ST 705 Linear models and variance components
Homework problem set 3

February 3, 2021

1. Prove that all norms on a finite-dimensional vector space V over $\mathbb{C}$ are *equivalent*. That is, show that for any two norms, say $\|\cdot\|_a$ and $\|\cdot\|_b$, defined on V, there exists real-valued positive constants c_1 and c_2 such that for every $x \in V$,

$$c_1\|x\|_b \leq \|x\|_a \leq c_2\|x\|_b.$$

   (a) First, show that it is without loss of generality to consider $\|\cdot\|_b = \|\cdot\|_1$.

   (b) Second, demonstrate that it suffices to only consider $x \in V$ with $\|x\|_1 = 1$.

   (c) Next, prove that any norm $\|\cdot\|_a$ is a continuous function under $\|\cdot\|_1$-distance.

   (d) Finally, apply a result from calculus such as the Bolzano-Weierstrass theorem or the extreme value theorem to finish your argument that all norms on a finite-dimensional vector space are *equivalent*.

   This notion of *equivalence* is in reference to the fact that if a sequence is convergent in *some* norm, then it is convergent in *all* norms. Note the assumption of a *finite*-dimensional vector space.

2. Let $A \in \mathbb{R}^{n\times p}$.

   (a) Prove that if A^g is a generalized inverse of A (i.e., only satisfying $AA^gA = A$), then $(A^g)'$ is a generalized inverse of A'. Conclude from this fact that $P_X := X(X'X)^gX'$ is symmetric.

   (b) Prove the existence **and** uniqueness of the Moore-Penrose generalized inverse, usually denoted A^+, of A.

   (c) Show that if A has full column rank, then $A^+ = (A'A)^{-1}A'$.

   (d) Show that if A has full row rank, then $A^+ = A'(AA')^{-1}$.

3. Let S be a nonempty subset of an inner product space V. The orthogonal complement to the set S is defined as

$$S^\perp := \{x \in V : \langle x, y\rangle = 0 \text{ for every } y \in S\}.$$

(a) Show that $S^\perp$ is a subspace of V for any $S \subseteq V$.

(b) Let $W \subseteq V$ be a finite-dimensional subspace, and let $y \in V$. Show that there exist **unique** vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$.

(c) Let $X \in \mathbb{R}^{n \times p}$. Verify that $\operatorname{col}(X)$ and $\operatorname{null}(X')$ are orthogonal complements.

4. Let $G : \mathbb{R}^p \to \mathbb{R}$ defined by $G(\beta) := (y - X\beta)'W(y - X\beta)$. Derive an expression for $\nabla_\beta G(\beta)$.

5. Let $x_i, y_i \in \mathbb{R}$ for $i \in \{1, \dots, n\}$, and show that

$$\frac{1}{n} \sum_{i=1}^{n} \sum_{j<i} (x_i - x_j)(y_i - y_j) = \sum_{i=1}^{n} (x_i - \bar{x}_n) y_i = \sum_{i=1}^{n} (x_i - \bar{x}_n)(y_i - \bar{y}_n).$$

Note the particular case when $x_i = y_i$ for every i.