Intro. to Complex Analysis

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1. Given $n \in \mathbb{N}, R_0 \in \mathbb{R}$, maximize $|z^{2n} + \alpha|$ where $|z| \leq R_0$. Find both *every* possible z and the maximum value. (10 pts.)

2. Calculate $2^{3\mathrm{i}}$ and its principal value. (5 pts.)
   Calculate the real part and the imaginary part of $\cos(2(x + \mathrm{i}y))$. (5 pts.)

3. Write down Cauchy's integral formula. Then prove

$$|f^{(n)}(z_0)| \leq M(r)n!/r^n,$$

   where $M(r) = \max\limits_{|z-z_0|=r} |f(z)|$. Finally prove Liouville's theorem. (10 pts.)

4. Calculate complex integral

$$I_1 = \oint_{|z|=1} \frac{1 - \cos z^3}{z^m} \mathrm{d}z, m \in \mathbb{Z}.$$

   (10 pts.)

5. Calculate real integral

$$I_2 = \int_0^{+\infty} \frac{x^3 \cdot \sin(2x)}{(x^2 + a^2) \cdot (x^2 + b^2)} \mathrm{d}x.$$

   (10 pts.)

6. Calculate real integral

$$I_3 = \int_0^{2\pi} \frac{\mathrm{d}\theta}{A + B\cos\theta}.$$

   HINT: Calculate $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{a^2 \cdot \sin^2\theta + b^2 \cdot \cos^2\theta}$ first (10 pts.)

7. Prove that the mapping $w = \dfrac{az + b}{cz + d}$ maps $\mathrm{Im}(z) > 0$ into $\mathrm{Im}(w) > 0$ where $ad - bc > 0$ and $a, b, c, d \in \mathbb{R}$. (5 pts.)
   Write down a possible mapping which maps $a < \mathrm{Re}(z) < b$ into $|w| < 1$. (5 pts.)

8. Write down *all* possible fractional linear mappings which map $|z-z_0| < r$ into $|w-w_0| < R$. (10 pts.)

9. Write down *all* possible fractional linear mappings which map $|z| < 1$ into $\mathrm{Im}(w) > 0$. (10 pts.)

10. Calculate one of the following integrals: (10 pts.)

$$I_{r,n} = \int_0^{+\infty} \frac{\mathrm{d}x}{r^{2n} + x^{2n}}, \quad r > 0, n \in \mathbb{N};$$
$$J_{r,n} = \int_0^{+\infty} \frac{\mathrm{d}x}{(r^2 + x^2)^n}, \quad r > 0, n \in \mathbb{N}.$$