Solution HW 4

September 30, 2019

Section 6.4 of Rice; Exercises 1, 2, 3, 6, 8, 9

6.1 Define $Y = \sqrt{U/n}$ and $T = Z/\sqrt{U/n}$. First we try to find the pdf of Y. According to the definition of Y, $U = nY^2$ and $|\frac{\partial U}{\partial Y}| = Y^{n-2}$. Thus

$$f_Y(y) = f_U(ny^2) \times 2ny = \frac{n^{n/2}y^{n-1}}{2^{n/2-1}\Gamma(n/2)}e^{-ny^2/2}$$

According to the density function of the quotient of two independent random variables derived in Section 3.6.1,

$$f_T(t) = f_T(t = z/y) = \int_{-\infty}^{\infty} |y| f_Y(y) f_Z(z = yt) dy$$

$$= \int_0^{\infty} y \frac{n^{n/2}y^{n-1}}{2^{n/2-1}\Gamma(n/2)}e^{-ny^2/2} \times \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2t^2}{2}} dy$$

$$= \frac{n^{n/2}}{2^{n/2-1}\Gamma(n/2)\sqrt{2\pi}} \int_0^{\infty} y^n \exp\{-\frac{ny^2 + t^2y^2}{2}\} dy$$

Let $v = y^2$

$$= \frac{n^{n/2}}{2^{n/2}\Gamma(n/2)\sqrt{2\pi}} \int_0^{\infty} v^{\frac{n-1}{2}} \exp\{-\frac{(n+t^2)v}{2}\} dv$$

The inner part of above integral is the kernel of $Gamma(\frac{n+1}{2}, \frac{n+t^2}{2})$

$$= \frac{n^{n/2}}{2^{n/2}\Gamma(n/2)\sqrt{2\pi}} \frac{\Gamma((n+1)/2)}{(\frac{n+t^2}{2})^{(n+1)/2}}$$

$$= \frac{\Gamma[\frac{n+1}{2}]}{\sqrt{n\pi}\Gamma(n/2)}(1 + \frac{t^2}{n})^{-\frac{n+1}{2}}$$

6.2 Define $X = U/m$ and $Y = V/n$. The density function of X is

$$f_X(x) = mf_U(U = mx) = \frac{m}{2^m/2\Gamma(m/2)}(mx)^{m/2-1}e^{-mx/2}.$$

Similarly,

$$f_Y(y) = nf_V(V = ny) = \frac{n}{2^n/2\Gamma(n/2)}(ny)^{n/2-1}e^{-ny/2}.$$

According to the density function of the quotient of two independent random variables derived in Section 3.6.1, the probability density of $W = X/Y$ is

$$\begin{aligned}
f_W(w) &= \int_0^\infty y f_Y(y) f_X(x = yw) dy \\
&= \int_0^\infty y \frac{n}{2^n/2\Gamma(n/2)}(ny)^{n/2-1}e^{-ny/2} \frac{m}{2^m/2\Gamma(m/2)}(myw)^{m/2-1}e^{-myw/2} dy \\
&= \frac{n^{n/2}m^{m/2}w^{m/2-1}}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} \int_0^\infty y^{\frac{n+m}{2}-1} \exp\{-\frac{n+mw}{2}y\} dy \\
&\text{The inner part of above integral is the kernel of } Gamma(\frac{n+m}{2}, \frac{n+mw}{2}) \\
&= \frac{n^{n/2}m^{m/2}w^{m/2-1}}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} \times \frac{\Gamma((n+m)/2)}{[\frac{n+mw}{2}]^{(n+m)/2}} \\
&= \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)}(\frac{m}{n})^{m/2}w^{m/2-1}(1+\frac{m}{n}w)^{-(m+n)/2}
\end{aligned}$$

6.3 The distribution of $\bar{X}$ is a normal distribution with $E(\bar{X}) = 0$ and $Var(X) = \frac{1}{16}$.

$$\begin{aligned}
&P(|\bar{X} < c|) = 0.5 \\
&\Rightarrow P(-c < \bar{X} < c) = 0.5 \\
&\Rightarrow P(-\frac{c}{\sqrt{1/16}} < \frac{\bar{X}}{\sqrt{1/16}} < \frac{c}{\sqrt{1/16}}) = 0.5 \\
&\bar{X}/16 \text{ has standard normal distribution} \\
&\Rightarrow \Phi(\frac{-c}{\sqrt{1/16}}) = 0.25 \\
&\Rightarrow \frac{-c}{\sqrt{1/16}} = -0.67 \Rightarrow c = 0.17
\end{aligned}$$

6.6 Let $Y = T^2$,

$$P(Y < y) = P(T^2 < y) = P(-\sqrt{y} < T < \sqrt{y})$$

Since the t-distribution is symmetric about 0,

$$P(Y < y) = 1 - 2P(T < -\sqrt{y}).$$

Taking derivative of both sides,

$$f_Y(y) = \frac{1}{\sqrt{y}} f_T(t = -\sqrt{y})$$

Thus

$$\begin{aligned} f_Y(y) &= \frac{1}{\sqrt{y}} f_T(T = -\sqrt{y}) \\ &= \frac{1}{\sqrt{y}} \frac{\Gamma[\frac{n+1}{2}]}{\sqrt{n\pi}\Gamma(n/2)}(1+\frac{y}{n})^{-\frac{n+1}{2}} \\ &= \frac{\Gamma[(1+n)/2]}{\Gamma(1/2)\Gamma(n/2)}(\frac{1}{n})^{1/2} y^{-1/2}(1+\frac{y}{n})^{-\frac{n+1}{2}}, \end{aligned}$$

which is the density of $F(1, n)$.

6.8

Solution 1:

According to the formula for the quotient of two continuous independent random variables ($Z = \frac{X}{Y}$) in page 98, the PDF of Z is given by

$$f_Z(z) = \int_{-\infty}^{\infty} |y| f_Y(y) f_x(yz) dy \tag{1}$$

In this problem, $X, Y \sim \exp(1)$ and are both greater than or equal to zero. Thus

$$\begin{aligned} (2) &= \int_0^{\infty} y e^{-y} e^{-yz} dy \\ &= \int_0^{\infty} y e^{-y(1+z)} dy \\ &= \frac{1}{(1+z)^2} \int_0^{\infty} (1+z)^2 y e^{-y(1+z)} dy \end{aligned}$$

Since $(1+z)^2 y e^{-y(1+z)}$ is the PDF for $Gamma(2, 1+z)$,

$$(2) = \frac{1}{(1+z)^2}, \quad z \geq 0$$

Thus $\frac{X}{Y}$ has distribution $F(2, 2)$.

Solution 2:

Since $X \sim \exp(1)$, $X \sim Gamma(1, 1)$ and $2X \sim Gamma(1, 2)$.

According to the relationship between Gamma distribution and chi-squared distribution, $2X \sim \chi^2(2)$. Similarly, $2Y \sim \chi^2(2)$.

Since $2X$ and $2Y$ are independent, $\frac{X}{Y} = \frac{2X/2}{2Y/2} \sim F(2, 2)$

6.9 According to theorem B in Section 6.3, $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$. Thus $E((n-1)S^2/\sigma^2) = n - 1$ and $Var((n-1)S^2/\sigma^2) = 2(n-1)$. So

$$E(S^2) = (n-1)\sigma^2/(n-1) = \sigma^2$$

and

$$Var(S^2) = \frac{2\sigma^4(n-1)}{(n-1)^2} = \frac{2\sigma^4}{n-1}$$