

Supplementary material

Supplement to “J P Williams, Y Xie, and J Hannig (2019+). The EAS approach for graphical selection consistency in vector autoregression models. *Inreview*.”

S1 Derivation of the generalized fiducial mass of G and the Jacobian

This section presents the details of deriving the generalized fiducial probability mass function (8) for each graph, G , beginning with (7). In particular, the closed-form mathematical expression of the Jacobian term, $J(Y, (\alpha_G, \{\sigma_j\}))$, in (7) is worked out.

From Hannig et al. (2016),

$$J(Y, (\alpha_G, \{\sigma_j\})) := |\mathcal{D}'_g \mathcal{D}_g|^{\frac{1}{2}} = \frac{n^{-\frac{|G|}{2}}}{\sigma_1 \cdots \sigma_p} |\tilde{\mathcal{D}}'_g \tilde{\mathcal{D}}_g|^{\frac{1}{2}},$$

where $\tilde{\mathcal{D}}$ is \mathcal{D} with the σ_i omitted and $n^{-\frac{1}{4}}$ factored out, the equality follows by the Cauchy-Binet formula, and

$$\mathcal{D} := n^{-\frac{1}{2}} \begin{pmatrix} \frac{\partial X^{(1)}}{\partial A_{11}} & \frac{\partial X^{(1)}}{\partial A_{12}} & \cdots & \frac{\partial X^{(1)}}{\partial A_{pp}} & \frac{\partial X^{(1)}}{\partial \sigma_1} & \cdots & \frac{\partial X^{(1)}}{\partial \sigma_p} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial X^{(n)}}{\partial A_{11}} & \frac{\partial X^{(n)}}{\partial A_{12}} & \cdots & \frac{\partial X^{(n)}}{\partial A_{pp}} & \frac{\partial X^{(n)}}{\partial \sigma_1} & \cdots & \frac{\partial X^{(n)}}{\partial \sigma_p} \end{pmatrix}.$$

Note that we have rescaled \mathcal{D} from its definition in Hannig et al. (2016) by the factor of $n^{-\frac{1}{2}}$. This scaling is necessary for controlling the asymptotic rate of growth of the Jacobian. The first p^2 columns are partial derivatives with respect to the components A_{ij} of the transition matrix over all observed time instances of data. For each $t \in \{1, \dots, n\}$, these partial derivatives are expressed as

$$\begin{aligned} \frac{\partial X^{(t)}}{\partial A_{ij}} &= \frac{\partial A}{\partial A_{ij}} X^{(t-1)} + A \frac{\partial X^{(t-1)}}{\partial A_{ij}} \\ &= J^{ij} X^{(t-1)} + A \left(\frac{\partial A}{\partial A_{ij}} X^{(t-2)} + A \frac{\partial X^{(t-2)}}{\partial A_{ij}} \right) \\ &= J^{ij} X^{(t-1)} + A J^{ij} X^{(t-2)} + A^2 \left(\frac{\partial A}{\partial A_{ij}} X^{(t-3)} + A \frac{\partial X^{(t-3)}}{\partial A_{ij}} \right) \\ &\vdots \\ &= J^{ij} X^{(t-1)} + A J^{ij} X^{(t-2)} + A^2 J^{ij} X^{(t-3)} + \cdots + A^{t-1} J^{ij} X^{(0)} + 0, \end{aligned}$$

where J^{ij} is the matrix whose only nonzero element is a 1 in the ij th coordinate. Thus, the

column of \mathcal{D} corresponding to the ij th partial derivative of A can be expressed as

$$\begin{pmatrix} J^{ij}X^{(0)} & + & \cdots & + & 0 & + & 0 & + & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{n-3}J^{ij}X^{(0)} & + & \cdots & + & J^{ij}X^{(n-3)} & + & 0 & + & 0 \\ A^{n-2}J^{ij}X^{(0)} & + & \cdots & + & AJ^{ij}X^{(n-3)} & + & J^{ij}X^{(n-2)} & + & 0 \\ A^{n-1}J^{ij}X^{(0)} & + & \cdots & + & A^2J^{ij}X^{(n-3)} & + & AJ^{ij}X^{(n-2)} & + & J^{ij}X^{(n-1)} \end{pmatrix} \\ = \Theta(I_n \otimes J^{ij})\text{vec}(\mathcal{X}),$$

where

$$\Theta := \begin{pmatrix} I_p & \cdots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ A^{n-3} & \cdots & I_p & 0 & 0 \\ A^{n-2} & \cdots & A & I_p & 0 \\ A^{n-1} & \cdots & A^2 & A & I_p \end{pmatrix}. \quad (\text{S1})$$

Note that the notation Θ_g will be taken to mean the matrix Θ with A_g replacing every occurrence of A in (S1).

Next, the remaining p columns of \mathcal{D} require the partial derivatives of the time instances of the data with respect to σ_i , for each $i \in \{1, \dots, p\}$. Accordingly, observe that

$$\begin{aligned} \frac{\partial X^{(t)}}{\partial \sigma_i} &= A \frac{\partial X^{(t-1)}}{\partial \sigma_i} + J^{ii}U^{(t)} \\ &= A \left(A \frac{\partial X^{(t-2)}}{\partial \sigma_i} + J^{ii}U^{(t-1)} \right) + J^{ii}U^{(t)} \\ &= A^2 \left(A \frac{\partial X^{(t-3)}}{\partial \sigma_i} + J^{ii}U^{(t-2)} \right) + AJ^{ii}U^{(t-1)} + J^{ii}U^{(t)} \\ &\vdots \\ &= A^{t-1}J^{ii}U^{(1)} + A^{t-2}J^{ii}U^{(2)} + \cdots + AJ^{ii}U^{(t-1)} + J^{ii}U^{(t)}, \end{aligned}$$

and so the column of \mathcal{D} corresponding to the partial derivative of σ_i can be expressed as

$$\begin{pmatrix} J^{ii}U^{(1)} & + & 0 & + & \cdots & + & 0 & + & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{n-3}J^{ii}U^{(1)} & + & A^{n-4}J^{ii}U^{(2)} & + & \cdots & + & 0 & + & 0 \\ A^{n-2}J^{ii}U^{(1)} & + & A^{n-3}J^{ii}U^{(2)} & + & \cdots & + & J^{ii}U^{(n-1)} & + & 0 \\ A^{n-1}J^{ii}U^{(1)} & + & A^{n-2}J^{ii}U^{(2)} & + & \cdots & + & AJ^{ii}U^{(n-1)} & + & J^{ii}U^{(n)} \end{pmatrix} \\ = \Theta(I_n \otimes J^{ii}\sigma_i^{-1})(Y - Z\alpha).$$

With expressions for all $p^2 + p$ columns of \mathcal{D} now derived, the Jacobian can be written in closed-form.

Keeping the derivation of $J(Y, (\alpha_G, \{\sigma_j\}))$ in mind, the marginal distribution,

$$\begin{aligned}
r(G, \{\sigma_j\} \mid Y) &= \int r(\alpha_G, \{\sigma_j\} \mid Y) d\alpha_G \\
&\propto \int f(Y \mid \alpha_G, \{\sigma_j\}) J(Y, (\alpha_G, \{\sigma_j\})) h(\alpha_G, \{\sigma_j\}) d\alpha_G \\
&= \int \frac{e^{-\frac{1}{2}(Y - Z_G \alpha_G)' \mathcal{W}^{-1}(Y - Z_G \alpha_G)}}{(2\pi)^{\frac{np}{2}} (\sigma_1^2 \dots \sigma_p^2)^{\frac{n+1}{2}} n^{\frac{|G|}{2}}} |\tilde{\mathcal{D}}'_g \tilde{\mathcal{D}}_g|^{\frac{1}{2}} h(\alpha_G, \{\sigma_j\}) d\alpha_G \\
&= \frac{e^{-\frac{1}{2} \mathcal{S}_G} (2\pi)^{-\frac{np}{2}} n^{-\frac{|G|}{2}}}{(\sigma_1^2 \dots \sigma_p^2)^{\frac{n+1}{2}}} \int e^{-\frac{1}{2}(\alpha_G - \hat{\alpha}_g)' Z'_G \mathcal{W}^{-1} Z_G (\alpha_G - \hat{\alpha}_g)} |\tilde{\mathcal{D}}'_g \tilde{\mathcal{D}}_g|^{\frac{1}{2}} h(\alpha_G, \{\sigma_j\}) d\alpha_G \\
&\propto \frac{e^{-\frac{1}{2} \mathcal{S}_G} n^{-\frac{|G|}{2}}}{(\sigma_1^2 \dots \sigma_p^2)^{\frac{n+1}{2}}} E_{\alpha_G \mid \{\sigma_j\}} \left(h(\alpha_G, \{\sigma_j\}) |\tilde{\mathcal{D}}'_g \tilde{\mathcal{D}}_g|^{\frac{1}{2}} \right) \cdot (2\pi)^{\frac{|G|}{2}} |Z'_G \mathcal{W}^{-1} Z_G|^{-\frac{1}{2}},
\end{aligned}$$

where $\hat{\alpha}_g := (Z'_G \mathcal{W}^{-1} Z_G)^{-1} Z'_G \mathcal{W}^{-1} Y$ is the weighted least squares estimator, $\mathcal{S}_G := Y' \mathcal{W}^{-1} (I_{np} - Z_G (Z'_G \mathcal{W}^{-1} Z_G)^{-1} Z'_G \mathcal{W}^{-1}) Y$ is the corresponding weighted sum-of-squared residuals, and the conditional expectation, $E_{\alpha_G \mid \{\sigma_j\}}(\cdot) := E(\cdot \mid \{\sigma_j\})$, is taken with respect to the $N_{|G|}(\hat{\alpha}_g, (Z'_G \mathcal{W}^{-1} Z_G)^{-1})$ distribution. Further, since \mathcal{W} is a diagonal matrix it can be factored out of $\hat{\alpha}_g$ and from parts of \mathcal{S}_G . This is a well-known fact from linear model theory, and the details are as follows.

$$\begin{aligned}
\mathcal{W}^{-1} Z_G &= (I_n \otimes \Sigma^{-1}) \begin{pmatrix} X^{(0)'} \otimes I_p \\ X^{(n-1)'} \otimes I_p \end{pmatrix}_G \\
&= \begin{pmatrix} X^{(0)'} \otimes \Sigma^{-1} \\ X^{(n-1)'} \otimes \Sigma^{-1} \end{pmatrix}_G \\
&= \begin{pmatrix} X^{(0)'} \otimes I_p \\ X^{(n-1)'} \otimes I_p \end{pmatrix}_G (I_p \otimes \Sigma^{-1})_{G,G} \\
&= Z_G (I_p \otimes \Sigma)_{G,G}^{-1},
\end{aligned}$$

which gives $Z'_G \mathcal{W}^{-1} = (\mathcal{W}^{-1} Z_G)' = (I_p \otimes \Sigma)_{G,G}^{-1} Z'_G$, and so

$$\hat{\alpha}_g = ((I_p \otimes \Sigma)_{G,G}^{-1} Z'_G Z_G)^{-1} (I_p \otimes \Sigma)_{G,G}^{-1} Z'_G Y = (Z'_G Z_G)^{-1} Z'_G Y,$$

and $\mathcal{S}_G = Y' \mathcal{W}^{-1} (I_{np} - H_g) Y$, where $H_g := Z_G (Z'_G Z_G)^{-1} Z'_G$.

Next, simplify $|\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G|$ as follows. Observe that

$$\begin{aligned} \mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G &= (\mathcal{X}' \otimes I_p)'_G (I_n \otimes \Sigma)^{-1} (\mathcal{X}' \otimes I_p)_G \\ &= \left((\mathcal{X}' \otimes I_p)' (I_n \otimes \Sigma^{-1}) (\mathcal{X}' \otimes I_p) \right)_{G,G} \\ &= (\mathcal{X}' \otimes \Sigma^{-\frac{1}{2}})'_G (\mathcal{X}' \otimes \Sigma^{-\frac{1}{2}})_G. \end{aligned}$$

Permuting the columns of $(\mathcal{X}' \otimes \Sigma^{-\frac{1}{2}})_G$ gives the matrix

$$\begin{pmatrix} \sigma_1^{-1} X_{r_1^g}^{(0)'} & & & \\ & \ddots & & \\ & & \sigma_p^{-1} X_{r_p^g}^{(0)'} & \\ & & \vdots & \\ \sigma_1^{-1} X_{r_1^g}^{(n-1)'} & & & \\ & \ddots & & \\ & & \sigma_p^{-1} X_{r_p^g}^{(n-1)'} & \end{pmatrix},$$

and so rearranging rows and columns gives

$$\begin{aligned} |\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G| &= \begin{vmatrix} \sigma_1^{-2} \sum_{t=1}^n X_{r_1^g}^{(t-1)} X_{r_1^g}^{(t-1)'} & & \\ & \ddots & \\ & & \sigma_p^{-2} \sum_{t=1}^n X_{r_p^g}^{(t-1)} X_{r_p^g}^{(t-1)'} \end{vmatrix} \\ &= (\sigma_1^2)^{-|r_1^g|} \dots (\sigma_p^2)^{-|r_p^g|} \prod_{j=1}^p \left| \sum_{t=1}^n X_{r_j^g}^{(t-1)} X_{r_j^g}^{(t-1)'} \right|. \end{aligned}$$

Note that rearranging the same number of rows as columns preserves the sign of the determinant. Accordingly,

$$r(G, \{\sigma_j\} \mid Y) \propto \frac{\prod_{j=1}^p \left| \sum_{t=1}^n X_{r_j^g}^{(t-1)} X_{r_j^g}^{(t-1)'} \right|^{-\frac{1}{2}} E_{\alpha_G \mid \{\sigma_j\}} \left(h(\alpha_G, \{\sigma_j\}) |\tilde{\mathcal{D}}'_g \tilde{\mathcal{D}}_g|^{\frac{1}{2}} \right)}{n^{\frac{|G|}{2}} (2\pi)^{-\frac{|G|}{2}} (\sigma_1^2 \dots \sigma_p^2)^{\frac{n+1}{2}} (\sigma_1^2)^{-\frac{|r_1^g|}{2}} \dots (\sigma_p^2)^{-\frac{|r_p^g|}{2}} e^{\frac{1}{2} \mathcal{S}_G}}.$$

To further simplify this expression, recall that $\Sigma = \text{diag}\{\sigma_1^2, \dots, \sigma_p^2\}$ and observe that

$$\mathcal{S}_G = \begin{pmatrix} X^{(1)'} \Sigma^{-1} & \dots & X^{(n)'} \Sigma^{-1} \end{pmatrix} \begin{pmatrix} (I_{np} - H_g)'_1 Y \\ \vdots \\ (I_{np} - H_g)'_{np} Y \end{pmatrix} = \sum_{j=1}^p \sigma_j^{-2} m_j^g,$$

where as in (5), $m_j^g := \sum_{t=1}^n X_j^{(t)} (I_{np} - H_g)'_{(t-1)p+j} Y$. Hence,

$$\begin{aligned} r(G \mid Y) &= \int_0^\infty \cdots \int_0^\infty r(G, \{\sigma_j\} \mid Y) d\sigma_1 \cdots d\sigma_p \\ &\quad \int_0^\infty \cdots \int_0^\infty E_{\alpha_G \mid \{\sigma_j\}} \left(h(\alpha_G, \{\sigma_j\}) |\tilde{\mathcal{D}}'_g \tilde{\mathcal{D}}_g|^{\frac{1}{2}} \right) \frac{e^{-\frac{m_1^g}{2} \sigma_1^{-2}} \cdots e^{-\frac{m_p^g}{2} \sigma_p^{-2}}}{(\sigma_1^2)^{\frac{n-|r_1^g|+1}{2}} \cdots (\sigma_p^2)^{\frac{n-|r_p^g|+1}{2}}} d\sigma_1 \cdots d\sigma_p \\ &\propto \frac{E_{\alpha_G \mid \{\sigma_j\}} \left(h(\alpha_G, \{\sigma_j\}) |\tilde{\mathcal{D}}'_g \tilde{\mathcal{D}}_g|^{\frac{1}{2}} \right) \prod_{j=1}^p \left(\frac{m_j^g}{2} \right)^{-\frac{n-|r_j^g|}{2}} \Gamma\left(\frac{n-|r_j^g|}{2}\right)}{\left(\frac{n}{2\pi} \right)^{\frac{|G|}{2}} \prod_{j=1}^p \left| \sum_{t=1}^n X_{r_j^g}^{(t-1)} X_{r_j^g}^{(t-1)'} \right|^{\frac{1}{2}}}} \\ &= \frac{E_{\alpha_G \mid \{\sigma_j\}} \left(h(\alpha_G, \{\sigma_j\}) |\tilde{\mathcal{D}}'_g \tilde{\mathcal{D}}_g|^{\frac{1}{2}} \right) \prod_{j=1}^p \left(\frac{m_j^g}{2} \right)^{-\frac{n-|r_j^g|}{2}} \Gamma\left(\frac{n-|r_j^g|}{2}\right)}{\left(\frac{n}{2\pi} \right)^{\frac{|G|}{2}} \prod_{j=1}^p \left| \sum_{t=1}^n X_{r_j^g}^{(t-1)} X_{r_j^g}^{(t-1)'} \right|^{\frac{1}{2}}}}, \end{aligned}$$

which gives (8).

S2 Additional lemmas

Lemma S2.1. Assume that Condition 3.1 holds. Then for all $n \geq N_2$,

$$P \left(\frac{\|(\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o}\|_2^2}{\lambda_{\min}^2(\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o})} \geq \frac{1}{n^{2-\frac{p}{2}} p^5} \right) \leq \frac{p^{\frac{7}{2}}}{n^{\frac{p}{4}}} \cdot \frac{(\sigma_{\max}^0)^2}{(\sigma_{\min}^0)^2} \cdot \frac{6\|\Gamma_n(0)\|_2/\delta}{(1 - |G_o|/n - 2/n)},$$

with probability exceeding $1 - V_2 - e^{-\frac{np}{4}}$, where V_2 is as in (10).

Lemma S2.2. Take any $G \subset \{1, \dots, p^2\}$. If $\|A_g\|_2 \leq c$, then

$$|\tilde{\mathcal{D}}'_g \tilde{\mathcal{D}}_g|^{\frac{1}{2}} < \exp \left\{ \frac{1}{2} (1-c)^{-2} (r_{\max}^g + (1+c)^2) \frac{\|Y\|^2}{\sqrt{n}} - \frac{|G|+p}{2} \right\},$$

where $r_{\max}^g := \max_{1 \leq j \leq p} |r_j^g|$.

Lemma S2.3. If $\|A\|_2 < 1$, then

$$\|f_X(\Theta_g)\|_2 \leq \frac{\sigma_{\max}^2}{2\pi} (1 - \|A\|_2)^{-2},$$

where f_X is the spectral density for a VAR(1) process with coefficient matrix A and error covariance matrix $\Sigma = \text{diag}\{\sigma_1^2, \dots, \sigma_p^2\}$.

Lemma S2.4. Assume Condition 3.1 holds. If $\|A_{g_o}\|_2 \leq c$, then for all $n \geq \max\{N_1, N_2\}$,

$$P_x \left(|\tilde{\mathcal{D}}'_{g_o} \tilde{\mathcal{D}}_{g_o}|^{\frac{1}{2}} \geq e^{\frac{|G_o|+p}{4}} \right) \geq 1 - V_3,$$

where V_3 is as in (11).

Lemma S2.5. Assume that Condition 3.1 holds. Then for all $n \geq N_2$,

$$P_x \left(\prod_{j=1}^p \frac{|(\mathcal{X}\mathcal{X}')_{r_j^{g_o}, r_j^{g_o}}|^{\frac{1}{2}}}{|(\mathcal{X}\mathcal{X}')_{r_j^g, r_j^g}|^{\frac{1}{2}}} \leq n^{\frac{|G_o| - |G|}{2}} e^{\frac{1}{2}(|G_o|[\delta + \lambda_{\max}(\Gamma_n(0))] + |G|2\delta^{-1})} \right) \geq 1 - 2V_2,$$

where V_2 is as in (10).

Lemma S2.6. Assume that Condition 3.1 holds, and that $1 \leq d \leq \min_{1 \leq j \leq p} m_j^{g_o}$. Then for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned} P_x \left(\prod_{j=1}^p \left[\frac{(m_j^{g_o})^{\frac{n - |r_j^{g_o}|}{2}}}{(m_j^g)^{\frac{n - |r_j^g|}{2}}} \right] \leq ((\sigma_{\max}^0)^2 3n)^{\frac{p^2}{2}} \cdot e^{(\sigma_{\max}^0)^2 p^2 \sqrt{n} \cdot \frac{n}{2q}} \right) \\ \geq 1 - e^{-\frac{np}{4}} - \frac{2(\sigma_{\max}^0)^2}{\delta(1 - c^2)\sqrt{n}} - V_2, \end{aligned}$$

where $q := \min_{1 \leq j \leq p} m_j$ with m_1, \dots, m_p corresponding to the full model.

S3 Proofs

Proof of Theorem 3.9. Recall that

$$h(\alpha_G, \{\sigma_j\}) := 1 \left\{ \frac{1}{2} \|\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G (\alpha_G - b_{\min})\|^2 \geq \varepsilon, \min_{1 \leq j \leq p} \{m_j^g\} \geq d, \|\mathcal{A}_g\|_2 \leq c \right\}$$

where b_{\min} solves $\min_{b \in \mathbb{R}^{|G|}} \frac{1}{2} \|\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G (\alpha_G - b)\|^2$ subject to $\|b\|_0 \leq |G| - 1$. By Lemma S2.2 the Jacobian term can be bounded to yield,

$$E \left(h(\alpha_G, \{\sigma_j\}) |\tilde{\mathcal{D}}'_g \tilde{\mathcal{D}}_g|^{\frac{1}{2}} \right) \leq e^{\frac{1}{2}(1-c)^{-2} (r_{\max}^g + (1+c)^2) \frac{\|Y\|^2}{\sqrt{n}} - \frac{|G|+p}{2}} \cdot E[h(\alpha_G, \{\sigma_j\})].$$

Further,

$$\begin{aligned} E[h(\alpha_G, \{\sigma_j\})] &= P \left(\frac{1}{2} \|\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G (\alpha_G - b_{\min})\|^2 \geq \varepsilon, \min_{1 \leq j \leq p} \{m_j^g\} \geq d \right) \\ &\leq P \left(\frac{1}{2} \|\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G (\alpha_G - \tilde{b})\|^2 \geq \varepsilon, \min_{1 \leq j \leq p} \{m_j^g\} \geq d \right), \end{aligned}$$

where \tilde{b} solves $\min_{b \in \mathbb{R}^{|G|}} \|(E_x(\mathcal{Z}'_G \mathcal{Z}_G))^{-1} E_x(\mathcal{Z}'_G Y) - b\|^2$ subject to $\|b\|_0 \leq |G| - 1$. Recall that the least squares estimate $\hat{\alpha}_g := (\mathcal{Z}'_G \mathcal{Z}_G)^{-1} \mathcal{Z}'_G Y$, and apply the triangle inequality to get the

following probabilistic bound.

$$\begin{aligned}
& E[h(\boldsymbol{\alpha}_G, \{\sigma_j\})] \\
& \leq P\left(\|\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G(\boldsymbol{\alpha}_G - \hat{\boldsymbol{\alpha}}_g)\| \geq \sqrt{2\varepsilon}/3, \min_{1 \leq j \leq p} \{m_j^g\} \geq d\right) \\
& \quad + P\left(\left\|\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G \left[\hat{\boldsymbol{\alpha}}_g - (E_x(\mathcal{Z}'_G \mathcal{Z}_G))^{-1} E_x(\mathcal{Z}'_G Y)\right]\right\| \geq \sqrt{2\varepsilon}/3, \min_{1 \leq j \leq p} \{m_j^g\} \geq d\right) \\
& \quad + P\left(\left\|\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G \left[(E_x(\mathcal{Z}'_G \mathcal{Z}_G))^{-1} E_x(\mathcal{Z}'_G Y) - \tilde{b}\right]\right\| \geq \sqrt{2\varepsilon}/3, \min_{1 \leq j \leq p} \{m_j^g\} \geq d\right) \\
& \leq \frac{3|G|\sqrt{\Lambda_g}}{\sqrt{\pi\varepsilon}} e^{-\frac{\varepsilon}{9\Lambda_g}} \\
& \quad + P\left(\sqrt{\Lambda_g} \cdot \|\hat{\boldsymbol{\alpha}}_g - (E_x(\mathcal{Z}'_G \mathcal{Z}_G))^{-1} E_x(\mathcal{Z}'_G Y)\| \geq \frac{\sqrt{2\varepsilon}}{3\sqrt{\Lambda_g}}, \min_{1 \leq j \leq p} \{m_j^g\} \geq d\right) \\
& \quad + P\left(\sqrt{\Lambda_g} \cdot \|(E_x(\mathcal{Z}'_G \mathcal{Z}_G))^{-1} E_x(\mathcal{Z}'_G Y) - \tilde{b}\| \geq \frac{\sqrt{2\varepsilon}}{3\sqrt{\Lambda_g}}, \min_{1 \leq j \leq p} \{m_j^g\} \geq d\right),
\end{aligned}$$

where the second inequality follows from Lemma 3.6, and recalling that $\Lambda_g := \text{tr}(\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G)$. Next, using the assumption that $\varepsilon = \Lambda_g \cdot \tilde{\varepsilon}$ for some $\tilde{\varepsilon}$ not depending on Σ or A_g , observe that

$$\begin{aligned}
& E[h(\boldsymbol{\alpha}_G, \{\sigma_j\})] \\
& \leq \frac{3|G|\sqrt{\Lambda_g}}{\sqrt{\pi\varepsilon}} e^{-\frac{\varepsilon}{9\Lambda_g}} + 2P\left(\sqrt{\Lambda_g} \geq \sqrt{n^{1+\frac{\rho}{2}} p^3}, \min_{1 \leq j \leq p} \{m_j^g\} \geq d\right) \\
& \quad + 1\left\{\left\|\hat{\boldsymbol{\alpha}}_g - (E_x(\mathcal{Z}'_G \mathcal{Z}_G))^{-1} E_x(\mathcal{Z}'_G Y)\right\| \geq \frac{\sqrt{2\varepsilon}}{3\sqrt{n^{1+\frac{\rho}{2}} p^3 \Lambda_g}}, \min_{1 \leq j \leq p} \{m_j^g\} \geq d\right\} \\
& \quad + 1\left\{\left\|(E_x(\mathcal{Z}'_G \mathcal{Z}_G))^{-1} E_x(\mathcal{Z}'_G Y) - \tilde{b}\right\| \geq \frac{\sqrt{2\varepsilon}}{3\sqrt{n^{1+\frac{\rho}{2}} p^3 \Lambda_g}}, \min_{1 \leq j \leq p} \{m_j^g\} \geq d\right\},
\end{aligned}$$

and applying Lemma 3.7 gives,

$$\begin{aligned}
E[h(\boldsymbol{\alpha}_G, \{\sigma_j\})] & \leq \frac{3|G|\sqrt{\Lambda_g}}{\sqrt{\pi\varepsilon}} e^{-\frac{\varepsilon}{9\Lambda_g}} + e^{-\left(\frac{d \cdot n^{\frac{\rho}{2}} p^2}{4\lambda_{\max}(\mathcal{X}\mathcal{X}'/n)} - \frac{np}{2}\right)} 2^{-\frac{|G|}{2}+1} \\
& \quad + 1\left\{\left\|\hat{\boldsymbol{\alpha}}_g - (E_x(\mathcal{Z}'_G \mathcal{Z}_G))^{-1} E_x(\mathcal{Z}'_G Y)\right\|^2 \geq \frac{2\varepsilon}{9n^{1+\frac{\rho}{2}} p^3 \Lambda_g}\right\} \\
& \quad + 1\left\{\left\|(E_x(\mathcal{Z}'_G \mathcal{Z}_G))^{-1} E_x(\mathcal{Z}'_G Y) - \tilde{b}\right\|^2 \geq \frac{2\varepsilon}{9n^{1+\frac{\rho}{2}} p^3 \Lambda_g}\right\}.
\end{aligned}$$

Hence, by Lemma 3.8 (and thus Condition 3.1) and Condition 3.3 it follows that for all $n \geq \max\{N_1, N_2\}$,

$$E[h(\boldsymbol{\alpha}_G, \{\sigma_j\})] \leq \frac{3|G|\sqrt{\Lambda_g}}{\sqrt{\pi\varepsilon}} e^{-\frac{\varepsilon}{9\Lambda_g}} + e^{-\left(\frac{d \cdot n^{\frac{\rho}{2}} p^2}{4\lambda_{\max}(\mathcal{X}\mathcal{X}'/n)} - \frac{np}{2}\right)} 2^{-\frac{|G|}{2}+1},$$

with probability exceeding $1 - V_1$ where V_1 is as defined in (9). ■

Proof of Theorem 3.10. Recall that

$$h(\alpha_{G_o}, \{\sigma_j\}) := 1 \left\{ \frac{1}{2} \|\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o} (\alpha_{G_o} - b_{\min})\|^2 \geq \varepsilon, \min_{1 \leq j \leq p} \{m_j^{g_o}\} \geq d, \|A_{g_o}\|_2 \leq c \right\}$$

where b_{\min} solves $\min_{b \in \mathbb{R}^{|G_o|}} \frac{1}{2} \|\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o} (\alpha_{G_o} - b)\|^2$ s. t. $\|b\|_0 \leq |G_o| - 1$. To show the desired result, let \tilde{b} be the solution to

$$\min_{b \in \mathbb{R}^{|G_o|}} \|(\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o} (\alpha_{G_o}^0 - b)\|^2 \text{ subject to } \|b\|_0 \leq |G_o| - 1.$$

Then

$$\begin{aligned} & \|(\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o} (\alpha_{G_o}^0 - \tilde{b})\|^2 \\ & \leq \|(\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o} (\alpha_{G_o}^0 - b_{\min})\|^2 \\ & = \|(\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o} (\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o})^{-1} \mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o} (\alpha_{G_o}^0 - b_{\min})\|^2 \\ & \leq \|(\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o} (\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o})^{-1}\|_2^2 \cdot \|\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o} (\alpha_{G_o}^0 - b_{\min})\|^2 \\ & \leq \frac{\|(\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o}\|_2^2}{\lambda_{\min}^2(\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o})} \cdot \|\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o} (\alpha_{G_o}^0 - b_{\min})\|^2, \end{aligned}$$

and so for $\rho \in (0, \frac{1}{2})$,

$$\begin{aligned} & 1 \left\{ \|(\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o} (\alpha_{G_o}^0 - \tilde{b})\|^2 \geq \frac{18\varepsilon}{n^{1-\rho} p^2 \Lambda_{g_o}} \right\} \cdot P(\|A_{g_o}\|_2 \leq c) \\ & \leq P \left(\frac{\|(\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o}\|_2^2}{\lambda_{\min}^2(\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o})} \geq \frac{1}{n^{1-\rho} p^2 \Lambda_{g_o}}, \|A_{g_o}\|_2 \leq c \right) \\ & \quad + P \left(\|\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o} (\alpha_{G_o}^0 - b_{\min})\|^2 \geq 18\varepsilon, \|A_{g_o}\|_2 \leq c \right) \\ & \leq P \left(\frac{\|(\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o}\|_2^2}{\lambda_{\min}^2(\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o})} \geq \frac{1}{n^{1-\rho} p^2 \Lambda_{g_o}} \right) \\ & \quad + P \left(\|\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o} (\alpha_{G_o}^0 - \hat{\alpha}_{g_o})\|^2 \geq 2\varepsilon \right) \\ & \quad + P \left(\|\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o} (\hat{\alpha}_{g_o} - \alpha_{G_o})\|^2 \geq 2\varepsilon \right) \\ & \quad + P \left(\|\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o} (\alpha_{G_o} - b_{\min})\|^2 \geq 2\varepsilon, \|A_{g_o}\|_2 \leq c \right) \end{aligned}$$

where $\hat{\alpha}_{g_o} := (\mathcal{Z}'_{G_o} \mathcal{Z}_{G_o})^{-1} \mathcal{Z}'_{G_o} Y$ is the least squares estimator. Applying Lemma 3.6, multiplying both sides of the inequality by $1 \left\{ \min_{1 \leq j \leq p} \{m_j^{g_o}\} \geq d \right\}$, and denoting $\omega := P(\|A_{g_o}\|_2 \leq c)$

gives,

$$\begin{aligned}
& 1 \left\{ \left\| (\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o} (\boldsymbol{\alpha}_{G_o}^0 - \tilde{b}) \right\|^2 \geq \frac{18\varepsilon}{n^{1-\rho} p^2 \Lambda_{g_o}}, \min_{1 \leq j \leq p} \{m_j^{g_o}\} \geq d \right\} \cdot \omega \\
& \leq P \left(\Lambda_{g_o} \cdot \frac{\left\| (\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o} \right\|_2^2}{\lambda_{\min}^2(\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o})} \geq \frac{1}{n^{1-\rho} p^2}, \min_{1 \leq j \leq p} \{m_j^{g_o}\} \geq d \right) \\
& \quad + P \left(\Lambda_{g_o} \cdot \left\| \hat{\boldsymbol{\alpha}}_{g_o} - \boldsymbol{\alpha}_{G_o}^0 \right\|^2 \geq \frac{2\varepsilon}{\Lambda_{g_o}}, \min_{1 \leq j \leq p} \{m_j^{g_o}\} \geq d \right) \\
& \quad + \frac{|G_o| \sqrt{\Lambda_{g_o}}}{\sqrt{\pi \varepsilon}} e^{-\frac{\varepsilon}{\Lambda_{g_o}}} + E[h(\boldsymbol{\alpha}_{G_o}, \{\sigma_j\})] \\
& \leq P \left(\frac{\left\| (\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o} \right\|_2^2}{\lambda_{\min}^2(\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o})} \geq \frac{1}{n^{2-\frac{\rho}{2}} p^5}, \min_{1 \leq j \leq p} \{m_j^{g_o}\} \geq d \right) \\
& \quad + 2P \left(\Lambda_{g_o} \geq n^{1+\frac{\rho}{2}} p^3, \min_{1 \leq j \leq p} \{m_j^{g_o}\} \geq d \right) \\
& \quad + 1 \left\{ \left\| \hat{\boldsymbol{\alpha}}_{g_o} - \boldsymbol{\alpha}_{G_o}^0 \right\|^2 \geq \frac{2\varepsilon}{9n^{1+\frac{\rho}{2}} p^3 \Lambda_{g_o}} \right\} \\
& \quad + \frac{|G_o| \sqrt{\Lambda_{g_o}}}{\sqrt{\pi \varepsilon}} e^{-\frac{\varepsilon}{\Lambda_{g_o}}} + E[h(\boldsymbol{\alpha}_{G_o}, \{\sigma_j\})].
\end{aligned}$$

Next, by Condition 3.2 the indicator function on the left side is equal to 1, and for all $n \geq N_2$, by Lemmas 3.7 and S2.1 (and accordingly Condition 3.1),

$$\begin{aligned}
\omega & \leq \frac{p^{\frac{7}{2}}}{n^{\frac{\rho}{4}}} \cdot \frac{(\sigma_{\max}^0)^2}{(\sigma_{\min}^0)^2} \cdot \frac{6\|\Gamma_n(0)\|_2/\delta}{(1 - |G_o|/n - 2/n)} + e^{-\left(\frac{d \cdot n^{\frac{\rho}{2}} p^2}{4\lambda_{\max}(\mathcal{X}\mathcal{X}'/n)} - \frac{np}{2}\right)} 2^{-\frac{|G_o|}{2}+1} \\
& \quad + 1 \left\{ \left\| \hat{\boldsymbol{\alpha}}_{g_o} - \boldsymbol{\alpha}_{G_o}^0 \right\|^2 \geq \frac{2\varepsilon}{9n^{1+\frac{\rho}{2}} p^3 \Lambda_{g_o}} \right\} + \frac{|G_o| \sqrt{\Lambda_{g_o}}}{\sqrt{\pi \varepsilon}} e^{-\frac{\varepsilon}{\Lambda_{g_o}}} + E[h(\boldsymbol{\alpha}_{G_o}, \{\sigma_j\})],
\end{aligned}$$

with probability exceeding $1 - V_2 - e^{-\frac{np}{4}}$, where V_2 is as in (10). Then, by observing that $\boldsymbol{\alpha}_{G_o}^0 = (E_x(\mathcal{Z}'_{G_o} \mathcal{Z}_{G_o}))^{-1} E_x(\mathcal{Z}'_{G_o} Y)$ and applying Lemma 3.8 yields for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned}
\omega & \leq \frac{p^{\frac{7}{2}}}{n^{\frac{\rho}{4}}} \cdot \frac{(\sigma_{\max}^0)^2}{(\sigma_{\min}^0)^2} \cdot \frac{6\|\Gamma_n(0)\|_2/\delta}{(1 - |G_o|/n - 2/n)} + e^{-\left(\frac{d \cdot n^{\frac{\rho}{2}} p^2}{4\lambda_{\max}(\mathcal{X}\mathcal{X}'/n)} - \frac{np}{2}\right)} 2^{-\frac{|G_o|}{2}+1} \\
& \quad + \frac{|G_o| \sqrt{\Lambda_{g_o}}}{\sqrt{\pi \varepsilon}} e^{-\frac{\varepsilon}{\Lambda_{g_o}}} + E[h(\boldsymbol{\alpha}_{G_o}, \{\sigma_j\})],
\end{aligned} \tag{S2}$$

with probability exceeding $1 - V_1 - V_2 - e^{-\frac{np}{4}}$, where V_1 is as in (9).

Finally, Condition 3.1 allows for the following probabilistic bound on ω .

$$\begin{aligned}
1 - \omega &:= P(\|A_{g_o}\|_2 > c) \\
&\leq P(\|A_{g_o} - \hat{A}_{g_o}\|_2 > c) + 1\{\|\hat{A}_{g_o} - A^0\|_2 > c\} + 1\{\|A^0\|_2 > c\} \\
&\leq P(\|A_{g_o} - \hat{A}_{g_o}\|_F > c) + 1\{\|\hat{A}_{g_o} - A^0\|_F > c\} \\
&= P(\|\alpha_{G_o} - \hat{\alpha}_{g_o}\|^2 > c^2) + 1\{\|\hat{\alpha}_{g_o} - \alpha_{G_o}^0\|^2 > c^2\} \\
&= P(\|\alpha_{G_o} - \hat{\alpha}_{g_o}\|^2 > c^2),
\end{aligned}$$

with probability exceeding $1 - \tilde{V}_1$ by Lemma 3.8 where \tilde{V}_1 is V_1 as in (9) with ε replaced by $c^2 \cdot \frac{9n^{1+\frac{p}{2}}p^3\Lambda_{g_o}}{2}$. Further,

$$\begin{aligned}
P(\|\alpha_{G_o} - \hat{\alpha}_{g_o}\|^2 > c^2) &= P(\|(\mathcal{Z}'_{G_o}\mathcal{W}^{-1}\mathcal{Z}_{G_o})^{-\frac{1}{2}}(\mathcal{Z}'_{G_o}\mathcal{W}^{-1}\mathcal{Z}_{G_o})^{\frac{1}{2}}(\alpha_{G_o} - \hat{\alpha}_{g_o})\|^2 > c^2) \\
&\leq P\left(\lambda_{\max}((\mathcal{Z}'_{G_o}\mathcal{W}^{-1}\mathcal{Z}_{G_o})^{-1})\|Z\|^2 > c^2\right)
\end{aligned}$$

where, as in the proof of Lemma 3.6, $Z := (\mathcal{Z}'_{G_o}\mathcal{W}^{-1}\mathcal{Z}_{G_o})^{\frac{1}{2}}(\alpha_{G_o} - \hat{\alpha}_{g_o}) \sim N_{|G_o|}(0, I_{|G_o|})$. Then

$$\begin{aligned}
P(\|\alpha_{G_o} - \hat{\alpha}_{g_o}\|^2 > c^2) &\leq P\left(\|Z\|^2 > c^2\lambda_{\min}(\mathcal{Z}'_{G_o}\mathcal{W}^{-1}\mathcal{Z}_{G_o})\right) \\
&\leq P\left(\|Z\|^2 > c^2\sqrt{n}\right) + P\left(\lambda_{\min}(\mathcal{Z}'_{G_o}\mathcal{W}^{-1}\mathcal{Z}_{G_o}) < \sqrt{n}\right).
\end{aligned}$$

Observe that

$$\begin{aligned}
P\left(\|Z\|^2 > c^2\sqrt{n}\right) &\leq P\left(|G_o| \max_{1 \leq i \leq |G_o|} \{Z_i^2\} > c^2\sqrt{n}\right) \\
&\leq \sum_{i=1}^{|G_o|} P\left(|Z_i| > cn^{\frac{1}{4}}|G_o|^{-\frac{1}{2}}\right) \\
&= \sum_{i=1}^{|G_o|} 2P\left(Z_i < -cn^{\frac{1}{4}}|G_o|^{-\frac{1}{2}}\right) \\
&\leq 2|G_o| \frac{|G_o|^{\frac{1}{2}}}{cn^{\frac{1}{4}}\sqrt{2\pi}} e^{-\frac{c^2\sqrt{n}}{2|G_o|}},
\end{aligned}$$

where the last inequality follows because for the standard normal CDF, Φ for $x > 0$, $\Phi(-x) \leq \frac{1}{x\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. Additionally, by the same arguments as in the proof of Lemma S2.1,

$$P\left(\lambda_{\min}(\mathcal{Z}'_{G_o}\mathcal{W}^{-1}\mathcal{Z}_{G_o}) < \sqrt{n}\right) \leq \frac{6(\sigma_{\max}^0)^2p}{\delta\sqrt{n}(1 - |G_o|/n - 2/n)},$$

with probability exceeding $1 - V_2 - e^{-\frac{np}{4}}$, where V_2 is as in (10). Thus,

$$\omega \geq 1 - \frac{\sqrt{2}|G_o|^{1+\frac{1}{2}}}{cn^{\frac{1}{4}}\sqrt{\pi}} e^{-\frac{c^2\sqrt{n}}{2|G_o|}} - \frac{6(\sigma_{\max}^0)^2p}{\delta\sqrt{n}(1 - |G_o|/n - 2/n)},$$

with probability exceeding $1 - \tilde{V}_1 - V_2 - e^{-\frac{np}{4}}$.

Therefore, substituting back into (S2) gives for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned} 1 \leq & \frac{\sqrt{2}|G_o|^{1+\frac{1}{2}}}{cn^{\frac{1}{4}}\sqrt{\pi}} e^{\frac{-c^2\sqrt{n}}{2|G_o|}} + \frac{6(\sigma_{\max}^0)^2 p}{\delta\sqrt{n}(1-|G_o|/n-2/n)} \\ & + \frac{p^{\frac{7}{2}}}{n^{\frac{p}{4}}} \cdot \frac{(\sigma_{\max}^0)^2}{(\sigma_{\min}^0)^2} \cdot \frac{6\|\Gamma_n(0)\|_2/\delta}{(1-|G_o|/n-2/n)} + e^{-\left(\frac{d \cdot n^{\frac{p}{2}} p^2}{4\lambda_{\max}(\mathcal{X}\mathcal{X}'/n)} - \frac{np}{2}\right)} 2^{-\frac{|G_o|}{2}+1} \\ & + \frac{|G_o|\sqrt{\Lambda_{g_o}}}{\sqrt{\pi\varepsilon}} e^{-\frac{\varepsilon}{\Lambda_{g_o}}} + E[h(\boldsymbol{\alpha}_{G_o}, \{\sigma_j\})], \end{aligned}$$

with probability exceeding $1 - V_1 - \tilde{V}_1 - 2V_2 - 2e^{-\frac{np}{4}}$. Lastly, for any fixed $K_3 \in (0, 1)$, by Condition 3.4, choose a positive constant N_3 such that for all $n \geq \max\{N_1, N_2, N_3\}$,

$$1 \leq K_3 + E[h(\boldsymbol{\alpha}_{G_o}, \{\sigma_j\})], \quad (\text{S3})$$

with probability exceeding $1 - V_1 - \tilde{V}_1 - 2V_2 - 2e^{-\frac{np}{4}}$. Multiplying both sides of the inequality by $e^{\frac{|G_o|+p}{4}}$, and applying Lemma S2.4 yields the desired result. \blacksquare

Proof of Lemma 3.6. It follows from the generalized fiducial distributional of $\boldsymbol{\alpha}_G$ that

$$Z := (\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G)^{\frac{1}{2}} (\boldsymbol{\alpha}_G - \hat{\boldsymbol{\alpha}}_g) \sim N_{|G|}(0, I_{|G|}).$$

Thus,

$$\|\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G (\boldsymbol{\alpha}_G - \hat{\boldsymbol{\alpha}}_g)\|^2 = \|LZ\|^2 = \|UDV'Z\|^2$$

where $L := (\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G)^{\frac{1}{2}}$ is a $|G| \times |G|$ matrix which has the singular value decomposition $L = QDV'$ for Q and V each orthogonal matrices. Since V is an orthogonal matrix and Z follows the standard multivariate normal distribution, $\tilde{Z} := V'Z \sim N(0, I_{|G|})$. Then

$$\|LZ\|^2 = \|UD\tilde{Z}\|^2 = \sum_{j=1}^{|G|} \tilde{Z}_j^2 \lambda_j \leq \Lambda_g \max \tilde{Z}_j^2$$

where $\Lambda_g = \sum_{j=1}^{|G|} \lambda_j$, and λ_j is the j th eigenvalue of $L'L$. In other words,

$$\Lambda_g = \text{tr}(L'L) = \text{tr}(\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G).$$

Then using the assumption that $\varepsilon = \Lambda_g \cdot \tilde{\varepsilon}$ for some $\tilde{\varepsilon}$ not depending on Σ or A_g ,

$$\begin{aligned}
P\left(\|LZ\|^2 \geq \varepsilon\right) &\leq P\left(\Lambda_g \max_j \tilde{Z}_j^2 \geq \varepsilon\right) \\
&\leq \sum_{j=1}^{|G|} P\left(|Z_j| \geq \Lambda_g^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}}\right) \\
&= 2 \sum_{j=1}^{|G|} P\left(Z_j \leq -\Lambda_g^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}}\right) \leq \frac{|G| \sqrt{2\Lambda_g}}{\sqrt{\pi\varepsilon}} e^{-\frac{\varepsilon}{2\Lambda_g}},
\end{aligned}$$

where the last inequality follows because for the standard normal CDF, Φ for $x > 0$, $\Phi(-x) \leq \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. \blacksquare

Proof of Lemma 3.7.

$$\begin{aligned}
&P\left(\Lambda_g \geq n^{1+\frac{\rho}{2}} p^3, \min_{1 \leq j \leq p} \{m_j^g\} \geq d\right) \\
&= P\left(\text{tr}(\mathcal{Z}'_G \mathcal{W}^{-1} \mathcal{Z}_G) \geq n^{1+\frac{\rho}{2}} p^3\right) 1\left\{\min_{1 \leq j \leq p} \{m_j^g\} \geq d\right\} \\
&\leq P\left(\text{tr}(\mathcal{Z}' \mathcal{W}^{-1} \mathcal{Z}) \geq n^{1+\frac{\rho}{2}} p^3\right) 1\left\{\min_{1 \leq j \leq p} \{m_j^g\} \geq d\right\} \\
&= P\left(\text{tr}((\mathcal{X} \mathcal{X}') \otimes (\Sigma^{-1})) \geq n^{1+\frac{\rho}{2}} p^3\right) 1\left\{\min_{1 \leq j \leq p} \{m_j^g\} \geq d\right\} \\
&= P\left(\text{tr}(\mathcal{X} \mathcal{X}') \sum_{j=1}^p \sigma_j^{-2} \geq n^{1+\frac{\rho}{2}} p^3\right) 1\left\{\min_{1 \leq j \leq p} \{m_j^g\} \geq d\right\} \\
&\leq P\left(\lambda_{\max}(\mathcal{X} \mathcal{X}') \sum_{j=1}^p \sigma_j^{-2} \geq n^{1+\frac{\rho}{2}} p^2\right) 1\left\{\min_{1 \leq j \leq p} \{m_j^g\} \geq d\right\}.
\end{aligned}$$

Then for any $t \in (0, \min_{1 \leq j \leq p} \{m_j^g\}/2)$, recalling from the generalized fiducial density that $\sigma_j^2 \sim \text{inv-gamma}((n - |r_j^g|)/2, m_j^g/2)$, the Chernoff bound gives,

$$\begin{aligned}
&P\left(\Lambda_g \geq n^{1+\frac{\rho}{2}} p^3, \min_{1 \leq j \leq p} \{m_j^g\} \geq d\right) \\
&\leq P\left(e^{t \sum_{j=1}^p \sigma_j^{-2}} \geq e^{\frac{t \cdot n^{1+\frac{\rho}{2}} p^2}{\lambda_{\max}(\mathcal{X} \mathcal{X}')}}\right) 1\left\{\min_{1 \leq j \leq p} \{m_j^g\} \geq d\right\} \\
&\leq e^{-\frac{t \cdot n^{1+\frac{\rho}{2}} p^2}{\lambda_{\max}(\mathcal{X} \mathcal{X}')}} E\left(e^{t \sum_{j=1}^p \sigma_j^{-2}}\right) 1\left\{\min_{1 \leq j \leq p} \{m_j^g\} \geq d\right\} \\
&= e^{-\frac{t \cdot n^{1+\frac{\rho}{2}} p^2}{\lambda_{\max}(\mathcal{X} \mathcal{X}')}} \prod_{j=1}^p \left(1 - \frac{2t}{m_j^g}\right)^{-\frac{n - |r_j^g|}{2}} 1\left\{\min_{1 \leq j \leq p} \{m_j^g\} \geq d\right\} \\
&\leq e^{-\frac{t \cdot n^{1+\frac{\rho}{2}} p^2}{\lambda_{\max}(\mathcal{X} \mathcal{X}')}} \left(1 - \frac{2t}{d}\right)^{-\frac{np - |G|}{2}}.
\end{aligned}$$

Next, taking $t = d/4$ yields the desired result,

$$\begin{aligned} P\left(\Lambda_g \geq n^{1+\frac{\rho}{2}}p^3, \min_{1 \leq j \leq p} \{m_j^g\} \geq d\right) &\leq e^{-\frac{d \cdot n^{1+\frac{\rho}{2}}p^2}{4\lambda_{\max}(\mathcal{X}\mathcal{X}')}} 2^{\frac{np-|G|}{2}} \\ &\leq e^{-\left(\frac{d \cdot n^{\frac{\rho}{2}}p^2}{4\lambda_{\max}(\mathcal{X}\mathcal{X}'/n)} - \frac{np}{2}\right)} 2^{-\frac{|G|}{2}}. \end{aligned}$$

■

Proof of Lemma 3.8. First consider the expressions,

$$\begin{aligned} E_x(\mathcal{Z}'Y) &= E_x(\mathcal{Z}'\mathcal{Z})\boldsymbol{\alpha}^0 + E_x(\mathcal{Z}'\text{vec}((\Sigma^0)^{\frac{1}{2}}\mathcal{U})) \\ &= (E_x(\mathcal{X}\mathcal{X}') \otimes I_p)\boldsymbol{\alpha}^0 + E_x(\text{vec}((\Sigma^0)^{\frac{1}{2}}\mathcal{U}\mathcal{X}')) \\ &= (n\Gamma_n(0) \otimes I_p)\boldsymbol{\alpha}^0 + \text{vec}\left((\Sigma^0)^{\frac{1}{2}} \sum_{t=1}^n E_x(U^{(t)}X^{(t-1)'})\right) \\ &= (n\Gamma_n(0) \otimes I_p)\boldsymbol{\alpha}^0, \end{aligned} \tag{S4}$$

and

$$\begin{aligned} E_x(Y'\mathcal{Z}\mathcal{Z}'Y) &= E_x\left((\boldsymbol{\alpha}^{0'}\mathcal{Z}'\mathcal{Z} + \text{vec}((\Sigma^0)^{\frac{1}{2}}\mathcal{U})'\mathcal{Z})(\mathcal{Z}'\mathcal{Z}\boldsymbol{\alpha}^0 + \mathcal{Z}'\text{vec}((\Sigma^0)^{\frac{1}{2}}\mathcal{U}))\right) \\ &= \boldsymbol{\alpha}^{0'} E_x(\mathcal{Z}'\mathcal{Z}\mathcal{Z}'\mathcal{Z})\boldsymbol{\alpha}^0 \\ &\quad + 2E_x(\text{vec}((\Sigma^0)^{\frac{1}{2}}\mathcal{U})'\mathcal{Z}\mathcal{Z}'\mathcal{Z})\boldsymbol{\alpha}^0 + E_x(\text{vec}((\Sigma^0)^{\frac{1}{2}}\mathcal{U})'\mathcal{Z}\mathcal{Z}'\text{vec}((\Sigma^0)^{\frac{1}{2}}\mathcal{U})) \\ &\leq \lambda_{\max}\left(E_x((\mathcal{X}\mathcal{X}')^2) \otimes I_p\right)\|\boldsymbol{\alpha}^0\|^2 \\ &\quad + 2E_x(\text{vec}((\Sigma^0)^{\frac{1}{2}}\mathcal{U}\mathcal{X}'\mathcal{X}\mathcal{X}'))'\boldsymbol{\alpha}^0 + E_x(\text{tr}(\mathcal{X}\mathcal{U}'\Sigma^0\mathcal{U}\mathcal{X}')) \\ &\leq \lambda_{\max}\left(E_x((\mathcal{X}\mathcal{X}')^2)\right)\|\boldsymbol{\alpha}^0\|^2 \\ &\quad + 2E_x(\text{tr}(\mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}'(\Sigma^0)^{\frac{1}{2}}A^0)) + (\sigma_{\max}^0)^2 E_x(\text{tr}(\mathcal{U}\mathcal{X}'\mathcal{X}\mathcal{U}')) \end{aligned} \tag{S5}$$

These will be needed shortly.

Notice that $E_x(\mathcal{Z}'_G\mathcal{Z}_G) = (n\Gamma_n(0) \otimes I_p)_{G,G}$. Then for $K_g := (n\Gamma_n(0) \otimes I_p)_{G,G}(\mathcal{Z}'_G\mathcal{Z}_G)^{-1}$,

$$\begin{aligned} P_x\left(\|\hat{\boldsymbol{\alpha}}_g - (E_x(\mathcal{Z}'_G\mathcal{Z}_G))^{-1}E_x(\mathcal{Z}'_GY)\|^2 \geq \frac{2\varepsilon}{9n^{1+\frac{\rho}{2}}p^3\Lambda_g}\right) \\ \leq P_x\left(\frac{\|K_g\mathcal{Z}'_GY - E_x(\mathcal{Z}'_GY)\|^2}{\lambda_{\min}^2((\Gamma_n(0) \otimes I_p)_{G,G})} \geq \frac{2n^{1-\frac{\rho}{2}}\varepsilon}{9p^3\Lambda_g}\right) \\ \leq P_x\left(\frac{\|K_g\mathcal{Z}'_GY - E_x(\mathcal{Z}'_GY)\|^2}{\delta^2} \geq \frac{2n^{1-\frac{\rho}{2}}\varepsilon}{9p^3\Lambda_g}\right), \end{aligned}$$

since by Condition 3.1,

$$\lambda_{\min}^2((\Gamma_n(0) \otimes I_p)_{G,G}) \geq \lambda_{\min}^2(\Gamma_n(0) \otimes I_p) = \lambda_{\min}^2(\Gamma_n(0)) \cdot 1 \geq \delta^2,$$

where the first inequality follows from the Poincaré separation theorem (see Rao (1979)) because $(\Gamma_n(0) \otimes I_p)_{G,G}$ is a principal submatrix of $\Gamma_n(0) \otimes I_p$.

Accordingly, for now set $\xi = \frac{2\delta^2}{9\Lambda_g}\varepsilon$. Then by the triangle and Markov inequalities,

$$\begin{aligned}
P_x\left(\|K_g \mathcal{Z}'_G Y - E_x(\mathcal{Z}'_G Y)\|^2 \geq \frac{n^{1-\frac{\rho}{2}}\xi}{p^3}\right) &\leq P_x\left(\|(K_g - I_{|G|})\mathcal{Z}'_G Y\|^2 \geq \frac{n^{1-\frac{\rho}{2}}\xi}{4p^3}\right) \\
&\quad + P_x\left(\|\mathcal{Z}'_G Y - E_x(\mathcal{Z}'_G Y)\|^2 \geq \frac{n^{1-\frac{\rho}{2}}\xi}{4p^3}\right) \\
&\leq P_x\left(\|K_g - I_{|G|}\|_2^2 \cdot \|\mathcal{Z}'_G Y\|^2 \geq \frac{n^{1-\frac{\rho}{2}}\xi}{4p^3}\right) \\
&\quad + \frac{4p^3}{n^{1-\frac{\rho}{2}}\xi} E_x(\|\mathcal{Z}'_G Y - E_x(\mathcal{Z}'_G Y)\|^2).
\end{aligned} \tag{S6}$$

Consider each right side quantity in turn. The first can be bounded as

$$\begin{aligned}
P_x\left(\|K_g - I_{|G|}\|_2^2 \cdot \|\mathcal{Z}'_G Y\|^2 \geq \frac{n^{1-\frac{\rho}{2}}\xi}{4p^3}\right) &= P_x\left(\left\|\left[(n\Gamma_n(0) \otimes I_p)_{G,G} - \mathcal{Z}'_G \mathcal{Z}_G\right](\mathcal{Z}'_G \mathcal{Z}_G)^{-1}\right\|_2^2 \cdot \|\mathcal{Z}'_G Y\|^2 \geq \frac{n^{1-\frac{\rho}{2}}\xi}{4p^3}\right) \\
&\leq P_x\left(\left\|(\Gamma_n(0) \otimes I_p)_{G,G} - \mathcal{Z}'_G \mathcal{Z}_G/n\right\|_2^2 \cdot \|(\mathcal{Z}'_G \mathcal{Z}_G/n)^{-1}\|_2^2 \geq n^{-2\rho}\right) \\
&\quad + P_x\left(\|\mathcal{Z}'_G Y\|^2 \geq \frac{n^{1+\frac{3\rho}{2}}\xi}{4p^3}\right) \\
&\leq P_x\left(\left\|(\Gamma_n(0) \otimes I_p)_{G,G} - \mathcal{Z}'_G \mathcal{Z}_G/n\right\|_2^2 \geq n^{-2\rho}\lambda_{\min}^2(\mathcal{Z}'_G \mathcal{Z}_G/n)\right) \\
&\quad + \frac{4p^3}{n^{1+\frac{3\rho}{2}}\xi} E_x(\|\mathcal{Z}'_G Y\|^2) \\
&\leq P_x\left(\left\|(\Gamma_n(0) \otimes I_p)_{G,G} - \mathcal{Z}'_G \mathcal{Z}_G/n\right\|_2^2 \geq n^{-2\rho}\delta^2/4\right) \\
&\quad + P_x\left(\lambda_{\min}^2(\mathcal{Z}'_G \mathcal{Z}_G/n) < \delta^2/4\right) + \frac{4p^3}{n^{1+\frac{3\rho}{2}}\xi} E_x(\|\mathcal{Z}'_G Y\|^2)
\end{aligned}$$

Then, again by the Poincaré separation theorem,

$$\begin{aligned}
P_x\left(\|K_g - I_{|G|}\|_2^2 \cdot \|\mathcal{Z}'_G Y\|^2 \geq \frac{n^{1-\frac{\rho}{2}}\xi}{4p^3}\right) &\leq P_x\left(\lambda_{\max}^2(\Gamma_n(0) \otimes I_p - \mathcal{Z}'\mathcal{Z}/n) \geq n^{-2\rho}\delta^2/4\right) \\
&\quad + P_x\left(\lambda_{\min}^2(\mathcal{Z}'\mathcal{Z}/n) < \delta^2/4\right) + \frac{4p^3}{n^{1+\frac{3\rho}{2}}\xi} E_x(\|\mathcal{Z}'_G Y\|^2) \\
&\leq P_x\left(\lambda_{\max}^2(\Gamma_n(0) - \mathcal{X}\mathcal{X}'/n) \cdot 1 \geq n^{-2\rho}\delta^2/4\right) \\
&\quad + P_x\left(\lambda_{\min}^2(\mathcal{X}\mathcal{X}'/n) \cdot 1 < \delta^2/4\right) + \frac{4p^3}{n^{1+\frac{3\rho}{2}}\xi} E_x(\|\mathcal{Z}'_G Y\|^2),
\end{aligned}$$

and by Lemma 3.19 (assuming Condition 3.1), for all $n \geq N_2$,

$$\begin{aligned}
& P_x \left(\|K_g - I_{|G|}\|_2^2 \cdot \|\mathcal{Z}'_G Y\|^2 \geq \frac{n^{1-\frac{\rho}{2}} \xi}{4p^3} \right) \\
& \leq P_x \left(\|\Gamma_n(0) - \mathcal{X}\mathcal{X}'/n\|_F^2 \geq n^{-2\rho} \delta^2 / 4 \right) \\
& \quad + 4\delta^{-2} \frac{(\sigma_{\max}^0)^4 (1+c^2)}{(1-c^2)^3} \cdot \frac{2 \min\{|G_o|, p\}^2}{n} + \frac{4p^3}{n^{1+\frac{3\rho}{2}} \xi} E_x(\|\mathcal{Z}'Y\|^2).
\end{aligned}$$

Next, applying the Markov inequality gives,

$$\begin{aligned}
P_x \left(\|\Gamma_n(0) - \mathcal{X}\mathcal{X}'/n\|_F^2 \geq n^{-2\rho} \delta^2 / 4 \right) & \leq 4\delta^{-2} n^{2\rho} E_x \left(\|\Gamma_n(0) - \mathcal{X}\mathcal{X}'/n\|_F^2 \right) \\
& = 4\delta^{-2} n^{2\rho} \text{tr} \left(\frac{1}{n^2} E_x((\mathcal{X}\mathcal{X}')^2) - \Gamma_n^2(0) \right) \\
& \leq 4\delta^{-2} n^{2\rho} \frac{(\sigma_{\max}^0)^4 (1+c^2)}{(1-c^2)^3} \cdot \frac{2 \min\{|G_o|, p\}^2}{n},
\end{aligned}$$

for all $n \geq N_2$ by Lemma 3.17. Hence, for all $n \geq N_2$, and by (S5),

$$\begin{aligned}
& P_x \left(\|K_g - I_{|G|}\|_2^2 \cdot \|\mathcal{Z}'_G Y\|^2 \geq \frac{n^{1-\frac{\rho}{2}} \xi}{4p^3} \right) \\
& \leq \left(\frac{4\delta^{-2}}{n} + \frac{4\delta^{-2} n^{2\rho}}{n} \right) \frac{(\sigma_{\max}^0)^4 (1+c^2)}{(1-c^2)^3} \cdot 2 \min\{|G_o|, p\}^2 + \frac{4p^3}{n^{1+\frac{3\rho}{2}} \xi} E_x(\|\mathcal{Z}'Y\|^2) \\
& \leq \frac{\delta^{-2} 16 (\sigma_{\max}^0)^4}{(1-c^2)^3} \cdot \frac{\min\{|G_o|, p\}^2}{n^{1-2\rho}} \\
& \quad + \frac{4p^3}{n^{1+\frac{3\rho}{2}} \xi} \left[\lambda_{\max} \left(E_x((\mathcal{X}\mathcal{X}')^2) \right) \|\alpha^0\|^2 + (\sigma_{\max}^0)^2 E_x(\text{tr}(\mathcal{U}\mathcal{X}'\mathcal{X}\mathcal{U}')) \right. \\
& \quad \left. + 2E_x(\text{tr}(\mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}'(\Sigma^0)^{\frac{1}{2}} A^0)) \right].
\end{aligned}$$

Thus, by Lemmas 3.16, 3.17, and a slight modification of the proof of Lemma 3.18, for all

$$n \geq \max\{N_1, N_2\},$$

$$\begin{aligned}
P_x \left(\|K_g - I_{|G|}\|_2^2 \cdot \|\mathcal{Z}'_G Y\|^2 \geq \frac{n^{1-\frac{\rho}{2}} \xi}{4p^3} \right) \\
\leq \frac{\delta^{-2} 16 (\sigma_{\max}^0)^4}{(1-c^2)^3} \cdot \frac{\min\{|G_o|, p\}^2}{n^{1-2\rho}} \\
+ \frac{4p^3 n^2 p^2}{n^{1+\frac{3\rho}{2}} \xi} \left[\left(\frac{\text{tr}(\Gamma_n^2(0))}{p^2} + \frac{4(\sigma_{\max}^0)^4 \min\{|G_o|, p\}^2}{(1-c^2)^3 n p^2} \right) \|\alpha^0\|^2 \right. \\
\left. + \frac{(\sigma_{\max}^0)^4 \min\{|G_o|, p\}}{(1-c^2) n p} + \frac{4(\sigma_{\max}^0)^4 \min\{|G_o|, p\}^2}{(1-c^2)^2 n p^2} \right] \\
\leq 16 \left[\frac{\delta^{-2} (\sigma_{\max}^0)^4 p^2}{n^{1-2\rho} (1-c^2)^3} + \frac{p^3 n^2 p^2}{n^{1+\frac{3\rho}{2}} \xi} \cdot \left(\frac{\|\Gamma_n(0)\|_2^2}{p} + \frac{(\sigma_{\max}^0)^4}{n} \cdot \frac{3+c^4}{(1-c^2)^3} \right) \right].
\end{aligned}$$

This bounds the first right side term in (S6).

The second right side term in (S6) is bounded as follows. From (S4) and (S5),

$$\begin{aligned}
& \frac{4p^3}{n^{1-\frac{\rho}{2}} \xi} E_x(\|\mathcal{Z}'_G Y - E_x(\mathcal{Z}'_G Y)\|^2) \\
& \leq \frac{4p^3}{n^{1-\frac{\rho}{2}} \xi} \left(E_x(Y' \mathcal{Z} \mathcal{Z}' Y) - E_x(Y' \mathcal{Z}) E_x(\mathcal{Z}' Y) \right) \\
& \leq \frac{4p^3}{n^{1-\frac{\rho}{2}} \xi} \left[\lambda_{\max} \left(E_x((\mathcal{X} \mathcal{X}')^2) \right) \|\alpha^0\|^2 - \lambda_{\max} \left(n^2 \Gamma_n^2(0) \right) \|\alpha^0\|^2 \right. \\
& \quad \left. + 2 E_x(\text{tr}(\mathcal{X} \mathcal{X}' \mathcal{X} \mathcal{U}' (\Sigma^0)^{\frac{1}{2}} A^0)) + (\sigma_{\max}^0)^2 E_x(\text{tr}(\mathcal{U} \mathcal{X}' \mathcal{X} \mathcal{U}')) \right],
\end{aligned}$$

and again by Lemmas 3.16, 3.17, and a slight modification of the proof of Lemma 3.18, for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned}
& \frac{4p^3}{n^{1-\frac{\rho}{2}} \xi} E_x(\|\mathcal{Z}'_G Y - E_x(\mathcal{Z}'_G Y)\|^2) \\
& \leq \frac{4p^3}{n^{1-\frac{\rho}{2}} \xi} \left(E_x(Y' \mathcal{Z} \mathcal{Z}' Y) - E_x(Y' \mathcal{Z}) E_x(\mathcal{Z}' Y) \right) \\
& \leq \frac{4p^3 n^2 p^2}{n^{1-\frac{\rho}{2}} \xi} \left[\frac{4(\sigma_{\max}^0)^4 \min\{|G_o|, p\}^2}{(1-c^2)^3 n p^2} \|\alpha^0\|^2 \right. \\
& \quad \left. + \frac{4(\sigma_{\max}^0)^4 \min\{|G_o|, p\}^2}{(1-c^2)^2 n p^2} + \frac{(\sigma_{\max}^0)^4 \min\{|G_o|, p\}}{(1-c^2) n p} \right] \\
& \leq \frac{p^3 n^2 p^2}{n^{1-\frac{\rho}{2}} \xi} \cdot \frac{16(\sigma_{\max}^0)^4 p}{n} \cdot \frac{3+c^4}{(1-c^2)^3}
\end{aligned}$$

Therefore, returning back to the original inequality (S6), for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned} P_x \left(\|K_g \mathcal{Z}'_G Y - E_x(\mathcal{Z}'_G Y)\|^2 \geq \frac{n^{1-\frac{\rho}{2}} \xi}{p^3} \right) \\ \leq 16(\sigma_{\max}^0)^4 \left[\frac{\delta^{-2} p^2}{(1-c^2)^3 n^{1-2\rho}} + \frac{p^6 n^{1-\frac{3\rho}{2}}}{\xi} \cdot \left(\frac{\|\Gamma_n(0)\|_2^2}{(\sigma_{\max}^0)^4 p} + \frac{(3+c^4)}{(1-c^2)^3 n} \right) \right. \\ \left. + \frac{(3+c^4)p^6 n^{\frac{\rho}{2}}}{(1-c^2)^3 \xi} \right] \end{aligned}$$

Recalling the expression for ξ yields the desired result. \blacksquare

Proof of Lemma 3.15. Recalling that $\delta > 0$, an application of Chebyshev's inequality gives,

$$\begin{aligned} P_x \left([\lambda_{\min}(\Omega - E_x(\Omega))]^2 > \delta^2 \right) &\leq P_x \left(\sum_{i=1}^p \lambda_i^2(\Omega - E_x(\Omega)) > \delta^2 \right) \\ &= P_x \left(\text{tr} \left((\Omega - E_x(\Omega))' (\Omega - E_x(\Omega)) \right) > \delta^2 \right) \\ &\leq \delta^{-2} E_x \left[\text{tr} \left((\Omega - E_x(\Omega))' (\Omega - E_x(\Omega)) \right) \right] \\ &= \delta^{-2} \text{tr} \left(E_x(\Omega^2) - E_x^2(\Omega) \right). \end{aligned}$$

Observe that

$$\begin{aligned} E_x(\Omega^2) - E_x^2(\Omega) \\ = \frac{1}{n^2} E_x \begin{pmatrix} (\mathcal{X}\mathcal{X}')^2 + \mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}' - n^2 \Gamma_n^2(0) & \mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}' + \mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{U}' \\ (\mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}')' + (\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{U}')' & (\mathcal{U}\mathcal{U}')^2 + \mathcal{U}\mathcal{X}'\mathcal{X}\mathcal{U}' - n^2 I_p \end{pmatrix}, \end{aligned}$$

and

$$\Gamma_n^2(0) := \frac{1}{n^2} E_x^2(\mathcal{X}\mathcal{X}') = \frac{1}{n^2} \sum_{t=1}^n \sum_{s=1}^n \sum_{k=0}^{t-2} \sum_{j=0}^{s-2} (A^0)^k \Sigma (A^0)^{k'} (A^0)^j \Sigma (A^0)^{j'}. \quad (\text{S7})$$

Then since

$$\begin{aligned} E_x(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{U}') &= \sum_{t=1}^n \sum_{s=1}^n E_x \left(X^{(t-1)} U^{(t)'} U^{(s)} U^{(s)'} \right) \\ &= 0 + \sum_{t \neq s} \sum_{k=0}^{t-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} U^{(t)'} U^{(s)} U^{(s)'} \right) \\ &= \sum_{t \neq s} \sum_{k=0}^{t-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(E_x \left(U^{(t-1-k)} U^{(t)'} \mid U^{(s)} \right) U^{(s)} U^{(s)'} \right) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned}
\frac{1}{n^2} E_x((\mathcal{U}\mathcal{U}')^2) &= \frac{1}{n^2} \sum_{t=1}^n \sum_{s=1}^n E_x \left(U^{(t)} U^{(t)'} U^{(s)} U^{(s)'} \right) \\
&= \frac{1}{n^2} \sum_{t=1}^n E_x \left(U^{(t)} U^{(t)'} U^{(t)} U^{(t)'} \right) + \frac{n^2 - n}{n^2} I_p \\
&= \frac{1}{n^2} \cdot n \cdot (2 + p) I_p + \frac{n - 1}{n} I_p \\
&= \frac{p + 1}{n} I_p + I_p,
\end{aligned}$$

it follows that

$$\begin{aligned}
&E_x(\Omega^2) - E_x^2(\Omega) \\
&= \begin{pmatrix} \frac{1}{n^2} E_x((\mathcal{X}\mathcal{X}')^2 + \mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}') - \Gamma_n^2(0) & \frac{1}{n^2} E_x(\mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}') \\ \frac{1}{n^2} E_x(\mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}')' & \frac{p+1}{n} I_p + \frac{1}{n^2} E_x(\mathcal{U}\mathcal{X}'\mathcal{X}\mathcal{U}') \end{pmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\text{tr} \left(E_x(\Omega^2) - E_x^2(\Omega) \right) &= \text{tr} \left(\frac{1}{n^2} E_x((\mathcal{X}\mathcal{X}')^2) - \Gamma_n^2(0) \right) \\
&\quad + \frac{2}{n^2} \text{tr} \left(E_x(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}') \right) + \frac{p(p+1)}{n}.
\end{aligned}$$

Applying Lemmas 3.16 and 3.17 gives the desired result. ■

Proof of Lemma 3.16. Begin by expressing

$$\begin{aligned}
&E_x(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}') \\
&= \sum_{t=1}^n \sum_{s=1}^n E_x \left(X^{(t-1)} U^{(t)'} U^{(s)} X^{(s-1)'} \right) \\
&= \sum_{t=1}^n \sum_{s=1}^n \sum_{k=0}^{t-2} \sum_{j=0}^{s-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} U^{(t)'} U^{(s)} U^{(s-1-j)'} \right) \Sigma^{\frac{1}{2}} (A^0)^{j'}.
\end{aligned}$$

Then

$$\begin{aligned}
& E_x(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}') \\
&= \sum_{t=1}^n \sum_{k=0}^{t-2} \sum_{j=0}^{t-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t)'} U^{(t)} \right) E_x \left(U^{(t-1-k)} U^{(t-1-j)'} \right) \Sigma^{\frac{1}{2}} (A^0)^{j'} \\
&\quad + \sum_{t>s} \sum_{k=0}^{t-2} \sum_{j=0}^{s-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} \right. \\
&\quad \quad \quad \times E_x \left(U^{(t)'} U^{(s)} U^{(s-1-j)'} | U^{(t-1-k)} \right) \left. \right) \Sigma^{\frac{1}{2}} (A^0)^{j'} \\
&\quad + \sum_{s>t} \sum_{k=0}^{t-2} \sum_{j=0}^{s-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(E_x \left(U^{(t-1-k)} U^{(t)'} U^{(s)} | U^{(s-1-j)} \right) \right. \\
&\quad \quad \quad \times U^{(s-1-j)'} \left. \right) \Sigma^{\frac{1}{2}} (A^0)^{j'},
\end{aligned}$$

which gives

$$E_x(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}') = p \sum_{t=1}^n \sum_{k=0}^{t-2} (A^0)^k \Sigma (A^0)^{k'},$$

and

$$\begin{aligned}
\frac{1}{n^2} \text{tr} \left(E_x(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}') \right) &= \frac{p}{n^2} \sum_{t=1}^n \sum_{k=0}^{t-2} \left\| (A^0)^k \Sigma^{\frac{1}{2}} \right\|_F^2 \\
&\leq \frac{p}{n^2} \sum_{t=1}^n \sum_{k=0}^{t-2} \text{rank}((A^0)^k \Sigma^{\frac{1}{2}}) \left\| (A^0)^k \Sigma^{\frac{1}{2}} \right\|_2^2 \\
&\leq \frac{p}{n^2} \sum_{t=1}^n \sum_{k=0}^{t-2} \text{rank}(A^0) \|A^0\|_2^{2k} \|\Sigma\|_2 \\
&\leq \frac{p}{n^2} \sum_{t=1}^n \sum_{k=0}^{t-2} \min\{|G_o|, p\} c^{2k} (\sigma_{\max}^0)^2.
\end{aligned}$$

Then using the formula for a geometric series,

$$\frac{1}{n^2} \text{tr} \left(E_x(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}') \right) \leq \frac{(\sigma_{\max}^0)^2}{(1-c^2)} \left(1 - \frac{1-c^{2n}}{n(1-c^2)} \right) \frac{\min\{|G_o|, p\} p}{n},$$

and so for all $n \geq N_1$,

$$\frac{1}{n^2} \text{tr} \left(E_x(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}') \right) \leq \frac{p(\sigma_{\max}^0)^2 \min\{|G_o|, p\}}{n(1-c^2)}.$$

■

Proof of Lemma 3.17.

$$\begin{aligned}\frac{1}{n^2}E_x((\mathcal{X}\mathcal{X}')^2) &= \frac{1}{n^2}\sum_{t=1}^n\sum_{s=1}^n E_x\left(X^{(t-1)}X^{(t-1)'}X^{(s-1)}X^{(s-1)'}\right) \\ &= \frac{1}{n^2}\sum_{t=1}^n\sum_{s=1}^n T(t, s),\end{aligned}$$

where for $t \neq s$,

$$\begin{aligned}T(t, s) &:= \sum_{k=0}^{t-2}\sum_{l=0}^{s-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x\left(U^{(t-1-k)}U^{(t-1-k)'}\Sigma^{\frac{1}{2}}(A^0)^{k'}\right. \\ &\quad \left.\times (A^0)^l \Sigma^{\frac{1}{2}} U^{(s-1-l)}U^{(s-1-l)'}\right) \Sigma^{\frac{1}{2}}(A^0)^{l'} \\ &\quad + \sum_{k=0}^{t-2}\sum_{l \neq m}^{s-2} 0 + \sum_{k \neq j}^{t-2}\sum_{l=0}^{s-2} 0 \\ &\quad + \sum_{k \neq j}^{t-2}\sum_{l \neq m}^{s-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x\left(U^{(t-1-k)}U^{(t-1-j)'}\Sigma^{\frac{1}{2}}(A^0)^{j'}\right. \\ &\quad \left.\times (A^0)^l \Sigma^{\frac{1}{2}} U^{(s-1-l)}U^{(s-1-m)'}\right) \Sigma^{\frac{1}{2}}(A^0)^{m'}.\end{aligned}$$

For $t > s$,

$$\begin{aligned}T(t, s) &= \sum_{l=0}^{s-2} (A^0)^{t-s+l} \Sigma (A^0)^{l'} (A^0)^{t-s+l} \Sigma (A^0)^{l'} \\ &\quad + \text{tr}((A^0)^l \Sigma (A^0)^{t-s+l'}) (A^0)^{t-s+l} \Sigma (A^0)^{l'} \\ &\quad + \sum_{l=0}^{s-2}\sum_{k=0}^{t-2} (A^0)^k \Sigma (A^0)^{k'} (A^0)^l \Sigma (A^0)^{l'} \\ &\quad + \sum_{l \neq m}^{s-2} (A^0)^{t-s+l} \Sigma (A^0)^{l'} (A^0)^{t-s+m} \Sigma (A^0)^{m'} \\ &\quad + \sum_{l \neq m}^{s-2} \text{tr}((A^0)^l \Sigma (A^0)^{t-s+l'}) (A^0)^{t-s+m} \Sigma (A^0)^{m'}\end{aligned}$$

which gives

$$\begin{aligned}
T(t, s) = & \sum_{l=0}^{s-2} \sum_{m=0}^{s-2} (A^0)^{t-s+l} \Sigma(A^0)^{l'} (A^0)^{t-s+m} \Sigma(A^0)^{m'} \\
& + \sum_{l=0}^{s-2} \sum_{m=0}^{s-2} \text{tr}((A^0)^l \Sigma(A^0)^{t-s+l'}) (A^0)^{t-s+m} \Sigma(A^0)^{m'} \\
& + \sum_{l=0}^{s-2} \sum_{k=0}^{t-2} (A^0)^k \Sigma(A^0)^{k'} (A^0)^l \Sigma(A^0)^{l'}.
\end{aligned}$$

For $s > t$,

$$\begin{aligned}
T(t, s) = & \sum_{k=0}^{t-2} (A^0)^k \Sigma(A^0)^{s-t+k'} (A^0)^k \Sigma(A^0)^{s-t+k'} \\
& + (A^0)^k \Sigma(A^0)^{s-t+k'} \text{tr}((A^0)^{s-t+k} \Sigma(A^0)^{k'}) \\
& + \sum_{k=0}^{t-2} \sum_{l=0}^{s-2} (A^0)^k \Sigma(A^0)^{k'} (A^0)^l \Sigma(A^0)^{l'} \\
& + \sum_{\substack{k=0 \\ k \neq j}}^{t-2} (A^0)^k \Sigma(A^0)^{s-t+k'} (A^0)^j \Sigma(A^0)^{s-t+j'} \\
& + \sum_{\substack{k=0 \\ k \neq j}}^{t-2} (A^0)^k \Sigma(A^0)^{s-t+k'} \text{tr}((A^0)^{s-t+j} \Sigma(A^0)^{j'})
\end{aligned}$$

which yields

$$\begin{aligned}
T(t, s) = & \sum_{k=0}^{t-2} \sum_{j=0}^{t-2} (A^0)^k \Sigma(A^0)^{s-t+k'} (A^0)^j \Sigma(A^0)^{s-t+j'} \\
& + \sum_{k=0}^{t-2} \sum_{j=0}^{t-2} (A^0)^k \Sigma(A^0)^{s-t+k'} \text{tr}((A^0)^{s-t+j} \Sigma(A^0)^{j'}) \\
& + \sum_{k=0}^{t-2} \sum_{l=0}^{s-2} (A^0)^k \Sigma(A^0)^{k'} (A^0)^l \Sigma(A^0)^{l'}.
\end{aligned}$$

And for $s = t$,

$$\begin{aligned}
T(t, t) &:= \sum_{k=0}^{t-2} \sum_{l=0}^{t-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} U^{(t-1-k)'} \Sigma^{\frac{1}{2}} (A^0)^{k'} (A^0)^l \Sigma^{\frac{1}{2}} \right. \\
&\quad \times E_x \left(U^{(t-1-l)} U^{(t-1-l)'} | U^{(t-1-k)} \right) \left. \right) \Sigma^{\frac{1}{2}} (A^0)^{l'} \\
&\quad + \sum_{k \neq j}^{t-2} \sum_{l=0}^{t-2} 0 \\
&\quad + \sum_{k \neq j}^{t-2} \sum_{l \neq m}^{t-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} U^{(t-1-j)'} \Sigma^{\frac{1}{2}} (A^0)^{j'} \right. \\
&\quad \times (A^0)^l \Sigma^{\frac{1}{2}} U^{(t-1-l)} U^{(t-1-m)'} \left. \right) \Sigma^{\frac{1}{2}} (A^0)^{m'}.
\end{aligned}$$

Then

$$\begin{aligned}
T(t, t) &= \sum_{k=0}^{t-2} (A^0)^k \Sigma^{\frac{1}{2}} \left(2 \Sigma^{\frac{1}{2}} (A^0)^{k'} (A^0)^k \Sigma^{\frac{1}{2}} \right. \\
&\quad \left. + I_p \text{tr}((A^0)^k \Sigma (A^0)^{k'}) \right) \Sigma^{\frac{1}{2}} (A^0)^{k'} \\
&\quad + \sum_{k \neq l}^{t-2} (A^0)^k \Sigma (A^0)^{k'} (A^0)^l \Sigma (A^0)^{l'} \\
&\quad + \sum_{k \neq j}^{t-2} \left((A^0)^k \Sigma (A^0)^{k'} (A^0)^j \Sigma (A^0)^{j'} \right. \\
&\quad \left. + (A^0)^k \Sigma (A^0)^{k'} \text{tr}((A^0)^j \Sigma (A^0)^{j'}) \right),
\end{aligned}$$

and so

$$\begin{aligned}
T(t, t) &= 2 \sum_{k=0}^{t-2} \sum_{j=0}^{t-2} (A^0)^k \Sigma (A^0)^{k'} (A^0)^j \Sigma (A^0)^{j'} \\
&\quad + \sum_{k=0}^{t-2} \sum_{j=0}^{t-2} (A^0)^k \Sigma (A^0)^{k'} \text{tr}((A^0)^j \Sigma (A^0)^{j'}).
\end{aligned}$$

Finally, putting all the pieces together gives,

$$\begin{aligned}
E_x((\mathcal{X}\mathcal{X}')^2) &= \sum_{t=1}^n T(t, t) + \sum_{t=1}^n \sum_{s=1}^{t-1} T(t, s) + \sum_{s=1}^n \sum_{t=1}^{s-1} T(t, s) \\
&= \tau_1 + \tau_2 + \sum_{t=1}^n \sum_{s=1}^n \sum_{k=0}^{t-2} \sum_{j=0}^{s-2} (A^0)^k \Sigma (A^0)^{k'} (A^0)^j \Sigma (A^0)^{j'},
\end{aligned}$$

where from expression (S7) the last term is equivalent to $n^2\Gamma_n^2(0)$, and

$$\begin{aligned}\tau_1 &:= \sum_{t=1}^n \sum_{s=1}^t \sum_{k=0}^{s-2} \sum_{j=0}^{s-2} \left[(A^0)^{t-s+k} \Sigma (A^0)^{k'} (A^0)^{t-s+j} \Sigma (A^0)^{j'} \right. \\ &\quad \left. + (A^0)^{t-s+j} \Sigma (A^0)^{j'} \text{tr}((A^0)^k \Sigma (A^0)^{t-s+k'}) \right] \\ \tau_2 &:= \sum_{s=1}^n \sum_{t=1}^{s-1} \sum_{k=0}^{t-2} \sum_{j=0}^{t-2} \left[(A^0)^k \Sigma (A^0)^{s-t+k'} (A^0)^j \Sigma (A^0)^{s-t+j'} \right. \\ &\quad \left. + (A^0)^k \Sigma (A^0)^{s-t+k'} \text{tr}((A^0)^{s-t+j} \Sigma (A^0)^{j'}) \right].\end{aligned}$$

To proceed, compute the trace of τ_1 and τ_2 .

$$\begin{aligned}\text{tr}(\tau_1) &= \sum_{t=1}^n \sum_{s=1}^t \sum_{k=0}^{s-2} \sum_{j=0}^{s-2} \left[\text{tr}((A^0)^{t-s+k} \Sigma (A^0)^{k'} (A^0)^{t-s+j} \Sigma (A^0)^{j'}) \right. \\ &\quad \left. + \text{tr}((A^0)^{t-s+j} \Sigma (A^0)^{j'}) \text{tr}((A^0)^k \Sigma (A^0)^{t-s+k'}) \right] \\ &\leq \sum_{t=1}^n \sum_{s=1}^t \sum_{k=0}^{s-2} \sum_{j=0}^{s-2} \left[\|(A^0)^k \Sigma (A^0)^{t-s+k'}\|_F \|(A^0)^{t-s+j} \Sigma (A^0)^{j'}\|_F \right. \\ &\quad \left. + \|\Sigma^{\frac{1}{2}}(A^0)^{t-s+j'}\|_F \|\Sigma^{\frac{1}{2}}(A^0)^{j'}\|_F \right. \\ &\quad \left. \times \|\Sigma^{\frac{1}{2}}(A^0)^{k'}\|_F \|\Sigma^{\frac{1}{2}}(A^0)^{t-s+k'}\|_F \right],\end{aligned}$$

and so

$$\begin{aligned}\text{tr}(\tau_1) &\leq \sum_{t=1}^n \sum_{s=1}^t \sum_{k=0}^{s-2} \sum_{j=0}^{s-2} \|\Sigma\|_2^2 \cdot \|A^0\|_2^{2(t-s+j+k)} \\ &\quad \times (\min\{|G_o|, p\} + \min\{|G_o|, p\}^2) \\ &\leq (\sigma_{\max}^0)^4 \sum_{t=1}^n \sum_{s=1}^t (c^2)^{t-s} \sum_{k=0}^{s-2} (c^2)^k \sum_{j=0}^{s-2} (c^2)^j \\ &\quad \times (\min\{|G_o|, p\} + \min\{|G_o|, p\}^2).\end{aligned}$$

The formula for a geometric series further reduces this expression as

$$\begin{aligned}
\text{tr}(\tau_1) &\leq (\sigma_{\max}^0)^4 \sum_{t=1}^n \sum_{s=1}^t (c^2)^{t-s} \left(\frac{1 - (c^2)^{s-1}}{1 - c^2} \right)^2 \\
&\quad \times (\min\{|G_o|, p\} + \min\{|G_o|, p\}^2) \\
&\leq \frac{(\min\{|G_o|, p\} + \min\{|G_o|, p\}^2)(\sigma_{\max}^0)^4}{(1 - c^2)^2} \sum_{t=1}^n (c^2)^t \sum_{s=1}^t \left(\frac{1}{c^2} \right)^s \\
&\leq \frac{(\min\{|G_o|, p\} + \min\{|G_o|, p\}^2)(\sigma_{\max}^0)^4}{(1 - c^2)^3} \left(n - \frac{c^2 - (c^2)^{n+1}}{1 - c^2} \right).
\end{aligned}$$

Similarly,

$$\text{tr}(\tau_2) \leq \frac{(\min\{|G_o|, p\} + \min\{|G_o|, p\}^2)(\sigma_{\max}^0)^4}{(1 - c^2)^3} \left(nc^2 - \frac{c^2 - (c^2)^{n+1}}{1 - c^2} \right),$$

and so

$$\begin{aligned}
\text{tr} \left(\frac{1}{n^2} E_x((\mathcal{X}\mathcal{X}')^2) - \Gamma_n^2(0) \right) \\
\leq \frac{(\sigma_{\max}^0)^4}{(1 - c^2)^3} \left(\frac{\min\{|G_o|, p\} + \min\{|G_o|, p\}^2}{n} \right) \\
\times \left[\left(1 - \frac{c^2 - (c^2)^{n+1}}{n(1 - c^2)} \right) + \left(c^2 - \frac{c^2 - (c^2)^{n+1}}{n(1 - c^2)} \right) \right].
\end{aligned}$$

Then for all $n \geq N_2$,

$$\begin{aligned}
\text{tr} \left(\frac{1}{n^2} E_x((\mathcal{X}\mathcal{X}')^2) - \Gamma_n^2(0) \right) \\
\leq \frac{(\sigma_{\max}^0)^4(1 + c^2)}{(1 - c^2)^3} \left(\frac{\min\{|G_o|, p\} + \min\{|G_o|, p\}^2}{n} \right).
\end{aligned}$$

■

Proof of Lemma 3.18. First,

$$\begin{aligned}
&E_x(\mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}') \\
&= \sum_{t=1}^n \sum_{s=1}^n E_x \left(X^{(t-1)} X^{(t-1)'} X^{(s-1)} U^{(s)'} \right) \\
&= 0 + \sum_{t \neq s} E_x \left(X^{(t-1)} X^{(t-1)'} X^{(s-1)} U^{(s)'} \right) \\
&= 0 + \sum_{t > s} \sum_{k=0}^{t-2} \sum_{j=0}^{t-2} \sum_{l=0}^{s-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} U^{(t-1-j)'} \Sigma^{\frac{1}{2}} (A^0)^{j'} \right. \\
&\quad \left. \times (A^0)^l \Sigma^{\frac{1}{2}} U^{(s-1-l)} U^{(s)'} \right).
\end{aligned}$$

Next,

$$\begin{aligned}
& E_x(\mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}') \\
&= \sum_{t>s} \sum_{k=0}^{t-2} \sum_{l=0}^{s-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} U^{(t-1-k)'} \Sigma^{\frac{1}{2}} (A^0)^{k'} (A^0)^l \Sigma^{\frac{1}{2}} \right. \\
&\quad \left. \times E_x \left(U^{(s-1-l)} U^{(s)'} \mid U^{(t-1-k)} \right) \right) \\
&\quad + \sum_{t>s} \sum_{k \neq j} \sum_{l=0}^{s-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} \right. \\
&\quad \left. \times E_x \left(U^{(t-1-j)'} \Sigma^{\frac{1}{2}} (A^0)^{j'} (A^0)^l \Sigma^{\frac{1}{2}} U^{(s-1-l)} U^{(s)'} \mid U^{(t-1-k)} \right) \right),
\end{aligned}$$

and since the first sum is zero, breaking up the second sum gives

$$\begin{aligned}
& E_x(\mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}') \\
&= 0 + \sum_{t=1}^n \sum_{k \neq j} \sum_{l=0}^{t-1-k-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} \right. \\
&\quad \left. \times E_x \left(U^{(t-1-j)'} \Sigma^{\frac{1}{2}} (A^0)^{j'} (A^0)^l \Sigma^{\frac{1}{2}} U^{(t-1-k-l-1)} \mid U^{(t-1-k)} \right) U^{(t-1-k)'} \right) \\
&\quad + \sum_{t=1}^n \sum_{k \neq j} \sum_{l=0}^{t-1-j-2} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} \right. \\
&\quad \left. \times E_x \left(U^{(t-1-j)'} \Sigma^{\frac{1}{2}} (A^0)^{j'} (A^0)^l \Sigma^{\frac{1}{2}} U^{(t-1-j-l-1)} U^{(t-1-j)'} \mid U^{(t-1-k)} \right) \right).
\end{aligned}$$

Further,

$$\begin{aligned}
& E_x(\mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}') \\
&= \sum_{t=1}^n \sum_{j>k} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} \right. \\
&\quad \left. \times E_x \left(U^{(t-1-j)'} \Sigma^{\frac{1}{2}} (A^0)^{j'} (A^0)^{j-k-1} \Sigma^{\frac{1}{2}} U^{(t-1-j)} \right) U^{(t-1-k)'} \right) \\
&\quad + \sum_{t=1}^n \sum_{k>j} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} \right. \\
&\quad \left. \times E_x \left(U^{(t-1-j)'} \Sigma^{\frac{1}{2}} (A^0)^{j'} (A^0)^{k-j-1} \Sigma^{\frac{1}{2}} U^{(t-1-k)} U^{(t-1-j)'} \mid U^{(t-1-k)} \right) \right),
\end{aligned}$$

and so

$$\begin{aligned}
& E_x(\mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}') \\
&= \sum_{t=1}^n \sum_{j>k} (A^0)^k \Sigma^{\frac{1}{2}} I_p \cdot E_x \left(U^{(t-1-j)'} \Sigma^{\frac{1}{2}} (A^0)^{j'} (A^0)^{j-k-1} \Sigma^{\frac{1}{2}} U^{(t-1-j)} \right) \\
&\quad + \sum_{t=1}^n \sum_{k>j} (A^0)^k \Sigma^{\frac{1}{2}} E_x \left(U^{(t-1-k)} U^{(t-1-k)'} \Sigma^{\frac{1}{2}} (A^0)^{k-j-1'} (A^0)^j \Sigma^{\frac{1}{2}} \right) \\
&= \sum_{t=1}^n \sum_{j>k} (A^0)^k \Sigma^{\frac{1}{2}} \text{tr} \left(\Sigma^{\frac{1}{2}} (A^0)^{j'} (A^0)^{j-k-1} \Sigma^{\frac{1}{2}} \right) \\
&\quad + \sum_{t=1}^n \sum_{k>j} (A^0)^k \Sigma (A^0)^{k-j-1'} (A^0)^j \Sigma^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \text{tr} \left(E_x(\mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}'A^0) \right) \\
&= \sum_{t=1}^n \sum_{k=0}^{t-2} \sum_{j=k+1}^{t-2} \text{tr} \left((A^0)^k \Sigma^{\frac{1}{2}} A^0 \right) \text{tr} \left(\Sigma (A^0)^{j'} (A^0)^{j-k-1} \right) \\
&\quad + \sum_{t=1}^n \sum_{j=0}^{t-2} \sum_{k=j+1}^{t-2} \text{tr} \left((A^0)^k \Sigma (A^0)^{k-j-1'} (A^0)^j \Sigma^{\frac{1}{2}} A^0 \right) \\
&\leq \min\{|G_o|, p\}^2 (\sigma_{\max}^0)^3 \sum_{t=1}^n \sum_{k=0}^{t-2} \sum_{j=k+1}^{t-2} \|A^0\|_2^{2j} \\
&\quad + \min\{|G_o|, p\} (\sigma_{\max}^0)^3 \sum_{t=1}^n \sum_{j=0}^{t-2} \sum_{k=j+1}^{t-2} \|A^0\|_2^{2k} \\
&\leq \left(\min\{|G_o|, p\}^2 + \min\{|G_o|, p\} \right) (\sigma_{\max}^0)^3 \sum_{t=1}^n \sum_{k=0}^{t-2} \sum_{j=k+1}^{t-2} c^{2j},
\end{aligned}$$

and applying the formula for a geometric series yields

$$\begin{aligned}
& \text{tr} \left(E_x(\mathcal{X}\mathcal{X}'\mathcal{X}\mathcal{U}'A^0) \right) \\
&\leq \left(\min\{|G_o|, p\}^2 + \min\{|G_o|, p\} \right) (\sigma_{\max}^0)^3 \sum_{t=1}^n \sum_{k=0}^{t-2} \frac{c^{2(k+1)} - c^{2(t-1)}}{1 - c^2} \\
&\leq \left(\min\{|G_o|, p\}^2 + \min\{|G_o|, p\} \right) \frac{(\sigma_{\max}^0)^3}{1 - c^2} \sum_{t=1}^n c^2 \frac{1 - c^{2(t-1)}}{1 - c^2} \\
&\leq \left(\min\{|G_o|, p\}^2 + \min\{|G_o|, p\} \right) \frac{n(\sigma_{\max}^0)^3 c^2}{(1 - c^2)^2}.
\end{aligned}$$

■

Proof of Lemma 3.19. Define $\widehat{\Sigma} := \frac{1}{n}\mathcal{X}\mathcal{X}'$. By Condition 3.1, $\lambda_{\min}(\Gamma_n(0)) > \delta$ which implies that

$$\begin{aligned}
P_x\left(\lambda_{\min}(\widehat{\Sigma}) \geq \delta/2\right) &\geq P_x\left(\lambda_{\min}(\Gamma_n(0)) + \lambda_{\min}(\widehat{\Sigma} - \Gamma_n(0)) \geq \delta - \delta/2\right) \\
&\geq P_x\left(\lambda_{\min}(\Gamma_n(0)) + \lambda_{\min}(\widehat{\Sigma} - \Gamma_n(0)) \geq \lambda_{\min}(\Gamma_n(0)) - \delta/2\right) \\
&= P_x\left(\lambda_{\min}(\widehat{\Sigma} - \Gamma_n(0)) \geq -\delta/2\right) \\
&\geq P_x\left(|\lambda_{\min}(\widehat{\Sigma} - \Gamma_n(0))| \leq \delta/2\right).
\end{aligned}$$

Further,

$$\begin{aligned}
P_x\left(|\lambda_{\min}(\widehat{\Sigma} - \Gamma_n(0))| > \delta/2\right) &\leq P_x\left(\sum_{i=1}^p \lambda_i^2(\widehat{\Sigma} - \Gamma_n(0)) > \delta^2/4\right) \\
&= P_x\left(\|\widehat{\Sigma} - \Gamma_n(0)\|_F^2 > \delta^2/4\right) \\
&\leq 4\delta^{-2}E_x\left(\|\widehat{\Sigma} - \Gamma_n(0)\|_F^2\right) \\
&= 4\delta^{-2}\text{tr}\left(E_x(\widehat{\Sigma}^2) - \Gamma_n^2(0)\right) \\
&\leq 4\delta^{-2}\frac{(\sigma_{\max}^0)^4(1+c^2)}{(1-c^2)^3} \cdot \frac{2\min\{|G_o|, p\}^2}{n}
\end{aligned} \tag{S8}$$

where the last inequality holds for all $n \geq N_2$ by Lemma 3.17. Therefore, for all $n \geq N_2$,

$$P_x\left(\lambda_{\min}(\widehat{\Sigma}) \geq \delta/2\right) \geq 1 - 4\delta^{-2}\frac{(\sigma_{\max}^0)^4(1+c^2)}{(1-c^2)^3} \cdot \frac{2\min\{|G_o|, p\}^2}{n}.$$

■

Proof of Lemma S2.1. By the Poincaré separation theorem (see Rao (1979)),

$$\begin{aligned}
P\left(\frac{\|(\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o}\|_2^2}{\lambda_{\min}^2(\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o})} \geq \frac{1}{n^{2-\frac{\rho}{2}} p^5}\right) & \leq P\left(\frac{\|\Gamma_n(0) \otimes (\Sigma^0)^{-1}\|_2^2}{\lambda_{\min}^2(\mathcal{Z}' \mathcal{W}^{-1} \mathcal{Z})} \geq \frac{n^{\frac{\rho}{2}}}{n^2 p^5}\right) \\
& = P\left(\frac{\lambda_{\min}(\mathcal{Z}' \mathcal{W}^{-1} \mathcal{Z})}{\|\Gamma_n(0)\|_2 (\sigma_{\min}^0)^{-2}} \leq \frac{np^{\frac{5}{2}}}{n^4}\right) \\
& \leq P\left(\frac{\min\{\sigma_j^{-2}\} \lambda_{\min}(\mathcal{Z}' \mathcal{Z})}{\|\Gamma_n(0)\|_2 (\sigma_{\min}^0)^{-2}} \leq \frac{np^{\frac{5}{2}}}{n^4}\right) \\
& \leq \sum_{j=1}^p P\left(\sigma_j^{-2} \leq \frac{np^{\frac{5}{2}} \|\Gamma_n(0)\|_2 (\sigma_{\min}^0)^{-2}}{\lambda_{\min}(\mathcal{Z}' \mathcal{Z})}\right) \\
& = \sum_{j=1}^p P\left(\sigma_j^2 \geq \frac{n^{\frac{\rho}{4}}}{np^{\frac{5}{2}}} \frac{\lambda_{\min}(\mathcal{Z}' \mathcal{Z})}{\|\Gamma_n(0)\|_2 (\sigma_{\min}^0)^{-2}}\right) \\
& \leq \frac{np^{\frac{5}{2}} \|\Gamma_n(0)\|_2 (\sigma_{\min}^0)^{-2}}{n^{\frac{\rho}{4}} \lambda_{\min}(\mathcal{Z}' \mathcal{Z})} \sum_{j=1}^p \frac{m_j^{g_o}}{n - |r_j^{g_o}| - 2}
\end{aligned}$$

by the Markov inequality, recalling that $\sigma_j^2 \sim \text{inv-gamma}((n - |r_j^{g_o}|)/2, m_j^{g_o}/2)$.

Next observe that

$$\sum_{j=1}^p \frac{m_j^{g_o}}{n - |r_j^{g_o}| - 2} \leq \frac{1}{n - |G_o| - 2} \sum_{j=1}^p m_j^{g_o},$$

and as in (S20),

$$P\left(\sum_{j=1}^p m_j^{g_o} > (\sigma_{\max}^0)^2 3np\right) \leq P\left(\sum_{t=1}^n \|U^{(t)}\|^2 > 3np\right) \leq P(\chi_{np}^2 > 3np) \leq e^{-\frac{np}{4}}.$$

Additionally, since $\lambda_{\min}(\mathcal{Z}' \mathcal{Z}) = \lambda_{\min}(\mathcal{X} \mathcal{X}') \cdot 1$, it follows from Lemma 3.19 (assuming that Condition 3.1 holds) that for all $n \geq N_2$,

$$P_x(\lambda_{\min}(\mathcal{Z}' \mathcal{Z}) \geq n\delta/2) \geq 1 - 4\delta^{-2} \frac{(\sigma_{\max}^0)^4 (1 + c^2)}{(1 - c^2)^3} \cdot \frac{2 \min\{|G_o|, p\}^2}{n}.$$

Therefore, for all $n \geq N_2$,

$$\begin{aligned}
P\left(\frac{\|(\Gamma_n(0) \otimes (\Sigma^0)^{-1})_{G_o, G_o}\|_2^2}{\lambda_{\min}^2(\mathcal{Z}'_{G_o} \mathcal{W}^{-1} \mathcal{Z}_{G_o})} \geq \frac{1}{n^{2-\frac{\rho}{2}} p^5}\right) & \leq \frac{np^{\frac{5}{2}}}{n^{\frac{\rho}{4}}} \frac{2 \|\Gamma_n(0)\|_2 (\sigma_{\min}^0)^{-2}}{n\delta} \frac{(\sigma_{\max}^0)^2 3np}{(n - |G_o| - 2)} \\
& \leq \frac{p^{\frac{7}{2}}}{n^{\frac{\rho}{4}}} \cdot \frac{(\sigma_{\max}^0)^2}{(\sigma_{\min}^0)^2} \cdot \frac{6 \|\Gamma_n(0)\|_2 / \delta}{(1 - |G_o|/n - 2/n)},
\end{aligned}$$

with probability exceeding

$$1 - 4\delta^{-2} \frac{(\sigma_{\max}^0)^4(1+c^2)}{(1-c^2)^3} \cdot \frac{2 \min\{|G_o|, p\}^2}{n} - e^{-\frac{np}{4}}.$$

■

Proof of Lemma S2.2. Consider the following upper bound on the natural logarithm of the determinant of the positive definite matrix, $\tilde{\mathcal{D}}_g' \tilde{\mathcal{D}}_g$,

$$\log(|\tilde{\mathcal{D}}_g' \tilde{\mathcal{D}}_g|) \leq \text{tr}(\tilde{\mathcal{D}}_g' \tilde{\mathcal{D}}_g - I_{|G|+p}) = \text{tr}(\tilde{\mathcal{D}}_g' \tilde{\mathcal{D}}_g) - |G| - p. \quad (\text{S9})$$

The trace can be bounded as follows.

$$\begin{aligned} \text{tr}(\tilde{\mathcal{D}}_g' \tilde{\mathcal{D}}_g) &= \frac{1}{\sqrt{n}} \sum_{j=1}^p \sum_{i \in r_j^g} \left(\frac{\partial Y}{\partial A_{ij}} \right)' \frac{\partial Y}{\partial A_{ij}} + \frac{1}{\sqrt{n}} \sum_{i=1}^p \left(\frac{\partial Y}{\partial \sigma_i} \right)' \frac{\partial Y}{\partial \sigma_i} \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^p \sum_{i \in r_j^g} \text{vec}(\mathcal{X})' (I_n \otimes J^{ji}) \Theta_g' \Theta_g (I_n \otimes J^{ij}) \text{vec}(\mathcal{X}) \\ &\quad + \frac{1}{\sqrt{n}} \sum_{i=1}^p (Y - \mathcal{Z}_G \alpha_G)' (I_n \otimes J^{ii}) \Theta_g' \Theta_g (I_n \otimes J^{ii}) (Y - \mathcal{Z}_G \alpha_G) \end{aligned} \quad (\text{S10})$$

where r_j^g is the set of active row indices of A_g for column $j \in \{1, \dots, p\}$, and Θ_g is as in (S1) with A_g in place of A .

Denote $\Upsilon_n := \text{Cov}(Y, Y)$, and note that $\tilde{\Upsilon}_n := \Theta_g \Theta_g'$ is the covariance matrix of a VAR(1) process with an identity contemporaneous error covariance matrix, and coefficient matrix A_g . Since Θ_g is a square matrix it must be the case that

$$\max_{v \in \mathbb{R}^{np}} \left\{ \frac{v' \Theta_g' \Theta_g v}{v' v} \right\} = \max_{v \in \mathbb{R}^{np}} \left\{ \frac{v' \Theta_g \Theta_g' v}{v' v} \right\} = \max_{v \in \mathbb{R}^{np}} \left\{ \frac{v' \tilde{\Upsilon}_n v}{v' v} \right\},$$

and so from (S10) it follows that

$$\begin{aligned} \text{tr}(\tilde{\mathcal{D}}_g' \tilde{\mathcal{D}}_g) &\leq \frac{1}{\sqrt{n}} \max_{v \in \mathbb{R}^{np}} \left\{ \frac{v' \tilde{\Upsilon}_n v}{v' v} \right\} \left\{ \sum_{j=1}^p \sum_{i \in r_j^g} \text{vec}(\mathcal{X})' (I_n \otimes J^{ji}) (I_n \otimes J^{ij}) \text{vec}(\mathcal{X}) \right. \\ &\quad \left. + \sum_{i=1}^p (Y - \mathcal{Z}_G \alpha_G)' (I_n \otimes J^{ii}) (I_n \otimes J^{ii}) (Y - \mathcal{Z}_G \alpha_G) \right\} \\ &= \frac{\lambda_{\max}(\tilde{\Upsilon}_n)}{\sqrt{n}} \left\{ \text{vec}(\mathcal{X})' \sum_{j=1}^p |r_j^g| (I_n \otimes J^{jj}) \text{vec}(\mathcal{X}) + \|Y - \mathcal{Z}_G \alpha_G\|^2 \right\} \\ &= \frac{\lambda_{\max}(\tilde{\Upsilon}_n)}{\sqrt{n}} \left\{ \sum_{j=1}^p |r_j^g| \sum_{t=0}^{n-1} (X_j^{(t)})^2 + \|Y - \mathcal{Z}_G \alpha_G\|^2 \right\}, \end{aligned}$$

where $|r_j|$ is the number of active rows of A_g for each column $j \in \{1, \dots, p\}$.

By Proposition 2.3 in Basu et al. (2015), and by Lemma S2.3,

$$\lambda_{\max}(\tilde{\Upsilon}_n) \leq 2\pi \operatorname{ess\,sup}_{\theta \in [-\pi, \pi]} \|f_{\tilde{X}}(\theta)\|_2 \leq (1 - \|A_g\|_2)^{-2},$$

recalling that $\tilde{\Upsilon}_n$ is the covariance matrix for the VAR(1) process with an identity contemporaneous error covariance matrix, and coefficient matrix A_g . The function $f_{\tilde{X}}$ denotes the spectral density of this process. Thus,

$$\operatorname{tr}(\tilde{\mathcal{D}}'_g \tilde{\mathcal{D}}_g) \leq \frac{(1 - \|A_g\|_2)^{-2}}{\sqrt{n}} \left\{ \sum_{j=1}^p |r_j^g| \sum_{t=0}^{n-1} (X_j^{(t)})^2 + \|Y - \mathcal{Z}_G \alpha_G\|^2 \right\}.$$

Then

$$\begin{aligned} \|Y - \mathcal{Z}_G \alpha_G\| &= \|Y - (I_n \otimes A_g) \operatorname{vec}(\mathcal{X})\| \\ &\leq \|Y\| + \|(I_n \otimes A_g) \operatorname{vec}(\mathcal{X})\| \\ &= \|Y\| + \sqrt{\sum_{t=0}^{n-1} X^{(t)'} A_g' A_g X^{(t)}} \\ &\leq \|Y\| + \sqrt{\max_{v \neq 0} \left\{ \frac{v' A_g' A_g v}{v' v} \right\} \sum_{t=0}^{n-1} X^{(t)'} X^{(t)}} \\ &= \|Y\| + \|A_g\|_2 \|\operatorname{vec}(\mathcal{X})\| \\ &\leq \|Y\| (1 + \|A_g\|_2). \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{tr}(\tilde{\mathcal{D}}'_g \tilde{\mathcal{D}}_g) &\leq \frac{(1 - \|A_g\|_2)^{-2}}{\sqrt{n}} \left\{ \sum_{j=1}^p |r_j^g| \sum_{t=1}^{n-1} (X_j^{(t)})^2 + \|Y\|^2 (1 + \|A_g\|_2)^2 \right\} \\ &\leq \frac{(1 - c)^{-2}}{\sqrt{n}} \left\{ \sum_{j=1}^p |r_j^g| \sum_{t=1}^{n-1} (X_j^{(t)})^2 + \|Y\|^2 (1 + c)^2 \right\} \\ &\leq (1 - c)^{-2} \left(r_{\max}^g + (1 + c)^2 \right) \frac{\|Y\|^2}{\sqrt{n}}. \end{aligned}$$

■

Proof of Lemma S2.3. For a stable VAR(1) process, i.e. $\|A\|_2 < 1$, the spectral density has the following form (Basu et al. 2015),

$$f_X(\theta) = \frac{1}{2\pi} (I_p - A e^{-i\theta})^{-1} \Sigma [(I_p - A e^{-i\theta})^{-1}]^*.$$

Then

$$\begin{aligned}
\|f_X(\theta)\|_2 &= \frac{1}{2\pi} \|(I_p - Ae^{-i\theta})^{-1} \Sigma [(I_p - Ae^{-i\theta})^{-1}]^* \|_2 \\
&\leq \frac{1}{2\pi} \|(I_p - Ae^{-i\theta})^{-1}\|_2 \|\Sigma\|_2 \|(I_p - Ae^{-i\theta})^{-1}\|_2^* \\
&= \frac{1}{2\pi} \|(I_p - Ae^{-i\theta})^{-1}\|_2^2 \max_{1 \leq i \leq p} \sigma_i^2,
\end{aligned}$$

for $\Sigma = \text{diag}\{\sigma_1^2, \dots, \sigma_p^2\}$.

Note that, for any matrix B satisfying $\|B\|_2 < 1$,

$$(I_p - B) \sum_{k=0}^{\infty} B^k = I_p - \lim_{r \rightarrow \infty} B^{r+1} = I_p,$$

and so

$$\|(I_p - B)^{-1}\|_2 = \left\| \sum_{k=0}^{\infty} B^k \right\|_2 \leq \sum_{k=0}^{\infty} \|B^k\|_2 \leq \sum_{k=0}^{\infty} \|B\|_2^k = \frac{1}{1 - \|B\|_2},$$

by the geometric series for real numbers. Therefore, taking $B = Ae^{-i\theta}$,

$$\|f_X(\theta)\|_2 \leq \frac{1}{2\pi} (1 - \|Ae^{-i\theta}\|_2)^{-2} \sigma_{\max}^2 = \frac{1}{2\pi} (1 - \|A\|_2)^{-2} \sigma_{\max}^2.$$

■

Proof of Lemma S2.4. Consider the inequality for positive definite matrices,

$$\log(|\tilde{\mathcal{D}}'_{g_o} \tilde{\mathcal{D}}_{g_o}|) \geq \text{tr}(I_{|G_o|+p} - (\tilde{\mathcal{D}}'_{g_o} \tilde{\mathcal{D}}_{g_o})^{-1}),$$

which is easily proven with the fact that $\log(x) \leq x - 1$ for all $x > 0$ implies $\log(y) \geq 1 - \frac{1}{y}$ for all $y > 0$. Thus,

$$\begin{aligned}
\log(|\tilde{\mathcal{D}}'_{g_o} \tilde{\mathcal{D}}_{g_o}|) &\geq |G_o| + p - \text{tr}((\tilde{\mathcal{D}}'_{g_o} \tilde{\mathcal{D}}_{g_o})^{-1}) \\
&\geq |G_o| + p - (|G_o| + p) \lambda_{\max}((\tilde{\mathcal{D}}'_{g_o} \tilde{\mathcal{D}}_{g_o})^{-1}) \\
&= (|G_o| + p) \left[1 - \lambda_{\min}^{-1}(\tilde{\mathcal{D}}'_{g_o} \tilde{\mathcal{D}}_{g_o}) \right].
\end{aligned} \tag{S11}$$

A lower bound on $\lambda_{\min}(\tilde{\mathcal{D}}'_{g_o} \tilde{\mathcal{D}}_{g_o})$ can be derived as follows. Decompose

$$\tilde{\mathcal{D}}'_{g_o} \tilde{\mathcal{D}}_{g_o} = \frac{1}{\sqrt{n}} \mathcal{B}'_{G_o} \Theta'_{g_o} \Theta_{g_o} \mathcal{B}_{G_o}$$

where Θ_{g_o} is as in (S1) with A_{g_o} in place of A ,

$$\begin{aligned} \mathcal{B}_{G_o} &:= \begin{pmatrix} (I_n \otimes J^{11})\text{vec}(\mathcal{X}) & (I_n \otimes J^{12})\text{vec}(\mathcal{X}) & \cdots & (I_n \otimes J^{pp})\text{vec}(\mathcal{U}) \end{pmatrix} \\ &= \begin{pmatrix} X_{c_1^{g_o}}^{(0)'} & & U_1^{(1)} & & \\ & \ddots & & \ddots & \\ & & X_{c_p^{g_o}}^{(0)'} & & U_p^{(1)} \\ & \vdots & & \vdots & \\ X_{c_1^{g_o}}^{(n-1)'} & & U_1^{(n)} & & \\ & \ddots & & \ddots & \\ & & X_{c_p^{g_o}}^{(n-1)'} & & U_p^{(n)} \end{pmatrix}, \end{aligned} \quad (\text{S12})$$

and $c_i^{g_o}$ is the set of active column indices of A_{g_o} for row $i \in \{1, \dots, p\}$. Then

$$\begin{aligned} \lambda_{\min}(\tilde{\mathcal{D}}'_{g_o} \tilde{\mathcal{D}}_{g_o}) &= \frac{1}{\sqrt{n}} \min_{v \neq 0} \left\{ \frac{v' \mathcal{B}'_{G_o} \Theta'_{g_o} \Theta_{g_o} \mathcal{B}_{G_o} v}{v' \mathcal{B}'_{G_o} \mathcal{B}_{G_o} v} \cdot \frac{v' \mathcal{B}'_{G_o} \mathcal{B}_{G_o} v}{v' v} \right\} \\ &\geq \frac{1}{\sqrt{n}} \min_{v \neq 0} \left\{ \frac{v' \mathcal{B}'_{G_o} \Theta'_{g_o} \Theta_{g_o} \mathcal{B}_{G_o} v}{v' \mathcal{B}'_{G_o} \mathcal{B}_{G_o} v} \right\} \min_{v \neq 0} \left\{ \frac{v' \mathcal{B}'_{G_o} \mathcal{B}_{G_o} v}{v' v} \right\} \\ &\geq \frac{1}{\sqrt{n}} \lambda_{\min}(\Theta'_{g_o} \Theta_{g_o}) \lambda_{\min}(\mathcal{B}'_{G_o} \mathcal{B}_{G_o}) \\ &\geq \frac{1}{\sqrt{n}} \lambda_{\min}(\Theta'_{g_o} \Theta_{g_o}) \lambda_{\min}(\mathcal{B}' \mathcal{B}), \end{aligned}$$

where \mathcal{B} is of the form (S12) corresponding to the graph with all components of A active, and the last inequality follows from the Poincaré separation theorem (see Rao (1979)) because $\mathcal{B}'_{G_o} \mathcal{B}_{G_o}$ is a principal submatrix of $\mathcal{B}' \mathcal{B}$.

The proof finishes by deriving lower bounds for both $\lambda_{\min}(\Theta'_{g_o} \Theta_{g_o})$ and $\lambda_{\min}(\mathcal{B}' \mathcal{B})$. First consider $\lambda_{\min}(\Theta'_{g_o} \Theta_{g_o})$, and observe that

$$\lambda_{\min}(\Theta'_{g_o} \Theta_{g_o}) = \frac{1}{\lambda_{\max}((\Theta'_{g_o} \Theta_{g_o})^{-1})}, \quad (\text{S13})$$

where

$$\Theta_{g_o}^{-1} = \begin{pmatrix} I_p & & & & \\ -A_{g_o} & I_p & & & \\ & -A_{g_o} & I_p & & \\ & & \ddots & \ddots & \\ & & & -A_{g_o} & I_p \end{pmatrix}.$$

Then for any $v := (v'_1, \dots, v'_n)' \in \mathbb{R}^{np}$,

$$\begin{aligned}
\frac{v'(\Theta'_{g_o} \Theta_{g_o})^{-1}v}{v'v} &= \frac{\|(\Theta'_{g_o})^{-1}v\|^2}{v'v} \\
&= \frac{\|v_n\|^2 + \sum_{i=1}^{n-1} \|v_i - A'_{g_o} v_{i+1}\|^2}{v'v} \\
&\leq \frac{\|v_n\|^2 + \sum_{i=1}^{n-1} (\|v_i\| + \|A'_{g_o} v_{i+1}\|)^2}{v'v} \\
&\leq \frac{2\|v_n\|^2 + \sum_{i=1}^{n-1} 2\|v_i\|^2 + 2\|A'_{g_o} v_{i+1}\|^2}{v'v} \\
&\leq \frac{2v'v + 2\|A_{g_o}\|_2^2 v'v}{v'v},
\end{aligned}$$

which gives

$$\lambda_{\max}((\Theta'_{g_o} \Theta_{g_o})^{-1}) \leq 2(1 + \|A_{g_o}\|_2^2) \leq 2(1 + c^2),$$

and thus from (S13),

$$\lambda_{\min}(\Theta'_{g_o} \Theta_{g_o}) \geq \frac{1}{2}(1 + c^2)^{-1}. \quad (\text{S14})$$

It remains to derive a lower bound for $\frac{1}{\sqrt{n}}\lambda_{\min}(\mathcal{B}'\mathcal{B})$. For any $v \in \mathbb{R}^{p^2+p}$, expressing $v = (\vec{v}_1, \dots, \vec{v}_p, v_{p^2+1}, \dots, v_{p^2+p})$, it follows that for all v with $\|v\| = 1$,

$$\begin{aligned}
v'\mathcal{B}'\mathcal{B}v &= \sum_{j=1}^p \sum_{t=1}^n \left(\begin{pmatrix} X^{(t-1)'} & U_j^{(t)} \end{pmatrix} \begin{pmatrix} \vec{v}_j \\ v_{p^2+j} \end{pmatrix} \right)^2 \\
&= \sum_{j=1}^p \sum_{t=1}^n \begin{pmatrix} \vec{v}_j' & v_{p^2+j} \end{pmatrix} \begin{pmatrix} X^{(t-1)} \\ U_j^{(t)} \end{pmatrix} \begin{pmatrix} X^{(t-1)'} & U_j^{(t)} \end{pmatrix} \begin{pmatrix} \vec{v}_j \\ v_{p^2+j} \end{pmatrix} \\
&= \sum_{j=1}^p \begin{pmatrix} \vec{v}_j' & 0 & \cdots & v_{p^2+j} \cdots & 0 \end{pmatrix} n\Omega \begin{pmatrix} \vec{v}_j \\ 0 \\ \vdots \\ v_{p^2+j} \\ \vdots \\ 0 \end{pmatrix} \\
&\geq n\lambda_{\min}(\Omega) \underbrace{\sum_{j=1}^p \begin{pmatrix} \vec{v}_j' & v_{p^2+j} \end{pmatrix} \begin{pmatrix} \vec{v}_j \\ v_{p^2+j} \end{pmatrix}}_{=1},
\end{aligned}$$

where

$$\Omega := \frac{1}{n} \begin{pmatrix} \sum_{t=1}^n X^{(t-1)} X^{(t-1)'} & \sum_{t=1}^n X^{(t-1)} U^{(t)'} \\ \sum_{t=1}^n U^{(t)} X^{(t-1)'} & \sum_{t=1}^n U^{(t)} U^{(t)'} \end{pmatrix} = \frac{1}{n} \begin{pmatrix} \mathcal{X}\mathcal{X}' & \mathcal{X}\mathcal{U}' \\ \mathcal{U}\mathcal{X}' & \mathcal{U}\mathcal{U}' \end{pmatrix}.$$

Then

$$\frac{1}{\sqrt{n}}\lambda_{\min}(\mathcal{B}'\mathcal{B}) \geq \sqrt{n}\lambda_{\min}(\Omega) \geq \sqrt{n}\left(\lambda_{\min}(E_x(\Omega)) + \lambda_{\min}(\Omega - E_x(\Omega))\right). \quad (\text{S15})$$

Having derived lower bounds for both $\lambda_{\min}(\Theta'_g\Theta_g)$ and $\frac{1}{\sqrt{n}}\lambda_{\min}(\mathcal{B}'\mathcal{B})$, from (S11) it follows that,

$$\begin{aligned} P_x\left(\log(|\tilde{\mathcal{D}}'_{g_o}\tilde{\mathcal{D}}_{g_o}|) \geq \frac{|G_o|+p}{2}\right) &\geq P_x\left(1 - \lambda_{\min}^{-1}(\tilde{\mathcal{D}}'_{g_o}\tilde{\mathcal{D}}_{g_o}) \geq \frac{1}{2}\right) \\ &= P_x\left(\lambda_{\min}(\tilde{\mathcal{D}}'_{g_o}\tilde{\mathcal{D}}_{g_o}) \geq 2\right) \\ &\geq P_x\left(\frac{1}{2}(1+c^2)^{-1}\frac{1}{\sqrt{n}}\lambda_{\min}(\mathcal{B}'\mathcal{B}) \geq 2\right), \end{aligned}$$

where the last inequality follows from (S14). Then by Condition 3.1,

$$\begin{aligned} P_x\left(\log(|\tilde{\mathcal{D}}'_{g_o}\tilde{\mathcal{D}}_{g_o}|) \geq \frac{|G_o|+p}{2}\right) &\geq P_x\left(\frac{1}{\sqrt{n}}\lambda_{\min}(\mathcal{B}'\mathcal{B}) \geq \sqrt{n}[\lambda_{\min}(E_x(\Omega)) - \delta]\right) \\ &\geq P_x\left(\lambda_{\min}(\Omega - E_x(\Omega)) \geq -\delta\right) \end{aligned}$$

where the final inequality comes from (S15). Finally, for all $n \geq \max\{N_1, N_2\}$, appealing to Lemma 3.15 yields the desired result,

$$\begin{aligned} P_x\left(\log(|\tilde{\mathcal{D}}'_{g_o}\tilde{\mathcal{D}}_{g_o}|) \geq \frac{|G_o|+p}{2}\right) &\geq 1 - \delta^{-2} \left[\frac{(\sigma_{\max}^0)^4(1+c^2)}{(1-c^2)^3} \left(\frac{\min\{|G_o|, p\} + \min\{|G_o|, p\}^2}{n} \right) \right. \\ &\quad \left. + \frac{2p(\sigma_{\max}^0)^2 \min\{|G_o|, p\}}{n(1-c^2)} + \frac{p(p+1)}{n} \right]. \end{aligned}$$

■

Proof of Lemma S2.5. Observe

$$\prod_{j=1}^p \frac{|(\mathcal{X}\mathcal{X}')_{r_j^{go}, r_j^{go}}|^{\frac{1}{2}}}{|(\mathcal{X}\mathcal{X}')_{r_j^g, r_j^g}|^{\frac{1}{2}}} = n^{\frac{|G_o|}{2} - \frac{|G|}{2}} \prod_{j=1}^p \frac{|\hat{\Sigma}_{r_j^{go}, r_j^{go}}|^{\frac{1}{2}}}{|\hat{\Sigma}_{r_j^g, r_j^g}|^{\frac{1}{2}}}, \quad (\text{S16})$$

where $\hat{\Sigma} := \frac{1}{n}\mathcal{X}\mathcal{X}'$. First, establish the following lower bound for the denominator. For each

$j \in \{1, \dots, p\}$,

$$\begin{aligned}
\log(|\widehat{\Sigma}_{r_j^g, r_j^g}|) &\geq \text{tr}\left(I_{|r_j^g|} - \widehat{\Sigma}_{r_j^g, r_j^g}^{-1}\right) \\
&\geq |r_j^g| - |r_j^g| \lambda_{\max}\left(\widehat{\Sigma}_{r_j^g, r_j^g}^{-1}\right) \\
&= |r_j^g| \left[1 - \lambda_{\min}^{-1}\left(\widehat{\Sigma}_{r_j^g, r_j^g}\right)\right] \\
&\geq |r_j^g| [1 - \lambda_{\min}^{-1}(\widehat{\Sigma})],
\end{aligned} \tag{S17}$$

where the last inequality follows from the Poincaré separation theorem (Rao 1979). Then for all $n \geq N_2$,

$$P_x\left(\log(|\widehat{\Sigma}_{r_j^g, r_j^g}|) \geq |r_j^g|(1 - 2\delta^{-1})\right) \geq P_x\left(\lambda_{\min}(\widehat{\Sigma}) \geq \delta/2\right) \geq 1 - V_2,$$

where the last inequality follows by Lemma 3.19 (and thus requires Condition 3.1) with V_2 as in (10).

Next, consider the numerator in (S16). The determinant can be bound from above as

$$\begin{aligned}
\log(|\widehat{\Sigma}_{r_j^{go}, r_j^{go}}|) &\leq \text{tr}\left(\widehat{\Sigma}_{r_j^{go}, r_j^{go}} - I_{|r_j^{go}|}\right) \\
&= \text{tr}\left(\widehat{\Sigma}_{r_j^{go}, r_j^{go}}\right) - |r_j^{go}| \\
&\leq |r_j^{go}| \left[\lambda_{\max}(\widehat{\Sigma} - \Gamma_n(0)) + \lambda_{\max}(\Gamma_n(0))\right] - |r_j^{go}|,
\end{aligned}$$

where the final inequality follows again by the Poincaré separation theorem. Further, by the same argument as in (S8), for all $n \geq N_2$ Lemma 3.17 gives,

$$\begin{aligned}
P_x\left(\log(|\widehat{\Sigma}_{r_j^{go}, r_j^{go}}|) \leq |r_j^{go}|[\delta + \lambda_{\max}(\Gamma_n(0)) - 1]\right) &\geq P_x\left(|\lambda_{\max}(\widehat{\Sigma} - \Gamma_n(0))| \leq \delta\right) \\
&\geq 1 - V_2.
\end{aligned} \tag{S18}$$

Therefore, substituting back into (S16) yields for all $n \geq N_2$,

$$\begin{aligned}
\prod_{j=1}^p \frac{|(\mathcal{X}\mathcal{X}')_{r_j^{go}, r_j^{go}}|^{\frac{1}{2}}}{|(\mathcal{X}\mathcal{X}')_{r_j^g, r_j^g}|^{\frac{1}{2}}} &= n^{\frac{|G_o|}{2} - \frac{|G|}{2}} \prod_{j=1}^p \frac{|\widehat{\Sigma}_{r_j^{go}, r_j^{go}}|^{\frac{1}{2}}}{|\widehat{\Sigma}_{r_j^g, r_j^g}|^{\frac{1}{2}}} \\
&\leq n^{\frac{|G_o|}{2} - \frac{|G|}{2}} \prod_{j=1}^p \frac{e^{\frac{1}{2}|r_j^{go}|[\delta + \lambda_{\max}(\Gamma_n(0)) - 1]}}{e^{\frac{1}{2}|r_j^g|(1 - 2\delta^{-1})}} \\
&\leq n^{\frac{|G_o|}{2} - \frac{|G|}{2}} e^{\frac{1}{2}(|G_o|[\delta + \lambda_{\max}(\Gamma_n(0))] + |G|2\delta^{-1})}
\end{aligned}$$

with probability exceeding $1 - 2V_2$. ■

Proof of Lemma S2.6.

Assuming $1 \leq d \leq \min_{1 \leq j \leq p} m_j^{go}$,

$$\prod_{j=1}^p \left[\frac{(m_j^{g_o})^{\frac{n-|r_j^{g_o}|}{2}}}{(m_j^g)^{\frac{n-|r_j^g|}{2}}} \right] \leq \prod_{j=1}^p \frac{(m_j^{g_o})^{\frac{n}{2}}}{(m_j^g)^{\frac{n-|r_j^g|}{2}}} \leq \prod_{j=1}^p \frac{(m_j^{g_o})^{\frac{n}{2}}}{m_j^{\frac{n-|r_j^g|}{2}}}$$

where m_1, \dots, m_p correspond to the full model. Then

$$\begin{aligned} \prod_{j=1}^p \left[\frac{(m_j^{g_o})^{\frac{n-|r_j^{g_o}|}{2}}}{(m_j^g)^{\frac{n-|r_j^g|}{2}}} \right] &\leq \prod_{j=1}^p \frac{(m_j^{g_o})^{\frac{n}{2}}}{m_j^{\frac{n}{2}}} \cdot m_j^{\frac{|r_j^g|}{2}} \\ &= \left[\prod_{j=1}^p m_j^{\frac{|r_j^g|}{2}} \right] \cdot \prod_{j=1}^p \left(\frac{m_j^{g_o}}{m_j} \right)^{\frac{n}{2}} \\ &\leq \left[\prod_{j=1}^p (m_j^{g_o})^{\frac{|r_j^g|}{2}} \right] \cdot \prod_{j=1}^p \left(\frac{m_j^{g_o} - m_j + m_j}{m_j} \right)^{\frac{n}{2}} \\ &\leq \left[\prod_{j=1}^p (m_j^{g_o})^{\frac{p}{2}} \right] \cdot \prod_{j=1}^p \left(\frac{m_j^{g_o} - m_j}{m_j} + 1 \right)^{\frac{n}{2}} \\ &\leq \left(\frac{1}{p} \sum_{j=1}^p m_j^{g_o} \right)^{\frac{p^2}{2}} \cdot \prod_{j=1}^p \left(\frac{m_j^{g_o} - m_j}{q} + 1 \right)^{\frac{n}{2}}, \end{aligned}$$

where the last inequality follows by the arithmetic-geometric inequality and $q := \min_{1 \leq j \leq p} m_j$.

Next, since $1 + x \leq e^x$ for real-valued x ,

$$\begin{aligned} \prod_{j=1}^p \left[\frac{(m_j^{g_o})^{\frac{n-|r_j^{g_o}|}{2}}}{(m_j^g)^{\frac{n-|r_j^g|}{2}}} \right] &\leq \left(\frac{1}{p} \sum_{j=1}^p m_j^{g_o} \right)^{\frac{p^2}{2}} \cdot \prod_{j=1}^p e^{\frac{m_j^{g_o} - m_j}{q} \cdot \frac{n}{2}} \\ &= (RSS_{g_o}/p)^{\frac{p^2}{2}} \cdot e^{(RSS_{g_o} - RSS) \cdot \frac{n}{2q}} \end{aligned}$$

where $RSS_{g_o} := \sum_{j=1}^p m_j^{g_o}$ and $RSS := \sum_{j=1}^p m_j$.

From (5),

$$\begin{aligned}
RSS_{g_o} &= \text{tr}((\mathcal{Y} - \hat{A}_{g_o}\mathcal{X})(\mathcal{Y} - \hat{A}_{g_o}\mathcal{X})') \\
&= \text{tr}\left(\sum_{t=1}^n (X^{(t)} - \hat{A}_{g_o}X^{(t-1)})(X^{(t)} - \hat{A}_{g_o}X^{(t-1)})'\right) \\
&= \sum_{t=1}^n \|X^{(t)} - \hat{A}_{g_o}X^{(t-1)}\|^2 \\
&= \|Y - \mathcal{Z}_{G_o}\hat{\alpha}_{g_o}\|^2 \\
&= \|(I_{np} - H_{g_o})Y\|^2 \\
&= \|(I_{np} - H_{g_o})(\mathcal{W}^0)^{\frac{1}{2}}\text{vec}(\mathcal{U})\|^2 \\
&\leq (\sigma_{\max}^0)^2 \sum_{t=1}^n \|U^{(t)}\|^2,
\end{aligned} \tag{S19}$$

where $H_{g_o} := \mathcal{Z}_{G_o}(\mathcal{Z}'_{G_o}\mathcal{Z}_{G_o})^{-1}\mathcal{Z}'_{G_o}$. Thus,

$$\begin{aligned}
P\left((RSS_{g_o}/p)^{\frac{p^2}{2}} > ((\sigma_{\max}^0)^2 3n)^{\frac{p^2}{2}}\right) &\leq P\left(\sum_{t=1}^n \|U^{(t)}\|^2 > 3np\right) \\
&\leq P(\chi_{np}^2 > 3np) \\
&\leq e^{-\frac{np}{4}},
\end{aligned} \tag{S20}$$

where the last inequality follows by the Chernoff bound (evaluating the moment generating function at 1/4).

Further, from (S19) it also follows that

$$\begin{aligned}
RSS_{g_o} - RSS &= \|(I_{np} - H_{g_o})(\mathcal{W}^0)^{\frac{1}{2}}\text{vec}(\mathcal{U})\|^2 - \|(I_{np} - H)(\mathcal{W}^0)^{\frac{1}{2}}\text{vec}(\mathcal{U})\|^2 \\
&\leq \|(\mathcal{W}^0)^{\frac{1}{2}}\text{vec}(\mathcal{U})\|^2 - \text{vec}(\mathcal{U})'(\mathcal{W}^0)^{\frac{1}{2}}(I_{np} - H)(\mathcal{W}^0)^{\frac{1}{2}}\text{vec}(\mathcal{U}) \\
&= \text{vec}(\mathcal{U})'(\mathcal{W}^0)^{\frac{1}{2}}H(\mathcal{W}^0)^{\frac{1}{2}}\text{vec}(\mathcal{U})
\end{aligned}$$

where $H := \mathcal{Z}(\mathcal{Z}'\mathcal{Z})^{-1}\mathcal{Z}'$. Then

$$\begin{aligned}
RSS_{g_o} - RSS &\leq \text{vec}(\mathcal{U})'(\mathcal{W}^0)^{\frac{1}{2}}\mathcal{Z}(\mathcal{Z}'\mathcal{Z})^{-1}\mathcal{Z}'(\mathcal{W}^0)^{\frac{1}{2}}\text{vec}(\mathcal{U}) \\
&\leq \lambda_{\max}((\mathcal{Z}'\mathcal{Z})^{-1})\text{vec}((\Sigma^0)^{\frac{1}{2}}\mathcal{U}\mathcal{X}')'\text{vec}((\Sigma^0)^{\frac{1}{2}}\mathcal{U}\mathcal{X}') \\
&\leq \frac{1}{\lambda_{\min}(\mathcal{X}'\mathcal{X}) \cdot 1}(\sigma_{\max}^0)^2\text{tr}(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}'),
\end{aligned}$$

and so

$$\begin{aligned}
P_x\left(RSS_{g_o} - RSS \geq (\sigma_{\max}^0)^2 p^2 \sqrt{n}\right) \\
&\leq P_x\left(\frac{1}{\lambda_{\min}(\mathcal{X}'\mathcal{X})}(\sigma_{\max}^0)^2 \text{tr}(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}') \geq (\sigma_{\max}^0)^2 p^2 \sqrt{n}\right) \\
&= P_x\left(\frac{1}{n} \text{tr}(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}') \geq p^2 \sqrt{n} \lambda_{\min}(\mathcal{X}'\mathcal{X}/n)\right) \\
&\leq P_x\left(\frac{1}{n} \text{tr}(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}') \geq p^2 \sqrt{n} \cdot \frac{\delta}{2}\right) \\
&\quad + P_x\left(\lambda_{\min}(\mathcal{X}\mathcal{X}'/n) < \delta/2\right) \\
&\leq \frac{1}{n} E_x\left(\text{tr}(\mathcal{X}\mathcal{U}'\mathcal{U}\mathcal{X}')\right) \frac{2}{p^2 \sqrt{n} \delta} + V_2
\end{aligned}$$

where the last inequality follows by the Markov inequality and Lemma 3.19 (assuming Condition 3.1) for all $n \geq N_2$, and V_2 is as in (10). Finally, by Lemma 3.16, for all $n \geq N_1$,

$$\begin{aligned}
P_x\left(RSS_{g_o} - RSS \geq (\sigma_{\max}^0)^2 p^2 \sqrt{n}\right) &\leq \frac{p(\sigma_{\max}^0)^2 \min\{|G_o|, p\}}{(1 - c^2)} \frac{2}{p^2 \sqrt{n} \delta} + V_2 \\
&\leq \frac{2(\sigma_{\max}^0)^2}{\delta(1 - c^2)\sqrt{n}} + V_2.
\end{aligned}$$

Therefore, for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned}
P_x\left(\prod_{j=1}^p \left[\frac{(m_j^{g_o})^{\frac{n - |r_j^{g_o}|}{2}}}{(m_j^g)^{\frac{n - |r_j^g|}{2}}} \right] \leq ((\sigma_{\max}^0)^2 3n)^{\frac{p^2}{2}} \cdot e^{(\sigma_{\max}^0)^2 p^2 \sqrt{n} \cdot \frac{n}{2q}} \right) \\
\geq 1 - e^{-\frac{np}{4}} - \frac{2(\sigma_{\max}^0)^2}{\delta(1 - c^2)\sqrt{n}} - V_2.
\end{aligned}$$

■

References

- Basu, S., Michailidis, G. et al. (2015), ‘Regularized estimation in sparse high-dimensional time series models’, *The Annals of Statistics* **43**(4), 1535–1567.
- Hannig, J., Iyer, H., Lai, R. C. S. & Lee, T. C. M. (2016), ‘Generalized fiducial inference: A review and new results’, *Journal of American Statistical Association* **111**(515), 1346–1361.
- Rao, C. R. (1979), ‘Separation theorems for singular values of matrices and their applications in multivariate analysis’, *Journal of Multivariate Analysis* **9**(3), 362–377.