ST 371 MIDTERM 2

October 19, 2020

NAME:

- You have **48 hours** to complete this exam.

1. Let $a, b, c \in (-\infty, \infty)$ with $a < c < b$, and for $x \in (-\infty, \infty)$, define

$$g(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{2(x-a)}{(b-a)(c-a)} & \text{if } a \leq x < c \\ \frac{2}{b-a} & \text{if } x = c \\ \frac{2(b-x)}{(b-a)(b-c)} & \text{if } c < x \leq b \\ 0 & \text{if } b < x \end{cases}$$

   (a) Is $g(\cdot)$ a probability mass function, a probability density function, or a cumulative distribution function? Prove your answer.

   (b) If X is a random variable with pdf $g(\cdot)$, derive the cdf of X.

2. Consider the experiment of dealing cards at random, one at a time from a deck of 52 cards, without replacement. What is the probability that the fifth card dealt is the second king to be dealt?

3. A defining property of a random variable X, in addition to its mass/density function and cdf, is called the moment generating function (mgf) of X. The mgf of X is defined for any $t \in (-\infty, \infty)$ as

$$m(t) := \mathrm{E}(e^{tX}),$$

   where $e^{(\cdot)}$ is the exponential function.

   (a) Assuming that $X \sim \text{Poisson}(\alpha)$, derive a closed-form expression of $m(t)$.

   (b) Using your answer to part (a), show that $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$, each evaluated at $t = 0$, are equal to $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$, respectively. Recall from your lecture notes what the $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$ are for a Poisson random variable.

4. Let $Z \sim \mathrm{N}(0, 1)$ and define $Y = Z^2$.

   (a) Derive the pdf of Y.

   (b) Evaluate $P(Y \leq 9)$.