

# ST 705 FINAL EXAM

May 4, 2020

NAME:

- You have **48 hours** to complete this exam.

1. (3 points) Consider the model  $Y_i = \beta_0 + \beta_1 \cdot i + U_i$  for  $i \in \{1, \dots, 5\}$ , where  $U_1, \dots, U_5 \stackrel{\text{iid}}{\sim} N(0, 1)$ . Find the power of the F-test for testing whether the slope is zero when testing at level  $\alpha = 0.05$  and the slope takes values 0.1, 0.2, and 0.3.
2. (3 points) Let  $Y \sim N_n(X\beta, \sigma^2 I_n)$ , where  $X$  is an  $n \times p$  design matrix. Show that the best linear unbiased predictor (BLUP) of an unobserved response  $Y_*$  at the design point  $x_*$  is uncorrelated with all unbiased estimators of zero.
3. Suppose that  $Y_1, \dots, Y_n \stackrel{\text{iid}}{\sim} \text{Uniform}(0, 2\theta)$ , and define  $U_i := Y_i - \theta$  for  $i \in \{1, \dots, n\}$ .
  - (a) (3 points) Find the mean and variance of  $U := (U_1, \dots, U_n)'$ .
  - (b) (3 points) Show that  $Y := (Y_1, \dots, Y_n)'$  is generated according to a linear model that satisfies the Gauss-Markov assumptions.
  - (c) (3 points) Find the BLUE of  $\theta$ , and denote the BLUE by  $\hat{\theta}_{\text{OLS}}$ .
  - (d) (3 points) Find  $c$  so that the estimator  $\hat{\theta} = cY_{(n)}$  is unbiased for  $\theta$ , where  $Y_{(i)}$  denotes the  $i$ th order statistic, and compute the variance of  $\hat{\theta}$ .
  - (e) (3 points) Compare the variances of  $\hat{\theta}_{\text{OLS}}$  and  $\hat{\theta}$ , and provide intuition for your finding.
4. Suppose that  $(X, Y)$  has a bivariate distribution (**not necessarily Gaussian**) with mean  $(\mu_X, \mu_Y)'$  and covariance matrix
$$\begin{pmatrix} \sigma_X^2 & \sigma_{X,Y} \\ \sigma_{Y,X} & \sigma_Y^2 \end{pmatrix}.$$
  - (a) (3 points) Show that if  $E(Y | X) = \beta_0 + \beta_1 X$ , then  $\beta_1 = \sigma_{Y,X}/\sigma_X^2$  and  $\beta_0 = \mu_Y - \beta_1 \mu_X$ .
  - (b) (3 points) Show that if  $E(Y | X) = \beta_0 + \beta_1 X$  and  $\text{Var}(Y | X) = \tau^2$ , then  $\tau^2 = \sigma_Y^2 - \sigma_{Y,X}^2/\sigma_X^2$ .
5. (3 points) Let

$$Y \sim N_2 \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$
$$A = \frac{1}{8} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

and  $B = (1, -2)'$ . Find the joint distribution of  $Y'AY$  and  $B'Y$ .