# Notes on Set Theory

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December 11, 2019

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# 1 Foreword

Notes for the entrance examination

# 2 Models of Set - Sertraline

## 2.1 Some mathematical logic

**Theorem 2.1** (Gödels second incompleteness theorem)**.** If a consistent recursive axiom set $T$ contains **ZFC** , then

$$T \nvdash \mathrm{Con}t$$

especially, **ZFC** $\nvdash$ Con**ZFC**

**Definition 2.2.** Suppose $(M, E_M)$ and $(N, E_N)$ are two models of set theory, then

1. if for any formula $\sigma$, $M \models \sigma$ if and only if $N \models \sigma$, then $M$ and $N$ are **elementary equivalent**, denoted by $M \equiv N$
2. If bijection $f : M \to N$ satisfies: for any $a, b \in M$, $aE_M b$ iff $f(a)E_N f(b)$, then $f : M \cong N$ is an **isomorphism**
3. If $M \subseteq N$ and $E_M = E_N \restriction M$, then $M$ is $N$'s submodel
4. If $M$ is isomorphic to a submodel of $N$ by injection $f$, and for any formula $\varphi(x_1, \ldots, x_n)$, for any $a_1, \ldots, a_n \in M$, $M \models \varphi[a_1, \ldots, a_n]$ iff $N \models \varphi[f(a_1), \ldots, f(a_n)]$, then $f$ is called an **elementary embedding** from $M$ to $N$, written as $f : M \prec N$
5. If $M \subseteq N$ and $M \prec N$, then $M$ is a **elementary submodel** of $N$

**Lemma 2.3.** Suppose $N \models$ **ZFC** , $M \subseteq N$, then $M \prec N$ iff $\forall \varphi(x, x_1, \ldots, x_n)$, $\forall (a_1, \ldots, a_n) \in M$, if $\exists a \in N$ s.t. $N \models \varphi[a, a_1, \ldots, a_n]$, then $\exists a \in M$ s.t. $M \models \varphi[a, a_1, \ldots, a_n]$

**Definition 2.4.** Suppose $(M, E) \models$ **ZFC**

1. $h_\varphi : M^n \to M$ is $\varphi$'s **Skolem function** if $\forall a_1, \ldots, a_n \in M$, if $\exists a \in M$ s.t. $M \models \varphi[a, a_1, \ldots, a_n]$, then $M \models \varphi[h_\varphi(a_1, \ldots, a_n), a_1, \ldots, a_n]$ . requires **AC**
2. Let $\mathcal{H} = \{h_\varphi \mid \varphi \text{is a formula on set theory}\}$. For any $S \subseteq M$, **Skolem hull** $\mathcal{H}(S)$ is the smallest set consisting of $S$ and closed under $\mathcal{H}$

**Lemma 2.5.** $N \models$ **ZFC** , $S \subseteq N$, if $M = \mathcal{H}(S)$, then $M \prec N$

**Theorem 2.6** (Löwenheim-Skolem theorem)**.** Suppose $N \models$ **ZFC** and is infinite, then there is a model $M$ s.t. $|M| = \omega$ and $M \prec N$

## 2.2 Cumulative Hierarchy

This section works in **ZF**$^-$(a.k.a. **ZF** $-$ axiom of foundation)

**Definition 2.7.** For any $\alpha$, define sequence $V_\alpha$

1. $V_0 = \emptyset$
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
3. For any limit ordinal $\lambda$, $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$

And **WF** $= \bigcup\limits_{\alpha \in \mathbf{On}} V_\alpha$

**Lemma 2.8.** For any ordinal $\alpha$

1. $V_\alpha$ is transitive
2. if $\xi \leq \alpha$, then $V_\xi \subseteq V_\alpha$
3. if $\kappa$ is inaccessible cardinal, then$|V_\kappa| = \kappa$

*Proof.* 1. Obviously $\kappa \leq V_\kappa$. Since $\kappa$ is inaccessible, then for any $\alpha < \kappa$, $|V_\alpha| < \kappa$.

□

**Definition 2.9.** For any set $x \in$ **WF** ,

$$\text{rank}(x) = \min\{\beta \mid x \in V_{\beta+1}\}$$

**Lemma 2.10.** 1. $V_\alpha = \{x \in \mathbf{WF} \mid \text{rank}(x) < \alpha\}$

2. **WF** is transitive
3. For any $x, y \in$ **WF** , if $x \in y$, then $\text{rank}(x) < \text{rank}(y)$
4. for any $y \in$ **WF** , $\text{rank}(y) = \sup\{\text{rank}(x) + 1 \mid x \in y\}$

**Lemma 2.11.** Supoose $\alpha$ is an ordinal

1. $\alpha \in$ **WF** and $\text{rank}(\alpha) = \alpha$

2. $V_\alpha \cap \mathbf{On} = \alpha$

**Lemma 2.12.** 1. If $x \in \mathbf{WF}$, then $\bigcup x, \mathcal{P}(x), \{x\} \in \mathbf{WF}$, and their ranks are all less than $\operatorname{rank}(x) + \omega$

2. If $x, y \in \mathbf{WF}$, then $x \times y, x \cup y, x \cap y, \{x, y\}, (x, y), x^y \in \mathbf{WF}$, and their ranks are all less than $\operatorname{rank}(x) + \operatorname{rank}(y) + \omega$

3. $\mathbb{Z}, \mathbb{Q}, \mathbb{R} \in V_{\omega+\omega}$

4. for any set $x$, $x \in \mathbf{WF}$ iff $x \subset \mathbf{WF}$

**Lemma 2.13.** Suppose **AC**

1. for any group $G$, there exists group $G' \cong G$ in **WF**

2. for any topological space $T$, there exists $T' \cong T$ in **WF**

**Definition 2.14.** Binary relation $<$ on set $A$ is **well-founded** if for any nonempty $X \subseteq A$, $X$ has minimal element under $<$

**Theorem 2.15.** If $A \in \mathbf{WF}$, then $\in$ is a well-founded relation on $A$

**Lemma 2.16.** If set $A$ is transitive and $\in$ is well-founded on $A$, then $A \in \mathbf{WF}$

**Lemma 2.17.** For any set $x$, there is a smallest transitive set $\operatorname{trcl}(x)$ s.t. $x \subseteq \operatorname{trcl}(x)$

*Proof.*

$$\begin{aligned} x_0 &= x \\ x_{n+1} &= \bigcup x_n \\ \operatorname{trcl}(x) &= \bigcup_{n<\omega} x_n \end{aligned}$$

□

$\operatorname{trcl}(x)$ is called **transitive closure** of $x$

**Lemma 2.18.** Without axiom of power set

1. if $x$ is transitive, then $\operatorname{trcl}(x) = x$

2. if $y \in x$, then $\operatorname{trcl}(y) \subseteq \operatorname{trcl}(x)$

3. $\operatorname{trcl}(x) = x \cup \bigcup \{\operatorname{trcl}(y) \mid y \in x\}$

**Theorem 2.19.** For any set $X$, the following are equivalent

1. $X \in \mathbf{WF}$
2. $\operatorname{trcl}(X) \in \mathbf{WF}$
3. $\in$ is a well-founded relation on $\operatorname{trcl}(X)$

**Theorem 2.20.** The following propositions are equivalent

1. Axiom of foundation
2. For any set $X$, $\in$ is a well-founded relation on $X$
3. $\mathbf{V} = \mathbf{WF}$

## 2.3 Relativization

**Definition 2.21.** Let $\mathbf{M}$ be a class $\varphi$ a formula, the **relativization** of $\varphi$ to $\mathbf{M}$ is $\varphi^{\mathbf{M}}$ defined inductively

$$\begin{aligned}(x \in y)^{\mathbf{M}} &\leftrightarrow x = y \\ (x \in y)^{\mathbf{M}} &\leftrightarrow x \in y \\ (\varphi \to \psi)^{\mathbf{M}} &\leftrightarrow \varphi^{\mathbf{M}} \to \psi^{\mathbf{M}} \\ (\neg\varphi)^{\mathbf{M}} &\leftrightarrow \neg\varphi^{\mathbf{M}} \\ (\forall x \varphi)^{\mathbf{M}} &\leftrightarrow (\forall x \in \mathbf{M})\varphi^{\mathbf{M}}\end{aligned}$$

Note $\varphi^{\mathbf{V}} = \varphi$ and

$$f^{\mathbf{M}} = \{(x_1, \dots, x_n, x_{n+1}) \in \mathbf{M} \mid \varphi^{\mathbf{M}}(x_1, \dots, x_n, x_{n+1})\}$$

**Definition 2.22.** For any theory $T$, any class $\mathbf{M}$, $\mathbf{M} \models T$ iff for any axiom $\varphi$ of $T$, $\varphi^{\mathbf{M}}$ holds

**Theorem 2.23 (ZF$^-$).** $\mathbf{WF} \models \mathbf{ZF}$

*Proof.*

- **Axiom of existence**

  $(\exists x(x = x))^{\mathbf{M}} \leftrightarrow \exists x \in \mathbf{M}\,(x = x)$, which is equivalent to $\mathbf{M}$ being nonempty

- **Axiom of extensionality**

$$\forall X \forall Y \forall u((u \in X \leftrightarrow u \in Y) \to X = Y)^{\mathbf{M}} \Leftrightarrow$$
$$\forall X \in \mathbf{M}\, \forall Y \in \mathbf{M}\, \forall u \in \mathbf{M}\,((u \in X \leftrightarrow u \in Y) \to X = Y)$$

**Lemma 2.24.** If **M** is transitive, then axiom of extensionality holds in **M**

- **Axiom schema of specification**

$$\forall X \in \mathbf{M}\, \exists Y \in \mathbf{M}\, \forall u \in \mathbf{M}\, (u \in Y \leftrightarrow u \in X \wedge \varphi^{\mathbf{M}}(u))$$

  Since for any $X \in$ **WF** , $\mathcal{P}(X) \subseteq$ **WF**

- **Axiom of paring**
- **Axiom of union**
- **Axiom of power set**

$$\forall X \in \mathbf{M}\, \exists Y \in \mathbf{M}\, \forall u \in \mathbf{M}\, (u \in Y \leftrightarrow (u \subseteq X)^{\mathbf{M}})$$

  and

$$(u \subseteq X)^{\mathbf{M}} \leftrightarrow \forall x \in \mathbf{M}\, (x \in u \to x \in X) \leftrightarrow u \cap \mathbf{M} \subseteq X$$

- **Axiom of foundation**
- **Axiom schema of replacement**

□

## 2.4 Absoluteness

**Definition 2.25.** For any formula $\psi(x_1, \ldots, x_n)$ and any class **M** ,**N** , **M** $\subseteq$ **N** , if

$$\forall x_1 \ldots \forall x_n \in \mathbf{M}\, (\psi^{\mathbf{M}}(x_1, \ldots, x_n) \leftrightarrow \psi^{\mathbf{N}}(x_1, \ldots, x_n))$$

then $\psi(x_1, \ldots, x_n)$ is **absolute** for **M** ,cn. If **N** $=$ **V** , then $\psi$ is **absolute** for **M**

**Lemma 2.26.** Suppose **M** $\subseteq$ **N** and $\varphi$,$\psi$ are formulas, then

1. if $\varphi$,$\psi$ are absolute for **M** ,cn, then so are $\neg\varphi, \varphi \to \psi$
2. if $\varphi$ doesn't contain any quantifiers, then $\varphi$ is absolute for any **M**
3. if **M** ,**N** are transitive and $\varphi$ is absolute for them, then so are $\forall x \in y\varphi$

**Definition 2.27.** $\Delta_0$ formula

1. $x = y, x \in y$ are $\Delta_0$ formulas

2. if $\varphi$,$\psi$ are $\Delta_0$, then so are $\neg\varphi, \varphi \to \psi$

3. if $\varphi$ is $\Delta_0$, $y$ is any set, then $(\forall x \in y)\varphi$ is $\Delta_0$

If $\varphi$ is $\Delta_0$, then $\exists x_1 \dots \exists x_n \varphi$ is $\Sigma_1$ formula, $\forall x_1 \dots \forall x_n \varphi$ is $\Pi_1$

**Lemma 2.28.** **M** $\subseteq$ **N** are both transitive, $\psi(x_0, \dots, x_n)$ is a formula, then

1. if $\psi$ is $\Delta_0$, then it's absolute for **M** ,cn

2. if $\psi$ is $\Sigma_1$, then

$$\forall x_1 \dots x_n (\psi^{\mathbf{M}} (x_1, \dots, x_n) \to \psi^{\mathbf{N}} (x_1, \dots, x_n))$$

3. if $\psi$ is $\Pi_1$, then

$$\forall x_1 \dots x_n (\psi^{\mathbf{N}} (x_1, \dots, x_n) \to \psi^{\mathbf{M}} (x_1, \dots, x_n))$$

**Lemma 2.29.** If **M** $\subseteq$ **N** , **M** $\models \Sigma$, **N** $\models \Sigma$ and

$$\Sigma \vdash \forall x_1 \dots \forall x_n (\varphi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n))$$

then $\varphi$ is absolute for **M** ,**N** if and only if $\psi$ is absolute for **M** ,**N**

**Definition 2.30.** Suppose **M** $\subseteq$ **N** , $f(x_1, \dots, x_n)$ is a function. $f$ is **absolute** for **M** and **N** if and only if $\varphi(x_1, \dots, x_n, x_{n+1})$ defining $f$ is absolute.

**Theorem 2.31.** Following relations and functions can be defined in **ZF**$^-$ $-$ Pow $-$ Inf and are equivalent to some $\Delta_0$ formulas. So they are absolute for any transitive model **M** on **ZF**$^-$ $-$ Pow $-$ Inf

1. $x \in y$

2. $x = y$

3. $x \subset y$

4. $\{x, y\}$

5. $\{x\}$

6. $(x, y)$

7. $\emptyset$

8. $x \cup y$

9. $x - y$

10. $x \cap y$

11. $x^+$

12. $x$ is a transitive set

13. $\bigcup x$

14. $\bigcap x$ ($\bigcap \emptyset = \emptyset$)

**Lemma 2.32.** Absoluteness is closed under operation composition

**Theorem 2.33.** Following relations and functions are absolute for any transitive model **M** on $\mathbf{ZF}^- - \mathrm{Pow} - \mathrm{Inf}$

1. $z$ is an ordered pair

2. $A \times B$

3. $R$ is a relation

4. $\mathrm{dom}(R)$

5. $\mathrm{ran}(R)$

6. $f$ is a function

7. $f(x)$

8. $f$ is injective

## 2.5 Relative consistence of the axiom of foundation

**Lemma 2.34.** Suppose transitive class $\mathbf{M} \models \mathbf{ZF}^- - \mathrm{Pow} - \mathrm{inf}$ and $\omega \in \mathbf{M}$, then the axiom of infinity is true in **M** . Hence the axiom of infinity is true in **WF**

**Theorem 2.35.** Let $T$ be a theory of set theory language and $\Sigma$ a set of sentences. Suppose **M** is a class and $T \vdash \mathbf{M} \neq \emptyset$, then if $\mathbf{M} \models_T \Sigma$, then

1. for any sentences $\varphi$, if $\Sigma \vdash \varphi$, then $T \vdash \varphi^{\mathbf{M}}$

2. if $T$ is consistent, then so is $\mathrm{Cn}(\Sigma)$

**Theorem 2.36.** The axiom of foundation is consistent with $\mathbf{ZF}^-$.

*Proof.* By 2.35, let T be $\mathbf{ZF}^-$, $\Sigma$ be $\mathbf{ZF}$ and $\mathbf{M}$ be $\mathbf{WF}$ $\square$

**Lemma 2.37** ($\mathbf{ZF}^-$)**.** Suppose transitive model $\mathbf{M} \models \mathbf{ZF}^- - \text{Pow} - \text{Inf}$. If $X, R \in \mathbf{M}$ and $R$ is a well-order on $X$, then

$$(R \text{ is a well-order on } X)^{\mathbf{M}}$$

**Theorem 2.38** ($\mathbf{ZF}^-$)**.** $V_\omega \models \mathbf{ZFC} - \text{Inf} + \neg\text{Inf}$

*Proof.* For any $X \in V_\omega$, $X$ is finite hence there is a well-ordering on $X$ $\square$

**Corollary 2.39.** $\text{Con}\mathbf{ZF}^- \to \text{Con}\mathbf{ZFC} - \text{Inf} + \neg\text{Inf}$

## 2.6 Induction and recursion based on well-order relation

**Definition 2.40.** $\boldsymbol{R}$ is a well-founded relation on $\boldsymbol{X}$ if and only if

$$\forall U \subset \boldsymbol{X}(U \neq \emptyset \to \exists y \in U(\neg\exists z \in U(z\boldsymbol{R}y)))$$

**Definition 2.41.** Relation $\boldsymbol{R}$ is **set-like** on $\boldsymbol{X}$ iff for any $x \in \boldsymbol{X}$,
$\{y \in \boldsymbol{X} \mid y\boldsymbol{R}x\}$ is a set

**Definition 2.42.** If $\boldsymbol{R}$ is a set-like relation on $\boldsymbol{X}$ and $x \in \boldsymbol{X}$, define

$$\begin{aligned} \text{pred}^0(\boldsymbol{X}, x, \boldsymbol{R}) &= \{y \in \boldsymbol{X} \mid y\boldsymbol{R}x\} \\ \text{pred}^{n+1}(\boldsymbol{X}, x, bR) &= \bigcup\{\text{pred}(\boldsymbol{X}, y, \boldsymbol{R}) \mid y \in \text{pred}^n(\boldsymbol{X}, x, \boldsymbol{R})\} \\ \text{cl}(\boldsymbol{X}, x, \boldsymbol{R}) &= \bigcup_{n\in\omega} \text{pred}^n(\boldsymbol{X}, x, \boldsymbol{R}) \end{aligned}$$

**Lemma 2.43.** If $\boldsymbol{R}$ is a set-like relation on $\boldsymbol{X}$, then for any $y \in \text{cl}(\boldsymbol{X}, x, \boldsymbol{R})$, $\text{pred}(\boldsymbol{X}, y, \boldsymbol{R}) \subseteq \text{cl}(\boldsymbol{X}, x, \boldsymbol{R})$

**Theorem 2.44** (Induction on well-founded set-like relation)**.** If $\boldsymbol{R}$ is a well-founded set-like relation on $\boldsymbol{X}$, then every nonempty $\boldsymbol{Y} \subseteq \boldsymbol{X}$ has minimal element under $\boldsymbol{R}$

**Theorem 2.45.** Suppose $\boldsymbol{R}$ is a well-founded set-like relation on $\boldsymbol{X}$. If $\boldsymbol{F} : \boldsymbol{X} \times \boldsymbol{V} \to \boldsymbol{V}$, then there is a unique $\boldsymbol{G} : \boldsymbol{X} \to \boldsymbol{V}$ s.t.

$$\forall x \in \boldsymbol{X}(\boldsymbol{G}(x) = \boldsymbol{F}(x, \boldsymbol{G}\restriction\text{pred}(\boldsymbol{X}, x, \boldsymbol{R})))$$

**Definition 2.46.** If $\boldsymbol{R}$ is a set-like well-founded relation on $\boldsymbol{X}$, define

$$\operatorname{rank}(x,\boldsymbol{X},\boldsymbol{R})=\sup\{\operatorname{rank}(y,\boldsymbol{X},\boldsymbol{R})+1\mid y\boldsymbol{R}x\wedge y\in\boldsymbol{X}\}$$

Note that

$$\boldsymbol{F}(x,h)=\sup\{\alpha+1\mid\alpha\in\operatorname{ran}(h)\}$$

**Lemma 2.47 ($\mathbf{ZF}^-$).** If $\boldsymbol{X}$ is transitive and $\in$ is well-founded on $\boldsymbol{X}$, then $\boldsymbol{X}\subseteq\mathbf{WF}$ and for any $x\in\boldsymbol{X}$, $\operatorname{rank}(x,\boldsymbol{X},\in)=\operatorname{rank}(x)$

**Definition 2.48.** $\boldsymbol{R}$ is a set-like well-founded relation on $\boldsymbol{X}$, **Mostowski function $\boldsymbol{G}$** on $(\boldsymbol{X},\boldsymbol{R})$ is

$$\boldsymbol{G}(x)=\{\boldsymbol{G}(y)\mid y\in\boldsymbol{X}\wedge y\boldsymbol{R}x\}$$

$\mathbf{M}=\operatorname{ran}(\boldsymbol{G})$ is called the **Mostowski collapse** of $(\boldsymbol{X},\boldsymbol{R})$

**Lemma 2.49.**

1. $\forall x,y\in\boldsymbol{X}(x\boldsymbol{R}y\to\boldsymbol{G}(x)\in\boldsymbol{G}(y))$
2. $\mathbf{M}$ is transitive
3. If the axiom of power set holds, $\mathbf{M}\subseteq\mathbf{WF}$
4. if the axiom of power set holds and $x\in\boldsymbol{X}$, then
   $\operatorname{rank}(x,\boldsymbol{X},\boldsymbol{R})=\operatorname{rank}(\boldsymbol{G}(x))$

**Definition 2.50.** $\boldsymbol{R}$ is extensional on $\boldsymbol{X}$ iff

$$\forall x,y\in\boldsymbol{X}(\forall z\in\boldsymbol{X}(z\boldsymbol{R}x\leftrightarrow z\boldsymbol{R}y)\to x=y)$$

**Lemma 2.51.** If $\boldsymbol{X}$ is transitive then $\in$ is extensional on $\boldsymbol{X}$

**Lemma 2.52.** Let $\boldsymbol{R}$ be a set-like well-founded relation on $\boldsymbol{X}$, $\boldsymbol{G}$ is a Mostowski function on it. If $\boldsymbol{R}$ is extensional, then $\boldsymbol{G}$ is an isomorphism

**Theorem 2.53** (Mostowski collapse theorem)**.** Suppose $\boldsymbol{R}$ is set-like well-founded extensional on $\boldsymbol{X}$, then there are unique transitive class $\mathbf{M}$ and bijection $\boldsymbol{G}:\boldsymbol{X}\to\mathbf{M}$ s.t. $\boldsymbol{G}:(\boldsymbol{X},\boldsymbol{R})\cong(\mathbf{M},\in)$

## 2.7 Absoluteness under the axiom of foundation

**Theorem 2.54.** The following relations and functions can be defined by formulas in **ZF** $-$ Pow and are equivalent to some $\Delta_0$ formulas

1. $x$ is an ordinal
2. $x$ is a limit ordinal
3. $x$ is a successor ordinal
4. $\omega$
5. $x$ is a finite ordinal
6. $0, 1, 2, \ldots, 20, \ldots$

**Theorem 2.55.** If transitive model $\mathbf{M} \models \mathbf{ZF} - \text{Pow}$, then every finite subset of $\mathbf{M}$ belongs to $\mathbf{M}$

*Proof.* prove

$$\forall x \subset \mathbf{M}\ (|x| = n \to x \in \mathbf{M}\ )$$

$\square$

**Theorem 2.56.** The following concepts are absolute for any transitive model of **ZF** $-$ Pow

1. $x$ is finite
2. $X^n$
3. $X^{<\omega}$
4. $R$ is a well-ordering on $X$
5. type$(X, R)$
6. $\alpha + 1$
7. $\alpha - 1$
8. $\alpha + \beta$
9. $\alpha \cdot \beta$

Class $\boldsymbol{X}$ is in fact a formula $\boldsymbol{X}(x)$. It's absolute for $\mathbf{M}$ if and only if $\forall x \in \mathbf{M}\,(\boldsymbol{X}^{\mathbf{M}}(x) \leftrightarrow \boldsymbol{X}(x))$, which is equivalent to $\{x \in \mathbf{M} \mid \boldsymbol{X}(x)\} = \{x \in \mathbf{M} \mid \boldsymbol{X}^{\mathbf{M}}(x)\}$. Hence $\boldsymbol{X}$ is absolute for $\mathbf{M}$ if and only if $\boldsymbol{X}^{\mathbf{M}} = \mathbf{M} \cap \boldsymbol{X}$

**Theorem 2.57.** Suppose $\boldsymbol{R}$ is a well-founded set-like relation on $\boldsymbol{X}$, $\boldsymbol{F} : \boldsymbol{X} \times \boldsymbol{V} \to \boldsymbol{V}$,

$$\forall x \in \boldsymbol{X}(\boldsymbol{G}(x) = \boldsymbol{F}(x, \boldsymbol{G}\restriction(\boldsymbol{X}, x, \boldsymbol{R})))$$

transitive model $\mathbf{M} \models \mathbf{ZF} - \mathrm{Pow}$ and

1. $\boldsymbol{F}$ is absolute for $\mathbf{M}$
2. $\boldsymbol{X}, \boldsymbol{R}$ are absolute for $\mathbf{M}$, $(\boldsymbol{R}$ is set-like on $\boldsymbol{X})^{\mathbf{M}}$ and

$$\forall x \in \mathbf{M}\,(\mathrm{pred}(\boldsymbol{X}, x, \boldsymbol{R}) \subseteq \mathbf{M})$$

then $\boldsymbol{G}$ is absolute for $\mathbf{M}$

**Theorem 2.58.** The following concept is absolute for any transitive model of $\mathbf{ZF} - \mathrm{Pow}$

1. $\alpha^{\beta}$
2. $\mathrm{rank}(x)$
3. $\mathrm{trcl}(x)$

**Lemma 2.59.** transitive $\mathbf{M} \models \mathbf{ZF}$

1. if $x \in \mathbf{M}$, then $\mathcal{P}^{\mathbf{M}}(x) = \mathcal{P}(x) \cap \mathbf{M}$
2. if $\alpha \in \mathbf{M}$, then $V_{\alpha}^{\mathbf{M}} = V_{\alpha} \cap \mathbf{M}$

## 2.8 Unaccessible cardinal and models of ZFC

$\boldsymbol{Z} = \mathbf{ZF} - \mathrm{Rep}, \mathbf{ZF}^{-} = \mathbf{ZFC} - \mathrm{Rep}$

**Theorem 2.60.** If $\gamma > \omega$ is a limit ordinal, then $V_{\gamma} \models_{\mathbf{ZF}} \boldsymbol{Z}$ and $V_{\gamma} \models_{\mathbf{ZFC}} \mathbf{ZC}$

**Corollary 2.61.** $V_{\omega+\omega}$ doesn't satisfies the axiom of replacement

*Proof.* □

**Theorem 2.62.** $\mathbf{ZC} \not\vdash \exists x(x = V_{\omega}), \mathbf{ZC} \not\vdash \forall x \exists y(\mathrm{trcl}(x) = y)$

**Theorem 2.63.** If $\kappa$ is an inaccessible cardinal, then $V_{\kappa} \models_{\mathbf{ZF}^{-}} \mathbf{ZF}$, $V_{\kappa} \models_{\mathbf{ZFC}^{-}} \mathbf{ZFC}$

*Proof.* Since $\kappa$ is inaccessible, $|V_\kappa| = \kappa$. For any $A \in V_\kappa$, $|A| < \kappa$. Since $\kappa$ is regular, any $f : A \to V_\kappa$ is bounded. Hence there exists $\alpha < \kappa$ s.t. $\mathrm{ran}(f) \subseteq V_\alpha$ □

**Corollary 2.64.** We cannot prove "there is some inaccessible cardinals" in **ZFC**

*Proof.* Suppose we could. Then we have $V_\kappa \models$ **ZFC** , which contradicts Gödels second incompleteness theorem □

**Lemma 2.65.** Suppose $\kappa$ is inaccessible. The following concepts are absolute for $V_\kappa$

1. $x$ is a cardinal
2. $x$ is a regular cardinal
3. $x$ is an inaccessible cardinal

**Lemma 2.66.** Con(**ZFC**) $\to$ Con(**ZFC** + "there is no inaccessible cardinal")

*Proof.* If $\kappa$ is the smallest inaccessible cardinal, then
$V_\kappa \models$ **ZFC** + "there is no inaccessible cardinal". Define

$$\mathbf{M} = \bigcap \{V_\kappa \mid \kappa \text{ is inaccessible}\}$$

□

If there are, then $\mathbf{M} = V_\kappa$

**Corollary 2.67.** Con(**ZFC**)$\neg\!\to$Con(**ZFC**+"there are some inaccessible cardinals")

**Definition 2.68.** For any infinite cardinal $\kappa$, $H_\kappa = \{x \mid |\mathrm{trcl}(x)| < \kappa\}$ is the collection of sets which **hereditarily have size less than** $\kappa$. Element of $H_\omega$ is called **hereditarily finite set**. Element of $H_{\omega_1}$ is called **hereditarily countable set**

**Lemma 2.69.** For any infinite cardinal $\kappa$, $H_\kappa \subseteq V_\kappa$

**Lemma 2.70.** If $\kappa$ is regular, then $H_\kappa = V_\kappa$ if and only if $\kappa$ is inaccessible

*Proof.* which implies $|V_\kappa| = \kappa$ □

**Lemma 2.71.** For any infinite cardinal $\kappa$

1. $H_\kappa$ is transitive

2. $H_\kappa \cap \mathbf{On} = \kappa$

3. If $x \in H_\kappa$, then $\bigcup x \in H_\kappa$

4. If $x, y \in H_\kappa$, then $\{x, y\} \in H_\kappa$

5. If $x \in H_\kappa, y \subseteq x$, then $y \in H_\kappa$

6. if $\kappa$ is regular, then $\forall x (x \in H_\kappa \leftrightarrow x \subset H_\kappa \wedge |x| < \kappa)$

**Theorem 2.72.** If $\kappa$ is uncountable regular cardinal, then $H_\kappa \models_{\mathbf{ZFC}} \mathbf{ZFC} -$ Pow

**Theorem 2.73.** If $\kappa$ is uncountable regular cardianl, then the following propositions are equivalent

1. $H_\kappa \models \mathbf{ZFC}$

2. $H_\kappa = V_\kappa$

3. $\kappa$ is inaccessible

**Corollary 2.74.** $\mathrm{Con}(\mathbf{ZFC}) \to \mathrm{Con}(\mathbf{ZFC} - \mathrm{pow} + \forall x (x \text{ is countable}))$

## 2.9 Reflection theorem

**Lemma 2.75.** $\mathbf{M} \subseteq \mathbf{N}$ are classes. $\varphi_1, \ldots, \varphi_n$ is a sequence closed under subformula, then the following propositions are equivalent

1. $\varphi_1, \ldots, \varphi_n$ are absolute for $\mathbf{M}$ and $\mathbf{N}$

2. if $\varphi_i = \exists \varphi_j(x, y_1, \ldots, y_m)$, then

   $$\forall y_1, \ldots, y_m \in \mathbf{M} \, (\exists x \in \mathbf{N} \, \varphi_j^{\mathbf{N}} \, (x, y_1, \ldots, y_m) \to \exists x \in \mathbf{M} \, \varphi_j^{\mathbf{M}} \, (x, y_1, \ldots, y_m))$$

**Theorem 2.76** (reflection theorem(**ZF**))**.** For any finite formula set $F = \{\varphi_1, \ldots, \varphi_n\}$, for any $V_\alpha$, there exists $V_\beta$ s.t. $V_\alpha \subseteq V_\beta$ and $\varphi_1, \ldots, \varphi_n$ are absolute for $V_\beta$

**Corollary 2.77 (ZF).** $F = \{\sigma_1, \ldots, \sigma_n\}$ are finite subsets of **ZF** , then

$$\forall \alpha \exists \beta > \alpha (\sigma_1^{V_\beta} \wedge \cdots \wedge \sigma_n^{V_\beta})$$

**Corollary 2.78.** $F = \{\sigma_1, \ldots, \sigma_n\}$ is a finite subset of **ZF** . Unless **ZF** is unconsistent, $F$ cannot prove all axioms of **ZF**

**Theorem 2.79 (ZFC).** For any finite formula set $F = \{\varphi_1, \dots, \varphi_n\}$, for any set $N$, there exists set $M$ s.t.

1. $N \subseteq M$
2. $\varphi_1, \dots, \varphi_n$ are absolute for $(M, \in)$
3. $|M| \leq |N| \cdot \omega$

**Corollary 2.80 (ZFC).** For any finite formula set $F = \{\varphi_1, \dots, \varphi_n\}$, for any set $N$, there exists set $M$ s.t.

1. $N \subseteq M$
2. $\varphi_1, \dots, \varphi_n$ are absolute for $(M, \in)$
3. $|M| \leq |N| \cdot \omega$
4. $M$ is transitive

# 3 Constructable Set - Venlafaxine

## 3.1 Definablity and Gödel operation

**Definition 3.1.** $M$ is a set, $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a formula, $X \subseteq M^n$ is **definable in $M$ from parameters from $\psi$** if and only if there are $y_1, \ldots, y_m \in M$ s.t.

$$X = \{(x_1, \ldots, x_n) \mid (\psi^M(x_1, \ldots, x_n, y_1, \ldots, y_m))\}$$

$$\text{Def}(M) = \{X \subseteq M \mid \exists\psi, X \text{ is definable in } M \text{ from } \psi\}$$

**Definition 3.2. Gödel operation**

1. $G_1(X, Y) = \{X, Y\}$
2. $G_2(X, Y) = X \times Y$
3. $G_3(X, Y) = \in \restriction X \times Y$
4. $G_4(X, Y) = X - Y$
5. $G_5(X, Y) = X \cap Y$
6. $G_6(X, Y) = \bigcap X$
7. $G_7(X, Y) = \text{dom}(X)$
8. $G_8(X, Y) = \{(x, y) \mid (y, x) \in X\}$
9. $G_9(X, Y) = \{(x, y, z) \mid (x, z, y) \in X\}$
10. $G_{10}(X, Y) = \{(x, y, z) \mid (y, z, x) \in X\}$

Class $C$ is closed under Gödel operation if for any $X, Y$, X,Y∈ C$ implies $G_i(X, Y) \in C$. For any set $M$, $\text{cl}_G(M)$ is the **closure under Gödel operation**

**Definition 3.3.** $\psi$ is a **normal form** if

1. only $\neg, \wedge, \exists$ are logical symbol
2. = doesn't appear
3. if $x_i \in x_j$ then $i \neq j$
4. $\exists$ only shown as: $\exists x_{m+1} \in x_i \varphi(x_1, \ldots, x_{m+1})$, $1 \leq i \leq m$

**Lemma 3.4.** Any $\Delta_0$ formula can be transformed into normal form

**Theorem 3.5.** For any $\Delta_0$ formula $\psi(x_1, \dots, x_n)$, there is Gödel operations' composition $G$ s.t. for any $X_1, \dots, X_n$

$$\begin{aligned} G(X_1, \dots, X_n) = &\{(x_1, \dots, x_n) \mid \\ & x_1 \in X_1 \land \cdots \land x_n \in X_n \land \psi(x_1, \dots, x_n)\} \end{aligned}$$

**Corollary 3.6.** If $M$ is transitive and $M = \mathrm{cl}_G(M)$, then for any $\Delta_0$ formula $\psi(x, y_1, \dots, y_m)$, any set $X \in M$, any $y_1, \dots, y_m \in M$ if

$$Y = \{x \in X \mid \psi(x, y_1, \dots, y_m)\}$$

then $Y \in M$. Hence $\Delta_0$ schema of specification holds in $M$

**Lemma 3.7.** If $G(X_1, \dots, X_n)$ is Gödel operations' composition, then $Z = G(X_1, \dots, X_n)$ is equivalent to a $\Delta_0$ formula

**Theorem 3.8.** For any transitive set $M$, $\mathrm{Def}(M) = \mathrm{cl}_G(M \cup \{M\}) \cap \mathcal{P}(M)$

**Lemma 3.9.** If transitive $\mathbf{M} \models \mathbf{ZF}$, then for any transitive set $M \in \mathbf{M}$, $\mathrm{Def}(M)$ is absolute for $\mathbf{M}$

**Lemma 3.10.** For any transitive set $M$

1. $\mathrm{Def}(M) \subseteq \mathcal{P}(M)$
2. $M \subseteq \mathrm{Def}(M)$
3. for any $X \subseteq M$, if $X$ is finite, then $X \in \mathrm{Def}(M)$
4. assume **AC** and $|M| \geq \omega$, then $\big|\mathrm{Def}(M)\big| = |M|$

## 3.2 Gödel's L

**Definition 3.11.** for any $\alpha$

1. $L_0 = \emptyset$
2. $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$
3. For any limit $\alpha$, $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$

$\mathbf{L} = \bigcup\limits_{\alpha \in \mathbf{On}} L_\alpha$. Element of $\mathbf{L}$ is called constructible set

**Lemma 3.12.** For any ordinal $\alpha$

1. $L_\alpha$ is transitive

2. If $\alpha < \beta$, then $L_\alpha \subseteq L_\beta$

3. $L_\alpha \subseteq V_\alpha$

**Definition 3.13.** $x \in \mathbf{L}$

$$\mathrm{rank}_{\mathbf{L}}(x) = \min\{\beta \mid x \in \mathbf{L}_{\beta+1}\}$$

**Lemma 3.14.** For any $\alpha$

$$L_\alpha = \{x \in \mathbf{L} \mid \mathrm{rank}_{\mathbf{L}}(x) < \alpha\}$$

**Lemma 3.15.** For any ordinal $\alpha$

1. $L_\alpha \cap \mathbf{On} = \alpha$

2. $\alpha \in \mathbf{L} \cap \mathrm{rank}_{\mathbf{L}}(\alpha) = \alpha$

*Proof.* since "$\alpha$ is a cardinal" is absolute for any transitive set.

$$\begin{aligned} \alpha = L_\alpha \cap \mathbf{On} &= \{\eta \in L_\alpha \mid \eta \text{ is a ordinal}\} \\ &= \{\eta \in L_\alpha \mid (\eta \text{ is an ordinal}^{L_\alpha})\} \in \mathrm{Def}(L_\alpha) \end{aligned}$$

□

**Lemma 3.16.** for any ordinal $\alpha$

1. $L_\alpha \in L_{\alpha+1}$

2. any finite subset of $L_\alpha$ belongs to $L_{\alpha+1}$

**Lemma 3.17.** 1. $\forall n \in \omega (L_n = V_n)$

2. $L_\omega = V_\omega$

**Lemma 3.18.** If **AC**, then for any $\alpha \geq \omega$, $|L_\alpha| = |\alpha|$

**Theorem 3.19.** $\mathbf{L} \models \mathbf{ZF}$

### 3.3 Axiom of constructibility and relativization

**Theorem 3.20** (Axiom of constructibility)**.** $\mathbf{V} = \mathbf{L}$

**Lemma 3.21.** function $\alpha \mapsto L_\alpha$ is absolute for any transitive model of **ZF**

**Theorem 3.22.** $\mathbf{L} \models \mathbf{ZF} + \mathbf{V} = \mathbf{L}$

*Proof.* $(\mathbf{V} = \mathbf{L})^{\mathbf{L}}$ is $\forall x \in \mathbf{L} \exists \alpha \in \mathbf{L}(x \in L_\alpha)^{\mathbf{L}}$. By 3.21, $(x \in L_\alpha)^{\mathbf{L}} \Leftrightarrow x \in L_\alpha$. Hence $\mathbf{L} \models \mathbf{V} = \mathbf{L}$ □

Hence

**Theorem 3.23.** $\mathrm{Con}(\mathbf{ZF}) \to \mathrm{Con}(\mathbf{ZF} + \mathbf{V} = \mathbf{L})$

**Theorem 3.24.** Suppose transitive proper class $\mathbf{M} \models \mathbf{ZF} - \mathrm{Pow}$, then $\mathbf{L} = \mathbf{L}^{\mathbf{M}} \subseteq \mathbf{M}$

*Proof.* For any ordinal $\alpha$, since $\mathbf{M}$ is proper, $\mathbf{M} \not\subseteq V_\alpha$. Hence there is $x \in \mathbf{M}$ s.t. $\mathrm{rank}(x) \geq \alpha$. Since rank is absolute, $\mathrm{rank}(x) \in \mathbf{M}$. And $\mathbf{M}$ is transitive, hence $\alpha \in \mathbf{M}$. By 3.21, $L_\alpha \in \mathbf{M}$

$$\begin{aligned}
\mathbf{L}^{\mathbf{M}} &= \{x \in \mathbf{M} \mid (\exists \alpha \in \mathbf{On}\ (x \in L_\alpha))^{\mathbf{M}}\} \\
&= \{x \mid \exists \alpha \in \mathbf{On} \cap \mathbf{M}(x \in L_\alpha \cap \mathbf{M})\} \\
&= \{x \mid \exists \alpha \in \mathbf{On}\ (x \in L_\alpha)\} \\
&= \mathbf{L}
\end{aligned}$$

□

**Definition 3.25.** If transitive model $\mathbf{M} \models \mathbf{ZF}$ contains all ordinals, then it's an **inner model**

**Lemma 3.26.** there is a finite set of axioms $\{\psi_1, \dots, \psi_n\}$ of $\mathbf{ZF} - \mathrm{Pow}$ s.t. ordinals,rank and $L_\alpha$ are absolute for any model of $\{\psi_1, \dots, \psi_n\}$

**Lemma 3.27.** If set $M$ is transitive, then $M \cap \mathbf{On}$ is a ordinall and is the least that doesn't belong to $M$, denoted by $\alpha^M$

**Theorem 3.28.** There is a finite subset $\{\psi_1, \dots, \psi_n\}$ of axioms of $\mathbf{ZF} - \mathrm{Pow}$ satisfying

$$\forall M(M \text{ is transitive } \wedge \psi_1^M \wedge \dots \wedge \psi_n^m \to (L_{\alpha^M} = \mathbf{L}^M \subseteq M))$$

**Theorem 3.29.** The is a finite subset $\{\psi_1, \dots, \psi_{n+1}\}$ of axioms of $\mathbf{ZF} - \mathrm{Pow} + \mathbf{V} = \mathbf{L}$ satisfying

1. If **M** is a transitive proper class and $\psi_1^{\mathbf{M}} \wedge \cdots \wedge \psi_{n+1}^{\mathbf{M}}$, then $\mathbf{M} = \mathbf{L}$

2. $\forall M(M$ is transitive $\wedge\, \psi_1^M \wedge \cdots \wedge \psi_n^m \to (L_{\alpha^M} = M))$

**Theorem 3.30.** There is a well-ordering on **L**. Hence $\mathbf{V} = \mathbf{L} \to \mathbf{AC}$

If $\mathbf{V} = \mathbf{L}$, hence $\aleph_\alpha \subseteq L_{\aleph_{\alpha+1}}$. Because $\left|L_{\alpha_{\alpha+1}}\right| = \aleph_{\alpha+1}$, $2^{\aleph_\alpha} \leq \aleph_{\alpha+1}$

**Theorem 3.31.** If $\mathbf{V} = \mathbf{L}$, then for any infinite ordinal $\alpha$, $\mathcal{P}(L_\alpha) \subseteq L_{|\alpha|^+}$

**Corollary 3.32 (ZF).** $(\mathbf{AC} + \mathbf{GCH})^{\mathbf{L}}$

**Theorem 3.33 (ZF).** $\mathrm{Con}(\mathbf{ZF}) \to \mathrm{Con}(\mathbf{ZFC} + \mathbf{GCH})$

**Theorem 3.34 (ZF).** Suppose $S_0 = \{\psi_1, \dots, \psi_n\} \subseteq \mathbf{ZF} + \mathbf{V} = \mathbf{L}$, then

$$\mathbf{ZF} \vdash \exists M(|M| = \omega \wedge M \text{ is transitive} \wedge (\psi_1^M \wedge \cdots \wedge \psi_n^M))$$

**Lemma 3.35.** Suppose $\mathbf{V} = \mathbf{L}$. For any uncountable regular cardinal $\kappa$, $L_\kappa = H_\kappa$

**Corollary 3.36.** If $\kappa$ is a uncountable regular cardinal, then ${}_\kappa \models \mathbf{ZF} - \mathrm{Pow} + \mathbf{V} = \mathbf{L}$. If $\kappa$ is inaccessible, then $L_\kappa \models \mathbf{ZF} + \mathbf{V} = \mathbf{L}$

## 4 The end

Learn and forget