ST 705 Linear models and variance components
Homework problem set 1

January 20, 2021

1. Prove or find a counter example to the following inequality.

$$1 \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2.$$

2. Show that the R^2 value for a simple linear regression can never achieve 1 if the observed data consists of repeated (different) observations of the response, y, at the same value of the predictor, x.

3. Prove that the eigenvalues of an upper triangular matrix M are the diagonal components of M.

4. Let $x = (x_1, \ldots, x_p)' \in \mathbb{R}^p$. Show that for $i \in \{1, \ldots, p\}$,

$$|x_i| \leq \|x\|_2 \leq \|x\|_1,$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ are the l_1 and l_2 vector norms, respectively.

5. Show that every eigenvalue of a real symmetric matrix is real.

6. Let U and V be random variables. Establish the following inequalities.

   (a) $P(|U+V| > a+b) \leq P(|U| > a) + P(|V| > b)$ for every $a, b \geq 0$.

   (b) $P(|UV| > a) \leq P(|U| > a/b) + P(|V| > b)$ for every $a \geq 0$ and $b > 0$.

7. The defining property of a projection matrix A is that $A^2 = A$ (recall the definition of the square of a matrix from your linear algebra course). Establish the following facts.

   (a) If A is a projection matrix, then all of its eigenvalues are either zero or one.

   (b) If $A \in \mathbb{R}^{p \times p}$ is a projection and symmetric (i.e., an orthogonal projection matrix), then for every vector v the projection Av is orthogonal to $v - Av$.

   (c) $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$.

   (d) $\text{tr}(AB) = \text{tr}(BA)$.