Logic and Game Theory

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1 Lec 2 An invitation to game theory

Every game has three main ingredients:

1. the set of players, $[n] = \{1, 2, \dots, n\}$
2. the rule of game
3. **outcomes** or winning conditions

The game arena, that gives the rules of the game, can be envisaged as a finite graph

- vertices denote game positions
- edges correspond to moves
- each vertex is labelled by the player whose turn it is to move
- winning conditions are not present in the arena

Game tree: the tree unfolding of a game arena is **extensive form**

Consider two-person zero-sum games of perfect information. $N = \{1, 2\}$. $\Sigma = \{a_1, a_2, \dots, a_m\}$ is a finite set of action symbols, representing move of players, common for both players. Such games are referred as **bipartisan games**

A **game arena** is a graph $\mathcal{G} = (W, \rightarrow, s_0)$ with $W = \bigcup_{i \in N} W^i \bigcup \left\{ W^{\text{leaf}} \right\}$. For $i \in N, W^i$ is the set of *game positions* for player i and W^{leaf} is the set of terminal game positions. s_0 is the initial node of the game. $\rightarrow: (W \times \Sigma) \rightarrow W$, is a partial function called the move function

If a player i owns the game position $s_0, s_0 \in W^i$ then she picks an action a' and moves to token to s', $s_0 \xrightarrow{a} s'$ A play in *calg* is a path $\rho : s_0 a_0 s_1 a_1 \dots$ where $\forall j : s_j \xrightarrow{a_j} s_{j+1}$

let $\mathcal{G}_t$ denote the tree unfolding of the arena $\mathcal{G}$. A **strategy** for player $i, \mu = (W_\mu, \rightarrow_\mu, s_0)$ is a maximal connected subtree of $\mathcal{G}_T$ where for each player i node, there is a unique outgoing edge and for the other player every move is included. That is, for $s \in W_\mu$, if $s \in W_\mu^i$ then there exists a unique $a \in \Sigma$ s.t. $s \xrightarrow{a}_\mu s'$ where we have $s \xrightarrow{a}_T s'$. If $s \in W_\mu^j (j \neq i)$, then for each s' s.t. $s \xrightarrow{a}_T s'$, we have $s \xrightarrow{a}_\mu s'$

A **strategy profile** is a pair $< \mu, \tau >$ that fixes a strategy for each player. A **winning strategy** σ for player i if every strategy τ of the opponent i wins.

A win/lose game is **determined** if starting from any game position, one of the players has a winning strategy

Theorem 1.1. *In every **finite** extensive form game of perfect information, we can compute whether player i can win. (Zermelo 1913)*

Proof. Backward induction □

1.1 Analysis of Nim

Lemma 1.1. *For all $m, n \geqslant 0, (m, n)$ is winning iff $m \neq n$*

Every finite extensive form game is of the form 0 or

$$g_1 + g_2 + \cdots + g_m$$

0 is the empty game, where no player can make any move , $g_1, g_2, \dots, g_m$ are subgames

If $g = g_1 + \cdots + g_m, h = h_1 + \cdots + h_m$, then

$$g + h = (g_1 + h) + \cdots + (g_m + h) + (g + h_1) + \cdots + (g + h_n)$$

When g is a subgame of h, we write $g \leqslant h$

$g_1 \equiv g_2$ if for all h, $g_1 + h$ is winning(losing) iff $g_2 + h$ is winning(losing).

$\equiv$ is an equivalence relation

Lemma 1.2 (The loser's lemma)**.** *If g is losing then $g \equiv 0$*

Proof.

1. fix a losing game g
2. prove: for all $h, g + h$ is losing iff h is losing
3. Assuming this, suppose h is winning, then there is a move to h' that is losing. Hence $g + h'$ is losing and $g + h$ is winning

□

Lemma 1.3. *If h and g are losing, so is $g + h$*

Proof. IH1(inductive hypothesis): for all $g' \leqslant g$, if h is losing, then so is $g' + h$

IH2: for all $h' \leqslant h$, if h is losing, then so is $g + h'$

Every initial move in $g + h$ is either in g or h. First consider the latter. h is losing, so every move in h to h' is winning □

Corollary 1.1. *if h is losing, then for all g, $g + h = g$*

2 lec3: Normal form games

$N = \{1, 2, \ldots, n\}$ the set of players

For each $i \in N$, a finite set $S_i = \{1, \ldots, m_i\}$ of pure strategies