

Notes on Set Theory

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1 Foreword

Notes for the entrance examination

2 Models of Set - Sertraline

2.1 Some mathematical logic

Theorem 2.1 (Gödel's second incompleteness theorem). If a consistent recursive axiom set T contains **ZFC**, then

$$T \not\vdash \text{Con}(T)$$

especially, **ZFC** $\not\vdash \text{Con}(\text{ZFC})$

Definition 2.2. Suppose (M, E_M) and (N, E_N) are two models of set theory, then

1. if for any formula σ , $M \models \sigma$ if and only if $N \models \sigma$, then M and N are **elementary equivalent**, denoted by $M \equiv N$
2. If bijection $f : M \rightarrow N$ satisfies: for any $a, b \in M$, $a E_M b$ iff $f(a) E_N f(b)$, then $f : M \cong N$ is an **isomorphism**
3. If $M \subseteq N$ and $E_M = E_N \upharpoonright M$, then M is N 's submodel
4. If M is isomorphic to a submodel of N by injection f , and for any formula $\varphi(x_1, \dots, x_n)$, for any $a_1, \dots, a_n \in M$, $M \models \varphi[a_1, \dots, a_n]$ iff $N \models \varphi[f(a_1), \dots, f(a_n)]$, then f is called an **elementary embedding** from M to N , written as $f : M \prec N$
5. If $M \subseteq N$ and $M \prec N$, then M is a **elementary submodel** of N

Lemma 2.3. Suppose $N \models \text{ZFC}$, $M \subseteq N$, then $M \prec N$ iff $\forall \varphi(x, x_1, \dots, x_n)$, $\forall (a_1, \dots, a_n) \in M$, if $\exists a \in N$ s.t. $N \models \varphi[a, a_1, \dots, a_n]$, then $\exists a \in M$ s.t. $M \models \varphi[a, a_1, \dots, a_n]$

Definition 2.4. Suppose $(M, E) \models \text{ZFC}$

1. $h_\varphi : M^n \rightarrow M$ is φ 's **Skolem function** if $\forall a_1, \dots, a_n \in M$, if $\exists a \in M$ s.t. $M \models \varphi[a, a_1, \dots, a_n]$, then $M \models \varphi[h_\varphi(a_1, \dots, a_n), a_1, \dots, a_n]$. requires **AC**
2. Let $\mathcal{H} = \{h_\varphi \mid \varphi \text{ is a formula on set theory}\}$. For any $S \subseteq M$, **Skolem hull** $\mathcal{H}(S)$ is the smallest set consisting of S and closed under \mathcal{H}

Lemma 2.5. $N \models \mathbf{ZFC}$, $S \subseteq N$, if $M = \mathcal{H}(S)$, then $M \prec N$

Theorem 2.6 (Löwenheim-Skolem theorem). Suppose $N \models \mathbf{ZFC}$ and is infinite, then there is a model M s.t. $|M| = \omega$ and $M \prec N$

2.2 Cumulative Hierarchy

This section works in \mathbf{ZF}^- (a.k.a. \mathbf{ZF} – axiom of foundation)

Definition 2.7. For any α , define sequence V_α

1. $V_0 = \emptyset$
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
3. For any limit ordinal λ , $V_\lambda = \bigcup_{\beta < \lambda} V_\beta$

And $\mathbf{WF} = \bigcup_{\alpha \in \mathbf{On}} V_\alpha$

Lemma 2.8. For any ordinal α

1. V_α is transitive
2. if $\xi \leq \alpha$, then $V_\xi \subseteq V_\alpha$
3. if κ is inaccessible cardinal, then $|V_\kappa| = \kappa$

Proof. 1. Obviously $\kappa \leq V_\kappa$. Since κ is inaccessible, then for any $\alpha < \kappa$, $|V_\alpha| < \kappa$.

□

Definition 2.9. For any set $x \in \mathbf{WF}$,

$$\text{rank}(x) = \min\{\beta \mid x \in V_{\beta+1}\}$$

Lemma 2.10. 1. $V_\alpha = \{x \in \mathbf{WF} \mid \text{rank}(x) < \alpha\}$

2. \mathbf{WF} is transitive
3. For any $x, y \in \mathbf{WF}$, if $x \in y$, then $\text{rank}(x) < \text{rank}(y)$
4. for any $y \in \mathbf{WF}$, $\text{rank}(y) = \sup\{\text{rank}(x) + 1 \mid x \in y\}$

Lemma 2.11. Suppose α is an ordinal

1. $\alpha \in \mathbf{WF}$ and $\text{rank}(\alpha) = \alpha$

$$2. V_\alpha \cap \mathbf{On} = \alpha$$

Lemma 2.12. 1. If $x \in \mathbf{WF}$, then $\bigcup x, \mathcal{P}(x), \{x\} \in \mathbf{WF}$, and their ranks are all less than $\text{rank}(x) + \omega$

2. If $x, y \in \mathbf{WF}$, then $x \times y, x \cup y, x \cap y, \{x, y\}, (x, y), x^y \in \mathbf{WF}$, and their ranks are all less than $\text{rank}(x) + \text{rank}(y) + \omega$

$$3. \mathbb{Z}, \mathbb{Q}, \mathbb{R} \in V_{\omega+\omega}$$

4. for any set x , $x \in \mathbf{WF}$ iff $x \subset \mathbf{WF}$

Lemma 2.13. Suppose AC

1. for any group G , there exists group $G' \cong G$ in \mathbf{WF}

2. for any topological space T , there exists $T' \cong T$ in \mathbf{WF}

Definition 2.14. Binary relation $<$ on set A is **well-founded** if for any nonempty $X \subseteq A$, X has minimal element under $<$

Theorem 2.15. If $A \in \mathbf{WF}$, then \in is a well-founded relation on A

Lemma 2.16. If set A is transitive and \in is well-founded on A , then $A \in \mathbf{WF}$

Lemma 2.17. For any set x , there is a smallest transitive set $\text{trcl}(x)$ s.t. $x \subseteq \text{trcl}(x)$

Proof.

$$\begin{aligned} x_0 &= x \\ x_{n+1} &= \bigcup x_n \\ \text{trcl}(x) &= \bigcup_{n < \omega} x_n \end{aligned}$$

□

$\text{trcl}(x)$ is called **transitive closure** of x

Lemma 2.18. Without axiom of power set

1. if x is transitive, then $\text{trcl}(x) = x$

2. if $y \in x$, then $\text{trcl}(y) \subseteq \text{trcl}(x)$

3. $\text{trcl}(x) = x \cup \bigcup \{\text{trcl}(y) \mid y \in x\}$

Theorem 2.19. For any set X , the following are equivalent

1. $X \in \mathbf{WF}$
2. $\text{trcl}(X) \in \mathbf{WF}$
3. \in is a well-founded relation on $\text{trcl}(X)$

Theorem 2.20. The following propositions are equivalent

1. Axiom of foundation
2. For any set X , \in is a well-founded relation on X
3. $\mathbf{V} = \mathbf{WF}$

2.3 Relativization

Definition 2.21. Let \mathbf{M} be a class φ a formula, the **relativization** of φ to \mathbf{M} is $\varphi^{\mathbf{M}}$ defined inductively

$$\begin{aligned} (x \in y)^{\mathbf{M}} &\leftrightarrow x = y \\ (x \in y)^{\mathbf{M}} &\leftrightarrow x \in y \\ (\varphi \rightarrow \psi)^{\mathbf{M}} &\leftrightarrow \varphi^{\mathbf{M}} \rightarrow \psi^{\mathbf{M}} \\ (\neg \varphi)^{\mathbf{M}} &\leftrightarrow \neg \varphi^{\mathbf{M}} \\ (\forall x \varphi)^{\mathbf{M}} &\leftrightarrow (\forall x \in \mathbf{M}) \varphi^{\mathbf{M}} \end{aligned}$$

Note $\varphi^{\mathbf{V}} = \varphi$ and

$$f^{\mathbf{M}} = \{(x_1, \dots, x_n, x_{n+1}) \in \mathbf{M} \mid \varphi^{\mathbf{M}}(x_1, \dots, x_n, x_{n+1})\}$$

Definition 2.22. For any theory T , any class \mathbf{M} , $\mathbf{M} \models T$ iff for any axiom φ of T , $\varphi^{\mathbf{M}}$ holds

Theorem 2.23 (ZF⁻). $\mathbf{WF} \models \mathbf{ZF}$

Proof. • **Axiom of existence**

$(\exists x(x = x))^{\mathbf{M}} \leftrightarrow \exists x \in \mathbf{M} (x = x)$, which is equivalent to \mathbf{M} being nonempty

• **Axiom of extensionality**

$$\begin{aligned} \forall X \forall Y \forall u ((u \in X \leftrightarrow u \in Y) \rightarrow X = Y)^{\mathbf{M}} &\Leftrightarrow \\ \forall X \in \mathbf{M} \forall Y \in \mathbf{M} \forall u \in \mathbf{M} ((u \in X \leftrightarrow u \in Y) \rightarrow X = Y) \end{aligned}$$

Lemma 2.24. If \mathbf{M} is transitive, then axiom of extensionality holds in \mathbf{M}

- **Axiom schema of specification**

$$\forall X \in \mathbf{M} \exists Y \in \mathbf{M} \forall u \in \mathbf{M} (u \in Y \leftrightarrow u \in X \wedge \varphi^{\mathbf{M}}(u))$$

Since for any $X \in \mathbf{WF}$, $\mathcal{P}(X) \subseteq \mathbf{WF}$

- **Axiom of paring**
- **Axiom of union**
- **Axiom of power set**

$$\forall X \in \mathbf{M} \exists Y \in \mathbf{M} \forall u \in \mathbf{M} (u \in Y \leftrightarrow (u \subseteq X)^{\mathbf{M}})$$

and

$$(u \subseteq X)^{\mathbf{M}} \leftrightarrow \forall x \in \mathbf{M} (x \in u \rightarrow x \in X) \leftrightarrow u \cap \mathbf{M} \subseteq X$$

- **Axiom of foundation**
- **Axiom schema of replacement**

□

2.4 Absoluteness

Definition 2.25. For any formula $\psi(x_1, \dots, x_n)$ and any class $\mathbf{M}, \mathbf{N}, \mathbf{M} \subseteq \mathbf{N}$, if

$$\forall x_1 \dots \forall x_n \in \mathbf{M} (\psi^{\mathbf{M}}(x_1, \dots, x_n) \leftrightarrow \psi^{\mathbf{N}}(x_1, \dots, x_n))$$

then $\psi(x_1, \dots, x_n)$ is **absolute** for \mathbf{M} , cn. If $\mathbf{N} = \mathbf{V}$, then ψ is **absolute** for \mathbf{M}

Lemma 2.26. Suppose $\mathbf{M} \subseteq \mathbf{N}$ and φ, ψ are formulas, then

1. if φ, ψ are absolute for \mathbf{M} , cn, then so are $\neg\varphi, \varphi \rightarrow \psi$
2. if φ doesn't contain any quantifiers, then φ is absolute for any \mathbf{M}
3. if \mathbf{M}, \mathbf{N} are transitive and φ is absolute for them, then so are $\forall x \in y \varphi$

Definition 2.27. Δ_0 formula

1. $x = y, x \in y$ are Δ_0 formulas
2. if φ, ψ are Δ_0 , then so are $\neg\varphi, \varphi \rightarrow \psi$
3. if φ is Δ_0 , y is any set, then $(\forall x \in y)\varphi$ is Δ_0

If φ is Δ_0 , then $\exists x_1 \dots \exists x_n \varphi$ is Σ_1 formula, $\forall x_1 \dots \forall x_n \varphi$ is Π_1

Lemma 2.28. $\mathbf{M} \subseteq \mathbf{N}$ are both transitive, $\psi(x_0, \dots, x_n)$ is a formula, then

1. if ψ is Δ_0 , then it's absolute for \mathbf{M}, cn
2. if ψ is Σ_1 , then

$$\forall x_1 \dots x_n (\psi^{\mathbf{M}}(x_1, \dots, x_n) \rightarrow \psi^{\mathbf{N}}(x_1, \dots, x_n))$$

3. if ψ is Π_1 , then

$$\forall x_1 \dots x_n (\psi^{\mathbf{N}}(x_1, \dots, x_n) \rightarrow \psi^{\mathbf{M}}(x_1, \dots, x_n))$$

Lemma 2.29. If $\mathbf{M} \subseteq \mathbf{N}, \mathbf{M} \models \Sigma, \mathbf{N} \models \Sigma$ and

$$\Sigma \vdash \forall x_1 \dots \forall x_n (\varphi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n))$$

then φ is absolute for \mathbf{M}, \mathbf{N} if and only if ψ is absolute for \mathbf{M}, \mathbf{N}

Definition 2.30. Suppose $\mathbf{M} \subseteq \mathbf{N}, f(x_1, \dots, x_n)$ is a function. f is **absolute** for \mathbf{M} and \mathbf{N} if and only if $\varphi(x_1, \dots, x_n, x_{n+1})$ defining f is absolute.

Theorem 2.31. Following relations and functions can be defined in $\mathbf{ZF}^- - \text{Pow} - \text{Inf}$ and are equivalent to some Δ_0 formulas. So they are absolute for any transitive model \mathbf{M} on $\mathbf{ZF}^- - \text{Pow} - \text{Inf}$

1. $x \in y$
2. $x = y$
3. $x \subset y$
4. $\{x, y\}$
5. $\{x\}$
6. (x, y)
7. \emptyset

8. $x \cup y$
9. $x - y$
10. $x \cap y$
11. x^+
12. x is a transitive set
13. $\bigcup x$
14. $\bigcap x$ ($\bigcap \emptyset = \emptyset$)

Lemma 2.32. Absoluteness is closed under operation composition

Theorem 2.33. Following relations and functions are absolute for any transitive model \mathbf{M} on $\mathbf{ZF}^- - \text{Pow} - \text{Inf}$

1. z is an ordered pair
2. $A \times B$
3. R is a relation
4. $\text{dom}(R)$
5. $\text{ran}(R)$
6. f is a function
7. $f(x)$
8. f is injective

2.5 Relative consistence of the axiom of foundation

Lemma 2.34. Suppose transitive class $\mathbf{M} \models \mathbf{ZF}^- - \text{Pow} - \text{inf}$ and $\omega \in \mathbf{M}$, then the axiom of infinity is true in \mathbf{M} . Hence the axiom of infinity is true in \mathbf{WF}

Theorem 2.35. Let T be a theory of set theory language and Σ a set of sentences. Suppose \mathbf{M} is a class and $T \vdash \mathbf{M} \neq \emptyset$, then if $\mathbf{M} \models_T \Sigma$, then

1. for any sentences φ , if $\Sigma \vdash \varphi$, then $T \vdash \varphi^{\mathbf{M}}$
2. if T is consistent, then so is $\text{Cn}(\Sigma)$

Theorem 2.36. The axiom of foundation is consistent with \mathbf{ZF}^- .

Proof. By 2.35, let \mathbf{T} be \mathbf{ZF}^- , Σ be \mathbf{ZF} and \mathbf{M} be \mathbf{WF} □

Lemma 2.37 (\mathbf{ZF}^-). Suppose transitive model $\mathbf{M} \models \mathbf{ZF}^- - \text{Pow} - \text{Inf}$. If $X, R \in \mathbf{M}$ and R is a well-order on X , then

$$(R \text{ is a well-order on } X)^{\mathbf{M}}$$

Theorem 2.38 (\mathbf{ZF}^-). $V_\omega \models \mathbf{ZFC} - \text{Inf} + \neg \text{Inf}$

Proof. For any $X \in V_\omega$, X is finite hence there is a well-ordering on X □

Corollary 2.39. $\text{ConZF}^- \rightarrow \text{ConZFC} - \text{Inf} + \neg \text{Inf}$

2.6 Induction and recursion based on well-order relation

Definition 2.40. R is a well-founded relation on X if and only if

$$\forall U \subset X (U \neq \emptyset \rightarrow \exists y \in U (\neg \exists z \in U (zRy)))$$

Definition 2.41. Relation R is **set-like** on X iff for any $x \in X$, $\{y \in X \mid yRx\}$ is a set

Definition 2.42. If R is a set-like relation on X and $x \in X$, define

$$\begin{aligned} \text{pred}^0(X, x, R) &= \{y \in X \mid yRx\} \\ \text{pred}^{n+1}(X, x, R) &= \bigcup \{\text{pred}(X, y, R) \mid y \in \text{pred}^n(X, x, R)\} \\ \text{cl}(X, x, R) &= \bigcup_{n \in \omega} \text{pred}^n(X, x, R) \end{aligned}$$

Lemma 2.43. If R is a set-like relation on X , then for any $y \in \text{cl}(X, x, R)$, $\text{pred}(X, y, R) \subseteq \text{cl}(X, x, R)$

Theorem 2.44 (Induction on well-founded set-like relation). If R is a well-founded set-like relation on X , then every nonempty $Y \subseteq X$ has minimal element under R

Theorem 2.45. Suppose R is a well-founded set-like relation on X . If $F : X \times V \rightarrow V$, then there is a unique $G : X \rightarrow V$ s.t.

$$\forall x \in X (G(x) = F(x, G \upharpoonright \text{pred}(X, x, R)))$$

Definition 2.46. If R is a set-like well-founded relation on X , define

$$\text{rank}(x, X, R) = \sup\{\text{rank}(y, X, R) + 1 \mid yRx \wedge y \in X\}$$

Note that

$$F(x, h) = \sup\{\alpha + 1 \mid \alpha \in \text{ran}(h)\}$$

Lemma 2.47 (ZF^-). If X is transitive and \in is well-founded on X , then $X \subseteq \mathbf{WF}$ and for any $x \in X$, $\text{rank}(x, X, \in) = \text{rank}(x)$

Definition 2.48. R is a set-like well-founded relation on X , **Mostowski function** G on (X, R) is

$$G(x) = \{G(y) \mid y \in X \wedge yRx\}$$

$\mathbf{M} = \text{ran}(G)$ is called the **Mostowski collapse** of (X, R)

Lemma 2.49. 1. $\forall x, y \in X (xRy \rightarrow G(x) \in G(y))$

2. \mathbf{M} is transitive

3. If the axiom of power set holds, $\mathbf{M} \subseteq \mathbf{WF}$

4. if the axiom of power set holds and $x \in X$, then
 $\text{rank}(x, X, R) = \text{rank}(G(x))$

Definition 2.50. R is extensional on X iff

$$\forall x, y \in X (\forall z \in X (zRx \leftrightarrow zRy) \rightarrow x = y)$$

Lemma 2.51. If X is transitive then \in is extensional on X

Lemma 2.52. Let R be a set-like well-founded relation on X , G is a Mostowski function on it. If R is extensional, then G is an isomorphism

Theorem 2.53 (Mostowski collapse theorem). Suppose R is set-like well-founded extensional on X , then there are unique transitive class \mathbf{M} and bijection $G : X \rightarrow \mathbf{M}$ s.t. $G : (X, R) \cong (\mathbf{M}, \in)$

2.7 Absoluteness under the axiom of foundation

Theorem 2.54. The following relations and functions can be defined by formulas in $\mathbf{ZF} - \text{Pow}$ and are equivalent to some Δ_0 formulas

1. x is an ordinal
2. x is a limit ordinal
3. x is a successor ordinal
4. ω
5. x is a finite ordinal
6. $0, 1, 2, \dots, 20, \dots$

Theorem 2.55. If transitive model $\mathbf{M} \models \mathbf{ZF} - \text{Pow}$, then every finite subset of \mathbf{M} belongs to \mathbf{M}

Proof. prove

$$\forall x \subset \mathbf{M} (|x| = n \rightarrow x \in \mathbf{M})$$

□

Theorem 2.56. The following concepts are absolute for any transitive model of $\mathbf{ZF} - \text{Pow}$

1. x is finite
2. X^n
3. $X^{<\omega}$
4. R is a well-ordering on X
5. $\text{type}(X, R)$
6. $\alpha + 1$
7. $\alpha - 1$
8. $\alpha + \beta$
9. $\alpha \cdot \beta$

Class X is in fact a formula $X(x)$. It's absolute for \mathbf{M} if and only if $\forall x \in \mathbf{M} (X^{\mathbf{M}}(x) \leftrightarrow X(x))$, which is equivalent to $\{x \in \mathbf{M} \mid X(x)\} = \{x \in \mathbf{M} \mid X^{\mathbf{M}}(x)\}$. Hence X is absolute for \mathbf{M} if and only if $X^{\mathbf{M}} = \mathbf{M} \cap X$

Theorem 2.57. Suppose R is a well-founded set-like relation on X , $F : X \times V \rightarrow V$,

$$\forall x \in X (G(x) = F(x, G \upharpoonright (X, x, R)))$$

transitive model $\mathbf{M} \models \mathbf{ZF} - \text{Pow}$ and

1. F is absolute for \mathbf{M}
2. X, R are absolute for \mathbf{M} , $(R \text{ is set-like on } X)^{\mathbf{M}}$ and

$$\forall x \in \mathbf{M} (\text{pred}(X, x, R) \subseteq \mathbf{M})$$

then G is absolute for \mathbf{M}

Theorem 2.58. The following concept is absolute for any transitive model of $\mathbf{ZF} - \text{Pow}$

1. α^β
2. $\text{rank}(x)$
3. $\text{trcl}(x)$

Lemma 2.59. transitive $\mathbf{M} \models \mathbf{ZF}$

1. if $x \in \mathbf{M}$, then $\mathcal{P}^{\mathbf{M}}(x) = \mathcal{P}(x) \cap \mathbf{M}$
2. if $\alpha \in \mathbf{M}$, then $V_\alpha^{\mathbf{M}} = V_\alpha \cap \mathbf{M}$

2.8 Unaccessible cardinal and models of ZFC

$$Z = \mathbf{ZF} - \text{Rep}, \mathbf{ZF}^- = \mathbf{ZFC} - \text{Rep}$$

Theorem 2.60. If $\gamma > \omega$ is a limit ordinal, then $V_\gamma \models_{\mathbf{ZF}} Z$ and $V_\gamma \models_{\mathbf{ZFC}} ZC$

Corollary 2.61. $V_{\omega+\omega}$ doesn't satisfies the axiom of replacement

Proof.

□

Theorem 2.62. $\mathbf{ZC} \not\models \exists x (x = V_\omega)$, $\mathbf{ZC} \not\models \forall x \exists y (\text{trcl}(x) = y)$

Theorem 2.63. If κ is an inaccessible cardinal, then $V_\kappa \models_{\mathbf{ZF}^-} \mathbf{ZF}$,
 $V_\kappa \models_{\mathbf{ZFC}^-} \mathbf{ZFC}$

Proof. Since κ is inaccessible, $|V_\kappa| = \kappa$. For any $A \in V_\kappa$, $|A| < \kappa$. Since κ is regular, any $f : A \rightarrow V_\kappa$ is bounded. Hence there exists $\alpha < \kappa$ s.t. $\text{ran}(f) \subseteq V_\alpha$ \square

Corollary 2.64. We cannot prove "there is some inaccessible cardinals" in **ZFC**

Proof. Suppose we could. Then we have $V_\kappa \models \mathbf{ZFC}$, which contradicts Gödel's second incompleteness theorem \square

Lemma 2.65. Suppose κ is inaccessible. The following concepts are absolute for V_κ

1. x is a cardinal
2. x is a regular cardinal
3. x is an inaccessible cardinal

Lemma 2.66. $\text{Con}(\mathbf{ZFC}) \rightarrow \text{Con}(\mathbf{ZFC} + \text{"there is no inaccessible cardinal"})$

Proof. If κ is the smallest inaccessible cardinal, then $V_\kappa \models \mathbf{ZFC} + \text{"there is no inaccessible cardinal"}$. Define

$$\mathbf{M} = \bigcap \{V_\kappa \mid \kappa \text{ is inaccessible}\}$$

\square

If there are, then $\mathbf{M} = V_\kappa$

Corollary 2.67. $\text{Con}(\mathbf{ZFC}) \rightarrow \text{Con}(\mathbf{ZFC} + \text{"there are some inaccessible cardinals"})$

Definition 2.68. For any infinite cardinal κ , $H_\kappa = \{x \mid |\text{trcl}(x)| < \kappa\}$ is the collection of sets which **hereditarily have size less than** κ . Element of H_ω is called **hereditarily finite set**. Element of H_{ω_1} is called **hereditarily countable set**

Lemma 2.69. For any infinite cardinal κ , $H_\kappa \subseteq V_\kappa$

Lemma 2.70. If κ is regular, then $H_\kappa = V_\kappa$ if and only if κ is inaccessible

Proof. which implies $|V_\kappa| = \kappa$ \square

Lemma 2.71. For any infinite cardinal κ

1. H_κ is transitive

2. $H_\kappa \cap \mathbf{On} = \kappa$
3. If $x \in H_\kappa$, then $\bigcup x \in H_\kappa$
4. If $x, y \in H_\kappa$, then $\{x, y\} \in H_\kappa$
5. If $x \in H_\kappa, y \subseteq x$, then $y \in H_\kappa$
6. if κ is regular, then $\forall x (x \in H_\kappa \leftrightarrow x \subset H_\kappa \wedge |x| < \kappa)$

Theorem 2.72. If κ is uncountable regular cardinal, then $H_\kappa \models_{\mathbf{ZFC}} \mathbf{ZFC} - \mathbf{Pow}$

Theorem 2.73. If κ is uncountable regular cardinal, then the following propositions are equivalent

1. $H_\kappa \models \mathbf{ZFC}$
2. $H_\kappa = V_\kappa$
3. κ is inaccessible

Corollary 2.74. $\text{Con}(\mathbf{ZFC}) \rightarrow \text{Con}(\mathbf{ZFC} - \text{pow} + \forall x (x \text{ is countable}))$

2.9 Reflection theorem

Lemma 2.75. $\mathbf{M} \subseteq \mathbf{N}$ are classes. $\varphi_1, \dots, \varphi_n$ is a sequence closed under subformula, then the following propositions are equivalent

1. $\varphi_1, \dots, \varphi_n$ are absolute for \mathbf{M} and \mathbf{N}
2. if $\varphi_i = \exists \varphi_j(x, y_1, \dots, y_m)$, then

$$\forall y_1, \dots, y_m \in \mathbf{M} (\exists x \in \mathbf{N} \varphi_j^{\mathbf{N}}(x, y_1, \dots, y_m) \rightarrow \exists x \in \mathbf{M} \varphi_j^{\mathbf{M}}(x, y_1, \dots, y_m))$$

Theorem 2.76 (reflection theorem(\mathbf{ZF})). For any finite formula set $F = \{\varphi_1, \dots, \varphi_n\}$, for any V_α , there exists V_β s.t. $V_\alpha \subseteq V_\beta$ and $\varphi_1, \dots, \varphi_n$ are absolute for V_β

Corollary 2.77 (\mathbf{ZF}). $F = \{\sigma_1, \dots, \sigma_n\}$ are finite subsets of \mathbf{ZF} , then

$$\forall \alpha \exists \beta > \alpha (\sigma_1^{V_\beta} \wedge \dots \wedge \sigma_n^{V_\beta})$$

Corollary 2.78. $F = \{\sigma_1, \dots, \sigma_n\}$ is a finite subset of \mathbf{ZF} . Unless \mathbf{ZF} is inconsistent, F cannot prove all axioms of \mathbf{ZF}

Theorem 2.79 (ZFC). For any finite formula set $F = \{\varphi_1, \dots, \varphi_n\}$, for any set N , there exists set M s.t.

1. $N \subseteq M$
2. $\varphi_1, \dots, \varphi_n$ are absolute for (M, \in)
3. $|M| \leq |N| \cdot \omega$

Corollary 2.80 (ZFC). For any finite formula set $F = \{\varphi_1, \dots, \varphi_n\}$, for any set N , there exists set M s.t.

1. $N \subseteq M$
2. $\varphi_1, \dots, \varphi_n$ are absolute for (M, \in)
3. $|M| \leq |N| \cdot \omega$
4. M is transitive

3 Constructable Set - Venlafaxine

3.1 Definability and Gödel operation

Definition 3.1. M is a set, $\psi(x_1, \dots, x_n, y_1, \dots, y_m)$ is a formula, $X \subseteq M^n$ is **definable in M from parameters from ψ** if and only if there are $y_1, \dots, y_m \in M$ s.t.

$$X = \{(x_1, \dots, x_n) \mid (\psi^M(x_1, \dots, x_n, y_1, \dots, y_m))\}$$

$$\text{Def}(M) = \{X \subseteq M \mid \exists \psi, X \text{ is definable in } M \text{ from } \psi\}$$

Definition 3.2. Gödel operation

1. $G_1(X, Y) = \{X, Y\}$
2. $G_2(X, Y) = X \times Y$
3. $G_3(X, Y) = \in \restriction X \times Y$
4. $G_4(X, Y) = X - Y$
5. $G_5(X, Y) = X \cap Y$
6. $G_6(X, Y) = \bigcap X$
7. $G_7(X, Y) = \text{dom}(X)$
8. $G_8(X, Y) = \{(x, y) \mid (y, x) \in X\}$
9. $G_9(X, Y) = \{(x, y, z) \mid (x, z, y) \in X\}$
10. $G_{10}(X, Y) = \{(x, y, z) \mid (y, z, x) \in X\}$

Class C is closed under Gödel operation if for any $X, Y, X, Y \in C$ implies $G_i(X, Y) \in C$. For any set M , $\text{cl}_G(M)$ is the **closure under Gödel operation**

Definition 3.3. ψ is a **normal form** if

1. only \neg, \wedge, \exists are logical symbol
2. $=$ doesn't appear
3. if $x_i \in x_j$ then $i \neq j$
4. \exists only shown as: $\exists x_{m+1} \in x_i \varphi(x_1, \dots, x_{m+1}), 1 \leq i \leq m$

Lemma 3.4. Any Δ_0 formula can be transformed into normal form

Theorem 3.5. For any Δ_0 formula $\psi(x_1, \dots, x_n)$, there is Gödel operations' composition G s.t. for any X_1, \dots, X_n

$$G(X_1, \dots, X_n) = \{(x_1, \dots, x_n) \mid x_1 \in X_1 \wedge \dots \wedge x_n \in X_n \wedge \psi(x_1, \dots, x_n)\}$$

Corollary 3.6. If M is transitive and $M = \text{cl}_G(M)$, then for any Δ_0 formula $\psi(x, y_1, \dots, y_m)$, any set $X \in M$, any $y_1, \dots, y_m \in M$ if

$$Y = \{x \in X \mid \psi(x, y_1, \dots, y_m)\}$$

then $Y \in M$. Hence Δ_0 schema of specification holds in M

Lemma 3.7. If $G(X_1, \dots, X_n)$ is Gödel operations' composition, then $Z = G(X_1, \dots, X_n)$ is equivalent to a Δ_0 formula

Theorem 3.8. For any transitive set M , $\text{Def}(M) = \text{cl}_G(M \cup \{M\}) \cap \mathcal{P}(M)$

Lemma 3.9. If transitive $\mathbf{M} \models \mathbf{ZF}$, then for any transitive set $M \in \mathbf{M}$, $\text{Def}(M)$ is absolute for \mathbf{M}

Lemma 3.10. For any transitive set M

1. $\text{Def}(M) \subseteq \mathcal{P}(M)$
2. $M \subseteq \text{Def}(M)$
3. for any $X \subseteq M$, if X is finite, then $X \in \text{Def}(M)$
4. assume **AC** and $|M| \geq \omega$, then $|\text{Def}(M)| = |M|$

3.2 Gödel's L

Definition 3.11. for any α

1. $L_0 = \emptyset$
2. $L_{\alpha+1} = \text{Def}(L_\alpha)$
3. For any limit α , $L_\alpha = \bigcup_{\beta < \alpha} L_\beta$

$\mathbf{L} = \bigcup_{\alpha \in \mathbf{On}} L_\alpha$. Element of \mathbf{L} is called constructible set

Lemma 3.12. For any ordinal α

1. L_α is transitive
2. If $\alpha < \beta$, then $L_\alpha \subseteq L_\beta$
3. $L_\alpha \subseteq V_\alpha$

Definition 3.13. $x \in \mathbf{L}$

$$\text{rank}_{\mathbf{L}}(x) = \min\{\beta \mid x \in \mathbf{L}_{\beta+1}\}$$

Lemma 3.14. For any α

$$L_\alpha = \{x \in \mathbf{L} \mid \text{rank}_{\mathbf{L}}(x) < \alpha\}$$

Lemma 3.15. For any ordinal α

1. $L_\alpha \cap \mathbf{On} = \alpha$
2. $\alpha \in \mathbf{L} \cap \text{rank}_{\mathbf{L}}(\alpha) = \alpha$

Proof. since " α is a cardinal" is absolute for any transitive set.

$$\begin{aligned} \alpha &= L_\alpha \cap \mathbf{On} = \{\eta \in L_\alpha \mid \eta \text{ is a ordinal}\} \\ &= \{\eta \in L_\alpha \mid (\eta \text{ is an ordinal}^{L_\alpha})\} \in \text{Def}(L_\alpha) \end{aligned}$$

□

Lemma 3.16. for any ordinal α

1. $L_\alpha \in L_{\alpha+1}$
2. any finite subset of L_α belongs to $L_{\alpha+1}$

Lemma 3.17. 1. $\forall n \in \omega (L_n = V_n)$

2. $L_\omega = V_\omega$

Lemma 3.18. If **AC**, then for any $\alpha \geq \omega, |L_\alpha| = |\alpha|$

Theorem 3.19. $\mathbf{L} \models \mathbf{ZF}$

3.3 Axiom of constructibility and relativization

Theorem 3.20 (Axiom of constructibility). $\mathbf{V} = \mathbf{L}$

Lemma 3.21. function $\alpha \mapsto L_\alpha$ is absolute for any transitive model of **ZF**

Theorem 3.22. $\mathbf{L} \models \mathbf{ZF} + \mathbf{V} = \mathbf{L}$

Proof. $(\mathbf{V} = \mathbf{L})^{\mathbf{L}}$ is $\forall x \in \mathbf{L} \exists \alpha \in \mathbf{L} (x \in L_\alpha)^{\mathbf{L}}$. By 3.21, $(x \in L_\alpha)^{\mathbf{L}} \Leftrightarrow x \in L_\alpha$. Hence $\mathbf{L} \models \mathbf{V} = \mathbf{L}$ \square

Hence

Theorem 3.23. $\text{Con}(\mathbf{ZF}) \rightarrow \text{Con}(\mathbf{ZF} + \mathbf{V} = \mathbf{L})$

Theorem 3.24. Suppose transitive proper class $\mathbf{M} \models \mathbf{ZF} - \text{Pow}$, then
 $\mathbf{L} = \mathbf{L}^{\mathbf{M}} \subseteq \mathbf{M}$

Proof. For any ordinal α , since \mathbf{M} is proper, $\mathbf{M} \not\subseteq V_\alpha$. Hence there is $x \in \mathbf{M}$ s.t. $\text{rank}(x) \geq \alpha$. Since rank is absolute, $\text{rank}(x) \in \mathbf{M}$. And \mathbf{M} is transitive, hence $\alpha \in \mathbf{M}$. By 3.21, $L_\alpha \in \mathbf{M}$

$$\begin{aligned} \mathbf{L}^{\mathbf{M}} &= \{x \in \mathbf{M} \mid (\exists \alpha \in \mathbf{On} (x \in L_\alpha))^{\mathbf{M}}\} \\ &= \{x \mid \exists \alpha \in \mathbf{On} \cap \mathbf{M} (x \in L_\alpha \cap \mathbf{M})\} \\ &= \{x \mid \exists \alpha \in \mathbf{On} (x \in L_\alpha)\} \\ &= \mathbf{L} \end{aligned}$$

\square

Definition 3.25. If transitive model $\mathbf{M} \models \mathbf{ZF}$ contains all ordinals, then it's an **inner model**

Lemma 3.26. there is a finite set of axioms $\{\psi_1, \dots, \psi_n\}$ of $\mathbf{ZF} - \text{Pow}$ s.t. ordinals, rank and L_α are absolute for any model of $\{\psi_1, \dots, \psi_n\}$

Lemma 3.27. If set M is transitive, then $M \cap \mathbf{On}$ is a ordinal and is the least that doesn't belong to M , denoted by α^M

Theorem 3.28. There is a finite subset $\{\psi_1, \dots, \psi_n\}$ of axioms of $\mathbf{ZF} - \text{Pow}$ satisfying

$$\forall M (M \text{ is transitive} \wedge \psi_1^M \wedge \dots \wedge \psi_n^M \rightarrow (L_{\alpha^M} = \mathbf{L}^M \subseteq M))$$

Theorem 3.29. There is a finite subset $\{\psi_1, \dots, \psi_{n+1}\}$ of axioms of $\mathbf{ZF} - \text{Pow} + \mathbf{V} = \mathbf{L}$ satisfying

1. If \mathbf{M} is a transitive proper class and $\psi_1^{\mathbf{M}} \wedge \cdots \wedge \psi_{n+1}^{\mathbf{M}}$, then $\mathbf{M} = \mathbf{L}$
2. $\forall M (M \text{ is transitive} \wedge \psi_1^M \wedge \cdots \wedge \psi_n^M \rightarrow (L_{\alpha^M} = M))$

Theorem 3.30. There is a well-ordering on \mathbf{L} . Hence $\mathbf{V} = \mathbf{L} \rightarrow \mathbf{AC}$

If $\mathbf{V} = \mathbf{L}$, hence $\aleph_\alpha \subseteq L_{\aleph_{\alpha+1}}$. Because $|L_{\alpha_{\alpha+1}}| = \aleph_{\alpha+1}$, $2^{\aleph_\alpha} \leq \aleph_{\alpha+1}$

Theorem 3.31. If $\mathbf{V} = \mathbf{L}$, then for any infinite ordinal α , $\mathcal{P}(L_\alpha) \subseteq L_{|\alpha|^+}$

Corollary 3.32 (ZF). $(\mathbf{AC} + \mathbf{GCH})^{\mathbf{L}}$

Theorem 3.33 (ZF). $\text{Con}(\mathbf{ZF}) \rightarrow \text{Con}(\mathbf{ZFC} + \mathbf{GCH})$

Theorem 3.34 (ZF). Suppose $S_0 = \{\psi_1, \dots, \psi_n\} \subseteq \mathbf{ZF} + \mathbf{V} = \mathbf{L}$, then

$$\mathbf{ZF} \vdash \exists M (|M| = \omega \wedge M \text{ is transitive} \wedge (\psi_1^M \wedge \cdots \wedge \psi_n^M))$$

Lemma 3.35. Suppose $\mathbf{V} = \mathbf{L}$. For any uncountable regular cardinal κ , $L_\kappa = H_\kappa$

Corollary 3.36. If κ is a uncountable regular cardinal, then $\kappa \models \mathbf{ZF} - \text{Pow} + \mathbf{V} = \mathbf{L}$. If κ is inaccessible, then $L_\kappa \models \mathbf{ZF} + \mathbf{V} = \mathbf{L}$

4 The end

Learn and forget