Solution HW 1

August 30, 2019

Section 1.8 of Rice; Exercises 2, 4, 5, 7, 8, 62, 64, 66, 68, 69

1.2

a.

$$S = \{(n_1, n_2)|1 \leq n_1, n_2 \leq 6\}$$

b.

$$\begin{aligned} A =& \{(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,2)(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\} \\ B =& \begin{array}{l}\{(2,1),(3,2),(3,1),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2),\\ (6,3),(6,4),(6,5)\}\end{array} \\ C =& \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\} \end{aligned}$$

c.

$$\begin{aligned} &A \cap C = C \\ &B \cup C = \begin{array}{l}\{(2,1),(3,2),(3,1),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4),(6,1),(6,2)\\ (6,3),(6,4),(6,4)\}\end{array} \\ &A \cap (B \cup C) = \begin{array}{l}\{(3,2),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(6,1),\\ (6,2),(6,3),(6,4),(6,5)\}\end{array} \end{aligned}$$

1.5

$$\Omega = (A \cap B)^c \cap (A \cup B)$$

1.7

$$\begin{aligned} &P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1 \\ &\Rightarrow P(A \cap B) \geq P(A) + P(B) - 1 \end{aligned}$$

1.8 According to the addition law in Property D,

$$
\begin{aligned}
P(\bigcup_{i=1}^{n} A_i) &= P(A_1) + P(\bigcup_{i=2}^{n} A_i) - P(A_1 \bigcap (\bigcup_{i=2}^{n} A_i)) \\
&= P(A_1) + P(A_2) + P(\bigcup_{i=3}^{n} A_i) - P(A_1 \bigcap (\bigcup_{i=2}^{n} A_i)) - P(A_2 \bigcap (\bigcup_{i=3}^{n} A_i)) \\
&\dots \\
&= \sum_{i=1}^{n} P(A_j) - \sum_{i=1}^{n-1} P(A_i \bigcap (\bigcup_{j=i+1}^{n} A_j))
\end{aligned}
$$

Since probability is always greater or equal to 0, $\sum_{i=1}^{n-1} P(A_i \bigcap (\bigcup\limits_{j=i+1}^{n} A_j)) \geq 0$. Thus

$$
P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_j)
$$

1.62 According to the law of total probability

$$
\begin{aligned}
P(A) &= P(A|E)P(E) + P(A|E^c)P(E^c) \\
&\geq P(B|E)P(E) + P(B|E^c)P(E^c) = P(B)
\end{aligned}
$$

1.64 There are three steps to verify that $Q(A) = P(A|B)$ satisfies the axioms for a probability measure.

(1) $Q(\Omega) = P(\Omega|B) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

(2) If $A \subset \Omega$, then

$$
Q(A) = P(A|B) = \frac{P(A \cap B)}{P(B)} \geq \frac{P(\emptyset \cap B)}{P(B)} = 0
$$

(3) If A_1 and A_2 are disjoint,

$$
\begin{aligned}
Q(A_1 \cup A_2) = P(A_1 \cup A_2|B) &= \frac{P((A_1 \cup A_2) \cap B)}{P(B)} \\
&= \frac{P\Big((A_1 \cap B) \cup (A_2 \cap B)\Big)}{P(B)} \text{(Distribution laws)}
\end{aligned}
$$

Since A_1 and A_2 are disjoint, $A_1 \cap B$ and $A_2 \cap B$ are also disjoint. Thus

$$
\frac{P\Big((A_1 \cap B) \cup (A_2 \cap B)\Big)}{P(B)} = \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} = Q(A_1) + Q(A_2)
$$

Finally, we proved that $Q(A_1 \cup A_c) = Q(A_1) + Q(A_2)$ if A_1 and A_2 are disjoint.

1.66 Recall the definition of independence. A and B are independent if $P(A \cap B) = P(A)P(B)$.

In this problem, for $\forall A$,

$$P(\emptyset \cap A) = P(\emptyset) = 0 = P(\emptyset)P(A).$$

Thus $\emptyset$ is independent of A for any A.

1.68 This statement is false.

Flip a coin and roll a die.

A=Head on the coin flip

B=Six on the dice roll

C=Tail on the coin flip

A and B are independent and B and C are independent but A and C are not independent.

1.69 If A and B are disjoint, $P(A \cap B) = P(\emptyset) = 0$. If A and B are further independent, we have $P(A \cap B) = P(A)P(B)$.

$P(A)P(B) = 0$ is true if A or B is $\emptyset$. Thus if A and B are disjoint, A and B are independent only if at least one of A and B is $\emptyset$.