Solution HW 10

December 2, 2019

Section 9.11 of Rice; Exercises 1, 3, 4, 7, 9, 12
9.1 -

(a)

$$\alpha = P(\text{Type I Error}) = P(\text{Reject } H_0 | H_0 \text{ true}) = P(0 \text{ or } 10 \text{ heads} | p = 1/2)$$

Define Y=# of heads, then $Y \sim Bin(10, p)$ and under $H_0 : p = 1/2$ the null distribution is $Y \overset{H_0}{\sim} Bin(10, 0.5)$.

$$\implies \alpha = P(Y = 0 \text{ or } Y = 10 | p = 1/2)$$

$$= \binom{10}{0}(1/2)^0(1-1/2)^{10-0} + \binom{10}{10}(1/2)^{10}(1-1/2)^{10-10} = 0.00195$$

(b)

$$PWR(0.1) = 1 - \beta(0.1) = 1 - P(\text{Type II Error})$$

$$= 1 - P(\text{Fail to reject } H_0 | H_A : p = 0.1) = P(\text{Reject } H_0 | H_A : p = 0.1)$$

$$= P(Y = 0 \text{ or } Y = 10 | p = 0.1) = \binom{10}{0}(0.1)^0(0.1)^{10-0} + \binom{10}{10}(0.1)^{10}(0.1)^{10-10} \approx 0.349$$

9.3

We have $X \sim Bin(100, p)$ with $H_0 : p = 0.5$ vs $H_A : p \neq 0.5$. We reject for $|X - 50| > 10$.

(a) What is α? Using the normal approximation we have $X \overset{H_0}{\sim} N(100(0.5), 100(0.5)(1-0.5)) \sim N(50, 25)$. Thus,

$$\alpha = P(\text{Reject } H_0 | H_0) = P\left(|X - 50| > 10 | p = 0.5\right)$$

$$= 1 - P\left(|X - 50| \leq 10 | p = 0.5\right) = 1 - P(-10 \leq X - 50 \leq 10 | p = 0.15)$$

$$= 1 - P(40 \leq X \leq 50 | p = 0.5)$$

Using our normal approximation we can find this via

$$1-(pnorm(60, mean = 50, sd = 5) - pnorm(40, mean = 50, sd = 5)) = 0.0455$$

Using the binomial distribution we have

$$\alpha = 1 - P(40 \leq X \leq 50|p = 0.5)$$

$= 1-(pbinom(50, size = 100, prob = 0.5)-pbinom(39, size = 100, prob = 0.5)) = 0.0352$

(b)

$$PWR(p) = P(\text{Reject } H_0|p) = 1 - P(40 \leq X \leq 60|p)$$

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#sequence of p's
p<-seq(from=0,to=1,by=0.01)

#Power given by 1-P(40<=X<=60|p)
#Normal approx
plot(p,1-(pnorm(60,mean=100*p,sd=sqrt(100*p*(1-p)))-
pnorm(40,mean=100*p,sd=sqrt(100*p*(1-p)))),type="l",
main="Power using Normal Approx",ylab="Power(p)",
xlab="p",col="Blue")

#Binomial
lines(p,1-(pbinom(60,size=100,prob=p)-
pbinom(39,size=100,prob=p)),type="l",
main="Power using Binomial",ylab="Power(p)",
xlab="p",col="Red")

#add legend
legend("bottomleft",legend=c("Normal Approx","Binomial"),
col=c("Blue","Red"),lty=c(1,1))
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9.4 -

(a)

X	H_0	H_A	Ratio
x_1	0.2	0.1	2
x_2	0.3	0.4	3/4
x_3	0.3	0.1	3
x_4	0.2	0.4	1/2

(b) $P(\Lambda(x) \leq c) = 0.2 \Rightarrow c \in [0.5, 0.75)$. $P(\Lambda(x) \leq c) = 0.5 \Rightarrow c \in [0.75, 2)$

(c) We would look at the ratio of posterior probabilities

$$\frac{P(H_0|x)}{P(H_A|x)} = \frac{P(x|H_0)P(H_0)}{P(x|H_A)P(H_A)} = \frac{P(x|H_0)}{P(x|H_A)}$$

Plugging in our values, we want to choose H_0 if this ratio is greater than 1:

X	H_0	H_A	Ratio	Decision
x_1	0.2	0.1	2	H_0
x_2	0.3	0.4	3/4	H_A
x_3	0.3	0.1	3	H_0
x_4	0.2	0.4	1/2	H_A

(d) When $\alpha = 0.2$, we reject H_0 only when $x = x_4$. Thus $\begin{cases} \frac{P(H_0)}{P(H_A)} * 0.5 < 1 \\ \frac{P(H_0)}{P(H_A)} * 3/4 > 1 \end{cases}$, which leads to $\frac{4}{3} < \frac{P(H_0)}{P(H_A)} < 2$.

Similarly, When $\alpha = 0.5$, we reject H_0 only when $x = x_2$ or x_4. Thus $\begin{cases} \frac{P(H_0)}{P(H_A)} * 0.5 < 1 \\ \frac{P(H_0)}{P(H_A)} * 3/4 < 1 \\ \frac{P(H_0)}{P(H_A)} * 2 > 1 \\ \frac{P(H_0)}{P(H_A)} * 3 > 1 \end{cases}$, which leads to $\frac{1}{2} < \frac{P(H_0)}{P(H_A)} < \frac{4}{3}$.

9.7 -

$$X_i \overset{iid}{\sim} Poi(\lambda), \qquad H_0 : \lambda = \lambda_0, H_A : \lambda = \lambda_1 \text{ where } \lambda_1 > \lambda_0$$

LRT says to reject for $\Lambda = \frac{L(\lambda_0)}{L(\lambda_1)} < c$.

$$\Lambda = \frac{e^{-n\lambda_0}\lambda_0^{\sum_{i=1}^n x_i}}{e^{-n\lambda_1}\lambda_1^{\sum_{i=1}^n x_i}} = e^{n(\lambda_1 - \lambda_0)} \left(\frac{\lambda_0}{\lambda_1}\right)^{\sum_{i=1}^n x_i} < c$$

This is equivalent to rejecting for

$$\left(\frac{\lambda_0}{\lambda_1}\right)^{\sum_{i=1}^n x_i} < c_2$$

$$\Leftrightarrow \sum_{i=1}^n x_i ln\left(\frac{\lambda_0}{\lambda_1}\right) < c_3$$

$$\Leftrightarrow \sum_{i=1}^n x_i > c_4 \text{ since } \lambda_1 > \lambda_0$$

Now to control α, we can use the fact that $\sum_{i=1}^n X_i \sim Poi(n\lambda)$.

9.9 -

Let $X_i \overset{iid}{\sim} N(\mu, 100)$ with n=25 and $\alpha = 0.1$. Hypotheses are $H_0 : \mu = 0$ vs $H_A : \mu = 1.5$. We should reject for large values of $\bar{X}$, say $\bar{X} > c$. We know the distribution of $\bar{X}$ is

$$\bar{X} \sim N(\mu, 100/25)$$

since we have a random sample from a normal parent population.
If H_0 is true,

$$P(\bar{X} > c|\mu = 0) = 0.1 \implies P\left(Z > \frac{c-0}{10/5}\right) = 0.1 \implies P(Z > c/2) = 0.1$$

This implies our c should be $2(1.282) = 2.564$.
The RR is then

$$\{\bar{x} : \bar{x} > 2.564\} \Leftrightarrow \{z : z > 1.282\}$$

Power? This is given by

$$PWR(1.5) = P(\text{Reject } H_0|\mu = 1.5) = P(\bar{X} > 2.564|\mu = 1.5)$$

$$P\left(Z > \frac{2.564 - 1.5}{10/5}\right) = P(Z > 0.532) = 0.2974$$

If $\alpha = 0.01$ then our c changes to 4.652 and the power becomes 0.0575.

9.12 -
$X_i \overset{iid}{\sim} exp(\theta)$. The LRT says reject for small values of Λ.
Since $H_0 : \theta = \theta_0$ vs $H_a : \theta \neq \theta_0$.
The maximum under H_0 can only be θ_0. The max under the whole space, Ω, is given by the usual MLE which we have found to be $\frac{1}{\bar{X}}$ previously. Thus,

$$\Lambda = \frac{\theta_0^n e^{-\theta_0 \sum_{i=1}^n x_i}}{\left(\frac{1}{\bar{x}}\right)^n e^{-\frac{1}{\bar{x}} \sum_{i=1}^n x_i}} = (\bar{x}\theta_0)^n \, e^{-\theta_0 \sum_{i=1}^n x_i + n}$$

The parts that don't depend on the data are just positive constants here so rejecting for small Λ is equivalent to rejecting for small

$$\bar{x}^n e^{-\theta_0 \sum_{i=1}^n x_i} = \bar{x}^n e^{-\theta_0 n \bar{x}} = \left(\bar{x} e^{-\theta_0 \bar{x}}\right)^n$$

Rejecting for small values of this statistic is equivalent to rejecting for small values of

$$\bar{x} e^{-\theta_0 \bar{x}}$$