Probability Theory
A
Comprehensive
Course

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Contents

1 Basic Measure Theory

1.1 Classes of Sets

Definition 1.1. A class of sets $\mathcal{A}$ is called

- $\cap$**-closed** or a π**-system** if $A \cap B \in \mathcal{A}$ whenever $A, B \in \mathcal{A}$
- σ**-**$\cap$**-closed** (closed under countable intersection)
- $\cup$**-closed** (closed under unions)
- σ-$\cup$**-closed**
- **\-closed**
- closed under complements

Definition 1.2 (σ-algebra)**.** A class of sets $\mathcal{A} \subset 2^{\Omega}$ is called a σ**-algebra** if it fullills the following three conditions

1. $\Omega \in \mathcal{A}$
2. $\mathcal{A}$ is closed under complements
3. $\mathcal{A}$ is closed under countable unions

Theorem 1.3. *If $\mathcal{A}$ is closed under complements, then we have the equivalence*

$$\begin{aligned} \mathcal{A} \textit{ is } \cap\textit{-closed} &\iff \mathcal{A} \textit{ is } \cup\textit{-closed} \\ \mathcal{A} \textit{ is } \sigma\text{-}\cap\textit{-closed} &\iff \mathcal{A} \textit{ is } \sigma\text{-}\cup\textit{-closed} \end{aligned}$$

Theorem 1.4. *Assume that $\mathcal{A}$ is $\setminus$-closed. Then the following statements hold:*

1. *$\mathcal{A}$ is $\cup$-closed*
2. *If in addition $\mathcal{A}$ is σ-$\cup$-closed, then $\mathcal{A}$ is σ-$\cup$-closed*
3. *Any countable (repectively finite) union of sets in $\mathcal{A}$ can be expressed as a countable (respectively finite) disjoint union of sets in $\mathcal{A}$*

Proof. 3. Assume that $A_1, A_2, \dots \in \mathcal{A}$

$$\bigcup_{n=1}^{\infty} A_n = A_1 \uplus (A_2 \setminus A_1) \uplus ((A_3 \setminus A_2) \setminus A_1) \uplus \dots$$

□

Definition 1.5. A class of sets $\mathcal{A} \subset 2^{\Omega}$ is called an **algebra** if the following three conditions are fulfilled

1. $\Omega \in \mathcal{A}$
2. $\mathcal{A}$ is $\setminus$-closed
3. $\mathcal{A}$ is $\cup$-closed

Theorem 1.6. *A class of sets $\mathcal{A} \subset 2^\Omega$ is an algebra if and only if the following three properties hold*

1. $\Omega \in \mathcal{A}$
2. *$\mathcal{A}$ is closed under complements*
3. *$\mathcal{A}$ is closed under intersections*

Definition 1.7. A class of sets $\mathcal{A} \subset 2^\Omega$ is called a **ring** if the following conditions hold

1. $\emptyset \in \mathcal{A}$
2. $\mathcal{A}$ is $\backslash$-closed
3. $\mathcal{A}$ is $\cup$-closed

Definition 1.8. A class of sets $\mathcal{A} \subset 2^\Omega$ is called a **semiring** if

1. $\emptyset \in \mathcal{A}$
2. for any two sets $A, B \in \mathcal{A}$ the difference set $B \backslash A$ is a finite union of mutually disjoint sets in $\mathcal{A}$
3. $\mathcal{A}$ is $\cap$-closed

Definition 1.9. A class of sets $\mathcal{A} \subset 2^\Omega$ is called a **λ-system** if

1. $\Omega \in \mathcal{A}$
2. for any two sets $A, B \in \mathcal{A}$ with $A \subset B$, $B \backslash A \in \mathcal{A}$
3. $\biguplus_{n=1}^\infty A_n \in \mathcal{A}$ for any choice of countably many pairwise disjoint sets $A_1, \dots \in \mathcal{A}$

Theorem 1.10.

1. *Every σ-algebra also is a λ-system, an algebra and a σ-ring*
2. *Every σ-ring is a ring, and every ring is a semiring*
3. *Every algebra is a ring. An algebra on a finite set Ω is a σ-algebra*

Definition 1.11 (liminf and limsup)**.** Let $A_1, A_2, \dots$ be a subset of Ω. The sets

$$\liminf_{n\to\infty} A_n := \bigcup_{n=1}^\infty \bigcap_{m=n}^\infty A_m \qquad \limsup_{n\to\infty} A_n := \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m$$

are called **limes inferior** and **limes superior**, respectively, of the sequence $(A_n)_{n\in\mathbb{N}}$

Remark.

1. $\liminf$ and $\limsup$ can be rewritten as

$$\liminf_{n\to\infty} A_n = \{\omega \in \Omega : \#\{n \in \mathbb{N} : \omega \notin A_n\} < \infty\}$$
$$\limsup_{n\to\infty} A_n = \{\omega \in \Omega : \#\{n \in \mathbb{N} : \omega \in A_n\} = \infty\}$$

In other words, limes inferior is the event where **eventually all** of the A_n occur. On the other hand, limes superior is the event where **infinitely many** of the A_n occur. In particular, $A_* := \liminf_{n\to\infty} A_n \subset A^* := \limsup_{n\to\infty} A_n$

2. We define the **indicator function** on the set A by

$$\mathbb{1}_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

With this notation

$$\mathbb{1}_{A_*} = \liminf_{n\to\infty} \mathbb{1}_{A_n} \quad \text{and} \quad \mathbb{1}_{A^*} = \limsup_{n\to\infty} \mathbb{1}_{A_n}$$

3. If $\mathcal{A} \subset 2^\Omega$ is a σ-algebra and if $A_n \in \mathcal{A}$ for every $n \in \mathbb{N}$, then $A_* \in \mathcal{A}$ and $A^* \in \mathcal{A}$

Theorem 1.12 (Intersection of classes of sets)**.** *Let I be an arbitrary index set, and assume that $\mathcal{A}_i$ is a σ-algebra for every $i \in I$. Hence the intersection*

$$\mathcal{A}_I := \{A \subset \Omega : A \in \mathcal{A}_i \text{ for every } i \in I\} = \bigcap_{i\in I} \mathcal{A}_i$$

is a σ-algebra. The analogous statement holds for rings, σ-rings, algebras and λ-systems. However, it fails for semirings

Theorem 1.13 (Generated σ-algebra)**.** *Let $\mathcal{E} \subset 2^\Omega$. Then there exists a smallest σ-algebra $\sigma(\mathcal{E})$ with $\mathcal{E} \subset \sigma(\mathcal{E})$*

$$\sigma(\mathcal{E}) := \bigcap_{\substack{\mathcal{A}\subset 2^\Omega \text{ is a } \sigma\text{-algebra} \\ \mathcal{A}\supset\mathcal{E}}} \mathcal{A}$$

$\sigma(\mathcal{E})$ is called the σ-algebra generated by $\mathcal{E}$. $\mathcal{E}$ is called a generator of $\sigma(\mathcal{E})$. Similarly, we define $\delta(\mathcal{E})$ as the λ-system generated by $\mathcal{E}$

Remark. The following three statements hold

1. $\mathcal{E} \subset \sigma(\mathcal{E})$
2. If $\mathcal{E}_1 \subset \mathcal{E}_2$, then $\sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$
3. $\mathcal{A}$ is a σ-algebra if and only if $\sigma(\mathcal{A}) = \mathcal{A}$

Theorem 1.14 ($\cap$-closed λ-system)**.** *Let $\mathcal{D} \subset 2^\Omega$ be a λ-system. Then*

$$\mathcal{D} \text{ is a } \pi\text{-system} \quad \Longleftrightarrow \quad \mathcal{D} \text{ is a } \sigma\text{-algebra}$$

Proof. "$\Longrightarrow$"

3. Let $A, B \in \mathcal{D}$. By assumption, $A \cap B \in \mathcal{D}$ and trivially $A \cap B \subset A$. Thus $A \backslash B = A \backslash (A \cap B) \in \mathcal{D}$. This implies that $\mathcal{D}$ is $\backslash$-closed. Thus by Theorem 1.4, works.

$\square$

Theorem 1.15 (Dynkin's π-λ theorem)**.** *If $\mathcal{E} \subset 2^{\Omega}$ is a π-system, then*

$$\sigma(\mathcal{E}) = \delta(\mathcal{E})$$

Proof.
1. $\supseteq$. $A^c = \Omega \backslash A$.
2. $\subseteq$. By Theorem 1.14, it is enough to show that $\delta(\mathcal{E})$ is a π-system. For any $B \in \delta(\mathcal{E})$ define

$$\mathcal{D}_B := \{A \in \delta(\mathcal{E}) : A \cap B \in \delta(\mathcal{E})\}$$

In order to show that $\delta(\mathcal{E})$ is a π system, it is enough to show that

$$\delta(\mathcal{E}) \subset \mathcal{D}_B \quad \text{for any } B \in \delta(\mathcal{E})$$

$\mathcal{D}_E$ is a λ-system

(a) $\Omega \cap E = E \in \delta(\mathcal{E})$. Hence $\Omega \in \mathcal{D}_E$

(b) For any $A, B \in \mathcal{D}_E$ with $A \subset B$, we have $(B \backslash A) \cap E = (B \cap E) \backslash (A \cap E) \in \delta(E)$

(c) Assume that $A_1, \ldots, \in \mathcal{D}_E$ are mutually disjoint. Hence

$$\left(\bigcup_{n=1}^{\infty} \right) \cap E = \biguplus_{n=1}^{\infty} (A_n \cap E) \in \delta(\mathcal{E})$$

By assumption, $A \cap E \in \mathcal{E}$ if $A, E \in \mathcal{E}$; thus $\mathcal{E} \subset \mathcal{E}_E$ if $E \in \mathcal{E}$. Hence $\delta(\mathcal{E}) \subset \delta(\mathcal{D}_E) = \mathcal{D}_E$ for any $E \in \mathcal{E}$. Hence we get that $B \cap E \in \delta(\mathcal{E})$ for any $B \in \delta(\mathcal{E})$ and $E \in \mathcal{E}$. This implies that $E \in \mathcal{E}_B$ for any $B \in \delta(\mathcal{E})$. Thus $\mathcal{E} \subset \mathcal{D}_B$ for any $B \in \delta(\mathcal{E})$. $\square$

Definition 1.16 (Topology)**.** Let $\Omega \neq \emptyset$ be an arbitrary set. A class of sets $\tau \subset 2^{\Omega}$ is called a **topology** if it has the following three properties:

1. $\emptyset, \Omega \in \tau$
2. $A \cap B \in \tau$ for any $A, B \in \tau$
3. $\bigcup_{A \in \mathcal{F}} A \in \tau$ for any $\mathcal{F} \subset \tau$

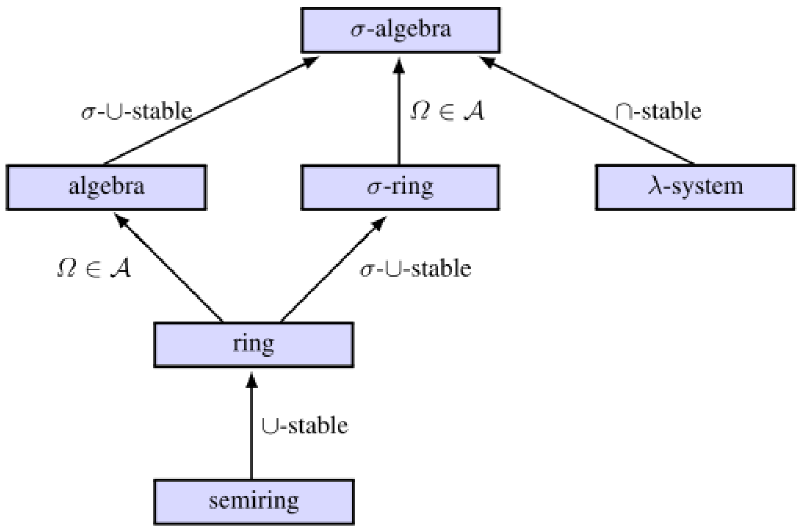

Figure 1: Inclusions between classes of sets $\mathcal{E} \subset 2^{\Omega}$

The pair (Ω, τ) is called a **topological space**. The sets $A \in \tau$ are called **open**, and the sets $A \subset \Omega$ with $A^c \in \tau$ are called closed

Let d be a metric on Ω, and denote the open ball with radius $r > 0$ centered at $x \in \Omega$ by

$$B_r(x) = \{y \in \Omega : d(x, y) < r\}$$

Then the usual class of open sets is the topology

$$\tau = \left\{ \bigcup_{(x,r) \in F} B_r(x) : F \subset \Omega \times (0, \infty) \right\}$$

Definition 1.17 (Borel σ-algebra)**.** Let (Ω, τ) be a topological space. The σ-algebra

$$\mathcal{B}(\Omega) := \mathcal{B}(\Omega, \tau) := \sigma(\tau)$$

that is generated by the open sets is called the **Borel σ-algebra** on Ω. The elements $A \in \mathcal{B}(\Omega, \tau)$ are called **Borel sets** or **Borel measuable sets**

For $a, b \in \mathbb{R}^n$, we write

$$a < b \quad \text{if } a_i < b_i \quad \text{for all } i = 1, \ldots, n$$

For $a < b$, we define the open **rectangle** as the Cartesian product

$$(a,b) := \bigtimes_{i=1}^{n} (a_i, b_i) := (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n x)$$

Definition 1.18 (Trace of a class of sets)**.** Let $\mathcal{A} \subset 2^\Omega$ be an arbitrary class of subsets of Ω and let $A \in 2^\Omega \backslash \{\emptyset\}$. The class

$$\mathcal{A}|_A := \{A \cap B : B \in \mathcal{A}\} \subset 2^A$$

is called the **trace** of $\mathcal{A}$ on A or the **restriction** of $\mathcal{A}$ on A

Theorem 1.19. *Let $A \subset \Omega$ be a nonempty set and let $\mathcal{A}$ be a σ-algebra on Ω (ring,semiring,). Then $\mathcal{A}|_A$ is a class of sets of the same type as $\mathcal{A}$; however on A instead of Ω. For λ-systems this is not true in general*

1.2 Set Functions

Definition 1.20. Let $\mathcal{A} \subset 2^\Omega$ and let $\mu : \mathcal{A} \to [0, \infty]$ be a set function. We say that μ is

1. **monotone** if $\mu(A) \leq \mu(B)$ for any two sets $A, B \in \mathcal{A}$ with $A \subset B$
2. **additive** if $\mu(\biguplus_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$ for any choice of finitely many mutually disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$ with $\bigcup_{i=1}^n A_i \in \mathcal{A}$
3. **σ-additive** if $\mu(\biguplus_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$ for any choice of countably many mutually disjoint sets $A_1, \dots \in \mathcal{A}$ with $\bigcup_{i=1}^\infty A_i \in \mathcal{A}$
4. **subadditive** if for any choice of finitely many sets $A, A_1, \ldots, A_n \in \mathcal{A}$ with $A \subset \bigcup_{i=1}^n A_i$, we have $\mu(A) \leq \sum_{i=1}^n \mu(A_i)$
5. σ**subadditive** if for any choice of countably many sets $A, A_1, \dots \in \mathcal{A}$ with $A \subset \bigcup_{i=1}^\infty A_i$, we have $\mu(A) \leq \sum_{i=1}^\infty \mu(A_i)$

Definition 1.21. Let $\mathcal{A}$ be a semiring and let $\mu : \mathcal{A} \to [0, \infty]$ be a set function with $\mu(\emptyset) = 0$, μ is called a

- **content** if μ is additive
- **premeasure** if μ is σ-additive
- **measure** if μ is a premeasure and $\mathcal{A}$ is a σ-algebra
- **probability measure** if μ is a measure and $\mu(\Omega) = 1$

Definition 1.22. Let $\mathcal{A}$ be a semiring. A content μ on $\mathcal{A}$ is called

1. **finite** if $\mu(A) < \infty$ for every $A \in \mathcal{A}$ and
2. **σ-finite** if there exists a sequence of sets $\Omega_1, \Omega_2, \ldots, \in \mathcal{A}$ s.t. $\Omega = \bigcup_{n=1}^\infty \Omega_n$ and s.t. $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$

Example 1.1 (Contents, measures)**.**

1. Let $\omega \in \Omega$ and $\delta_\omega(A) = \mathbb{1}_A(\omega)$. Then δ_ω is a probability measure on any σ-algebra $\mathcal{A} \subset 2^\Omega$. δ_ω is called the **Dirac measure** for the point ω
2. Let Ω be a finite nonempty set. By

$$\mu(A) := \frac{\#A}{\#B} \quad \text{for } A \subset \Omega$$

we define a probability measure on $\mathcal{A} = 2^\Omega$. This μ is called the **uniform distribution** on Ω. For this distribution, we introduce the symbol $\mathcal{U}_\Omega := \mu$. The resulting triple $(\Omega, \mathcal{A}, \mathcal{U}_\Omega)$ is called a **Laplace space**

Lemma 1.23 (Properties of contents)**.** *Let $\mathcal{A}$ be a semiring and let μ be a content on $\mathcal{A}$. Then the following statements hold.*

1. *If $\mathcal{A}$ is a ring, then $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$*
2. *μ is monotone. If $\mathcal{A}$ is a ring, then $\mu(B) = \mu(A) + \mu(B\mathbb{A})$ for any two sets $A, B \in \mathcal{A}$ with $A \subset B$*
3. *μ is subadditive. If μ is σ-additive, then μ is also σ-subadditive*
4. *If $\mathcal{A}$ is a ring then $\sum_{n=1}^{\infty} \mu(A_n) \leq \mu(\bigcup_{n=1}^{\infty} A_n)$ for any choice of countably many mutually disjoint sets $A_1, \dots \in \mathcal{A}$ with $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$*

#+END$_{\text{definition}}$