Real Analysis: Measure Theory, Integration, and Hilbert Spaces

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1 Measure Theory

1.1 Preliminaries

The **open ball** in $\mathbb{R}^d$ centered at x and of radius r is defined by

$$B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r\}$$

A subset $E \subset \mathbb{R}^d$ is **open** if for every $x \in E$, there exists $r > 0$ with $B_r(x) \subset E$. A set is **closed** if its complement is open.

A set E is **bounded** if it's contained in some ball of finite radius. A bounded set is **compact** if it's also closed. Compact sets enjoy the Heine-Borel covering property:

- Assume E is compact, $E \subset \bigcup_\alpha \mathcal{O}_\alpha$, and each $\mathcal{O}_\alpha$ is open. Then there are finitely many of the open sets $\mathcal{O}_{\alpha_1}, \dots, \mathcal{O}_{\alpha_N}$ s.t. $E \subset \bigcup_{j=1}^N \mathcal{O}_{\alpha_j}$

Lemma 1.1. *If $R, R_1, \dots, R_N$ are rectangles, and $R \subset \bigcup_{k=1}^N R_k$, then*

$$|R| \leq \sum_{k=1}^N |R_k|$$

Theorem 1.2. *Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals*

Proof. For every $x \in \mathcal{O}$, let

$$a_x = \inf\{a < x : (a, x) \subset \mathcal{O}\} \quad b_x = \sup\{b > x : (x, b) \subset \mathcal{O}\}$$

and $I_x = (a_x, b_x)$. Then $\mathcal{O} = \bigcup_{x \in \mathcal{O}} I_x$. Now suppose that two intervals I_x and I_y intersects. Then $(I_x \cup I_y) \subset I_x$ and $(I_x \cup I_y) \subset I_x$. This can happen only if $I_x = I_y$. Therefore any two disjoint intervals in the collection $\mathcal{I} = \{I_x\}_{x \in \mathcal{O}}$. Since every open interval I_x contains a rational number. $\square$

Theorem 1.3. *Every open subset $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.*

1.2 The exterior measure

Definition 1.4. If E is *any* subset of $\mathbb{R}^d$, the **exterior measure** of E is

$$m_*(E) = \inf \sum_{j=1}^\infty |Q_j|$$

where the infimum is taken over all countable coverings $E \subset \bigcup_{j=1}^\infty Q_j$ by closed cubes .

Example 1.1. The exterior measure of a point is zero. This is clear once we observe that a point is a cube with volume zero.

Example 1.2. The exterior measure of a closed cube is equal to its volume. Indeed suppose Q is a closed cube in $\mathbb{R}^d$. Since Q covers itself, we must have $m_*(Q) \leq |Q|$. Therefore, it suffices to prove the reverse inequality.

We consider an arbitrary covering $Q \subset \bigcup_{j=1}^{\infty} Q_j$ by cubes, and note that it suffices to prove that

$$|Q| \leq \sum_{j=1}^{\infty} |Q_j|$$

For a fixed $\epsilon > 0$ we choose for each j an open cube S_j which contains Q_j and s.t. $|S_j| \leq (1+\epsilon)|Q_j|$. From the open covering $\bigcup_{j=1}^{\infty} S_j$ of the compact set Q, we may select a finite subcovering which, after possibly renumbering the rectangles, we may write as $Q \subset \bigcup_{j=1}^{N} S_j$. We may apply Lemma 1.1 to conclude that $|Q| \leq \sum_{j=1}^{N} |S_j|$. Consequently,

$$|Q| \leq (1+\epsilon) \sum_{j=1}^{N} |Q_j| \leq (1+\epsilon) \sum_{j=1}^{\infty} |Q_j|$$

Since ϵ is arbitrary, the inequality holds; thus $|Q| \leq m_*(Q)$

Example 1.3. If Q is an open cube, the result $m_*(Q) = |Q|$ still holds. Since Q is covered by its closure $\overline{Q}$ and $\left|\overline{Q}\right| = |Q|$, we immediately see that $m_*(Q) \leq |Q|$. Note that if Q_0 is a closed cube contained in Q, then $m_*(Q_0) \leq m_*(Q)$, since any covering of Q by a countable number of closed cubes is also a covering of Q_0. Hence $|Q_0| \leq m_*(Q)$, and since we can choose Q_0 with a volume as close as we wich to $|Q|$, we must have $|Q| \leq m_*(Q)$

Example 1.4. The exteriormeasure of a rectangle R is equal to its volume. To obtain $|R| \leq m_*(R)$, consider a grid in $\mathbb{R}^d$ formed by cubes of side length $1/k$. Then if $\mathcal{Q}$ consists of the (finite) collection of all cubes entirely contained in R, and $\mathcal{Q}'$ the (fintie) collection of all cubes that intersect the complement of R, we first note that $R \subset \bigcup_{Q \in (\mathcal{Q} \cup \mathcal{Q}')} Q$. Also a simple argument yields

$$\sum_{Q \in \mathcal{Q}} |Q| \leq |R|$$

Moreover, there are $O(k^{d-1})$ cubes in $\mathcal{Q}'$ and these cubes have volume k^{-d}, so that $\sum_{Q \in \mathcal{Q}'} |Q| = O(1/k)$. Hence

$$\sum_{Q \in \mathcal{Q} \cup \mathcal{Q}'} |Q| \leq |R| + O(1/k)$$

and letting k tend to infinity yields $m_*(R) \leq |R|$

Example 1.5. The exterior measure of $\mathbb{R}^d$ is infinite. This follows from the fact that any covering of $\mathbb{R}^d$ is also a covering of any cube $Q \subset \mathbb{R}^d$ hence $|Q| \leq m_*(\mathbb{R}^d)$.

Example 1.6. The Cantor set $\mathcal{C}$ has exterior measure 0. From the construction of $\mathcal{C}$, we know that $\mathcal{C} \subset C_k$, where each C_k is a dijoint union of 2^k closed intervals, each of length 3^{-k}. Consequently, $m_*(\mathcal{C}) \leq (2/3)^k$ for all k, hence $m_*(\mathcal{C}) = 0$

Proposition 1.5. *For every $\epsilon > 0$, there exists a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$ with*

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon$$

Proposition 1.6 (Monotonicity)**.** *If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$*

Proposition 1.7 (Countable sub-additivity)**.** *If $E = \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$*

Proof. First we may assume that each $m_*(E_j) < \infty$ for otherwise the inequality clearly holds. For any $\epsilon > 0$ the definition of the exterior measure yields for each j a covering $E_j \subset \bigcup_{k=1}^{\infty} Q_{k,j}$ by closed cubes with

$$\sum_{k=1}^{\infty} |Q_{k,j}| \leq m_*(E_j) + \frac{\epsilon}{2^j}$$

Then, $E \subset \bigcup_{j,k=1}^{\infty} Q_{k,j}$ is a covering of E by closed cubes and therefore

$$\begin{aligned} m_*(E) \leq \sum_{j,k} |Q_{k,j}| &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |Q_{k,j}| \\ &\leq \sum_{j=1}^{\infty} (m_*(E_j) + \frac{\epsilon}{2^j}) \\ &= \sum_{j=1}^{\infty} m_*(E_j) + \epsilon \end{aligned}$$

□

Proposition 1.8. *If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf m_*(\mathcal{O})$ where the infimum is taken over all open sets $\mathcal{O}$ containing E*

Proof. By monotonicity, it is clear that $m_*(E) \leq \inf m_*(\mathcal{O})$ holds. For the reverse inequality, let $\epsilon > 0$ and choose cubes Q_j s.t. $E \subset \bigcup_{j=1}^{\infty} Q_j$ with

$$\sum_{j=1}^{\infty} \left|Q_j\right| \leq m_*(E) + \frac{\epsilon}{2}$$

Let Q_j^0 denote an open cube containing Q_j, and s.t. $\left|Q_j^0\right| \leq \left|Q_j\right| + \frac{\epsilon}{2^{j+1}}$. Then $\mathcal{O} = \bigcup_{j=1}^{\infty} Q_j^0$ is open, and by Proposition 1.7

$$\begin{aligned} m_*(\mathcal{O}) \leq \sum_{j=1}^{\infty} m_*(Q_j^0) &= \sum_{j=1}^{\infty} \left|Q_j^0\right| \\ &\leq \sum_{j=1}^{\infty} (\left|Q_j\right| + \frac{\epsilon}{2^{j+1}}) \\ &\leq \sum_{j=1}^{\infty} \left|Q_j\right| + \frac{\epsilon}{2} \\ &\leq m_*(E) + \epsilon \end{aligned}$$

□

Proposition 1.9. *If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$, then*

$$m_*(E) = m_*(E_1) + m_*(E_2)$$

Proof. By Proposition 1.7, we already know that $m_*(E) \leq m_*(E_1) + m_*(E_2)$. First select $d(E_1, E_2) > \delta > 0$. Next we choose a covering $E \subset \bigcup_j = 1^{\infty} Q_j$ by closed cubes with $\sum_{j=1}^{\infty} \left|Q_j\right| \leq m_*(E) + \epsilon$ □