# Numerical Analysis

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## Contents

# 1 Chap1 Mathematical Preliminaries

## 1.1 1.2 Roundoff Errors and Computer Arithmetic

**Truncation Error** : the error involved in using a truncated, or finite, summation to approximate the sum of an infinite series

**Roundoff Error**: the error produced when performing real number calculations. It occurs because the arithmetic performed in a machine involves numbers with only a finite number of digits.

Suppose $y = 0.d_1d_2\ldots d_k d_{k+1}d_{k+2}\ldots \times 10^n$, then

$$fl(y) = \begin{cases} 0.d_1d_2\ldots d_k \times 10^n & \text{chopping} \\ chop(y + 5\times 10^{n-(k+1)}) = 0.\delta_1\delta_2\ldots\delta_k \times 10^n & \text{Rounding} \end{cases}$$

**Definition 1.1.** *If $p*$ is an approximation to $p$, the absolute error is $|p-p*|$, and the relative error is $\frac{|p-p*|}{|p|}$, provided that $p \neq 0$*

**Definition 1.2.** *The number $p*$ is said to approximate $p$ to $t$ significant digits if $t$ is the largest nonnegative integer for which $\frac{|p-p*|}{|p|} < 5\times 10^{-t}$*

**chopping** $|\frac{y-fl(y)}{y}| = |\frac{0.d_1d_2\ldots d_kd_{k+1}\cdots\times 10^n - 0.d_1d_2\ldots d_k\times 10^n}{0.d_1d_2\ldots d_kd_{k+1}\times 10^n}| = |\frac{0.d_{k+1}\cdots}{0.d_1d_2\ldots}|\times 10^{-k} \leqslant \frac{1}{0.1}\times 10^{-k} = 10^{-k+1}$

**rounding** $|\frac{y-fl(y)}{y}| \leqslant \frac{0.5}{0.1}\times 10^{-k} = 0.5\times 10^{-k+1}$

**Finite digit arithmetic**

- $x \oplus y = fl(fl(x) + fl(y))$
- $x \otimes y = fl(fl(x) \times fl(y))$
- $x \ominus y = fl(fl(x) - fl(y))$
- $x \oplus y = fl(fl(x) \div fl(y))$

### 1.2 1.3 ALgorithms and Convergence

An algorithm that satisfies that small changes in the initial data produce correspondingly small changes in the final results is called **stable**; otherwise it is **unstable**. An algorithm is called **conditionally stable** if it is stable only for certain choices of initial data.

Suppose that E > 0 denotes an initial error and En represents the magnitude of an error after n subsequent operations. If $E_n \approx CnE_0$, where C is a constant independent of n, then the growth of error is said to be **linear**. If $E_n \approx C^n E_0$, for some C > 1, then the growth of error is called **exponential**

Suppose $\{\beta_n\}_{n=1}^{\infty}, \lim\limits_{n\to\infty} \beta_n = 0, \{\alpha_n\}_{n=1}^{\infty}, \lim\limits_{n\to\infty} \alpha_n = \alpha$. If a positive constant K exists with $|\alpha_n - \alpha| \leqslant K|\beta_n|$ for large n, then $\{\alpha_n\}_{n=1}^{\infty}$ converges to with **rate, or order, of convergence** $O(\beta_n)$

Suppose $\lim\limits_{h\to 0} G(h) = 0, \lim\limits_{h\to 0} F(h) = L$ and $|F(h) - L| \leqslant K|G(h)|$ for sufficiently small h, then we write $F(h) = L + O(G(h))$

# 2 Chap2 Solutions of equations in one variable

## 2.1 2.1 Bisection method

**Theorem 2.1.** *Intermediate Value Theorem If $f \in C[a,b]$, $K \in (f(a), f(b))$, then there exists a number $p \in (a,b)$ for which $f(p) = K$*

**Theorem 2.2.** *Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The bisection method generates a sequence $\{p_n\}, n = 0, 1, \dots$ approximating a zero $p$ of $f$ with*

$$|p_n - p| \leqslant \frac{b-a}{2^n}, \quad \textit{when } n \geqslant 1$$

## 2.2 2.2 Fixed-Point Iteration

$f(x) = 0 \xleftrightarrow{\text{equivalent}} x = f(x) + x = g(x)$

**Theorem 2.3.** *Fixed-Point Theorem Let $g \in C[a,b]$ be s.t. $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose that $g'$ exists on $(a,b)$ and that a constant $0 < k < 1$ exists with $|g'(x)| \leqslant k$ for all $x \in (a,b)$ (hence $g'$ can't converge to 1). Then for any number $p_0$ in $[a,b]$, the sequence defined by $p_n = g(p_{n-1}), n \geqslant 1$ converges to the unique point $p$ in $[a,b]$*

**Corollary 2.1.** $|p_n - p| \leqslant \frac{1}{1-k}|p_{n+1} - p_n|$ *and* $|p_n - p| \leqslant \frac{k^n}{1-k}|p_1 - p_0|$

## 2.3 2.3 Newton's method

Linearize a nonlinear function using **Taylor's expansion**

Let $p_0 \in [a,b]$ be an approximation to $p$ s.t. $f'(p_0) \neq 0$, hence $f(x) = f(p_0) + f'(p_0)(x-p_0) + \frac{f''(\xi_x)}{2!}(x-p_0)^2$, then $0 = f(p) \approx f(p_0) + f'(p_0)(p-p0) \to p \approx p_0 - \frac{f(p_0)}{f'(p_0)}$ $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for} n \geqslant 1$

**Theorem 2.4.** *Let $f \in C^2[a,b]$. If $p \in [a,b]$ is s.t. $f(p) = 0, f'(p) \neq 0$, then there exists a $\delta > 0$ s.t. Newton's method generates a sequence $\{p_n\}, n \in \mathbb{N}\backslash\{0\}$ converging to $p$ for any initial approximation $p \in [p-\delta, p+\delta]$.*

## 2.4 2.4 Error analysis for iterative methods

**Definition 2.1.** *Suppose $\{p_n\}(n = 0,1,\dots)$ is a sequence that converges to $p$ with $p_n \neq p$ for all $n$. If positive constants $\alpha$ and $\lambda$ exist with*

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lambda$$

*then $\{p_n\}(n = 0,1,\dots)$ converges to $p$ of order $\alpha$, with asymptotic error constant $\lambda$*

**Theorem 2.5.** *Let $p$ be a fixed point of $g(x)$. If there exists some constant $\alpha \geqslant 2$ s.t. $g \in C^\alpha[p-\delta, p+\delta]$, $g'(p) = \cdots = g^{\alpha-1}(p) = 0$ and $g^\alpha(p) \neq 0$. Then the iterations with $p_n = g(p_{n-1})$, $n \geqslant 1$ is of order $\alpha$*

$$p_{n+1} = g(p_n) = g(p) + g'(p)(p_n - p) + \cdots + \frac{g^\alpha(\xi_n)}{\alpha!}(p_n-p)^\alpha$$

**Theorem 2.6.** *Let $g \in C[a,b]$ be s.t. $g(x) \in [a,b]$ for all $x \in [a,b]$. Suppose in addition that $g'$ is continuous on $(a,b)$ and a positive constant $k < 1$ exists with*

$$|g'(x)| \leqslant k, \quad \textit{for all } x \in (a,b)$$

*If $g'(p) \neq 0$, then for any number $p_0 \neq p$ in $[a,b]$, the sequence*

$$p_n = g(p_{n-1}), \quad \textit{for } n \geqslant 1$$

*converges only linearly to the unique fixed point in $[a,b]$*

*Proof.*

$$\begin{aligned}\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|} &= \lim_{n\to\infty}\frac{|g(p_n)-p|}{|p_n-p|}\\ &= \lim_{n\to\infty}\frac{|g'(\xi)(p_n-p)|}{|p_n-p|}\\ &= |g'(p)|\end{aligned}$$

□

**Theorem 2.7.** *Let $p$ be a solution of the equation $x = g(x)$. Suppose that $g'(p) = 0$ and g" is continuous with $|g''(x)| < M$ on an open interval $I$ containing $p$. Then there exists a $\delta > 0$ s.t. for $p_0 \in [p-\delta, p+\delta]$, the sequence defined by $p_n = g(p_{n-1})$, when $n \geqslant 1$ converges at least quadratically to $p$. Moreover, for sufficiently large values of $n$,*

$$|p_{n+1}-p| < \frac{M}{2}|p_n-p|^2$$

*Proof.* Choose $k \in (0,1), \delta > 0$ s.t. $[p-\delta, p+\delta] \subseteq I$ and $|g'(x)| < k$ and $g''$ is continuous.

$$g(x) = g(p) + g'(p)(x-p) + \frac{g''(\xi)}{2}(x-p)^2$$

Hence $g(x) = p + \frac{g''(\xi)}{2}(x-p)^2$. $p_{n+1} = g(p_n) = p + \frac{g''(\xi_n)}{2}(p_n-p)^2$. Thus $p_{n+1} - p = \frac{g''(\xi_n)}{2}(p_n-p)^2$. We get

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^2} = \frac{g''(p)}{2}$$

□

**Definition 2.2.** *A solution $p$ of $f(x) = 0$ is a zero of multiplicity $m$ of $f$ if for $x \neq p$, $f(x) = (x-p)^m q(x)$ where $\lim_{x\to p} q(x) \neq 0$*

**Theorem 2.8.** *The function $f \in C^m[a,b]$ has a zero of multiplicity $m$ at $p$ in $(a,b)$ if and only if*

$$0 = f(p) = f'(p) = \cdots = f^{(m-1)}(p), \quad \textit{but } f^{(m)}(p) \neq 0$$

To handle the problem of multiple roots of a function $f$ is to define $\mu(x) = \frac{f(x)}{f'(x)}$.

If p is a zero of f of multiplicity m with $f(x) = (x-p)^m q(x)$, then

$$
\begin{aligned}
\mu(x) &= \frac{(x-p)^m q(x)}{m(x-p)^{m-1}q(x) + (x-p)^m q'(x)} \\
&= (x-p)\frac{q(x)}{mq(x) + (x-p)q'(x)}
\end{aligned}
$$

And $q(x) \neq 0$.

Now Newton's method:

$$
\begin{aligned}
g(x) &= x - \frac{\mu(x)}{\mu'(x)} \\
&= x - \frac{f(x)/f'(x)}{(f'(x)^2 - f(x)f''(x))/f'(x)^2} \\
&= x - \frac{f(x)f'(x)}{f'(x)^2 - f(x)f''(x)}
\end{aligned}
$$

# 3 Chap3 Interpolation and polynomial approximation

## 3.1 3.1 Interpolation and the Lagrange polynomial

$P_n(x) = \sum_{i=0}^{n} L_{n,i}(x)y_i$. Find $L_{n,i}(x)$ for $i = 0, \ldots, n$ s.t. $L_{n,j}(x_j) = \delta_{ij}$. $\delta_{ij}$ Kronecker delta. Each $L_{n,i}$ has n roots $x_0, \ldots, \hat{x_i}, \ldots, x_n$. $L_{n,j}(x) = C_i(x-x_0)\ldots(x \hat{-} x_i)\ldots(x-x_n) = C_i \prod_{\substack{j\neq i \\ j=0}}^{n}(x-x_j)$. $L_{n,j}(x_i) = 1 \to C_i = \prod_{j\neq i}\frac{1}{x_i - x_j}$. Hence $L_{n,i}(x) = \prod_{\substack{j\neq i \\ j=0}}^{n}\frac{x-x_j}{x_i-x_j}$

**Theorem 3.1.** *If $x_0, x_1, \ldots, x_n$ are n+1 distinct numbers and $f$ is a function whose values are given at these numbers, then the n-th Lagrange interpolating polynomial is unique*

**Analyze the remainder**. Suppose $a \leqslant x_0 < x_1 < \cdots < x_n \leqslant b$ and $f \in C^{n+1}[a,b]$. Consider $R_n(x) = f(x) - P_n(x)$. $R_n(x)$ has at least n+1 roots => $R_n(x) = K(x)\prod_{i=0}^{n}(x-x_i)$. For any $x \neq x_i$. Define $g(t) = R_n(t) - K(x)\prod_{i=0}^{n}(t-x_i)$. $g(x)$ has n+2 distinct roots $x_0 \dots x_n x$. Hence $g^{(n+1)}(\xi_x) = 0, \xi_x \in (a,b)$. $f^{(n+1)}(\xi_x) - Pn^{(n+1)}(\xi_x) - K(x)(n+1)! = R_n^{(n+1)}(\xi_x) - K(x)(n+1)!$. Thus $R_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)$.

**Definition 3.1.** *Let $f$ be a function defined at $x_0, \dots, x_n$ and suppose $m_1, \dots, m_k$ are k distinct integers with $0 \leqslant m_i \leqslant n$ for each i. The Lagrange polynomial that agrees with $f(x)$ at the k points $x_{m_1}, \dots, x_{m_k}$ denoted by $P_{m_1,;m_k}(x)$*

**Theorem 3.2.** *Let $f$ be defined at $x_0, \dots, x_k$ and let $x_i$ and $x_j$ be two distinct numbers in this set. Then*

$$P(x) = \frac{(x-x_j)P_{0,1,\dots,j-1,j+1,\dots,k(x)} - (x-x_i)P_{0,\dots,i-1,i+1,\dots,k(x)}}{x_i - x_j}$$

*describes the k-th Lagrange polynomial that interpolates $f$ at the k+1 points $x_0, \dots, x_k$*

**Neville's Method** n $x_0$ & $P_0$ & & &
$x_1$ & $P_1$ & $P_{0,1}$ & &
$x_2$ & $P_2$ & $P_{1,2}$ & $P_{0,1,2}$ &
$x_3$ & $P_3$ & $P_{2,3}$ & $P_{1,2,3}$ & \$P$_{0,1,2,3}$\$ n

## 3.2 3.2 Divied differences

$f[x_i, x_j] = \frac{f(x_i)-f(x_j)}{x_i-x_j}(i \neq j, x_i \neq x_j)$. $f[x_i, x_j, x_k] = \frac{f[x_i,x_j]-f[x_j,x_k]}{x_i-x_k}$.

## 3.3 Additional Newton Interpolation

### 3.3.1 Simple idea

Given $x_0, \dots, x_n$

1. Fitting $x_0$ first: $f(x) \approx f_0, f_0 = f(x_0)$
2. Add one more point $x_1$, $f_1 = f(x_1)$

$$f(x) \approx f_0 + \alpha_1(x-x_0), \alpha_1 = \frac{f_1 - f_0}{x_1 - x_0}$$

3. More points $f(x) \approx f_0 + \alpha_1(x - x_0) + \alpha_2(x - x_0)(x - x_1)$

**The pattern and coefficients**. $f(x) = \sum_{i=0}^{n} \alpha_i \prod_{j=0}^{j<i}(x-x_j) = \sum_{i=0}^{n} \alpha_i N^{(i)}(x)$

$$\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} N^{(0)}(x_0) & N^{(1)}(x_0) & \dots & N^{(n)}(x_0) \\ N^{(0)}(x_1) & N^{(1)}(x_1) & \dots & N^{(n)}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ N^{(0)}(x_n) & N^{(1)}(x_n) & \dots & N^{(n)}(x_n) \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$N^{(i)}(x_k) = \begin{cases} 0 & k < i \\ \prod_{j=0}^{j<i}(x_k - x_j) & k \geqslant i \end{cases}$ with $N^{(0)}(x) = 1$. Newton interpolation matrix is lower triangular. Lagrange matrix is identity.

### 3.3.2 Basis transformation

$$\begin{pmatrix} 1 \\ (x - x_0) \\ (x - x_0)(x - x_1) \\ \vdots \end{pmatrix} = (?) \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \end{pmatrix}$$

Hence $(\Phi_B)^T = (T_A^B)^T (\Phi_A)^T$. $\Phi_B = \Phi_A T_A^B$

$$\begin{aligned} (\Phi_A)(\alpha_A) = (f) &= (\Phi_B)(\alpha_B) \\ &= (\Phi_A)(T_A^B)(\alpha_B) \\ &\Rightarrow \\ (\alpha_A) &= (T_A^B)(\alpha_B) \\ (\alpha_B) &= (T_A^B)^{-1}(\alpha_A) \\ &= (T_B^A)(\alpha_A) \end{aligned}$$

## 3.4 3.3 Hermite interpolation

Find the **osculating polynomial** $P(x)$ s.t. $P(x_i) = f(x_i), P'(x_i) = f'(x_i), \dots, P^{(m_i)}(x_i) = f^{(m_i)}(x_i)$ for all $i = 0, 1, \dots, n$.

Just the Taylor polynomial $P(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(m_0)}(x_0)}{m_0!}(x - x_0)^{m_0}$ with remainder $R(x) = f(x) - \varphi(x) = \frac{f^{(m_0+1)}(\xi)}{(m_0+1)!}(x - x_0)^{(m_0+1)}$

$m_i = 1$ gives **Hermite polynomial**

**Example 3.1.** *Suppose $x_0 \neq x_1 \neq x_2$. Given $f(x_0), f(x_1), f(x_2), f'(x_1)$ find the polynomial $P(x)$ s.t. $P(x_i) = f(x_i), P'(x_1) = f'(x_1)$ and analyze the errors.*

*Proof.* $P_3(x) = \sum_{i=0}^{2} f(x_i)h_i(x) + f'(x_1)\hat{h}_1(x)$ where $h_i(x_j) = \delta_{ij}, h_i'(x_i) = 0, \hat{h}_i(x_i) = 0, \hat{h}_i'(x_1) = 1$.

- $h_0(x)$. Has roots $x_1, x_2$ and $x_1$ is a multiple root. $h_0(x) = C_0(x - x_1)^2(x - x_2)$ and $h_0(x_0) = 1 \Longrightarrow C_0$
- $\hat{h}_1(x)$ has root $x_0, x_1, x_2 \Longrightarrow \hat{h}_1(x) = C_1(x - x_0)(x - x_1)(x - x_2)$

□

In general, given $x_0, \ldots, x_n; y_0, \ldots, y_n$ and $y_0', \ldots, y_n'$. The Hermite polynomial $H_{2n+1}(x)$ satisfies $H_{2n+1}(x_i) = y_i$ and $H_{2n+1}'(x_i) = y_i'$

*Solution.* $H_{2n+1}(x) = \sum_{i=0}^{n} y_i h_i(x) + \sum_{i=0}^{n} y_i' \hat{h}_i(x)$

## 3.5 3.4 Cubic spline interpolation

**Piecewise linear interpolation**. Approximate $f(x)$ by linear polynomials on each subinterval $[x_i, x_{i+1}]$.

$f \approx P_1(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}} y_i + \frac{x - x_i}{x_{i+1} - x_i} y_{i+1}$ for $x \in [x_i, x_{i+1}]$

Let $h = \max|x_{i+1} - x_i|$. Then $P_1^h(x) \xrightarrow{uniform} f(x)$ as $h \to 0$ However, this is no longer smooth.

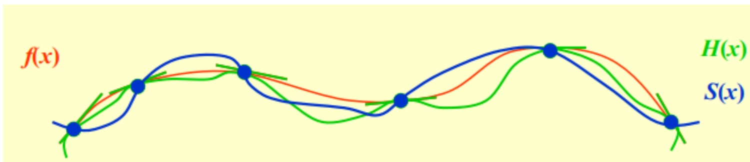
**Hermite piecewise polynomials**. Given $x_0, \ldots, x_n; y_0, \ldots, y_n, y_0', \ldots, y_n'$, construct the Hermite polynomial of degree 3 with $y$ and $y'$ on the two endpoints of $[x_i, x_{i+1}]$

**Cubic Spline**.

**Definition 3.2.** *Given a function $f$ define on $[a, b]$ and a set of nodes $a = x_0 < x_1 < \cdots < x_n = b$, cubic spline interpolant $S$ for $f$ is a function that satisfies the following conditions*

- *$S(x)$ is a cubic polynomial, denoted by $S_i(x)$ on the subinterval $[x_i, x_{i+1}]$ for each $i = 0, \ldots, n - 1$*
- *$S(x_i) = f(x_i)$ for each $i = 0, \ldots, n$*
- *$S_{i+1}(x_{i+1}) = S_i(x_{i+1})$*

- $S'_{i+1}(x_{i+1}) = S'_i(x_{i+1})$
- $S''_{i+1}(x_{i+1}) = S''_i(x_{i+1})$




**Method of Bending moment**. Let $h_j = x_j - x_{j-1}$ and $S(x) = S_j(x)$ for $x \in [x_{j-1}, x_j]$. Then $S''_j$ is a polynomial of degree 1, which can be determined by the values of f on 2 nodes .

Assume $S''_j(x_{j-1}) = M_{j-1}, S''_j(x_j) = M_j$. Then for all $x \in [x_{j-1}, x_j]$, $S''_j(x) = M_{j-1}\frac{x_j - x}{h_j} + M_j\frac{x - x_{j-1}}{h_j}$. Hence we get

$$S'_j(x) = -M_{j-1}\frac{(x_j - x)^2}{2h_j} + M_j\frac{(x - x_{j-1})^2}{2h_j} + A_j$$
$$S_j(x) = M_{j-1}\frac{(x_j - x)^3}{6h_j} + M_j\frac{(x - x_{j-1})^3}{6h_j} + A_j x + B_j$$

Solve this by $S_j(x_{j-1}) = y_{j-1}, S_j(x_j) = y_j$, we get

$$A_j = \frac{y_j - y_{j-1}}{h_j} - \frac{M_j - M_{j-1}}{6}h_j$$
$$A_j x + B_j = (y_{i-1} - \frac{M_{j-1}}{6}h_j^2)\frac{x_j - x}{h_j} + (y_j - \frac{M_j}{6}h_j^2)\frac{x - x_{j-1}}{h_j}$$

Now solve for $M_j$: Since $S'$ is continuous at $x_j$

$$[x_{j-1}, x_j] : S'_j(x) = -M_{j-1}\frac{(x_j - x)^2}{2h_j} + M_j\frac{(x - x_{j-1})^2}{2h_j} + f[x_{j-1}, x_j] - \frac{M_j - M_{j-1}}{6}h_j$$
$$[x_j, x_{j+1}] : S'_{j+1}(x) = -M_j\frac{(x_{j+1} - x)^2}{2h_{j+1}} + M_{j+1}\frac{(x - x_j)^2}{2h_{j+1}} + f[x_j, x_{j+1}] - \frac{M_{j+1} - M_j}{6}h_{j+1}$$

From $S'_j(x_j) = S'_{j+1}(x_j)$, let $\lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}}, \mu_j = 1 - \lambda_j, g_j = \frac{6}{h_j + h_{j+1}}(f[x_j, x_{j+1}] - f[x_{j-1}, x_j])$ we get

$$\mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = g_j \quad \text{for } 1 \leqslant j \leqslant n-1$$

$$\begin{pmatrix} \mu_1 & 2 & \lambda_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu_{n-1} & 2 & \lambda_{n-1} \end{pmatrix} \begin{pmatrix} M_0 \\ \vdots \\ \vdots \\ M_n \end{pmatrix} = \begin{pmatrix} g_1 \\ \vdots \\ g_{n-1} \end{pmatrix}$$

And $S'(a) = y_0', S'(b) = y_n'$

# 4 Chap6 Direct Methods for Solving Linear Systems

## 4.1 6.1 Linear Systems of Equations

**Gaussian elimination with backward substitution**

## 4.2 6.2 Pivoting Strategies

**Problem**: small pivot element may cause trouble

**Paritial Pivoting**: Determine the smallest pk s.t. $|a_{pk}^{(k)}| = \max\limits_{k \leqslant j \leqslant n} |a_{ik}^{(k)}|$ and interchange the pth and the kth rows

**Scaled Partial Pivoting**:

1. Define a scale factor $s_i$ for each row as $s_i = \max\limits_{1 \leqslant j \leqslant n} |a_{ij}|$
2. Determine the smallest $p \geqslant k$ s.t. $\frac{|a_{pk}^{(k)}|}{s_p} = \max\limits_{k \leqslant i \leqslant n} \frac{|a_{ik}^{(k)}|}{s_i}$ and interchange the pth and the kth rows

**Complete Pivoting**: Search all the entries $a_{ij}$ to find the entry with the largest magnitude

## 4.3 6.5 Matrix Factorization

$m_{ik} = a_{ik}/a_{kk}$

$$L_k = \begin{pmatrix} 1 & & & & \\ & \ddots & & \Large 0 & \\ & & 1 & & \\ & & -m_{k+1,k} & & \\ & & \vdots & \ddots & \\ & & -m_{n,k} & & 1 \end{pmatrix}$$

Hence

$$L_1^{-1}L_2^{-1}\dots L_{n-1}^{-1} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ m_{i,j} & & & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & \dots & a_{2n} \\ & & \dots & \vdots \\ & & & a_{nn} \end{pmatrix}$$

$A = LU$

## 4.4 6.6 Special Types of Matrices

**Strictly Diagonally Dominant Matrix**. $|a_{ii}| > \sum_{\substack{j=1, \\ j \neq i}}^{n} |a_{ij}| \quad \text{for each } i = 1, \dots, n$

**Theorem 4.1.** *A strictly diagonally dominant matrix A is nonsingular. Moreover, Gaussian elimination can be performed without row or column interchanges, and the computations will be stable w.r.t. the growth of roundoff errors*

**Choleski's Method for Positive Definite Matrix**:

**Definition 4.1.** *A matrix A is positive definite if ti's symmetric and if* $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ *for every n-dimensional vector* $\mathbf{x} \neq 0$

**Lemma 4.1.** *A is positive definite*

1. $A^{-1}$ *is positive definite as well, and* $a_{ii} > 0$
2. $\sum |a_{ij}| \leqslant \max |a_{kk}|$; $(a_{ij})^2 < a_{ii}a_{jj}$ *for each i j*
3. *Each of /A's leading principal submatrices* $A_k$*/ has a positive determinant*

$$U = \begin{pmatrix} & & \\ & u_{ij} & \\ & & \end{pmatrix} = \begin{pmatrix} u_{11} & & \\ & \ddots & \\ & & u_{nn} \end{pmatrix} \begin{pmatrix} 1 & & u_{ij}/u_{ii} \\ & 1 & \\ & & 1 \end{pmatrix} = D\tilde{U}$$

A is symmetric, hence

$$L = \tilde{U}^t, A = LDL^t$$

Let

$$D^{1/2} = \begin{pmatrix} \sqrt{u_{11}} & & \\ & \ddots & \\ & & \sqrt{u_{nn}} \end{pmatrix}, \tilde{L} = LD^{1/2/}, A = \tilde{L}\tilde{L}^t$$

**Crout Reduction for tridiagonal Linear System**

$$\begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix}$$

$$A = \begin{pmatrix} \alpha_1 & & & \\ \gamma_2 & \ddots & & \\ & \ddots & \ddots & \\ & & \gamma_n & \alpha_n \end{pmatrix} \begin{pmatrix} 1 & \beta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \beta_{n-1} \\ & & & 1 \end{pmatrix}$$

# 5 Chap7 Iterative techiniques in Matrix algebra

## 5.1 7.1 Norms of vectors and matrices

**Definition 5.1.** *A vector norm on $R^n$ is a function $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ with following properties for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n, \alpha \in C$*

1. $||\mathbf{x}|| \leqslant 0$; $||\mathbf{x}|| = 0 \Longleftrightarrow \mathbf{x} = \mathbf{0}$
2. $||\alpha\mathbf{x}|| = |\alpha| \cdot ||\mathbf{x}||$
3. $||\mathbf{x} + \mathbf{y}|| \leqslant ||\mathbf{x}|| + ||\mathbf{y}||$

$||\mathbf{x}||_1 = \sum_{i=1}^{n} |x_i|$. $||\mathbf{x}_p|| = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$

**Definition 5.2.** *A sequence $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ of vectors in $R^n$ converge to $\mathbf{x}$ w.r.t the norm $||\cdot||$ if given any $\epsilon > 0$ there exists an integer $N(\epsilon)$ s.t. $||\mathbf{x}^{(k)} - \mathbf{x}|| < \epsilon$ for all $k \geqslant N(\epsilon)$*

**Theorem 5.1.** *The sequence of vectors $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x} \in R^n$ w.r.t. $||\cdot||$ if and only if $\lim_{k\to\infty} \mathbf{x}_i^{(k)} = x_i$ for each $i = 1, 2, \ldots, n$*

**Definition 5.3.** *If there exist positive constants $C_1, C_2$ s.t. $C_1||\mathbf{x}||_B \leqslant ||\mathbf{x}||_A \leqslant C_2||\mathbf{x}|_B|$. Then $||\cdot||_A, ||\cdot||_B$ are equivalent*

**Theorem 5.2.** *All the vector norm in $R^n$ are equivalent*

**Definition 5.4.** *A matrix norm on the set of $n \times n$:*

1. $||\mathbf{A}|| \geqslant 0; ||\mathbf{A}|| = 0 \Longleftrightarrow \mathbf{A} = \mathbf{0}$
2. $||\alpha\mathbf{A}|| = |\alpha| \cdot ||\mathbf{A}||$
3. $||\mathbf{A} + \mathbf{B}|| \leqslant ||\mathbf{A}|| + ||\mathbf{B}||$
4. $||\mathbf{AB}|| \leqslant ||\mathbf{A}|| \cdot ||\mathbf{B}||$

**Frobenius Norm**: $||\mathbf{A}||_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2}$

**Natural Norm**: $||\mathbf{A}||_p = \max_{\mathbf{x}\neq\mathbf{0}} \frac{||\mathbf{Ax}||_p}{||\mathbf{x}||_p} = \max_{\mathbf{z}\neq\mathbf{0}} ||\mathbf{A}\frac{\mathbf{z}}{||\mathbf{z}||}|| = \max_{||\mathbf{x}||_p=1} ||\mathbf{Ax}||_p$

$||\mathbf{A}||_\infty = \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |a_{ij}|$, $||\mathbf{A}||_1 = \max_{1\leqslant j\leqslant n} \sum_{i=1}^{n} |a_{ij}|$, $||\mathbf{A}||_2 = \sqrt{\lambda_{\max}(\mathbf{A}^T\mathbf{A})}$

## 5.2 7.2 Eigenvalues and Eigenvectors

**spectral radius**.

**Definition 5.5.** *The spectral radius $\rho(A)$ of a matrix $A$ is defined as $\rho(A) = \max|\lambda|$ where $\lambda$ is an eigenvalue of $A$*

**Theorem 5.3.** *If $A$ is an $n \times n$ matrix, then $\rho(A) \leqslant ||A||$ for any natural norm*

*Proof.* $|\lambda| \cdot ||\boldsymbol{x}|| = ||\lambda\boldsymbol{x}|| = ||A\boldsymbol{x}|| \leqslant ||A|| \cdot ||\boldsymbol{x}||$ □

**Definition 5.6.** *We call an $n \times n$ matrix $A$ convergent if for all $i, j = 1, \ldots, n$ $\lim_{k\to\infty} (A^k)_{ij} = 0$*

## 5.3 7.3 Iterative techniques for solving linear systems

**Jacobi iterative method**.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \implies \begin{cases} x_1 = \frac{1}{a_{11}}(-a_{12}x_2 - \cdots - a_{1n}x_n + b_1) \\ x_2 = \frac{1}{a_{22}}(-a_{21}x_1 - \cdots - a_{2n}x_n + b_2) \\ \cdots \\ x_1 = \frac{1}{a_{nn}}(-a_{n2}x_1 - \cdots - a_{nn-1}x_{n-1} + b_n) \end{cases}$$

In matrix form,

$$A = \begin{pmatrix} D & -U & -U \\ -L & D & -U \\ -L & -L & D \end{pmatrix}$$

$$\begin{aligned} A\boldsymbol{x} = \boldsymbol{b} &\Leftrightarrow (D - L - U)\boldsymbol{x} = \boldsymbol{b} \\ &\Leftrightarrow D\boldsymbol{x} = (L + U)\boldsymbol{x} + \boldsymbol{b} \\ &\Leftrightarrow \boldsymbol{x} = \underbrace{D^{-1}(L + U)}_{T_j}\boldsymbol{x} + \underbrace{D^{-1}}_{\boldsymbol{c}_j}\boldsymbol{b} \end{aligned}$$

. $T_j$ is Jacobi iterative matrix. $\boldsymbol{x}^{(k)} = T_j\boldsymbol{x}^{(k-1)} + \boldsymbol{c}_j$
*Gauss-Seidel iterative method*

$$\begin{aligned} \boldsymbol{x}^{(k)} &= D^{-1}(L\boldsymbol{x}^{(k)} + U\boldsymbol{x}^{(k-1)}) + D^{-1}\boldsymbol{b} \\ \Leftrightarrow (D - L)\boldsymbol{x}^{(k)} &= U\boldsymbol{x}^{(k-1)} + \boldsymbol{b} \\ \Leftrightarrow \boldsymbol{x}^{(k)} &= \underbrace{(D - L)^{-1}U\boldsymbol{x}^{(\boldsymbol{k-1})}}_{T_g} + \underbrace{(D - L)^{-1}\boldsymbol{b}}_{\boldsymbol{c}_g} \end{aligned}$$

**convergence of iterative methods**

**Theorem 5.4.** *the following are equivalent:*

1. *$A$ is a convergent matrix*
2. $\lim_{n\to\infty} ||A^n|| = 0$ *for some natural norm*
3. $\lim_{n\to\infty} ||A^n|| = 0$ *for all natural norms*
4. $\rho(A) < 1$
5. $\lim_{n\to\infty} A^n\boldsymbol{x} = \boldsymbol{0}$ *for every* $\boldsymbol{x}$

$\boldsymbol{e}^{(k)} = \boldsymbol{x}^{(k)} - \boldsymbol{x}^* = (T\boldsymbol{x}^{(k-1)} + \boldsymbol{c}) - (T\boldsymbol{x}^* + \boldsymbol{c}) = T(\boldsymbol{x}^{(k-1)} - \boldsymbol{x}^*) = T\boldsymbol{e}^{(k-1)} \Rightarrow \boldsymbol{e}^{(k)} = T^k\boldsymbol{e}^{(0)}$. $||\boldsymbol{e}^{(k)} \leqslant ||T|| \cdot ||\boldsymbol{e}^{(k-1)}|| \leqslant \cdots \leqslant ||T||^k \cdot ||ble^{(0)}||$

**Theorem 5.5.** *For any $\boldsymbol{x}^{(0)} \in R^n$, the sequence $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ defined by $\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c}$ for each $k$, converges to the unique solution of $\boldsymbol{x} = T\boldsymbol{x} + \boldsymbol{c}$ if and only if $\rho(T) < 1$*

$$\rho(T) < 1 \Longrightarrow (I-T)^{-1} = \sum_{j=0}^{\infty} T^j$$

**Theorem 5.6.** *If $\|T\| < 1$ for any natural matrix norm and $\boldsymbol{c}$ is a given vector, then the sequence $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ defined by $\boldsymbol{x}^{(k)} = T\boldsymbol{x}^{(k-1)} + \boldsymbol{c}$ converges for any $\boldsymbol{x}^{(0)} \in R^n$ to a vector $\boldsymbol{x}$. And the following error bounds hold*

1. $\left\|\boldsymbol{x} - \boldsymbol{x}^{(k)}\right\| \leqslant \|T\|^k \left\|\boldsymbol{x} - \boldsymbol{x}^{(0)}\right\|$
2. $\left\|\boldsymbol{x} - \boldsymbol{x}^{(k)}\right\| \leqslant \frac{\|T\|^k}{1-\|T\|} \left\|\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}\right\|$

**Theorem 5.7.** *If $A$ is a strictly diagonally dominant, then for any choice of $\boldsymbol{x}^{(0)}$, both the Jacobi and Gauss-Seidel methods give sequences $\{\boldsymbol{x}^{(k)}\}_{k=0}^{\infty}$ that converges to the unique solution*

**relaxation methods**. $x_i^{(k)} = \frac{1}{a_{ii}}(b_i - \sum_{j=1}^{i-1} a_{ij}x_i^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}) = x_i^{(k-1)} + \dfrac{r_i^{(k)}}{a_{ii}}$ and relaxation method is $x_i^{(k)} = x_i^{(k-1)} + \omega\frac{r_i^{(k)}}{a_{ii}}$

**Theorem 5.8.** *(kahan) If $a_{ii} \neq 0$ for each $i$. Then $\rho(T_\omega) \geqslant |\omega - 1|$.*

This implies the SOR method can converge only if $0 < \omega < 2$

**Theorem 5.9.** *(Ostrowski-Reich) If $A$ is positive definite and $0 < \omega < 2$, the SOR converges*

**Theorem 5.10.** *If $A$ is positive definite and tridiagonal, then $\rho(T_g) = (\rho(T_j))^2 < 1$, and the optimal choice of $\omega$ for the SOR method is $\omega = \frac{2}{1+\sqrt{1-(\rho(T_j))^2}}$. With this choice of $\omega$, we have $\rho(T_\omega) = \omega - 1$*

## 5.4 7.4 Error bounds and iterative refinement

Assume that A is accurate and $\boldsymbol{b}$ has the error $\delta\boldsymbol{b}$, then $\boldsymbol{A}(\boldsymbol{x} + \delta\boldsymbol{x}) = \boldsymbol{b} + \delta\boldsymbol{b}$

**Theorem 5.11.** *Suppose $\tilde{\boldsymbol{x}}$ is an approximation to the solution of $\boldsymbol{Ax} = \boldsymbol{b}$ A is nonsingular matrix. Then for any natural norm,*

$$||\boldsymbol{x} - \tilde{\boldsymbol{x}}|| \leqslant ||\boldsymbol{r}|| \cdot ||A^{-1}||$$

*and if $\boldsymbol{x} \neq \boldsymbol{0}, \boldsymbol{b} \neq \boldsymbol{0}$,*

$$\frac{||\delta\boldsymbol{x}||}{||\boldsymbol{x}||} \leqslant ||\boldsymbol{A}|| \cdot ||\boldsymbol{A}^{-1}|| \cdot \frac{||\delta\boldsymbol{b}||}{||\boldsymbol{b}||}$$

*Proof.* $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\tilde{\boldsymbol{x}} = A\boldsymbol{x} - A\tilde{\boldsymbol{x}}$ and A is nonsingular. Hence $\boldsymbol{x} - \tilde{\boldsymbol{x}} = A^{-1}\boldsymbol{r}$. Since $\frac{||A^{-1}\boldsymbol{r}||}{||\boldsymbol{r}||} \leqslant ||A^{-1}||$ , $||\boldsymbol{x} - \tilde{\boldsymbol{x}}|| = ||A^{-1}\boldsymbol{x}|| \leqslant ||A^{-}1|| \cdot ||\boldsymbol{r}||$. Also $||\boldsymbol{b}|| \leqslant ||A|| \cdot ||\boldsymbol{x}||$. So $1/||\boldsymbol{x}|| \leqslant ||A||/||\boldsymbol{b}||$ □

**Theorem 5.12.** *If a matrix $B$ satisfies $||B|| < 1$ for some natural norm, then*

1. *$I \pm B$ is nonsingular*
2. *$||(I \pm B)^{-1}|| \leqslant \frac{1}{1-||B||}$*

Assume $\boldsymbol{b}$ is accurate, A has the error $\delta A$, then $(A + \delta A)(\boldsymbol{x} + \delta\boldsymbol{x}) = \boldsymbol{b}$. Hence $\frac{||\delta\boldsymbol{x}||}{||\boldsymbol{x}||} \leqslant \frac{||A^{-1}|| \cdot ||\delta A||}{1-||A^{-1}|| \cdot ||\delta A||} = \frac{||A|| \cdot ||A^{-1}||}{1-||A|| \cdot ||A^{-1}|| \cdot \frac{||\delta A||}{||A||}}$

**condition number K(A)** is $||A|| \cdot ||A^{-1}||$

**Theorem 5.13.** *Suppose $A$ is nonsingular and $||\delta A|| \leqslant \frac{1}{||A^{-1}||}$. The solution $\boldsymbol{x} + \delta\boldsymbol{x}$ to $(A + \delta A)(\boldsymbol{x} + \delta\boldsymbol{x})$ approximates the solution $\boldsymbol{x}$ of $A\boldsymbol{x} = \boldsymbol{b}$ with the error estimate*

$$\frac{||\delta\boldsymbol{x}||}{||\boldsymbol{x}||} \leqslant \frac{K(A)}{1 - K(A)||\delta A||/||A||}\left(\frac{||\delta A||}{||A||} + \frac{||\delta\boldsymbol{b}||}{||\boldsymbol{b}||}\right)$$

note:

1. If A is symmetric, then $K(A)_2 = \frac{\max|\lambda|}{\min|\lambda|}$
2. $K(A)_p \geqslant 1$ for all natural norm
3. $K(\alpha A)_= K(A)$ for any $\alpha \in R$
4. $K(A)_2 = 1$ if A is orthogonal
5. $K(RA)_2 = K(AR)_2 = K(A)_2$ for all orthogonal matrix R_

**iterative refinement**:

**Theorem 5.14.** *Suppose $\boldsymbol{x}^*$ is an approximation to the solution of $A\boldsymbol{x} = \boldsymbol{b}$, $A$ is nonsingular matrix and $\boldsymbol{r} = \boldsymbol{b} - A\boldsymbol{x}$. Then for any natural norm, $||\boldsymbol{x} - \boldsymbol{x}^* \leqslant ||\boldsymbol{r}|| \cdot ||A^{-1}||$, and if $\boldsymbol{x}, \boldsymbol{b} \neq \boldsymbol{0}$*

$$\frac{||\boldsymbol{x} - \boldsymbol{x}^*||}{||\boldsymbol{x}||} \leqslant K(A)\frac{||\boldsymbol{r}||}{||\boldsymbol{b}||}$$

**refinement**

1. $A\boldsymbol{x} = \boldsymbol{b}$ => approximation $\boldsymbol{x}_1$
2. $\boldsymbol{r}_1 = \boldsymbol{b} - A\boldsymbol{x}_1$
3. $A\boldsymbol{d}_1 = \boldsymbol{r}_1$ => $\boldsymbol{d}_1$
4. $\boldsymbol{x}_2 = \boldsymbol{x}_1 + \boldsymbol{d}_1$

# 6 chap9 Approximating Eigenvalues

## 6.1 9.3 the power method

**the original method** Assumptions: A is an $n \times n$ matrix with eigenvalues satisfying $|\lambda_1| > |\lambda_2| \geqslant \cdots \geqslant |\lambda_n| \geqslant 0$

$$\begin{aligned}
&\boldsymbol{x}^{(0)} = \sum_{j=1}^{n} \beta_j \boldsymbol{v}_j, \quad \beta_1 \neq 0 \\
&\boldsymbol{x}^{(1)} = A\boldsymbol{x}^{(0)} = \sum_{j=1}^{n} \beta_j \lambda_j \boldsymbol{v}_j \\
&\boldsymbol{x}^{(2)} = A\boldsymbol{x}^{(1)} = \sum_{j=1}^{n} \beta_j \lambda_j^2 \boldsymbol{v}_j \\
&\dots \\
&\boldsymbol{x}^{(k)} \approx \lambda_1^k \beta_1 \boldsymbol{v}_1, \quad \lambda_1 \approx \frac{\boldsymbol{x}_i^{(k)}}{\boldsymbol{x}_i^{(k-1)}}
\end{aligned}$$

**Normalization**. Suppose $||\boldsymbol{x}||_\infty = 1$. Let $||\boldsymbol{x}^{(k)}||_\infty = |x_{p_k}^{(k)}|$.Then $\boldsymbol{u}^{(k-1)} = \frac{\boldsymbol{x}^{(k-1)}}{|x_{p_{k-1}}^{(k-1)}|}$ and $\boldsymbol{x}^{(k)} = A\boldsymbol{u}^{(k-1)}$. Then $\boldsymbol{u}^{(k)} = \frac{\boldsymbol{x}^{(k)}}{|x_{p_k}^{(k)}|} \to \boldsymbol{v}_1$. $\lambda_1 \approx \frac{\boldsymbol{x}_i^{(k)}}{\boldsymbol{u}_i^{(k-1)}} = \boldsymbol{x}_{p_{k-1}}^{(k)}$

Note:

1. the method works for **multiple** eigenvalues $\lambda_1 = \lambda_2 = \cdots = \lambda_r$
2. the method fails to converge if $\lambda_1 = -\lambda_2$
3. Aitken's $\Delta^2$ can be used

**Rate of convergence**. $\boldsymbol{x}^{(k)} = A\boldsymbol{x}^{(k-1)} = \lambda_1^k \sum_{j=1}^{n} \beta_j (\frac{\lambda_j}{\lambda_1})^k \boldsymbol{v}_j$. Make $|\lambda_2/\lambda_1|$ as small as possible. Assume $\lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_n, |\lambda_2| > |\lambda_n|$. Let $B = A - pI$, then $|\lambda I - A| = |\lambda I - (B + pI)| = |(\lambda - p)I - B|$. Hence $\lambda_A - p = \lambda_B$. Since $\frac{|\lambda_2 - p|}{|\lambda_1 - p|} < \frac{|\lambda_2|}{|\lambda_1|}$ . The iteration is fast

**Inverse power method**. If A has $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots > |\lambda_n|$, then $A^{-1}$ has $|\frac{1}{\lambda_n}| > |\frac{1}{\lambda_{n-1}}| \geqslant \cdots \geqslant |\frac{1}{\lambda_1}|$