

Markov chain Monte Carlo simple example

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MCMC Example (Metropolis-Hastings)

Suppose $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Exp}(\lambda)$. Then the likelihood function of the data is

$$\ell(x_1, \dots, x_n | \lambda) = \prod_{i=1}^n \lambda e^{-\lambda x_i}.$$

Assume the prior density $\pi(\lambda) := \text{Gamma}(\lambda | a, b)$.

The posterior density is then given by

$$\pi(\lambda | x_1, \dots, x_n) = \frac{\ell(x_1, \dots, x_n | \lambda) \cdot \pi(\lambda)}{\int \ell(x_1, \dots, x_n | \lambda) \cdot \pi(\lambda) d\lambda} \propto \underbrace{\ell(x_1, \dots, x_n | \lambda) \cdot \pi(\lambda)}_{=: f(\lambda)}.$$

MCMC Example

Outline of a random walk Metropolis-Hastings algorithm:

Step 1. Given current $\lambda^{(t)}$, propose a new $\lambda^* \sim N(\cdot | \lambda^{(t)}, \sigma^2)$

Step 2. Set

$$\lambda^{(t+1)} = \begin{cases} \lambda^* & \text{w.p. } \rho(\lambda^*, \lambda^{(t)}) \\ \lambda^{(t)} & \text{w.p. } 1 - \rho(\lambda^*, \lambda^{(t)}) \end{cases}$$

where

$$\begin{aligned} \rho(\lambda^*, \lambda^{(t)}) &= \min \left\{ \frac{\pi(\lambda^* | x_1, \dots, x_n) \cdot N(\lambda^{(t)} | \lambda^*, \sigma^2)}{\pi(\lambda^{(t)} | x_1, \dots, x_n) \cdot N(\lambda^* | \lambda^{(t)}, \sigma^2)}, 1 \right\} \\ &= \min \left\{ \frac{f(\lambda^*)}{f(\lambda^{(t)})}, 1 \right\}. \end{aligned}$$

This is called the Metropolis-Hastings acceptance ratio.

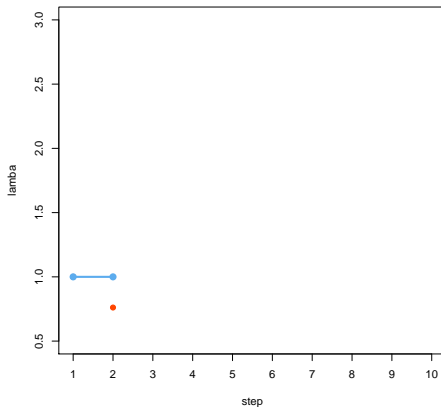
MCMC Example

Current $\lambda = 1$

Proposed $\lambda (\sim 1 + N(0, 0.5^2)) = 0.7613$

MH Ratio $= 1e - 78$

Coin-flip $(\sim U(0, 1)) = 0.2788 \implies \text{Reject}$



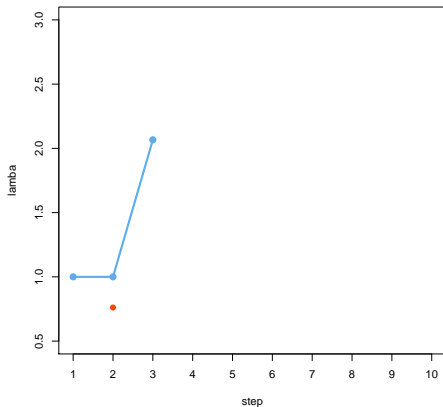
MCMC Example

Current $\lambda = 1$

Proposed $\lambda (\sim 1 + N(0, 0.5^2)) = 2.0667$

MH Ratio $= 4e + 134$

Coin-flip $(\sim U(0, 1)) = 0.5027 \implies \text{Accept}$



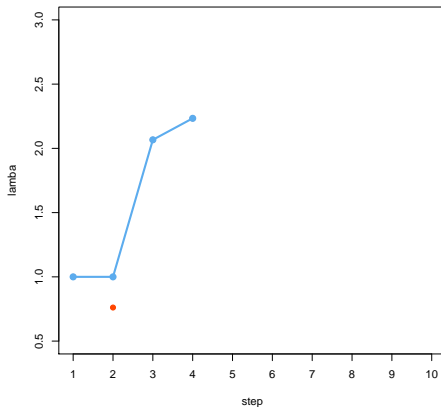
MCMC Example

Current $\lambda = 2.0667$

Proposed $\lambda (\sim 2.0667 + N(0, 0.5^2)) = 2.2337$

MH Ratio $= 3e + 05$

Coin-flip $(\sim U(0, 1)) = 0.3707 \implies \text{Accept}$



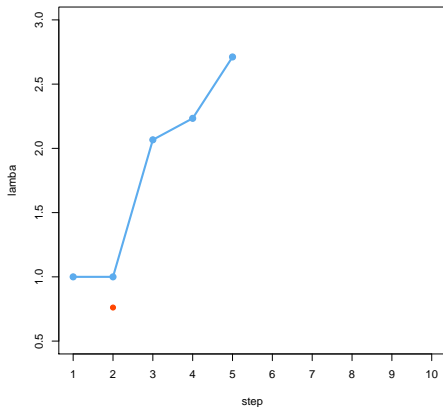
MCMC Example

Current $\lambda = 2.2337$

Proposed $\lambda (\sim 2.2337 + N(0, 0.5^2)) = 2.7115$

MH Ratio = 1964

Coin-flip ($\sim U(0, 1)$) = 0.2875 \implies Accept



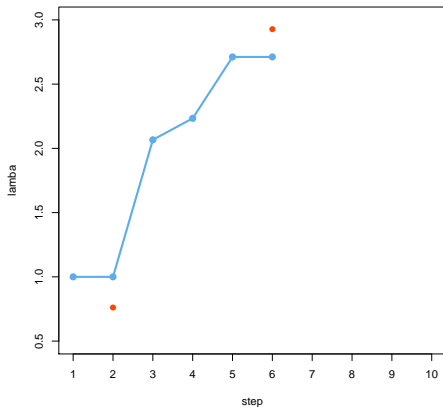
MCMC Example

Current $\lambda = 2.7115$

Proposed $\lambda (\sim 2.7115 + N(0, 0.5^2)) = 2.9276$

MH Ratio = 0.0005

Coin-flip ($\sim U(0, 1)$) = 0.1298 \implies Reject



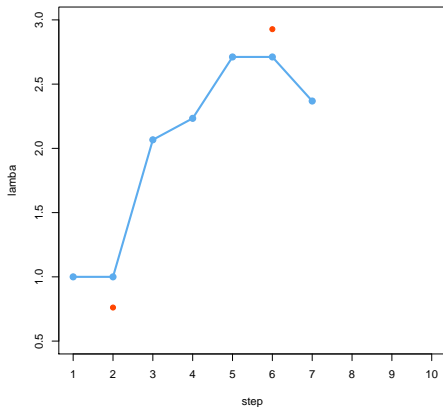
MCMC Example

Current $\lambda = 2.7115$

Proposed $\lambda (\sim 2.7115 + N(0, 0.5^2)) = 2.3685$

MH Ratio = 0.2142

Coin-flip ($\sim U(0, 1)$) = 0.1653 \implies Accept



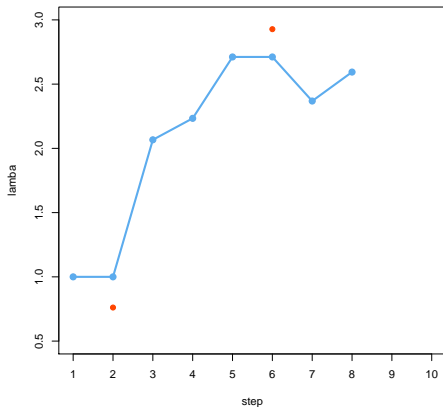
MCMC Example

Current $\lambda = 2.3685$

Proposed $\lambda (\sim 2.3685 + N(0, 0.5^2)) = 2.5939$

MH Ratio = 21

Coin-flip ($\sim U(0, 1)$) = 0.0457 \implies Accept



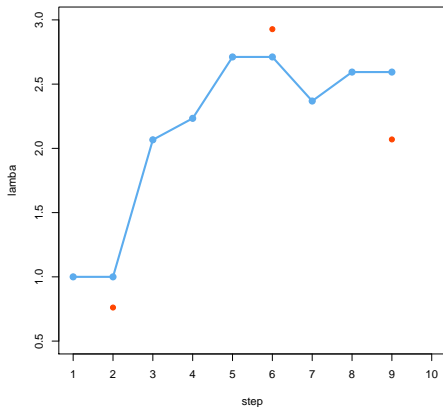
MCMC Example

Current $\lambda = 2.5939$

Proposed $\lambda (\sim 2.5939 + N(0, 0.5^2)) = 2.0695$

MH Ratio $= 5e - 10$

Coin-flip $(\sim U(0, 1)) = 0.8348 \implies \text{Reject}$



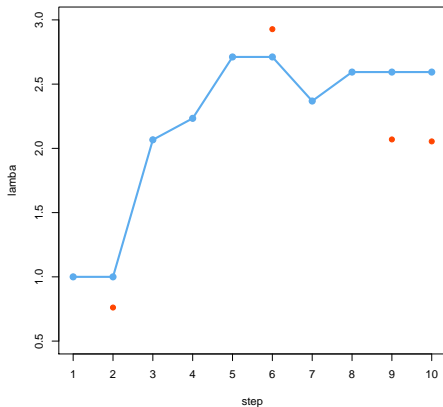
MCMC Example

Current $\lambda = 2.5939$

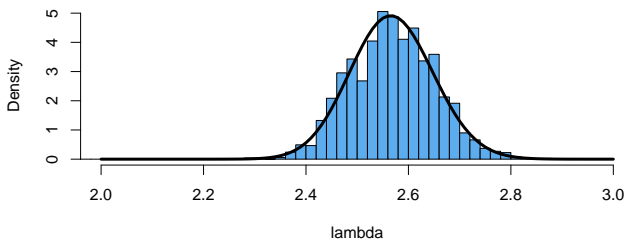
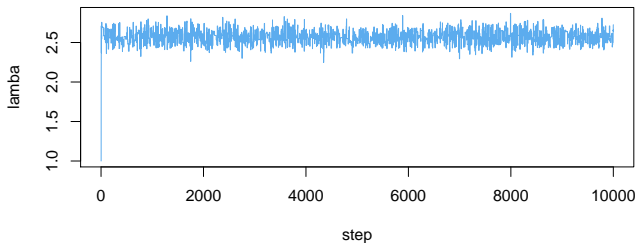
Proposed $\lambda (\sim 2.5939 + N(0, 0.5^2)) = 2.0542$

MH Ratio $= 1e - 10$

Coin-flip $(\sim U(0, 1)) = 0.3117 \implies \text{Reject}$



MCMC Example



The End