Probability Theory
A
Comprehensive
Course

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Contents

1 Basic Measure Theory

1.1 Classes of Sets

Definition 1.1. A class of sets $\mathcal{A}$ is called

- $\cap$**-closed** or a π**-system** if $A \cap B \in \mathcal{A}$ whenever $A, B \in \mathcal{A}$
- σ**-**$\cap$**-closed** (closed under countable intersection)
- $\cup$**-closed** (closed under unions)
- σ-$\cup$**-closed**
- **\\-closed**
- closed under complements

Definition 1.2 (σ-algebra)**.** A class of sets $\mathcal{A} \subset 2^{\Omega}$ is called a σ**-algebra** if it fullills the following three conditions

1. $\Omega \in \mathcal{A}$
2. $\mathcal{A}$ is closed under complements
3. $\mathcal{A}$ is closed under countable unions

Theorem 1.3. *If $\mathcal{A}$ is closed under complements, then we have the equivalence*

$$\begin{aligned} \mathcal{A} \textit{ is } \cap\textit{-closed} &\iff \mathcal{A} \textit{ is } \cup\textit{-closed} \\ \mathcal{A} \textit{ is } \sigma\text{-}\cap\textit{-closed} &\iff \mathcal{A} \textit{ is } \sigma\text{-}\cup\textit{-closed} \end{aligned}$$

Theorem 1.4. *Assume that $\mathcal{A}$ is $\setminus$-closed. Then the following statements hold:*

1. *$\mathcal{A}$ is $\cup$-closed*
2. *If in addition $\mathcal{A}$ is σ-$\cup$-closed, then $\mathcal{A}$ is σ-$\cup$-closed*
3. *Any countable (repectively finite) union of sets in $\mathcal{A}$ can be expressed as a countable (respectively finite) disjoint union of sets in $\mathcal{A}$*

Proof. 3. Assume that $A_1, A_2, \dots \in \mathcal{A}$

$$\bigcup_{n=1}^{\infty} A_n = A_1 \uplus (A_2 \setminus A_1) \uplus ((A_3 \setminus A_2) \setminus A_1) \uplus \dots$$

□

Definition 1.5. A class of sets $\mathcal{A} \subset 2^{\Omega}$ is called an **algebra** if the following three conditions are fulfilled

1. $\Omega \in \mathcal{A}$
2. $\mathcal{A}$ is $\setminus$-closed
3. $\mathcal{A}$ is $\cup$-closed

Theorem 1.6. *A class of sets $\mathcal{A} \subset 2^\Omega$ is an algebra if and only if the following three properties hold*

1. $\Omega \in \mathcal{A}$
2. *$\mathcal{A}$ is closed under complements*
3. *$\mathcal{A}$ is closed under intersections*

Definition 1.7. A class of sets $\mathcal{A} \subset 2^\Omega$ is called a **ring** if the following conditions hold

1. $\emptyset \in \mathcal{A}$
2. $\mathcal{A}$ is $\backslash$-closed
3. $\mathcal{A}$ is $\cup$-closed

Definition 1.8. A class of sets $\mathcal{A} \subset 2^\Omega$ is called a **semiring** if

1. $\emptyset \in \mathcal{A}$
2. for any two sets $A, B \in \mathcal{A}$ the difference set $B \backslash A$ is a finite union of mutually disjoint sets in $\mathcal{A}$
3. $\mathcal{A}$ is $\cap$-closed

Definition 1.9. A class of sets $\mathcal{A} \subset 2^\Omega$ is called a **λ-system** if

1. $\Omega \in \mathcal{A}$
2. for any two sets $A, B \in \mathcal{A}$ with $A \subset B$, $B \backslash A \in \mathcal{A}$
3. $\biguplus_{n=1}^\infty A_n \in \mathcal{A}$ for any choice of countably many pairwise disjoint sets $A_1, \dots \in \mathcal{A}$

Theorem 1.10.

1. *Every σ-algebra also is a λ-system, an algebra and a σ-ring*
2. *Every σ-ring is a ring, and every ring is a semiring*
3. *Every algebra is a ring. An algebra on a finite set Ω is a σ-algebra*

Definition 1.11 (liminf and limsup)**.** Let $A_1, A_2, \dots$ be a subset of Ω. The sets

$$\liminf_{n\to\infty} A_n := \bigcup_{n=1}^\infty \bigcap_{m=n}^\infty A_m \qquad \limsup_{n\to\infty} A_n := \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m$$

are called **limes inferior** and **limes superior**, respectively, of the sequence $(A_n)_{n\in\mathbb{N}}$

Remark.

1. $\liminf$ and $\limsup$ can be rewritten as

$$\liminf_{n\to\infty} A_n = \{\omega \in \Omega : \#\{n \in \mathbb{N} : \omega \notin A_n\} < \infty\}$$
$$\limsup_{n\to\infty} A_n = \{\omega \in \Omega : \#\{n \in \mathbb{N} : \omega \in A_n\} = \infty\}$$

In other words, limes inferior is the event where **eventually all** of the A_n occur. On the other hand, limes superior is the event where **infinitely many** of the A_n occur. In particular, $A_* := \liminf_{n\to\infty} A_n \subset A^* := \limsup_{n\to\infty} A_n$

2. We define the **indicator function** on the set A by

$$\mathbb{1}_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

With this notation

$$\mathbb{1}_{A_*} = \liminf_{n\to\infty} \mathbb{1}_{A_n} \quad \text{and} \quad \mathbb{1}_{A^*} = \limsup_{n\to\infty} \mathbb{1}_{A_n}$$

3. If $\mathcal{A} \subset 2^\Omega$ is a σ-algebra and if $A_n \in \mathcal{A}$ for every $n \in \mathbb{N}$, then $A_* \in \mathcal{A}$ and $A^* \in \mathcal{A}$

Theorem 1.12 (Intersection of classes of sets)**.** *Let I be an arbitrary index set, and assume that $\mathcal{A}_i$ is a σ-algebra for every $i \in I$. Hence the intersection*

$$\mathcal{A}_I := \{A \subset \Omega : A \in \mathcal{A}_i \text{ for every } i \in I\} = \bigcap_{i\in I} \mathcal{A}_i$$

is a σ-algebra. The analogous statement holds for rings, σ-rings, algebras and λ-systems. However, it fails for semirings

Theorem 1.13 (Generated σ-algebra)**.** *Let $\mathcal{E} \subset 2^\Omega$. Then there exists a smallest σ-algebra $\sigma(\mathcal{E})$ with $\mathcal{E} \subset \sigma(\mathcal{E})$*

$$\sigma(\mathcal{E}) := \bigcap_{\substack{\mathcal{A}\subset 2^\Omega \text{ is a } \sigma\text{-algebra} \\ \mathcal{A}\supset\mathcal{E}}} \mathcal{A}$$

$\sigma(\mathcal{E})$ is called the σ-algebra generated by $\mathcal{E}$. $\mathcal{E}$ is called a generator of $\sigma(\mathcal{E})$. Similarly, we define $\delta(\mathcal{E})$ as the λ-system generated by $\mathcal{E}$

Remark. The following three statements hold

1. $\mathcal{E} \subset \sigma(\mathcal{E})$
2. If $\mathcal{E}_1 \subset \mathcal{E}_2$, then $\sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$
3. $\mathcal{A}$ is a σ-algebra if and only if $\sigma(\mathcal{A}) = \mathcal{A}$

Theorem 1.14 ($\cap$-closed λ-system)**.** *Let $\mathcal{D} \subset 2^\Omega$ be a λ-system. Then*

$$\mathcal{D} \text{ is a } \pi\text{-system} \quad \Longleftrightarrow \quad \mathcal{D} \text{ is a } \sigma\text{-algebra}$$

Proof. "$\Longrightarrow$"

3. Let $A, B \in \mathcal{D}$. By assumption, $A \cap B \in \mathcal{D}$ and trivially $A \cap B \subset A$. Thus $A \backslash B = A \backslash (A \cap B) \in \mathcal{D}$. This implies that $\mathcal{D}$ is $\backslash$-closed. Thus by Theorem 1.4, works.

$\square$

Theorem 1.15 (Dynkin's π-λ theorem)**.** *If $\mathcal{E} \subset 2^{\Omega}$ is a π-system, then*

$$\sigma(\mathcal{E}) = \delta(\mathcal{E})$$

Proof. 1. $\supseteq$. $A^c = \Omega \backslash A$.

2. $\subseteq$. By Theorem 1.14, it is enough to show that $\delta(\mathcal{E})$ is a π-system. For any $B \in \delta(\mathcal{E})$ define

$$\mathcal{D}_B := \{A \in \delta(\mathcal{E}) : A \cap B \in \delta(\mathcal{E})\}$$

In order to show that $\delta(\mathcal{E})$ is a π system, it is enough to show that

$$\delta(\mathcal{E}) \subset \mathcal{D}_B \quad \text{for any } B \in \delta(\mathcal{E})$$

$\mathcal{D}_E$ is a λ-system

(a) $\Omega \cap E = E \in \delta(\mathcal{E})$. Hence $\Omega \in \mathcal{D}_E$

(b) For any $A, B \in \mathcal{D}_E$ with $A \subset B$, we have $(B \backslash A) \cap E = (B \cap E) \backslash (A \cap E) \in \delta(E)$

(c) Assume that $A_1, \ldots, \in \mathcal{D}_E$ are mutually disjoint. Hence

$$\left(\bigcup_{n=1}^{\infty} \right) \cap E = \biguplus_{n=1}^{\infty} (A_n \cap E) \in \delta(\mathcal{E})$$

By assumption, $A \cap E \in \mathcal{E}$ if $A, E \in \mathcal{E}$; thus $\mathcal{E} \subset \mathcal{E}_E$ if $E \in \mathcal{E}$. Hence $\delta(\mathcal{E}) \subset \delta(\mathcal{D}_E) = \mathcal{D}_E$ for any $E \in \mathcal{E}$. Hence we get that $B \cap E \in \delta(\mathcal{E})$ for any $B \in \delta(\mathcal{E})$ and $E \in \mathcal{E}$. This implies that $E \in \mathcal{E}_B$ for any $B \in \delta(\mathcal{E})$. Thus $\mathcal{E} \subset \mathcal{D}_B$ for any $B \in \delta(\mathcal{E})$. $\square$