Music 320
Autumn 2010–2011
Homework #4
DFT, Fourier Theorems
140 points
Due in one week (10/21/2010)

Theory Problems

1. (30 points) If $Y(k)$ denotes the kth element of the length N DFT of y, show that:

Solution:

$$\begin{aligned} y &\longleftrightarrow Y \\ \overline{y} &\longleftrightarrow \text{FLIP}(\overline{Y}) \\ \text{FLIP}(y) &\longleftrightarrow \text{FLIP}(Y) \end{aligned}$$

(a) $\text{im}\{y\} = 0 \Longleftrightarrow Y(k) = \overline{Y[N-k]}$ (DFT{real} is *Hermitian*)
Solution:
   i. $(\Longrightarrow)$ $\text{Im}\{y\} = 0 \longrightarrow \ y = \overline{y}$
   Therefore,

$$\begin{aligned} Y &= \text{DFT}(y) = \text{DFT}(\overline{y}) = \text{FLIP}(\overline{Y}) \\ &= \overline{Y[N-k]} \end{aligned}$$

   ii. $(\Longleftarrow)$

$$\begin{aligned} Y[k] &= \overline{Y[N-k]} = \text{FLIP}(\overline{Y}) \\ \Longrightarrow y &= \overline{y} \\ \Longrightarrow y - \overline{y} &= 2\text{Im}\{y\} = 0 \end{aligned}$$

(b) $\text{re}\{y\} = 0 \Longleftrightarrow Y(k) = -\overline{Y[N-k]}$ (*anti-Hermitian*)
Solution: $\text{Re}\{y\} = 0 \longrightarrow \ y = -\overline{y}$
Therefore,

$$\begin{aligned} Y &= \text{DFT}(y) = \text{DFT}(-\overline{y}) = -\text{FLIP}(\overline{Y}) \\ &= -\overline{Y[N-k]} \end{aligned}$$

(c) y even $\Longleftrightarrow Y$ even

Solution: y even $\longrightarrow y[n] = y[-n] = \text{FLIP}(y)$
Therefore,

$$\begin{aligned} Y &= \text{DFT}(y) = \text{DFT}(\text{FLIP}(y)) \\ &= \text{FLIP}(Y) = Y[-k] \text{ (even)} \end{aligned}$$

We can also prove this explicitly with the DFT definition:

$$\begin{aligned} Y[k] &= \sum_{n=0}^{N-1} y[n] e^{\frac{-j2\pi kn}{N}} \\ &= \sum_{n=0}^{N-1} y[-n] e^{\frac{-j2\pi kn}{N}} \quad \text{(Using } y[n] = y[-n]\text{)} \\ &= \sum_{m=0}^{-(N-1)} y[m] e^{\frac{-j2\pi k(-m)}{N}} \quad \text{(Using } m = -n,\ -m = n\text{)} \\ &= \sum_{m=0}^{N-1} y[m] e^{\frac{-j2\pi(-k)m}{N}} \quad \text{(Using periodicity of } y,\ y[N-n] = y[n]\text{)} \\ &= Y[-k] \end{aligned}$$

(d) y odd $\Longleftrightarrow Y$ odd
Solution: y odd $\longrightarrow y[n] = -y[-n] = -\text{FLIP}(y)$
Therefore,

$$\begin{aligned} Y &= \text{DFT}(y) = \text{DFT}(-\text{FLIP}(y)) \\ &= -\text{FLIP}(Y) = -Y[-k] \text{ (odd)} \end{aligned}$$

(e) y real, even $\Longleftrightarrow Y$ real,even
Solution:

i. ($\Longrightarrow$) From (c), Y is even.
Also, from (a) and (c),

$$\begin{aligned} Y[k] &= \overline{Y[-k]} = Y[-k] \\ \Longrightarrow Y - \overline{Y} &= 2\text{Im}\{Y\} = 0 \text{ (Y: real)} \end{aligned}$$

Therefore, Y is real and even.

ii. ($\Longleftarrow$) From (c), y is even.
Also,

$$\begin{aligned} Y(\text{even}) &\Longrightarrow Y[k] = Y[-k] \\ Y(\text{real}) &\Longrightarrow Y[k] = \overline{Y[k]} \end{aligned}$$

So,

$$Y[k] = \overline{Y[-k]} = \overline{Y[N-k]}$$

which, from (a), shows that y is real.
Therefore, y is even and real.

(f) y real, odd $\Longleftrightarrow Y$ imag, odd
Solution: From (d), Y is odd.
Also, from (a) and (d),

$$\begin{aligned} Y[k] &= \overline{Y[-k]} = -Y[-k] \\ \Longrightarrow Y + \overline{Y} &= 2\text{Re}\{Y\} = 0 \text{ (Y: imaginary)} \end{aligned}$$

Therefore, Y is imaginary and odd.

(g) y imag, even $\Longleftrightarrow Y$ imag, even
Solution: From (c), Y is even.
Also, from (b) and (c),

$$\begin{aligned} Y[k] &= -\overline{Y[-k]} = Y[-k] \\ \Longrightarrow Y + \overline{Y} &= 2\text{Re}\{Y\} = 0 \text{ (Y: imaginary)} \end{aligned}$$

Therefore, Y is imaginary and even.

(h) y imag, odd $\Longleftrightarrow Y$ real, odd
Solution: From (d), Y is odd.
Also, from (b) and (d),

$$\begin{aligned} Y[k] &= -\overline{Y[-k]} = -Y[-k] \\ \Longrightarrow Y - \overline{Y} &= 2\text{Im}\{Y\} = 0 \text{ (Y: real)} \end{aligned}$$

Therefore, Y is real and odd.

2. (10 pts) [Hermitian] A spectrum $X[k]$ is said to be *Hermitian* if $X[-k] = \overline{X[k]}$, i.e., its real part is *even* and its imaginary part is *odd*. Given a Hermitian spectrum, determine whether its magnitude and phase are even, odd, or other.

Solution:

$$\begin{aligned} X(k) &\triangleq |X(k)|e^{j\angle X(k)} \\ X(-k) &= \overline{X(k)} \quad \text{(since X(k) is Hermitian)} \\ &= \overline{|X(k)|e^{j\angle X(k)}} \\ &= \overline{|X(k)|}\,\overline{e^{j\angle X(k)}} \\ &= |X(k)|e^{-j\angle X(k)} \\ &= |X(-k)|e^{j\angle X(-k)} \end{aligned}$$

Therefore, $|X(-k)| = |X(k)|$, and $\angle X(-k) = -\angle X(k)$, which proves its magnitude part is *even* and phase part is *odd.*

3. (10 pts) [Fourier Theorems] Show that $\text{DFT}(\text{DFT}(y)) = N \cdot \text{FLIP}(y)$, where $\text{FLIP}_n(y) \triangleq y[-n] = y[N-n]$. What does this say about DFT(DFT(DFT(DFT(y))))?
Solution:

$$\begin{aligned}
\text{DFT}\left(\text{DFT}(y)\right) &= \text{DFT}\left(\sum_{n=0}^{N-1} y(n)e^{-j2\pi kn/N}\right) \\
&= \text{DFT}\left(Y(\omega_k)\right) \\
&= \sum_{n=0}^{N-1} Y(\omega_k)e^{-j2\pi kn/N} \\
&= N\left(\frac{1}{N}\sum_{n=0}^{N-1} Y(\omega_k)e^{j2\pi k(-n)/N}\right) \\
&= Ny(-n) \\
&= N\text{FLIP}_n(y)
\end{aligned}$$

4. (10 pts) [DFT of a pair of impulses]

(a) Find the length 8 DFT of a pair of symmetric unit-amplitude impulses:

$$x(n) = [0, 0, 1, 0, 0, 0, 1, 0]$$

Solution:

$$\begin{aligned}
X(k) &= \sum_{n=0}^{7} x(n)e^{-j2\pi nk/8} \\
&= e^{-j2\pi 2k/8} + e^{-j2\pi 6k/8} \\
&= e^{-j\pi k/2} + e^{-j\pi 3k/2} \\
&= e^{-j\pi k/2} + e^{j\pi k/2} \\
&= \cos\left(\frac{\pi k}{2}\right) - j\sin\left(\frac{\pi k}{2}\right) + \cos\left(\frac{\pi k}{2}\right) + j\sin\left(\frac{\pi k}{2}\right) \\
&= 2\cos\left(\frac{\pi k}{2}\right)
\end{aligned}$$

(b) Find the length 8 DFT of a pair of anti-symmetric unit-amplitude impulses:

$$x(n) = [0, 0, 1, 0, 0, 0, -1, 0]$$

Solution:

$$
\begin{aligned}
X(k) &= \sum_{n=0}^{7} x(n) e^{-j2\pi nk/8} \\
&= e^{-j2\pi 2k/8} + e^{-j2\pi 6k/8} \\
&= e^{-j\pi k/2} - e^{-j\pi 3k/2} \\
&= e^{-j\pi k/2} - e^{j\pi k/2} \\
&= \cos\left(\frac{\pi k}{2}\right) - j\sin\left(\frac{\pi k}{2}\right) - \cos\left(\frac{\pi k}{2}\right) - j\sin\left(\frac{\pi k}{2}\right) \\
&= -2j\sin\left(\frac{\pi k}{2}\right)
\end{aligned}
$$

5. (15 pts) [DFT of a rectanglar pulse] Find the length 8 DFT of the unit-amplitude, zero-centered rectangular pulse of length 5:

$$x = [1, 1, 1, 0, 0, 0, 1, 1];$$

Note that, since x is real and even, its DFT $X(k)$ is also real and even. [Hint: Change the DFT summation order from $[0:7]$ to $[-4:3]$ via the shift theorem, making the nonzero signal appearing from $[-2:2]$].

For reference, here is the analogous answer for the Fourier transform of the unit-amplitude, five-second rectangular pulse centered on time $t = 0$:

$$X_{FT}(\omega) = \int_{-2.5}^{2.5} e^{-j\omega t} dt = 5\,\text{sinc}\left(\frac{5}{2}\omega\right),$$

where $\text{sinc}(x) \triangleq \sin(x)/x$.

Solution:

$$
\begin{aligned}
X(k) &\triangleq \sum_{n=0}^{7} x(n) e^{-j2\pi nk/8} = \sum_{n=-2}^{2} e^{-j2\pi nk/8} = e^{j2\omega_k T} \sum_{n=0}^{4} e^{-jn\omega_k T} \\
&= e^{j2\omega_k T} \frac{1 - e^{-j5\omega_k T}}{1 - e^{-j\omega_k T}} = e^{j2\omega_k T} \frac{e^{-j2.5\omega_k T}\left[e^{j2.5\omega_k T} - e^{-j2.5\omega_k T}\right]}{e^{-j0.5\omega_k T}\left[e^{j0.5\omega_k T} - e^{-j0.5\omega_k T}\right]} = \frac{e^{j2.5\omega_k T} - e^{-j2.5\omega_k T}}{e^{j0.5\omega_k T} - e^{-j0.5\omega_k T}} \\
&= \frac{\sin\left(\frac{5}{2}\omega_k T\right)}{\sin\left(\frac{1}{2}\omega_k T\right)} \triangleq 5\,\text{asinc}_5(\omega_k T)
\end{aligned}
$$

6. (5 pts) [DFT of a rectangular pulse] Find the length 16 DFT of the unit-amplitude, zero-centered rectangular pulse of length 5.

Solution: For even k, the DFT is the same as in the previous problem. For odd k, we have new samples that interpolate the even-k values:

$$X(k) = \text{asinc}_5(\omega_k T)$$

where now $\omega_k T \triangleq 2\pi k/16$.

Lab Assignments NOTE: change of file naming conventions!

For all lab assignments, submit your M-file scripts, functions, and figures in one zip file through coursework[1]. Within coursework, upload the zip file using the Drop Box menu.

The zip file should be named with your initials and the homework number. Within the zip file, please have a single folder named hw4. Within the hw4 folder, you can titled each question as q1, q2, etc. So, for John Doe's zip file, the file should be titled `jd_hw4.zip`. For John Doe's answer to question 2 on homework 4, the file would be titled `hw4/q2.m`. For any required functions, please leave the functions as titled by their function name.

1. (10 pts) [Plotting magnitude spectrum] Write a function that will quickly plot a magnitude spectrum of a signal.

```
function plotspec (x, fs)

% function plotspec (x, fs)
% A function to quickly plot the spectrum of a time domain signal
%
% plotspec(x)
%   when no sampling rate is specified, normalize frequency
%
% plotspec(x, fs)
%   when a sampling rate is given, set the frequency axis in [Hz]
%
% Your Name / Lab 4-1
```

   (a) Use `[dB]` scale for magnitude spectrum.

   (b) If there is no input for `fs`, use normalized frequency [-0.5, 0.5] for the horizontal axis.

   (c) If necessary, use the following structure to support optional arguments.

```
if nargin == 1
    ...
else
    ....
end
```

   (d) Label your plot carefully (using `title`, `xlabel`, and `ylabel`).

[1] http://coursework.stanford.edu

Solution:

```
function plotspec (x, fs)

% function plotspec (x, fs)
% A function to quickly plot the spectrum of a time domain signal
%
% plotspec(x)
%   when no sampling rate is specified, normalize frequency
%
% plotspec(x, fs)
%   when a sampling rate is given, set the frequency axis in [Hz]

N=2^(nextpow2(x));
X=fft(x,N);
X=fftshift(X);

if nargin == 1
fs=1;
    xLabelText = 'Normalized Frequency (-0.5 ~ 0.5)';
else
xLabelText = ['Frequency (-fs/2 ~ fs/2, fs = ', num2str(fs), ' [Hz])'];
end

xi=[-fs/2:fs/N:(fs/2-fs/N)];

figure
plot(xi,20*log10(abs(X))-max(20*log10(abs(X))))
grid
xlabel (xLabelText);
ylabel('Amplitude[dB]')
title('Spectrum')

return
```

2. (40 points) [Spectrum of a "time-slice"] Write a function that plots the magnitude spectrum of an input signal at a specific time t.

```
function plotspec_st (x, fs, time, Nf)

% function plotspec_st (x, fs, time, Nf)
% A function to quickly plot the spectrum of a signal
```

```
% segment centered at a given time
%
% x: input signal (assume a row vector)
% fs: sampling rate of x
% time: the time at which you want to see the spectrum
% Nf: frame (or slice) size
%
% Your Name / Lab ##
```

Remember to do the followings:

(a) Apply a hann window (same size) to the sliced short signal.

(b) Zero-pad your windowed signal with zpf of 8.

(c) Plot the magnitude spectra of the windowed, zero-padded signal using your `plotspec` function.

Solution:

```
function X=plotspec_st(x, fs, time, Nf)

% function plotspec_st (x, fs, time, Nf)
% A function to quickly plot the spectrum of a signal
% segment centered at a given time
%
% x: input signal (assume a row vector)
% fs: sampling rate of x
% time: the time at which you want to see the spectrum
% Nf: frame (or slice) size

x_t=x(time*fs+1-Nf/2:time*fs+Nf/2);

zpf=8;
win_handle=@hann;
plotspec2 (x_t, fs, zpf, win_handle)

return

function plotspec2 (x, fs, zpf, win_handle)

% function plotspec2 (x, fs, zpf, win_handle)
%
% A function to quickly plot the spectrum of a time domain signal, applying
```

```
% zero padding and windowing
%
% INPUT
% x          = signal in time domain
% fs         = sampling rate in [Hz](if sampling rate is not given, specify
%              fs=1 for normalized frecuency)
% zpf        = zero padding factor
% win_handle = windows type handle, from the list below
%   @bartlett       - Bartlett window.
%   @barthannwin    - Modified Bartlett-Hanning window.
%   @blackman       - Blackman window.
%   @blackmanharris - Minimum 4-term Blackman-Harris window.
%   @bohmanwin      - Bohman window.
%   @chebwin        - Chebyshev window.
%   @flattopwin     - Flat Top window.
%   @gausswin       - Gaussian window.
%   @hamming        - Hamming window.
%   @hann           - Hann window.
%   @kaiser         - Kaiser window.
%   @nuttallwin     - Nuttall defined minimum 4-term Blackman-Harris window.
%   @parzenwin      - Parzen (de la Valle-Poussin) window.
%   @rectwin        - Rectangular window.
%   @tukeywin       - Tukey window.
%   @triang         - Triangular window.

N=length(x);
win = window(win_handle,N)';
x = win .* x;
x=z_pad(x,zpf);

N=2^(nextpow2(x));
X=fft(x,N);
X=fftshift(X);

if fs == 1;
xLabelText = 'Normalized Frequency (-0.5 ~ 0.5)';
else
xLabelText = ['Frequency (-fs/2 ~ fs/2, fs = ', num2str(fs), ' [Hz])'];
end

xi=[-fs/2:fs/N:(fs/2-fs/N)];
```

```
plot(xi,20*log10(abs(X))-max(20*log10(abs(X))))
grid
xlabel (xLabelText);
ylabel('Amplitude[dB]')
title(['Spectrum windowed with ', func2str(win_handle)])

return


function x_zp=z_pad(x,zpf);

N=length(x);
M=N*zpf;
x_zp=[x zeros(1,M)];

return
```

3. (10 points) Use the following Matlab command sequence to generate a chirp signal going up:

```
w1=100; w2=3000; %(Hz)
T=3; %(sec)
fs=8000; %(Hz)
dT=1/fs;
t=(0:dT:T);
up = chirp(t,w1,T,w2);
```

Using your function from the previous problem, plot the spectrum of the `up` signal just generated at 0.1 [sec], 1.5 [sec], and 2.9 [sec]. Submit these three plots in one figure (as a Matlab figure file); in the title of each plot, indicate the time it was taken. Do they verify that your function works as expected?

Solution:

```
clear all
close all

%UP
%----------------------------
w1=100; %(Hz)
w2=3000; %(Hz)
T=3; %(sec)
fs=8000; %(Hz)
```

```
dT=1/fs;
t=(0:dT:T);

up = chirp(t,w1,T,w2);

Nf=256;
subplot(311)
plotspec_st (up, fs, .1, Nf)
title('Spectrum windowed with hann, T=0.1')
subplot(312)
plotspec_st (up, fs, 1.5, Nf)
title('Spectrum windowed with hann, T=1.5')
subplot(313)
plotspec_st (up, fs, 2.9, Nf)
title('Spectrum windowed with hann, T=2.9')
```