Sheaves in Geometry and Logic

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1 Categorical Preliminaries

A **category C** consists of a collection of **objects**, a collection of **morphisms** and four operations; two of these operations associate with each morphism f of **C** its **domain** $\mathrm{dom}(f)$ or $\mathrm{d}_0(f)$ and its **codomain** $\mathrm{cod}(f)$ or $\mathrm{d}_1(f)$, respectively, both of which are objects of **C**. The other two operations are operation which associates with each object C of **C** a morphism 1_C (or id_C) of **C** called the **identity morphism** of C and an operation of **C** s.t. $\mathrm{d}_0(f) = \mathrm{d}_1(g)$ another morphism $f \circ g$. These operations are required to satisfy the following axioms

1. $\mathrm{d}_0(1_C) = C = \mathrm{d}_1(1_C)$
2. $\mathrm{d}_0(f \circ g) = \mathrm{d}_0(g), \mathrm{d}_1(f \circ g) = \mathrm{d}_1(f)$
3. $1_D \circ f = f, f \circ 1_C = f$
4. $(f \circ g) \circ h = f \circ (g \circ h)$

In an arbitrary category **C**, a morphism $f : C \to D$ in **C** is called an **isomorphism** if there exists a morphism $g : D \to C$ s.t. $f \circ g = 1_D$ and $g \circ f = 1_C$. If such a morphism f exists, one says that C is isomorphic to D and one writes $f : C \xrightarrow{\sim} D$ and $C \cong D$

A morphism $f : C \to D$ is called an **epi(morphism)** if for any object E and any two parallel morphisms $g, h : D \rightrightarrows E$ in **C**, $gf = hf$ implies $g = h$; one writes $f : C \twoheadrightarrow D$ to indicate that f is an epimorphism. Dually, $f : C \to D$ is called a **mono(morphism)** if for any object B and any two parallel morphisms $g, h : B \rightrightarrows C$ in **C**, $fg = fh$ implies $g = h$; in this case, one writes $f : C \rightarrowtail D$. Two monomorphisms $f : A \rightarrowtail D$ and $g : B \rightarrowtail D$ with a common codomain D are called **equivalent** if there exists an isomorphism $h : A \xrightarrow{\sim} B$ with $gh = f$. A **subobject** of D is an equivalence class of monomorphisms into D. The collection $\mathrm{Sub}_{\mathbf{C}}(D)$ of subobjects of D carries a natural partial order defined by $[f] \leq [g]$ iff there is an $h : A \to B$ s.t. $f = gh$, where $[f]$ and $[g]$ are the classes of $f : A \rightarrowtail D$ and $g : B \rightarrowtail D$

$$\begin{tikzcd}
A \arrow[r, tail, "f"] \arrow[d, "h"] & D \\
B \arrow[ur, tail, "g"'] &
\end{tikzcd}$$

If **C** is a category, we sometimes write $\mathbf{C}_0$ for its collection of objects and $\mathbf{C}_1$ for its collection of mophisms. For two objects C and D, the collection of morphisms with domain C and codomain D is denoted by one of the following three symbols

$$\mathrm{Hom}_{\mathbf{C}}(C,D), \quad \mathrm{Hom}(C,D), \quad \mathbf{C}(C,D)$$

We shall tacitly assume we are working in some fixed universe U of sets. Members of U are then called **small** sets, whereas a collection of members of U which doesnot itself belong to U will sometimes be referred to as a **large** set. Given such an ambient universe U, a category **C** is **locally small** if for any two objects C and D of **C** the hom-set $\mathrm{Hom}_{\mathbf{C}}(C,D)$ is a small set, while **C** is called **small** if both $\mathbf{C}_0$ and $\mathbf{C}_1$ are small sets.

Given a category **C**, one can form a new category $\mathbf{C}^{\mathrm{op}}$, called the **opposite** or **dual** category of **C**, by taking the same objects but reversing the direction of all the morphisms and the order of all compositions.

Given a category **C** and an object C of **C**, one can construct the **comma category** or the **slice category** $\mathbf{C}/C$ (read: **C** over C): object of $\mathbf{C}/C$ are morphisms of **C** with codomain C, and morphisms in $\mathbf{C}/C$ from one such object $f : D \to C$ to another $g : E \to C$ are commutative triangles in **C**

$$\begin{tikzcd} D \arrow[rr, "h"] \arrow[dr, "f"'] & & E \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

Given two categories **C** and **D**, a **functor** from **C** to **D** is an operation F which assigns to each objects C of **C** an object $F(C)$ of **D** and to each morphism f of **C** a morphism $F(f)$ of **D** in such a way that F respects the domain and codomain as well as the identities and compositions.

For a category **C** there is an **identity functor** $\mathrm{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$, and for two functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$ one can form a new functor $G \circ F : \mathbf{C} \to \mathbf{E}$ by **composition**

Let F and G be two functors from a category **C** to a category **D**. A **natural transformation** α from F to G, written $\alpha : F \to G$, is an operation associating with each object C of **C** a morphism $\alpha_C : FC \to GC$ of **D** in such a way that for any morphism $f : C' \to C$ in **C**, the diagram

$$\begin{tikzcd} FC' \arrow[r, "\alpha_{C'}"] \arrow[d, "F(f)"'] & GC' \arrow[d, "G(f)"] \\ FC \arrow[r, "\alpha_C"] & GC \end{tikzcd}$$

commutes. The morphism α_C is called the **component** of α at C. If every component of α is an isomorphism, α is said to be a **natural isomorphism**.

If $\alpha : F \to G$ and $\beta : G \to H$ are two natural transformation between functors $\mathbf{C} \to \mathbf{D}$, one can define composite natural transformation $\beta \circ \alpha$ by setting

$$(\beta \circ \alpha)_C = \beta_{G(C)} \circ \alpha_C$$

By fixed categories **C** and **D** this yields a new category $\mathbf{D}^{\mathbf{C}}$: the objects of $\mathbf{D}^{\mathbf{C}}$ are functors from **C** to **D** while the morphisms of $\mathbf{D}^{\mathbf{C}}$ are natural transformations between such functors. Categories so constructed are called **functor categories**

For categories **C** and **D**, a functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ is also called a **contravariant functor** from **C** to **D**. In contrast, ordinary functors from **C** to **D** are sometimes called *covariant. Thus $C' \mapsto \mathrm{Hom}_{\mathbf{C}}(C', C)$ for fixed C yields a contravariant functor from **C** to **Sets**, while $C \mapsto \mathrm{Hom}_{\mathbf{C}}(C', C)$ for fixed C' is the covariant Hom-functor.

$$\begin{tikzcd}
C' \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{C}}(C', C) \arrow[d] \\
C'' \arrow[r] & \mathrm{Hom}_{\mathbf{C}}(C'', C)
\end{tikzcd}$$

A functor $F : \mathbf{C} \to \mathbf{D}$ is called **full** (respectively **faithful**) if for any two objects C and C' of **C**, the operation

$$\mathrm{Hom}_{\mathbf{C}}(C', C) \to \mathrm{Hom}_{\mathbf{D}}(FC', FC); \quad f \mapsto F(f)$$

induced by F is surjective (respectively injective). A functor $F : \mathbf{C} \to \mathbf{D}$ is called an **equivalence of categories** if F is full and faithful and if any object of **D** is isomorphic to an object in the image of F. For example, if $F : \mathbf{C} \to \mathbf{D}$ is a functor s.t. there exists a functor $G : \mathbf{D} \to \mathbf{C}$ and natural isomorphism $\alpha : F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathbf{D}}$ and $\beta : G \circ F \xrightarrow{\sim} \mathrm{id}_{\mathbf{C}}$, then F is an equivalence (and G is sometimes called a **quasi-inverse** for F).

We say that an object X equipped with morphsims $\pi_1 : X \to A$ and $\pi_2 : X \to B$ is a **product** of A and B if for any other object Y and any two maps $f : Y \to A$ and $g : Y \to B$ there is a **unique** map $h : Y \to X$ s.t. $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$ [this unique is denoted by $(f, g) : Y \to X$ or sometimes $\langle f, g \rangle$]

$$\begin{tikzcd}
 & Y \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"', "!"] \arrow[dr, "g"] & \\
A & X \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B
\end{tikzcd}$$

A product of an I-indexed family A_i is written $\prod_i A_i$. For a poset $(P, \leq)$ viewed as a category in the way explained above, the product of two objects p and q is their infimum, which may or may not exist.

The singleton set $\{*\}$ is the set S, unique up to isomorphism, for which there is exactly one morphism $A \to S$ from any other set A into S. In an arbitrary category **C**, an object C with the property that for any other object D of **C** there is one and only one morphism from D to C is called a **terminal object** of C. It's often denoted by 1 or by $1_{\mathbf{C}}$

Given two functions $f : B \to A$ and $g : C \to A$ between sets, one may construct their **fibered product** (or **pullback**) as the set

$$B \times_A C = \{(b, c) \in B \times C \mid f(b) = g(c)\}$$

Thus $B \times_A C$ is a subset of the product, and therefore comes equipped with two **projections** $\pi_1 : B \times_A C \to B$ and $\pi_2 : B \times_A C \to C$ which fit into a commutative diagram

$$\begin{tikzcd} B \times_A C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

This diagram has the property that given any other set X and functions $\beta : X \to B$ and $\gamma : X \to C$ s.t. $f\beta = g\gamma$, there is a unique function $\delta : X \to B \times_A C$ with $\pi_1\delta = \beta$ and $\pi_2\delta = \gamma$ [namely $\delta(x) = (\beta(x), \gamma(x))$]

In a general category **C**, one says that a commutative square

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & C \arrow[d, "g"] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

is a **pullback** (**square**) or a **fibered product** if it has the property just described for sets: given any object X of **C** and morphisms $\beta : X \to B$ and $\gamma : X \to C$ with $f\beta = g\gamma$, there is a unique $\delta : X \to P$ s.t. $p\delta = \beta$ and $q\delta = \gamma$

$$\begin{tikzcd} X \arrow[dr, dashed, "!\delta"] \arrow[drr, bend left, "\gamma"] \arrow[ddr, bend right, "\beta"'] & & \\ & P \arrow[r, "q"] \arrow[d, "p"'] & C \arrow[d, "g"] \\ & B \arrow[r, "f"'] & A \end{tikzcd}$$

This unique map δ is usually denoted by (β, γ). Given $f : B \to A$ and $g : C \to A$, the pullback P with its projections P and q is uniquely determined up to isomorphism and one usually writes $B \times_A C$ for this pullback. We also says that the arrow p is the pullback of g **along** f. Notice that p is a monomorphism if g is. A morphism $f : B \to A$ in a category **C** is a monomorphism iff the pullback of f along itself is an isomorphism, iff the square

$$\begin{tikzcd} B \arrow[r, "1"] \arrow[d, "1"'] & B \arrow[d, "f"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

is a pullback

Proof. $1 \to 2$. Consider

$$\begin{tikzcd} B \arrow[dr, dashed, "g'"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & & \\ & C \arrow[r, "g"] \arrow[d, "g"] & B \arrow[d, "f"] \\ & B \arrow[r, "f"] & A \end{tikzcd} \qquad \begin{tikzcd} C \arrow[dr, dashed, "1"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "g"'] & & \\ & C \arrow[r, "g"] \arrow[d, "g"] & B \arrow[d, "f"] \\ & B \arrow[r, "f"] & A \end{tikzcd}$$

We have $g(g'g) = gg'g = (gg')g = g$, hence $g'g = 1$

$2 \to 3$. λ has inverse g

$$\begin{tikzcd} B \arrow[dr, dashed, "\lambda"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right] & & \\ & C \arrow[r, "g", "\cong"'] \arrow[d] & B \arrow[d, "f"] \\ & B \arrow[r] & A \end{tikzcd}$$

$3 \to 1$. □

There is an important "pasting-lemma" for pullback squares. Given a commutative diagram of the form

$$\begin{array}{ccccc} Q & \longrightarrow & P & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & B & \longrightarrow & A \end{array}$$

the outer rectangle is a pullback if both inner squares are pullbacks; and conversely, if the outer rectangle as well as the right-hand square pullbacks, then so is the left-hand square

For two parallel arrows $f : A \to B$ and $g : A \to B$ in a category **C**, the **equalizer** of f and g is a morphism $e : E \to A$ s.t. $fe = ge$ and which is universal with this property; that is, given any other morphism $u : X \to A$ in **C** s.t. $fu = gu$, there is a unique $v : X \to E$ s.t. $ev = u$

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \\ X \arrow[u, dashed, "v"] \arrow[ur, "u"'] & & \end{tikzcd}$$

Equalizer need not always exists. However, in the category of sets the equalizer of any pair of functions $f, g \rightrightarrows B$ exists, and can be constructed be the set

$$E = \{a \in A \mid f(a) = g(a)\}$$

where e is set inclusion

Consider two categories **A** and **X** and two functors between them in opposite directions, say

$$F : \mathbf{X} \to \mathbf{A} \quad G : \mathbf{A} \to \mathbf{X}$$

One says that G is **right adjoint** to F (and that F is **left adjoint** to G, notation $F \dashv G$) when for any two objects X from **X** and A from **A** there is a natural bijection between morphisms

$$\frac{X \xrightarrow{f} GA}{FX \xrightarrow{h} A} \tag{1}$$

in the sense that each morphism f uniquely determines a morphism h, and conversely. This bijection is to be natural in the following sense: given any morphisms $\alpha : A \to A'$ in **A** and $\xi : X' \to X$ in **X**, and corresponding arrows f and h composites also correspond

$$\frac{X' \xrightarrow{\xi} X \xrightarrow{f} GA \xrightarrow{G\alpha} GA'}{FX' \xrightarrow{F\xi} FX \xrightarrow{h} A \xrightarrow{\alpha} A'}$$

If we write this bijective correspondence as

$$\theta : \mathrm{Hom}_{\mathbf{X}}(X, GA) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{A}}(FX, A) \tag{2}$$

then this naturality condition can be expressed by the equation

$$\theta(G(\alpha) \circ f \circ \xi) = \alpha \circ \theta(f) \circ F(\xi)$$

Given θ as in (2), and an object X in **X**, setting $A = FX$ gives a unique map

$$\eta = \eta_X : X \to GFX$$

s.t. $\theta(\eta_X) = 1_{F(X)}$. This map η_X is called the **unit** of the adjunction (at X). If one takes $\xi = 1_X, A = FX, f = \eta, \alpha = 1_A$ and $A' = A$, then

$$\frac{X \xrightarrow{1_X} X \xrightarrow{\eta} GFX \xrightarrow{Gh} GA}{FX \xrightarrow{F1_X} FX \xrightarrow{h} A \xrightarrow{1_A} A}$$

In short, η determined the adjunction, since h corresponds to $G(h) \circ \eta_X$ under the correspondence (1). This means that each f determines uniquely an h which makes the following triangle commutes.

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, "f"'] & GFX \arrow[d, dashed, "Gh"] & FX \arrow[d, dashed] \\ & GA & A \end{tikzcd}$$

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