1. **Solution:**

(a) B
If $P(A) = 0$ or $P(A) = 1$, then $P(A)P(\bar{A}) = 0 = P(A\bar{A})$.
If $0 < P(A) < 1$, then $P(A)P(\bar{A}) \neq 0 = P(A\bar{A})$

(b) D
If $a = 3, 2, 1/2$, then $P\{X = i\} \leq 0$, therefore A,B,C cannot be the answer.
Another method:

$$\sum_{i=1}^{\infty} p\{X = i\} = 1 \Rightarrow \frac{2(1-2a)}{1-a} = 1(a < 1) \Rightarrow a = 1/3$$

(c) A
The pdf of Y should be function of y^2, therefore B,D cannot be the answer. And

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} ce^{-\frac{y^2}{8}} I_{(y>0)} \mathrm{d}y \\ &= c\int_{0}^{\infty} e^{-\frac{y^2}{8}} \mathrm{d}y \\ &= 2c\int_{0}^{\infty} e^{-\frac{u^2}{2}} \mathrm{d}u \quad (u = y/2) \\ &= 2c\sqrt{\frac{\pi}{2}} \end{aligned}$$

$\Rightarrow c = \frac{1}{\sqrt{2\pi}}$
Another method:

$$\begin{aligned} f_Y(y) &= f_X(x)\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| I_{(x>0)} + f_X(x)\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| I_{(x<0)} \\ &= \frac{1}{2} f_X\left(\frac{y}{2}\right) I_{(y>0)} + \frac{1}{2} f_X\left(-\frac{y}{2}\right) I_{(y>0)} \\ &= f_X\left(\frac{y}{2}\right) I_{(y>0)} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{8}} \end{aligned}$$

(d) D

$$P(\bar{A}) = 1/3, P(\bar{B}) = 2/3$$

(e) C

$$Var\left(\frac{X-EX}{VarX}\right) = Var\left(\frac{X-EX}{\sqrt{VarX}\sqrt{VarX}}\right) = \frac{1}{(\sqrt{VarX})^2} Var\left(\frac{X-EX}{\sqrt{VarX}}\right) = \frac{1}{VarX} \times 1$$

(f) B

Let $X \sim N(0,1)$, then

$$\begin{aligned}
E(|X|) &= E(|X||X>0)P(X>0) + E(|X||X<0)P(X<0) \\
&= \frac{1}{2}E(X|X>0) - \frac{1}{2}E(X|X<0) \\
&= E(X|X>0) \\
&= \frac{1}{P(X>0)}\int_0^\infty x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\mathrm{d}x \\
&= 2\times\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-t}\mathrm{d}t \quad (t=x^2) \\
&= \sqrt{\frac{2}{\pi}}
\end{aligned}$$

(g) A
EX exists, and the pdf is symmetrical, therefore $EX = 0$.

(h) A

$$p_Y(y) = \int_0^1 p_{X|Y}(y|x)p_X(x)\mathrm{d}x = \frac{2}{3}(1+y)$$

Then

$$\frac{1}{3}(1+4x) = p_X(x) = \int_0^1 p_{Y|X}(x|y)p_Y(y)\mathrm{d}y = \int_0^1 p_{Y|X}(x|y)\frac{2}{3}(1+y)\mathrm{d}y$$

Therefore, the denominator of $p_{Y|X}(x|y)$ should be $1+y$. Consider A and C,

$$\frac{1}{3}(1+4x) = \int_0^1 \frac{y+cx}{y+1}\frac{2}{3}(y+1)\mathrm{d}y = \frac{2}{3}\int_0^1 (y+cx)\mathrm{d}y = \frac{1}{3}(1+2cx)$$

$\Rightarrow c = 2$

(i) B
If $a^2 > b^2$, then A:$b^2 - a^2$ and D:$\frac{1}{2}(b^2-a^2)$ both are incompatible with $VarX \geq 0$.
Assume the answer is C:$VarX = (b-a)^2$, we could let $a=0, b=1, EX=\mu$, then

$$E(X^2) = VarX + (EX)^2 = 1 + \mu^2$$

is incompatible with $X \leq 1$.
Another method:
$P(X=a) = P(X=b) = 1/2$ yields the maximum value of $VarX$.

2. **Solution:**

(a)

$$P(X=1|hit) = \frac{P(hit|X=1)P(X=1)}{P(hit|X=0)P(X=0)+P(hit|X=1)P(X=1)} = 1/2$$

(b)

$$\begin{aligned}
P(\sum_{i=1}^{10} Y_i = 5) &= P(\sum_{i=1}^{10} Y_i = 5|X=0)P(X=0) + P(\sum_{i=1}^{10} Y_i = 5|X=1)P(X=1) \\
&= 0.2\binom{10}{5}0.8^5 0.2^5 + 0.8\binom{10}{5}0.2^5 0.8^5 \\
&= \binom{10}{5}0.8^5 0.2^5
\end{aligned}$$

3. **Solution:**

   (a) $X_i = 1, 2...6 (i \geq 1)$, Y represents the number of steps to stop.

$$\begin{aligned} EY &= E(Y|X_1 \neq 6)P(X_1 \neq 6) + E(Y|X_1 = 6, X_2 \neq 6)P(X_1 = 6, X_2 \neq 6) \\ &\quad + E(Y|X_1 = 6, X_2 = 6)P(X_1 = 6, X_2 = 6) \\ &= \frac{5}{6}(EY + 1) + \frac{5}{36}(EY + 2) + \frac{1}{36} \times 2 \end{aligned}$$

   $\Rightarrow EY = 42$

   (b) Y_1 represents the number of steps to 6-6 and Y_2, 1-6.

$$\begin{aligned} P(Y_1 < Y_2) &= \frac{2}{36}P(Y_1 < Y_2|X_1 = 1or6, X_2 = 1) + \frac{1}{36}P(Y_1 < Y_2|X_1 = 1, X_2 = 6) \\ &\quad + \frac{1}{36}P(Y_1 < Y_2|X_1 = 6, X_2 = 6) + \frac{32}{36}P(Y_1 < Y_2|X_1 \neq 1or6, X_2 \neq 1or6) \\ &= \frac{1}{18}P(Y_1 < Y_2|X_1 = 1) + \frac{1}{36} \times 0 + \frac{1}{36} \times 1 + \frac{8}{9}P(Y_1 < Y_2) \\ &= \frac{1}{18}P(Y_1 < Y_2|X_1 = 1) + \frac{1}{36} + \frac{8}{9}P(Y_1 < Y_2) \end{aligned}$$

$$\begin{aligned} P(Y_1 < Y_2|X_1 = 1) &= P(Y_1 < Y_2|X_1 = 1, X_2 = 1)P(X_2 = 1|X_1 = 1) \\ &\quad + P(Y_1 < Y_2|X_1 = 1, X_2 = 6)P(X_2 = 6|X_1 = 1) \\ &\quad + P(Y_1 < Y_2|X_1 = 1, X_2 \neq 1or6)P(X_2 \neq 1or6|X_1 = 1) \\ &= \frac{1}{6}P(Y_1 < Y_2|X_1 = 1) + \frac{1}{6} \times 0 + \frac{2}{3}P(Y_1 < Y_2) \end{aligned}$$

   $\Rightarrow P(Y_1 < Y_2) = \frac{5}{12}, P(Y_1 > Y_2) = \frac{7}{12}$

4. **Proof:**

   (a) $X \sim Ge(p)$

$$\begin{aligned} P(X > m) &= \sum_{i=m+1}^{\infty} P(X = i) \\ &= \sum_{i=m+1}^{\infty} p(1-p)^{i-1} \\ &= (1-p)^m \sum_{i=1}^{\infty} p(1-p)^{i-1} \\ &= (1-p)^m \\ P(X > n + m) &= (1-p)^{n+m} \\ P(X > n + m|X > m) &= \frac{P(X > n + m, X > m)}{P(X > m)} \\ &= \frac{P(X > n + m)}{P(X > m)} \\ &= \frac{(1-p)^{n+m}}{(1-p)^m} \\ &= (1-p)^n \\ &= P(X > n) \end{aligned}$$

(b) $Y \sim Exp(\lambda)$

$$\begin{aligned} P(Y > s) &= 1 - P(Y \leq s) \\ &= 1 - (1 - \exp(-\lambda s)) \\ &= \exp(-\lambda s) \\ P(Y > s+t) &= \exp(-\lambda(s+t)) \\ P(Y > s+t|Y > s) &= \frac{P(Y > s+t, Y > t)}{P(Y > s)} \\ &= \frac{P(Y > s+t)}{P(Y > s)} \\ &= \frac{\exp(-\lambda(s+t))}{\exp(-\lambda s)} \\ &= \exp(-\lambda t) \\ &= P(Y > t) \end{aligned}$$

5. (a) **Proof:**

$$\begin{aligned} EX &= \sum_{i=1}^{\infty} iP(X = i) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} P(X = i) \\ &= \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} P(X = i) \\ &= \sum_{k=1}^{\infty} P(X \geq k) \end{aligned}$$

(b) **Solution:** $X \sim Ge(p)$

$$EX = \sum_{k=1}^{\infty} P(X \geq k) = \sum_{k=1}^{\infty} P(X > k-1) = \sum_{k=1}^{\infty} (1-p)^{k-1} = \frac{1}{1-(1-p)} = \frac{1}{p}$$

6. **Solution:**

(a)

$$\begin{aligned} P(U = 1, V = 1) &= P(X = 1, Y = 1) = P(X = 1)P(Y = 1) = 4/9 \\ P(U = 1, V = 2) &= 0 \\ P(U = 2, V = 1) &= P(X = 1, Y = 2) + P(X = 2, Y = 1) \\ &= P(X = 1)P(Y = 2) + P(X = 2)P(Y = 1) = 4/9 \\ P(U = 2, V = 2) &= P(X = 2, Y = 2) = 1/9 \end{aligned}$$

(b)

$$\begin{aligned} P(U = 1) &= P(U = 1, V = 1) + P(U = 1, V = 2) = 4/9 \\ P(U = 2) &= P(U = 2, V = 1) + P(U = 2, V = 2) = 5/9 \end{aligned}$$

$$EU = 14/9 \quad DU = 20/81$$

$$P(V=1) = P(U=1, V=1) + P(U=2, V=1) = 8/9$$
$$P(V=2) = P(U=1, V=2) + P(U=2, V=2) = 1/9$$

$$EV = 10/9 \quad DV = 8/81$$

7. **Solution:**

(a) $\forall 0 < x < 1$

$$p_X(x) = \int_{-x}^{x} \mathrm{d}y = 2x$$

Otherwise, $p_X(x) = 0$.

$$p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)} = \frac{1}{2x}, \quad \forall 0 < x < 1$$

$\forall -1 < y < 1$

$$p_Y(y) = \int_{|y|}^{1} \mathrm{d}x = 1 - |y|$$

Otherwise, $p_Y(y) = 0$.

$$p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)} = \frac{1}{1-|y|}, \quad \forall -1 < y < 1$$

(b) $\forall 0 < x < 1$

$$E(Y|X=x) = \int_{-x}^{x} y p_{Y|X}(y|x)\mathrm{d}y = \int_{-x}^{x} \frac{y}{2x}\mathrm{d}y = x/2$$

$\Rightarrow E(Y|X) = X/2 \quad (0 < X < 1)$
$\forall -1 < y < 1$

$$E(X|Y=y) = \int_{|y|}^{1} x p_{X|Y}(x|y)\mathrm{d}x = \int_{|y|}^{1} \frac{x}{1-|y|}\mathrm{d}x = (1+|y|)/2$$

$\Rightarrow E(X|Y) = (1+|Y|)/2 \quad (-1 < Y < 1)$

(c)

$$P\left\{|Y| < \frac{1}{3}\middle| X = \frac{1}{2}\right\} = \int_{-1/3}^{1/3} p_{Y|X}(y|1/2)\mathrm{d}y = 2/3$$

$p_{X|Y}(x|-1/2) = 2 \quad \forall 1/2 < x < 1$, otherwise $p_{X|Y}(x|-1/2) = 0$.

$$P\left\{X < \frac{1}{3}\middle| Y = -\frac{1}{2}\right\} = \int_{-\infty}^{1/3} p_{X|Y}(x|-1/2)\mathrm{d}x = 0$$

8. X_n represents the number of steps, Y represent the position of the maximum integer.

$$
\begin{aligned}
EX_n &= \sum_{i=1}^{n} E(X_n|Y=i)P(Y=i) \\
&= \frac{1}{n}\sum_{i=1}^{n} E(X_n|Y=i) \\
&= \frac{1}{n}\sum_{i=1}^{n} [E(X_n|Y=n)+n-i] \\
&= \frac{1}{n}\sum_{i=1}^{n} (EX_{n-1}+n-i) \\
&= EX_{n-1}+\frac{n-1}{2} \\
&\dots \\
&= \frac{n(n-1)}{4}
\end{aligned}
$$