ST 705 Linear models and variance components
Lab practice problem set 1

January 6, 2020

1. Prove or find a counter example to the following inequalities.

$$1 \le \sum_{k=1}^{\infty} \frac{1}{k^2} \le 2.$$

2. Let $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ be sequences of real numbers. Show that

$$\min\{a_i\} + \min\{b_i\} \le \min\{a_i + b_i\} \le \min\{a_i\} + \max\{b_i\}.$$

3. Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Is A diagonalizable? If so, find the eigenvalues and eigenvectors of A.

4. Show that if $Q \in \mathbb{R}^{p \times p}$ is orthogonal and $\|Qx\|_2 = \|x\|_2$ for every $x \in \mathbb{R}^p$, then every real eigenvalue of Q is -1 or 1.

5. Use Jensen's inequality to establish the arithmetic-geometric mean inequality. That is, show that if $a_1, \dots, a_n$ are positive constants, then

$$\frac{1}{n} \sum_{i=1}^{n} a_i \ge \left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}}.$$