

# ST 705 Linear models and variance components

## Homework problem set 1

January 8, 2020

1. Prove the following theorem. Let  $V$  be a vector space and  $B = \{u_1, \dots, u_n\}$  be a subset of  $V$ . Then  $B$  is a basis if and only if each  $v \in V$  can be expressed *uniquely* as

$$v = a_1 u_1 + \dots + a_n u_n$$

for some set of scalars  $\{a_1, \dots, a_n\}$ .

2. Prove that the eigenvalues of an upper triangular matrix  $M$  are the diagonal components of  $M$ . Write pseudo-code for an algorithm that exploits this fact for more efficiently computing the matrix exponential  $e^M$  (recall the definition of the matrix exponential from your linear algebra course).
3. The defining property of a projection matrix  $A$  is that  $A^2 = A$  (recall the definition of the square of a matrix from your linear algebra course). Establish the following facts.
  - (a) If  $A$  is a projection matrix, then all of its eigenvalues are either zero or one.
  - (b) If  $A \in \mathbb{R}^{p \times p}$  is a projection and symmetric (i.e., an orthogonal projection matrix), then for every vector  $v$  the projection  $Av$  is orthogonal to  $v - Av$ .
  - (c)  $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$ .
  - (d)  $\text{tr}(AB) = \text{tr}(BA)$ .

4. Let  $A \in \mathbb{R}^{p \times p}$  be symmetric. Use the spectral decomposition of  $A$  to show that

$$\sup_{x \in \mathbb{R}^p \setminus \{0\}} \frac{x'Ax}{x'x} = \lambda_{\max},$$

where  $\lambda_{\max}$  is the largest eigenvalue of  $A$ . Observe that this is a special case of the Courant-Fischer theorem (see [https://en.wikipedia.org/wiki/Min-max\\_theorem](https://en.wikipedia.org/wiki/Min-max_theorem)).

5. Let  $x = (x_1, \dots, x_p)' \in \mathbb{R}^p$ . Show that for  $i \in \{1, \dots, p\}$ ,

$$|x_i| \leq \|x\|_2 \leq \|x\|_1,$$

where  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are the  $l_1$  and  $l_2$  vector norms, respectively.

6. Show that every eigenvalue of a real symmetric matrix is real.
7. Show that if  $X \sim N_p(\mu, \Sigma)$  and  $Y = X'AX$ , then  $E(Y) = \text{tr}(A\Sigma) + \mu' A \mu$ .
8. Let  $U$  and  $V$  be random variables. Establish the following inequalities.
  - (a)  $P(|U + V| > a + b) \leq P(|U| > a) + P(|V| > b)$  for every  $a, b \geq 0$ .
  - (b)  $P(|UV| > a) \leq P(|U| > a/b) + P(|V| > b)$  for every  $a \geq 0$  and  $b > 0$ .