Measure Theory

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1 Introduction: a non-measurable set

Suppose we want a measure that satisfies:

0. $\lambda : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+ \cup \{+\infty\}$
1. $\lambda((a, b]) = b - a$
2. $A \subseteq \mathbb{R}, A + x = \{x + y : y \in A\}$

$$\forall A \subseteq \mathbb{R} \forall x \in \mathbb{R}, \lambda(A + x) = \lambda(A)$$

3. $A = \bigcup_{j\geq 1} A_j, A_j \cap A_k = \emptyset$

$$\lambda(A) = \sum_{k\geq 1} \lambda(A_k)$$

Define $x \sim y$ for $x, y \in \mathbb{R}$ if $y - x \in \mathbb{Q}$. $\Lambda = \mathbb{R}/\sim$ and $\alpha, \beta \in \Lambda$. Γ is uncountable since each equivalent class is countable.

By the **Axiom of Choice**, we have a $\Omega \subseteq \mathbb{R}$ s.t. for each $[x] \in \mathbb{R}/\sim$, there is a $x \in [x]$ s.t. $x \in \Omega$. Hence we can assume $\Omega \subseteq (0, 1)$.

Claim: For $p, q \in \mathbb{Q}$, either $\Omega + p = \Omega + q$ or $\Omega + p \cap \Omega + q = \emptyset$.

Proof. Assume $(\Omega + p) \cap (\Omega + q) \neq \emptyset$, $x = \alpha + p = \beta + q$. Hence $\alpha - \beta = q - p \in \mathbb{Q}$, which implies $\alpha = \beta$. □

Claim: $\Omega + q \subseteq (-1, 2)$ since $-1 < q < 1$.

In particular,

$$\bigcup_{\substack{q\in\mathbb{Q} \\ -1<q<1}} (\Omega + q) \subseteq (-1, 2)$$

Claim: If $E \subseteq F$, then $\lambda(E) \leq \lambda(F)$

Proof. $\lambda(F) = \lambda(E \cup (F - E)) = \lambda(E) + \lambda(F - E)$ □

If $q \neq p$,

$$\lambda(\bigcup_{\substack{q\in\mathbb{Q} \\ -1<q<1}} (\Omega + q)) = \sum_{\substack{q\in\mathbb{Q} \\ -1<q<1}} \lambda(\Omega + q) = \sum_{\substack{q\in\mathbb{Q} \\ -1<q<1}} \lambda(\Omega) \leq \lambda((-1, 2)) = 3$$

Hence $\lambda(\Omega) = 0$

Claim: $(0, 1) \subseteq \sum_{q\in\mathbb{Q},-1<q<1} (\Omega + q)$

Proof. Fix $x \in [0, 1]$, $\exists \alpha \in [x] \cap \Omega$ and $\alpha \in (0, 1)$. Hence $\alpha - x = q \in \mathbb{Q}$. Then $x \in \Omega + q$ □

Hence we have a contradiction and there is no such λ function.

2 Classes of subsets

Definition 2.1. For $\mathcal{S} \subseteq \mathcal{P}(\Omega)$, $\mathcal{S}$ is a **semi-algebra** if

1. $\Omega \in \mathcal{S}$
2. If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$
3. For all $A \in \mathcal{S}$, there are $E_1, \ldots, E_n \in \mathcal{S}$ s.t. $A^c = \sqcup E_j$

Example 2.1. If $\Omega = \mathbb{R}$ and

$$\begin{aligned} \mathcal{S} = \mathbb{R} &\cup \{(a, b] : a < b, a, b \in \mathbb{R}\} \\ &\cup \{(-\infty, b] : b \in \mathbb{R}\} \\ &\cup \{(a, \infty) : a \in \mathbb{R}\} \\ &\cup \emptyset \end{aligned}$$

then $\mathcal{S}$ is a semi-algebra

Definition 2.2. Take $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, $\mathcal{A}$ is an **algebra** if

1. $\Omega \in \mathcal{A}$
2. If $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$
3. If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$

If $\mathcal{A}$ is an algebra, then it is also semi-algebra.

Definition 2.3. $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a σ**-algebra** if

1. $\Omega \in \mathcal{F}$
2. If $A_j \in \mathcal{F}$ for $j \geq 1$, then $\bigcap_{j \geq 1} A_j \in \mathcal{F}$
3. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

Proposition 2.4. *Suppose $\mathcal{A}_\alpha \subseteq \mathcal{P}(\Omega)$, $\mathcal{A}_\alpha$ is an algebra, $\alpha \in I$. Then $\mathcal{A} = \bigcap_{\alpha \in I} \mathcal{A}_\alpha$ is an algebra*