ST 705 Linear models and variance components
Homework problem set 2

January 28, 2021

1. Let A be a positive definite matrix, and show that

$$\operatorname{tr}(I - A^{-1}) \leq \log\det(A) \leq \operatorname{tr}(A - I).$$

2. Show that the covariance function defined for $X, Y \in \mathbb{R}^p$ by

$$\operatorname{Cov}(X, Y) := E[(X - E(X))(Y - E(Y))']$$

satisfies the following properties. For random variables $X, Y, Z \in \mathbb{R}^p$ with finite covariance, and any $c \in \mathbb{R}$,

   (a) $\operatorname{Cov}(X + Y, Z) = \operatorname{Cov}(X, Z) + \operatorname{Cov}(Y, Z)$

   (b) $\operatorname{Cov}(cX, Y) = c \cdot \operatorname{Cov}(X, Y)$

   (c) $\operatorname{Cov}(X, Y)^* = \operatorname{Cov}(Y, X)$

   (d) $\operatorname{Cov}(X, X) \geq 0$ for all X, and $\operatorname{Cov}(X, X) = 0$ implies that X is constant a.s.

Then, deduce that if $p = 1$ the covariance is an inner product over some (quotient) vector space, and if $p > 1$ the the function $f(X, Y) := \operatorname{tr}\big(\operatorname{Cov}(X, Y)\big)$ is an inner product.

3. Exercise A.50 from Monahan.

4. For matrices $A \in \mathbb{R}^{p \times q}$, the *spectral* norm is defined as,

$$\|A\|_2 := \sqrt{\sup_{x \neq 0} \frac{x'A'Ax}{x'x}}.$$

Further, the eigenvalues of $A'A$ are the squares of the *singular values* of A, so sometimes the definition of the spectral norm is expressed as

$$\|A\|_2 := \sigma_{\max}(A),$$

where $\sigma_{\max}$ denotes the largest singular value of A.

(a) Verify that the spectral norm is a norm. Recall that a norm must satisfy the following axioms for any $A, B, C \in \mathbb{R}^{p\times q}$ and any $\alpha \in \mathbb{R}$.

   i. $\|\alpha A\| = |\alpha|\|A\|$

   ii. $\|A + B\| \leq \|A\| + \|B\|$

   iii. $\|A\| \geq 0$ with equality if and only if $A = 0$.

(b) Show that the spectral norm is sub-multiplicative for square matrices. That is, for $A, B \in \mathbb{R}^{p\times p}$, $\|AB\|_2 \leq \|A\|_2\|B\|_2$.

5. Suppose you do not know that the rank of a matrix is equal to the number of nonzero singular values. Show that the rank of a projection matrix is equal to its trace. First think about how to show this in the symmetric case, and then consider the more general case of a non-symmetric idempotent matrix.

6. Show that if $\text{rank}(BC) = \text{rank}(B)$, then $\text{col}(BC) = \text{col}(B)$, where $\text{col}(\cdot)$ denotes the column space.