4 (4.4)**.** *1. If $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$ and $\boldsymbol{P}' \supset \boldsymbol{P}$, then $\boldsymbol{P}' \Rightarrow \boldsymbol{Q}$*

2. If $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$ and $\boldsymbol{Q}' \subset \boldsymbol{Q}$ then $\boldsymbol{P} \Rightarrow \boldsymbol{Q}'$

3. $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$ and $\boldsymbol{Q} \Rightarrow \boldsymbol{R}$ imply $\boldsymbol{P} \Rightarrow \boldsymbol{R}$

4. $\boldsymbol{P} \Rightarrow \boldsymbol{R}$ and $\boldsymbol{Q} \Rightarrow \boldsymbol{R}$ imply $\boldsymbol{P} \cup \boldsymbol{Q} \Rightarrow \boldsymbol{R}$

5. $\boldsymbol{P} \Rightarrow \boldsymbol{R} \cup \boldsymbol{Q}$ implies $\boldsymbol{P} \Rightarrow \boldsymbol{R}$ and $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$

6. $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$ and $\boldsymbol{R} \Rightarrow \boldsymbol{T}$ imply $\boldsymbol{P} \cup \boldsymbol{R} \Rightarrow \boldsymbol{Q} \cup \boldsymbol{T}$

7. $\boldsymbol{P} \Rightarrow \boldsymbol{Q}$ and $\boldsymbol{R} \Rightarrow \boldsymbol{T}$ imply $\boldsymbol{P} \cup \boldsymbol{R} \Rightarrow \boldsymbol{Q} \cup \boldsymbol{T}$

4.2 Partial dependency of knowledge

Let $K = (U, \boldsymbol{R})$ be the knowledge base and $\boldsymbol{P}, \boldsymbol{Q} \subset \boldsymbol{R}$. Knowledge $\boldsymbol{Q}$ **depends in a degree** $k (0 \leq k \leq 1)$ from knowledge $\boldsymbol{P}$, symbolically $\boldsymbol{P} \Rightarrow_k \boldsymbol{Q}$ if and only if

$$k = \gamma_{\boldsymbol{P}}(\boldsymbol{Q}) = \frac{card\, POS_{\boldsymbol{P}}(\boldsymbol{Q})}{card\, U}$$

If $k = 1$, $\boldsymbol{Q}$ **totally depends from** $\boldsymbol{P}$. If $0 < k < 1$, $\boldsymbol{Q}$ **roughly depends from** $\boldsymbol{P}$. If $k = 0$, $\boldsymbol{Q}$ is **totally independent from** $\boldsymbol{P}$

Ability to classify objects.

Proposition 4.5 (4.5)**.**

1. *If $\boldsymbol{R} \Rightarrow_k \boldsymbol{P}$ and $\boldsymbol{Q} \Rightarrow_l \boldsymbol{P}$, then $\boldsymbol{R} \cup \boldsymbol{Q} \Rightarrow \boldsymbol{P}$ for some $m \geq \max(k, l)$*
2. *If $\boldsymbol{R} \cup \boldsymbol{P} \Rightarrow_k \boldsymbol{Q}$, then $\boldsymbol{R} \Rightarrow_l \boldsymbol{Q}$ and $\boldsymbol{P} \Rightarrow_m \boldsymbol{Q}$ for some $l, m \leq k$*
3. *If $\boldsymbol{R} \Rightarrow_k \boldsymbol{Q}$ and $\boldsymbol{R} \Rightarrow_l \boldsymbol{P}$ then $\boldsymbol{R} \Rightarrow_m \boldsymbol{Q} \cup \boldsymbol{P}$ for some $m \leq \min(k, l)$*
4. *If $\boldsymbol{R} \Rightarrow_k \boldsymbol{Q} \cup \boldsymbol{P}$ then $\boldsymbol{R} \Rightarrow_l \boldsymbol{Q}$ and $\boldsymbol{R} \Rightarrow_m \boldsymbol{P}$ for some $l, m \geq k$*
5. *If $\boldsymbol{R} \Rightarrow_k \boldsymbol{P}$ and $\boldsymbol{P} \Rightarrow_l \boldsymbol{Q}$ then $\boldsymbol{R} \Rightarrow_m \boldsymbol{Q}$ for some $m \geq l + k - 1$*

5 Knowledege prepresentation

5.1 Formal definition

Knowledge representation system is a pair $S = (U, A)$ where U is a nonempty finite set called the **universe**, and A is a nonempty finite set of **primitive attributes**

Every primitive attribute $a \in A$ is a total function $a : U \to V_a$ where V_a is the **domain** of a

With every subset of attributes $B \subseteq A$ we associate a binary relation $IND(B)$ called and **indiscernibility relation**

$$IND(B) = \Big\{(x, y) \in U^2 : \text{for every } a \in B, a(x) = a(y)\Big\}$$

$IND(B)$ is an euivalence relation and

$$IND(B) = \bigcap_{a \in B} IND(a)$$

Every subset $B \subseteq A$ will be called an **attribute**. If B is a single element set, then B is called **primitive** otherwise **compound**

$a(x)$ can be viewed as a name of $[x]_{IND(a)}$. The name of an elementary category of attribute $B \subseteq A$ containing object x is a set of pairs $\{a, a(x) : a \in B\}$

There is a one-to-one correspondence between knowledge bases and knowledge representation system up to isomorphism of attributes and attribute names

Suppose

U	a	b	c	d	e
1	1	0	2	2	0
2	0	1	1	1	2
3	2	0	0	1	1
4	1	1	0	2	2
5	1	0	2	0	1
6	2	2	0	1	1
7	2	1	1	1	2
8	0	1	1	0	1

The universe $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$. $V = V_a = \cdots = V_e = \{0, 1, 2\}$

$$U/IND\{a\} = \{\{2, 8\}, \{1, 4, 5\}, \{3, 6, 7\}\}$$
$$U/IND\{c, d\} = \{\{1\}, \{3, 6\}, \{2, 7\}, \{4\}, \{5\}, \{8\}\}$$

5.2 Discernibility matrix

Let $S = (U, A)$ be a knowledge representation system with $U = \{x_1, x_2, \ldots, x_n\}$. By an **discernibility matrix of** S is

$$M(S) = (c_{ij}) = \{a \in A : a(x_i) \neq a(x_j)\} \quad \text{for } i, j = 1, 2, \ldots, n$$

Now the core can be defined as the set of all single element entries of the discernibility matrix

$B \subseteq A$ is the reduct of A if B is the minimal subset of A s.t.

$$B \cap c \neq \emptyset \text{ for any nonempty entry } c(c \neq \emptyset) \text{ in } M(S)$$

6 Decision tables

6.1 Formal definition and some properties

Let $K = (U, A)$ vbe a knowledge representation system and let $C, D \subset A$ be two subsets of attributes called **condition** and **decision attributes** repectively. KR-system with distinguished condition ad decision attributes will be called a **decision table** and will be denoted by $T = (U, A, C, D)$ or in short CD

Equivalence classes of the relations $IND(C)$ and $IND(D)$ will be called **condition** and **decision classes**

With every $x \in U$ we associate a function $d_x : A \to V$ s.t. $d_x(a) = a(x)$ for every $a \in C \cup D$. The function d_x will be called a **decision rule**

If d_x is a decision rule, then the restriction of d_x to C, denoted $d_x|C$ and the restriction of d_x to D, denoted $d_x|D$ will be called **conditions** and **decisions** of d_x

The decision rule d_x is **consistent** if for every $y \neq x, d_x|C = d_y|C$ implies $d_x|D = d_y|D$. Otherwise **inconsistent**

A decision table is **consistent** if al its decision rules are consistent

Proposition 6.1 (6.1). *A decision table $T = (U, A, C, D)$ is consistent if and only if $C \Rightarrow D$*

Proposition 6.2 (6.2). *Each decision table $T = (U, A, C, D)$ can be uniquely decomposed into two decision tables $T_1 = (U, A, C, D)$ and $T_2 = (U, A, C, D)$ s.t. $C \Rightarrow_1 D$ in T_1 and $C \Rightarrow_0 D$ in T_2 where $U_1 = POS_C(D)$ and $U_2 = \bigcup_{X \in U/IND(D)} BN_C(X)$*