Notes on Set Theory

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Notes for the entrance examination

2 Models of Set - Sertraline

2.1 Some mathematical logic

Theorem 2.1 (Gödels second incompleteness theorem)**.** If a consistent recursive axiom set T contains **ZFC**, then

$$T \nvdash \mathrm{Con}(t)$$

especially, **ZFC** $\nvdash \mathrm{Con}(\textbf{ZFC})$

Definition 2.2. Suppose (M, E_M) and (N, E_N) are two models of set theory, then

1. if for any formula σ, $M \models \sigma$ if and only if $N \models \sigma$, then M and N are **elementary equivalent**, denoted by $M \equiv N$
2. If bijection $f : M \to N$ satisfies: for any $a, b \in M$, $aE_M b$ iff $f(a)E_N f(b)$, then $f : M \cong N$ is an **isomorphism**
3. If $M \subseteq N$ and $E_M = E_N \restriction M$, then M is N's submodel
4. If M is isomorphic to a submodel of N by injection f, and for any formula $\varphi(x_1, \ldots, x_n)$, for any $a_1, \ldots, a_n \in M$, $M \models \varphi[a_1, \ldots, a_n]$ iff $N \models \varphi[f(a_1), \ldots, f(a_n)]$, then f is called an **elementary embedding** from M to N, written as $f : M \prec N$
5. If $M \subseteq N$ and $M \prec N$, then M is a **elementary submodel** of N

Lemma 2.3. Suppose $N \models$ **ZFC**, $M \subseteq N$, then $M \prec N$ iff $\forall \varphi(x, x_1, \ldots, x_n)$, $\forall (a_1, \ldots, a_n) \in M$, if $\exists a \in N$ s.t. $N \models \varphi[a, a_1, \ldots, a_n]$, then $\exists a \in M$ s.t. $M \models \varphi[a, a_1, \ldots, a_n]$

Definition 2.4. Suppose $(M, E) \models$ **ZFC**

1. $h_\varphi : M^n \to M$ is φ's **Skolem function** if $\forall a_1, \ldots, a_n \in M$, if $\exists a \in M$ s.t. $M \models \varphi[a, a_1, \ldots, a_n]$, then $M \models \varphi[h_\varphi(a_1, \ldots, a_n), a_1, \ldots, a_n]$. requires **AC**
2. Let $\mathcal{H} = \{h_\varphi \mid \varphi \text{is a formula on set theory}\}$. For any $S \subseteq M$, **Skolem hull** $\mathcal{H}(S)$ is the smallest set consisting of S and closed under $\mathcal{H}$

Lemma 2.5. $N \models$ **ZFC**$, S \subseteq N$, if $M = \mathcal{H}(S)$, then $M \prec N$

Theorem 2.6 (Löwenheim-Skolem theorem)**.** Suppose $N \models$ **ZFC** and is infinite, then there is a model M s.t. $|M| = \omega$ and $M \prec N$

2.2 Cumulative Hierarchy

This section works in **ZF**$^{-}$(a.k.a. **ZF** $-$ axiom of foundation)

Definition 2.7. For any α, define sequence V_α

1. $V_0 = \emptyset$
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
3. For any limit ordinal λ, $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$

And **WF** $= \bigcup\limits_{\alpha \in \mathbf{On}} V_\alpha$

Lemma 2.8. For any ordinal α

1. V_α is transitive
2. if $\xi \leq \alpha$, then $V_\xi \subseteq V_\alpha$
3. if κ is inaccessible cardinal, then$|V_\kappa| = \kappa$

Proof. 1. Obviously $\kappa \leq V_\kappa$. Since κ is inaccessible, then for any $\alpha < \kappa$, $|V_\alpha| < \kappa$.

□

Definition 2.9. For any set $x \in$ **WF**,

$$\mathrm{rank}(x) = \min\{\beta \mid x \in V_{\beta+1}\}$$

Lemma 2.10. 1. $V_\alpha = \{x \in \mathbf{WF} \mid \mathrm{rank}(x) < \alpha\}$

2. **WF**is transitive
3. For any $x, y \in$ **WF**, if $x \in y$, then $\mathrm{rank}(x) < \mathrm{rank}(y)$
4. for any $y \in$ **WF**, $\mathrm{rank}(y) = \sup\{\mathrm{rank}(x) + 1 \mid x \in y\}$

Lemma 2.11. Supoose α is an ordinal

1. $\alpha \in$ **WF** and $\mathrm{rank}(\alpha) = \alpha$

2. $V_\alpha \cap \mathbf{On} = \alpha$

Lemma 2.12. 1. If $x \in \mathbf{WF}$, then $\bigcup x, \mathcal{P}(x), \{x\} \in \mathbf{WF}$, and their ranks are all less than $\operatorname{rank}(x) + \omega$

2. If $x, y \in \mathbf{WF}$, then $x \times y, x \cup y, x \cap y, \{x, y\}, (x, y), x^y \in \mathbf{WF}$, and their ranks are all less than $\operatorname{rank}(x) + \operatorname{rank}(y) + \omega$

3. $\mathbb{Z}, \mathbb{Q}, \mathbb{R} \in V_{\omega+\omega}$

4. for any set x, $x \in \mathbf{WF}$ iff $x \subset \mathbf{WF}$

Lemma 2.13. Suppose **AC**

1. for any group G, there exists group $G' \cong G$ in **WF**

2. for any topological space T, there exists $T' \cong T$ in **WF**

Definition 2.14. Binary relation $<$ on set A is **well-founded** if for any nonempty $X \subseteq A$, X has minimal element under $<$

Theorem 2.15. If $A \in \mathbf{WF}$, then $\in$ is a well-founded relation on A

Lemma 2.16. If set A is transitive and $\in$ is well-founded on A, then $A \in \mathbf{WF}$

Lemma 2.17. For any set x, there is a smallest transitive set $\operatorname{trcl}(x)$ s.t. $x \subseteq \operatorname{trcl}(x)$

Proof.

$$\begin{aligned} x_0 &= x \\ x_{n+1} &= \bigcup x_n \\ \operatorname{trcl}(x) &= \bigcup_{n<\omega} x_n \end{aligned}$$

$\square$

$\operatorname{trcl}(x)$ is called **transitive closure** of x

Lemma 2.18. Without axiom of power set

1. if x is transitive, then $\operatorname{trcl}(x) = x$

2. if $y \in x$, then $\operatorname{trcl}(y) \subseteq \operatorname{trcl}(x)$

3. $\operatorname{trcl}(x) = x \cup \bigcup \{\operatorname{trcl}(y) \mid y \in x\}$

Theorem 2.19. For any set X, the following are equivalent

1. $X \in \mathbf{WF}$
2. $\mathrm{trcl}(X) \in \mathbf{WF}$
3. $\in$ is a well-founded relation on $\mathrm{trcl}(X)$

Theorem 2.20. The following propositions are equivalent

1. Axiom of foundation
2. For any set X, $\in$ is a well-founded relation on X
3. $\mathbf{V} = \mathbf{WF}$

2.3 Relativization

Definition 2.21. Let $\mathbf{M}$ be a class φ a formula, the **relativization** of φ to $\mathbf{M}$ is $\varphi^{\mathbf{M}}$ defined inductively

$$\begin{aligned}(x \in y)^{\mathbf{M}} &\leftrightarrow x = y\\(x \in y)^{\mathbf{M}} &\leftrightarrow x \in y\\(\varphi \to \psi)^{\mathbf{M}} &\leftrightarrow \varphi^{\mathbf{M}} \to \psi^{\mathbf{M}}\\(\neg\varphi)^{\mathbf{M}} &\leftrightarrow \neg\varphi^{\mathbf{M}}\\(\forall x\varphi)^{\mathbf{M}} &\leftrightarrow (\forall x \in \mathbf{M})\varphi^{\mathbf{M}}\end{aligned}$$

Note $\varphi^{\mathbf{V}} = \varphi$ and

$$f^{\mathbf{M}} = \{(x_1, \dots, x_n, x_{n+1}) \in \mathbf{M} \mid \varphi^{\mathbf{M}}(x_1, \dots, x_n, x_{n+1})\}$$

Definition 2.22. For any theory T, any class $\mathbf{M}$, $\mathbf{M} \models T$ iff for any axiom φ of T, $\varphi^{\mathbf{M}}$ holds

Theorem 2.23 ($\mathbf{ZF}^-$). $\mathbf{WF} \models \mathbf{ZF}$

Proof.

- **Axiom of existence**

  $(\exists x(x = x))^{\mathbf{M}} \leftrightarrow \exists x \in \mathbf{M}(x = x)$, which is equivalent to $\mathbf{M}$being nonempty

- **Axiom of extensionality**

$$\forall X \forall Y \forall u((u \in X \leftrightarrow u \in Y) \to X = Y)^{\mathbf{M}} \Leftrightarrow$$
$$\forall X \in \mathbf{M} \forall Y \in \mathbf{M} \forall u \in \mathbf{M}((u \in X \leftrightarrow u \in Y) \to X = Y)$$

Lemma 2.24. If **M** is transitive, then axiom of extensionality holds in **M**

- **Axiom schema of specification**

$$\forall X \in \mathbf{M} \exists Y \in \mathbf{M} \forall u \in \mathbf{M}(u \in Y \leftrightarrow u \in X \wedge \varphi^{\mathbf{M}}(u))$$

  Since for any $X \in$ **WF**, $\mathcal{P}(X) \subseteq$ **WF**

- **Axiom of paring**
- **Axiom of union**
- **Axiom of power set**

$$\forall X \in \mathbf{M} \exists Y \in \mathbf{M} \forall u \in \mathbf{M}(u \in Y \leftrightarrow (u \subseteq X)^{\mathbf{M}})$$

  and

$$(u \subseteq X)^{\mathbf{M}} \leftrightarrow \forall x \in \mathbf{M}(x \in u \rightarrow x \in X) \leftrightarrow u \cap \mathbf{M} \subseteq X$$

- **Axiom of foundation**
- **Axiom schema of replacement**

□

2.4 Absoluteness

Definition 2.25. For any formula $\psi(x_1, \dots, x_n)$ and any class **M**,**N**, $\mathbf{M} \subseteq \mathbf{N}$, if

$$\forall x_1 \dots \forall x_n \in \mathbf{M}(\psi^{\mathbf{M}}(x_1, \dots, x_n) \leftrightarrow \psi^{\mathbf{N}}(x_1, \dots, x_n))$$

then $\psi(x_1, \dots, x_n)$ is **absolute** for **M**,cn. If $\mathbf{N} = \mathbf{V}$, then ψ is **absolute** for **M**

Lemma 2.26. Suppose $\mathbf{M} \subseteq \mathbf{N}$ and φ,ψ are formulas, then

1. if φ,ψ are absolute for **M**,cn, then so are $\neg\varphi, \varphi \rightarrow \psi$
2. if φ doesn't contain any quantifiers, then φ is absolute for any **M**
3. if **M**,**N**are transitive and φ is absolute for them, then so are $\forall x \in y \varphi$

Definition 2.27. Δ_0 formula

1. $x = y, x \in y$ are Δ_0 formulas

2. if φ,ψ are Δ_0, then so are $\neg\varphi, \varphi \to \psi$
3. if φ is Δ_0, y is any set, then $(\forall x \in y)\varphi$ is Δ_0

If φ is Δ_0, then $\exists x_1 \dots \exists x_n \varphi$ is Σ_1 formula, $\forall x_1 \dots \forall x_n \varphi$ is Π_1

Lemma 2.28. **M** $\subseteq$ **N** are both transitive, $\psi(x_0, \dots, x_n)$ is a formula, then

1. if ψ is Δ_0, then it's absolute for **M**,cn
2. if ψ is Σ_1, then

$$\forall x_1 \dots x_n(\psi^{\mathbf{M}}(x_1, \dots, x_n) \to \psi^{\mathbf{N}}(x_1, \dots, x_n))$$

3. if ψ is Π_1, then

$$\forall x_1 \dots x_n(\psi^{\mathbf{N}}(x_1, \dots, x_n) \to \psi^{\mathbf{M}}(x_1, \dots, x_n))$$

Lemma 2.29. If **M** $\subseteq$ **N**, **M** $\models \Sigma$, **N** $\models \Sigma$ and

$$\Sigma \vdash \forall x_1 \dots \forall x_n(\varphi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n))$$

then φ is absolute for **M**,**N**if and only if ψ is absolute for **M**,**N**

Definition 2.30. Suppose **M** $\subseteq$ **N**, $f(x_1, \dots, x_n)$ is a function. f is **absolute** for **M**and **N**if and only if $\varphi(x_1, \dots, x_n, x_{n+1})$ defining f is absolute.

Theorem 2.31. Following relations and functions can be defined in **ZF**$^-$ $-$ Pow $-$ Inf and are equivalent to some Δ_0 formulas. So they are absolute for any transitive model **M**on **ZF**$^-$ $-$ Pow $-$ Inf

1. $x \in y$
2. $x = y$
3. $x \subset y$
4. $\{x, y\}$
5. \x\
6. (x, y)
7. $\emptyset$
8. $x \cup y$

9. $x - y$
10. $x \cap y$
11. x^+
12. x is a transitive set
13. $\bigcup x$
14. $\bigcap x$ ($\bigcap \emptyset = \emptyset$)

Lemma 2.32. Absoluteness is closed under operation composition

Theorem 2.33. Following relations and functions are absolute for any transitive model $\mathbf{M}$on $\mathbf{ZF}^- - \mathrm{Pow} - \mathrm{Inf}$

1. z is an ordered pair
2. $A \times B$
3. R is a relation
4. $\mathrm{dom}(R)$
5. $\mathrm{ran}(R)$
6. f is a function
7. $f(x)$
8. f is injective

2.5 Relative consistence of axiom of foundation