

Advanced Modern Algebra

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February 11, 2020

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1 Things Past

1.1 Some Number Theory

Least Integer Axiom (Well-ordering Principle). There is a smallest integer in every nonempty subset C of \mathbb{N}

1.2 Roots of Unity

Proposition 1.1 (Polar Decomposition). *Every complex number z has a factorization*

$$z = r(\cos \theta + i \sin \theta)$$

where $r = |z| \geq 0$ and $0 \leq \theta \leq 2\pi$

Proposition 1.2 (Addition Theorem). *If $z = \cos \theta + i \sin \theta$ and $w = \cos \psi + i \sin \psi$, then*

$$zw = \cos(\theta + \psi) + i \sin(\theta + \psi)$$

Theorem 1.3 (De Moivre). $\forall x \in \mathbb{R}, n \in \mathbb{N}$

$$\cos(nx) + i \sin(nx) = (\cos x + i \sin x)^n$$

Theorem 1.4 (Euler). $e^{ix} = \cos x + i \sin x$

Definition 1.5. If $n \in \mathbb{N} \geq 1$, an **n th root of unity** is a complex number ξ with $\xi^n = 1$

Corollary 1.6. *Every n th root of unity is equal to*

$$e^{2\pi ik/n} = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right)$$

for $k = 0, 1, \dots, n-1$

$$x^n - 1 = \prod_{\xi^n=1} (x - \xi)$$

If ξ is an n th root of unity and if n is the smallest, then ξ is a **primitive n th root of unity**

Definition 1.7. If $d \in \mathbb{N}^+$, then the **d th cyclotomic polynomial** is

$$\Phi_d(x) = \prod (x - \xi)$$

where ξ ranges over all the *primitive d th* roots of unity

Proposition 1.8. For every integer $n \geq 1$

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

Definition 1.9. Define **Euler ϕ -function** as the degree of the n th cyclotomic polynomial

$$\phi(n) = \deg(\Phi_n(x))$$

Proposition 1.10. If $n \geq 1$ is an integer, then $\phi(n)$ is the number of integers k with $1 \leq k \leq n$ and $(k, n) = 1$

Proof. Suffice to prove $e^{2\pi i k/n}$ is a primitive n th root of unity if and only if k and n are relatively prime \square

Corollary 1.11. For every integer $n \geq 1$, we have

$$n = \sum_{d|n} \phi(d)$$

2 Group I

2.1 Permutations

Definition 2.1. A **permutation** of a set X is a bijection from X to itself.

Definition 2.2. The family of all the permutations of a set X , denoted by S_X is called the **symmetric group** on X . When $X = \{1, 2, \dots, n\}$, S_X is usually denoted by S_n and is called the **symmetric group on n letters**

Definition 2.3. Let i_1, i_2, \dots, i_r be distinct integers in $\{1, 2, \dots, n\}$. If $\alpha \in S_n$ fixes the other integers and if

$$\alpha(i_1) = i_2, \alpha(i_2) = i_3, \dots, \alpha(i_{r-1}) = i_r, \alpha(i_r) = i_1$$

then α is called an **r -cycle**. α is a cycle of **length r** and denoted by

$$\alpha = (i_1 \ i_2 \ \dots \ i_r)$$

2-cycles are also called the **transpositions**.

Definition 2.4. Two permutations $\alpha, \beta \in S_n$ are **disjoint** if every i moved by one is fixed by the other.

Lemma 2.5. *Disjoint permutations $\alpha, \beta \in S_n$ commute*

Proposition 2.6. *Every permutation $\alpha \in S_n$ is either a cycle or a product of disjoint cycles.*

Proof. Induction on the number k of points moved by α □

Definition 2.7. A **complete factorization** of a permutation α is a factorization of α into disjoint cycles that contains exactly one 1-cycle (i) for every i fixed by α

Theorem 2.8. *Let $\alpha \in S_n$ and let $\alpha = \beta_1 \dots \beta_t$ be a complete factorization into disjoint cycles. This factorization is unique except for the order in which the cycles occur*

Proof. for all i , if $\beta_t(i) \neq i$, then $\beta_t^k(i) \neq \beta_t^{k-1}(i)$ for any $k \geq 1$ □

Lemma 2.9. *If $\gamma, \alpha \in S_n$, then $\alpha\gamma\alpha^{-1}$ has the same cycle structure as γ . In more detail, if the complete factorization of γ is*

$$\gamma = \beta_1 \beta_2 \dots (i_1 \ i_2 \ \dots) \dots \beta_t$$

then $\alpha\gamma\alpha^{-1}$ is permutation that is obtained from γ by applying α to the symbols in the cycles of γ

Example 2.1. Suppose

$$\beta = (1 \ 2 \ 3)(4)(5)$$

$$\gamma = (5 \ 2 \ 4)(1)(3)$$

then we can easily find the α

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix}$$

and so $\alpha = (1 \ 5 \ 3 \ 4)$. Now $\alpha \in S_5$ and $\gamma = (\alpha 1 \ \alpha 2 \ \alpha 3)$

Theorem 2.10. *Permutations γ and σ in S_n has the same cycle structure if and only if there exists $\alpha \in S_n$ with $\sigma = \alpha\gamma\alpha^{-1}$*

Proposition 2.11. *If $n \geq 2$ then every $\alpha \in S_n$ is a product of transpositions*

Proof. $(1 \ 2 \ \dots \ r) = (1 \ r)(1 \ r-1) \dots (1 \ 2)$ □

Example 2.2. The **15-puzzle** has a **starting position** that is a 4×4 array of the numbers between 1 and 15 and a symbol #, which we interpret as "blank". For example, consider the following starting position

3	15	4	8
10	11	1	9
2	5	13	12
6	7	14	#

A **simple move** interchanges the blank with a symbol adjacent to it. We win the game if after a sequence of simple moves, the starting position is transformed into the standard array $1, 2, \dots, 15, \#$.

To analyze this game, note that the given array is really a permutation $\alpha \in S_{16}$. For example, the given starting position is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 3 & 15 & 4 & 8 & 10 & 11 & 1 & 9 & 2 & 5 & 13 & 12 & 6 & 7 & 14 & 16 \end{pmatrix}$$

To win the game, we need special transpositions τ_1, \dots, τ_m so that

$$\tau_m \dots \tau_1 \alpha = (1)$$

Definition 2.12. A permutation $\alpha \in S_n$ is **even** if it can be factored into a product of an even number of transpositions. Otherwise **odd**

Definition 2.13. If $\alpha \in S_n$ and $\alpha = \beta_1 \dots \beta_t$ is a complete factorization, then **signum** α is defined by

$$\text{sgn}(\alpha) = (-1)^{n-t}$$

Theorem 2.14. For all $\alpha, \beta \in S_n$

$$\text{sgn}(\alpha\beta) = \text{sgn}(\alpha) \text{sgn}(\beta)$$

Theorem 2.15. 1. Let $\alpha \in S_n$; if $\text{sgn}(\alpha) = 1$ then α is even. otherwise odd
2. A permutation α is odd if and only if it's a product of an odd number of transpositions

Corollary 2.16. Let $\alpha, \beta \in S_n$. If α and β have the same parity, then $\alpha\beta$ is even while if α and β have distinct parity, $\alpha\beta$ is odd

Example 2.3. An analysis of the 15-puzzle shows that if $\alpha \in S_{16}$ is the starting position, then the game can be won if and only if α is an even permutation that fixes 16.

The blank 16 starts in position 16. Each simple move takes 16 up, down, left or right. Thus the total number m of moves is $u + d + l + r$. If 16 is to return home, each one of these must be undone. Thus the total number of moves is even: $m = 2u + 2r$. Hence $\alpha = \tau_1 \dots \tau_m$ and so α is an even permutation. In example

$$\alpha = (1\ 3\ 4\ 8\ 9\ 2\ 15\ 14\ 7)(5\ 10)(6\ 11\ 13)(12)(16)$$

Now $\text{sgn}(\alpha) = (-1)^{16-5} = -1$.

2.2 Groups

Definition 2.17. A **binary operation** on a set G is a function

$$* : G \times G \rightarrow G$$

Definition 2.18. A **group** is a set G equipped with a binary operation $*$ s.t.

1. the **associative law** holds
2. **identity**
3. every $x \in G$ has an **inverse**, there is a $x' \in G$ with $x * x' = e = x' * x$

Definition 2.19. A group G is called **abelian** if it satisfies the **commutative law**

Lemma 2.20. Let G be a group

1. The **cancellation laws** holds: if either $x * a = x * b$ or $a * x = b * x$, then $a = b$
2. e is **unique**
3. Each $x \in G$ has a **unique inverse**
4. $(x^{-1})^{-1} = x$

Definition 2.21. An expression $a_1 a_2 \dots a_n$ **needs no parentheses** if all the ultimate products it yields are equal

Theorem 2.22 (Generalized Associativity). If G is a group and $a_1, a_2, \dots, a_n \in G$ then the expression $a_1 a_2 \dots a_n$ needs no parentheses

Definition 2.23. Let G be a group and let $a \in G$. If $a^k = 1$ for some $k > 1$ then the smallest such exponent $k \geq 1$ is called the **order** or a ; if no such power exists, then one says that a has **infinite order**

Proposition 2.24. If G is a finite group, then every $x \in G$ has finite order

Definition 2.25. A **motion** is a distance preserving bijection $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. If π is a polygon in the plane, then its **symmetry group** $\Sigma(\pi)$ consists of all the motions φ for which $\varphi(\pi) = \pi$. The elements of $\Sigma(\pi)$ are called the **symmetries** of π

Let π_4 be a square. Then the group $\Sigma(\pi_4)$ is called the **dihedral group** with 8 elements, denoted by D_8

Definition 2.26. If π_n is a regular polygon with n vertices v_1, \dots, v_n and center O , then the symmetry group $\Sigma(\pi_n)$ is called the {dihedral group} with $2n$ elements, and it's denoted by D_{2n}

Exercise 2.2.1. If G is a group with an even number of elements, prove that the number of elements in G of order 2 is odd. In particular, G must contain an element of order 2.

Proof. 1 is an element of order 1. □

2.3 Lagrange's theorem

Definition 2.27. A subset H of a group G is a **subgroup** if

1. $1 \in H$
2. if $x, y \in H$, then $xy \in H$
3. if $x \in H$, then $x^{-1} \in H$

If H is a subgroup of G , we write $H \leq G$. If H is a proper subgroup, then we write $H < G$

The four permutations

$$\mathbf{V} = \{(1), (12)(34), (13)(24), (14)(23)\}$$

form a group because $\mathbf{V} \leq S_4$

Proposition 2.28. A subset H of a group G is a subgroup if and only if H is nonempty and whenever $x, y \in H$, $xy^{-1} \in H$

Proposition 2.29. A nonempty subset H of a finite group G is a subgroup if and only if H is closed; that is, if $a, b \in H$, then $ab \in H$

Example 2.4. The subset A_n of S_n , consisting of all the even permutations, is a subgroup called the **alternating group** on n letters

Definition 2.30. If G is a group and $a \in G$

$$\langle a \rangle = \{a^n : n \in \mathbb{Z}\} = \{\text{all powers of } a\}$$

$\langle a \rangle$ is called the **cyclic subgroup** of G **generated** by a . A group G is called **cyclic** if there exists $a \in G$ s.t. $G = \langle a \rangle$, in which case a is called the **generator**

Definition 2.31. The **integers mod m** , denoted by \mathbb{I}_m is the family of all congruence classes mod m

Proposition 2.32. Let $m \geq 2$ be a fixed integer

1. If $a \in \mathbb{Z}$, then $[a] = [r]$ for some r with $0 \leq r < m$
2. If $0 \leq r' < r < m$, then $[r'] \neq [r]$
3. \mathbb{I}_m has exactly m elements

Theorem 2.33. 1. If $G = \langle a \rangle$ is a cyclic group of order n , then a^k is a generator of G if and only if $(k, n) = 1$
2. If G is a cyclic group of order n and $\text{gen}(G) = \{\text{all generators of } G\}$, then

$$|\text{gen}(G)| = \phi(n)$$

where ϕ is the Euler ϕ -function

Proof. 1. there is $t \in \mathbb{N}$ s.t. $a^{kt} = a$ hence $a^{kt-1} = 1$ and $n \mid kt - 1$

□

Proposition 2.34. Let G be a finite group and let $a \in G$. Then the order of a is $|\langle a \rangle|$.

Definition 2.35. If G is a finite group, then the number of elements in G , denoted by $|G|$ is called the **order** of G

Proposition 2.36. The intersection $\bigcap_{i \in I} H_i$ of any family of subgroups of a group G is again a subgroup of G

Corollary 2.37. If X is a subset of a group G , then there is a subgroup $\langle X \rangle$ of G containing X that is **smallest** in the sense that $\langle X \rangle \leq H$ for every subgroup H of G that contains X

Definition 2.38. If X is a subset of a group G , then $\langle X \rangle$ is called the **subgroup generated by X**

A **word** on X is an element $g \in G$ of the form $g = x_1^{e_1} \dots x_n^{e_n}$ where $x_i \in X$ and $e_i = \pm 1$ for all i

Proposition 2.39. If X is a nonempty subset of a group G , then $\langle X \rangle$ is the set of all words on X

Definition 2.40. If $H \leq G$ and $a \in G$, then the **coset** aH is the subset aH of G , where

$$aH = \{ah : h \in H\}$$

aH **left coset**, Ha **right coset**

Lemma 2.41. $H \leq G$, $a, b \in G$

1. $aH = bH$ if and only if $b^{-1}a \in H$
2. if $aH \cap bH \neq \emptyset$, then $aH = bH$
3. $|aH| = |H|$ for all $a \in G$

Proof. define a relation $a \equiv b$ if $b^{-1}a \in H$ □

Theorem 2.42 (Lagrange's Theorem). If H is a subgroup of a finite group G , then $|H|$ is a divisor of $|G|$

Proof. Let $\{a_1H, a_2H, \dots, a_tH\}$ be the family of all the distinct cosets of H in G . Then

$$G = a_1H \cup a_2H \cup \dots \cup a_tH$$

hence

$$|G| = |a_1H| + \dots + |a_tH|$$

But $|a_iH| = |H|$ for all i . Hence $|G| = t|H|$ □

Definition 2.43. The **index** of a subgroup H in G denoted by $[G : H]$, is the number of left cosets of H in G

Note that $|G| = [G : H]|H|$

Corollary 2.44. If G is a finite group and $a \in G$, then the order of a is a divisor of $|G|$

Corollary 2.45. If G is a finite group, then $a^{|G|} = 1$ for all $a \in G$

Corollary 2.46. If p is a prime, then every group G of order p is cyclic

Proposition 2.47. The set $U(\mathbb{I}_m)$, defined by

$$U(\mathbb{I}_m) = \{[r] \in \mathbb{I}_m : (r, m) = 1\}$$

is a multiplicative group of order $\phi(m)$. If p is a prime, then $U(\mathbb{I}_p) = \mathbb{I}_p^\times$, the nonzero elements of \mathbb{I}_p .

Proof. $(r, m) = 1 = (r', m)$ implies $(rr', m) = 1$. Hence $U(\mathbb{I}_m)$ is closed under multiplication. If $(x, m) = 1$, then $rs + sm = 1$. Therefore $(r, m) = 1$. Each of them have inverse. \square

Corollary 2.48 (Fermat). *If p is a prime and $a \in \mathbb{Z}$, then*

$$a^p \equiv a \pmod{p}$$

Proof. suffices to show $[a^p] = [a]$ in \mathbb{I}_p . If $[a] = [0]$, then $[a^p] = [a]^p = [0]$. Else, since $|\mathbb{I}_p^\times| = p - 1$, $[a]^{p-1} = [1]$ \square

Theorem 2.49 (Euler). *If $(r, m) = 1$, then*

$$r^{\phi(m)} \equiv 1 \pmod{m}$$

Proof. Since $|U(\mathbb{I}_m)| = \phi(m)$. Lagrange's theorem gives $[r]^{\phi(m)} = [1]$ for all $[r] \in U(\mathbb{I}_m)$.

In fact we construct a group to prove this. \square

Theorem 2.50 (Wilson's Theorem). *An integer p is a prime if and only if*

$$(p - 1)! \equiv -1 \pmod{p}$$

Proof. Assume that p is a prime. If a_1, \dots, a_n is a list of all the elements of finite abelian group, then product $a_1 a_2 \dots a_n$ is the same as the product of all elements a with $a^2 = 1$. Since p is prime, \mathbb{I}_p^\times has only one element of order 2, namely $[-1]$. It follows that the product of all the elements in \mathbb{I}_p^\times namely $[(p - 1)!]$ is equal to $[-1]$.

Conversely assume that m is composite: there are integers a and b with $m = ab$ and $1 < a \leq b < m$. If $a < b$ then $m = ab$ is a divisor of $(m - 1)!$. If $a = b$, then $m = a^2$. if $a = 2$, then $(a^2 - 1)! \equiv 2 \pmod{4}$. If $2 < a$, then $2a < a^2$ and so a and $2a$ are factors of $(a^2 - 1)!$ \square

2.4 Homomorphisms

Definition 2.51. If $(G, *)$ and (H, \circ) are groups, then a function $f : G \rightarrow H$ is a **homomorphism** if

$$f(x * y) = f(x) \circ f(y)$$

for all $x, y \in G$. If f is also a bijection, then f is called an **isomorphism**. G and H are called **isomorphic**, denoted by $G \cong H$

Lemma 2.52. Let $f : G \rightarrow H$ be a homomorphism

1. $f(1) = 1$
2. $f(x^{-1}) = f(x)^{-1}$
3. $f(x^n) = f(x)^n$ for all $n \in \mathbb{Z}$

Definition 2.53. If $f : G \rightarrow H$ is a homomorphism, define

$$\ker f = \{x \in G : f(x) = 1\}$$

and

$$\operatorname{im} f = \{h \in H : h = f(x) \text{ for some } x \in G\}$$

Proposition 2.54. Let $f : G \rightarrow H$ be a homomorphism

1. $\ker f$ is a subgroup of G and $\operatorname{im} f$ is a subgroup of H
2. if $x \in \ker f$ and if $a \in G$, then $axa^{-1} \in \ker f$
3. f is an injection if and only if $\ker f = \{1\}$

Proof. 3. $f(a) = f(b) \Leftrightarrow f(ab^{-1}) = 1$

□

Definition 2.55. A subgroup K of a group G is called a **normal subgroup** if $k \in K$ and $g \in G$ imply $gkg^{-1} \in K$, denoted by $K \triangleleft G$

Definition 2.56. If G is a group and $a \in G$, then a **conjugate** of a is any element in G of the form

$$gag^{-1}$$

where $g \in G$

Definition 2.57. If G is a group and $g \in G$, define **conjugation** $\gamma_g : G \rightarrow G$ by

$$\gamma_g(a) = gag^{-1}$$

for all $a \in G$

Proposition 2.58. 1. If G is a group and $g \in G$, then conjugation $\gamma_g : G \rightarrow G$ is an isomorphism
2. Conjugate elements have the same order

Proof. 1. bijection: $\gamma_g \circ \gamma_{g^{-1}} = 1 = \gamma_{g^{-1}} \circ \gamma_g$.

□

Example 2.5. Define the **center** of a group G , denoted by $Z(G)$, to be

$$Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}$$

Example 2.6. If G is a group, then an **automorphism** of G is an isomorphism $f : G \rightarrow G$. For example, every conjugation γ_g is an automorphism of G (it is called an **inner automorphism**), for its inverse is conjugation by g^{-1} . The set $\text{Aut}(G)$ of all the automorphism of G is itself a group.

$$\text{Inn}(G) = \{\gamma_g : g \in G\}$$

is a subgroup of $\text{Aut}(G)$

Proposition 2.59. 1. If H is a subgroup of index 2 in a group G , then $g^2 \in H$ for every $g \in G$
 2. If H is a subgroup of index 2 in a group G , then H is a normal subgroup of G

Definition 2.60. The group of **quaternions** is the group \mathbf{Q} of order 8 consisting of the following matrices in $GL(2, \mathbb{C})$

$$\mathbf{Q} = \{I, A, A^2, A^3, B, BA, BA^2, BA^3\}$$

where I is the identity matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Example 2.7. \mathbf{Q} is normal. By Lagrange's theorem the only possible orders of subgroups are 1, 2, 4 or 8. The only subgroup of order 2 is $\langle -I \rangle$ since $-I$ is the only element of order 2

Proposition 2.61. The alternating group A_4 is a group of order 12 having no subgroup of order 6

Exercise 2.4.1. Show that if there is a bijection $f : X \rightarrow Y$, then there is an isomorphism $\varphi : S_X \rightarrow S_Y$

Proof. If $\alpha \in S_X$, define $\varphi(\alpha) = f \circ \alpha \circ f^{-1}$. Since f, α, f^{-1} are bijections, $\varphi(\alpha)$ is a bijection. φ is a homomorphism. $\forall \beta \in S_Y$, we have $\alpha = f^{-1} \circ \beta \circ f$ \square

2.5 Quotient group

$\mathcal{S}(G)$ is the set of all nonempty subsets of a group G . If $X, Y \in \mathcal{S}(G)$, define

$$XY = \{xy : x \in X \text{ and } y \in Y\}$$

Lemma 2.62. $K \leq G$ is normal if and only if

$$gK = Kg$$

A natural question is that whether HK is a subgroup when H and K are subgroups. The answer is no. Let $G = S_3$, $H = \langle (1\ 2) \rangle$, $K = \langle (1\ 3) \rangle$

Proposition 2.63. 1. If H and K are subgroups of a group G , and if one of them is normal, then $HK \leq G$ and $HK = KH$
 2. If $H, K \triangleleft G$, then $HK \triangleleft G$

Theorem 2.64. Let G/K denote the family of all the left cosets of a subgroup K of G . If $K \triangleleft G$, then

$$aKbK = abK$$

for all $a, b \in G$ and G/K is a group under this operation

Proof. $aKbK = abKK = abK$ □

G/K is called the **quotient group** $G \bmod K$

Corollary 2.65. Every $K \triangleleft G$ is the kernel of some homomorphism

Proof. Define the **natural map** $\pi : G \rightarrow G/K$, $a \mapsto aK$ □

Theorem 2.66 (First Isomorphism Theorem). If $f : G \rightarrow H$ is a homomorphism, then

$$\ker f \triangleleft G \quad \text{and} \quad G/\ker f \cong \text{im } f$$

If $\ker f = K$ and $\varphi : G/K \rightarrow \text{im } f \leq H$, $aK \mapsto f(a)$, then φ is an isomorphism

Remark.
$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ & \searrow \pi & \nearrow \varphi \\ & G/K & \end{array}$$

Example 2.8. What's the quotient group \mathbb{R}/\mathbb{Z} ? Define $f : \mathbb{R} \rightarrow S^1$ where S^1 is the circle group by

$$f : x \mapsto e^{2\pi i x}$$

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

Proposition 2.67 (Product Formula). If H and K are subgroups of a finite group G , then

$$|HK||H \cap K| = |H||K|$$

Proof. Define a function $f : H \times K \rightarrow HK, (h, k) \mapsto hk$. Show that $|f^{-1}(x)| = |H \cap K|$.

Claim that if $x = hk$, then

$$f^{-1}(x) = \{(hd, d^{-1}k) : d \in H \cap K\}$$

□

Theorem 2.68 (Second Isomorphism Theorem). *If $H \triangleleft G, K \leq G$, then $HK \leq G, H \cap K \triangleleft G$ and*

$$K/(H \cap K) \cong HK/H$$

Proof. $hkH = kk^{-1}hkH = kh'H = kH$

□

Theorem 2.69 (Third Isomorphism Theorem). *If $H, K \triangleleft G$ with $K \leq H$, then $H/K \triangleleft G/K$ and*

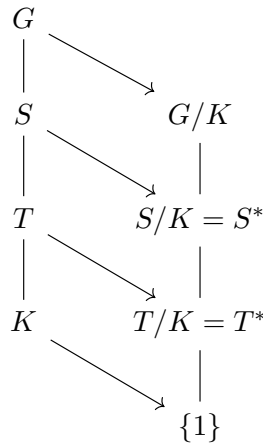
$$(G/K)/(H/K) \cong G/H$$

Theorem 2.70 (Correspondence Theorem). *If $K \triangleleft G, \pi : G \rightarrow G/K$ is the natural map, then*

$$S \mapsto \pi(S) = S/K$$

is a bijection between $\text{Sub}(G; K)$, the family of all those subgroups S of G that contain K , and $\text{Sub}(G/K)$, the family of all the subgroups of G/K . If we denote S/K by S^ , then*

1. $T \leq S \leq G$ if and only if $T^* \leq S^*$, in which case $[S : T] = [S^* : T^*]$
2. $T \triangleleft S$ if and only if $T^* \triangleleft S^*$, in which case $S/T \cong S^*/T^*$



Proof. Use $\pi^{-1}\pi = 1$ and $\pi\pi^{-1} = 1$ to prove injectivity and surjectivity respectively.

For $[S : T] = [S^* : T^*]$, show there is a bijection between the family of all cosets of the form sT and the family of all the cosets of the form s^*T^* .

injective:

$$\begin{aligned}\pi(m)T^* = \pi(n)T^* &\Leftrightarrow \pi(m)\pi(n)^{-1} \in T^* \\ &\Leftrightarrow mn^{-1}K \in T/K \\ &\Rightarrow mn^{-1}t^{-1} \in K \\ &\Rightarrow mn^{-1} = tk \in T \\ &\Leftrightarrow mT = nT\end{aligned}$$

surjective:

If G is finite, then

$$\begin{aligned}[S^* : T^*] &= |S^*| / |T^*| \\ &= |S/K| / |T/K| \\ &= (|S|/|K|) / (|T|/|K|) \\ &= |S| / |T| \\ &= [S : T]\end{aligned}$$

If $T \triangleleft S$, by third isomorphism theorem, $T/S \cong (T/K)/(S/K) = T^*/S^*$

If $T^* \triangleleft S^*$,

$$\pi(sts^{-1}) \in \pi(s)T^*\pi(s)^{-1} = T^*$$

so that $sts^{-1} \in \pi^{-1}(T^*) = T$ □

Proposition 2.71. If G is a finite abelian group and d is a divisor of $|G|$, then G contains a subgroup of order d

Proof. Abelian group's subgroup is normal and hence we can build quotient groups. p90 for proof. Use the correspondence theorem □

Definition 2.72. If H and K are groups, then their **direct product**, denoted by $H \times K$, is the set of all ordered pairs (h, k) with the operation

$$(h, k)(h', k') = (hh', kk')$$

Proposition 2.73. Let G and G' be groups and $K \triangleleft G, K' \triangleleft G'$. Then $K \times K' \triangleleft G \times G'$ and

$$(G \times G') / (K \times K') \cong (G/K) \times (G'/K')$$

Proof. □

Proposition 2.74. *If G is a group containing normal subgroups H and K and $H \cap K = \{1\}$ and $HK = G$, then $G \cong H \times K$*

Proof. Note $|HK||H \cap K| = |H||K|$. Consider $\varphi : G \rightarrow H \times K$. Show it's homo and bijective. □

Theorem 2.75. *If m, n are relatively prime, then*

$$\mathbb{I}_{mn} \cong \mathbb{I}_m \times \mathbb{I}_n$$

Proof.

$$\begin{aligned} f : \mathbb{Z} &\rightarrow \mathbb{I}_m \times \mathbb{I}_n \\ a &\mapsto ([a]_m, [a]_n) \end{aligned}$$

is a homo. $\mathbb{Z}/\langle mn \rangle \cong \mathbb{I}_m \times \mathbb{I}_n$ □

Proposition 2.76. *Let G be a group, and $a, b \in G$ be commuting elements of orders m, n . If $(m, n) = 1$, then ab has order mn*

Corollary 2.77. *If $(m, n) = 1$, then $\phi(mn) = \phi(m)\phi(n)$*

Proof. Theorem 2.75 shows that $f : \mathbb{I}_{mn} \cong \mathbb{I}_m \times \mathbb{I}_n$. The result will follow if we prove that $f(U(\mathbb{I}_{mn})) = U(\mathbb{I}_m) \times U(\mathbb{I}_n)$, for then

$$\begin{aligned} \phi(mn) &= |U(\mathbb{I}_{mn})| = |f(U(\mathbb{I}_{mn}))| \\ &= |U(\mathbb{I}_m) \times U(\mathbb{I}_n)| = |U(\mathbb{I}_m)| \cdot |U(\mathbb{I}_n)| \end{aligned}$$

If $[a] \in U(\mathbb{I}_{mn})$, then $[a][b] = [1]$ for some $[b] \in \mathbb{I}_{mn}$ and

$$\begin{aligned} f([ab]) &= ([ab]_m, [ab]_n) = ([a]_m[b]_m, [a]_n[b]_n) \\ &= ([a]_m, [a]_n)([b]_m, [b]_n) = ([1]_m, [1]_n) \end{aligned}$$

Hence $f([a]) = ([a]_m, [a]_n) \in U(\mathbb{I}_m) \times U(\mathbb{I}_n)$

For the reverse inclusion, if $f([c]) = ([c]_m, [c]_n) \in U(\mathbb{I}_m) \times U(\mathbb{I}_n)$, then we must show that $[c] \in U(\mathbb{I}_{mn})$. There is $[d]_m \in \mathbb{I}_m$ with $[c]_m[d]_m = [1]_m$, and there is $[e]_n \in \mathbb{I}_n$ with $[c]_n[e]_n = [1]_n$. Since f is surjective, there is $b \in \mathbb{Z}$ with $([b]_m, [b]_n) = ([d]_m, [e]_n)$, so that

$$f([1]) = ([1]_m, [1]_n) = ([c]_m[b]_m, [c]_n[b]_n) = f([c][b])$$

Since f is an injection, $[1] = [c][b]$ and $[c] \in U(\mathbb{I}_{mn})$ □

Corollary 2.78. 1. If p is a prime, then $\phi(p^e) = p^e - p^{e-1} = p^e(1 - \frac{1}{p})$
 2. If $n = p_1^{e_1} \dots p_t^{e_t}$, then

$$\phi(n) = n(1 - \frac{1}{p_1}) \dots (1 - \frac{1}{p_t})$$

Lemma 2.79. A cyclic group of order n has a unique subgroup of order d , for each divisor d of n , and this subgroup is cyclic.

Define an equivalence relation on a group G by $x \equiv y$ if $\langle x \rangle = \langle y \rangle$. Denote the equivalence class containing x by $\text{gen}(C)$, where $C = \langle x \rangle$. Equivalence classes form a partition and we get

$$G = \coprod_C \text{gen}(C)$$

where C ranges over all cyclic subgroups of G . Note $|\text{gen}(C)| = \phi(n)$

Theorem 2.80. A group G of order n is cyclic if and only if for each divisor d of n , there is at most one cyclic subgroup of order d

Theorem 2.81. If G is an abelian group of order n having at most one cyclic subgroup of order p for each prime divisor p of n , then G is cyclic

Exercise:

- 2.71 Suppose $H \leq G, |H| = |K|$. Since $|H| = [H : K]|K|$, $[H : K] = 1$. Hence $H = K$
- 2.67 1. $\text{Inn}(S_3) \cong S_3/Z(S_3) \cong S_3$ and $|\text{Aut}(S_3)| \leq 6$. Hence $\text{Aut}(S_3) = \text{Inn}(S_3)$

Exercise 2.5.1. Prove that if G is a group for which $G/Z(G)$ is cyclic, then G is abelian

Proof. Suppose $G/Z(G) = \langle a \rangle$, let $g = a^k z^{-1}, g' = a^{k'} z'^{-1}$, then $gg' = a^k z^{-1} z^{k'} z'^{-1} = a^{k+k'} z'^{-1} z^{-1} = g'g$. Hence G is abelian. \square

2.6 Group Actions

Theorem 2.82 (Cayley). Every group G is isomorphic to a subgroup of the symmetric group S_G . In particular, if $|G| = n$, then G is isomorphic to a subgroup of S_n

Proof. For each $a \in G$, define $\tau_a(x) = ax$ for every $x \in G$. τ_a is a bijection for its inverse is $\tau_{a^{-1}}$

$$\tau_a \tau_{a^{-1}} = \tau_1 = \tau_{a^{-1}} \tau_a$$

\square

Theorem 2.83 (Representation on Cosets). *Let G be a group and $H \leq G$ having finite index n . Then there exists a homomorphism $\varphi : G \rightarrow S_n$ with $\ker \varphi \leq H$*

Proof. We still denote the family of all the cosets of H in G by G/H

For each $a \in G$, define "translation" $\tau_a : G/H \rightarrow G/H$ by $\tau_a(xH) = axH$ for every $x \in G$. For $a, b \in G$

$$(\tau_a \circ \tau_b)(xH) = a(bxH) = (ab)xH$$

so that

$$\tau_a \tau_b = \tau_{ab}$$

It follows that each τ_a is a bijection and so $\tau_a \in S_{G/H}$. Define $\varphi : G \rightarrow S_{G/H}$ by $\varphi(a) = \tau_a$. Rewriting

$$\varphi(a)\varphi(b) = \tau_a \tau_b = \tau_{ab} = \varphi(ab)$$

so that φ is a homomorphism. Finally if $a \in \ker \varphi$, then $\varphi(a) = 1_{G/H}$, so that $\tau_a(xH) = xH$, in particular, when $x = 1$, this gives $aH = H$ and $a \in H$. And $S_{G/H} \cong S_n$ \square

When $H = \{1\}$, this is the Cayley theorem.

Four-group $\mathbf{V} = \{1, (12)(34), (13)(24), (14)(23)\}$

Proposition 2.84. *Every group G of order 4 is isomorphic to either \mathbb{I}_4 or the four-group \mathbf{V} . And $\mathbb{I}_4 \not\cong \mathbf{V}$*

Proof. By Lagrange's theorem, every element in G other than 1 has order 2 or 4. If 4, then G is cyclic.

Suppose $x, y \neq 1$, then $xy \neq x, y$. Hence $G = \{1, x, y, xy\}$. \square

Proposition 2.85. *If G is a group of order 6, then G is isomorphic to either \mathbb{I}_6 or S_3 . Moreover $\mathbb{I}_6 \not\cong S_3$*

Proof. If G is not cyclic, since $|G|$ is even, it has some elements having order 2, say t by exercise 2.2.1

If G is abelian. Suppose it has another different element a with order 2. Then $H = \{1, a, t, at\}$ is a subgroup which contradicts. Hence it must contain an element b of order 3. Then bt has order 6 and G is cyclic.

If G is not abelian. If G doesn't have elements of order 3, then it's abelian. Hence G has an element s of order 3.

Now $|\langle s \rangle| = 3$, so $[G : \langle s \rangle] = |G|/|\langle s \rangle| = 2$ and $\langle s \rangle$ is normal. Since $t = t^{-1}$, $tst \in \langle s \rangle$. If $tst = s^0 = 1$, $s = 1$. If $tst = s$, $|\langle st \rangle| = 6$. Therefore $tst = s^2 = s^{-1}$.

Let $H = \langle t \rangle$, $\varphi : G \rightarrow S_{G/\langle t \rangle}$ given by

$$\varphi(g) : x\langle t \rangle \mapsto gx\langle t \rangle$$

By representation on cosets, $\ker \varphi \leq \langle t \rangle$. Hence $\ker \varphi = \{1\}$ or $\ker \varphi = \langle t \rangle$. Since

$$\varphi(t) = \begin{pmatrix} \langle t \rangle & s\langle t \rangle & s^2\langle t \rangle \\ t\langle t \rangle & ts\langle t \rangle & ts^2\langle t \rangle \end{pmatrix}$$

If $\varphi(t)$ is the identity permutation, then $ts\langle t \rangle = s\langle t \rangle$, so that $s^{-1}ts \in \langle t \rangle = \{1, t\}$. But now $s^{-1}ts = t$. Therefore $t \notin \ker \varphi$ and $\ker \varphi = \{1\}$. Therefore φ is injective. Because $|G| = |S_3|$, $G \cong S_3$ \square

Definition 2.86. If X is a set and G is a group, then G **acts** on X if there is a function $G \times X \rightarrow X$, denoted by $(g, x) \rightarrow gx$ s.t.

1. $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$
2. $1x = x$ for all $x \in X$

X is a **G -set** if G acts on X

If a group G acts on a set X , then fixing the first variable, say g , gives a function $\alpha_g : X \rightarrow X$, namely, $\alpha_g : x \mapsto gx$. This function is a permutation of X , for its inverse is $\alpha_{g^{-1}}$

$$\alpha_g \alpha_{g^{-1}} = 1 = \alpha_{g^{-1}} \alpha_g$$

It's easy to see that $\alpha : G \rightarrow S_X$ defined by $\alpha : g \mapsto \alpha_g$ is a homomorphism. Conversely, given any homomorphism $\varphi : G \rightarrow S_X$, define $gx = \varphi(g)(x)$. Thus an action of a group G on a set X is another way of viewing a homomorphism.

Definition 2.87. If G acts on X and $x \in X$, then the **orbit** of x , denoted by $\mathcal{O}(x)$, is the subset of X

$$\mathcal{O}(x) = \{gx : g \in G\} \subseteq X$$

the **stabilizer** of x , denoted by G_x , is the subgroup

$$G_x = \{g \in G : gx = x\} \leq G$$

Example 2.9. 1. Cayley's theorem says that G acts on itself by translation: $\tau_g : a \mapsto ga$. We say G acts **transitively** on X if there is only one orbit.
2. When G acts on G/H by translation $\tau_g : aH \mapsto gaH$, then the orbit $\mathcal{O}(aH) = G/H$

3. When a group G acts on itself by conjugation, then the orbit $\mathcal{O}(x)$ is

$$\{y \in G : y = axa^{-1} \text{ for some } a \in G\}$$

in this case, $\mathcal{O}(x)$ is called the **conjugacy class** of x , and it is commonly denoted by x^G .

centralizer $C_G(x) = \{g \in G : gxg^{-1} = x\}$

4. Let $X = \{1, 2, \dots, n\}$, let $\alpha \in S_n$ and regard the cyclic group $G = \langle \alpha \rangle$ as acting on X . If $i \in X$, then

$$\mathcal{O}(i) = \{\alpha^k(i) : k \in \mathbb{Z}\}$$

Let the complete factorization of α be $\alpha = \beta_1 \dots \beta_{t(\alpha)}$, and let $i = i_1$ be moved by α . If the cycle involving i_1 is $\beta_j = (i_1 i_2 \dots i_r)$,

$$\mathcal{O}(i) = \{i_1, \dots, i_r\}$$

where $i = i_1$. It follows that $|\mathcal{O}(i)| = r$. The stabilizer G_l of a number l is G if α fixes l

Normalizer

$$N_G(H) = \{g \in G : gHg^{-1} = H\}$$

Proposition 2.88. *If G acts on a set X , then X is the disjoint union of the orbits. If X is finite, then*

$$|X| = \sum_i |\mathcal{O}(x_i)|$$

where x_i is chosen from each orbit

Proof. $x \equiv y \Leftrightarrow$ there exists $g \in G$ with $y = gx$ is an equivalence relation \square

Theorem 2.89. *If G acts on a set X and $x \in X$ then*

$$|\mathcal{O}(x)| = [G : G_x]$$

Proof. Let G/G_x denote the family of cosets. Construct a bijection $\varphi : G/G_x \rightarrow \mathcal{O}(x)$ \square

Corollary 2.90. *If a finite group G acts on a set X , then the number of elements in any orbit is a divisor of $|G|$.*

Corollary 2.91. *If x lies in a finite group G , then the number of conjugates of x is the index of its centralizer*

$$|x^G| = [G : C_G(x)]$$

and hence it's a divisor of $|G|$

Proof. x^G is the orbit, $C_G(x)$ is the stabilizer □

Proposition 2.92. *If H is a subgroup of a finite group G , then the number of conjugates of H in G is $[G : N_G(H)]$*

Proof. Similar to theorem 2.89 □

Theorem 2.93 (Cauchy). *If G is a finite group whose order is divisible by a prime p , then G contains an element of order p*

Proof. Prove by induction on $m \geq 1$, where $|G| = mp$. If $m = 1$, it's obvious.

If $x \in G$, then $|x^G| = [G : C_G(x)]$. If $x \notin Z(G)$, then x^G has more than one element, so $|C_G(x)| < |G|$. If $p \mid |C_G(x)|$, by inductive hypothesis, we are done. Else if $p \nmid |C_G(x)|$ for all noncentral x and $|G| = [G : C_G(x)]|C_G(x)|$, we have

$$p \mid [G : C_G(x)]$$

$Z(G)$ consists of all those elements with $|x^G| = 1$, we have

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$$

Hence $p \mid |Z(G)|$ and by proposition 2.71 □

Definition 2.94. The **class equation** of a finite group G is

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$$

where each x_i is selected from each conjugacy class having more than one element

Definition 2.95. If p is a prime, then a finite group G is called a **p-group** if $|G| = p^n$ for some $n \geq 0$

Theorem 2.96. *If p is a prime and G is a p -group, then $Z(G) \neq \{1\}$*

Proof. Consider

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$$

□

Corollary 2.97. *If p is a prime, then every group G of order p^2 is abelian*

Proof. If G is not abelian, then $Z(G)$ has order p . The center is always normal, and so $G/Z(G)$ is defined; it has order p and is cyclic by Lagrange's theorem. This contradicts Exercise 2.5.1 \square

Example 2.10. Cauchy's theorem and Fermat's theorem are special cases of some common theorem.

If G is a finite group and p is a prime, define

$$X = \{(a_0, a_1, \dots, a_{p-1}) \in G^p : a_0 a_1 \dots a_{p-1} = 1\}$$

Note that $|X| = |G|^{p-1}$, for having chosen the last $p-1$ entries arbitrarily, the 0th entry must equal $(a_1 a_2 \dots a_{p-1})^{-1}$. Introduce an action of \mathbb{I}_p on X by defining, for $0 \leq i \leq p-1$,

$$[i](a_0, \dots, a_{p-1}) = (a_{i+1}, \dots, a_{p-1}, a_0, \dots, a_i)$$

The product of the new p -tuple is a conjugate of $a_0 a_1 \dots a_{p-1}$

$$a_{i+1} \dots a_{p-1} a_0 \dots a_i = (a_0 \dots a_i)^{-1} (a_0 \dots a_{p-1}) (a_0 \dots a_i)$$

This conjugate is 1 for $g^{-1}1g = 1$, and so $[i](a_0, \dots, a_{p-1}) \in X$. By Corollary 2.90, the size of every orbit of X is a divisor of $|\mathbb{I}_p| = p$. Now orbits with just one element consists of a p -tuple all of whose entries a_i are equal, for all cyclic permutations of the p -tuple are the same. In other words, such an orbit corresponds to an element $a \in G$ with $a^p = 1$. Clearly $(1, 1, \dots, 1)$ is such an orbit; if it were the only such, then we would have

$$|G|^{p-1} = |X| = 1 + kp$$

That is, $|G|^{p-1} \equiv 1 \pmod{p}$. If p is a divisor of $|G|$, then we have a contradiction and thus proved Cauchy's theorem.

Proposition 2.98. If G is a group of order $|G| = p^e$ then G has a normal subgroup of order p^k for every $k \leq e$

Proof. We prove the result by induction on $e \geq 0$.

By Theorem 2.96, $Z(G) \neq \{1\}$. Let $Z \leq Z(G)$ be a subgroup of order p and Z is normal. If $k \leq e$, then $p^{k-1} \leq p^{e-1} = |G/Z|$. By induction, G/Z has a normal subgroup H^* of order p^{k-1} . The correspondence theorem says there is a subgroup H of G containing Z with $H^* = H/Z$; moreover $H^* \triangleleft G/Z$ implies $H \triangleleft G$. But $|H/Z| = p^{k-1}$ implies $|H| = p^k$ as desired. \square

Definition 2.99. A group $G \neq \{1\}$ is called **simple** if G has no normal subgroups other than $\{1\}$ and G itself.

Proposition 2.100. *An abelian group G is simple if and only if it is finite and of prime order*

Proof. Assume G is simple. Since G is abelian, every subgroup is normal, and so G has no subgroups other than $\{1\}$ and G . Choose $x \in G$ with $x \neq 1$. Since $\langle x \rangle \leq G$, we have $\langle x \rangle = G$. If x has infinite order, then all the powers of x are distinct, and so $\langle x^2 \rangle < \langle x \rangle$ is a forbidden subgroup of $\langle x \rangle$, a contradiction. Therefore every $x \in G$ has finite order. If x has order m and if m is composite, say $m = kl$, then $\langle x^k \rangle$ is a proper subgroup of $\langle x \rangle$, a contradiction. Therefore $G = \langle x \rangle$ has prime order. \square

Suppose that an element $x \in G$ has k conjugates, that is

$$|x^G| = |\{gxg^{-1} : g \in G\}| = k$$

If there is a subgroup $H \leq G$ with $x \in H \leq G$, how many conjugates does x have in H ?

Since

$$x^H = \{h x h^{-1} : h \in H\} \subseteq x^G$$

we have $|x^H| \leq |x^G|$. It is possible that there is a strict inequality $|x^H| < |x^G|$. For example, take $G = S_3$, $x = (1\ 2)$, and $H = \langle x \rangle$. Now let us consider this question, in particular, for $G = S_5$, $x = (1\ 2\ 3)$, $H = A_5$

Lemma 2.101. *All 3-cycles are conjugate in A_5*

Proof. Let $G = S_5$, $\alpha = (1\ 2\ 3)$, $H = A_5$. We know that $|\alpha^{S_5}| = 20$, for there are 20 3-cycles in S_5 . Therefore, $20 = |S_5| / |C_{S_5}(\alpha)|$ by Corollary 2.91, so that $|C_{S_5}(\alpha)| = 6$. Here they are

$$(1), (1\ 2\ 3), (1\ 3\ 2), (4\ 5), (4\ 5)(1\ 2\ 3), (4\ 5)(1\ 3\ 2)$$

The last three of these are odd permutations, so that $|C_{A_5}(\alpha)| = 3$. We conclude that

$$|\alpha^{A_5}| = |A_5| / |C_{A_5}(\alpha)| = 20$$

that is all 3-cycles are conjugate to α in A_5 \square

Lemma 2.102. *If $n \geq 3$, every element in A_n is a 3-cycle or a product of 3-cycles*

Proof. Since each β equals $\tau_1 \dots \tau_{2q}$ \square

Theorem 2.103. *A_5 is a simple group*

Proof. If $H \triangleleft A_5$ and $H \neq \{(1)\}$. Now if H contains a 3-cycle, then normality forces H to contain all its conjugates. Therefore it suffices to prove that H contains 3-cycle.

Since $\sigma \in H$, we may assume, after a harmless relabeling, that either $\sigma = (1\ 2\ 3)$, $\sigma = (1\ 2)(3\ 4)$ or $\sigma = (1\ 2\ 3\ 4\ 5x)$

If $\sigma = (1\ 2)(3\ 4)$, define $\tau = (1\ 2)(3\ 5)$. Now $(3\ 5\ 4) = (\tau\sigma\tau^{-1})\sigma^{-1} \in H$. If $\sigma = (1\ 2\ 3\ 4\ 5)$, define $\rho = (1\ 3\ 2)$ and $(1\ 3\ 4) = \rho\sigma\rho^{-1}\sigma^{-1} \in H$ \square

A_4 is not simple for $\mathbf{V} \triangleleft A_4$.

Lemma 2.104. A_6 is a simple group

Proof. Let $\{1\} \neq H \triangleleft A_6$; we must show that $H = A_6$. Assume that there is some $\alpha \in H$ with $\alpha \neq (1)$ that fixes some i , where $1 \leq i \leq 6$. Define

$$F = \{\sigma \in A_6 : \sigma(i) = i\}$$

Note that $\alpha \in H \cap F$, so that $H \cap F \neq \{(1)\}$. The second isomorphism theorem gives $H \cap F \triangleleft F$. But F is simple for $F \cong A_5$, we have $H \cap F = F$: that is $F \leq H$. It follows that H contains a 3-cycle, and so $H = A_6$ by Exercise 2.6.2.

If there is no $\alpha \in H$ with $\alpha \neq \{1\}$ that fixes some i with $1 \leq i \leq 6$. If we consider the cycle structures of permutations in A_6 , however, any such α must have cycle structure $(1\ 2)(3\ 4\ 5\ 6)$ or $(1\ 2\ 3)(4\ 5\ 6)$. In the first case $\alpha^2 \in H$, $\alpha^2 \in H$ fixes 1. In the second case $\alpha(\beta\alpha^{-1}\beta^{-1})$ where $\beta = (2\ 3\ 4)$ fixes 1. \square

Theorem 2.105. A_n is a simple group for all $n \geq 5$

Proof. If H is a nontrivial normal subgroup of A_n , then we must show that $H = A_n$. By Exercise 2.6.2 it suffices to prove that H contains a 3-cycle. If $\beta \in H$ is nontrivial, then there exists some i that β moves: say, $\beta(i) = j \neq i$. Choose a 3-cycle α that fixes i and moves j . The permutations α and β do not commute. It follows that $\gamma = (\alpha\beta\alpha^{-1})\beta^{-1}$ is a nontrivial element of H . But $\beta\alpha^{-1}\beta^{-1}$ is a 3-cycle, and so $\gamma = \alpha(\beta\alpha^{-1}\beta^{-1})$ is a product of two 3-cycles. Hence γ moves at most 6 symbols, say i_1, \dots, i_6 . Define

$$F = \{\sigma \in A_n : \sigma \text{ fixes all } i \neq i_1, \dots, i_6\}$$

Now $F \cong A_6$ and $\gamma \in H \cap F$. Hence $H \cap F \triangleleft F$. But F is simple, and so $H \cap F = F$; that is $F \leq H$. Therefore H contains a 3-cycle \square

Theorem 2.106 (Burnside's Lemma). *Let G act on a finite set X . If N is the number of orbits, then*

$$N = \frac{1}{|G|} \sum_{\tau \in G} \text{Fix}(\tau)$$

where $\text{Fix}(\tau)$ is the number of $x \in X$ fixed by τ

Proof. List the elements of X as follows: Choose $x_1 \in X$ and then list all the elements x_1, \dots, x_r in the orbit $\mathcal{O}(x_1)$; then choose $x_{r+1} \notin \mathcal{O}(x_1)$, and so on until all the elements of X are listed. Now list the elements τ_1, \dots, τ_n of G and form the following array, where

$$f_{i,j} = \begin{cases} 1 & \text{if } \tau_i \text{ fixes } x_j \\ 0 & \text{if } \tau_i \text{ moves } x_j \end{cases}$$

	x_1	x_2	\dots	x_{r+1}	x_{r+2}	\dots
τ_1	$f_{1,1}$	$f_{1,2}$	\dots	$f_{1,r+1}$	$f_{1,r+2}$	\dots
\vdots						
τ_n	$f_{n,1}$	$f_{n,2}$	\dots	$f_{n,r+1}$	$f_{n,r+2}$	\dots

Now $\text{Fix}(\tau_i)$ is the number of 1's in the i th row. therefore $\sum_{\tau \in G} \text{Fix}(\tau)$ is the total number of 1's in the array. The number of 1's in column 1 is $|G_{x_1}|$. By Exercise 2.6.3 $|G_{x_1}| = |G_{x_2}|$. By Theorem 2.89 the number of 1's in the r columns labels by the $x_i \in \mathcal{O}(x_i)$ is thus

$$r|G_{x_1}| = |\mathcal{O}(x_1)| \cdot |G_{x_1}| = (|G|/|G_{x_1}|)|G_{x_1}| = |G|$$

Therefore

$$\sum_{\tau \in G} \text{Fix}(\tau) = N|G|$$

□

We are going to use Burnside's lemma to solve problems of the following sort. How many striped flags are there having six stripes each of which can be colored red, white or blue?

r	w	b	r	w	b
---	---	---	---	---	---

b	w	r	b	w	r
---	---	---	---	---	---

Let X be the set of all 6-tuples of colors: if $x \in X$, then

$$x = (c_1, c_2, c_3, c_4, c_5, c_6)$$

Let τ be the permutation that reserves all the indices:

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = (1\ 6)(2\ 5)(3\ 4)$$

(thus τ turns over each 6-tuple x of colored stripes). The cyclic group $G = \langle \tau \rangle$ acts on X ; since $|G| = 2$, the orbit of any 6-tuple x consists of either 1 or 2 elements. Since a flag is unchanged by turning it over, it is reasonable to identify a flag with an orbit of 6-tuple. For example, the orbit consisting of the 6-tuples

$$(r, w, b, r, w, b) \text{ and } (b, w, r, b, w, r)$$

above. The number of flags is thus the number N of orbits; by Burnside's lemma, $N = \frac{1}{2}[Fix((1)) + Fix(\tau)]$. The identity permutation (1) fixes every $x \in X$, and so $Fix((1)) = 3^6$. Now τ fixes a 6-tuple x if it's a "palindrome". It follows that $Fix(x) = 3^3$. The number of flags is thus

$$N = \frac{1}{2}(3^6 + 3^3) = 378$$

Definition 2.107. If a group G acts on $X = \{1, \dots, n\}$ and if \mathcal{C} is a set of q colors, then G acts on the set \mathcal{C}^n of all n -tuples of colors by

$$\tau(c_1, \dots, c_n) = (c_{\tau 1}, \dots, c_{\tau n}) \text{ for all } \tau \in G$$

An orbit of $(c_1, \dots, c_n) \in \mathcal{C}^n$ is called a (q, G) -**coloring** of X .

Example 2.11. Color each square in a 4×4 grid red or black.

If X consists of the 16 squares in the grid and if \mathcal{C} consists of the two colors red and black, then the cyclic group $G = \langle R \rangle$ of order 4 acts on X , where R is a clockwise rotation by 90° ;

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

13	9	5	1
14	10	6	2
15	11	7	3
16	12	8	4

Figure shows how R acts: the right square is R 's action on the left square. In cycle notation

$$R = (1, 4, 16, 13)(2, 8, 15, 9)(3, 12, 14, 5)(6, 7, 11, 10)$$

$$R^2 = (1, 16)(4, 13)(2, 15)(8, 9)(3, 14)(12, 5)(6, 11)(7, 10)$$

$$R^3 = (1, 13, 16, 4)(2, 9, 15, 8)(3, 5, 14, 12)(6, 10, 11, 7)$$

By Burnside's lemma, the number of chessboards is

$$\frac{1}{4}[Fix((1)) + Fix(R) + Fix(R^2) + Fix(R^3)]$$

Exercise 2.6.1. Prove that if p is a prime and G is a finite group in which every element has order a power of p , then G is a p -group. (A possibly infinite group G is called a **p -group** if every element in G has order a power of p)

Proof. By Cauchy's theorem 2.93 □

Exercise 2.6.2. 1. For all $n \geq 5$, prove that all 3-cycles are conjugate in A_n
 2. Prove that if a normal subgroup $H \triangleleft A_n$ contains a 3-cycle, where $n \geq 5$, then $H = A_n$

Proof. 1. If $(1\ 2\ 3)$ and $(i\ j\ k)$ are not disjoint. As Example 2.1 illustrated, $\alpha \in S_5$

If they are disjoint, simple

2. By lemma 2.102 □

Exercise 2.6.3. 1. Let a group G act on a set X , and suppose that $x, y \in X$ lie in the same orbit: $y = gx$ for some $g \in G$. Prove that $G_y = gG_xg^{-1}$

2. Let G be a finite group acting on a set X ; prove that if $x, y \in X$ lie in the same orbit, then $|G_x| = |G_y|$

Proof. 1. If $f \in G_x$, then $gfg^{-1}(y) = gfg^{-1}gx = gx = y$

2. There is a bijection. □

3 Commutative Rings I

3.1 First Properties

Definition 3.1. A **commutative ring** R is a set with two binary operations, addition and multiplication s.t.

1. R is an abelian group under addition
2. (**commutativity**) $ab = ba$ for all $a, b \in R$
3. (**associativity**) $a(bc) = (ab)c$ for every $a, b, c \in R$
4. there is an element $1 \in R$ with $1a = a$ for every $a \in R$
5. (**distributivity**) $a(b + c) = ab + ac$ for every $a, b, c \in R$

The element 1 in a ring R has several names: it is called **one**, the **unit** of R , or the **identity** in R

Example 3.1. 1. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and \mathbb{C} are commutative rings with the usual addition and multiplication
2. Consider the set R of all real numbers x of the form

$$x = a + b\omega$$

where $a, b \in \mathbb{Q}$ and $\omega = \sqrt[3]{2}$. R is closed under ordinary addition. However, if R is closed under multiplication, then $\omega^2 \in R$ and there are rationals a and b with

$$\omega^2 = a + b\omega$$

$$2 = a\omega + b\omega^2$$

$$b\omega^2 = ab + b^2\omega$$

Hence $2 - a\omega = ab + b^2\omega$ and so

$$2 - ab = (b^2 + a)\omega$$

A contradiction.

Proposition 3.2. Let R be a commutative ring.

1. $0 \cdot a = 0$ for every $a \in R$
2. If $1 = 0$ then R consists of the single element 0. In this case R is called the **zero ring**
3. If $-a$ is the additive inverse of a , then $(-1)(-a) = a$
4. $(-1)a = -a$ for every $a \in R$
5. If $n \in \mathbb{N}$ and $n1 = 0$, then $na = 0$ for all $a \in R$

6. The binomial theorem holds: if $a, b \in R$, then

$$(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^r b^{n-r}$$

Proof. 6. $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

□

Definition 3.3. A subset S of a commutative ring R is a **subring** of R if

1. $1 \in S$
2. if $a, b \in S$ then $a - b \in S$
3. if $a, b \in S$, then $ab \in S$

Notation. The tradition in ring theory is to write $S \subseteq R$ for a subring

Proposition 3.4. A subring S of a commutative ring R is itself a commutative ring.

Definition 3.5. A **domain** (often called an **integral domain**) is a commutative ring R that satisfies two extra axioms: first

$$1 \neq 0$$

second, the **cancellation law** for multiplication: for all $a, b, c \in R$

$$\text{if } ca = cb \text{ and } c \neq 0, \text{ then } a = b$$

Proposition 3.6. A nonzero commutative ring R is a domain if and only if the product of any two nonzero elements of R is nonzero

Proof. $ab = ac$ if and only if $a(b - c) = 0$

□

Proposition 3.7. The commutative ring \mathbb{I}_m is a domain if and only if m is a prime

Proof. If $m = ab$, where $1 < a, b < m$, then $[a], [b] \neq [0]$ yet $[a][b] = [m] = [0]$

Conversely, if m is a prime and $[a][b] = [ab] = [0]$, then $m \mid ab$ □

Example 3.2. 1. Let $\mathcal{F}(\mathbb{R})$ be the set of all the function $\mathbb{R} \rightarrow \mathbb{R}$ equipped with the operations of **point-wise addition** and **point-wise multiplication**: Given $f, g \in \mathcal{F}(\mathbb{R})$, define functions $f + g$ and fg by

$$f + g : a \mapsto f(a) + f(b) \quad \text{and} \quad fg : a \mapsto f(a)g(a)$$

We claim that $\mathcal{F}(\mathbb{R})$ with these operations is a commutative ring. The zero element is the constant function z with value 0. $\mathcal{F}(\mathbb{R})$ is not a domain by

$$f(a) = \begin{cases} a & \text{if } a \leq 0 \\ 0 & \text{if } a > 0 \end{cases} \quad g(a) = \begin{cases} 0 & \text{if } a \leq 0 \\ a & \text{if } a > 0 \end{cases}$$

Definition 3.8. Let a and b be elements of a commutative ring R . Then a **divides** b in R (or a is a **divisor** of b or b is a **multiple** of a), denoted by $a \mid b$, if there exists an element $c \in R$ with $b = ca$

Definition 3.9. An element u in a commutative ring R is called a **unit** if $u \mid 1$ in R .

Proposition 3.10. Let R be a domain, and let $a, b \in R$ be nonzero. Then $a \mid b$ and $b \mid a$ if and only if $b = ua$ for some unit $u \in R$

Proposition 3.11. If a is an integer, then $[a]$ is a unit in \mathbb{I}_m if and only if a and m are relatively prime.

Corollary 3.12. If p is a prime, then every nonzero $[a]$ in \mathbb{I}_p is a unit.

Definition 3.13. If R is a commutative ring, then the **group of units** of R is

$$U(R) = \{\text{all units in } R\}$$

Definition 3.14. A **field** F is a commutative ring in which $1 \neq 0$ and every nonzero element a is a unit; that is, there is $a^{-1} \in F$ with $a^{-1}a = 1$

A commutative ring R is a field if and only if $U(R) = R^\times$, the nonzero elements of R .

Proposition 3.15. Every field F is a domain

Proof. $ab = ac, b = a^{-1}ab = a^{-1}(ac) = c$ □

Proposition 3.16. The commutative ring \mathbb{I}_m is a field if and only if m is prime

Theorem 3.17. If R is a domain then there is a field F containing R as a subring. Moreover, F can be chosen so that for each $f \in F$, there are $a, b \in R$ with $b \neq 0$ and $f = ab^{-1}$

Proof. Let $X = \{(a, b) \in R \times R : b \neq 0\}$ and define a relation \equiv on X by $(a, b) \equiv (c, d)$ if $ad = bc$. We claim that \equiv is an equivalence relation. If $(a, b) \equiv (c, d)$ and $(c, d) \equiv (e, f)$, then $ad = bc, cf = de$ and $adf = b(cf) = bde$, gives $af = be$

Denote the equivalence class of (a, b) by $[a, b]$, define F as the set of all equivalence classes $[a, b]$ and equip F with the following addition and multiplication

$$\begin{aligned}[a, b] + [c, d] &= [ad + bc, bd] \\ [a, b][c, d] &= [ac, bd]\end{aligned}$$

Show addition and multiplication are well-defined. \square

Definition 3.18. The field F constructed from R in Theorem 3.17 is called the **fraction field** of R , denoted by $\text{Frac}(R)$, and we denote $[a, b] \in \text{Frac}(R)$ by a/b

Note that $\text{Frac}(\mathbb{Z}) = \mathbb{Q}$

3.2 Polynomials

Definition 3.19. If R is a commutative ring, then a **sequence** σ in R is

$$\sigma = (s_0, s_1, \dots, s_i, \dots)$$

the entries $s_i \in R$ for all $i \geq 0$ are called the **coefficients** of σ

Definition 3.20. A sequence $\sigma = (s_0, \dots, s_i, \dots)$ in a commutative ring R is called a **polynomial** if there is some integer $m \geq 0$ with $s_i = 0$ for all $i > m$; that is

$$\sigma = (s_0, \dots, s_m, 0, \dots)$$

A polynomial has only finitely many nonzero coefficients. The **zero polynomial**, denoted by $\sigma = 0$

Definition 3.21. If $\sigma(s_0, \dots, s_n, 0, \dots) \neq 0$ is a polynomial, we call s_n the **leading coefficient** of σ , we call n the **degree** of σ , and we denote n by $\deg(\sigma)$

Notation. If R is a commutative ring, then the set of all polynomials with coefficients in R is denoted by $R[x]$

Proposition 3.22. If R is a commutative ring, then $R[x]$ is a commutative ring that contains R as a subring

Proof. $\sigma = (s_0, s_1, \dots), \tau = (t_0, t_1, \dots)$

$$\sigma + \tau = (s_0 + t_0, s_1 + t_1, \dots)$$

$$\sigma\tau = (c_0, c_1, \dots)$$

where $c_k = \sum_{i+j=k} s_i t_j = \sum_{i=0}^k s_i t_{k-i}$. □

Lemma 3.23. *Let R be a commutative ring and let $\sigma, \tau \in R[x]$ be nonzero polynomials.*

1. *Either $\sigma\tau = 0$ or $\deg(\sigma\tau) \leq \deg(\sigma) + \deg(\tau)$*
2. *If R is a domain, then $\sigma\tau \neq 0$ and*

$$\deg(\sigma\tau) = \deg(\sigma) + \deg(\tau)$$

3. *If R is a domain, then $R[x]$ is a domain*

Proof. $\sigma = (s_0, s_1, \dots), \tau = (t_0, t_1, \dots)$ have degrees m and n respectively.

1. *if $k > m + n$, then each term in $\sum_i s_i t_{k-i}$ is 0*
2. *Each term in $\sum_i s_i t_{m+n-i}$ is 0 with the possible exception of $s_m t_n$.*
Since R is a domain, $s_m \neq 0$ and $t_n \neq 0$ imply $s_m t_n \neq 0$.

□

Definition 3.24. *If R is a commutative ring, then $R[x]$ is called the **ring of polynomials over R***

Definition 3.25. *Define the element $x \in R[x]$ by*

$$x = (0, 1, 0, 0, \dots)$$

Lemma 3.26. 1. *If $\sigma = (s_0, \dots)$, then*

$$x\sigma = (0, s_0, s_1, \dots)$$

2. *If $n \geq 1$, then x^n is the polynomial having 0 everywhere except for 1 in the n th coordinate*
3. *If $r \in R$, then*

$$(r, 0, \dots)(s_0, s_1, \dots, s_j, \dots) = (rs_0, rs_1, \dots, rs_j, \dots)$$

Proposition 3.27. *If $\sigma = (s_0, \dots, s_n, 0, \dots)$, then*

$$\sigma = s_0 + s_1 x + s_2 x^2 + \dots + s_n x^n$$

where each element $s \in R$ is identified with the polynomial $(s, 0, \dots)$

As a customary, we shall write

$$f(x) = s_0 + s_1x + \cdots + s_nx^n$$

instead of σ . s_0 is called its **constant term**. If $s_n = 1$, then $f(x)$ is called **monic**.

Corollary 3.28. *Polynomials $f(x) = s_0 + \cdots + s_nx^n$ and $g(x) = t_0 + \cdots + t_mx^m$ are equal if and only if $n = m$ and $s_i = t_i$ for all i .*

If R is a commutative ring, each polynomial $f(x) = s_0 + \cdots + s_nx^n$ defines a **polynomial function** $f : R \rightarrow R$ by evaluation: If $a \in R$, define $f(a) = s_0 + \cdots + s_na^n \in R$.

Definition 3.29. Let k be a field. The fraction field of $k[x]$, denoted by $k(x)$, is called the **field of rational function** over k

Proposition 3.30. *If k is a field, then the elements of $k(x)$ have the form $f(x)/g(x)$ where $f(x), g(x) \in k[x]$ and $g(x) \neq 0$*

Proposition 3.31. *If p is a prime, then the field of rational functions $\mathbb{I}_p(x)$ is a infinite field containing \mathbb{I}_p as a subfield.*

Proof. By Lemma 3.23 (3), $\mathbb{I}_p[x]$ is an infinite domain for the powers x^n for $n \in \mathbb{N}$ are distinct. Thus its fraction field $\mathbb{I}_p(x)$ is an infinite field containing $\mathbb{I}_p[x]$ as a subring. But $\mathbb{I}_p[x]$ contains \mathbb{I}_p as a subring, by Proposition 3.22. \square

$R[x]$ is often called the ring of all **polynomials over R in one variable**. If we write $A = R[x]$, then $A[y]$ is called the ring of all **polynomials over R in two variables x and y** , and it is denoted by $R[x, y]$.

Exercise 3.2.1. Show that if R is a commutative ring, then $R[x]$ is never a field

Proof. If $R[x]$ is a field, then $x^{-1} \in R[x]$ and $x^{-1} = \sum_i c_i x^i$. However

$$\deg(xx^{-1}) = \deg(1) = 1 = \deg(x) + \deg(x^{-1})$$

A contradiction. \square

Exercise 3.2.2. Show that the polynomial function defined by $f(x) = x^p - x \in \mathbb{I}_p[x]$ is identically zero.

Proof. By Fermat's theorem 2.48, $a^p \equiv a \pmod{p}$ \square

3.3 Greatest Common Divisors

Theorem 3.32 (Division Algorithm). *Assume that k is a field and that $f(x), g(x) \in k[x]$ with $f(x) \neq 0$. Then there are unique polynomials $q(x), r(x) \in k[x]$ with*

$$g(x) = q(x)f(x) + r(x)$$

and either $r(x) = 0$ or $\deg(r) < \deg(f)$

Proof. We first prove the existence of such q and r . If $f \mid g$, then $g = qf$ for some q ; define the remainder $r = 0$. If $f \nmid g$, then consider all polynomials of the form $g - qf$ as q varies over $k[x]$. The least integer axiom provides a polynomial $r = g - qf$ having least degree among all such polynomials. Since $g = qf + r$, it suffices to show that $\deg(r) < \deg(f)$. Write $f(x) = s_n x^n + \dots + s_1 x + s_0$ and $r(x) = t_m x^m + \dots + t_0$. Now $s_n \neq 0$ implies that s_n is a unit because k is a field and so $s_n^{-1} \in k$. If $\deg(r) \geq \deg(f)$, define

$$h(x) = r(x) - t_m s_n^{-1} x^{m-n} f(x)$$

that is, if $\text{LT}(f) = s_n x^n$, where LT abbreviates **leading term**, then

$$h = r - \frac{\text{LT}(r)}{\text{LT}(f)} f$$

note that $h = 0$ or $\deg(h) < \deg(r)$. If $h = 0$, then $r = [\text{LT}(r)/\text{LT}(f)]f$ and

$$\begin{aligned} g &= qf + r = qf + \frac{\text{LT}(r)}{\text{LT}(f)} f \\ &= \left[q + \frac{\text{LT}(r)}{\text{LT}(f)} \right] f \end{aligned}$$

contradicting $f \nmid g$. If $h \neq 0$, then $\deg(h) < \deg(r)$ and

$$g - qf = r = h + \frac{\text{LT}(r)}{\text{LT}(f)} f$$

Thus $g - [q + \text{LT}(r)/\text{LT}(f)]f = h$, contradicting r being a polynomial of least degree having this form. Therefore $\deg(r) < \deg(f)$

To prove uniqueness of $q(x)$ and $r(x)$ assume that $g = q'f + r'$, where $\deg(r') < \deg(f)$. Then

$$(q - q')f = r' - r$$

If $r' \neq r$, then each side has a degree. But $\deg((q - q')f) = \deg(q - q') + \deg(f) \geq \deg(f)$, while $\deg(r' - r) \leq \max\{\deg(r'), \deg(r)\} < \deg(f)$, a contradiction. Hence $r' = r$ and $(q - q')f = 0$. As $k[x]$ is a domain and $f \neq 0$, it follows that $q - q' = 0$ and $q = q'$ \square

Definition 3.33. If $f(x)$ and $g(x)$ are polynomials in $k[x]$, where k is a field, then the polynomials $q(x)$ and $r(x)$ occurring in the division algorithm are called the **quotient** and the **remainder** after dividing $g(x)$ by $f(x)$

The hypothesis that k is a field is much too strong: long division can be carried out in $R[x]$ for every commutative ring R as long as the leading coefficient of $f(x)$ is a unit in R ; in particular, long division is always possible when $f(x)$ is monic.

Corollary 3.34. Let R be a commutative ring and let $f(x) \in R[x]$ be a monic polynomial. If $g(x) \in R[x]$, then there exists $q(x), r(x) \in R[x]$ with

$$g(x) = q(x)f(x) + r(x)$$

where either $r(x) = 0$ or $\deg(r) < \deg(f)$

Proof. Note that $\text{LT}(r)/\text{LT}(f) \in R$ because $f(x)$ is monic □

Definition 3.35. If $f(x) \in k[x]$, where k is a field, then a **root** of $f(x)$ in k is an element $a \in k$ with $f(a) = 0$

Lemma 3.36. Let $f(x) \in k[x]$, where k is a field, and let $u \in k$. Then there is $q(x) \in k[x]$ with

$$f(x) = q(x)(x - u) + f(u)$$

Proof. The division algorithm gives

$$f(x) = q(x)(x - u) + r$$

Now evaluate

$$f(u) = q(u)(u - u) + r$$

and so $r = f(u)$ □

Proposition 3.37. If $f(x) \in k[x]$, where k is a field, then a is a root of $f(x)$ in k if and only if $x - a$ divides $f(x)$ in $k[x]$

Proof. If a is a root of $f(x)$ in k , then $f(a) = 0$ and the lemma gives $f(x) = q(x)(x - a)$. □

Theorem 3.38. Let k be a field and let $f(x) \in k[x]$. If $f(x)$ has degree n , then $f(x)$ has at most n roots in k

Proof. We prove the statement by induction on $n \geq 0$. If $n = 0$, then $f(x)$ is a nonzero constant, and so the number of its roots in k is zero. Now let $n > 0$. If $f(x)$ has no roots in k , then we are done. Otherwise we may assume that there is $a \in k$ with a a root of $f(x)$; hence by Proposition 3.37

$$f(x) = q(x)(x - a)$$

moreover, $q(x) \in k[x]$ has degree $n - 1$. □

Example 3.3. Theorem 3.38 is not true for polynomials with coefficients in an arbitrary commutative ring R . For example, if $R = \mathbb{I}_8$, then the quadratic polynomial $x^2 - 1$ has 4 roots: $[1], [3], [5], [7]$

Corollary 3.39. Every n th root of unity in \mathbb{C} is equal to

$$e^{2\pi i k/n} = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right)$$

where $k = 0, 1, \dots, n - 1$

Corollary 3.40. Let k be an infinite field and let $f(x)$ and $g(x)$ be polynomials in $k[x]$. If $f(x)$ and $g(x)$ determine the same polynomial function, then $f(x) = g(x)$

Proof. If $f(x) \neq g(x)$, then the polynomial $h(x) = f(x) - g(x)$ is nonzero, so that it has some degree, say n . Now every element of k is a root of $h(x)$; since k is infinite, $h(x)$ has more than n roots, a contradiction. □

Theorem 3.41. If k is a field and G is a finite subgroup of the multiplicative group k^\times , then G is cyclic. In particular, if k itself is finite, then k^\times is cyclic.

Proof. Let d be a divisor of $|G|$. If there are two subgroups of G of order d , say S and T , then $|S \cup T| > d$. But each $a \in S \cup T$ satisfies $a^d = 1$ and hence it's a root of $x^d - 1$, a contradiction. Thus G is cyclic, by Theorem 2.80. □

Definition 3.42. If k is a finite field, a generator of the cyclic group k^\times is called a **primitive element** of k

Definition 3.43. If $f(x)$ and $g(x)$ are polynomials in $k[x]$, where k is a field, then a **common divisor** is a polynomial $c(x) \in k[x]$ with $c(x) \mid f(x)$ and $c(x) \mid g(x)$. If $f(x)$ and $g(x)$ in $k[x]$ are not both 0, define their **greatest common divisor**, abbreviated gcd, to be the monic common divisor having largest degree. If $f(x) = 0 = g(x)$, define their gcd = 0. The gcd of $f(x)$ and $g(x)$ is often denoted by (f, g)

Theorem 3.44. If k is a field and $f(x), g(x) \in k[x]$, then their gcd $d(x)$ is a nonlinear combination of $f(x)$ and $g(x)$; that is there are $s(x), t(x) \in k[x]$ with

$$d(x) = s(x)f(x) + t(x)g(x)$$

Corollary 3.45. Let k be a field and let $f(x), g(x) \in k[x]$. A monic common divisor $d(x)$ is the gcd if and only if $d(x)$ is divisible by every common divisor

Definition 3.46. An element p in a domain R is **irreducible** if p is neither 0 nor a unit and in any factorization $p = uv$ in R , either u or v is a unit. Elements $a, b \in R$ are **associates** if there is a unit $u \in R$ with $b = ua$

For example, a prime p is irreducible in \mathbb{Z}

Proposition 3.47. If k is a field, then a polynomial $p(x) \in k[x]$ is irreducible if and only if $\deg(p) = n \geq 1$ and there is no factorization in $k[x]$ of the form $p(x) = g(x)h(x)$ in which both factors have degree smaller than n

Proof. We show first that $h(x) \in k[x]$ is a unit if and only if $\deg(h) = 0$. If $h(x)u(x) = 1$, then $\deg(h) + \deg(u) = \deg(1) = 0$, we have $\deg(h) = 0$. Conversely if $\deg(h) = 0$, then $h(x)$ is a nonzero constant; that is, $h \in k$; since k is a field, h has an inverse

If $p(x)$ is irreducible, then its only factorization are of the form $p(x) = g(x)h(x)$ where $g(x)$ or $h(x)$ is a unit; that is, either $\deg(g) = 0$ or $\deg(h) = 0$.

Conversely, if $p(x)$ is reducible, then it has factorization $p(x) = g(x)h(x)$ where neither $g(x)$ nor $h(x)$ is a unit; \square

Corollary 3.48. Let k be a field and let $f(x) \in k[x]$ be a quadratic or cubic polynomial. Then $f(x)$ is irreducible in $k[x]$ if and only if $f(x)$ does not have a root in k

Proof. If $f(x) = g(x)h(x)$, then $\deg(f) = \deg(g) + \deg(h)$ \square

Example 3.4. 1. We determine the irreducible polynomials in $\mathbb{I}_2[x]$ of small degree.

As always, the linear polynomials x and $x + 1$ are irreducible

There are four quadratics: $x^2, x^2 + x, x^2 + 1, x^2 + x + 1$

Lemma 3.49. Let k be a field, let $p(x), f(x) \in k[x]$, and let $d(x) = (p, f)$. If $p(x)$ is a monic irreducible polynomial, then

$$d(x) = \begin{cases} 1 & \text{if } p(x) \nmid f(x) \\ p(x) & \text{if } p(x) \mid f(x) \end{cases}$$

Theorem 3.50 (Euclid's Lemma). *Let k be a field and let $f(x), g(x) \in k[x]$. If $p(x)$ is an irreducible polynomial in $k[x]$, and $p(x) \mid f(x)g(x)$, then either*

$$p(x) \mid f(x) \quad \text{or} \quad p(x) \mid g(x)$$

More generally, if $p(x) \mid f_1(x) \cdots f_n(x)$, then $p(x) \mid f_i(x)$ for some i

Proof. Assume $p \mid fg$ but that $p \nmid f$. Since p is irreducible, $(p, f) = 1$, and so $1 = sp + tf$ for some polynomials s and t . Therefore

$$g = spg + tfg$$

and so $p \mid g$ □

Definition 3.51. Two polynomials $f(x), g(x) \in k[x]$ where k is a field, are called **relatively prime** if their gcd is 1

Corollary 3.52. *Let $f(x), g(x), h(x) \in k[x]$, where k is a field and let $h(x)$ and $f(x)$ be relatively prime. If $h(x) \mid f(x)g(x)$, then $h(x) \mid g(x)$*

Definition 3.53. If k is a field, then a rational function $f(x)/g(x) \in k(x)$ is in **lowest terms** if $f(x)$ and $g(x)$ are relatively prime

Proposition 3.54. *If k is a field, every nonzero $f(x)/g(x) \in k(x)$ can be put in lowest terms*

Theorem 3.55 (Euclidean Algorithm). *If k is a field and $f(x), g(x) \in k[x]$, then there are algorithms for computing $\gcd(f, g)$ as well as for finding a pair of polynomials $s(x)$ and $t(x)$ with*

$$(f, g) = s(x)f(x) + t(x)g(x)$$

Proof.

$$g = q_1f + r_1$$

$$f = q_2r_1 + r_2$$

$$r_1 = q_3r_2 + r_3$$

$$\vdots$$

$$r_{n-4} = q_{n-2}r_{n-3} + r_{n-2}$$

$$r_{n-3} = q_{n-1}r_{n-2} + r_{n-1}$$

$$r_{n-2} = q_nr_{n-1} + r_n$$

$$r_{n-1} = q_{n+1}r_n$$

Since the degrees of the remainders are strictly decreasing, this procedure must stop after a finite number of steps. The claim is that $d = r_n$ is the gcd. If c is any common divisor of f and g , then $c \mid r_i$ for every i . Also

$$\begin{aligned}
r_n &= r_{n-2} - q_n r_{n-1} \\
&= r_{n-2} - q_n(r_{n-3} - q_{n-1} r_{n-2}) \\
&= (1 + q_{n-1})r_{n-2} - q_n r_{n-3} \\
&= (1 + q_{n-1})(r_{n-4} - q_{n-2} r_{n-3}) - q_n r_{n-3} \\
&= (1 + q_{n-1})r_{n-4} - [(1 + q_{n-1})q_{n-2} + q_n]r_{n-3} \\
&\vdots \\
&= sf + tg
\end{aligned}$$

□

Corollary 3.56. *Let k be a subfield of a field K , so that $k[x]$ is a subring of $K[x]$. If $f(x), g(x) \in k[x]$, then their gcd in $k[x]$ is equal to their gcd in $K[x]$.*

Proof. The division algorithm in $K[x]$ gives

$$g(x) = Q(x)f(x) + R(x)$$

$k[x]$ gives

$$g(x) = q(x)f(x) + r(x)$$

and this also holds in $K[x]$. So that uniqueness of quotient and remainder gives $Q(x) = q(x), R(x) = r(x)$. □

Theorem 3.57 (Unique Factorization). *If k is a field, then every polynomial $f(x) \in k[x]$ of degree ≥ 1 is a product of a nonzero constant and monic irreducibles. Moreover, if $f(x)$ has two such factorizations*

$$f(x) = ap_1(x) \dots p_m(x) \quad \text{and} \quad f(x) = bq_1(x) \dots q_n(x)$$

then $a = b, m = n$ and the q 's may be reindexed so that $q_i = p_i$ for all i

Proof. We prove the existence of a factorization for a polynomial $f(x)$ by induction on $\deg(f) \geq 1$. If $\deg(f) = 1$, then $f(x) = ax + c = a(x + a^{-1}c)$. As every linear polynomial, $x + a^{-1}c$ is irreducible.

Assume now that $\deg(f) \geq 1$. If $f(x)$ is irreducible and its leading coefficient is a , write $f(x) = a(a^{-1}f(x))$; we are done. If $f(x)$ is not irreducible, then $f(x) = g(x)h(x)$, where $\deg(g) < \deg(f)$ and $\deg(h) < \deg(f)$. By the

inductive hypothesis, $g(x) = bp_1(x) \dots p_m(x)$ and $h(x) = cq_1(x) \dots q_n(x)$. It follows that

$$f(x) = (bc)p_1(x) \dots p_m(x)q_1(x) \dots q_n(x)$$

We now prove by induction on $M = \max\{m, n\} \geq 1$ if there is an equation

$$ap_1(x) \dots p_m(x) = bq_1(x) \dots q_n(x)$$

where a and b are nonzero constants and the p 's and q 's are monic irreducibles. For the inductive step, $p_m(x) \mid q_1(x) \dots q_n(x)$. By Euclid's lemma, there is i with $p_m(x) \mid q_i(x)$. But $q_i(x)$ are monic irreducible, so that $q_i(x) = p_m(x)$. Canceling this factor we will use inductive hypothesis \square

Let k be a field and assume that there are $a, r_1, \dots, r_n \in k$ with

$$f(x) = a \prod_{i=1}^n (x - r_i)$$

If r_1, \dots, r_s where $s \leq n$ are the distinct roots of $f(x)$, then collecting terms gives

$$f(x) = a(x - r_1)^{e_1} \dots (x - r_s)^{e_s}$$

where r_j are distinct and $e_j \geq 1$. We call e_j the **multiplicity** of the root r_j .

Theorem 3.58. *Let $f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathbb{Z}[x] \subseteq \mathbb{Q}[x]$. Every rational root r of $f(x)$ has the form b/c , where $b \mid a_0$ and $c \mid a_n$*

Proof. We may assume that $r = b/c$ is in lowest form.

$$\begin{aligned} 0 &= f(b/c) = a_0 + a_1(b/c) + \dots + a_n(b/c)^n \\ 0 &= a_0c^n + a_1bc^{n-1} + \dots + a_nb^n \end{aligned}$$

Hence $a_0c^n = b(-a_1c^{n-1} - \dots - a_nb^{n-1})$, that is $b \mid a_0c^n$. \square

Definition 3.59. A complex number α is called an **algebraic integer** if α is a root of a monic $f(x) \in \mathbb{Z}[x]$

Corollary 3.60. *A rational number z that is an algebraic integer must lie in \mathbb{Z} . More precisely, if $f(x) \in \mathbb{Z}[x] \subseteq \mathbb{Q}[x]$ is a monic polynomial, then every rational root of $f(x)$ is an integer that divides the constant term*

Proof. $a_n = 1$ in Theorem 3.58 \square

For example, consider $f(x) = x^3 + 4x^2 - 2x - 1 \in \mathbb{Q}[x]$. By Corollary 3.48, this cubic is irreducible if and only if it has no rational root. As $f(x)$ is monic, the candidates for rational roots are ± 1 , for these are the only divisor of -1 in \mathbb{Z} . Thus $f(x)$ has no roots in \mathbb{Q} and hence $f(x)$ is irreducible in $\mathbb{Q}[x]$

3.4 Homomorphisms

Definition 3.61. If A and R are (commutative) rings, a **(ring) homomorphism** is a function $f : A \rightarrow R$ s.t.

1. $f(1) = 1$
2. $f(a + a') = f(a) + f(a')$
3. $f(aa') = f(a)f(a')$

Example 3.5. 1. Let R be a domain and let $F = \text{Frac}(R)$. $R' = \{[a, 1] : a \in R\} \subseteq F$, then the function $f : R \rightarrow R'$ given by $f(a) = [a, 1]$, is an isomorphism

2. Complex conjugation $z = a + ib \mapsto \bar{z} = a - ib$ is an isomorphism $\mathbb{C} \rightarrow \mathbb{C}$.
3. Let R be a commutative ring, and let $a \in R$. Define the **evaluation homomorphism** $e_a : R[x] \rightarrow R$ by $e_a(f(x)) = f(a)$.

Lemma 3.62. If $f : A \rightarrow R$ is a ring homomorphism, then for all $a \in A$

1. $f(a^n) = f(a)^n$
2. if a is a unit, then $f(a)$ is a unit and $f(a^{-1}) = f(a)^{-1}$
3. if $f : A \rightarrow R$ is a ring homomorphism, then

$$f(U(A)) \leq U(R)$$

where $U(A)$ is the group of units of A ; if f is an isomorphism, then

$$U(A) \cong U(R)$$

Proposition 3.63. If R and S are commutative rings and $\varphi : R \rightarrow S$ is a ring homomorphism, then there is a ring homomorphism $\varphi^* : R[x] \rightarrow S[x]$ given by

$$\varphi^* : r_0 + r_1x + r_2x^2 + \cdots \mapsto \varphi(r_0) + \varphi(r_1)x + \varphi(r_2)x^2 + \cdots$$

Definition 3.64. If $f : A \rightarrow R$ is a ring homomorphism, then its **kernel** is

$$\ker f = \{a \in A : f(a) = 0\}$$

and its **image** is

$$\text{im } f = \{r \in R : \exists a \in A \text{ } r = f(a)\}$$

The kernel of a group homomorphism is not merely a subgroup; it is a **normal** subgroup. Similarly, the kernel of a ring homomorphism is almost a subring ($1 \notin \ker f$) and is closed under multiplication.

Definition 3.65. An **ideal** in a commutative ring R is a subset I of R s.t.

1. $0 \in I$
2. if $a, b \in I$, then $a + b \in I$
3. if $a \in I$ and $r \in R$, then $ra \in I$

An ideal $I \neq R$ is called a **proper ideal**

Example 3.6. If $b_1, \dots, b_n \in R$, then the set of all linear combinations

$$I = \{r_1b_1 + \dots + r_nb_n : r_i \in R\}$$

is an ideal in R . We write $I = (b_1, \dots, b_n)$ in this case and we call I the **ideal generated by** b_1, \dots, b_n . In particular, if $n = 1$, then

$$I = (b) = \{rb : r \in R\}$$

is an ideal in R ; (b) consists of all the multiplies of b and it is called the **principal ideal** generated by b . Notice that R and $\{0\}$ are always principal ideals: $R = (1)$, $\{0\} = (0)$

Proposition 3.66. *If $f : A \rightarrow R$ is a ring homomorphism, then $\ker f$ is an ideal in A and $\text{im } f$ is a subring of R . Moreover, if A and R are not zero rings, then $\ker f$ is a proper ideal.*

Example 3.7. 1. If an ideal I in a commutative ring R contains 1, then

$$I = R$$
2. it follows from