Notes on Set Theory

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December 9, 2019

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Notes for the entrance examination

2 Models of Set - Sertraline

2.1 Some mathematical logic

Theorem 2.1 (Gödels second incompleteness theorem)**.** If a consistent recursive axiom set T contains **ZFC**, then

$$T \nvdash \mathrm{Con}(t)$$

especially, **ZFC** $\nvdash \mathrm{Con}(\textbf{ZFC})$

Definition 2.2. Suppose (M, E_M) and (N, E_N) are two models of set theory, then

1. if for any formula σ, $M \models \sigma$ if and only if $N \models \sigma$, then M and N are **elementary equivalent**, denoted by $M \equiv N$
2. If bijection $f : M \to N$ satisfies: for any $a, b \in M$, $aE_M b$ iff $f(a)E_N f(b)$, then $f : M \cong N$ is an **isomorphism**
3. If $M \subseteq N$ and $E_M = E_N \restriction M$, then M is N's submodel
4. If M is isomorphic to a submodel of N by injection f, and for any formula $\varphi(x_1, \ldots, x_n)$, for any $a_1, \ldots, a_n \in M$, $M \models \varphi[a_1, \ldots, a_n]$ iff $N \models \varphi[f(a_1), \ldots, f(a_n)]$, then f is called an **elementary embedding** from M to N, written as $f : M \prec N$
5. If $M \subseteq N$ and $M \prec N$, then M is a **elementary submodel** of N

Lemma 2.3. Suppose $N \models$ **ZFC**, $M \subseteq N$, then $M \prec N$ iff $\forall\varphi(x, x_1, \ldots, x_n)$, $\forall(a_1, \ldots, a_n) \in M$, if $\exists a \in N$ s.t. $N \models \varphi[a, a_1, \ldots, a_n]$, then $\exists a \in M$ s.t. $M \models \varphi[a, a_1, \ldots, a_n]$

Definition 2.4. Suppose $(M, E) \models$ **ZFC**

1. $h_\varphi : M^n \to M$ is φ's **Skolem function** if $\forall a_1, \ldots, a_n \in M$, if $\exists a \in M$ s.t. $M \models \varphi[a, a_1, \ldots, a_n]$, then $M \models \varphi[h_\varphi(a_1, \ldots, a_n), a_1, \ldots, a_n]$. requires **AC**
2. Let $\mathcal{H} = \{h_\varphi \mid \varphi \text{is a formula on set theory}\}$. For any $S \subseteq M$, **Skolem hull** $\mathcal{H}(S)$ is the smallest set consisting of S and closed under $\mathcal{H}$

Lemma 2.5. $N \models$ **ZFC**$, S \subseteq N$, if $M = \mathcal{H}(S)$, then $M \prec N$

Theorem 2.6 (Löwenheim-Skolem theorem)**.** Suppose $N \models$ **ZFC** and is infinite, then there is a model M s.t. $|M| = \omega$ and $M \prec N$

2.2 Cumulative Hierarchy

This section works in **ZF**(a.k.a. **ZF** $-$ axiom of foundation)

Definition 2.7. For any α, define sequence V_α

1. $V_0 = \emptyset$
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
3. For any limit ordinal λ, $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$

And **WF** $= \bigcup\limits_{\alpha \in \mathbf{On}} V_\alpha$

Lemma 2.8. For any ordinal α

1. V_α is transitive
2. if $\xi \leq \alpha$, then $V_\xi \subseteq V_\alpha$
3. if κ is inaccessible cardinal, then$|V_\kappa| = \kappa$

Proof. 1. Obviously $\kappa \leq V_\kappa$. Since κ is inaccessible, then for any $\alpha < \kappa$, $|V_\alpha| < \kappa$.

□

Definition 2.9. For any set $x \in$ **WF** ,

$$\text{rank}(x) = \min\{\beta \mid x \in V_{\beta+1}\}$$

Lemma 2.10. 1. $V_\alpha = \{x \in \mathbf{WF} \mid \text{rank}(x) < \alpha\}$

2. **WF** is transitive
3. For any $x, y \in$ **WF** , if $x \in y$, then $\text{rank}(x) < \text{rank}(y)$
4. for any $y \in$ **WF** , $\text{rank}(y) = \sup\{\text{rank}(x) + 1 \mid x \in y\}$

Lemma 2.11. Supoose α is an ordinal

1. $\alpha \in$ **WF** and $\text{rank}(\alpha) = \alpha$

2. $V_\alpha \cap \mathbf{On} = \alpha$

Lemma 2.12. 1. If $x \in \mathbf{WF}$, then $\bigcup x, \mathcal{P}(x), \{x\} \in \mathbf{WF}$, and their ranks are all less than $\operatorname{rank}(x) + \omega$

2. If $x, y \in \mathbf{WF}$, then $x \times y, x \cup y, x \cap y, \{x, y\}, (x, y), x^y \in \mathbf{WF}$, and their ranks are all less than $\operatorname{rank}(x) + \operatorname{rank}(y) + \omega$

3. $\mathbb{Z}, \mathbb{Q}, \mathbb{R} \in V_{\omega+\omega}$

4. for any set x, $x \in \mathbf{WF}$ iff $x \subset \mathbf{WF}$

Lemma 2.13. Suppose **AC**

1. for any group G, there exists group $G' \cong G$ in **WF**

2. for any topological space T, there exists $T' \cong T$ in **WF**

Definition 2.14. Binary relation $<$ on set A is **well-founded** if for any nonempty $X \subseteq A$, X has minimal element under $<$

Theorem 2.15. If $A \in \mathbf{WF}$, then $\in$ is a well-founded relation on A

Lemma 2.16. If set A is transitive and $\in$ is well-founded on A, then $A \in \mathbf{WF}$

Lemma 2.17. For any set x, there is a smallest transitive set $\mathrm{trcl}(x)$ s.t. $x \subseteq \mathrm{trcl}(x)$

Proof.

$$\begin{aligned} x_0 &= x \\ x_{n+1} &= \bigcup x_n \\ \mathrm{trcl}(x) &= \bigcup_{n<\omega} x_n \end{aligned}$$

□

$\mathrm{trcl}(x)$ is called **transitive closure** of x

Lemma 2.18. Without axiom of power set

1. if x is transitive, then $\mathrm{trcl}(x) = x$

2. if $y \in x$, then $\mathrm{trcl}(y) \subseteq \mathrm{trcl}(x)$

3. $\mathrm{trcl}(x) = x \cup \bigcup \{\mathrm{trcl}(y) \mid y \in x\}$

Theorem 2.19. For any set X, the following are equivalent

1. $X \in \mathbf{WF}$
2. $\operatorname{trcl}(X) \in \mathbf{WF}$
3. $\in$ is a well-founded relation on $\operatorname{trcl}(X)$

Theorem 2.20. The following propositions are equivalent

1. Axiom of foundation
2. For any set X, $\in$ is a well-founded relation on X
3. $\mathbf{V} = \mathbf{WF}$

2.3 Relativization

Definition 2.21. Let $\mathbf{M}$ be a class φ a formula, the **relativization** of φ to $\mathbf{M}$ is $\varphi^{\mathbf{M}}$ defined inductively

$$\begin{aligned}
(x \in y)^{\mathbf{M}} &\leftrightarrow x = y \\
(x \in y)^{\mathbf{M}} &\leftrightarrow x \in y \\
(\varphi \to \psi)^{\mathbf{M}} &\leftrightarrow \varphi^{\mathbf{M}} \to \psi^{\mathbf{M}} \\
(\neg\varphi)^{\mathbf{M}} &\leftrightarrow \neg\varphi^{\mathbf{M}} \\
(\forall x \varphi)^{\mathbf{M}} &\leftrightarrow (\forall x \in \mathbf{M}\,)\varphi^{\mathbf{M}}
\end{aligned}$$

Note $\varphi^{\mathbf{V}} = \varphi$ and

$$f^{\mathbf{M}} = \{(x_1, \dots, x_n, x_{n+1}) \in \mathbf{M} \mid \varphi^{\mathbf{M}}(x_1, \dots, x_n, x_{n+1})\}$$

Definition 2.22. For any theory T, any class $\mathbf{M}$, $\mathbf{M} \models T$ iff for any axiom φ of T, $\varphi^{\mathbf{M}}$ holds

Theorem 2.23 (ZF). $\mathbf{WF} \models \mathbf{ZF}$

Proof.

- **Axiom of existence**

  $(\exists x(x = x))^{\mathbf{M}} \leftrightarrow \exists x \in \mathbf{M}\,(x = x)$, which is equivalent to $\mathbf{M}$ being nonempty

- **Axiom of extensionality**

$$\forall X \forall Y \forall u((u \in X \leftrightarrow u \in Y) \to X = Y)^{\mathbf{M}} \Leftrightarrow$$
$$\forall X \in \mathbf{M}\, \forall Y \in \mathbf{M}\, \forall u \in \mathbf{M}\, ((u \in X \leftrightarrow u \in Y) \to X = Y)$$

Lemma 2.24. If **M** is transitive, then axiom of extensionality holds in **M**

- **Axiom schema of specification**

$$\forall X \in \mathbf{M}\, \exists Y \in \mathbf{M}\, \forall u \in \mathbf{M}\, (u \in Y \leftrightarrow u \in X \wedge \varphi^{\mathbf{M}}(u))$$

  Since for any $X \in$ **WF** , $\mathcal{P}(X) \subseteq$ **WF**

- **Axiom of paring**
- **Axiom of union**
- **Axiom of power set**

$$\forall X \in \mathbf{M}\, \exists Y \in \mathbf{M}\, \forall u \in \mathbf{M}\, (u \in Y \leftrightarrow (u \subseteq X)^{\mathbf{M}})$$

  and

$$(u \subseteq X)^{\mathbf{M}} \leftrightarrow \forall x \in \mathbf{M}\, (x \in u \to x \in X) \leftrightarrow u \cap \mathbf{M} \subseteq X$$

- **Axiom of foundation**
- **Axiom schema of replacement**

□

2.4 Absoluteness

Definition 2.25. For any formula $\psi(x_1, \ldots, x_n)$ and any class **M** ,**N** , **M** $\subseteq$ **N** , if

$$\forall x_1 \ldots \forall x_n \in \mathbf{M}\, (\psi^{\mathbf{M}}(x_1, \ldots, x_n) \leftrightarrow \psi^{\mathbf{N}}(x_1, \ldots, x_n))$$

then $\psi(x_1, \ldots, x_n)$ is **absolute** for **M** ,cn. If **N** $=$ **V** , then ψ is **absolute** for **M**

Lemma 2.26. Suppose **M** $\subseteq$ **N** and φ,ψ are formulas, then

1. if φ,ψ are absolute for **M** ,cn, then so are $\neg\varphi, \varphi \to \psi$
2. if φ doesn't contain any quantifiers, then φ is absolute for any **M**
3. if **M** ,**N** are transitive and φ is absolute for them, then so are $\forall x \in y\varphi$

Definition 2.27. Δ_0 formula

1. $x = y, x \in y$ are Δ_0 formulas

2. if φ,ψ are Δ_0, then so are $\neg\varphi, \varphi \to \psi$

3. if φ is Δ_0, y is any set, then $(\forall x \in y)\varphi$ is Δ_0

If φ is Δ_0, then $\exists x_1 \dots \exists x_n \varphi$ is Σ_1 formula, $\forall x_1 \dots \forall x_n \varphi$ is Π_1

Lemma 2.28. $\mathbf{M} \subseteq \mathbf{N}$ are both transitive, $\psi(x_0, \dots, x_n)$ is a formula, then

1. if ψ is Δ_0, then it's absolute for **M** ,cn

2. if ψ is Σ_1, then

$$\forall x_1 \dots x_n (\psi^{\mathbf{M}}(x_1, \dots, x_n) \to \psi^{\mathbf{N}}(x_1, \dots, x_n))$$

3. if ψ is Π_1, then

$$\forall x_1 \dots x_n (\psi^{\mathbf{N}}(x_1, \dots, x_n) \to \psi^{\mathbf{M}}(x_1, \dots, x_n))$$

Lemma 2.29. If $\mathbf{M} \subseteq \mathbf{N}$, $\mathbf{M} \models \Sigma, \mathbf{N} \models \Sigma$ and

$$\Sigma \vdash \forall x_1 \dots \forall x_n (\varphi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n))$$

then φ is absolute for **M** ,**N** if and only if ψ is absolute for **M** ,**N**

Definition 2.30. Suppose $\mathbf{M} \subseteq \mathbf{N}$, $f(x_1, \dots, x_n)$ is a function. f is **absolute** for **M** and **N** if and only if $\varphi(x_1, \dots, x_n, x_{n+1})$ defining f is absolute.

Theorem 2.31. Following relations and functions can be defined in $\mathbf{ZF}^- -$ Pow $-$ Inf and are equivalent to some Δ_0 formulas. So they are absolute for any transitive model **M** on $\mathbf{ZF}^- -$ Pow $-$ Inf

1. $x \in y$

2. $x = y$

3. $x \subset y$

4. $\{x, y\}$

5. $\{x\}$

6. (x, y)

7. $\emptyset$

8. $x \cup y$

9. $x - y$

10. $x \cap y$

11. x^+

12. x is a transitive set

13. $\bigcup x$

14. $\bigcap x$ ($\bigcap \emptyset = \emptyset$)

Lemma 2.32. Absoluteness is closed under operation composition

Theorem 2.33. Following relations and functions are absolute for any transitive model $\mathbf{M}$ on $\mathbf{ZF}^- - \mathrm{Pow} - \mathrm{Inf}$

1. z is an ordered pair
2. $A \times B$
3. R is a relation
4. $\mathrm{dom}(R)$
5. $\mathrm{ran}(R)$
6. f is a function
7. $f(x)$
8. f is injective

2.5 Relative consistence of the axiom of foundation

Lemma 2.34. Suppose transitive class $\mathbf{M} \models \mathbf{ZF}^- - \mathrm{Pow} - \mathrm{inf}$ and $\omega \in \mathbf{M}$, then the axiom of infinity is true in $\mathbf{M}$. Hence the axiom of infinity is true in $\mathbf{WF}$

Theorem 2.35. Let T be a theory of set theory language and Σ a set of sentences. Suppose $\mathbf{M}$ is a class and $T \vdash \mathbf{M} \neq \emptyset$, then if $\mathbf{M} \models_T \Sigma$, then

1. for any sentences φ, if $\Sigma \vdash \varphi$, then $T \vdash \varphi^{\mathbf{M}}$
2. if T is consistent, then so is $\mathrm{Cn}(\Sigma)$

Theorem 2.36. The axiom of foundation is consistent with **ZF**.

Proof. By 2.35, let T be **ZF**, Σ be **ZF** and **M** be **WF** $\square$

Lemma 2.37 (**ZF**$^-$)**.** Suppose transitive model $\mathbf{M} \models \mathbf{ZF}^- - \text{Pow} - \text{Inf}$. If $X, R \in \mathbf{M}$ and R is a well-order on X, then

$$(R \text{ is a well-order on } X)^{\mathbf{M}}$$

Theorem 2.38 (**ZF**$^-$)**.** $V_\omega \models \mathbf{ZFC} - \text{Inf} + \neg\text{Inf}$

Proof. For any $X \in V_\omega$, X is finite hence there is a well-ordering on X $\square$

Corollary 2.39. $\text{Con}(\mathbf{ZF}^-) \to \text{Con}(\mathbf{ZFC} - \text{Inf} + \neg\text{Inf})$

2.6 Induction and recursion based on well-order relation

Definition 2.40. $\boldsymbol{R}$ is a well-founded relation on $\boldsymbol{X}$ if and only if

$$\forall U \subset \boldsymbol{X}(U \neq \emptyset \to \exists y \in U(\neg\exists z \in U(z\boldsymbol{R}y)))$$

Definition 2.41. Relation $\boldsymbol{R}$ is **set-like** on $\boldsymbol{X}$ iff for any $x \in \boldsymbol{X}$,
$\{y \in \boldsymbol{X} \mid y\boldsymbol{R}x\}$ is a set

Definition 2.42. If $\boldsymbol{R}$ is a set-like relation on $\boldsymbol{X}$ and $x \in \boldsymbol{X}$, define

$$\begin{aligned}
\text{pred}^0(\boldsymbol{X}, x, \boldsymbol{R}) &= \{y \in \boldsymbol{X} \mid y\boldsymbol{R}x\} \\
\text{pred}^{n+1}(\boldsymbol{X}, x, bR) &= \bigcup\{\text{pred}(\boldsymbol{X}, y, \boldsymbol{R}) \mid y \in \text{pred}^n(\boldsymbol{X}, x, \boldsymbol{R})\} \\
\text{cl}(\boldsymbol{X}, x, \boldsymbol{R}) &= \bigcup_{n\in\omega} \text{pred}^n(\boldsymbol{X}, x, \boldsymbol{R})
\end{aligned}$$

Lemma 2.43. If $\boldsymbol{R}$ is a set-like relation on $\boldsymbol{X}$, then for any $y \in \text{cl}(\boldsymbol{X}, x, \boldsymbol{R})$, $\text{pred}(\boldsymbol{X}, y, \boldsymbol{R}) \subseteq \text{cl}(\boldsymbol{X}, x, \boldsymbol{R})$

Theorem 2.44 (Induction on well-founded set-like relation)**.** If $\boldsymbol{R}$ is a well-founded set-like relation on $\boldsymbol{X}$, then every nonempty $\boldsymbol{Y} \subseteq \boldsymbol{X}$ has minimal element under $\boldsymbol{R}$

Theorem 2.45. Suppose $\boldsymbol{R}$ is a well-founded set-like relation on $\boldsymbol{X}$. If $\boldsymbol{F} : \boldsymbol{X} \times \boldsymbol{V} \to \boldsymbol{V}$, then there is a unique $\boldsymbol{G} : \boldsymbol{X} \to \boldsymbol{V}$ s.t.

$$\forall x \in \boldsymbol{X}(\boldsymbol{G}(x) = \boldsymbol{F}(x, \boldsymbol{G}{\restriction}\text{pred}(\boldsymbol{X}, x, \boldsymbol{R})))$$

Definition 2.46. If $\boldsymbol{R}$ is a set-like well-founded relation on $\boldsymbol{X}$, define

$$\text{rank}(x, \boldsymbol{X}, \boldsymbol{R}) = \sup\{\text{rank}(y, \boldsymbol{X}, \boldsymbol{R}) + 1 \mid y\boldsymbol{R}x \wedge y \in \boldsymbol{X}\}$$

Note that

$$\boldsymbol{F}(x, h) = \sup\{\alpha + 1 \mid \alpha \in \text{ran}(h)\}$$

Lemma 2.47 ($\mathbf{ZF}^-$). If $\boldsymbol{X}$ is transitive and $\in$ is well-founded on $\boldsymbol{X}$, then $\boldsymbol{X} \subseteq \mathbf{WF}$ and for any $x \in \boldsymbol{X}$, $\text{rank}(x, \boldsymbol{X}, \in) = \text{rank}(x)$

Definition 2.48. $\boldsymbol{R}$ is a set-like well-founded relation on $\boldsymbol{X}$, **Mostowski function $\boldsymbol{G}$** on $(\boldsymbol{X}, \boldsymbol{R})$ is

$$\boldsymbol{G}(x) = \{\boldsymbol{G}(y) \mid y \in \boldsymbol{X} \wedge y\boldsymbol{R}x\}$$

$\mathbf{M} = \text{ran}(\boldsymbol{G})$ is called the **Mostowski collapse** of $(\boldsymbol{X}, \boldsymbol{R})$

Lemma 2.49.

1. $\forall x, y \in \boldsymbol{X}(x\boldsymbol{R}y \to \boldsymbol{G}(x) \in \boldsymbol{G}(y))$
2. $\mathbf{M}$ is transitive
3. If the axiom of power set holds, $\mathbf{M} \subseteq \mathbf{WF}$
4. if the axiom of power set holds and $x \in \boldsymbol{X}$, then
   $\text{rank}(x, \boldsymbol{X}, \boldsymbol{R}) = \text{rank}(\boldsymbol{G}(x))$

Definition 2.50. $\boldsymbol{R}$ is extensional on $\boldsymbol{X}$ iff

$$\forall x, y \in \boldsymbol{X}(\forall z \in \boldsymbol{X}(z\boldsymbol{R}x \leftrightarrow z\boldsymbol{R}y) \to x = y)$$

Lemma 2.51. If $\boldsymbol{X}$ is transitive then $\in$ is extensional on $\boldsymbol{X}$

Lemma 2.52. Let $\boldsymbol{R}$ be a set-like well-founded relation on $\boldsymbol{X}$, $\boldsymbol{G}$ is a Mostowski function on it. If $\boldsymbol{R}$ is extensional, then $\boldsymbol{G}$ is an isomorphism

Theorem 2.53 (Mostowski collapse theorem)**.** Suppose $\boldsymbol{R}$ is set-like well-founded extensional on $\boldsymbol{X}$, then there are unique transitive class $\mathbf{M}$ and bijection $\boldsymbol{G} : \boldsymbol{X} \to \mathbf{M}$ s.t. $\boldsymbol{G} : (\boldsymbol{X}, \boldsymbol{R}) \cong (\mathbf{M}, \in)$