Sheaves jn Geometry and Logic

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1 Categorical Preliminaries

A category C consists of a collection of **objects**, a collection of **morphisms** and four operations; two of these operations associate with each morphism f of C its **domain** $\operatorname{dom}(f)$ or $\operatorname{d}_0(f)$ and its **codomain** $\operatorname{cod}(f)$ or $\operatorname{d}_1(f)$, respectively, both of which are objects of C. The other two operations are operation which associates with each object C of C a morphism 1_C (or id_C) of C called the **identity morphism** of C and an operation of C s.t. $\operatorname{d}_0(f) = \operatorname{d}_1(g)$ another morphism $f \circ g$. These operations are required to satisfy the following axioms

- 1. $d_0(1_C) = C = d_1(1_C)$
- 2. $d_0(f \circ g) = d_0(g), d_1(f \circ g) = d_1(f)$
- 3. $1_D \circ f = f, f \circ 1_C = f$
- 4. $(f \circ g) \circ h = f \circ (g \circ h)$

In an arbitrary category ${\bf C}$, a morphism $f:C\to D$ in ${\bf C}$ is called an **isomorphism** if there exists a morphism $g:D\to C$ s.t. $f\circ g=1_D$ and $g\circ f=1_C$. If such a morphism f exists, one says that C is isomorphic to D and one writes $f:C\xrightarrow{\sim} D$ and $C\cong D$

A morphism $f:C\to D$ is called an **epi(morphism)** if for any object E and any two parallel morphisms $g,h:D\rightrightarrows E$ in ${\bf C},\,gf=hf$ implies g=h; one writes $f:C\to D$ to indicate that f is an epimorphism. Dually, $f:C\to D$ is called a **mono(morphism)** if for any object B and any two parallel morphisms $g,h:B\rightrightarrows C$ in ${\bf C},\,fg=fh$ implies g=h; in this case, one writes $f:C\to D$. Two monomorphisms $f:A\to D$ and $g:B\to D$ with a common codomain D are called **equivalent** if there exists an isomorphism $h:A\overset{\sim}{\to}B$ with gh=f. A **subobject** of D is an equivalence class of monomorphisms into D. The collection $\operatorname{Sub}_{\bf C}(D)$ of subobjects of D carries a natural partial order defined by $[f]\leq [g]$ iff there is an $h:A\to B$ s.t. f=gh, where [f] and [g] are the classes of $f:A\to D$ and $g:B\to D$



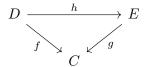
If C is a category, we sometimes write C_0 for its collection of objects and C_1 for its collection of mophisms. For two objects C and D, the collection of morphisms with domain C and codomain D is denoted by one of the following three symbols

$$\operatorname{Hom}_{\mathbf{C}}(C,D), \quad \operatorname{Hom}(C,D), \quad \mathbf{C}(C,D)$$

We shall tacitly assume we are working in some fixed universe U of sets. Members of U are then called **small** sets, whereas a collection of members of U which doesnot itself belong to U will sometimes be referred to as a **large** set. Given such an ambient universe U, a category \mathbf{C} is **locally small** if for any two objects C and D of \mathbf{C} the hom-set $\mathrm{Hom}_{\mathbf{C}}(C,D)$ is a small set, while \mathbf{C} is called **small** if both \mathbf{C}_0 and \mathbf{C}_1 are small sets.

Given a category C, one can form a new category C^{op} , called the **opposite** or **dual** category of C, by taking the same objects but reversing the direction of all the morphisms and the order of all compositions.

Given a category ${\bf C}$ and an object C of ${\bf C}$, one can construct the **comma category** or the **slice category** ${\bf C}/C$ (read: ${\bf C}$ over C): object of ${\bf C}/C$ are morphisms of ${\bf C}$ with codomain C, and morphisms in ${\bf C}/C$ from one such object $f:D\to C$ to another $g:E\to C$ are commutative triangles in ${\bf C}$



Given two categories ${\bf C}$ and ${\bf D}$, a **functor** from ${\bf C}$ to ${\bf D}$ is an operation F which assigns to each objects C of ${\bf C}$ an object F(C) of ${\bf D}$ and to each morphism f of ${\bf C}$ a morphism F(f) of ${\bf D}$ in such a way that F respects the domain and codomain as well as the identities and compositions.

For a category C there is an **identity functor** $id_C : C \to C$, and for two functors $F : C \to D$ and $G : D \to E$ one can form a new functor $G \circ F : C \to E$ by **composition**

Let F and G be two functors from a category \mathbf{C} to a category \mathbf{D} . A **natural transformation** α from F to G, written $\alpha: F \to G$, is an operation associating with each object C of \mathbf{C} a morphism $\alpha_C: FC \to GC$ of \mathbf{D} in such a way that for any morphism $f: C' \to C$ in \mathbf{C} , the diagram

$$FC' \xrightarrow{\alpha_{C'}} GC'$$

$$F(f) \downarrow \qquad \qquad \downarrow G(f)$$

$$FC \xrightarrow{\alpha_C} GC$$

commutes. The morphism α_C is called the **component** of α at C. If every component of α is an isomorphism, α is said to be a **natural isomorphism**.

If $\alpha: F \to G$ and $\beta: G \to H$ are two natural transformation between functors $\mathbf{C} \to \mathbf{D}$, one can define composite natural transformation $\beta \circ \alpha$ by setting

$$(\beta \circ \alpha)_C = \beta_{G(C)} \circ \alpha_C$$

By fixed categories C and D this yields a new category D^C : the objects of D^C are functors from C to D while the morphisms of D^C are natural transformations between such functors. Categories so constructed are called functor categories

For categories **C** and **D**, a functor $F: \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ is also called a **contravariant functor** from **C** to **D**. In contrast, ordinary functors from **C** to **D** are sometimes called *covariant. Thus $C' \mapsto \mathrm{Hom}_{\mathbf{C}}(C',C)$ for fixed C yields a contravariant functor from **C** to **Sets**, while $C \mapsto \mathrm{Hom}_{\mathbf{C}}(C',C)$ for fixed C' is the covariant Hom-functor.

$$C' \longrightarrow \operatorname{Hom}_{\mathbf{C}}(C', C)$$

$$\downarrow \qquad \qquad \downarrow$$

$$C'' \longrightarrow \operatorname{Hom}_{\mathbf{C}}(C'', C)$$

A functor $F : \mathbf{C} \to \mathbf{D}$ is called **full** (respectively **faithful**) if for any two objects C and C' of \mathbf{C} , the operation

$$\operatorname{Hom}_{\mathbf{C}}(C',C) \to \operatorname{Hom}_{\mathbf{D}}(FC',FC); \quad f \mapsto F(f)$$

induced by F is surjective (respectively injective). A functor $F: \mathbf{C} \to \mathbf{D}$ is called an **equivalence of categories** if F is full and faithful and if any object of \mathbf{D} is isomorphic to an object in the image of F. For example, if $F: \mathbf{C} \to \mathbf{D}$ is a functor s.t. there exists a functor $G: \mathbf{D} \to \mathbf{C}$ and natural isomorphism $\alpha: F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathbf{D}}$ and $\beta: G \circ F \xrightarrow{\sim} \mathrm{id}_{\mathbf{C}}$, then F is an equivalence (and G is sometimes called a **quasi-inverse** for F).

We say that an object X equipped with morphsims $\pi_1: X \to A$ and $\pi_2: X \to B$ is a **product** of A and B if for any other object Y and any two maps $f: Y \to A$ and $g: Y \to B$ there is a **unique** map $h: Y \to X$ s.t. $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$ [this unique is denoted by $(f,g): Y \to X$ or sometimes $\langle f,g \rangle$]

$$A \xleftarrow{f} f, g \downarrow \downarrow \qquad g$$

$$A \xleftarrow{\pi_1} X \xrightarrow{\pi_2} B$$

A product of an I-indexed family A_i is written $\prod_i A_i$. For a poset (P, \leq) viewed as a category in the way explained above, the product of two objects p and q is their infimum, which may or may not exist.

The singleton set $\{*\}$ is the set S, unique up to isomorphism, for which there is exactly one morphism $A \to S$ from any other set A into S. In an arbitrary category \mathbf{C} , an object C with the property that for any other object D of \mathbf{C} there is one and only one morphism from D to C is called a **terminal object** of C. It's often denoted by 1 or by $1_{\mathbf{C}}$

Given two functions $f: B \to A$ and $g: C \to A$ between sets, one may construct their **fibered product** (or **pullback**) as the set

$$B \times_A C = \{(b, c) \in B \times C \mid f(b) = g(c)\}\$$

Thus $B \times_A C$ is a subset of the product, and therefore comes equipped with two **projections** $\pi_1: B \times_A C \to B$ and $\pi_2: B \times_A C \to C$ which fit into a commutative diagram

$$\begin{array}{ccc} B \times_A C & \xrightarrow{\pi_2} & C \\ \downarrow^{\pi_1} & & \downarrow^g \\ B & \xrightarrow{f} & A \end{array}$$

This diagram has the property that given any other set X and functions $\beta: X \to B$ and $\gamma: X \to C$ s.t. $f\beta = g\gamma$, there is a unique function $\delta: X \to B \times_A C$ with $\pi_1 \delta = \beta$ and $\pi_2 \delta = \gamma$ [namely $\delta(x) = (\beta(x), \gamma(x))$]

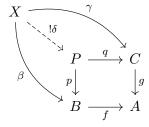
In a general category C, one says that a commutative square

$$P \xrightarrow{q} C$$

$$\downarrow g$$

$$B \xrightarrow{f} A$$

is a **pullback** (square) or a fibered product if it has the property just described for sets: given any object X of $\mathbf C$ and morphisms $\beta: X \to B$ and $\gamma: X \to C$ with $f\beta = g\gamma$, there is a unique $\delta: X \to P$ s.t. $p\delta = \beta$ and $q\delta = \gamma$

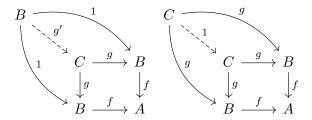


This unique map δ is usually denoted by (β,γ) . Given $f:B\to A$ and $g:C\to A$, the pullback P with its projections P and q is uniquely determined up to isomorphism and one usually writes $B\times_A C$ for this pullback. We also says that the arrow p is the pullback of g along f. Notice that p is a monomorphism if g is. A morphism $f:B\to A$ in a category ${\bf C}$ is a monomorphism iff the pullback of f along itself is an isomorphism, iff the square

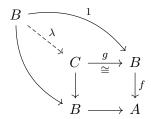
$$\begin{array}{ccc}
B & \xrightarrow{1} & B \\
\downarrow^{1} & & \downarrow^{f} \\
B & \xrightarrow{f} & A
\end{array}$$

is a pullback

Proof. $1 \rightarrow 2$. Consider



We have g(g'g) = gg'g = (gg')g = g, hence g'g = 1 $2 \rightarrow 3$. λ has inverse g



 $3 \rightarrow 1$.

There is an important "pasting-lemma" for pullback squares. Given a commutative diagram of the form

$$\begin{array}{ccc}
Q \longrightarrow P \longrightarrow D \\
\downarrow & & \downarrow \\
C \longrightarrow B \longrightarrow A
\end{array}$$

the outer rectangle is a pullback if both inner squares are pullbacks; and conversely, if the outer rectangle as well as the right-hand square pullbacks, then so is the left-hand square

For two parallel arrows $f:A\to B$ and $g:A\to B$ in a category ${\bf C}$, the **equalizer** of f and g is a morphism $e:E\to A$ s.t. fe=ge and which is universal with this property; that is, given any other morphism $u:X\to A$ in ${\bf C}$ s.t. fu=gu, there is a unique $v:X\to E$ s.t. ev=u

$$E \xrightarrow{e} A \xrightarrow{g} B$$

$$\downarrow v \downarrow u$$

$$X$$

Equalizer need not always exists. However, in the category of sets the equalizer of any pair of functions $f,g \rightrightarrows B$ exists, and can be constructed be the set

$$E = \{a \in A \mid f(a) = g(a)\}$$

where e is set inclusion

Consider two categories \boldsymbol{A} and \boldsymbol{X} and two functors between them in opposite directions, say

$$F: \mathbf{X} \to \mathbf{A} \quad G: \mathbf{A} \to \mathbf{X}$$

One says that G is **right adjoint** to F (and that F is **left adjoint** to G, notation $F \dashv G$) when for any two objects X from X and A from A there is a natural bijection between morphisms

$$\frac{X \xrightarrow{f} GA}{FX \xrightarrow{h} A} \tag{1}$$

in the sense that each morphism f uniquely determines a morphism h, and conversely. This bijection is to be natural in the following sense: given any morphisms $\alpha:A\to A'$ in $\mathbf A$ and $\xi:X'\to X$ in $\mathbf X$, and corresponding arrows f and h composites also correspond

$$\frac{X' \xrightarrow{\xi} X \xrightarrow{f} GA \xrightarrow{G\alpha} GA'}{FX' \xrightarrow{F\xi} FX \xrightarrow{h} A \xrightarrow{\alpha} A'}$$

If we write this bijective correspondence as

$$\theta: \operatorname{Hom}_{\mathbf{X}}(X, GA) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{A}}(FX, A)$$
 (2)

then this naturality condition can be expressed by the equation

$$\theta(G(\alpha) \circ f \circ \xi) = \alpha \circ \theta(f) \circ F(\xi)$$

Given θ as in (2), and an object X in X, setting A = FX gives a unique map

$$\eta = \eta_X : X \to GFX$$

s.t. $\theta(\eta_X) = 1_{F(X)}$. This map η_X is called the **unit** of the adjunction (at X). If one takes $\xi = 1_X$, A = FX, $f = \eta$, $\alpha = 1_A$ and A' = A, then

$$\frac{X \xrightarrow{1_X} X \xrightarrow{\eta} GFX \xrightarrow{Gh} GA}{FX \xrightarrow{F1_X} FX \xrightarrow{h} A \xrightarrow{1_A} A}$$

In short, η determined the adjunction, since h corresponds to $G(h) \circ \eta_X$ under the correspondence (1). This means that each f determines uniquely an h which makes the following triangle commutes.

$$X \xrightarrow{\eta} GFX \quad FX$$

$$\downarrow_{Gh} \quad \downarrow$$

$$GA \quad A$$

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