Model Theory: An Introduction

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1 Structures and Theories

1.1 Languages and Structures

Definition 1.1. A language $\mathcal{L}$ is given by specifying the following data

1. A set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$
2. a set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$
3. a set of constant symbols $\mathcal{C}$

Definition 1.2. An $\mathcal{L}$-structure $\mathcal{M}$ is given by the following data

1. a nonempty set M called the **universe**, **domain** or **underlying set** of $\mathcal{M}$
2. a function $f^{\mathcal{M}} : M^{n_f} \to M$ for each $f \in \mathcal{F}$
3. a set $R^{\mathcal{M}} \subseteq M^{n_R}$ for each $R \in \mathcal{R}$
4. an element $c^{\mathcal{M}} \in M$ for each $c \in \mathcal{C}$

We refer to $f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}}$ as the **interpretations** of the symbols f, R and c. We often write the structure as $\mathcal{M} = (M, f^{\mathcal{M}}, R^{\mathcal{M}}, c^{\mathcal{M}} : f \in \mathcal{F}, R \in \mathcal{R}, c \in \mathcal{C})$

Definition 1.3. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures with universes M and N respectively. An $\mathcal{L}$**-embedding** $\eta : \mathcal{M} \to \mathcal{N}$ is a one-to-one map $\eta : M \to N$ that

1. $\eta(f^{\mathcal{M}}(a_1, \ldots, a_{n_f})) = f^{\mathcal{N}}(\eta(a_1), \ldots, \eta(a_{n_f}))$ for all $f \in \mathcal{F}$ and $a_1, \ldots, a_{n_f} \in M$
2. $(a_1, \ldots, a_{m_R}) \in R^{\mathcal{M}}$ if and only if $(\eta(a_1), \ldots, \eta(a_{m_R})) \in R^{\mathcal{N}}$ for all $R \in \mathcal{R}$ and $a_1, \ldots, a_{m_R} \in M$
3. $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for $c \in \mathcal{C}$

A bijective $\mathcal{L}$-embedding is called an $\mathcal{L}$**-isomorphism**. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-embedding, we say either $\mathcal{M}$ is a **substrcture** of $\mathcal{N}$ or that $\mathcal{N}$ is an **extension** of $\mathcal{M}$

The **cardinality** of $\mathcal{M}$ is $|M|$, the cardinality of the universe of $\mathcal{M}$

Definition 1.4. The set of $\mathcal{L}$**-terms** is the smallest set $\mathcal{T}$ s.t.

1. $c \in \mathcal{T}$ for each constant symbol $c \in \mathcal{C}$
2. each variable symbol $v_i \in \mathcal{T}$ for $i = 1, 2, \ldots$
3. if $t_1, \ldots, t_{n_f} \in \mathcal{T}$ and $f \in \mathcal{F}$ then $f(t_1, \ldots, n_{n_f}) \in \mathcal{T}$

Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and that t is a term built using variables from $\bar{v} = (v_{i_1}, \ldots, v_{i_m})$. We want to interpret t as a function $t^{\mathcal{M}} : M^m \to M$. For s a subterm of t and $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M$, we inductively define $s^{\mathcal{M}}(\bar{a})$ as follows.

1. If s is a constant symbol c, then $s^{\mathcal{M}}(\bar{a}) = c^{\mathcal{M}}$
2. If s is the variable v_{i_j}, then $s^{\mathcal{M}}(\bar{a}) = a_{i_j}$
3. If s is the term $f(t_1, \ldots, t_{n_f})$, where f is a function symbol of $\mathcal{L}$ and $t_1, \ldots, t_{n_f}$ are terms, then $s^{\mathcal{M}}(\bar{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_f}^{\mathcal{M}}(\bar{a}))$

The function $t^{\mathcal{M}}$ is defined by $\bar{a} \mapsto t^{\mathcal{M}}(\bar{a})$

Definition 1.5. ϕ is an **atomic $\mathcal{L}$-formula** if ϕ is either
1. $t_1 = t_2$ where t_1 and t_2 are terms
2. $R(t_1, \ldots, t_{n_R})$

The set of **$\mathcal{L}$-formulas** is the smallest set $\mathcal{W}$ containing the atomic formulas s.t.
1. if $\phi \in \mathcal{W}$, then $\neg\phi \in \mathcal{W}$
2. if $\phi, \psi \in \mathcal{W}$, then $(\phi \wedge \psi), (\phi \vee \psi) \in \mathcal{W}$
3. if $\phi \in \mathcal{W}$, then $\exists v_i \phi, \forall v_i \phi \in \mathcal{W}$

We say a variable v **occurs freely** in a formula ϕ if it is not inside a $\exists v$ or $\forall v$ quantifier; otherwise we say that it's **bound**. We call a formula a **sentence** if it has no free variables. We often write $\phi(v_1, \ldots, v_n)$ to make explicit the free variables in ϕ

Definition 1.6. Let ϕ be a formula with free variables from $\bar{v} = (v_{i_1,\ldots,v_{i_m}})$ and let $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M^m$. We inductively define $\mathcal{M} \models \phi\bar{a}$ as follows
1. If ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$
2. If ϕ is $R(t_1, \ldots, t_{m_R})$ then $\mathcal{M} \models \phi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{m_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$
3. If ϕ is $\neg\psi$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \not\models \psi(\bar{a})$
4. If ϕ is $(\psi \wedge \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \theta(\bar{a})$
5. If ϕ is $(\psi \vee \theta)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ or $\mathcal{M} \models \theta(\bar{a})$
6. If ϕ is $\exists v_j \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if there is $b \in M$ s.t. $\mathcal{M} \models \psi(\bar{a}, b)$
7. If ϕ is $\forall v_j \psi(\bar{v}, v_j)$ then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a}, b)$ for all $b \in M$

If $\mathcal{M} \models \phi(\bar{a})$ we say that $\mathcal{M}$ **satisfies** $\phi(\bar{a})$ or $\phi(\bar{a})$ is **true** in $\mathcal{M}$

Proposition 1.7. *Suppose that $\mathcal{M}$ is a substrcture of $\mathcal{N}$, $\bar{a} \in M$ and $\phi(\bar{v})$ is a quantifier-free formula. Then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \psi(\bar{a})$*

Proof. **Claim** If $t(\bar{v})$ is a term and $\bar{b} \in M$ then $t^{\mathcal{M}}(\bar{b}) = t^{\mathcal{N}}(\bar{b})$. □

Definition 1.8. We say that two $\mathcal{L}$-strctures $\mathcal{M}$ and $\mathcal{N}$ are **elementarily equivalent** and write $\mathcal{M} \equiv \mathcal{N}$ if

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{N} \models \phi$$

for all $\mathcal{L}$-sentences ϕ

We let $\mathrm{Th}(\mathcal{M})$, the **full theory** of $\mathcal{M}$ be the set of $\mathcal{L}$-sentences ϕ s.t. $\mathcal{M} \models \phi$

Theorem 1.9. *Suppose that $j : \mathcal{M} \to \mathcal{N}$ is an isomorphism. Then $\mathcal{M} \equiv \mathcal{N}$*

Proof. Show by induction on formulas that $\mathcal{M} \models \phi(a_1, \ldots, a_n)$ if and only if $\mathcal{N} \models \phi(j(a_1), \ldots, j(a_n))$ for all formulas ϕ □

1.2 Theories

Let $\mathcal{L}$ be a language. An **$\mathcal{L}$-theory** T is a set of $\mathcal{L}$-sentences. We say that $\mathcal{M}$ is a **model** of T and write $\mathcal{M} \models T$ if $\mathcal{M} \models \phi$ for all sentences $\phi \in T$. A theory is **satisfiable** if it has a model.

A class of $\mathcal{L}$-structures $\mathcal{K}$ is an **elementary class** if there is an $\mathcal{L}$-theory T s.t. $\mathcal{K} = \{\mathcal{M} : \mathcal{M} \models T\}$

Example 1.1 (Groups)**.** Let $\mathcal{L} = \{\cdot, e\}$ where $\cdot$ is a binary function symbol and e is a constant symbol. The class of groups is axiomatized by

$$\begin{aligned}
&\forall x \; e \cdot x = x \cdot e = x \\
&\forall x \forall y \forall z \; x \cdot (y \cdot z) = (x \cdot y) \cdot z \\
&\forall x \exists y \; x \cdot y = y \cdot x = e
\end{aligned}$$

Example 1.2 (Left R-modules)**.** Let R be a ring with multiplicative identity 1. Let $\mathcal{L} = \{+, 0\} \cup \{r : r \in R\}$ where $+$ is a binary function symbol, 0 is a constant, and r is a unary function symbol for $r \in R$. In an R-module, we will interpret r as scalar multiplication by R. The axioms for R-modules are

$$\begin{aligned}
&\forall x \;\; r(x + y) = r(x) + r(y) \text{ for each } r \in R \\
&\forall x \;\; (r + s)(x) = r(x) + s(x) \text{ for each } r, s \in R \\
&\forall x \;\; r(s(x)) = rs(x) \text{ for } r, s \in R \\
&\forall x \;\; 1(x) = x
\end{aligned}$$

Example 1.3 (Rings and Fields)**.** Let $\mathcal{L}_r$ be the language of rings $\{+, -, \cdot, 0, 1\}$, where $+, -$ and $\cdot$ are binary function symbols and 0 and 1 are constants.

The axioms for rings are given by

$$\begin{aligned}
&\forall x \forall y \forall z \ (x - y = z \leftrightarrow x = y + z) \\
&\forall x \ x \cdot 0 = 0 \\
&\forall x \forall y \forall z \ x \cdot (y \cdot z) = (x \cdot y) \cdot z \\
&\forall x \ x \cdot 1 = 1 \cdot x = x \\
&\forall x \forall y \forall z \ x \cdot (y + z) = (x \cdot y) + (x \cdot z) \\
&\forall x \forall y \forall z \ (x + y) \cdot z = (x \cdot z) + (y \cdot z)
\end{aligned}$$

We axiomatize the class of fields by adding

$$\begin{aligned}
&\forall x \forall y \ x \cdot y = y \cdot x \\
&\forall x \ (x \neq 0 \rightarrow \exists y \ x \cdot y = 1)
\end{aligned}$$

We axiomatize the class of algebraically closed fields by adding to the field axioms the sentences

$$\forall a_0 \ldots \forall a_{n-1} \exists x \ x^n + \sum_{i=1}^{n-1} a_i x^i = 0$$

for $n = 1, 2, \ldots$. Let ACF be the axioms for algebraically closed fields.

Let ψ_p be the $\mathcal{L}_r$-sentence $\forall x \ \underbrace{x + \cdots + x}_{p\text{-times}} = 0$, which asserts that a field has characteristic p. For $p > 0$ a prime, let $\mathrm{ACF}_p = \mathrm{ACF} \cup \{\psi_p\}$ and $\mathrm{ACF}_0 = \mathrm{ACF} \cup \{\neg\psi_p : p > 0\}$ be the theories of algebraically closed fields of characteristic p and zero respectively

Definition 1.10. Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence. We say that ϕ is a **logical consequence** of T and write $T \models \phi$ if $\mathcal{M} \models \phi$ whenever $\mathcal{M} \models T$

Proposition 1.11.

1. *Let $\mathcal{L} = \{+, <, 0\}$ and let T be the theory of ordered abelian groups. Then $\forall x (x \neq 0 \rightarrow x + x \neq 0)$ is a logical consequence of T*
2. *Let T be the theory of groups where every element has order 2. Then $T \not\models \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_1 \neq x_3)$*

Proof.

1. $\mathbb{Z}/2\mathbb{Z} \models T \wedge \neg \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \wedge x_2 \neq x_3 \wedge x_1 \neq x_3)$

□

1.3 Definable Sets and Interpretability

Definition 1.12. Let $\mathcal{M} = (M, \dots)$ be an $\mathcal{L}$-structure. We say that $X \subseteq M^n$ is **definable** if and only if there is an $\mathcal{L}$-formula $\phi(v_1, \dots, v_n, w_1, \dots, w_m)$ and $\bar{b} \in M^b$ s.t. $X = \{\bar{a} \in M^n : \mathcal{M} \models \phi(\bar{a}, \bar{b})\}$. We say that $\phi(\bar{v}, \bar{b})$ **defines** X. We say that X is A**-definable** or **definable over** A if there is a formula $\psi(\bar{v}, w_1, \dots, w_l)$ and $\bar{b} \in A^l$ s.t. $\psi(\bar{v}, \bar{b})$ defines X

A number of examples using $\mathcal{L}_r$, the language of rings

- Let $\mathcal{M} = (R, +, -, \cdot, 0, 1)$ be a ring. Let $p(X) \in R[X]$. Then $Y = \{x \in R : p(x) = 0\}$ is definable. Suppose that $p(X) = \sum_{i=0}^{m} a_i X^i$. Let $\phi(v, w_0, \dots, w_n)$ be the formula

$$w_n \cdot \underbrace{v \cdots v}_{n\text{-times}} + \cdots + w_1 \cdot v + w_0 = 0$$

  Then $\phi(v, a_0, \dots, a_n)$ defines Y. Indeed, Y is A-definable for any $A \supseteq \{a_0, \dots, a_n\}$
- Let $\mathcal{M} = (\mathbb{R}, +, -, \cdot, 0, 1)$ be the field of real numbers. Let $\phi(x, y)$ be the formula

$$\exists z(z \neq 0 \wedge y = x + z^2)$$

  Because $a < b$ if and only if $\mathcal{M} \models \phi(a, b)$, the ordering is $\emptyset$-definable
- Consider the natural numbers $\mathbb{N}$ as an $\mathcal{L} = \{+, \cdot, 0, 1\}$ structure. There is an $\mathcal{L}$-formula $T(e, x, s)$ s.t. $\mathbb{N} \models T(e, x, s)$ if and only if the Turing machine with program coded by e halts on input x in at most s steops. Thus the Turing machine with program e halts on input x if and only if

$\mathbb{N} \models \exists s\, T(e, x, s)$. So the halting computations is definable

Proposition 1.13. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Suppose that D_n is a collection of subsets of M^n for all $n \geq 1$ and $\mathcal{D} = (D_n : n \geq 1)$ is the smallest collection s.t.*

1. $M^n \in D_n$
2. *for all n-ary function symbols f of $\mathcal{L}$, the graph of $f^{\mathcal{M}}$ is in D_{n+1}*
3. *for all n-ary relation symbols R of $\mathcal{L}$, $R^{\mathcal{M}} \in D_n$*
4. *for all $i, j \leq n$, $\{(x_1, \dots, x_n) \in M^n : x_i = x_j\} \in D_n$*
5. *if $X \in D_n$, then $M \times X \in D_{n+1}$*
6. *each D_n is cloed under complement, union and intersection*
7. *if $X \in D_{n+1}$ and $\pi : M^{n+1} \to M^n$ is the projection $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$, then $\pi(X) \in D_n$*
8. *if $X \in D_{n+m}$ and $b \in M^m$, then $\{a \in M^n : (a, b) \in X\} \in D_n$*

Thus $X \subseteq M^n$ is definable if and only if $X \in D_n$

Proposition 1.14. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. If $X \subset M^n$ is A-definable, then every $\mathcal{L}$-automorphism of $\mathcal{M}$ that fixes A pointwise fixes X setwise(that is, if σ is an automorphism of M and $\sigma(a) = a$ for all $a \in A$, then $\sigma(X) = X$)*

Proof.

$$\mathcal{M} \models \psi(\bar{v}, \bar{a}) \leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{v}), \sigma(\bar{a})) \leftrightarrow \mathcal{M} \models \psi(\sigma(\bar{v}), \bar{a})$$

In other words, $\bar{b} \in X$ if and only if $\sigma(\bar{b}) \in X$ $\square$

Definition 1.15. A subset S of a field L is **algebraically independent** over a subfield K if the elements of S do not satisfy any non-trivial polynomial equation with coefficients in K

Corollary 1.16. *The set of real numbers is not definable in the field of complex numbers*

Proof. If $\mathbb{R}$ where definable, then it would be definable over a finite $A \subset \mathbb{C}$. Let $r, s \in \mathbb{C}$ be algebraically independent over A with $r \in \mathbb{R}$ and $s \notin \mathbb{R}$. There is an automorphism σ of $\mathbb{C}$ s.t. $\sigma|A$ is the identity and $\sigma(r) = s$. Thus $\sigma(\mathbb{R}) \neq \mathbb{R}$ and $\mathbb{R}$ is not definable over A $\square$

We say that an $\mathcal{L}_0$-structure $\mathcal{N}$ is **definably interpreted** in an $\mathcal{L}$-structure $\mathcal{M}$ if and only if we can find a definable $X \subseteq M^n$ for some n and we can interpret the symbols of $\mathcal{L}_0$ as definable subsets and functions on X so that the resulting $\mathcal{L}_0$-structure is isomorphic to $\mathcal{M}$

For example, let K be a field and G be $\mathrm{GL}_2(K)$, the group of invertible 2×2 matrices over K. Let $X = \{(a, b, c, d) \in K^4 : ad - bc \neq 0\}$. Let $f : X^2 \to X$ by

$$f((a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2)) = \\ (a_1 a_2 + b_1 c_2, a_1 b_2 + b_1 d_2, c_1 a_2 + d_1 c_2, c_1 b_2 + d_1 d_2)$$

X and f are definable in $(K, +, \cdot)$, and the set X with operation f is isomorphic to $\mathrm{GL}_2(K)$, where the identity element of X is $(1, 0, 0, 1)$

Clearly, $(\mathrm{GL}_n(K), \cdot, e)$ is definably interpreted in $(K, +, \cdot, 0, 1)$. A **linear algebraic group** over K is a subgroup of $\mathrm{GL}_n(K)$ defined by polynomial equations over K. Any linear algebraic group over K is definably interpreted in K

Let F be an infinite field and let G be the group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a, b \in F, a \neq 0$. This group is isomorphic to the group of affine transformations $x \mapsto ax + b$, where $a, b \in F$ and $a \neq 0$

We will show that F is definably interpreted in the group G. Let

$$\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix}$$

where $\tau \neq 0$. Let

$$A = \{g \in G : g\alpha = \alpha g\} = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F\}$$

$$B = \{g \in G : g\beta = \beta g\} = \{\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} : x \neq 0\}$$

Clearly A, B are definable using parameters α and β

B acts on A by conjugation

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{y}{x} \\ 0 & 1 \end{pmatrix}$$

We can define the map $i : A \backslash \{1\} \to B$ by $i(a) = b$ if and only if $b^{-1}ab = \alpha$, that is

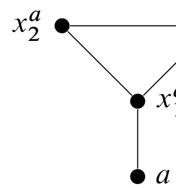
$$i \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

Define an operation $*$ on A by

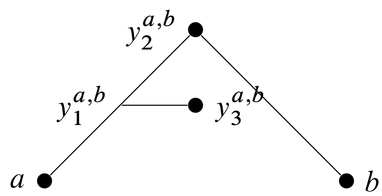
$$a * b = \begin{cases} i(b)a(i(b))^{-1} & \text{if } b \neq I \\ 1 & \text{if } b = I \end{cases}$$

where I is the identity matrix. Now $(F, +, \cdot, 0, 1) \cong (A, \cdot, *, 1, \alpha)$

Very complicated structures can often be interpreted in seemingly simpler ones. For example, any structure in a countable language can be interpreted in a graph. Let $(A, <)$ be a linear order. For each $a \in A$, G_A will have vertices a, x_1^a, x_2^a, x_3^a and contain the subgraph

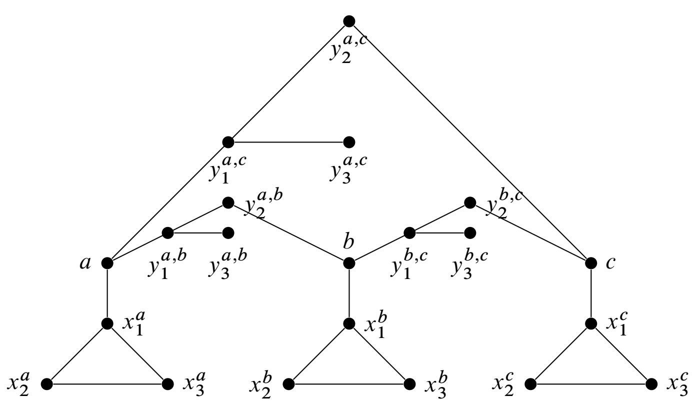


If $a < b$, then G_A will have vertices $y_1^{a,b}, y_2^{a,b}, y_3^{a,b}$ and contain the subgraph



Let $V_A = A \cup \{x_1^a, x_2^a, x_3^a : a \in A\} \cup \{y_1^{a,b}, y_2^{a,b}, y_3^{a,b} : a, b \in A \text{ and } a < b\}$, and let R_A be the smallest symmetric relation containing all edges drawn above.

For example, if A is the three-element linear order $a < b < c$, then G_A is the graph



Let $\mathcal{L} = \{R\}$ where R is a binary relation. Let $\phi(x, u, v, w)$ be the formula asserting that x, u, v, w are distinct, there are edges $(x, u), (u, v), (v, w), (u, w)$ and these are the only edges involving u, v, w. $G_A \models \phi(a, x_1^a, x_2^a, x_3^a)$ for all $a \in A$.

$\psi(x, y, u, v, w)$ asserts that x, y, u, v, w are distinct. $(x, u), (u, v), (u, w), (v, y)$

Define $\theta_i(z)$ as follows:

$$\begin{aligned}
\theta_0(z) &:= \exists u \exists v \exists w\ \phi(z, u, v, w) \\
\theta_1(z) &:= \exists x \exists v \exists w\ \phi(x, z, v, w) \\
\theta_2(z) &:= \exists u \exists u \exists w\ \phi(x, u, z, w) \\
\theta_3(z) &:= \exists x \exists y \exists v \exists w\ \psi(x, y, z, v, w) \\
\theta_4(z) &:= \exists x \exists y \exists u \exists w\ \psi(x, y, u, z, w) \\
\theta_5(z) &:= \exists x \exists y \exists u \exists v\ \psi(x, y, u, v, z)
\end{aligned}$$

If $a, b \in A$ and $a < b$, then

$$G_A \models \theta_0(a) \wedge \theta_1(x_1^a) \wedge \theta_2(x_2^a) \wedge \theta_2(x_3^a)$$

and

$$G_A \models \theta_3(y_1^{a,b}) \wedge \theta_4(y_2^{a,b}) \wedge \theta_5(y_3^{a,b})$$

Lemma 1.17. *If $(A, <)$ is a linear order, then for all vertices x in G, there is a unique $i \leq 5$ s.t. $G_A \models \theta_i(x)$*

Let T be the $\mathcal{L}$-theory with the following axioms

1. R is symmetric and irreflexive
2. for all x, exactly one θ_i holds
3. if $\theta_0(x)$ and $\theta_0(y)$ then $\neg R(x, y)$
4. if $\exists u \exists v \exists w\ \psi(x, y, u, v, w)$ then $\forall u_1 \forall v_1 \forall w_1 \neg \psi(y, x, u_1, v_1, w_1)$
5. if $\exists u \exists v \exists w\ \psi(x, y, u, v, w)$ and $\exists u \exists v \exists w\ \psi(y, z, u, v, w)$ then $\exists u \exists v \exists w\ \psi(x, z, u, v, w)$
6. if $\theta_0(x)$ and $\theta_0(y)$, then either $x = y$ or $\exists u \exists v \exists w\ \psi(x, y, u, v, w)$ or $\exists u \exists v \exists w\ \psi(y, x, u, v, w)$
7. if $\phi(x, u, v, w) \wedge \phi(x, u', v', w')$, then $u = u', v = v', w = w'$
8. if $\psi(x, y, u, v, w) \wedge \psi(x, y, u', v', w')$, then $u' = u, v = v', w = w'$

If $(A, <)$ is a linear order, then $G_A \models T$

Suppose $G \models T$. Let $X_G = \{x \in G : G \models \theta_0(x)\}$

Lemma 1.18. *If $(A, <)$ is a linear order, then $(X_{G_A}, <_{G_A}) \cong (A, <)$. Moreover, $G_{X_G} \cong G$ for all $G \models T$*

Definition 1.19. An $\mathcal{L}_0$-structure $\mathcal{N}$ is **interpretable** in an $\mathcal{L}$-structure M if there is a definable $X \subseteq M^n$, a definable equivalence relation E on X, and for each symbol of $\mathcal{L}_0$ we can find definable E-invariant sets on X s.t. X/E with the induced structure is isomorphic to $\mathcal{N}$

1.4 Answers to Exercises

Exercise 1.4.1.

1. transform ψ to CNF
2. prenex normal form

Exercise 1.4.2.

1. $$\begin{array}{cc} s & rs \\ \bullet & \bullet \\ e & r \\ \bullet & \bullet \end{array}$$
2. enumerate $\mathcal{M}$'s functions, relations and constants

Exercise 1.4.3. [1] Note that every $\mathcal{L}$-structure $\mathcal{M}$ of size κ is isomorphic to an $\mathcal{L}$-structure with domain κ. For each relation symbols, we have 2^κ options. If the language has size λ, this is at most $(2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^{\max(\lambda,\kappa)}$

Exercise 1.4.4.

$$\begin{aligned} T \models \phi &\Leftrightarrow \forall \mathcal{M}\ \mathcal{M} \models T \to \mathcal{M} \models \phi \\ &\Leftrightarrow \forall \mathcal{M}\ \mathcal{M} \models T' \to \mathcal{M} \models \phi \\ &\Leftrightarrow T' \models \phi \end{aligned}$$

Exercise 1.4.5. Follow the definition

Exercise 1.4.6. Since there is no model $\mathcal{M}$ s.t. $\mathcal{M} \models T$. It's true that $T \models \phi$

Exercise 1.4.7.

1. Suppose $\mathcal{M} \models \phi$, then $E^{\mathcal{M}}$ is an equivalent relation and each equivalence class's cardinality is 2
2. follows from number theory
3. [**?**]

Exercise 1.4.8. TBD

Exercise 1.4.9. $G(f) = \{(\bar{x}, \bar{y}) \in M^{n+m} \mid \phi(\bar{x}, \bar{y})\}$ and $G(g) = \{(\bar{y}, \bar{z}) \in M^{m+l} \mid \psi(\bar{y}, \bar{z})\}$. Hence $G(g \circ f) = \{(\bar{x}, \bar{z}) \in M^{n+l} \mid \phi(\bar{x}, \bar{y}) \wedge \psi(\bar{y}, \bar{z})\}$

Exercise 1.4.10. $\phi(\bar{a}, b)$ really defines a function and since $\phi(\bar{a}, y) \to y = b$

2 Basic Techniques

2.1 The Compactness Theorem

A language $\mathcal{L}$ is **recursive** if there is an algorithm that decides whether a sequence of symbols is an $\mathcal{L}$-formula. An $\mathcal{L}$-theory T is **recursive** if there is an algorithm that when given an $\mathcal{L}$-sentence ϕ as input, decides whether $\phi \in T$

[1] stackexchange

Proposition 2.1. *If $\mathcal{L}$ is a recursive language and T is a recursive $\mathcal{L}$-theory, then $\{\phi : T \vdash \phi\}$ is recursively enumerable; that is, there is an algorithm that when given ϕ as input will halt accepting if $T \vdash \phi$ and not halt if $T \nvdash \phi$*

Proof. There is $\sigma_0, \sigma_1, \ldots$ a computable listing of all finite sequence of $\mathcal{L}$-formulas. At stage i, we check to see whether σ_i is a proof of ψ from T. If it is, then halt. □

Theorem 2.2 (Gödel's Completeness Theorem)**.** *Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence, then $T \models \phi$ if and only if $T \vdash \phi$*

We say that an $\mathcal{L}$-theory T is **inconsistent** if $T \vdash (\phi \wedge \neg\phi)$ for some sentence ϕ.

Corollary 2.3. *T is consistent if and only if T is satisfiable*

Proof. Supose that T is not satisfiable, then every model of T is a model of $\phi \wedge \neg\phi$. Thus by the Completeness theorem $T \vdash (\phi \wedge \neg\phi)$ □

Theorem 2.4 (Compactness Theorem)**.** *T is satisfiable if and only if every finite subset of T is satisfiable*

Proof. If T is not satisfiable, then T is inconsistent. Let σ be a proof of a contradiction from T. Because σ is finite, only finitely many assumptions from T are used in the proof. Thus there is a finite $T_0 \subseteq T$ s.t. σ is a proof of a contradiction from T_0 □

2.1.1 Henkin Constructions

A theory T is **finitely satisfiable** if every finite subset of T is satisfiable. We will show that every finitely satisfiable theory T is satisfiable.

Definition 2.5. We say that an $\mathcal{L}$-theory T has the **witness property** if whenever $\phi(v)$ is an $\mathcal{L}$-formula with one free variable v, then there is a constant symbol $c \in \mathcal{L}$ s.t. $T \vdash (\exists v \phi(v)) \rightarrow \phi(c)$

An $\mathcal{L}$-theory T is **maximal** if for all ϕ either $\phi \in T$ or $\neg\phi \in T$

Lemma 2.6. *Suppose T is a maximal and finitely satisfiable $\mathcal{L}$-theory. If $\Delta \subseteq T$ is finite and $\Delta \models \psi$, then $\psi \in T$*

Proof. If $\psi \notin T$, then $\neg\psi \in T$ but $\Delta \cup \{\psi\}$ is unsatisfiable □

Lemma 2.7. *Suppose that T is a maximal and finitely satisfiable $\mathcal{L}$-theory with the witness property. Then T has a model. In fact, if κ is a cardinal and $\mathcal{L}$ has at most κ constant symbols, then there is $\mathcal{M} \models T$ with $|\mathcal{M}| \leq \kappa$*

Proof. Let $\mathcal{C}$ be the set of constant symbols of $\mathcal{L}$. For $c, d \in \mathcal{C}$, we say $c \sim d$ if $T \models c = d$

Claim 1 $\sim$ is an equivalence relation.

The universe of our model will be $M = \mathcal{C}/\sim$. Clearly$|M| \leq \kappa$. We let c^* denote the equivalence class of c and interprete c as its equivalence class, that is, $c^{\mathcal{M}} = c^*$

Suppose that R is an n-ary relation symbol of $\mathcal{L}$

Claim 2 Suppose that $c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathcal{C}$ and $c_i \sim d_i$ for $i = 1, \ldots, n$, then $R(\bar{c})$ if and only if $R(\bar{d})$

$$R^{\mathcal{M}} = \{(c_1^*, \ldots, c_n^*) : R(c_1, \ldots, c_n) \in T\}$$

Suppose that f is an n-ary function symbol of $\mathcal{L}$ and $c_1, \ldots, c_n \in \mathcal{C}$. Because $\underline{\emptyset \models \exists v f(c_1, \ldots, c_n) = v}$, and T has the witness property, then there is $c_{n+1} \in \mathcal{C}$ s.t. $f(c_1, \ldots, c_n) = c_{n+1} \in T$. As above, if $d_i \sim c_i$ for $i = 1, \ldots, n+1$, then $f(d_1, \ldots, d_n) = d_{n+1} \in T$. Thus we get a well-defined function $f^{\mathcal{M}} : M^n \to M$ by

$$f^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^* \text{ if and only if } f(c_1, \ldots, c_n) = d \in T$$

Claim 3 Suppose that t is a term using free variables from $v_1, \ldots, v_n$. If $c_1, \ldots, c_n, d \in \mathcal{C}$, then $t(c_1, \ldots, c_n) = d \in T$ if and only if $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$

($\Leftarrow$) Suppose $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$. By the witness property, there is a $e \in \mathcal{C}$ s.t. $t(c_1, \ldots, c_n) = e \in T$. Using the ($\Rightarrow$) direction of the proof, $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = e^*$. Thus $e^* = d^*$ and $e = d \in T$

Claim 4 For all $\mathcal{L}$-formulas $\phi(v_1, \ldots, v_n)$ and $c_1, \ldots, c_n \in \mathcal{C}$, $\mathcal{M} \models \phi(\bar{c}^*)$ if and only if $\phi(\bar{c}) \in T$ $\square$

Lemma 2.8. *Let T be a finitely satisfiable $\mathcal{L}$-theory. There is a language $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ a finitely satisfiable $\mathcal{L}^*$-theory s.t. any $\mathcal{L}^*$-theory extending T^* has the witness property. We can choose $\mathcal{L}^*$ s.t. $\left|\mathcal{L}^*\right| = |\mathcal{L}| + \aleph_0$*

Proof. We first show that there is a language $\mathcal{L}_1 \supseteq \mathcal{L}$ and a finitely satisfiable $\mathcal{L}_1$-theory $\mathcal{L}_1 \supseteq T$ s.t. for any $\mathcal{L}$-formula $\phi(v)$ there is an $\mathcal{L}_1$-constant symbol c s.t. $T_1 \models (\exists v \phi(v)) \to \phi(c)$. For each $\mathcal{L}$-formula $\phi(v)$, let c_ϕ be a new constant symbol and let $\mathcal{L}_1 = \mathcal{L} \cup \{c_\phi : \phi(v) \text{ an } \mathcal{L}\text{-formula}\}$. For each $\mathcal{L}$-formula $\phi(v)$, let Θ_ϕ be the $\mathcal{L}_1$-sentence $(\exists v \phi(v)) \to \phi(c_\phi)$. Let $T_1 = T \cup \{\Theta_\phi : \phi(v) \text{ an } \mathcal{L}\text{-formula}\}$

Claim T_1 is finitely satisfiable

Suppose that Δ is a finite subset of T_1. Then $\Delta = \Delta_0 \cup \{\Theta_{\phi_1}, \ldots, \Theta_{\phi_n}\}$ where Δ_0 is a finite subset of T and there is $\mathcal{M} \models \Delta_0$. We will make $\mathcal{M}$ into an $\mathcal{L} \cup \{c_{\phi_1}, \ldots, c_{\phi_n}\}$-structure $\mathcal{M}'$. If $\mathcal{M} \models \exists v \phi(v)$, choose a_i some element of M s.t. $\mathcal{M} \models \phi(a_i)$ and let $c_{\phi_i}^{\mathcal{M}'} = a_i$. Otherwise, let $c_{\phi_i}^{\mathcal{M}'}$ be any element of $\mathcal{M}$. Clearly $\mathcal{M}' \models \Theta_{\phi_i}$ for $i \leq n$. Thus T_1 is finitely satisfiable.

We now iterate the construction above to build a sequence of languages $\mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \ldots$ and a sequence of finitely satisfiable $\mathcal{L}_i$-theories $T \subseteq T_1 \subseteq T_2 \subseteq \ldots$ s.t. if $\phi(v)$ is an $\mathcal{L}_i$-formula then there is a constant symbol $c \in \mathcal{L}_{i+1}$ s.t. $T_{i+1} \models (\exists v \phi(v)) \to \phi(c)$

Let $\mathcal{L}^* = \bigcup \mathcal{L}_i$ and $T^* = \bigcup T_i$. And by induction, $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$ □

Lemma 2.9. *Suppose that T is a finitely satisfiable $\mathcal{L}$-theory and ϕ is an $\mathcal{L}$-sentence, then either $T \cup \{\phi\}$ or $T \cup \{\neg\phi\}$ is finitely satisfiable*

Corollary 2.10. *If T is a finitely satisfiable $\mathcal{L}$-theory, then there is a maximal finitely satisfiable $\mathcal{L}$-theory $T' \supseteq T$*

Proof. Let I be the set of all finitely satisfiable $\mathcal{L}$-theory containing T. We partially order I by inclusion. If $C \subseteq I$ is a chain, let $T_C = \bigcup\{\Sigma : \Sigma \in C\}$. If Δ is a finite subset of T_C, then there is a $\Sigma \in C$ s.t. $\Delta \subseteq \Sigma$, so T_C is finitely satisfiable and $T_C \supseteq \Sigma$ for all $\Sigma \in C$. Thus every chain in I has an upper bound, and we can apply Zorn's lemma to find a $T' \in I$ maximal w.r.t. the partial order. □

Theorem 2.11 (stengthening of Compactness Theorem)**.** *If T is a finitely satisfiable $\mathcal{L}$-theory and κ is an infinite cardinal with $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality at most κ*

Proposition 2.12. *Let $\mathcal{L} = \{\cdot, +, <, 0, 1\}$ and let* Th($\mathbb{N}$) *be the full $\mathcal{L}$-theory of the natural numbers. There is $\mathcal{M} \models$* Th($\mathbb{N}$) *and $a \in M$ s.t. a is larget than every natural number*

Proof. Let $\mathcal{L}^* = \mathcal{L} \cup \{c\}$ where c is a new constant symbol and let

$$T = \mathrm{Th}(\mathbb{N})] \cup \{\underbrace{1 + 1 + \cdots + 1}_{n\text{-times}} < c : \text{for } n = 1, 2, \ldots\}$$

If Δ is a finite subset of T we can make $\mathbb{N}$ a model of Δ by interpreting c as a suitably large natural number. Thus T is finitely satisfiable and there is $\mathcal{M} \models T$. □

Lemma 2.13. *If $T \models \phi$, then $\Delta \models T$ for some finite $\Delta \subseteq T$*

Proof. Suppose not. Let $\Delta \subseteq T$ be finite. Because $\Delta \not\models \phi$, $\Delta \cup \{\neg\phi\}$ is satisfiable. Thus $T \cup \{\neg\phi\}$ is finitely satisfiable and by the compactness theorem, $T \not\models \phi$ □

2.2 Complete Theories

Definition 2.14. An $\mathcal{L}$-theory T is called **complete** if for any $\mathcal{L}$-sentence ϕ either $T \models \phi$ or $T \models \neg\phi$

For $\mathcal{M}$ an $\mathcal{L}$-structure, then the full theory

$$\mathrm{Th}(\mathcal{M}) = \{\phi : \phi \text{ is an } \mathcal{L}\text{-sentence and } \mathcal{M} \models \phi\}$$

is a complete theory.

Proposition 2.15. *Let T be an $\mathcal{L}$-theory with infinite models. If κ is an infinite cardinal and $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality κ*

Proof. Let $\mathcal{L}^* = \mathcal{L} \cup \{c_\alpha : \alpha < \kappa\}$, where each c_α is new constant symbol, and let T^* be the $\mathcal{L}^*$-theory $T \cup \{c_\alpha \neq c_\beta : \alpha, \beta < \kappa, \alpha \neq \beta\}$. Clearly if $\mathcal{M} \models T^*$, then $\mathcal{M}$ is a model of T of cardinality at least κ. Thus by Theorem 2.11, it suffices to show that T^* is finitely satisfiable. But if $\Delta \subseteq T^*$ is finite, then $\Delta \subseteq T \cup \{c_\alpha \neq c_\beta : \alpha \neq \beta, \alpha, \beta \in I\}$, where I is a finite subset of κ. Let $\mathcal{M}$ be an infinite model of T. We can interpret the symbols $\{c_\alpha : \alpha \in I\}$ as $|I|$ distinct elements of M. Because $\mathcal{M} \models \Delta$, T^* is finitely satisfiable. $\square$

Definition 2.16. Let κ be an infinite cardinal and let T be a theory with models of size κ. We say that T is **κ-categorical** if any two models of T of cardinality κ are isomorphic.

Let $\mathcal{L} = \{+, 0\}$ be the language of additive groups and let T be the $\mathcal{L}$-theory of torsion-free divisible Abelian groups. The axioms of T are the axioms for Abelian groups together with the axioms

$$\forall x (x \neq 0 \to \underbrace{x + \cdots + x}_{n\text{-times}} \neq 0)$$
$$\forall y \exists x \underbrace{x + \cdots + x}_{n\text{-times}} = y$$

for $n = 1, 2, \ldots$

Proposition 2.17. *The theory of torsion-free divisible Abelian groups is κ-categorical for all $\kappa > \aleph_0$*

Proof. We first argue that models of T are essentially vector spaces over the field of rational numbers $\mathbb{Q}$. If V is any vector space over $\mathbb{Q}$, then the underlying additive group V is a model of T. Check StackExchange. On the other hand, if $G \models T$, $g \in G$ and $n \in \mathbb{N}$ with $g > 0$, we can find $h \in G$

s.t. $nh = g$. If $nk = g$, then $n(h - k) = 0$. Because G is torsion-free there is a unique $h \in G$ s.t. $nh = g$. We call this element g/n. We can view G as a $\mathbb{Q}$-vector space under the action $\frac{m}{n}g = m(g/n)$

Two $\mathbb{Q}$-vector spaces are isomorphic if and only if they have the same dimension. Thus the model of T are determined up to isomorphism by their dimension. If G has dimension λ, then $|G| = \lambda + \aleph_0$. If κ is uncountable and G has cardinality κ, then G has dimension κ. Thus for $\kappa > \aleph_0$ any two models of T of cardinality κ are isomorphic □

Proposition 2.18. *ACF_p is κ-categorical for all uncountable cardinals κ*

Proof. □

Theorem 2.19 (Vaught's Test)**.** *Let T be a satisfiable theory with no finite models that is κ-categorical for some infinite cardinal $\kappa \geq |\mathcal{L}|$. Then T is complete*

Proof. Suppose T is not complete. Then there is a sentence ϕ s.t. $T \not\models \phi$ and $T \not\models \neg\phi$. Because $T \not\models \psi$ if and only if $T \cup \{\neg\psi\}$ is satisfiable, the theories $T_0 = T \cup \{\phi\}$ and $T_1 = T \cup \{\neg\phi\}$ are satisfiable. Because T has no finite models, both T_0 and T_1 have infinite models. By Proposition 2.15 we can find $\mathcal{M}_0$ and $\mathcal{M}_1$ of cardinality κ with $\mathcal{M}_i \models T_i$. Because $\mathcal{M}_0$ and $\mathcal{M}_1$ disagree about ϕ, they are not elementarily equivalent, and hence by Theorem 1.9, nonisomorphic. □

Definition 2.20. We say that an $\mathcal{L}$-theory T is **decidable** if there is an algorithm that when given an $\mathcal{L}$-sentence ϕ as input decides whether $T \models \phi$

Lemma 2.21. *Let T be a recursive complete satisfiable theory in a recursive language $\mathcal{L}$. Then T is decidable*

Proof. Because T is satisfiable $A = \{\phi : T \models \phi\}$ and $B = \{\phi : T \models \neg\phi\}$ are disjoint. Because T is consistent $A \cup B$ is the set of all $\mathcal{L}$-sentences. By the Completeness Theorem, $A = \{\phi : T \vdash \phi\}$ and $B = \{\phi : T \vdash \neg\phi\}$. By Proposition 2.1 A and B are recursively enumerable. But any recursively enumerable set with a recursively enumerable complement is recursive. □

Corollary 2.22. *For $p = 0$ or p prime, ACF_p is decidable. In particular,* Th($\mathbb{C}$), *the first-order theory of the field of complex numbers, is decidable*

Corollary 2.23. *Let ϕ be a sentence in the language of rings. The following are equivalent*

1. *ϕ is true in the complex number*
2. *ϕ is true in every algebraically closed field of characteristic zero*

3. ϕ *is true in some algebraically closed field of characteristic zero*
4. *There are arbitrarily large primes* p *s.t.* ϕ *is true in some algebraically closed field of characteristic* p
5. *There is an* m *s.t. for all* $p > m$, ϕ *is true in all algebraically closed fields of characteristic* p

3 Reference

References

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