Understanding Topic Modeling: From Multivariate OLS to LDA

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Columbus Machine Learners meetup

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Agenda

- Introduction
- Prerequisite
- Non Negative Matrix Factorization
- Principal Component Analysis
- Latent Semantic Analysis
- Probabilistic Latent Semantic Analysis
- Latent Dirichlet Allocation
- Take home message

Introduction

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- PCA is a matrix factorization algorithm (MF).
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- LDA is a Bayesian approach to pLSA.
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- LSA is equivalent to PCA
- PCA is a matrix factorization algorithm (MF).
- MF is an application of OLS.
- The general idea of these algorithms is that:

$$W_{D\times V} \simeq Z_{D\times K} B_{K\times V}$$

where $K << V$

Introduction: practical example

Collapse a $W_{596\times1034}$ words counts into a $Z_{596\times2}$ matrix:

Table 1: Example of topics distributions when K = 2

	Topic.1	Topic.2
Alabama_2001_D_1.txt	0.75	0.25
Alabama_2002_D_2.txt	0.65	0.35
Alabama_2003_R_3.txt	0.26	0.74
Alabama_2004_R_4.txt	0.38	0.62
Alabama_2005_R_5.txt	0.50	0.50
Alabama_2006_R_6.txt	0.45	0.55

Z

Table 2: Words relative importance when K = 2

	Topic.1	Topic.2
abil	0.0004	0.001
abus	0.001	0.0004
academ	0.001	0.0000
acceler	0.0004	0.0000
accept	0.0002	0.001
access	0.003	0.0000
accomplish	0.001	0.001
accord	0.0000	0.001
account	0.001	0.002
achiev	0.003	0.001

B^T

Introduction: practical example

Table 3: List of words ordered by their relative importance for their respective topics. The list is used to infer the meaning of the topic.

Topic 1	Topic 2
school	budget
educ	fund
work	govern
help	peopl
econom	million
children	work
famili	make
health	public
busi	propos
nation	servic
make	chang
creat	program
student	know
teach	spend
invest	come

Prerequisite

Prerequisite: basic rules

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$$\begin{pmatrix} 4 & 3 & 1 \\ 2 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

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- Transpose of a matrix product:

$$(AB)^T = B^T A^T$$

Prerequisite: Simple OLS

- Extended form:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \\ \vdots & \vdots \\ X_{n,1} & X_{n,2} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

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  - Assuming $X^T X$ invertible, $(X^T X)^{-1} X^T y = \hat{\beta}$

Prerequisite: Multivariate OLS

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- Note: no distributional assumption is required.

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  - $\ell(B,\alpha) \simeq -\frac{\alpha}{2}[B^T(X^T X)B - B^T(X^T Y)]$
  - $$\frac{\partial \ell}{\partial B} = 0 \Longrightarrow (X^T X)B - (X^T Y) = 0$$

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Prerequisite: Bayesian Regression vs OLS

Prerequesite take home,

Method	Parameter estimate
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Bayesian	$\hat{B}_{bayes} = (X^T X + V_0)^{-1}[(X^T X)\hat{B}_{ols} + V_0 m_0]$

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Method	Parameter estimate	Topic Models
OLS	$\hat{B}_{ols} = (X^T X)^{-1} X^T Y$	MF/NMF
MLE	$\hat{B}_{mle} = (X^T X)^{-1} X^T Y$	pLSA
Bayesian	$\hat{B}_{bayes} = (X^T X + V_0)^{-1}[(X^T X)\hat{B}_{ols} + V_0 m_0]$	LDA

Non Negative Matrix Factorization (NMF)

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  - K is an arbitrary number.

NMF: Matrix Factorization (MF)

Table 6: Example matrix of words counts

	college	education	family	health	medicaid
document.1	4	6	0	2	2
document.2	0	0	4	8	12
document.3	6	9	1	5	6
document.4	2	3	3	7	10
document.5	0	0	3	6	9
document.6	4	6	1	4	5

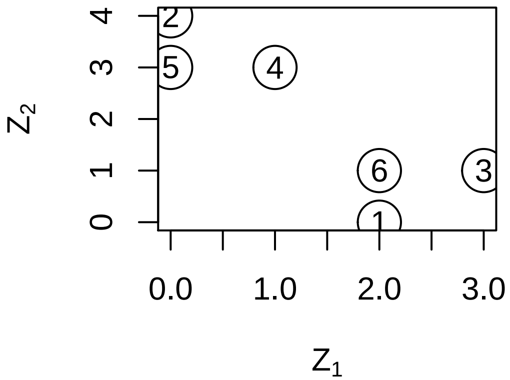
NMF: Matrix Factorization (MF)

Example:

$$\underbrace{\begin{bmatrix} 4 & 6 & 0 & 2 & 2 \\ 0 & 0 & 4 & 8 & 12 \\ 6 & 9 & 1 & 5 & 6 \\ 2 & 3 & 3 & 7 & 10 \\ 0 & 0 & 3 & 6 & 9 \\ 4 & 6 & 1 & 4 & 5 \end{bmatrix}}_{\mathbf{W_{6\times 5}}} \simeq \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 3 & 1 \\ 1 & 3 \\ 0 & 3 \\ 2 & 1 \end{bmatrix}}_{\mathbf{Z_{6\times 2}}} \underbrace{\begin{bmatrix} 2 & 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}}_{\mathbf{B_{2\times 5}}}$$

NMF: Matrix Factorization (MF)

Scatterplot based on the Z matr



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- But, we do not have Z; however, we can write:

$$\hat{Z}_{D\times K} = W B^T [B B^T]^{-1}$$

NMF: Matrix Factorization (MF)

MF: Iterative Multivariate Least Square algorithm

- Write:

$$W_{D\times V} = Z_{D\times K} B_{K\times V} + \epsilon_{D\times V}$$

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- Initialize random Z, and itteratively solve for B and Z.

Non Negative Matrix Factorization (NMF)

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- Impose constraints such that: $Z_{i,j} \geq 0$, and $B_{i,j} \geq 0$

$$W_{D \times V} \simeq Z^{nmf} B^{nmf}$$

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- Let $D_{W_{d,d}} = \sum_{v=1}^{V} W_{d,v}$ and $D_{B_{k,k}} = \sum_{v=1}^{V} B_{k,v}$ be some normalizing matrices.
- Then, Z^* and B^* can be interpreted as probabilities:

$$D_W^{-1} W = \left[D_W^{-1} Z D_B \right] \left[D_B^{-1} B \right]$$

$$\Longleftrightarrow$$

$$W^* = Z^* B^*$$

NMF: Matrix Factorization (MF)

Observation:

- $$\hat{B}_{K \times V} = \left[Z^T Z \right]^{-1} Z^T W = P_{K \times D} W_{D \times V}$$
- $$\hat{B} = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,V} \\ B_{2,1} & B_{2,2} & \cdots & B_{1,V} \\ \vdots & \vdots & \ddots & \vdots \\ B_{K,1} & B_{K,2} & \cdots & B_{K,V} \end{pmatrix}$$
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- $$\hat{Z}_{d,k} = \sum_{v=1}^{V} Q_{v,k} W_{d,v}$$

Principal Component Analysis (PCA)

PCA: Spectral decomposition

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- Solution: find an $\perp$ matrix $\tilde{B}$ such that $Z = W\tilde{B}$ is $\perp$.

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  - We want Z to be non correlated (orthogonal($\perp$));
  - We also want to preserve the variance of the W matrix.
- Solution: find an $\perp$ matrix $\tilde{B}$ such that $Z = W\tilde{B}$ is $\perp$.
- Observe that if $Z = W\tilde{B}$, then:

$$
\begin{aligned}
C_Z &= \frac{1}{n-1} Z^T Z \\
&= \frac{1}{n-1} \left[\tilde{B}^T W^T W \tilde{B} \right] \\
&= \tilde{B}^T \left[\frac{1}{n-1} W^T W \right] \tilde{B} \\
&= \tilde{B}^T C_W \tilde{B}
\end{aligned}
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  - Then compute $Z = WE = W\tilde{B}$

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- To check if Z is $\perp$, use the theorem and set $\tilde{B} = E$,

$$\begin{aligned}
C_Z &= \tilde{B}^T C_W \tilde{B} \\
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- **Definition:** The total variance is the trace of the covariance matrix

$$\begin{aligned} tr(C_Z) &= tr(D) \\ &= tr(\tilde{B}^T C_W \tilde{B}) \\ &= tr(E^T C_W E) \\ &= tr(EE^T C_W) \\ &= tr(C_W) \end{aligned}$$

PCA: Spectral decomposition

- As a dimension reduction method, we hope that there is a $K << V$ such that $\sum_{k=1}^{K} d_{k,k} \simeq tr(C_W)$; in which case, $Z_{D\times K} \simeq W_{D\times V} E_{V\times K}$ approximates $W_{D\times V}$.

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- Where $B_{K\times V} = E^T_{K\times V}$ and $Z_{D\times K} = W_{D\times V} E_{V\times K}$

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- U, V are orthonormal matrices, i.e. $U^T U = UU^T = I_D$, $V^T V = VV^T = I_V$. S is a diagonal matrix containing the $r = min(D, V)$ singular values $\sigma_k \geq 0$ on the main diagonal, with 0s filling the rest of the matrix.

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- If $W_{D\times V}$ are zero means V variables, the covariance matrix:

$$\begin{aligned} C_W &= \frac{1}{n-1} W^T W \\ &= \frac{1}{n-1} VSU^T USV^T \\ &= \frac{1}{n-1} VS^2 V^T \\ &= VDV^T \end{aligned}$$

PCA: Singular Value Decomposition (SVD)

- If there is a K such that $\sigma_{K+i} \simeq 0$, for $i = 1, 2, \cdots, V - K$, we can approximate $W_{D\times V}$, by

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- Along the spirit of $W \simeq ZB$, let's define $Z = US$, and $B = V^T$. Then, we can write:

$$W \simeq ZB$$

Latent Semantic Analysis (LSA)

LSA

- LSA is an application of SVD to a matrix of words counts.

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- Example

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  - It does not assume any distribution for the data.
- pLSA was proposed to address these concerns

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- Assume $p(w_v|d_i)$ is the probability of observing the word w_v in the document d_i.

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- Then:

$$\begin{aligned} p(w_v|d_i) &= \sum_{z\in Z} p(w_v, z|d_i) \\ &= \sum_{z\in Z} p(w_v|z, d_i)p(z|d_i) \\ &= \sum_{z\in Z} p(w_v|z)p(z|d_i) \end{aligned}$$

Probabilistic Latent Semantic Analysis (pLSA)

- A document is a collection of $N_{d_i} = \sum_v^V n_{d_i, w_v}$ words, assumed independent. Therefore:

$$p(w_1, w_2, \cdots, w_V | d_i) = \prod_{v=1}^{V} p(w_v | d_i)^{n(d_i, w_v)}$$

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- Assuming D independent documents,

$$L(\theta | W) = p(W | D) = \prod_{d=1}^{D} \prod_{v=1}^{V} p(w_v | d_i)^{n(d_i, w_v)}$$

$$\mathcal{L}(\theta | W) = \sum_{d=1}^{D} \sum_{v=1}^{V} n(d_i, w_v) log \left(\sum_{z \in Z} p(w_v | z) p(z | d_i) \right)$$

Probabilistic Latent Semantic Analysis (pLSA)

- $$p(z_k|d_i, w_v) = \frac{p(w_v|z_k)p(z_k|d_i)}{\sum_{l=1}^{K} p(w_v|z_l)p(z_l|d_i)}$$

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Probabilistic Latent Semantic Analysis (pLSA)

- $$p(w_v, d_i) = \sum_{Z} p(z)p(w_v|z)p(d_i|z) = \sum_{z_k=1}^{K} p(d_i|z_k)p(z_k)p(w_v|z_k)$$
- Let's define $U = [p(d_i|z_k)]_{D\times K}$, $V^T = [p(w_v|z_k)]_{K\times V}$, and $S = [p(z_k)]_{K\times K}$.
- Then, it follows that:

$$\begin{aligned}
[p(w_v, d_i)]_{D\times V} &= \sum_{z_k=1}^{K} p(d_i|z_k)p(z_k)p(w_v|z_k) \\
&= [p(d_i|z_k)]_{D\times K} [p(z_k)]_{K\times K} [p(w_v|z_k)]_{K\times V} \\
&= USV^T
\end{aligned}$$

Latent Dirichlet Allocation (LDA)

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- LDA is a Bayesian treatment of pLSA

$$p(z_k|d) = \theta_{d,k}$$
$$p(w_v|z_k) = \phi_{k,v}$$

$$\theta_d \sim Dirichlet_K(\alpha)$$
$$\phi_k \sim Dirichlet_V(\beta_k)$$

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- By VB,

$$\theta_d | w_d, \tilde{\boldsymbol{\alpha}} \sim Dirichlet_K(\tilde{\boldsymbol{\alpha}}_{\boldsymbol{d}})$$

$$\phi_k | w, \tilde{\boldsymbol{\beta}} \sim Dirichlet_V(\tilde{\boldsymbol{\beta}}_{\boldsymbol{k}})$$

Latent Dirichlet Allocation (LDA)

$$E(z_{d,v,\cdot}) = exp(E(log(\theta_{d,\cdot})) + E(log(\phi_{\cdot,v})))$$

$$E(\theta_d|\tilde{\alpha_d}) = \frac{\alpha + \sum_{v=1}^{V} n_{d,v} \times E(z_{d,v,.})}{\sum_{k=1}^{K}[\alpha + \sum_{v=1}^{V} E(z_{d,v,k})]}$$

$$E(\phi_k|\tilde{\beta_k}) = \frac{\beta + \sum_{d=1}^{D} n_{d,v} \times E(z_{d,\cdot,k})}{\sum_{v=1}^{V}(\beta + \sum_{d=1}^{D} n_{d,v} \times E(z_{d,v,k}))}$$

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