# ST 705 MIDTERM

March 8, 2021

**NAME:**

**STUDENT ID:**

- You have **3.5 hours** to complete this exam.
- Communication with other individuals is **not** permitted during this exam.

1. (3 points) Consider the vector space, $P_3(\mathbb{R})$, of polynomials over $\mathbb{R}$ with degree at most 3, and with the inner product $\langle f, g \rangle := \int_{-1}^{1} f(t)g(t)\, dt$. Beginning with the standard basis, $\{1, x, x^2, x^3\}$, construct an orthonormal basis for $P_3(\mathbb{R})$.

2. Let $X \in \mathbb{R}^{n\times p}$, $u \in \mathbb{R}^n$, and $v \in \mathbb{R}^p$.

   (a) (3 points) Prove that

   $$|u'Xv| \leq \left( \max_{1\leq j\leq p} \left\{ \sum_{i=1}^{n} |X_{i,j}| \right\} \right)^{\frac{1}{2}} \left( \max_{1\leq i\leq n} \left\{ \sum_{j=1}^{p} |X_{i,j}| \right\} \right)^{\frac{1}{2}} \cdot \|u\|_2 \cdot \|v\|_2.$$

   (b) (3 points) Show that the Cauchy-Schwarz inequality is a special case of the inequality given in part (a).

3. (3 points) Let $Q = X(X'V^{-1}X)^g X'V^{-1}$, with $V > 0$ and symmetric, and show that $Q$ is a projection onto $\text{col}(X)$.

4. Suppose that $y \in \mathbb{R}^n$ is nonzero, $X \in \mathbb{R}^{n\times n}$ is invertible, $\beta \in \mathbb{R}^n$, and $y = X\beta$. Denoting by $\sigma_{\min}$ and $\sigma_{\max}$ the minimum and maximum singular values of $X$, respectively, prove the following statements.

   (a) (3 points)
   $$\frac{\sigma_{\min}}{\sigma_{\max}} \cdot \frac{\|\Delta y\|_2}{\|y\|_2} \leq \frac{\|\Delta\beta\|_2}{\|\beta\|_2} \leq \frac{\sigma_{\max}}{\sigma_{\min}} \cdot \frac{\|\Delta y\|_2}{\|y\|_2},$$
   where $\Delta y$ and $\Delta\beta$ are vectors that satisfy $y + \Delta y = X(\beta + \Delta\beta)$.

   (b) (3 points) The ratio $\dfrac{\sigma_{\max}}{\sigma_{\min}} = 1$ if and only if $X$ is a scalar multiple of an orthogonal matrix.