HW9 Solution

11/7/2019

Problem 1

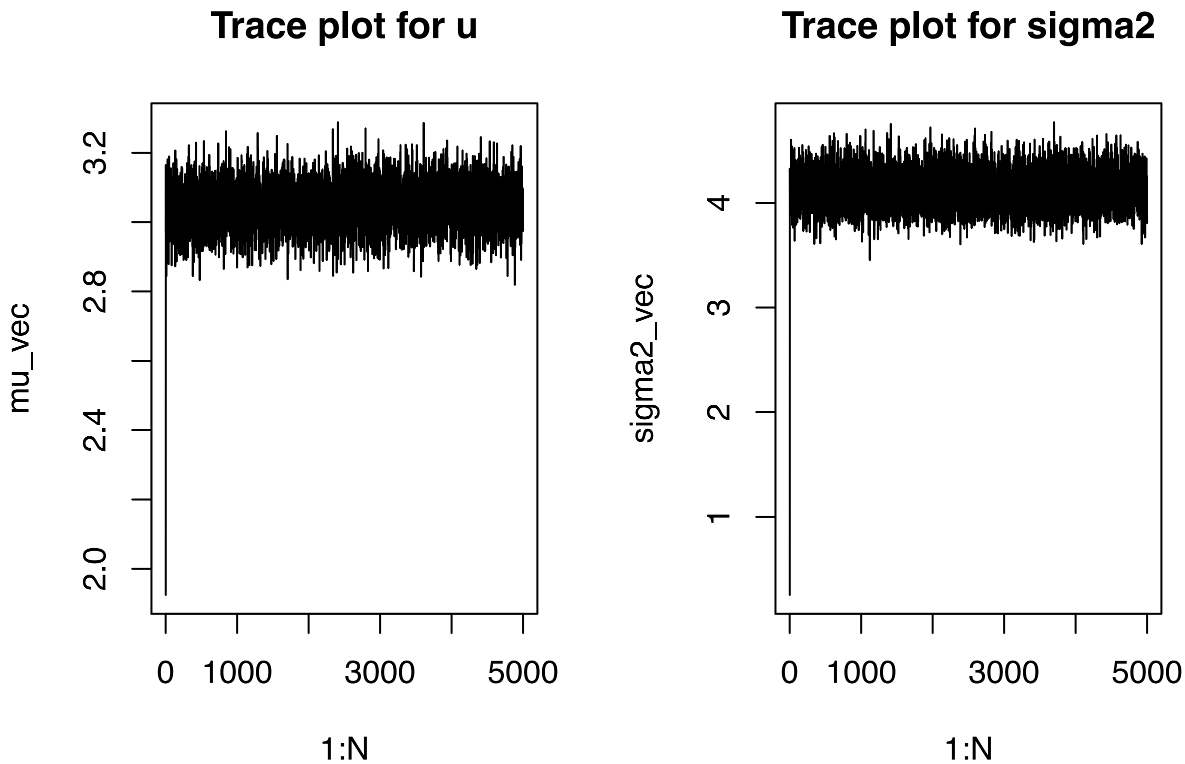
Model Setting

$$X_1, \dots, X_n \sim N(u, \sigma^2)$$
$$u \sim N(0, \tau^2)$$
$$\sigma^2 \sim IG(\alpha, \beta)$$

Conditional posterior distribution

$$u|X, \sigma^2 \sim N(\frac{n\bar{X}_n}{n + \sigma^2/\tau^2}, (\frac{n}{\sigma^2} + \frac{1}{\tau^2})^{-1})$$
$$\sigma^2|X, u \sim IG(\alpha + \frac{n}{2}, \frac{1}{2}\sum_{i=1}^{n}(x_i - u)^2 + \beta)$$

```
library(invgamma)
# Generate 1000 data points from N(3,2^2)
n=1000
data<-rnorm(1000,3,2)

# Specify prior distribution for mean and variance.
# Let tau=5, alpha=1, beta=1
tau=5
alpha=1
beta=1

# Gibbs sampler
# Initialization
mu<-rnorm(1)
sigma2<-runif(1)
mu_vec<-mu
sigma2_vec<-sigma2

# Number of iterations
N=5000

# Number of burnins
burnin=2000

for (i in 2:N){
  mu<-rnorm(1,mean=sum(data)/(n+sigma2/tau^2),sd=sqrt(1/(n/sigma2+1/tau^2)))
  sigma2<-rinvgamma(1,shape=alpha+n/2,rate=0.5*sum((data-mu)^2)+beta)
  mu_vec<-c(mu_vec,mu)
  sigma2_vec<-c(sigma2_vec,sigma2)
}
par(mfrow=c(1,2))
plot(1:N,mu_vec,type="l",main="Trace plot for u")
plot(1:N,sigma2_vec,type="l",main="Trace plot for sigma2")
```




```r
# Estimated u and sigma2 through posterior mean.
mean(mu_vec[(burnin+1):N])
```

```
## [1] 3.04837
```

```r
mean(sigma2_vec[(burnin+1):N])
```

```
## [1] 4.13898
```

Problem 2

(a)

$$L(\theta) = \frac{1}{\theta^n} \quad \theta \geq Y_{(n)}$$

(b) The conjugate prior for uniform distribution is Pareto distribution, i.e., $Pareto(\alpha, k)$

$$f(\theta) = \frac{k\alpha^k}{\theta^{k+1}}, \quad k > 0,\ \theta > \alpha > 0$$

The resulting posterior distribution is $Pareto(max\{Y_{(n)}, \alpha\}, n + k)$, i.e.,

$$f(\theta|Y) \propto \frac{k\alpha^k}{\theta^{n+k+1}} \quad \theta > max\{Y_{(n)}, \alpha\}$$

(c) Metropolis Hastings algorithm

```r
library(sads)
```

```
## Loading required package: bbmle
## Loading required package: stats4
```

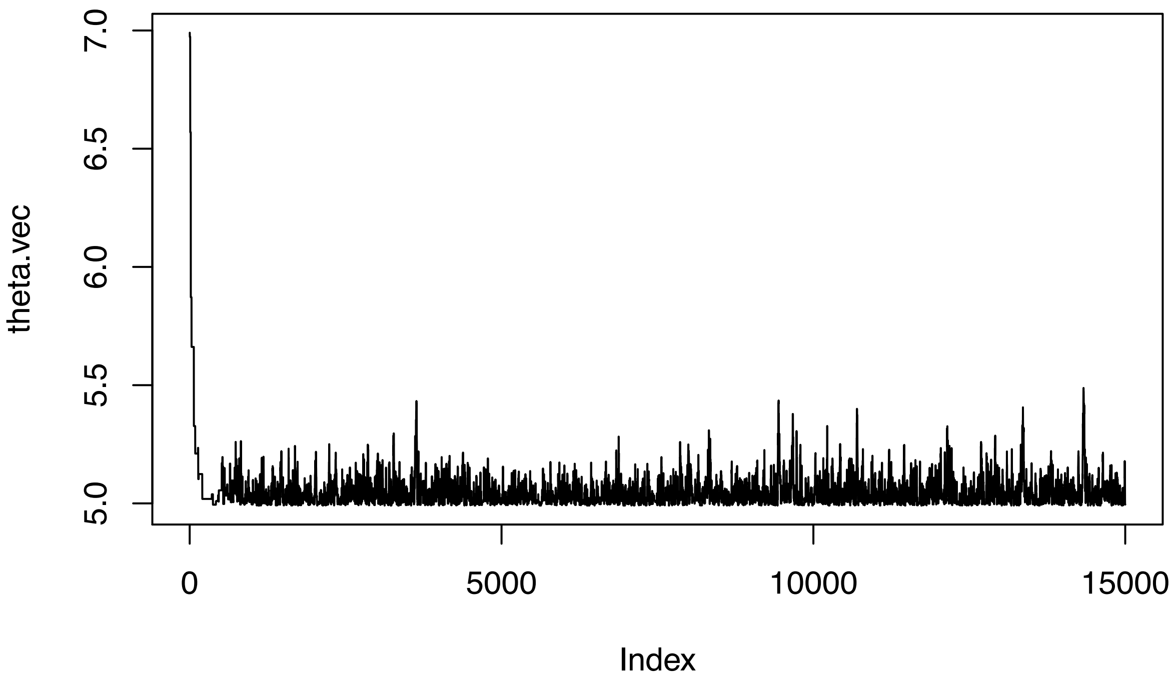
```r
#Generate n data points from unifrom(0,5)
n <- 100
Y <- runif(n,0,5)

#Specify pareto prior
k=1
alpha=0.05

#Specify the variance for the proposal distribution.
tau<-5

#Number of MCMC iterations and burnins
N=15000
burnin=5000

#Initialization
theta<-max(Y)+2
theta.vec<-theta
acc<-0
att<-0
for(i in 1:N){
    att<-att+1
    #Draw condidate
    theta.new<-rnorm(1,theta,tau)
   #Since the the proposal distribution is symmetric, the MH ratio is the
    #posterior probability ratio. Note that if theta<max(Y)  or theta<alpha,
    #the posterior probability is 0.
    ratio<-(theta/theta.new)^(n+k+1)
    if(theta.new<max(Y)||theta.new<alpha) ratio=0
    if(ratio>runif(1)) {
      theta<-theta.new
      acc<-acc+1
    }
    theta.vec<-c(theta.vec,theta)

  #tuning tau to make the acceptance ratio betwen 0.3 and 0.5
  if(i<burnin/2&att>50){
    if(acc/att<0.3) tau<-0.8*tau
    if(acc/att>0.5) tau<-1.2*tau
    acc=att=0
  }
  #Plot the results thus far:
  #if(i %%500==0){
  #  plot(theta.vec[1:i],type="l")
  #}
}
#Final chosen tau and acceptance ratio
print(tau)
```

```
## [1] 0.06768437
```

```r
print(acc/att)
```

```
## [1] 0.437325
```

```r
#Trace plot for theta.
plot(theta.vec,type="l")
```




```r
#Overlay the histogram of MCMC samples with the true posterior distribution.
hist(theta.vec[(burnin+1):15000],breaks=20,freq=FALSE,ylim=c(0,20),xlim=c(4.90,5.45))
curve(sads::dpareto(x,scale=max(Y,alpha),shape=n+k),add=TRUE,from=4.90,to=5.45)
```