ST 705 Linear models and variance components
Lab practice problem set 6

February 17, 2020

1. Let X be an $n \times p$ matrix with $\operatorname{rank}(X) = r$, and let C be a $(p-r) \times p$ matrix. If

   (i) $\operatorname{rank}(C) = p - r$ and

   (ii) $\operatorname{column}(X') \cap \operatorname{column}(C') = \{0\}$,

   then
   $$\operatorname{rank}\begin{pmatrix} X \\ C \end{pmatrix} = p.$$

   Proof. Let $\{u_1, \ldots, u_r\}$ be a basis for $\operatorname{column}(X')$, and $\{u_{r+1}, \ldots, u_p\}$ be a basis for $\operatorname{column}(C')$. Next, show that the vectors in $\{u_1, \ldots, u_r\}$ are linearly independent of the vectors in $\{u_{r+1}, \ldots, u_p\}$. Let $j \in \{r+1, \ldots, p\}$ and suppose that $u_j = \sum_{i=1}^{r} a_i u_i$, for some coefficients $a_1, \ldots, a_r$. Accordingly, $a_j \in \operatorname{column}(X')$, but by construction $u_j \in \operatorname{column}(C')$. Thus, $u_j \in \operatorname{column}(X') \cap \operatorname{column}(C') = \{0\}$. Since $u_1, \ldots, u_r, u_{r+1}, \ldots, u_p$ are all basis vectors they cannot be the zero vector, and so the vectors in $\{u_1, \ldots, u_r\}$ must be linearly independent of the vectors in $\{u_{r+1}, \ldots, u_p\}$.

   Moreover, since $u_1, \ldots, u_p \in \mathbb{R}^p$ are a set p linearly independent vectors, they form a basis for $\mathbb{R}^p$. Therefore, any $v \in \mathbb{R}^p$ can be expressed as
   $$v = \sum_{i=1}^{p} b_i u_i = \underbrace{\sum_{i=1}^{r} b_i u_i}_{\in \operatorname{column}(X')} + \underbrace{\sum_{i=r+1}^{p} b_i u_i}_{\in \operatorname{column}(C')},$$
   for some coefficients $b_1, \ldots, b_p$, and so there exist vectors z and w such that
   $$v = X'z + C'w = \begin{pmatrix} X' & C' \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} \in \operatorname{column}\begin{pmatrix} X' & C' \end{pmatrix}.$$

   Hence, $\mathbb{R}^p \subseteq \operatorname{column}\begin{pmatrix} X' & C' \end{pmatrix} \subseteq \mathbb{R}^p$, and so $p = \operatorname{rank}\begin{pmatrix} X' & C' \end{pmatrix} = \operatorname{rank}\begin{pmatrix} X \\ C \end{pmatrix}$. ■

2. Denote by W a matrix with $\text{column}(W) = \text{null}(P')$. Show that $\text{null}(W') = \text{column}(P)$.

3. Consider the restricted linear model $Y = X\beta + U$ over the constrained parameter space $\{P'\beta = \delta\}$, for some full-column rank matrix P. Set up the Langrangian function and derive the *restricted normal equations* (RNE),

$$\begin{pmatrix} X'X & P \\ P' & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \theta \end{pmatrix} = \begin{pmatrix} X'y \\ \delta \end{pmatrix}.$$

4. Prove that there exists a solution to the RNE.