Solution HW 6

October 24, 2019

Section 8.10 of Rice; Exercises 1, 3, 4(a,b,c,d), 31, 33, 53
8.1 -

$$\hat{\lambda} = \frac{4436 + 1800 \times 2 + 534 \times 3 + 111 \times 4 + 21 \times 5}{5267 + 4436 + 1800 + 534 + 111 + 21} = 0.837$$

and

$$N = 5267 + 4436 + 1800 + 534 + 111 + 21 = 12169$$

n	Observed	Expected
0	5267	$\frac{0.837^0 e^{-0.837}}{0!} \times 12169 = 5269$
1	4436	$\frac{0.837^1 e^{-0.837}}{1!} \times 12169 = 4410$
2	1800	$\frac{0.837^2 e^{-0.837}}{2!} \times 12169 = 1846$
3	534	$\frac{0.837^3 e^{-0.837}}{3!} \times 12169 = 515$
4	111	$\frac{0.837^4 e^{-0.837}}{4!} \times 12169 = 108$
5+	21	$12169 - 5369 - 4410 - 1846 - 515 - 108 = 21$

8.3

(a) Using the MOM estimator, $\hat{\lambda}_{MOM} = \bar{Y}$ we have our estimates for concentration 1 ($\hat{\lambda}_{MOM} = \bar{y} = 0.6825$),concentration 2 ($\hat{\lambda}_{MOM} = \bar{y} = 1.3225$), concentration 3 ($\hat{\lambda}_{MOM} = \bar{y} = 1.7775$) concentration 4 ($\hat{\lambda}_{MOM} = \bar{y} = 4.68$).

(b) A large sample distribution for $\hat{\lambda}_{MOM}$ can be found using the CLT

$$\hat{\lambda}_{MOM} \overset{\bullet}{\sim} N(\lambda, \lambda/n)$$

So a 95% CI for λ is approximately given by

$$\hat{\lambda}_{MOM} \pm 1.96\sqrt{\hat{\lambda}_{MOM}/n}$$

For concentration 1 we get $0.6825 \pm 1.96\sqrt{0.6825/400}$. Similar conclusions could be gotten for concentration 2,3,4.

(c)

8.4

(a)

$$E(X) = 0*2/3*\theta + 1*1/3*\theta + 2*2/3*(1-\theta) + 3*1/3*(1-\theta) = 7/3 - 2*\theta$$

This implies that

$$\hat{\theta}_{MOM} = \frac{7/3 - \bar{X}}{2}$$

The point estimate based on the data is then

$$\hat{\theta}_{MOM} = \frac{7/3 - 1.5}{2} = 5/12$$

(b)

$$Var(\hat{\theta}_{MOM}) = Var\left(\frac{7/3 - \bar{X}}{2}\right) = \frac{1}{4}Var(\bar{X}) = \frac{1}{4n}Var(X)$$

To find $Var(X)$ we can use the variance computing formula, $Var(X) = E(X^2) - [E(X)]^2$.

$$E(X^2) = 0^2(2/3)\theta + 1^2(1/3)\theta + 2^2(2/3)(1-\theta) + 3^2*1/3*(1-\theta) = 17/3 - (16/3)\theta$$

This implies

$$Var(X) = 17/3 - (16/3)\theta - (7/3 - 2\theta)^2 = 2/9 + 4\theta - 4\theta^2$$

So, $Var(\hat{\theta}_{MOM}) = \frac{1}{4n}(2/9 + 4\theta - 4\theta^2)$. The estimate of this variance is then

$$Var(\hat{\theta}_{MOM}) = \frac{1}{4n}(2/9 + 4(5/12) - 4(5/12)^2)$$

For this problem, n=10 which gives 0.02986 and this gives an estimated standard error of 0.1728.

(c) The likelihood is the joint distribution of the observed data. Here that corresponds to

$$L(\theta|x_1 = 3, x_2 = 0, ..., x_{10} = 1) = (1/3)(1-\theta)(2/3)\theta(2/3)(1-\theta)\cdots(1/3)\theta$$

this is proportional to (notice we don't care about the constant for maximization purposes)

$$\propto \theta^5(1-\theta)^5$$

The log-likelihood is then

$$l(\theta) = 5ln(\theta) + 5ln(1-\theta)$$

Maximizing this we have

$$\hat{\theta}_{MLE} = 1/2$$

which is a maximum as the 2nd derivative is negative for all θ.

(d) To do this problem generically we can recognize a multinomial distribution. Define your sample size as n and $Y_1 = \#$ of values taking on 0, $Y_2 = \#$ of values taking on 1, $Y_3 = \#$ of values taking on 2, and $Y_4 = \#$ of values taking on 3. Then we have

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} \sim Multinomial(n, (2/3)\theta, (1/3)\theta, (2/3)(1-\theta), (1/3)(1-\theta))$$

The likelihood is then proportional to

$$L(\theta) \propto \theta^{x_1+x_2}(1-\theta)^{x_3+x_4}$$

and

$$\ell(\theta) = (x_1 + x_2)ln(\theta) + (x_3 + x_4)ln(1-\theta)$$

Differentiating we have

$$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{x_1 + x_2}{\theta} - \frac{x_3 + x_4}{1-\theta}$$

Setting equal to zero and solving for θ we get

$$\hat{\theta} = \frac{x_1 + x_2}{x_1 + x_2 + x_3 + x_4}$$

The second derivative yields

$$-\frac{x_1 + x_2}{\theta^2} - \frac{x_3 + x_4}{(1-\theta)^2}$$

(which is negative for all θ). To find the asymptotic variance we want to find one divided by the negative of the expected value of this.

$$I_n(\theta) = -E\left(-\frac{X_1 + X_2}{\theta^2} - \frac{X_3 + X_4}{(1-\theta)^2}\right)$$

$$= \frac{n(2/3)\theta + n(1/3)\theta}{\theta^2} + \frac{n(2/3)(1-\theta) + n(1/3)(1-\theta)}{(1-\theta)^2}$$

$$= n\left(1/\theta + 1/(1-\theta)\right) = \frac{n}{\theta(1-\theta)}$$

Thus, the asymptotic variance of the MLE is

$$AVAR\left(\hat{\theta}_{MLE}\right) = \frac{1}{I_n(\theta)} = \frac{\theta(1-\theta)}{n}$$

The estimate of this could be found by plugging in our $\hat{\theta}$:

$$A\hat{V}AR\left(\hat{\theta}_{MLE}\right) = (1/2)^2/10 = 1/40$$

8.31

(a)

$$L(\theta) = (1-\theta)^3 \times (1-\theta)^3 \times \theta = \theta(1-\theta)^6$$

(b)

$$l(\theta) = \log(\theta) + 6\log(1-\theta)$$

Let $\frac{\partial l(\theta)}{\partial \theta} = 0$ we have $\hat{\theta} = \frac{1}{7}$. The second derivative is

$$-\frac{1}{\theta^2} - \frac{6}{(1-\theta)^2}$$

, which is always negative. Thus the MLE for θ is $\frac{1}{7}$

8.33

The value c would need to be $t_{n-1,0.05}S/\sqrt{n}$. This can be found by a similar argument to how we derive the two-sided confidence interval:

$$P\left(-t_{n-1,0.05} < \frac{\bar{X}-\mu}{S/\sqrt{n}} < \infty\right) = 0.95$$

$$P\left(-t_{n-1,0.05}S/\sqrt{n} < \bar{X}-\mu < \infty\right) = 0.95$$

$$P\left(-\infty < \mu < \bar{X} + t_{n-1,0,05}S/\sqrt{n}\right) = 0.95$$

8.53

(a) The MOM estimator of θ is

$$\hat{\theta}_{MOM} = 2\bar{X}$$

The mean of the estimator is

$$E(\hat{\theta}_{MOM}) = E(2\bar{X}) = 2E(X) = 2\theta/2 = \theta$$

The variance of the estimator is

$$Var(\hat{\theta}_{MOM}) = Var(2\bar{X}) = 4Var(X)/n = \frac{4}{n}\frac{(\theta-0)^2}{12} = \frac{\theta^2}{3n}$$

(b)

$$L(\theta) = \prod_{i=1}^{n} \frac{X_i}{\theta} \mathbb{1}\{X_{(n)} \leq \theta\},$$

where $\mathbb{1}\{\}$ is the indicator function and $X_{(n)}$ is the maximum among $X_1, X_2, \ldots, X_n$. When $X_{(n)} \leq \theta$, $L(\theta)$ is monotonically decreasing with θ. Thus $L(\theta)$ is maximized when $\theta = X_{(n)}$, i.e., $\hat{\theta}_{MLE} = X_{(n)}$

(c) Distribution of the max is given by

$$f_{X_{(n)}}(x) = n\,[F_X(x)]^{n-1}\, f_X(x) = n\left(\frac{x}{\theta}\right)^{n-1}\frac{1}{\theta} = n\frac{x^{n-1}}{\theta^n}$$

where $0 < x < \theta$. The mean of the max is then the mean of this distribution

$$E(\hat{\theta}_{MLE}) = E(X_{(n)}) = \int_0^\theta x\frac{nx^{n-1}}{\theta^n}dx = \frac{nx^{n+1}}{(n+1)\theta^n}|_0^\theta = \frac{n\theta}{n+1}$$

Variance can be found via the variance computing formula

$$E(\hat{\theta}^2_{MLE}) = E(X^2_{(n)}) = \int_0^\theta x^2\frac{nx^{n-1}}{\theta^n}dx = \frac{nx^{n+2}}{(n+2)\theta^n}|_0^\theta = \frac{n\theta^2}{n+2}$$

This implies the variance is

$$Var(\hat{\theta}_{MLE}) = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2 = \theta^2\frac{n}{(n+2)(n+1)^2}$$

The MOM bias is 0, here the bias is $\theta\left(\frac{n}{n+1} - 1\right) = \left(\frac{-1}{n+1}\right)\theta$. This bias gets very small as n grows.
The variance of the MLE and of the MOM estimator go to zero as n grows.

(d) $\hat{\theta}_{MLE}\frac{n+1}{n}$ will be unbiased for θ