ST 705 Linear models and variance components
Homework problem set 6

February 11, 2020

1. (2 points) Exercise 3.9 from Monahan. Not necessary to do the "(More practice)" item.

2. (2 points) Consider the model $Y_{ijk} = \mu + \alpha_i + \beta_j + \theta_{ij} + U_{ijk}$, with $k \in \{1, \ldots, m_{ij}\}$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, and $E(U_{ijk}) = 0$. Find necessary and sufficient conditions for which $\lambda'\gamma$ is estimable for
$$\gamma = \begin{pmatrix} \mu & \alpha_1 & \cdots & \alpha_n & \beta_1 & \cdots & \beta_m & \theta_{11} & \cdots & \theta_{nm} \end{pmatrix}'.$$

3. (2 points) Prove the converse to Result 3.2 (recall that we saw two proofs of Result 3.2 in lecture). That is, if the least squares estimator $\lambda'\widehat{\beta}$ is the same for all solutions $\widehat{\beta}$ to the normal equations, then $\lambda'\beta$ is estimable.

4. (2 points) Let V be a finite-dimensional inner product space over $\mathbb{C}$, and let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for V.

   (a) Show that for any $x, y \in V$,
   $$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$
   This is called Parseval's identity.

   (b) For $V = \mathbb{R}^2$ use Parseval's identity to prove the Pythagorean theorem. Generalize this result to $\mathbb{R}^n$.

5. (2 points) Let V be an inner product space over $\mathbb{C}$, and let $\{v_1, \ldots, v_n\} \subset V$ be orthonormal.

   (a) Prove that for any $x \in V$,
   $$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2.$$
   This is called Bessel's inequality.

   (b) Show that Bessel's inequality is an equality if and only if $x \in \operatorname{span}\{v_1, \ldots, v_n\}$.