Markov chain Monte Carlo simple example

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MCMC Example (Metropolis-Hastings)

Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Exp}(\lambda)$. Then the likelihood function of the data is

$$\ell(x_1, \ldots, x_n | \lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}.$$

Assume the prior density $\pi(\lambda) := \text{Gamma}(\lambda | a, b)$.

The posterior density is then given by

$$\pi(\lambda | x_1, \ldots, x_n) = \frac{\ell(x_1, \ldots, x_n | \lambda) \cdot \pi(\lambda)}{\int \ell(x_1, \ldots, x_n | \lambda) \cdot \pi(\lambda)\ d\lambda} \propto \underbrace{\ell(x_1, \ldots, x_n | \lambda) \cdot \pi(\lambda)}_{=: f(\lambda)}.$$

Outline of a random walk Metropolis-Hastings algorithm:

Step 1. Given current $\lambda^{(t)}$, propose a new $\lambda^\star \sim \mathrm{N}(\cdot|\lambda^{(t)}, \sigma^2)$

Step 2. Set

$$\lambda^{(t+1)} = \begin{cases} \lambda^\star & \text{w.p. } \rho(\lambda^\star, \lambda^{(t)}) \\ \lambda^{(t)} & \text{w.p. } 1 - \rho(\lambda^\star, \lambda^{(t)}) \end{cases}$$

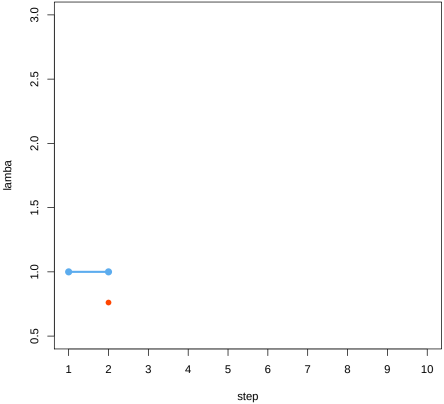
where

$$\begin{aligned} \rho(\lambda^\star, \lambda^{(t)}) &= \min\left\{ \frac{\pi(\lambda^\star|x_1, \ldots, x_n) \cdot \mathrm{N}(\lambda^{(t)}|\lambda^\star, \sigma^2)}{\pi(\lambda^{(t)}|x_1, \ldots, x_n) \cdot \mathrm{N}(\lambda^\star|\lambda^{(t)}, \sigma^2)}, 1 \right\} \\ &= \min\left\{ \frac{f(\lambda^\star)}{f(\lambda^{(t)})}, 1 \right\}. \end{aligned}$$

This is called the Metropolis-Hastings acceptance ratio.

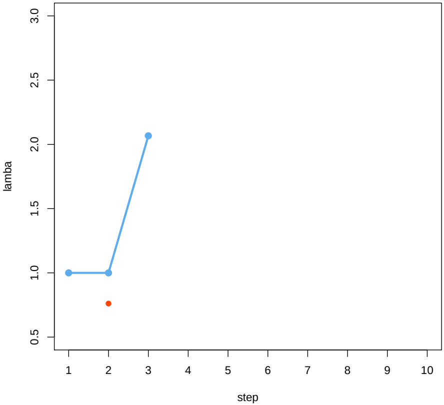
MCMC Example

$$\begin{aligned} \text{Current } \lambda &= 1 \\ \text{Proposed } \lambda &\ (\sim 1 + \mathrm{N}(0, 0.5^2)) = 0.7613 \\ \text{MH Ratio} &= 1e - 78 \\ \text{Coin-flip} &\ (\sim \mathrm{U}(0,1)) = 0.2788 \implies \text{Reject} \end{aligned}$$



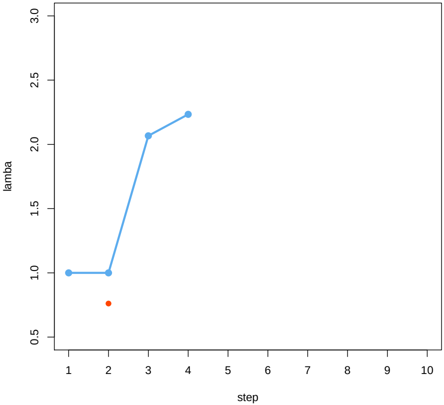
MCMC Example

$$\begin{aligned} \text{Current } \lambda &= 1 \\ \text{Proposed } \lambda & (\sim 1 + \mathrm{N}(0, 0.5^2)) = 2.0667 \\ \text{MH Ratio} &= 4e + 134 \\ \text{Coin-flip} & (\sim \mathrm{U}(0,1)) = 0.5027 \implies \text{Accept} \end{aligned}$$



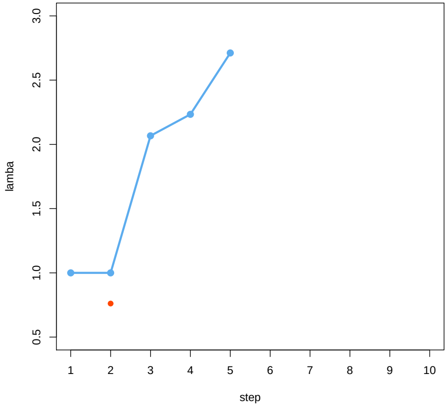
MCMC Example

$$\begin{aligned}
\text{Current } \lambda &= 2.0667 \\
\text{Proposed } \lambda & (\sim 2.0667 + \mathrm{N}(0, 0.5^2)) = 2.2337 \\
\text{MH Ratio} &= 3e + 05 \\
\text{Coin-flip} & (\sim \mathrm{U}(0,1)) = 0.3707 \implies \text{Accept}
\end{aligned}$$



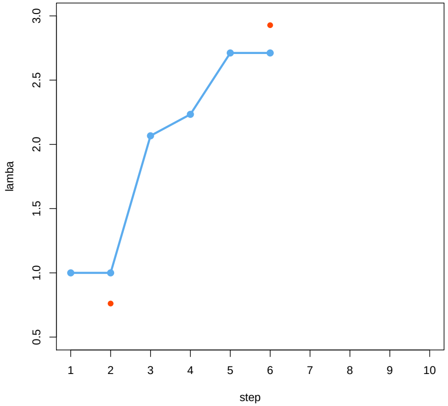
MCMC Example

$$
\begin{aligned}
\text{Current } \lambda &= 2.2337 \\
\text{Proposed } \lambda &\ (\sim 2.2337 + \mathrm{N}(0, 0.5^2)) = 2.7115 \\
\text{MH Ratio } &= 1964 \\
\text{Coin-flip } &\ (\sim \mathrm{U}(0,1)) = 0.2875 \implies \text{Accept}
\end{aligned}
$$



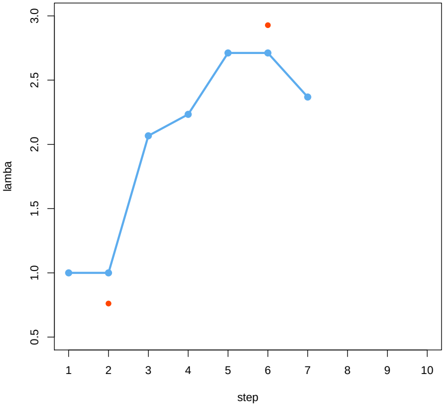
MCMC Example

$$\begin{aligned}
\text{Current } \lambda &= 2.7115 \\
\text{Proposed } \lambda &\ (\sim 2.7115 + \text{N}(0, 0.5^2)) = 2.9276 \\
\text{MH Ratio} &= 0.0005 \\
\text{Coin-flip} &\ (\sim \text{U}(0,1)) = 0.1298 \implies \text{Reject}
\end{aligned}$$



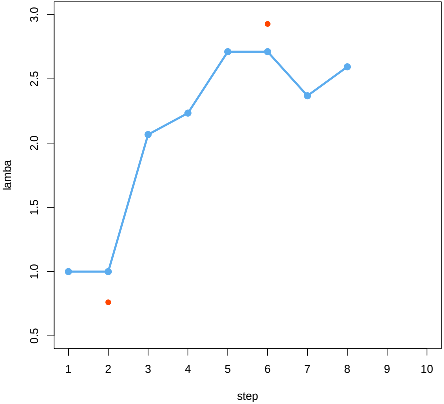
MCMC Example

$$
\begin{aligned}
\text{Current } \lambda &= 2.7115 \\
\text{Proposed } \lambda &\ (\sim 2.7115 + N(0, 0.5^2)) = 2.3685 \\
\text{MH Ratio} &= 0.2142 \\
\text{Coin-flip} &\ (\sim U(0,1)) = 0.1653 \implies \text{Accept}
\end{aligned}
$$



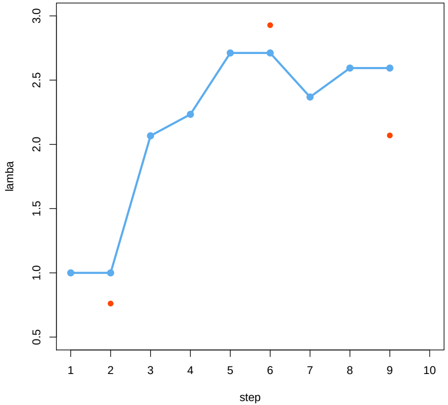
MCMC Example

$$\begin{aligned} \text{Current } \lambda &= 2.3685 \\ \text{Proposed } \lambda & (\sim 2.3685 + \mathrm{N}(0, 0.5^2)) = 2.5939 \\ \text{MH Ratio} &= 21 \\ \text{Coin-flip} & (\sim \mathrm{U}(0,1)) = 0.0457 \implies \text{Accept} \end{aligned}$$



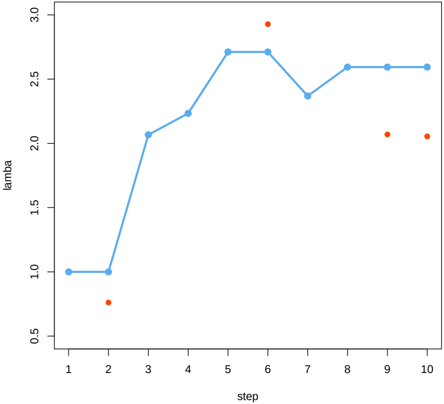
MCMC Example

$$\begin{aligned} \text{Current } \lambda &= 2.5939 \\ \text{Proposed } \lambda &\ (\sim 2.5939 + \mathrm{N}(0, 0.5^2)) = 2.0695 \\ \text{MH Ratio} &= 5e - 10 \\ \text{Coin-flip} &\ (\sim \mathrm{U}(0,1)) = 0.8348 \implies \text{Reject} \end{aligned}$$

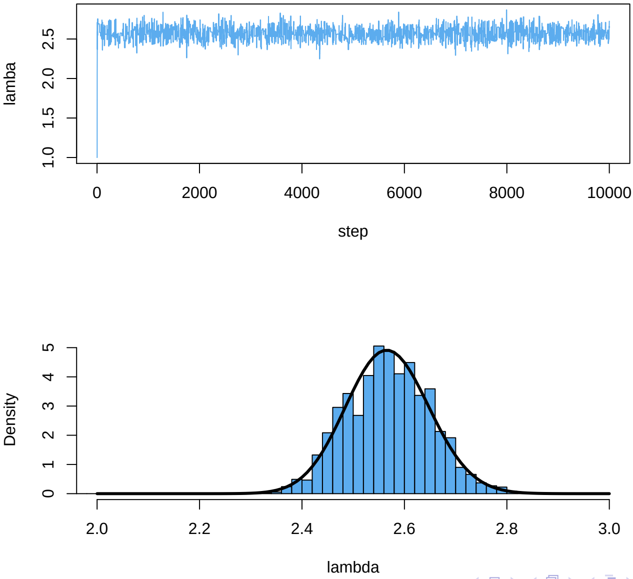


MCMC Example

$$\begin{aligned} \text{Current } \lambda &= 2.5939 \\ \text{Proposed } \lambda &(\sim 2.5939 + \mathrm{N}(0, 0.5^2)) = 2.0542 \\ \text{MH Ratio} &= 1e - 10 \\ \text{Coin-flip} &(\sim \mathrm{U}(0,1)) = 0.3117 \implies \text{Reject} \end{aligned}$$



MCMC Example



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