Solution HW 8

November 10, 2019

Section 8.10 of Rice; Exercises 13, 24, 55, 68-75
8.13

(a) $E(\hat{\alpha}) = E(3\bar{X}) = 3E(\bar{X}) = 3E(X)$ since iid $= 3\alpha/3 = \alpha$

(b) $Var(\hat{\alpha}) = Var(3\bar{X}) = 9Var(X)/n$ since iid . Now to find the variance we do some integration:

$$\begin{aligned} Var(X) = E(X^2) - (E(X))^2 &= \int_{-1}^{1} x^2 \frac{1+\alpha x}{2} dx - (\alpha/3)^2 \\ &= \frac{1}{2}\int_{-1}^{1} (x^2 + \alpha x^3) dx - \alpha^2/9 \\ &= \frac{1}{2}(x^3/3 + \alpha x^4/4|_{-1}^{1}) - \alpha^2/9 \\ &= \frac{1}{2}\left[(1/3 + \alpha/4) - (-1/3 + \alpha/4)\right] - \alpha^2/4 \\ &= 1/3 - \alpha^2/4 = \frac{3-\alpha^2}{9} \end{aligned}$$

This gives $Var(\hat{\alpha}) = \frac{3-\alpha^2}{n}$.

(c) $\hat{\alpha} = 3\bar{X}$, by the CLT $\bar{X} \overset{\bullet}{\sim} N\left(\alpha/3, \frac{3-\alpha^2}{9n}\right)$. This implies

$$\hat{\alpha} = 3\bar{X} \overset{\bullet}{\sim} N\left(\alpha, \frac{3-\alpha^2}{n}\right)$$

If n=25, $\alpha = 0$,

$$P(|\hat{\alpha}| > 0.5) = 1 - P(|\hat{\alpha}| \leq 0.5) = 1 - P(-0.5 < \hat{\alpha} < 0.5)$$

$$\approx 1 - P\left(\frac{-0.5-0}{\sqrt{3/25}} < Z < \frac{0.5-0}{\sqrt{3/25}}\right) = 1 - P(-1.44 < Z < 1.44) = 1 - 0.8501 = 0.1499$$

8.24 For the first case,

$$L(\pi) = \binom{n}{\sum_{i=1}^{n} X_i} \pi^{\sum_{i=1}^{n} X_i}(1-\pi)^{(n-\sum_{i=1}^{n} X_i)} = \binom{20}{12}\pi^{12}(1-\pi)^{8}$$

.

Taking log of both sides,

$$l(\pi) = log\{\binom{20}{12}\} + 12\log\pi + 8\log(1-\pi)$$

. Thus we can graph the log likelihood of π using R.

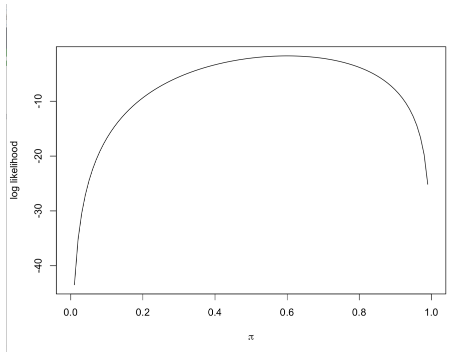

Figure 1: 8.24(1)

For the second case, $N = 18$ from experiment. Thus

$$L(\theta) = \binom{17}{9}\pi^{10}(1-\pi)^8$$

$$l(\theta) = \log\{\binom{17}{9}\} + 10\log\pi + 8\log(1-\pi)$$

Thus we can graph the log likelihood of π using R.

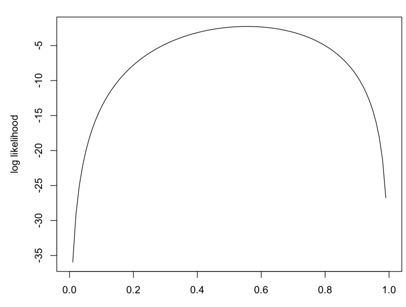

Figure 2: 8.24(1)

8.55

(a) The likelihood assumes we know our data. Here (N_1, N_2, N_3, N_4) follow a multinomial, thus,

$$L(\theta) = [0.25(2+\theta)]^{n_1}[0.25(1-\theta)]^{n_2}[0.25(1-\theta)]^{n_3}[0.25\theta]^{n_4}$$

The log-likelihood is then

$$l(\theta) = n_1 ln(0.25(2+\theta)) + n_2 ln(0.25(1-\theta)) + n_3 ln(0.25(1-\theta)) + n_4 ln(0.25\theta)$$

Maximizing this we get

$$\frac{\partial l(\theta)}{\partial \theta} = n_1 \frac{1}{2+\theta} + n_2 \frac{-1}{1-\theta} + n_3 \frac{-1}{1-\theta} + n_4 \frac{1}{\theta} = 0$$

Solving we get a quadratic equation of θ

$$(n_1 + n_2 + n_3 + n_4)\theta^2 - (n_1 - 2n_2 - 2n_3 - n_4)\theta - 2n_4 = 0$$

Since it is quadratic we can use the quadratic formula to find the root. We want the positive root of this since θ must be between 0 and 1.

$$\hat{\theta}_{MLE} = \frac{(n_1 - 2n_2 - 2n_3 - n_4) + \sqrt{(n_1 - 2n_2 - 2n_3 - n_4)^2 - 4(n_1 + n_2 + n_3 + n_4)(-2n_4)}}{2(n_1 + n_2 + n_3 + n_4)}$$

Checking this is a max can be done using the second derivative:

$$\frac{\partial^2 l(\theta)}{\partial \theta^2} = \frac{-n_1}{(2+\theta)^2} + \frac{-n_2}{(1-\theta)^2} + \frac{-n_3}{(1-\theta)^2} + \frac{-n_4}{\theta^2} < 0$$

Plugging in our observed counts, we get our MLE estimate:

$$\hat{\theta}_{MLE} = \frac{-1655 + \sqrt{(1655)^2 - 4(3839)(-64)}}{2(3839)} = 0.0357$$

To find the asymptotic variance, we can use the information for the sample

$$I_n(\theta) = -E(\frac{\partial^2 l(\theta)}{\partial \theta^2}) = -E\left[\frac{-N_1}{(2+\theta)^2} + \frac{-N_2}{(1-\theta)^2} + \frac{-N_3}{(1-\theta)^2} + \frac{-N_4}{\theta^2}\right]$$

As the N_i's are multinomial we know their expectations are (n=total # of values)$\times$probability

$$\begin{aligned} I_n(\theta) &= \frac{n(0.25)(2+\theta)}{(2+\theta)^2} + \frac{n(0.25)(1-\theta)}{(1-\theta)^2} + \frac{n(0.25)(1-\theta)}{(1-\theta)^2} + \frac{n(0.25)\theta}{\theta^2} \\ &= \frac{n}{4}\left[\frac{1}{(2+\theta)} + \frac{1}{(1-\theta)} + \frac{1}{(1-\theta)} + \frac{1}{\theta}\right] \end{aligned}$$

The asymptotic variance is then

$$Var(\hat{\theta}_{MLE}) = \frac{1}{I_n(\theta)} = \frac{2}{n}\frac{\theta(1-\theta)(2+\theta)}{1+2\theta}$$

Plugging in for θ, the estimate of the variance simplifies to

$$\frac{2}{3839}\frac{0.0357(1-0.0357)(2+0.0357)}{1+2(0.0357)} = 3.407645e-05$$

The estimated standard error is the square root of this, 0.0058.

(b) Using the asymptotic normality of the MLE estimator we can form an approximate $(1-\alpha)\%$ CI

$$\hat{\theta}_{MLE} \pm z_{\alpha/2}\hat{SE}(\hat{\theta}_{MLE})$$

Plugging in we get

$$0.0357 \pm 1.96(0.0058) = (0.0243, 0.0471)$$

(c) and (d)

```
theta<-0.0357
prob<-c(0.25*(2+theta),0.25*(1-theta),0.25*(1-theta),0.25*theta)
n<-1997+906+904+32
N<-1000
mle_vec<-NULL
for(i in 1:N){
  boot_sam<-sample(c(1,2,3,4),size=n,replace=TRUE,prob=prob)
  n1<-length(which(boot_sam==1))
  n2<-length(which(boot_sam==2))
  n3<-length(which(boot_sam==3))
  n4<-length(which(boot_sam==4))
  mle<-((n1-2*n2-2*n3-n4)+sqrt((n1-2*n2-2*n3-n4)^2-4*n*(-2*n4)))/2/n
  mle_vec<-c(mle_vec,mle)
}
boot_sd<-sd(mle_vec)
boot_mean<-mean(mle_vec)
c(boot_mean-1.96*boot_sd,boot_mean+1.96*boot_sd)
```

8.68

(a)

$$T \sim Poi(n\lambda)$$

. Thus

$$\begin{aligned} P(X_1,\ldots,X_n|T) &= \frac{P(X_1,\ldots,X_n,T)}{P(T)} \\ &= \frac{\lambda^t e^{-n\lambda}/\prod_{i=1}^n X_i!}{(n\lambda)^t e^{-n\lambda}/t!} = \frac{t!}{n^t\prod_{i=1}^n x_i!} \end{aligned}$$

The conditional distribution is independent of λ, thus T is sufficient for λ.

(b)

$$P(X_1,\ldots,X_n|X_1) = \frac{\lambda^t e^{-n\lambda}/\prod_{i=1}^n X_i!}{\lambda^{X_1}e^{-\lambda}/X_1}$$

The conditional distribution depends on λ, thus X_1 is not sufficient.

(c)

$$L(\lambda) = \frac{\lambda^{\sum_{i=1}^n X_i}e^{-n\lambda}}{\prod_{i=1}^n X_i!}$$

Let $g(T,\lambda) = \lambda^{\sum_{i=1}^n X_i}e^{-n\lambda} = \lambda^T e^{-n\lambda}$ and $h(X) = \frac{1}{\prod_{i=1}^n X_i!}$. Thus according to Theorem A, T is sufficient.

8.69

$$L(p) = (1-p)^{\sum_{i=1}^n X_i - n}p^n = (1-p)^{\sum_{i=1}^n X_i}(\frac{p}{1-p})^n$$

According to Theorem A, $T = \sum_{i=1}^n X_i$ is a sufficient statistic for p.

8.70

(Since the support doesn't depend on the parameter we leave off the indicator functions.) Using the factorization theorem we have

$$f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) = \prod_{i=1}^n \lambda e^{-\lambda y_i}$$

$$= \lambda^n e^{-\lambda\sum_{i=1}^n y_i}$$

giving $h(y_1,\ldots,y_n) = 1$ and $g(T,\lambda) = \lambda^n e^{-\lambda T}$, where $T = \sum_{i=1}^n y_i$. Thus, $T = \sum_{i=1}^n y_i$ is sufficient for λ.

8.71

(Since the support doesn't depend on the parameter we leave off the indicator functions.) Using the factorization theorem we have

$$f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) = \prod_{i=1}^n \frac{\theta}{(1+y_i)^{\theta+1}}$$

$$= \frac{\theta^n}{\left[\prod_{i=1}^n(1+y_i)\right]^{\theta+1}}$$

giving $h(y_1,\ldots,y_n) = 1$ and $g(T,\theta) = \frac{\theta^n}{[T]^{\theta+1}}$, where $T = \prod_{i=1}^n(1+y_i)$. Thus, $T = \prod_{i=1}^n(1+y_i)$ is sufficient for θ.

8.72

$Y \sim gamma(\alpha,\lambda)$ then $f_Y(y) = \frac{\lambda^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\lambda y}$ where $y > 0$.

$Y_i \stackrel{iid}{\sim} gamma(\alpha,\lambda)$, we can use the factorization theorem:

$$f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n|\alpha,\lambda) = \frac{\lambda^{n\alpha}}{(\Gamma(\alpha))^n}\left(\prod_{i=1}^n y_i\right)^{\alpha-1} e^{-\lambda\sum_{i=1}^n y_i}(1)$$

We can now write $g(T,\lambda,\alpha) = \frac{\lambda^{n\alpha}}{(\Gamma(\alpha))^n} \left(\prod_{i=1}^n y_i\right)^{\alpha-1} e^{-\lambda \sum_{i=1}^n y_i}$ where

$$T = \left(\prod_{i=1}^n Y_i, \sum_{i=1}^n Y_i\right)$$

and $h(y_1, ..., y_n) = 1$. Note: any 1-1 function of T is still a sufficient statistic!

8.73

(Since the support doesn't depend on the parameter we leave off the indicator functions.) Using the factorization theorem we have

$$f_{Y_1,...,Y_n}(y_1, ..., y_n) = \prod_{i=1}^n \frac{x_i}{\theta^2} e^{-x_i^2/(2\theta^2)}$$

$$= \left(\prod_{i=1}^n x_i\right) \frac{1}{\theta^{2n}} e^{-\sum_{i=1}^n x_i^2/(2\theta^2)}$$

giving $h(y_1, ..., y_n) = \prod_{i=1}^n x_i$ and $g(T,\theta) = \frac{1}{\theta^{2n}} e^{-T/(2\theta^2)}$, where $T = \sum_{i=1}^n x_i^2$. Thus, $T = \sum_{i=1}^n x_i^2$ is sufficient for θ.

8.74

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \exp\{x \log(\frac{p}{1-p}) + n \log(1-p) + \log[\binom{n}{x}]\}$$

8.75

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

$$= \exp\{[\alpha - 1, -\beta] \begin{bmatrix} \log x \\ x \end{bmatrix} + \alpha \log \beta - \log(\Gamma(\alpha)\}$$