Solution HW 2

September 9, 2019

Section 2.5 of Rice; Exercises 1, 3, 5, 6, 38, 40, 50, 51, 54, 57, 62

2.3 $p(1) = 0.1$, $p(2) = 0.2$, $p(3) = 0.4$, $p(4) = 0.1$, $p(5) = 0.2$

2.5 (a) If X is discrete and X_0 is the minimal value of X,

$$F(v) - F(u) = \sum_{k=x_0}^{v} P(X = k) - \sum_{k=x_0}^{u} P(X = k) = \sum_{v+1}^{u} P(X = k) = P(u < X \leq v)$$

(b) If X is continuous,

$$F(v) - F(u) = \int_{-\infty}^{v} f(x)dx - \int_{-\infty}^{u} f(x)dx = \int_{u}^{v} f(x)dx = P(u < X \leq v)$$

2.6 For any x,

$$I_{A\cap B}(x) = \begin{cases} 1 & \text{if } x \in A \cap B \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$\begin{aligned} I_A(x)I_B(x) = 1 &\Leftrightarrow I_A(x) = 1 \text{ and } I_B(x) = 1 \\ &\Leftrightarrow x \in A \text{ and } x \in B \\ &\Leftrightarrow x \in A \cap B \end{aligned}$$

$$min(I_A(x), I_B(x)) = 1 \Leftrightarrow I_A(x) = 1 \text{ and } I_B(x) = 1 \Leftrightarrow x \in A \cap B$$

Thus $I_{A\cap B} = I_A I_B = min(I_A, I_B)$

Similarly, for any x,

$$I_{A\cup B}(x) = \begin{cases} 1 & \text{if } x \in A \cup B \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$max(I_A, I_B) = 1 \Leftrightarrow I_A = 1 \text{ or} I_B = 1 \Leftrightarrow x \in A \cup B$$

Thus $I_{A\cup B} = max(I_A, I_B)$

2.38 Since f and g are densities, we know for any x, $f(x) \geq 0$ and $g(x) \geq 0$; $\int_{-\infty}^{\infty} f(x)dx = 1$ and $\int_{-\infty}^{\infty} g(x)dx = 1$.

(1) For any x and $0 \leq \alpha \leq 1$,

$$\alpha f(x) + (1-\alpha)g(x) \geq 0$$

(2)

$$\int_{-\infty}^{\infty} \alpha f(x) + (1-\alpha)g(x)dx = \alpha \int_{-\infty}^{\infty} f(x)dx + (1-\alpha) \int_{-\infty}^{\infty} g(x)dx = \alpha + 1 - \alpha = 1$$

(3) Since f and g are both piecewise continuous, the linear combination of f and g, $\alpha f + (1-\alpha)g$ is also piecewise continuous.

Thus $\alpha f + (1-\alpha)g$ is a density.

2.40

a.

$$\int_0^1 cx^2 = 1 \Rightarrow \frac{c}{3} = 1 \Rightarrow c = 3$$

b.For any $0 \leq x \leq 1$,

$$F(x) = P(X \leq x) = \int_0^x 3y^2 dy = x^3$$

.

Thus

$$F(x) = \begin{cases} 0 & x \leq 0 \\ x^3 & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

c.

$$P(0.1 \leq X < 0.5) = F(0.5) - F(0.1) = 0.5^3 - 0.1^3 = 0.124$$

2.50 Let $z = 2\log t \Rightarrow t = \exp(\frac{z}{2})$. The support of z is $(-\infty, \infty)$

$$\begin{aligned}
\Gamma(x) &= 2\int_0^{\infty} t^{2x-1} e^{-t^2} dt \\
&= 2\int_{-\infty}^{\infty} \exp(\frac{z}{2})^{2x-1} \exp(-\exp(\frac{z}{2})^2) d(\exp(z/2)) \\
&= 2\int_{-\infty}^{\infty} \exp(zx - \frac{z}{2}) \exp(-\exp(z)) \exp(z/2) \frac{1}{2} dz \\
&= \int_{-\infty}^{\infty} e^{xz} e^{-e^z} dz
\end{aligned}$$

2.51 Let $z = \frac{x-u}{\sigma}$, thus

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{(x-u)^2}{2\sigma^2}\} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp^{-z^2/2} dz \qquad (3) \\
&= \frac{1}{\sqrt{2\pi}} \sqrt{\Big(\int_{-\infty}^{\infty} \exp^{-x^2/2} dx\Big)\Big(\int_{-\infty}^{\infty} \exp^{-y^2/2} dy\Big)}
\end{aligned}$$

Let $x = \gamma \cos t$ and $y = \gamma \sin t$, $t \in [0, 2\pi]$ and $\gamma \geq 0$. The Jacobian matrix of transforming (x, y) to (γ, t) is $J = \begin{bmatrix} \cos t & -\gamma \sin t \\ \sin t & \gamma \cos t \end{bmatrix}$ and $|J| = \gamma$.

Thus

$$
\begin{aligned}
\Big(\int_{-\infty}^{\infty} \exp^{-x^2/2}\, dx\Big)\Big(\int_{-\infty}^{\infty} \exp^{-y^2/2}\, dy\Big) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{1}{2}(x^2+y^2)} dx dy \\
&= \int_0^{\infty}\int_0^{2\pi} e^{-\frac{1}{2}\gamma^2} \gamma dt d\gamma \\
&= 2\pi \int_0^{\infty} e^{-\frac{1}{2}\gamma^2} \gamma d\gamma \\
&\text{let } s = \frac{1}{2}\gamma^2 \\
&= 2\pi \int_0^{\infty} e^{-s} ds \\
&2\pi(0 + e^0) = 2\pi
\end{aligned}
$$

Thus (3) = 1.

2.54

$$
\begin{aligned}
P(|X - u| \leq 0.675\sigma) &= P(-0.675\sigma \leq X - u \leq 0.675\sigma) \\
&= P(-0.675 \leq \frac{X-u}{\sigma} \leq 0.675) = \Phi(0.675) - \Phi(-0.675) = 0.5
\end{aligned}
$$

2.57

$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P(X \geq \frac{y-b}{a}) = 1 - P(X < \frac{y-b}{a}).$$

Take derivative of both sides,

$$f_Y(y) = -\frac{1}{a} f_X(\frac{y-b}{a}) = \frac{1}{\sqrt{2\pi}|a|\sigma} \exp\{-\frac{(y-b-au)^2}{2a^2\sigma^2}\}.$$

This is the pdf distribution for $N(au + b, a^2\sigma^2)$.

2.62 Let F_X be the cumulative density function of X. Thus

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(aX + b \leq y) \\
&= \begin{cases} P(X \leq \frac{y-b}{a}) & \text{if } a > 0 \\ 1 - P(X < \frac{y-b}{a}) & \text{if } a < 0 \end{cases}
\end{aligned}
$$

Taking derivation of both sides,

$$f_Y(y) = \begin{cases} \frac{1}{a} f_X(\frac{y-b}{a}) & \text{if } a > 0 \\ -\frac{1}{a} f_X(\frac{y-b}{a}) & \text{if } a < 0 \end{cases}$$

Thus $f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$