Solution HW 7

October 31, 2019

Section 8.10 of Rice; Exercises 16(a,b,c), 21(a,b), 47(a,b,c), 50, 51
8.16ab

(a) Note that the support is $-\infty < x < \infty$.
First, we need to find the theoretical moment of this distribution.

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{2\sigma} exp\left(-\frac{|x|}{\sigma}\right) dx$$

This is an odd function integrated over the whole real line so this gives 0. Thus, we will have to try the second theoretical moment.

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{2\sigma} exp\left(-\frac{|x|}{\sigma}\right) dx$$

This is an even function so we can rewrite this as

$$= 2\int_{0}^{\infty} x^2 \frac{1}{2\sigma} exp\left(-\frac{|x|}{\sigma}\right) dx = \int_{0}^{\infty} \frac{x^2}{\sigma} exp\left(-\frac{x}{\sigma}\right) dx = \frac{1}{\sigma}\int_{0}^{\infty} x^2 exp\left(-\frac{x}{\sigma}\right) dx$$

Now we can recognize this as the kernel of a gamma distribution with $\alpha = 3$ and $\lambda = \frac{1}{\sigma}$. Thus we have

$$= \frac{1}{\sigma}\frac{\Gamma(3)}{(1/\sigma)^3} = 2\sigma^2$$

For method of moments, we now set this equal to the second sample moment and solve

$$\frac{1}{n}\sum_{i=1}^{n} Y_i^2 = 2\sigma^2 \implies \hat{\sigma} = \sqrt{(\frac{1}{n}\sum_{i=1}^{n} Y_i^2)/2}$$

(b) The likelihood is given by

$$L(\sigma) = \prod_{i=1}^{n} \frac{1}{2\sigma} exp\left(-|x_i|/\sigma\right) = \frac{1}{(2\sigma)^n} exp\left(-\sum_{i=1}^{n} |x_i|/\sigma\right)$$

To maximize this, we can consider the log-likelihood

$$l(\sigma) = -nln(2\sigma) - \sum_{i=1}^{n} |x_i|/\sigma$$

Taking the derivative and setting equal to zero we have

$$-n/\sigma + \sum_{i=1}^{n} |x_i|/\sigma^2 = 0 \implies \hat{\sigma} = \sum_{i=1}^{n} |x_i|/n$$

This is clearly in the appropriate range for σ as the quantity will always be positive. Further, it is a maximum as the second derivative is

$$n/\sigma^2 - 2\sum_{i=1}^{n} |x_i|/\sigma^3$$

which is negative when you plug in the potential MLE. Thus,

$$\hat{\sigma}_{MLE} = \sum_{i=1}^{n} |X_i|/n$$

(c)

$$AVar(\hat{\sigma}_{MLE}) = 1/I_n(\sigma)$$

where

$$I_n(\sigma) = -E\left(\frac{\partial^2 l(\sigma)}{\partial \sigma^2}\right)$$

From part (b), the 2nd derivative of the log-likelihood is given by

$$n/\sigma^2 - 2\sum_{i=1}^{n} |x_i|/\sigma^3$$

Therefore,

$$I_n(\sigma) = -n/\sigma^2 + 2\sum_{i=1}^{n} E|X_i|/\sigma^3$$

Now

$$E|X| = \int_{-\infty}^{\infty} |x| \frac{1}{2\sigma} e^{-|x|/\sigma} dx = \int_{-\infty}^{0} \frac{-x}{2\sigma} e^{x/\sigma} + \int_{0}^{\infty} \frac{x}{2\sigma} e^{-x/\sigma} dx$$

$$= \int_{0}^{\infty} \frac{x}{\sigma} e^{-x/\sigma} dx = \sigma$$

This gives

$$I_n(\sigma) = -n/\sigma^2 + 2n/\sigma^2 = n/\sigma^2$$

Thus,

$$AVar(\hat{\sigma}_{MLE}) = \sigma^2/n$$

8.21ab

(a)

$$\begin{aligned}
E(X) &= \int_\theta^\infty x f(x|\theta) dx \\
&= \int_\theta^\infty x e^{(x-\theta)} dx \\
&= \int_0^\infty (y+\theta) e^{-y} dy \quad [Replacing\ x = y+\theta] \\
&= \int_0^\infty y e^{-y} dy + \theta \int_0^\infty e^{-y} dy \\
&= 1+\theta.
\end{aligned}$$

Thus, $1+\theta_{MOM} = \overline{X}$, i.e., $\theta_{MOM} = \overline{X} - 1$.

(b) The likelihood function is given by

$$L(\theta|x_1, \cdots, x_n) = \prod_{i=1}^n e^{(x_i-\theta)} I_{(x_i \geq \theta)}(x_1) = e^{\sum_{i=1}^n (x_i-\theta)} I_{(x_1 \geq \theta, \cdots, x_n \geq \theta)}(x_1, ..., x_n)$$

$$= e^{\sum_{i=1}^n (x_i-\theta)} I_{(x_{(1)} \geq \theta)}(x_1, ..., x_n)$$

Thus, $L(\theta|x_1, \cdots, x_n)$ is an increasing function of θ and the likelihood is maximized for the maximum possible value of θ. Now, for the likelihood to be positive, the condition is $\theta \leq x_1, \cdots, x_n$, i.e. $\theta \leq x_{(1)} = min\{x_1, \cdots, x_n\}$. Thus, $\hat{\theta}_{MLE} = X_{(1)}$.

8.47a

(a) We will need to know the first raw moment (mean) of the distribution.

$$\begin{aligned}
E(X) &= \int_{x_0}^\infty x \theta x_0^\theta x^{-\theta-1} dx \\
&= \int_{x_0}^\infty \theta x_0^\theta x^{-\theta} dx \\
&= \theta x_0^\theta x^{-\theta+1}/(-\theta+1)|_{x_0}^\infty \\
&= 0 - \theta x_0^\theta x_0^{-\theta+1}/(-\theta+1) = \frac{\theta x_0}{\theta-1}
\end{aligned}$$

as $\theta > 1$.

Now we set this equal to $\bar{X}$ and solve for θ yielding

$$\hat{\theta}_{MOM} = \frac{\bar{X}}{\bar{X} - x_0}$$

.

(b) The likelihood is given by

$$L(\theta|x_1, \ldots, x_n) = \prod_{i=1}^{n} \theta x_0^{\theta} x_i^{-\theta-1} = \theta^n x_0^{n\theta} \prod_{i=1}^{n} x_i^{-\theta-1}, \;\; x_1, \ldots, x_n > x_0, \; \theta > 1$$

To maximize this, we can consider the log-likelihood

$$l(\theta) = n \log \theta + n\theta \log(x_0) - (\theta + 1) \sum_{i=1}^{n} \log x_i$$

Taking the derivative and setting equal to zero we have

$$\frac{n}{\theta} + n \log(x_0) - \sum_{i=1}^{n} \log x_i = 0 \Rightarrow \hat{\theta} = \frac{n}{\sum_{i=1}^{n} \log(x_i) - n \log(x_0)}$$

Further, it is a maximum as the second derivative is $-\frac{n}{\theta^2}$ which is always negative. Thus

$$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} \log(x_i) - n \log(x_0)}$$

.

(c)

$$AVar(\hat{\sigma}_{MLE}) = 1/I_n(\theta)$$

where

$$I_n(\theta) = -E\left(\frac{\partial^2 l(\sigma)}{\partial \sigma^2}\right)$$

From part (b), the 2nd derivative of the log-likelihood is given by

$$-\frac{n}{\theta^2}$$

Thus,

$$AVar(\hat{\theta}_{MLE}) = \frac{\theta^2}{n}$$

8.50

(a)

$$E(\theta) = \int_0^\infty \frac{x^2}{\theta^2} e^{-x^2/(2\theta^2)} dx$$

Let $t = \theta^2$,

$$E(\theta) = \frac{1}{2\theta^2} \int_0^\infty t^{1/2} e^{-t/(2\theta^2)} dt$$

The inner part of the above integral is the kernel of $Gamma(\frac{3}{2}, \frac{1}{2\theta^2})$. Thus

$$E(\theta) = \frac{1}{2\theta^2} \times \frac{\Gamma(3/2)}{(\frac{1}{2\theta^2})^{3/2}} = \sqrt{\frac{\theta^2\pi}{2}}.$$

Let $E(\theta) = \bar{X}$, we have

$$\sqrt{\frac{\theta^2\pi}{2}} = \bar{X} \Rightarrow \hat{\theta}_{MOM} = \bar{X}\sqrt{\frac{2}{\pi}}$$

(b) The likelihood is given by

$$L(\theta) = \prod_{i=1}^n \frac{x_i}{\theta^2} e^{-x_i^2/(2\theta^2)} = \frac{\prod_{i=1}^n x_i}{\theta^{2n}} \exp\{-\frac{\sum_{i=1}^n x_i^2}{2\theta^2}\}.$$

The log-likelihood is given by

$$l(\theta) = \sum_{i=1}^n \ln x_i - 2n \ln \theta - \frac{\sum_{i=1}^n x_i^2}{2\theta^2}$$

Taking the derivative and setting equal to 0 we have

$$-\frac{2n}{\theta} + \frac{\sum_{i=1}^n x_i^2}{\theta^3} = 0 \Rightarrow \hat{\theta} = \sqrt{\frac{\sum_{i=1}^n x_i^2}{2n}}$$

Checking this is a max could be down using the second derivative:

$$\frac{\partial^2 l(\theta)}{\partial \theta^2} = \frac{2n}{\theta^2} - \frac{3\sum_{i=1}^n x_i^2}{\theta^4}$$

Plugging $\hat{\theta}$ we have

$$\frac{\partial^2 l(\theta)}{\partial \theta^2}|_{\theta=\hat{\theta}} = \frac{4n^2}{\sum_{i=1}^n x_i^2} - \frac{12n^2}{\sum_{i=1}^n x_i^2} = \frac{-8n^2}{\sum_{i=1}^n x_i^2} < 0$$

Thus $\hat{\theta}$ is a max. Since $\hat{\theta}$ is also greater than 0, $\hat{\theta}$ is the MLE for θ.

(c) To find the asymptotic variance, we can use the information for the sample

$$I_n(\theta) = -E(\frac{\partial^2 l(\theta)}{\partial \theta^2}) = -E(\frac{2n}{\theta^2} - \frac{3\sum_{i=1}^n x_i^2}{\theta^4}) = -\frac{2n}{\theta^2} + \frac{3\sum_{i=1}^n E(x_i^2)}{\theta^4}.$$

For $x_i,\ i = 1, \ldots, n$,

$$E(x^2) = \int_0^{\infty} x^2 \times \frac{x}{\theta^2} e^{-x^2/(2\theta^2)} dx$$

Let $u = \frac{x^2}{2\theta^2}$,

$$E(x^2) = 2\theta^2 \int u e^{-u} du$$

The integral part is the kernel of $Gamma(2, 1)$. Thus

$$E(x^2) = \tau(2) \times 2\theta^2 = 2\theta^2$$

Thus

$$I_n(\theta) = -\frac{2n}{\theta^2} + \frac{6n\theta^2}{\theta^4} = \frac{4n}{\theta^2}$$

and

$$Avar(\hat{\theta}) = \frac{1}{I_n(\theta)} = \frac{\theta^2}{4n}$$

8.51

$$L(\theta) = \frac{1}{2^n} \exp\{-\sum_{i=1}^{n} |x_i - \theta|\}$$

$$l(\theta) = -n \log 2 - \sum_{i=1}^{n} |x_i - \theta| = -n \log 2 - \sum_{i=1}^{n} |x_{(i)} - \theta|,$$

where $x_{(i)}$ is the ith order statistics for x.

Suppose

$$x_{(1)} \leq x_{(2)} \leq \cdots < x_{(k)} \leq \theta \leq x_{(k+1)} \leq \cdots \leq x_n$$

If θ move to the right with step size being a, then the change of $\sum_{i=1}^{n} |x_{(i)} - \theta|$ is $ka - (n-k)a$; conversely, if θ move to the left with step size being a, then the change of $\sum_{i=1}^{n} |x_{(i)} - \theta|$ is $(n-k)a - ka$. Thus if $k < n - k$, in order to maximize $l(\theta)$, θ would continue to the right. Otherwise, it would continue to the left. The maximam is achieved when $n - k = k$, thus $\hat{\theta}_{MLE}$ is the median of the sample.