Real Analysis: Measure Theory, Integration, and Hilbert Spaces

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1 Measure Theory

1.1 Preliminaries

The **norm** of x is denoted by $|x|$ and is defined to be the standard Euclidean norm given by

$$|x| = (x_1^2 + \cdots + x_d^2)^{1/2}$$

The **distance** between two points x and y is then simply $|x - y|$

The **distance** between two sets E and F is defined by

$$d(E, F = \inf|x - y|)$$

where the infimum is taken over all $x \in E$ and $y \in F$

The **open ball** in $\mathbb{R}^d$ centered at x and of radius r is defined by

$$B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r\}$$

A subset $E \subset \mathbb{R}^d$ is **open** if for every $x \in E$, there exists $r > 0$ with $B_r(x) \subset E$. A set is **closed** if its complement is open.

A set E is **bounded** if it's contained in some ball of finite radius. A bounded set is **compact** if it's also closed. Compact sets enjoy the Heine-Borel covering property:

- Assume E is compact, $E \subset \bigcup_\alpha \mathcal{O}_\alpha$, and each $\mathcal{O}_\alpha$ is open. Then there are finitely many of the open sets $\mathcal{O}_{\alpha_1}, \ldots, \mathcal{O}_{\alpha_N}$ s.t. $E \subset \bigcup_{j=1}^N \mathcal{O}_{\alpha_j}$

Lemma 1.1. *If $R, R_1, \ldots, R_N$ are rectangles, and $R \subset \bigcup_{k=1}^N R_k$, then*

$$|R| \leq \sum_{k=1}^N |R_k|$$

Theorem 1.2. *Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals*

Proof. For every $x \in \mathcal{O}$, let

$$a_x = \inf\{a < x : (a, x) \subset \mathcal{O}\} \quad b_x = \sup\{b > x : (x, b) \subset \mathcal{O}\}$$

and $I_x = (a_x, b_x)$. Then $\mathcal{O} = \bigcup_{x \in \mathcal{O}} I_x$. Now suppose that two intervals I_x and I_y intersects. Then $(I_x \cup I_y) \subset I_x$ and $(I_x \cup I_y) \subset I_x$. This can happen only if $I_x = I_y$. Therefore any two disjoint intervals in the collection $\mathcal{I} = \{I_x\}_{x \in \mathcal{O}}$. Since every open interval I_x contains a rational number. $\square$

Theorem 1.3. *Every open subset $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.*

1.2 The exterior measure

Definition 1.4. If E is *any* subset of $\mathbb{R}^d$, the **exterior measure** of E is

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$

where the infimum is taken over all countable coverings $E \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes .

Example 1.1. The exterior measure of a point is zero. This is clear once we observe that a point is a cube with volume zero.

Example 1.2. The exterior measure of a closed cube is equal to its volume. Indeed suppose Q is a closed cube in $\mathbb{R}^d$. Since Q covers itself, we must have $m_*(Q) \leq |Q|$. Therefore, it suffices to prove the reverse inequality.

We consider an arbitrary covering $Q \subset \bigcup_{j=1}^{\infty} Q_j$ by cubes, and note that it suffices to prove that

$$|Q| \leq \sum_{j=1}^{\infty} |Q_j|$$

For a fixed $\epsilon > 0$ we choose for each j an open cube S_j which contains Q_j and s.t. $|S_j| \leq (1+\epsilon)|Q_j|$. From the open covering $\bigcup_{j=1}^{\infty} S_j$ of the compact set Q, we may select a finite subcovering which, after possibly renumbering the rectangles, we may write as $Q \subset \bigcup_{j=1}^{N} S_j$. We may apply Lemma 1.1 to conclude that $|Q| \leq \sum_{j=1}^{N} |S_j|$. Consequently,

$$|Q| \leq (1+\epsilon) \sum_{j=1}^{N} |Q_j| \leq (1+\epsilon) \sum_{j=1}^{\infty} |Q_j|$$

Since ϵ is arbitrary, the inequality holds; thus $|Q| \leq m_*(Q)$

Example 1.3. If Q is an open cube, the result $m_*(Q) = |Q|$ still holds. Since Q is covered by its closure $\overline{Q}$ and $\left|\overline{Q}\right| = |Q|$, we immediately see that $m_*(Q) \leq |Q|$. Note that if Q_0 is a closed cube contained in Q, then $m_*(Q_0) \leq m_*(Q)$, since any covering of Q by a countable number of closed cubes is also a covering of Q_0. Hence $|Q_0| \leq m_*(Q)$, and since we can choose Q_0 with a volume as close as we wich to $|Q|$, we must have $|Q| \leq m_*(Q)$

Example 1.4. The exteriormeasure of a rectangle R is equal to its volume. To obtain $|R| \leq m_*(R)$, consider a grid in $\mathbb{R}^d$ formed by cubes of side length

$1/k$. Then if $\mathcal{Q}$ consists of the (finite) collection of all cubes entirely contained in R, and $\mathcal{Q}'$ the (fintie) collection of all cubes that intersect the complement of R, we first note that $R \subset \bigcup_{Q \in (\mathcal{Q} \cup \mathcal{Q}')} Q$. Also a simple argument yields

$$\sum_{Q \in \mathcal{Q}} |Q| \leq |R|$$

Moreover, there are $O(k^{d-1})$ cubes in $\mathcal{Q}'$ and these cubes have volume k^{-d}, so that $\sum_{Q \in \mathcal{Q}'} |Q| = O(1/k)$. Hence

$$\sum_{Q \in \mathcal{Q} \cup \mathcal{Q}'} |Q| \leq |R| + O(1/k)$$

and letting k tend to infinity yields $m_*(R) \leq |R|$

Example 1.5. The exterior measure of $\mathbb{R}^d$ is infinite. This follows from the fact that any covering of $\mathbb{R}^d$ is also a covering of any cube $Q \subset \mathbb{R}^d$ hence $|Q| \leq m_*(\mathbb{R}^d)$.

Example 1.6. The Cantor set $\mathcal{C}$ has exterior measure 0. From the construction of $\mathcal{C}$, we know that $\mathcal{C} \subset C_k$, where each C_k is a dijoint union of 2^k closed intervals, each of length 3^{-k}. Consequently, $m_*(\mathcal{C}) \leq (2/3)^k$ for all k, hence $m_*(\mathcal{C}) = 0$

Proposition 1.5. *For every $\epsilon > 0$, there exists a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$ with*

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon$$

Proposition 1.6 (Monotonicity)**.** *If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$*

Proposition 1.7 (Countable sub-additivity)**.** *If $E = \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$*

Proof. First we may assume that each $m_*(E_j) < \infty$ for otherwise the inequality clearly holds. For any $\epsilon > 0$ the definition of the exterior measure yields for each j a covering $E_j \subset \bigcup_{k=1}^{\infty} Q_{k,j}$ by closed cubes with

$$\sum_{k=1}^{\infty} |Q_{k,j}| \leq m_*(E_j) + \frac{\epsilon}{2^j}$$

Then, $E \subset \bigcup_{j,k=1}^{\infty} Q_{k,j}$ is a covering of E by closed cubes and therefore

$$\begin{aligned} m_*(E) \leq \sum_{j,k} |Q_{k,j}| &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |Q_{k,j}| \\ &\leq \sum_{j=1}^{\infty} (m_*(E_j) + \frac{\epsilon}{2^j}) \\ &= \sum_{j=1}^{\infty} m_*(E_j) + \epsilon \end{aligned}$$

□

Proposition 1.8. *If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf m_*(\mathcal{O})$ where the infimum is taken over all open sets $\mathcal{O}$ containing E*

Proof. By monotonicity, it is clear that $m_*(E) \leq \inf m_*(\mathcal{O})$ holds. For the reverse inequality, let $\epsilon > 0$ and choose cubes Q_j s.t. $E \subset \bigcup_{j=1}^{\infty} Q_j$ with

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \frac{\epsilon}{2}$$

Let Q_j^0 denote an open cube containing Q_j, and s.t. $\left|Q_j^0\right| \leq |Q_j| + \frac{\epsilon}{2^{j+1}}$. Then $\mathcal{O} = \bigcup_{j=1}^{\infty} Q_j^0$ is open, and by Proposition 1.7

$$\begin{aligned} m_*(\mathcal{O}) \leq \sum_{j=1}^{\infty} m_*(Q_j^0) &= \sum_{j=1}^{\infty} \left|Q_j^0\right| \\ &\leq \sum_{j=1}^{\infty} (|Q_j| + \frac{\epsilon}{2^{j+1}}) \\ &\leq \sum_{j=1}^{\infty} |Q_j| + \frac{\epsilon}{2} \\ &\leq m_*(E) + \epsilon \end{aligned}$$

□

Proposition 1.9. *If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$, then*

$$m_*(E) = m_*(E_1) + m_*(E_2)$$

Proof. By Proposition 1.7, we already know that $m_*(E) \leq m_*(E_1)+m_*(E_2)$. First select $d(E_1, E_2) > \delta > 0$. Next we choose a covering $E \subset \bigcup_j = 1^\infty Q_j$ by closed cubes with $\sum_{j=1}^\infty |Q_j| \leq m_*(E) + \epsilon$. We may, after subdividing the cubes Q_j, assume that each Q_j has a diameter less than δ. In this case, each Q_j can intersect at most one of the two sets E_1 or E_2. If we denote by J_1 and J_2 the sets of those indices j for which Q_j intersects E_1 and E_2, respectively, then $J_1 \cap J_2$ is empty, and we have

$$E_1 \subset \bigcap_{j \in J_1}^{\infty} Q_j \quad \text{as well as} \quad E_2 \subset \bigcap_{j \in J_2}^{\infty} Q_j$$

Therefore,

$$\begin{aligned} m_*(E_1) + m_*(E_2) &\leq \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j| \\ &\leq \sum_{j=1}^{\infty} |Q_j| \\ &\leq m_*(E) + \epsilon \end{aligned}$$

□

Proposition 1.10. *If a set E is the countable union of almost disjoint cubes $E = \bigcup_{j=1}^\infty Q_j$, then*

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j|$$

Proof. Let $\tilde{Q}_j$ dentoe a cube strictly contained in Q_j s.t. $|Q_j| \leq |Q_j| + \epsilon/2^j$, where ϵ is arbitrary but fixed. Then for every N, the cubes $\tilde{Q}_1, \dots, \tilde{Q}_N$ are disjoint, hence at a finite distance from one another, and repeated applications of Proposition 1.9 imply

$$m_*(\bigcup_{j=1}^{N} \tilde{Q}_j) = \sum_{j=1}^{N} \left|\tilde{Q}_j\right| \geq \sum_{j=1}^{N} (|Q_j| - \epsilon/2^j)$$

Since $\bigcup_{j=1}^N \tilde{Q}_j \subset E$, we conclude that for every integer N,

$$m_*(E) \geq \sum_{j=1}^{N} |Q_j| - \epsilon$$

In the limit as N tends to infinity we deduce $\sum_{j=1}^\infty |Q_j| \leq m_*(E) + \epsilon$ for every $\epsilon > 0$ □

1.3 Measurable sets and the Lebesgue measure

Definition 1.11. A subset E of $\mathbb{R}^d$ is **Lebesgue measurable** or simply **measurable**, if for any $\epsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and

$$m_*(\mathcal{O} - E) \leq \epsilon$$

If E is measurable, we define its **Lebesgue measure** (or **measure**) $m(E)$ by

$$m(E) = m_*(E)$$

Proposition 1.12. *Every open set in $\mathbb{R}^d$ is measurable*

Proof. $m_*(E - E) = 0 \leq \epsilon$ □

Proposition 1.13. *If $m_*(E) = 0$, then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable.*

Proof. By proposition 1.8, for every $\epsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m_*(\mathcal{O}) \leq \epsilon$. Since $(\mathcal{O} - E) \subset \mathcal{O}$, monotonicity implies $m_*(\mathcal{O} - E) \leq \epsilon$ □

As a consequence, the Cantor set $\mathcal{C}$ is measurable.

Proposition 1.14. *A countable union of measurable sets is measurable*

Proof. Suppose $E = \bigcup_{j=1}^{\infty} E_j$ where each E_j is measurable. Given $\epsilon > 0$, we may choose for each j an open set $\mathcal{O}_j$ with $E_j \subset \mathcal{O}_j$ and $m_*(\mathcal{O}_j - E_j) \leq \epsilon/2^j$. Then the union $\mathcal{O} = \bigcup_{j=1}^{\infty} \mathcal{O}_j$ is open, $E \subset \mathcal{O}$ and $\mathcal{O} - E \subset \bigcup_{j=1}^{\infty}(\mathcal{O}_j - E_j)$, so monotonicity and sub-additivity of the exterior measure imply

$$m_*(\mathcal{O} - E) \leq \sum_{j=1}^{\infty} m_*(\mathcal{O}_j - E_j) \leq \epsilon$$

□

Proposition 1.15. *Closed sets are measurable*

Proof. First we observe that it suffices to prove that compact sets are measurable. Indeed any closed set F can be written as the union of compact sets, say $F = \bigcup_{k=1}^{\infty} F \cap B_k$, where B_k denotes the closed ball of radius k centered at the origin; then Proposition 1.14 applies.

So suppose F is compact (so that in particular $m_*(F) < \infty$), and let $\epsilon > 0$. By Proposition 1.8 we can select an open set $\mathcal{O}$ with $F \subset \mathcal{O}$ and

$m_*(\mathcal{O}) \leq m_*(F) + \epsilon$. Since F is closed, the difference $\mathcal{O} - F$ is open, and by Theorem 1.3 we may write this difference as countable union of almost disjoint cubes

$$\mathcal{O} - F = \bigcup_{j=1}^{\infty} Q_j$$

For a fixed N, the finite union $K = \bigcup_{j=1}^{N} Q_j$ is compact; therefore $d(K, F) > 0$. Since $(K \cup F) \subset \mathcal{O}$

$$\begin{aligned} m_*(\mathcal{O}) &\geq m_*(F) + m_*(K) \\ &= m_*(F) + \sum_{j=1}^{N} m_*(Q_j) \end{aligned}$$

Hence $\sum_{j=1}^{N} m_*(Q_j) \leq m_*(\mathcal{O}) - m_*(F) \leq \epsilon$, and this also holds in the limit as N tends to be infinite. Hence

$$m_*(\mathcal{O} - F) \leq \sum_{j=1}^{\infty} m_*(Q_j) \leq \epsilon$$

□

Lemma 1.16. *If F is closed, K is compact, and these sets are disjoint, then $d(F, K) > 0$*

Proof. Since F is closed, for each point $x \in K$, there exists $\delta_x > 0$ so that $d(x, F) > 3\delta_x$. Since $\bigcup_{x \in K} B_{2\delta_x}(x)$ covers K, and K is compact, we may find a subcover, which we denote by $\bigcup_{j=1}^{N} B_{2\delta_j}(x_j)$. If we let $\delta = \min(\delta_1, \ldots, \delta_N)$, then we must have $d(K, F) \geq \delta > 0$. Indeed, if $x \in K$ and $y \in F$, then for some j we have $\left|x_j - x\right| \leq 2\delta_j$ and by construction $\left|y - x_j\right| \geq 3\delta_j$. Therefore

$$|y - x| \geq \left|y - x_j\right| - \left|x_j - x\right| \geq 3\delta_j - 2\delta_j \geq \delta$$

□

Proposition 1.17. *The complement of a measurable set is measurable*

Proof. If E is measurable, then for every positive integer n we may choose an open set $\mathcal{O}_n$ with $E \subset \mathcal{O}_n$ and $m_*(\mathcal{O}_n - E) \leq 1/n$. The complement $\mathcal{O}_n^c$ is closed, hence measurable, which implies that the union $S = \bigcup_{n=1}^{\infty} \mathcal{O}_n^c$ is also measurable. Now we simply note that $S \subset E^c$ and

$$(E^c - S) \subset (\mathcal{O}_n - E)$$

s.t. $m_*(E^c - S) \leq 1/n$ for all n. Therefore $m_*(E^c - S) = 0$ and $E^c - S$ is measurable. Hence $E^c = S \cup (E^c - S)$ is measurable □

Proposition 1.18. *A countable intersection of measurable sets is measurable*

Proof.

$$\bigcap_{j=1}^{\infty} E_j = (\bigcup_{j=1}^{\infty} E_j^c)^c$$

□