ST 705 FINAL EXAM

May 4, 2020

NAME:

- You have **48 hours** to complete this exam.

1. Consider the model $Y_i = \beta_0 + \beta_1 \cdot i + U_i$ for $i \in \{1, \dots, 5\}$, where $U_1, \dots, U_5 \overset{\text{iid}}{\sim} \mathrm{N}(0, 1)$. Find the power of the F-test for testing whether the slope is zero when testing at level $\alpha = 0.05$ and the slope takes values $0.1, 0.2$, and 0.3.

2. (3 points) Let $Y \sim \mathrm{N}_n(X\beta, \sigma^2 I_n)$, where X is an $n \times p$ design matrix. Show that the best linear unbiased predictor (BLUP) of an unobserved response Y_* at the design point x_* is uncorrelated with all unbiased estimators of zero.

3. (3 points) Suppose that $Y_1, \dots, Y_n \overset{\text{iid}}{\sim} \text{Uniform}(0, 2\theta)$, and define $U_i := Y_i - \theta$ for $i \in \{1, \dots, n\}$.

   (a) Find the mean and variance of $U := (U_1, \dots, U_n)'$.

   (b) Show that $Y := (Y_1, \dots, Y_n)'$ is generated according to a linear model that satisfies the Gauss-Markov assumptions.

   (c) Find the BLUE of θ, and denote the BLUE by $\hat{\theta}_{\text{OLS}}$.

   (d) Find c so that the estimator $\hat{\theta} = cY_{(n)}$ is unbiased for θ, where $Y_{(i)}$ denotes the ith order statistic, and compute the variance of $\hat{\theta}$.

   (e) Compare the variances of $\hat{\theta}_{\text{OLS}}$ and $\hat{\theta}$, and provide intuition for your finding.

4. (3 points) Suppose that (X, Y) has a bivariate distribution (**not necessarily Gaussian**) with mean $(\mu_X', \mu_Y')'$ and covariance matrix
$$\begin{pmatrix} \sigma_X^2 & \sigma_{X,Y} \\ \sigma_{Y,X} & \sigma_Y^2 \end{pmatrix}.$$

   (a) Show that if $E(Y \mid X) = \beta_0 + \beta_1 X$, then $\beta_1 = \sigma_{Y,X}/\sigma_X^2$ and $\beta_0 = \mu_Y - \beta_1 \mu_X$.

   (b) Show that if $E(Y \mid X) = \beta_0 + \beta_1 X$ and $\mathrm{Var}(Y \mid X) = \tau^2$, then $\tau^2 = \sigma_Y^2 - \sigma_{Y,X}^2/\sigma_X^2$.

5. (3 points) Let
$$Y \sim \mathrm{N}_2\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$
$$A = \frac{1}{8}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$
and $B = (1, -2)'$. Find the joint distribution of $Y'AY$ and $B'Y$.