ST 705 Linear models and variance components
Lab practice problem set 2

January 13, 2020

1. Show that the covariance function defined for $X, Y \in \mathbb{R}^p$ by

$$\text{Cov}(X, Y) := E[(X - E(X))(Y - E(Y))']$$

satisfies the following properties. For any $X, Y, Z \in \mathbb{R}^p$ and any $c \in \mathbb{R}$,

   (a) $\text{Cov}(X + Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$
   (b) $\text{Cov}(cX, Y) = c \cdot \text{Cov}(X, Y)$
   (c) $\text{Cov}(X, Y)^* = \text{Cov}(Y, X)$
   (d) $\text{Cov}(X, X) > 0$ if $X \neq 0$ a.s.

Then, deduce that if $p = 1$ the covariance is an inner product, and if $p > 1$ the the function $f(X, Y) := \text{tr}\big(\text{Cov}(X, Y)\big)$ is an inner product.

2. For matrices $A \in \mathbb{R}^{p \times q}$, the *spectral* norm is defined as

$$\|A\|_2 := \sqrt{\sup_{x \neq 0} \frac{x'A'Ax}{x'x}}.$$

By a previous homework problem it follows that $\sup_{x \neq 0} \frac{x'A'Ax}{x'x} = \lambda_{\max}(A'A)$, where $\lambda_{\max}$ denotes the larges eigenvalue of $A'A$. Further, eigenvalues of $A'A$ are the squares of the *singular values* of A, so sometimes the definition of the spectral norm is expressed as

$$\|A\|_2 := \sigma_{\max}(A),$$

where $\sigma_{\max}$ denotes the largest singular value of A. More about singular values in the homework assignment for this week. For now,

   (a) Verify that the spectral norm is a norm. Recall that a norm must satisfy the following axioms for any $A, B, C \in \mathbb{R}^{p \times q}$ and any $\alpha \in \mathbb{R}$.
      i. $\|\alpha A\| = |\alpha| \|A\|$

ii. $\|A+B\| \leq \|A\| + \|B\|$

iii. $\|A\| \geq 0$ with equality if and only if $A = 0$.

(b) Show that the spectral norm is sub-multiplicative for square matrices. That is, for $A, B \in \mathbb{R}^{p\times p}$, $\|AB\|_2 \leq \|A\|_2 \|B\|_2$.

3. Let V be a convex subset of some vector space. Recall that a function $f : V \to \mathbb{R}$ is said to be *convex* if for every $x, y \in V$ and every $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Show, by definition, that the sum of squared errors function

$$Q(\beta) := \|Y - X\beta\|_2^2$$

is convex.

4. If P is a symmetric and idempotent matrix, show that the Pythagorean relationship holds:

$$\|y\|^2 = \|Py\|^2 + \|(I-P)y\|^2.$$