Measure Theory

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1 Introduction: a non-measurable set

Suppose we want a measure that satisfies:

0. $\lambda : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+ \cup \{+\infty\}$
1. $\lambda((a,b]) = b - a$
2. $A \subseteq \mathbb{R}, A + x = \{x + y : y \in A\}$

$$\forall A \subseteq \mathbb{R} \forall x \in \mathbb{R}, \lambda(A + x) = \lambda(A)$$

3. $A = \bigcup_{j\geq 1} A_j, A_j \cap A_k = \emptyset$

$$\lambda(A) = \sum_{k\geq 1} \lambda(A_k)$$

Define $x \sim y$ for $x, y \in \mathbb{R}$ if $y - x \in \mathbb{Q}$. $\Lambda = \mathbb{R}/\sim$ and $\alpha, \beta \in \Lambda$. Γ is uncountable since each equivalent class is countable.

By the **Axiom of Choice**, we have a $\Omega \subseteq \mathbb{R}$ s.t. for each $[x] \in \mathbb{R}/\sim$, there is a $x \in [x]$ s.t. $x \in \Omega$. Hence we can assume $\Omega \subseteq (0, 1)$.

Claim: For $p, q \in \mathbb{Q}$, either $\Omega + p = \Omega + q$ or $\Omega + p \cap \Omega + q = \emptyset$.

Proof. Assume $(\Omega + p) \cap (\Omega + q) \neq \emptyset$, $x = \alpha + p = \beta + q$. Hence $\alpha - \beta = q - p \in \mathbb{Q}$, which implies $\alpha = \beta$. □

Claim: $\Omega + q \subseteq (-1, 2)$ since $-1 < q < 1$.

In particular,

$$\bigcup_{\substack{q\in\mathbb{Q} \\ -1<q<1}} (\Omega + q) \subseteq (-1, 2)$$

Claim: If $E \subseteq F$, then $\lambda(E) \leq \lambda(F)$

Proof. $\lambda(F) = \lambda(E \cup (F - E)) = \lambda(E) + \lambda(F - E)$ □

If $q \neq p$,

$$\lambda(\bigcup_{\substack{q\in\mathbb{Q} \\ -1<q<1}} (\Omega + q)) = \sum_{\substack{q\in\mathbb{Q} \\ -1<q<1}} \lambda(\Omega + q) = \sum_{\substack{q\in\mathbb{Q} \\ -1<q<1}} \lambda(\Omega) \leq \lambda((-1, 2)) = 3$$

Hence $\lambda(\Omega) = 0$

Claim: $(0, 1) \subseteq \sum_{q\in\mathbb{Q},-1<q<1}(\Omega + q)$

Proof. Fix $x \in [0, 1]$, $\exists \alpha \in [x] \cap \Omega$ and $\alpha \in (0, 1)$. Hence $\alpha - x = q \in \mathbb{Q}$. Then $x \in \Omega + q$ □

Hence we have a contradiction and there is no such λ function.

2 Classes of subsets

Definition 2.1. For $\mathcal{S} \subseteq \mathcal{P}(\Omega)$, $\mathcal{S}$ is a **semi-algebra** if

1. $\Omega \in \mathcal{S}$
2. If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$
3. For all $A \in \mathcal{S}$, there are $E_1, \ldots, E_n \in \mathcal{S}$ s.t. $A^c = \sqcup E_j$

Example 2.1. If $\Omega = \mathbb{R}$ and

$$\begin{aligned}\mathcal{S} = \mathbb{R} &\cup \{(a, b] : a < b, a, b \in \mathbb{R}\} \\ &\cup \{(-\infty, b] : b \in \mathbb{R}\} \\ &\cup \{(a, \infty) : a \in \mathbb{R}\} \\ &\cup \emptyset\end{aligned}$$

then $\mathcal{S}$ is a semi-algebra

Definition 2.2. Take $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, $\mathcal{A}$ is an **algebra** if

1. $\Omega \in \mathcal{A}$
2. If $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$
3. If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$

If $\mathcal{A}$ is an algebra, then it is also semi-algebra.

Definition 2.3. $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a σ**-algebra** if

1. $\Omega \in \mathcal{F}$
2. If $A_j \in \mathcal{F}$ for $j \geq 1$, then $\bigcap_{j \geq 1} A_j \in \mathcal{F}$
3. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

Proposition 2.4. *Suppose $\mathcal{A}_\alpha \subseteq \mathcal{P}(\Omega)$, $\mathcal{A}_\alpha$ is an (σ-)algebra, $\alpha \in I$. Then $\mathcal{A} = \bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is an (σ-)algebra*

Definition 2.5. Suppose class $\mathcal{C} \subseteq \mathcal{P}(\Omega)$. An ($\sigma$-)algebra $\mathcal{A}(\mathcal{C})$ **generated by** $\mathcal{C}$ is the smallest (σ-)algebra s.t. for any (σ-)algebra $\mathcal{B} \supseteq \mathcal{C}$ and $\mathcal{B}$ is an (σ-)algebra, then $\mathcal{B} \supseteq \mathcal{A}$. Hence $\mathcal{A}(\mathcal{C}) = \bigcap_\alpha \mathcal{A}_\alpha$

Lemma 2.6. *Let $\mathcal{S}$ be a semi-algebra and $\mathcal{S} \subseteq \mathcal{P}(\Omega)$. $A \in \mathcal{A}(\mathcal{S})$ iff there exists $1 \leq j \leq n, E_j \in \mathcal{S}$ s.t.*

$$A = \sum_{j=1}^{n} E_j$$

Proof. Suppose $A = \sum_{j=1}^n E_j, E_j \in \mathcal{S} \subseteq \mathcal{A}(\mathcal{S})$

- If $E, F \in \mathcal{A}$ then $E \cup F \in \mathcal{A}$ since $E \cup F = (E^c \cap F^c)^c$

Suppose $A \in \mathcal{A}(\mathcal{S})$, let $\mathcal{B} = \{\sum_{j=1}^n F_j : F_j \in \mathcal{S}\} \subseteq \mathcal{P}(\Omega)$
- $\mathcal{B}$ is an algebra
- $\mathcal{B} \supseteq \mathcal{S}$

□

Definition 2.7. $\mathcal{C} \subseteq \mathcal{P}(\Omega), \emptyset \in \mathcal{C}$ and $\mu : \mathcal{C} \to \mathbb{R}_+ \cup \{+\infty\}$, μ is **additive** if
1. $\mu(\emptyset) = 0$
2. $E_1, \dots, E_n \in \mathcal{C}, E = \sum_{j=1}^n E_j \in \mathcal{C}$, then $\mu(E) = \sum_{j=1}^n \mu(E_j)$

Remark.
1. Assume there is a set $A \in \mathcal{C}, \mu(A) < \infty$, then $A = A \cup \emptyset$ and $\mu(\emptyset) = 0$
2. If $E \subseteq F$ and $F - E \in \mathcal{C}$, if $\mu(E) = +\infty$, $F = E \cup (F - E)$, hence $\mu(F) = \mu(E) + \mu(F - E) = +\infty$. If $\mu(E) < \infty, \mu(F - E) = \mu(F) - \mu(E)$. We can conclude $\mu(E) \leq \mu(F)$

Example 2.2. Discrete measure. Suppose we have $\{X_j : j \geq 1\}, \{P_j : j \geq 1\}$ and $\mu(A) = \sum_j P_j 1\{X_j \in A\}$ (indicator function), then μ is additive

Definition 2.8. $\emptyset \in \mathcal{C} \subseteq \mathcal{P}(\Omega), \mu : \mathcal{C} \to \mathbb{R}_+ \cup \{+\infty\}$. μ is σ**-additive** if
1. $\mu(\emptyset) = 0$
2. $E = \sum_{j \geq 1} E_j \in \mathcal{C}, \mu(E) = \sum_{j \geq 1} \mu(E_j)$

Example 2.3. $\Omega = (0, 1), \mathcal{C} = \{(a, b] : 0 \leq a < b < 1\}$

$$\mu((a, b]) = \begin{cases} +\infty & a = 0 \\ b - a & b > a \end{cases}$$

$(a, b] = \sum_{j=1}^n (a_j, b_j]$. μ is additive but not sigma-additive. If x_j converges to 0, $x_0 = 1/2$, then $\mu((0, 1/2]) = +\infty, \mu(\sum_{j \geq 1}(x_{j+1}, x_j]) = 1/2$

3 Set functions

Definition 3.1.
1. μ is **continuous from below** at E if for all $(E_n)_{n \geq 1}$, $E_n \in \mathcal{C}, E_n \subseteq E_{n+1}, \bigcup_{n \geq 1} E_n = E, \lim_{n \to \infty} \mu(E_n) = \mu(E)$
2. μ is **continuous from above** at E if $E_n \supseteq E_{n+1}, \bigcap_{n \geq 1} E_n = E, \exists n \; \mu(E_{n_0}) < \infty$, then $\lim_{n \to \infty} \mu(E_n) = \mu(E)$

Lemma 3.2. *Algebra $\mathcal{A} \subseteq \mathcal{P}(\Omega), \mu : \mathcal{A} \to \mathbb{R}_+ \cup \{+\infty\}$ additive, then*
1. *If μ is σ-additive, then μ is continuous at E for all $E \in \mathcal{A}$*
2. *If μ is continuous from below, then μ is σ-additive*
3. *If μ is continuous from above at $\emptyset$ and μ is finite, then μ is σ-additive*

Proof. 1. Suppose $E_n \uparrow E$, define $F_1 = E_1, F_2 = E_2 - E_1, \dots, F_n = E_n - E_{n-1}$, then $\bigcup E_n = \sum F_n = E$,$\mu(E) = \sum \mu(F_k) = \lim_{n\to\infty} \sum_{k=1}^n \mu(F_k) = \lim \mu(\sum F_k) = \lim \mu(E_n)$

Suppose $E, E_n \in \mathcal{A}, E_n \downarrow E, \mu(E_{n_0}) < \infty$. Define $G_1 = E_{n_0} - E_{n_0+1}, G_2 = E_{n_0} - E_{n_0+2}, \dots, G_k = E_{n_0} - E_{n_0+k}$, we know $G_k \uparrow E_{n_0} - E$. By the first part, $\mu(G_k) \uparrow \mu(E_{n_0} - E)$. $\mu(E_{n_0} - E) = \mu(E_{n_0}) - \mu(E) = \lim_k \mu(E_{n_0} - E_{n_0+k}) = \lim_k(\mu(E_{n_0}) - \mu(E_{n_0+k}))$

2. $\displaystyle\sum_{k=1}^n E_k \subseteq E$, $\mu(\sum E_k) \leq \mu(E)$, $\sum(\mu(E_k)) \leq \mu(E)$, hence $\sum \mu(E_k) \leq \mu(E)$. Let $F_n = \displaystyle\sum_{k=1}^n E_k \in \mathcal{A}$, $F_n \uparrow E$. σ-additivity follows.

3. $F_n = \sum_{k\geq n} E_k$. $\mu(E) = \mu(\displaystyle\sum_{k=1}^n E_k \cup \sum_{k>n} E_k) = \sum_{k=1}^n \mu(E_k) + \mu(F_{n+1})$ converges to $\displaystyle\sum_{k\geq 1} \mu(E_k)$

□

Example 3.1. $(a,b], 0 \leq a < b < 1$,

$$\mu((a,b]) = \begin{cases} b-a & a > 0 \\ +\infty & a = 0 \end{cases}$$

Take $E_n \downarrow \emptyset$, μ won't be finite in some cases

Theorem 3.3. *Suppose we have a semi-algebra $\mathcal{S} \subseteq \mathcal{P}(\Omega)$, μ is additive, there is a ν s.t. $\nu : \mathcal{A}(\mathcal{S}) \to \mathbb{R}_+ \cup \{+\infty\}$*

1. *ν is additive*
2. *$\nu(A)$=$\mu(A)$ for all $A \in \mathcal{S}$*
3. *If $\mu_1, \mu_2 : \mathcal{A}(\mathcal{S}) \to \mathbb{R}_+ \cup \{+\infty\}$,$\forall A \in \mathcal{S}, \mu_1(A) = \mu_2(A)$, then $\forall E \in \mathcal{A}(\mathcal{S}), \mu_1(E) = \mu_2(E)$*
4. *ν is σ-additive*

Proof. If $A \in \mathcal{A}(\mathcal{S})$, then $A = \displaystyle\sum_{j=1}^n E_j, E_j \in \mathcal{S}$. $\nu(A) = \displaystyle\sum_{j=1}^n \nu(E_j) = \sum_{j=1}^n \mu(E_j)$

1. ν is well-defined

   If $A = \displaystyle\sum_{k=1}^m F_k, F_k \in \mathcal{S}$, $E_j = E_j \cap \displaystyle\sum_{k=1}^m F_k = \sum_{k=1}^m E_j \cap F_k$ and $\mu(E_j) = \displaystyle\sum_{k=1}^m \mu(E_j \cap F_k)$. Hence $\nu(A) = \displaystyle\sum_{j=1}^n \sum_{k=1}^m \mu(E_j \cap F_k)$

2. ν is additive
3. unique
4. $A = \sum_{j\geq 1} A_j, A, A_j \in \mathcal{A}(\mathcal{S})$, $A = \sum_{j=1}^{n} E_j, E_j \in \mathcal{S}$, $A_k = \sum_{l=1}^{m_k} E_{k,l} \in \mathcal{S}$.

$$\begin{aligned} E_j &= E_j \cap A = E_j \cap (\sum_{k\geq 1} A_k) \\ &= E_j \cap (\sum_{k\geq 1}\sum_{l=1}^{m_k} E_{k,l}) = \sum_{k\geq 1}\sum_{l=1}^{m_k} E_j \cap E_{k,l} \end{aligned}$$

$\mu(E_j) = \sum_{k\geq 1}\sum_{l=1}^{m_k} \mu(E_j \cap E_{k,l})$ and $\nu(A) = \sum_{j=1}^{n} \mu(E_j) = \sum_{j=1}^{n}\sum_{k\geq 1}\sum_{l=1}^{m_k} \mu(E_j \cap E_{k,l})$.

$$E_{k,l} = E_{k,l} \cap A = \sum_{j=1}^{n} E_{k,l} \cap E_j$$

$\mu(E_{k,l}) = \sum_{j=1}^{n} \mu(E_{k,l} \cap E_j)$. Finish.

□