ST 705 Linear models and variance components
Homework problem set 2

February 1, 2020

1. (2 points) Let $x, \mu_1, \mu_2 \in \mathbb{R}^p$ and $\Sigma_1, \Sigma_2 \in \mathbb{R}^{p\times p}$ be positive definite and symmetric. Derive expressions for $\widetilde{\mu} \in \mathbb{R}^p$, $\widetilde{\Sigma} \in \mathbb{R}^{p\times p}$, and $c \in \mathbb{R}$ that satisfy

$$-(x-\mu_1)'\Sigma_1^{-1}(x-\mu_1) - (x-\mu_2)'\Sigma_2^{-1}(x-\mu_2) = -(x-\widetilde{\mu})'\widetilde{\Sigma}^{-1}(x-\widetilde{\mu}) + c,$$

where c does not depend on x.

2. (2 points) Let A be a positive definite matrix, and show that

$$\operatorname{tr}(I - A^{-1}) \le \log\det(A) \le \operatorname{tr}(A - I).$$

3. (2 points) Suppose you do not know that the rank of a matrix is equal to the number of nonzero singular values. Show that the rank of a projection matrix is equal to its trace. First think about how to show this in the symmetric case, and then consider the more general case of a non-symmetric idempotent matrix.

4. (2 points) Show that if $\operatorname{rank}(BC) = \operatorname{rank}(B)$, then $\operatorname{column}(BC) = \operatorname{column}(B)$, where $\operatorname{column}(\cdot)$ denotes the column space.

5. (2 points) Exercise A.50 from Monahan.

6. (2 points) Let A, B, C, and D be real valued matrices of dimension $p \times p$, $p \times q$, $q \times p$, and $q \times q$, respectively. Show that if D is invertible, then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\cdot\det(A - BD^{-1}C).$$