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SYMMETRIC-TRIANGULAR DECOMPOSITION AND ITS APPLICATIONS PART I: THEOREMS AND ALGORITHMS *

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Abstract.

A new decomposition of a nonsingular matrix, the Symmetric times Triangular (ST) decomposition, is proposed. By this decomposition, every nonsingular matrix can be represented as a product of a symmetric matrix S and a triangular matrix T. Furthermore, S can be made positive definite. Two numerical algorithms for computing the ST decomposition with positive definite S are presented.

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1 Introduction.

It is well-known that symmetry and positive-definiteness are very nice properties for performing matrix computations. For symmetric and positive definite matrices, there is the Cholesky decomposition, and for systems of equations there are excellent iterative methods, e.g., the conjugate gradient method [12]. Also there are many methods for computing the generalized eigenvalues for symmetric or skew symmetric matrices, cf. [6, 7, 14, 17, 19], and these methods have very good convergence properties.

For the nonsymmetric or the symmetric indefinite case, we are not as fortunate as in the symmetric and positive definite case even though there exist many direct methods and iterative methods [2, Chapters 2, 7], [3, Chapters 2, 6], [9, Chapters 3–5, 9–10]. We still find many difficulties in solving nonsymmetric or symmetric indefinite problems. An important application, for example, is the augmented system

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, \tag{1.1}$$

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from various areas, such as, constrained optimization [1, 2, 18], generalized least squares problems [10, 15, 16, 20, 21, 22], and numerical methods for partial differential equations [4, 5, 8].

Since the properties of nonsymmetric matrices are quite different from that of the symmetric case, we have serious difficulties. This motivates us to find some way to make use of symmetry or positive-definiteness for solving nonsymmetric problems. In this paper, we shall introduce a symmetric and triangular (ST) decomposition to transform nonsymmetric (or symmetric indefinite) matrices to symmetric matrices, or symmetric and positive definite matrices.

We show that for every nonsingular matrix, whose leading principal submatrices are nonsingular, there exist a unit triangular matrix T and a symmetric matrix S such that $A = TS$. We also prove that for such a decomposition, S may be positive definite if T is just triangular. The great advantage of the new decomposition is to make use of properties of symmetric and positive definite matrices to study nonsymmetric matrices. Numerical experiments showing the efficacy of the methods will be described in another paper.

The outline of this paper is as follows. We present our main result, the ST decomposition, in Section 2. We prove that every nonsingular matrix A with nonsingular leading principal submatrices, can be factored as $A = TS$, where S is symmetric (or symmetric and positive definite) and T triangular. Two numerical algorithms for such decompositions are presented in Section 3. Some remarks on stability and uniqueness of algorithms are given too.

Throughout this paper we assume that the principle leading minors of the matrix A are non-zero unless otherwise noted. In Part II of this paper, we will give a number of applications [11]. We note that there is a similar decomposition in [12] to the one presented here.

2 The ST decomposition.

We shall prove our main results on symmetric and triangular decomposition in this section.

THEOREM 2.1. *For every nonsingular and nonsymmetric $n \times n$ matrix A, whose leading principal submatrices are nonsingular, there exists a decomposition $A = ST$ where S is symmetric and T is unit triangular.*

PROOF. We shall prove the result by induction on n. Without loss of generality, suppose that the triangular matrix T is unit upper triangular. For $n = 2$, and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

we can obtain

$$S = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} - a_{21}t_{12} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & t_{12} \\ 0 & 1 \end{pmatrix}$$

such that $A = ST$, where $t_{12} = (a_{12} - a_{21})/a_{11}$. Here, S is symmetric and nonsingular from the nonsingularity of A.

Suppose that $A_n = S_n T_n$ holds for $n = k$. Now we like to show that it is still true for $n = k + 1$. For $n = k + 1$ we write

$$A = \begin{pmatrix} A_k & a_{k+1} \\ \\ \tilde{a}_{k+1}^T & \alpha_{k+1} \end{pmatrix} \tag{2.1}$$

and take

$$S = \begin{pmatrix} S_k & s_{k+1} \\ \\ s_{k+1}^T & \beta_{k+1} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} T_k & t_{k+1} \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

It follows from $A = ST$ that

$$\begin{cases} S_k T_k = A_k, \\ s_{k+1}^T T_k = \tilde{a}_{k+1}^T, \\ S_k t_{k+1} + s_{k+1} = a_{k+1}, \\ \beta_{k+1} + s_{k+1}^T t_{k+1} = \alpha_{k+1}. \end{cases} \tag{2.3}$$

Since A_k is nonsingular, S_k and T_k are nonsingular by the induction assumption. Hence, we get the unique solution from (2.3) as follows:

$$\begin{cases} s_{k+1} = T_k^{-T} \tilde{a}_{k+1}, \\ t_{k+1} = S_k^{-1}(a_{k+1} - s_{k+1}), \\ \beta_{k+1} = \alpha_{k+1} - s_{k+1}^T t_{k+1}. \end{cases} \tag{2.4}$$

Therefore, $A = ST$ is well defined for $n = k + 1$. From the nonsingularity of A, it is easy to check that S is nonsingular. By induction, we have proved the theorem. □

Similarly, we can prove the following decomposition.

THEOREM 2.2. *For every nonsingular and nonsymmetric $n \times n$ matrix A, whose leading principal submatrices are nonsingular, there exists a decomposition $A = TS$ where S is symmetric and T is unit triangular.*

The following example illustrates existence of the new decomposition.

EXAMPLE 2.1. Suppose that the matrix A is a p-cyclic matrix and is given by

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & \cdots & a_{1p} \\ a_{21} & a_{22} & \ddots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a_{p,p-1} & a_{pp} \end{pmatrix}.$$

Here we consider $p = 4$. Then we have $AT = S$, where

$$S = \begin{pmatrix} a_{11} & a_{21} & 0 & 0 \\ a_{21} & \hat{a}_{22} & a_{23} & 0 \\ 0 & a_{23} & \hat{a}_{33} & a_{34} \\ 0 & 0 & a_{34} & \hat{a}_{44} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & t_{12} & 0 & t_{14} \\ 0 & 1 & t_{23} & t_{24} \\ 0 & 0 & 1 & t_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\begin{aligned}
&t_{12} = \frac{a_{21}}{a_{11}}, \qquad t_{14} = -\frac{a_{14}}{a_{11}}, \\
&t_{23} = \frac{a_{23}}{a_{22}}, \qquad t_{24} = -\frac{t_{14}a_{21}}{a_{22}}, \qquad t_{34} = \frac{a_{34} - a_{33}t_{34}}{a_{33}}, \\
&\hat{a}_{22} = a_{22} + t_{12}a_{21}, \\
&\hat{a}_{33} = a_{33} + t_{23}a_{23}, \qquad a_{44} = a_{44} + a_{34}t_{34}.
\end{aligned}$$

If we allow the matrix T to change from a unit triangular to a triangular matrix then S can be made symmetric and *positive definite*.

THEOREM 2.3. *For every nonsingular and nonsymmetric $n \times n$ matrix A, whose leading principal submatrices are nonsingular, there exist S and T such that $A = TS$, where T is triangular and S is symmetric and positive definite.*

PROOF. We prove the theorem by induction. For simplicity of proof, we suppose that T is a lower triangular matrix. To show that S is symmetric positive definite it is sufficient to show that it has a decomposition $S = LL^T$ with L nonsingular and lower triangular. For $n = 1$, we take $t_{11} = \operatorname{sign}(a_{11})$ and $S = |a_{11}|$. Then, T is triangular and S is symmetric and positive definite. Now we consider $n = 2$ and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Since $a_{11} \neq 0$ and $\det(A) \neq 0$ we can define $t_{11} = a_{11}$, $t_{22} = \det(A)$, $t_{21} = a_{21} - a_{12}\det(A)/a_{11}$, $s_{11} = 1$, $s_{12} = s_{21} = a_{12}/a_{11}$ and $s_{22} = 1/a_{11} + (a_{12}/a_{11})^2$. It is easy to check that $A = TS$ and S is positive definite.

Now, suppose that the result is true for $n = k$. For $n = k + 1$, we

$$A = \begin{pmatrix} A_k & a_{k+1} \\ \\ \tilde{a}_{k+1}^T & \alpha \end{pmatrix}.$$

We partition the matrices T, S and L similarly as follows

$$T = \begin{pmatrix} T_k & 0 \\ \\ t_{k+1}^T & \beta \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} L_k & 0 \\ \\ l_{k+1}^T & \tau \end{pmatrix} \begin{pmatrix} L_k^T & l_{k+1} \\ \\ 0 & \tau \end{pmatrix}.$$

By the induction hypothesis, $A_k = T_k L_k L_k^T$. It follows from $A = TS$ that

$$\begin{aligned}
l_{k+1} &= L_k^{-1} T_k^{-1} a_{k+1}, \\
t_{k+1} &= L_k^{-T}(l_{k+1} - \beta L_k^{-T} \tilde{a}_{k+1}), \\
\tau &= \sqrt{\frac{\alpha - \tilde{a}_{k+1}^T L_k^{-T} l_{k+1}}{\beta}},
\end{aligned}$$

where we choose $\beta \neq 0$ such that $(\alpha - \tilde{a}_{k+1}^T L_k^{-T} l_{k+1})/\beta \geq 0$. To show that S is positive definite, we must prove that $\alpha - \tilde{a}_{k+1}^T L_k^{-T} l_{k+1} = \alpha - \tilde{a}_{k+1}^T A_k^{-1} a_{k+1} \neq 0$.

Since A_k is nonsingular,

$$\bar{A} = \begin{pmatrix} A_k^{-1} & 0 \\ -\tilde{a}_{k+1}^T A_k^{-1} & 1 \end{pmatrix}$$

is nonsingular. From the nonsingularity of A it follows that

$$\bar{A}A = \begin{pmatrix} I & A_k^{-1} a_{k+1} \\ 0 & \alpha - \tilde{a}_{k+1}^T A_k^{-1} a_{k+1} \end{pmatrix}$$

is nonsingular. Thus, $\alpha - \tilde{a}_{k+1}^T A_k^{-1} a_{k+1} \neq 0$ and therefore, $\tau > 0$ and S is positive definite. By the induction, we obtain the desired result. $\square$

Note that the matrices S and T in Theorem 2.3 are positive definite and triangular respectively, but just symmetric and unit triangular respectively in Theorem 2.2. Similarly we can show the following results as well.

THEOREM 2.4. *For every nonsingular and nonsymmetric $n \times n$ matrix A, whose leading principal submatrices are nonsingular, there exist T and S such that $A = ST$, where T is triangular and S is symmetric and positive definite.*

THEOREM 2.5. *For every nonsingular and nonsymmetric $n \times n$ matrix A, whose leading principal submatrices are nonsingular, there exist T and S such that $TA = S = LL^T$ or $AT = S = LL^T$, where T is triangular and S is symmetric and positive definite.*

3 Algorithms.

Since symmetric and positive definite matrices have very nice properties, we shall derive an algorithm to reduce a nonsingular and nonsymmetric matrix A to a symmetric and positive definite matrix S.

Since S is symmetric and positive definite, there exists a lower triangular matrix L such that $S = LL^T$. Suppose that there exist lower triangular matrices T and L such that $TA = LL^T$. Assume that

$$T = \begin{pmatrix} T_k & 0 & 0 \\ t_{k+1}^T & \beta & 0 \\ \bar{T}_k & \bar{t}_{k+1} & T_{n-k} \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} L_k & 0 & 0 \\ l_{k+1}^T & \alpha & 0 \\ \bar{L}_k & \bar{l}_{k+1} & L_{n-k} \end{pmatrix}.$$

Consequently, A has the following form:

$$A = \begin{pmatrix} A_k & a_{k+1} & \hat{A}_k \\ \tilde{a}_{k+1} & \tau & \bar{a}_{k+1}^T \\ A_{n-k} & a_{n-k} & \bar{A}_{n-k} \end{pmatrix}.$$

It follows from $LL^T = TA$ that

$$L_k L_k^T = T_k A_k, \tag{3.1}$$

$$L_k l_{k+1} = T_k a_{k+1}, \tag{3.2}$$

$$A_k^T t_{k+1} = L_k l_{k+1} - \beta \tilde{a}_{k+1}, \tag{3.3}$$

and

$$l_{k+1}^T L_k^T A_k^{-1} a_{k+1} + (\tau - \tilde{a}_{k+1}^T A_k^{-1} a_{k+1})\beta = \alpha^2 + \|l_{k+1}\|_2^2. \tag{3.4}$$

In terms of (3.2), we obtain

$$l_{k+1} = L_k^{-1} T_k a_{k+1}. \tag{3.5}$$

It follows from (3.3) and (3.4) by (3.1) that

$$\beta(\tau - \tilde{a}_{k+1}^T A_k^{-1} a_{k+1}) = \alpha^2, \tag{3.6}$$

and

$$t_{k+1} = A_k^{-T}(L_k l_{k+1} - \beta \tilde{a}_{k+1}). \tag{3.7}$$

By the nonsingularity of A, $\tau - \tilde{a}_{k+1}^T A_k^{-1} a_{k+1} \neq 0$. So, we take $\beta \neq 0$ such that $\beta(\tau - \tilde{a}_{k+1}^T A_k^{-1} a_{k+1}) > 0$. Then, the method is well defined. Therefore, we have the following algorithm to decompose A as product of lower triangular matrix T and symmetric and positive definite matrix $S = LL^T$. The method to find T and L is called the *escalator method* (see [13, p. 127]).

Algorithm 3.1. Set t_{11} such that $t_{11}a_{11} > 0$, and $l_{11} = \sqrt{t_{11}a_{11}}$;

For $k = 1, \ldots, n-1$
  $l_{k+1} = L_k^{-1} T_k a_{k+1}$,
  $\hat{l}_{k+1} = L_k^{-1} \tilde{a}_{k+1}$,
  $s = a_{k+1,k+1} - \hat{l}_{k+1}^T l_{k+1}$,
  Choose $t_{k+,k+1} \neq 0$ such that $\gamma = t_{k+1,k+1} s > 0$ (or big enough),
  $l_{k+1,k+1} = \sqrt{\gamma}$,
  $t_{k+1} = L_k^{-T}(l_{k+1} - t_{k+1,k+1}\hat{l}_{k+1})$.

We can also derive another version of the decomposition as $TLL^T = A$ as follows:

Algorithm 3.2. Set t_{11} such that $t_{11}a_{11} > 0$, and $l_{11} = \sqrt{t_{11}a_{11}}$;

For $k = 1, \ldots, n-1$ $\quad$ $l_{k+1} = L_k^{-1} T_k^{-1} a_{k+1}$,
  $\hat{l}_{k+1} = L_k^{-1} \tilde{a}_{k+1}$,
  $s = a_{k+1,k+1} - \hat{l}_{k+1}^T l_{k+1}$,
  Choose $t_{k+,k+1} \neq 0$ such that $\gamma = s / t_{k+1,k+1} > 0$ (or big enough),
  $l_{k+1,k+1} = \sqrt{\gamma}$,
  $t_{k+1} = L_k^{-T}(l_{k+1} - t_{k+1,k+1}\hat{l}_{k+1})$.

Both these algorithms require $2n^3/3$ flops.

For general nonsingular matrices, we need permutations such that the leading principal submatrices are nonsingular. For example, if A is 2×2 matrix with $a_{11} = 0$, and $a_{12}a_{21} \neq 0$, then, we cannot apply the algorithm to A directly. But we can apply the algorithm to find the desired decomposition after a one step permutation. Since we do not need to keep symmetry of A to get the ST decomposition, we can use pivoting to obtain a stable decomposition. For example, the symmetric indefinite matrix

$$A = \begin{pmatrix} \epsilon & 1 \\ 1 & 0 \end{pmatrix}$$

is not stable for LDL^T decomposition. After we apply permutation (or pivoting), we can find a stable symmetric positive definite and triangular decomposition as follows:

$$PA = \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where} \quad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that the symmetric and triangular decomposition is not unique, especially for the symmetric positive definite and triangular decomposition and it is also not necessary to keep the symmetry of the original matrices. This gives us a chance to choose the right diagonal elements of T so that the decomposition is stable. The following example illustrates the case.

Example 3.1. Let

$$A = \begin{pmatrix} \epsilon_1 & 1 \\ 1 & \epsilon_2 \end{pmatrix},$$

where ϵ_1, $\epsilon_2 << 1$. Then, we have the decomposition as follows:

$$\begin{pmatrix} \alpha & 0 \\ \alpha\epsilon_2 - \frac{\beta\epsilon_1}{1-\epsilon_1\epsilon_2} & \frac{\beta}{1-\epsilon_1\epsilon_2} \end{pmatrix} P \begin{pmatrix} \epsilon_1 & 1 \\ 1 & \epsilon_2 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha\epsilon_2 \\ \alpha\epsilon_2 & \beta + \alpha\epsilon_2^2 \end{pmatrix},$$

where

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$\alpha > 0$ and $\beta > 0$. The matrix in the right side is symmetric and positive definite. In fact, P is pivoting matrix. The decomposition is stable. [11].

Some applications of the new decomposition will be given in Part II.

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