# Solution HW 3

September 16, 2019

Section 5.4 of Rice; Exercises 1, 2, 5, 6, 21, 27, 29

5.1 $E(\bar{X}) = u$ and $Var(\bar{X}) = \frac{\sum_{i=1}^{n}\sigma_i^2}{n^2}$. According to Chebyshev's inequality, for any $\epsilon > 0$,

$$P(|\bar{X}_n - u| > \epsilon) \leq \frac{Var(\bar{X}_n)}{\epsilon^2} = \frac{\sum_{i=1}^{n}\sigma_i^2}{n^2\epsilon^2} \to 0$$

Thus $\bar{X} \to u$ in probability.

5.2 We have $E(X_i) = u_i$ which implies

$$E(\bar{X}_n) = \frac{\sum_{i=1}^{n} u_i}{n}.$$

By definition $\bar{X}_n \xrightarrow{p} u$ if

$$\lim_{n\to\infty} P(|\bar{X}_n - u| > \epsilon) = 0$$

This is equivalent to

$$\lim_{n\to\infty} P((\bar{X}_n - u)^2 > \epsilon^2) \leq \lim_{n\to\infty} \frac{E[(\bar{X}_n - u)^2]}{\epsilon^2}$$

by Markov's inequality. We want to show this limit is 0. Let's add and subtract $E(\bar{X}_n)$,

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{E[(\bar{X}_n - E(\bar{X}_n) + E(\bar{X}_n) - u)^2]}{\epsilon^2} \\
&= \lim_{n\to\infty} \frac{E[(\bar{X}_n - E(\bar{X}_n))^2]}{\epsilon^2} + \frac{2E[(\bar{X}_n - E(\bar{X}_n))(E(\bar{X}_n - u)]}{\epsilon^2} + \frac{E[(E(\bar{X}_n) - u)^2]}{\epsilon^2} \\
&= \lim_{n\to\infty} \frac{Var(\bar{X}_n)}{\epsilon^2} + \lim_{n\to\infty} + \frac{2E[(\bar{X}_n - E(\bar{X}_n))(E(\bar{X}_n - u)]}{\epsilon^2} + \lim_{n\to} \frac{E[(E(\bar{X}_n) - u)^2]}{\epsilon^2}
\end{aligned}$$

The first term is 0 by 5.1, the second term is 0 because $E(\bar{X}_n) - E(\bar{X}_n) = 0$, and the third term foes to 0 since $E(\bar{X}_n) = \frac{\sum_{i=1}^{n} u_i}{n} \to u \Rightarrow (E(\bar{X}_n) - u)^2 \to 0$

5.5 The moment generating function for the binomial distribution is

$$(1 - p + pe^t)^n = \exp\{n \log(1 - p + pe^t)\} \tag{1}$$

Using Taylor expansion of $\log(1-x)$,

$$(1) = \exp\{n(-p + pe^t - \frac{(p-pe^t)^2}{2} - \frac{(p-pe^t)^3}{3!} - \dots)\}.$$

Since $n \to \infty$, $p \to 0$ and $np \to \lambda$, the limit of above formula is

$$\exp\{-\lambda + \lambda e^t\},$$

which is the moment generating function of the Poisson distribution with parameter $\lambda$.

5.6 The moment generating function for this gamma distribution is

$$M_X(t) = (1-\lambda t)^{-\alpha} \quad t < \frac{1}{\lambda}.$$

Since $E(X) = \lambda\alpha$ and $Var(X) = \lambda^2\alpha$, the standardized version of $X$ is

$$Y = \frac{X - \lambda\alpha}{\lambda\sqrt{\alpha}}.$$

The moment generating of $Y$ is

$$\begin{aligned} M_Y(t) &= E[e^{tY}] = E[e^{t(X/(\lambda\sqrt{\alpha})-\sqrt{\alpha}}] \\ &= e^{-t\sqrt{\alpha}} M_X(\frac{t}{\lambda\sqrt{\alpha}}) \\ &= e^{-t\sqrt{\alpha}}(1 - \frac{t}{\sqrt{\alpha}})^{-\alpha} \quad t < \sqrt{\alpha}. \end{aligned}$$

Thus

$$\lim_{\alpha\to\infty} M_Y(t) = \lim_{\alpha\to\infty} e^{-t\sqrt{\alpha}}(1 - \frac{t}{\sqrt{\alpha}})^{-\alpha}$$

Let's consider the limit of the log version, i.e,

$$\begin{aligned} \lim_{\alpha\to\infty} \log M_Y(t) &= \lim_{\alpha\to\infty} -t\sqrt{\alpha} - \alpha\log(1 - \frac{t}{\sqrt{\alpha}}) \\ &\text{Using the Taylor expansion of } \log(1-x) \\ &= \lim_{\alpha\to\infty} -t\sqrt{\alpha} - \alpha(-\frac{t}{\sqrt{\alpha}} - \frac{t^2}{2\alpha} - \frac{t^3}{3\alpha^{3/2}} - \dots) \\ &= \lim_{\alpha\to\infty} -t\sqrt{\alpha} + t\sqrt{\alpha} + \frac{t^2}{2} + 0 \\ &= t^2/2 \end{aligned}$$

Thus

$$\lim_{\alpha\to\infty} M_Y(t) = e^{\frac{t}{2}},$$

which is the moment generating function of the standard normal distribution.

5.21 (a)

$$E(\hat{I}(f)) = \frac{1}{n}\sum_{i=1}^{n}\int_a^b \frac{f(X_i)}{g(X_i)} g(X_i) dX_i$$
$$= \frac{1}{n} n \int_a^b f(X) dX = I(f)$$

(b)First we try to find $Var(\frac{f(X)}{g(X)})$.

$$Var(\frac{f(X)}{g(X)}) = \int_a^b \frac{f(x)^2}{g(x)^2} g(x)dx - (\int_a^b f(x)dx)^2 = \int_a^b \frac{f(x)^2}{g(x)} dx - I(f)^2$$

Since $\frac{f(X_i)}{g(X_i)}$, $i = 1, \ldots, n$ are independent,

$$Var(\hat{I}(f)) = \frac{1}{n}[\int_a^b \frac{f(x)^2}{g(x)} dx - I(f)^2]$$

Finite example: $f(x) = x$ and $g(x) = 1$ when $x \in (0, 1)$.
Infinite example: $f(x) = 1$ and $g(x) = 2x$ when $x \in (0, 1)$
(c)When $g$ is chosen to be uniform distribution,

$$Var(\hat{I}(f)) = \frac{1}{n}[\int_0^1 f(x)^2 dx - I(f)^2]$$

Let $g^*(x) = \dfrac{f(x)}{\int_0^1 f(x)dx}$ and $g^*(x)$ could be verified as a density function on $[0, 1]$.

$$Var(\hat{I}^*(f)) = \frac{1}{n}[\int_0^1 \frac{f(x)^2}{f(x)}(\int_0^1 f(x)dx)dx - I(f)^2] = \frac{1}{n}[(\int_0^1 f(x)dx)^2 - I(f)^2]$$

According to Jensen's inequality, $\int_0^1 f(x)^2 dx \geq (\int_0^1 f(x)dx)^2$. Thus

$$Var(\hat{I}(f)) \geq Var(\hat{I}^*(f))$$

Thus this estimate could be improved by choosing $g^*$.

5.27 This problem is equivalent to prove

$$n \log(1 + a_n/n) \to a$$

Using the Taylor expansion of $\log(1 + x)$,

$$\lim_{\alpha\to\infty} n \log(1 + a_n/n) = \lim_{\alpha\to\infty} n(\frac{a_n}{n} - \frac{a_n^2}{2n^2} + \frac{a_n^3}{3n^3} - \ldots)$$
$$= \lim_{\alpha\to\infty} a_n - \frac{a_n^2}{2n} + \frac{a_n^3}{3n^2} - \ldots$$
$$= a$$

5.29 Fisrt we try to find the cdf of $U_{(n)}$,

$$F_{U_{(n)}}(x) = [F_X(x)]^n = x^n \quad x \in [0,1]$$

We can then get $E(U_{(n)}) = \int_0^1 nx^{n-1}xdx = \frac{n}{n+1}$, $E(U_{(n)}^2) = \frac{n}{n+2}$ and $Var(U_{(n)}^2) = E(U_{(n)}^2) - E(U_{(n)})^2 = \frac{n}{(n+2)(n+1)^2}$.

The standardized $U_{(n)}$ is $Z_n = \frac{U_{(n)} - n/(n+1)}{\sqrt{n/[(n+2)(n+1)^2]}} = \frac{(n+1)U_n - n}{\sqrt{n/(n+2)}}$.

$$\begin{aligned} P(Z_n \leq z) &= P(\frac{(n+1)U_{(n)} - n}{\sqrt{n/(n+2)}} \leq z) \\ &= P(U_{(n)} \leq \frac{z\sqrt{n/(n+2)} + n}{n+1}) \\ &= (\frac{\sqrt{\frac{n}{n+2}}z + n}{n+1})^n \end{aligned}$$

Take limit of both sides,

$$\lim_{n\to\infty} P(Z_n \leq z) = \lim_{n\to\infty}(\frac{\sqrt{\frac{n}{n+2}}z + n}{n+1})^n = \lim_{n\to\infty}\{1 + \frac{z}{n}\}^n = e^z$$