ST 705 Linear models and variance components
Lab practice problem set 1

January 24, 2021

1. Prove the following theorem. Let V be a vector space and $B = \{u_1, \ldots, u_n\}$ be a subset of V. Then B is a basis if and only if each $v \in V$ can be expressed *uniquely* as

$$v = a_1 u_1 + \cdots + a_n u_n$$

   for some set of scalars $\{a_1, \ldots, a_n\}$.

2. Let $A \in \mathbb{R}^{p \times p}$ be symmetric. Use the spectral decomposition of A to show that

$$\sup_{x \in \mathbb{R}^p \setminus \{0\}} \frac{x' A x}{x' x} = \lambda_{\max},$$

   where $\lambda_{\max}$ is the largest eigenvalue of A. Observe that this is a special case of the Courant-Fischer theorem (see `https://en.wikipedia.org/wiki/Min-max_theorem`).

3. Show that if X is a p-dimensional random vector with mean μ and covariance Σ, A is a $p \times p$ matrix, and $Y = X'AX$, then $E(Y) = \operatorname{tr}(A\Sigma) + \mu' A \mu$.