THE MATHEMATICS OF LANGUAGE

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April 9, 2020

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1 Fundamental Structures

1.1 Algebra and Structures

We define $M + N := M \times \{0\} \cup N \times \{1\}$. This is called the **disjoint union**

If $R \subseteq M \times N$ is a relation, we write $R^{\smile} := \{\langle x, y\rangle : yRx\}$ for the so-called **converse of** R. We put $\Delta_M := \{\langle x, x\rangle : x \in M\}$ and call this set the **diagonal on** M. Now put

$$R^0 := \Delta_M \qquad R^{n+1} := R \circ R^n$$
$$R^+ := \bigcup_{0<i\in\omega} R^i \qquad R^* := \bigcup_{i\in\omega} R^i$$

R^+ is called the **transitive closure of** R.

An n**-ary relation** on M is a subset of M^n, an n**-ary function** on M is a function $f : M^n \to M$. A 0-ary function on M is a function $c : 1 \to M$. We also call it a **constant**.

Now let F be a set and $\Omega : F \to \omega$. The pair $\langle F, \Omega\rangle$ also denoted by Ω alone, is called a **signature** and F the set of **function symbols**

Definition 1.1. Let $\Omega : F \to \omega$ be a signature and A a nonempty set. Furthur, let Π be a mapping which assigns to every $f \in F$ an $\Omega(f)$-ary function on A. Then we call the pair $\mathfrak{A} := \langle A, \Pi\rangle$ and Ω**-algebra**.

In order not to get drowned in notation we write $f^{\mathfrak{A}}$ for the function $\Pi(f)$.

The set of Ω-terms is the smallest set Tm_Ω s.t. if $f \in F$ and $t_i \in \mathrm{Tm}_\Omega$, $i < \Omega(f)$, also $f(t_0, \ldots, t_{\Omega(f)-1}) \in \mathrm{Tm}_\Omega$. To begin with we define the **level** of a term. If $\Omega(f) = 0$, then $f()$ is a term of level 0, which we also denote by 'f'. If $t_i, i < \Omega(f)$, are terms of level n_i, then $f(t_0, \ldots, t_{\Omega(f)-1})$ is a term of level $1 + \max\{n_i : i < \Omega(f)\}$. We therefore speak about *induction on the construction of the term*. Two terms u and v are equal, in symbols $u = v$, if they have identical level and either they are both of level 0 and there is an $f \in F$ such $u = v = f()$ or there is an $f \in F$ and terms $s_i, t_i, i < \Omega(f)$, s.t. $u = f(s_0, \ldots, s_{\Omega(f)-1})$ and $v = f(t_0, \ldots, t_{\Omega(f)-1})$ as well as $s_i = t_i$ for all $i < \Omega(f)$