ST 705 Linear models and variance components
Homework problem set 1

January 10, 2020

1. Prove the following theorem. Let V be a vector space and $B = \{u_1, \ldots, u_n\}$ be a subset of V. Then B is a basis if and only if each $v \in V$ can be expressed *uniquely* as

$$v = a_1 u_1 + \cdots + a_n u_n$$

for some set of scalars $\{a_1, \ldots, a_n\}$.

2. Prove that the eigenvalues of an upper triangular matrix M are the diagonal components of M.

3. The defining property of a projection matrix A is that $A^2 = A$ (recall the definition of the square of a matrix from your linear algebra course). Establish the following facts.

   (a) If A is a projection matrix, then all of its eigenvalues are either zero or one.

   (b) If $A \in \mathbb{R}^{p \times p}$ is a projection and symmetric (i.e., an orthogonal projection matrix), then for every vector v the projection Av is orthogonal to $v - Av$.

   (c) $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.

   (d) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

4. Let $A \in \mathbb{R}^{p \times p}$ be symmetric. Use the spectral decomposition of A to show that

$$\sup_{x \in \mathbb{R}^p \setminus \{0\}} \frac{x' A x}{x' x} = \lambda_{\max},$$

where $\lambda_{\max}$ is the largest eigenvalue of A. Observe that this is a special case of the Courant-Fischer theorem (see `https://en.wikipedia.org/wiki/Min-max_theorem`).

5. Let $x = (x_1, \ldots, x_p)' \in \mathbb{R}^p$. Show that for $i \in \{1, \ldots, p\}$,

$$|x_i| \leq \|x\|_2 \leq \|x\|_1,$$

where $\| \cdot \|_1$ and $\| \cdot \|_2$ are the l_1 and l_2 vector norms, respectively.

6. Show that every eigenvalue of a real symmetric matrix is real.

7. Show that if $X \sim \mathrm{N}_p(\mu, \Sigma)$ and $Y = X'AX$, then $E(Y) = \mathrm{tr}(A\Sigma) + \mu' A \mu$.

8. Let U and V be random variables. Establish the following inequalities.

   (a) $P(|U + V| > a + b) \leq P(|U| > a) + P(|V| > b)$ for every $a, b \geq 0$.

   (b) $P(|UV| > a) \leq P(|U| > a/b) + P(|V| > b)$ for every $a \geq 0$ and $b > 0$.