Music 320

Autumn 2010–2011

Homework #1

Complex Numbers, Polynomials, Trigonometry

50 points

Due in one week (09/30/2010)

Theory Problems

1. (10 pts) For the complex number $z = x + jy$, where x and y are real, find:

   (a) real part

   Solution: $\mathrm{re}\{\mathrm{z}\} = x$

   (b) imaginary part

   Solution: $\mathrm{im}\{\mathrm{z}\} = y$

   (c) modulus

   Solution: $|z| = \sqrt{x^2 + y^2}$

   (d) phase

   Solution:

   $$\angle z = \tan^{-1}\left(\frac{y}{x}\right)$$

   (e) complex conjugate

   Solution: $\overline{z} = x - jy$

   (f) reciprocal in rectangular form

   Solution:

   $$\frac{1}{z} = \frac{1}{x + jy} = \frac{x - jy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - j\frac{y}{x^2 + y^2}$$

   (g) reciprocal in polar form

   Solution:

   $$\frac{1}{z} = \frac{1}{x + jy} = \frac{x - jy}{x^2 + y^2} = \frac{1}{\sqrt{x^2 + y^2}}e^{-j\tan^{-1}\left(\frac{y}{x}\right)}$$

2. (5 pts) Using DeMoivre's formula, find $(3/5 + j4/5)^{100}$ in polar form.

   Solution: $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5} \rightarrow \theta = \arccos\frac{3}{5}$.

   Therefore, $(\frac{3}{5} + j\frac{4}{5})^{100} = \cos(100\theta) + j\sin(100\theta) = e^{100\theta} = e^{100\cdot\arccos\frac{3}{5}}$

3. (10 pts) Convert the following expressions to both Cartesian and polar forms (a, b, c, and d are real):

$(a)\ (1+j)^2$	$(d)\ \sqrt{1+j}$	$(g)\ \ln(j)$
$(b)\ (a+jb)/(c+jd)$	$(e)\ e^{e^{j\theta}}$	$(h)\ j^j$
$(c)\ e^{j\pi}+1$	$(f)\ (-1)^{1/10}$	$(i)\ \tan(\frac{1+j}{1-j})$

Solution:

(a) $(1+j)^2 = (1+j)(1+j) = 1+2j-1 = 0+j2$ (cartesian)
$= 2e^{j\frac{\pi}{2}}$ (polar)

(b) $\frac{a+jb}{c+jd} = \frac{a+jb}{c+jd} \cdot \frac{c-jd}{c-jd} = \frac{ac-jad+jcb-j^2bd}{c^2-(jd)^2} = \frac{ac+bd+j(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + j\frac{bc-ad}{c^2+d^2}$ (cartesian)
$= \sqrt{\frac{(ac+bd)^2+(bc-ad)^2}{(c^2+d^2)^2}}e^{j\tan^{-1}(\frac{bc-ad}{ac+bd})} = \sqrt{\frac{a^2+b^2}{c^2+d^2}}e^{j\tan^{-1}(\frac{bc-ad}{ac+bd})}$ (polar)

(c) $e^{j\pi}+1 = -1+1 = 0$ (cartesian)
$= 0e^{j0}$ (polar)

(d) $\sqrt{1+j} = \left(\sqrt{2}e^{j\frac{\pi}{4}}\right)^{\frac{1}{2}} = 2^{\frac{1}{4}}e^{j\frac{\pi}{8}}$ (polar)
$= 2^{\frac{1}{4}}\cos\frac{\pi}{8} + j2^{\frac{1}{4}}\sin\frac{\pi}{8}$ (cartesian)

(e) $e^{e^{j\theta}} = e^{\cos\theta + j\sin\theta} = e^{\cos\theta}e^{j\sin\theta}$ (polar)
$= e^{\cos\theta}(\cos(\sin\theta) + j\sin(\sin\theta))$
$= e^{\cos\theta}\cos(\sin\theta) + je^{\cos\theta}\sin(\sin\theta)$ (cartesian)

(f) $(-1)^{\frac{1}{10}} = (e^{j\pi})^{\frac{1}{10}} = 1e^{j\frac{\pi}{10}}$ (polar)
$\cos(\frac{\pi}{10}) + j\sin(\frac{\pi}{10})$ (cartesian)

(g) $\ln(j) = \ln(e^{j\frac{\pi}{2}}) = j\frac{\pi}{2}$ (cartesian)
$= \frac{\pi}{2}e^{j\frac{\pi}{2}}$ (polar)

(h) $j^j = \left(e^{j\frac{\pi}{2}}\right)^j = e^{j^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}}$ (cartesian)
$= e^{-\frac{\pi}{2}}e^{j0}$ (polar)

(i) Note: $\tan x = -j\left(\frac{e^{2jx}-1}{e^{2jx}+1}\right)$
$\tan\left(\frac{1+j}{1-j}\right) = \tan j = -j\left(\frac{e^{-2}-1}{e^{-2}+1}\right) = j\left(\frac{1-e^{-2}}{1+e^{-2}}\right) = j\tanh(1) = j0.7616$ (cartesian)
$\tanh(1)e^{j\frac{\pi}{2}} = 0.7616e^{j\frac{\pi}{2}}$ (polar)

4. (5 pts)

(a) For real numbers a and b, find a relationship between a and b so that the complex number $a+jb$ lies on a unit-radius circle in the complex plane centered at the origin (the *unit circle*).
Solution: $\sqrt{a^2+b^2} = 1$

(b) Describe the conditions on the real numbers A and φ such that the complex number $Ae^{j\varphi}$ lies on the unit circle in the complex plane.

Solution: $|A| = 1$

5. (10 pts) Derive the identities

$$\begin{aligned}
\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\
\sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b)
\end{aligned}$$

using Euler's identity and the basic rule of exponents

$$e^{j(a+b)} = e^{ja}e^{jb}.$$

Solution: Using $\cos\theta = (e^{j\theta} + e^{-j\theta})/2$,

$$\begin{aligned}
\cos(a+b) &= \frac{e^{j(a+b)} + e^{-j(a+b)}}{2} \\
&= \frac{1}{2}(e^{ja} \cdot e^{jb} + e^{-ja} \cdot e^{-jb}) \\
&= \frac{1}{2}\{(\cos(a) + j\sin(a))(\cos(b) + j\sin(b)) + (\cos(a) - j\sin(a))(\cos(b) - j\sin(b))\} \\
&= \frac{1}{2}(2\cos(a)\cos(b) - 2\sin(a)\sin(b)) \\
&= \cos(a)\cos(b) - \sin(a)\sin(b)
\end{aligned}$$

Similarly, using $\sin\theta = (e^{j\theta} - e^{-j\theta})/(j2)$,

$$\begin{aligned}
\sin(a+b) &= \frac{e^{j(a+b)} - e^{-j(a+b)}}{j2} \\
&= \frac{1}{j2}(e^{ja} \cdot e^{jb} - e^{-ja} \cdot e^{-jb}) \\
&= \frac{1}{j2}\{(\cos(a) + j\sin(a))(\cos(b) + j\sin(b)) - (\cos(a) - j\sin(a))(\cos(b) - j\sin(b))\} \\
&= \frac{1}{j2}(j2\sin(a)\cos(b) + j2\cos(a)\sin(b)) \\
&= \sin(a)\cos(b) + \cos(a)\sin(b)
\end{aligned}$$

Alternatively, we can solve the two simultaneously,

$$\begin{aligned}
e^{j(a+b)} &= e^{ja}e^{jb} \\
\cos(a+b) + j\sin(a+b) &= (\cos(a) + j\sin(a))(\cos(b) + j\sin(b)) \\
\cos(a+b) + j\sin(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) + j(\sin(a)\cos(b) + \cos(a)\sin(b))
\end{aligned}$$

We then see that taking the real part of each side of the final equation proves the $\cos(a+b)$ identity, and the imaginary part proves the $\sin(a+b)$ identity.

6. (10 pts) Find the roots of the following polynomials ($j = \sqrt{-1}$, a and b are real):

$$\begin{array}{lll} (a)\ x^2+2x+1 & (d)\ x^2+9 & (g)\ x^3-3x^2+7x \\ (b)\ 6x^2+5x+1 & (e)\ jx^2+jx+j & (h)\ x^3+4x^2+3x+12 \\ (c)\ 5x^2-2x+1 & (f)\ ax^2+bx+j & (i)\ x^3+6x^2+11x+6 \end{array}$$

Solution:

(a) $x^2 + 2x + 1 = (x+1)^2$.
$x = -1$ (double root)

(b) $6x^2 + 5x + 1 = (2x+1)(3x+1)$.
$x = -\frac{1}{2}, -\frac{1}{3}$

(c) $5x^2 - 2x + 1$.
$x = \frac{1 \pm j2}{5}$

(d) $x^2 = -9$.
$x = \pm j3$

(e) $jx^2 + jx + j = j(x^2 + x + 1) = 0$.
$\Rightarrow x^2 + x + 1 = 0$.
$x = \frac{-1 \pm j\sqrt{3}}{2}$

(f) $ax^2 + bx + j$.
$x = \frac{-b \pm \sqrt{b^2 - j4a}}{2a}$

(g) $x^3 - 3x^2 + 7x = x(x^2 - 3x + 7)$.
$x = 0, \frac{3 \pm j\sqrt{19}}{2}$

(h) $x^3 + 4x^2 + 3x + 12 = x^2(x+4) + 3(x+4) = (x^2+3)(x+4)$.
$x = -4, \pm j\sqrt{3}$

(i) $x^3 + 6x^2 + 11x + 6 = (x+1)(x^2+5x+6) = (x+1)(x+2)(x+3)$.
$x = -1, -2, -3$

Lab Assignments

1. For this assignment, the lab is simply to get comfortable with Matlab. Spend some time using the help function (`>> help functionName`) on each of the functions listed below. You should code an example using each of the operators/functions. There is nothing to be turned in for this Lab.

   (a) **operators**: `* .* + - / ./ ' .' : ; ^ .^`

   (b) **math constants**: `1i, 1j, pi, exp(1)`

(c) **simple math functions**: `angle, conj, abs, real, imag, min, max, sum , exp, log, log10, sin, cos, tan, asin, acos, atan, sqrt`

(d) **math concepts**: vector/matrix vs. scalar operators, creating vectors and matrices

(e) **generators**: `ones, zeros, eye, rand, randn, linspace`

(f) **plotting**: `plot, figure, subplot, xlabel, ylabel, title, legend, grid, axis, hold`

(g) **audio functions**: `wavread, wavwrite, sound, soundsc`

(h) **general programming concepts**: functions, plotting, command line vs. scripting vs. functions, control statements (loops and conditional statements using `==, =, <, >, <=, >=`)

(i) **other useful commands**: `help, clear all, clc, close all, size, length, % (for comments), whos`

(j) **less useful, but come up**: `eps, format, fliplr, flipud, pause`

(k) **storing your work**: `disp, print, saveas, save, diary`