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A Course In Universal Algebra

Stanley Burris & H. P. Sankappanavar

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Contents

1 Lattices

1.1 Definitions of Lattices

Definition 1.1 () *A nonempty set L together with two binary operations $\vee$ and $\wedge$ (read "join" and "meet" respectively) on L is called a **lattice** if it satisfies the following identities*

L1: *(a)* $x \vee y \approx y \vee x$

(b) $x \wedge y \approx y \wedge x$ *(commutative laws)*

L2: *(a)* $x \vee (y \vee z) \approx (x \vee y) \vee z$

(b) $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$ *(associate laws)*

L3: *(a)* $x \vee x \approx x$

(b) $x \wedge x \approx x$ *(idempotent laws)*

L4: (a) $x \approx x \vee (x \wedge y)$

(b) $x \approx x \wedge (x \vee y)$ *(absorption laws)*

Definition 1.2 () *Let A be a subset of a poset P. An element p in P is an* ***upper bound*** *for A if $a \leq p$ for every a in A. An element p in P is the* ***least upper bound*** *of A (l.u.b. of A) or* ***supremum*** *of A ($\sup A$.*

For a, b in P we say b ***covers*** *a, or a is* ***covered by*** *b if $a < b$ and whenever $a \leq c \leq b$ it follows that $a = c$ or $c = b$. We use the notation $a \prec b$ to denote a is covered by b.*

Definition 1.3 () *A poset L is a lattice iff for every a, b in L both* $\sup\{a, b\}$ *and* $\inf\{a, b\}$ *exist*

1. If L is a lattice by the first definition, then define $\leq$ on L by $a \leq b$ iff

   $a = a \wedge b$

2. If L is a lattice by the second definition, then define $\vee$ and $\wedge$ by

   $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$

1.2 Isomorphism Lattices, and Sublattices

Definition 1.4 () *Two lattices L_1 and L_2 are* ***isomorphic*** *if there is a bijection α from L_1 to L_2 s.t. for every a, b in L_1 the following two equation hold: $\alpha(a \vee b) = \alpha(a) \vee \alpha(b)$ and $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$. Such an α is called* *an **isomorphism***

Definition 1.5 () *If P_1 and P_2 are two posets and α is a map from P_1 to P_2, then we say α is **order-preserving** if $\alpha(a) \leq \alpha(b)$ holds in P_2 whenever $a \leq b$ holds in P_1*

Theorem 1.6 () *Two lattices L_1 and L_2 are isomorphic iff there is a bijection α from L_1 to L_2 s.t. both α and α^{-1} are order-preserving*

Definition 1.7 () *If L is a lattice and $L' \neq \emptyset$ is a subset of L s.t. for every pair of elements a, b in L' both $a \vee b$ and $a \wedge b$ are in L', where $\wedge, \vee$ are the lattice operations of L, then we say that L' with the same operations is a*

sublattice *of* L

Definition 1.8 () *A lattice* L_1 *can be* ***embedded*** *into a lattice* L_2 *if there is a sublattice of* L_2 *isomorphic to* L_1*; in this case we also say that* L_2 ***contains a copy of*** L_1 ***as a sublattice***

1.3 Distributive and Modular Lattices

Definition 1.9 () *A* ***distributive lattice*** *is a lattice which satisfies either of the distributive laws,*

D1: $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$

D2: $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$

Theorem 1.10 () *A lattice L satisfies D1 iff it satisfies D2*

$$x \vee (y \wedge z) \approx (x \vee (x \wedge z)) \vee (y \wedge z) \qquad \text{(by L4(a))}$$

$$\approx x \vee ((x \wedge z) \vee (y \wedge z))$$

$$\approx x \vee ((z \wedge x) \vee (z \wedge y))$$

$$\approx x \vee (z \wedge (x \vee y))$$

$$\approx x \vee ((x \vee y) \wedge z)$$

$$\approx (x \wedge (x \vee y)) \vee (x \vee y \wedge z)$$

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$$\approx (x \vee y) \wedge (x \vee z)$$