《复变函数引论》期末考试部分解答
(2004.1)

3. 计算积分(20分)

$$\int_{-\infty}^{+\infty} \frac{x-1}{x^5-1}\mathrm{d}x$$

解:

$R(z)=\frac{z-1}{z^5-1}$在上半平面的极点是$z_k=e^{\frac{2k\pi i}{5}},k=1,2.$

$\mathrm{Res}[R(z),z_k]=\frac{z_k-1}{5z_k}=\frac{1}{5}(z_k^2-z_k).$

积分$=\ 2\pi i\sum_{k=1}^{2}\mathrm{Res}[R(z),z_k]\ =\ \frac{2\pi i}{5}(z_2^2-z_2+z_1^2-z_1)\ =\ \frac{2\pi i}{5}(z_2^2-z_1)\ =\ \frac{2\pi i}{5}(e^{\frac{8\pi i}{5}}-e^{\frac{2\pi i}{5}})=\frac{2\pi i}{5}(e^{\frac{-2\pi i}{5}}-e^{\frac{2\pi i}{5}})=\frac{4\pi}{5}\sin\frac{2\pi}{5}.$

4. 假设$f(z)$在包含简单闭曲线γ的区域内解析,证明

$$\int_\gamma \overline{f(z)}f'(z)\mathrm{d}z$$

是纯虚数.(10分)

证明:

设$f=u+iv$,则$f'=u_x+v_x$,

$\mathrm{Re}(\int_\gamma \bar{f}f'\mathrm{d}z)=\int_\gamma(uu_x+vv_x)\mathrm{d}x+(u_xv-uv_x)\mathrm{d}y=\int_\gamma(uu_x+vv_x)\mathrm{d}x+(v_yv+uu_y)\mathrm{d}y=\int_\gamma \mathrm{d}(\frac{u^2+v^2}{2})=0.$

6. (20分)

(1) 把$\ln\frac{\sin z}{z}$展成z的幂级数,直到z^6;

解:

$\frac{\sin z}{z}=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^6}{7!}+o(z^6)$

$\ln\frac{\sin z}{z}=\ln[1-(\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}+o(z^6))=-\sum_1^\infty\frac{1}{n}(\frac{z^2}{3!}-\frac{z^4}{5!}+\frac{z^6}{7!}+o(z^6))^n=-\frac{z^2}{6}-\frac{z^4}{180}-\frac{z^6}{2835}+o(z^6).$

(2) 把函数$f(z)=\frac{z^2-2z+5}{(z-2)(z^2+1)}$在区域$0<|z-2|<\sqrt{5}$内展成Laurent级数.

解:

$f(z)=\frac{1}{z-2}+i(\frac{1}{z-i}-\frac{1}{z+i})$.
$\frac{1}{z-i}=\frac{1}{z-2+2-i}=\frac{1}{2-i}\cdot\frac{1}{1+\frac{z-2}{2-i}}=-\sum_0^\infty\frac{(z-2)^n}{(-2+i)^{n+1}}$,
$\frac{1}{z+i}=\frac{1}{z-2+2+i}=\frac{1}{2+i}\cdot\frac{1}{1+\frac{z-2}{2+i}}=-\sum_0^\infty\frac{(z-2)^n}{(-2-i)^{n+1}}$.
所以
$f(z)=\frac{1}{z-2}+i\sum_0^\infty[\frac{1}{(-2-i)^{n+1}}-\frac{1}{(-2+i)^{n+1}}](z-2)^n]$
$=\frac{1}{z-2}-2\sum_0^\infty 5^{-\frac{n+1}{2}}\sin[(n+1)\arccos(-\frac{2}{\sqrt{5}})](z-2)^n$.

7. (20分)
(2)找一个共形映射把区域$\mathbb{C}\setminus\{z:|z|=1,\mathrm{Im}z\geq 0\}$映到单位圆$|w|>1$外,并把$\infty$映到$\infty$.
解:
作映射$z_1=i\frac{z+1}{z-1}$把区域$\mathbb{C}\setminus\{z:|z|=1,\mathrm{Im}z\geq 0\}$映到$D_1=\mathbb{C}\setminus\{z:\mathrm{Re}z\geq 0,\mathrm{Im}z=0\}$, $\infty\mapsto i$;
映射$z_2=\sqrt{z_1}$把D_1映到上半平面, $i\mapsto\frac{1+i}{\sqrt{2}}$;
映射$w=\frac{z_2-\frac{1-i}{\sqrt{2}}}{z_2-\frac{1+i}{\sqrt{2}}}$把上半平面映到单位圆外, $\frac{1+i}{\sqrt{2}}\mapsto\infty$.
$w=\frac{\sqrt{i\frac{z+1}{z-1}}-\frac{1-i}{\sqrt{2}}}{\sqrt{i\frac{z+1}{z-1}}-\frac{1+i}{\sqrt{2}}}$就是所要的共形映射.