Music 320
Autumn 2010–2011
Homework #2
Sinusoids, Complex Sinusoids
135 points
Due in one week (10/7/2010)

Theory Problems

1. (15 pts) The phase of a sinusoid can be related to time shift as follows:

$$x(t) = A\cos(2\pi f_0 t + \phi) = A\cos(2\pi f_0 (t - t_1))$$

In the following parts, assume that the period of the sinusoidal wave is $T_0 = 8$ sec

(a) When $t_1 = -2$ sec, the value of the phase is $\phi = \pi/2$. Explain whether this is true or false.
Solution:

$$\begin{aligned} T_0 &= 8 \text{ sec} \\ f_0 &= \frac{1}{8} \text{ Hz} \\ \phi &= -2\pi f_0 t_1 \end{aligned}$$

$$\begin{aligned} \phi &= -2\pi \frac{1}{8}(-2) \\ \phi &= -\frac{\pi}{4}(-2) \\ &= \frac{\pi}{2} \text{ (true)} \end{aligned}$$

(b) When $t_1 = 3$ sec, the value of the phase is $\phi = 3\pi/4$. Explain whether this is true or false.
Solution:

$$\begin{aligned} \phi &= -2\pi \frac{1}{8}(3) \\ &= -\frac{\pi}{4}(3) \\ &= -\frac{3\pi}{4} \text{ (false)} \end{aligned}$$

(c) When $t_1 = 7$ sec, the value of the phase is $\phi = \pi/4$. Explain whether this is true or false.
Solution:

$$\begin{aligned}
\phi &= -2\pi\frac{1}{8}(7) \\
&= -\frac{\pi}{4}(7) \\
&= -\frac{7\pi}{4} \\
A\cos(2\pi f_0 t + \phi) &= A\cos(2\pi f_0 t - \frac{7\pi}{4}) \\
&= A\cos\left(2\pi f_0 t - 2\pi + \frac{\pi}{4}\right) \\
&= A\cos\left(2\pi f_0 t + \frac{\pi}{4}\right) \text{ (true)}
\end{aligned}$$

2. (15 pts) [Sinusoids]

(a) For a sinusoid with a period $T_0 = 8.0$ seconds, what is the frequency f_0 in Hz?

(b) Define $x(t)$ as

$$x(t) = A\sin(w(t - \tau))$$

Write an expression for the phase in terms of the frequency and time delay.

(c) For $x(t)$ defined as above, find the phase at $t = 0$ for a time delay of $\tau = .25$ seconds and frequencies of 2 Hz and 3 Hz.

Solution:

(a) For a sinusoid with a period $T_0 = 8.0$ seconds, the frequency $f_0 = 1/8$ Hz.

(b) Using the standard notation from https://ccrma.stanford.edu/ jos/mdft/Sinusoids.html,

$$x(t) = A\sin(w(t - \tau))$$
$$x(t) = A\sin(wt - w\tau))$$

Therefore, the phase $\phi = -w\tau$ rads.

(c) For f = 2 Hz,

$$x(t) = A\sin((2)(2)(0)\pi - 2 * \pi(2)(.25)) \tag{1}$$

Therefore, the phase $\phi = -\pi$ rads.
For f = 3 Hz,

$$x(t) = A\sin((2)(2)(0)\pi - 2 * \pi(3)(.25))$$

Therefore, the phase $\phi = -\frac{3\pi}{2}$ rads.

3. (35 pts) [Complex Sinusoids] Define the discrete-time generalized sinusoid $x[n] = Xz_0^n$ for $n = 0, 1, 2, \ldots$ where

$$X = 2e^{j\pi/4}$$

$$z_0 = 0.9e^{j\pi/8}$$

(a) (5 pts) What is the amplitude of this sinusoid? What is the phase in radians? What is the phase in cycles? What is the phase in degrees?

$$\begin{aligned} x(n) &= Xz_0^n = 2e^{j\pi/4}(0.9e^{j\pi/8})^n \\ &= 2e^{j\pi/4}(0.9)^n e^{jn\pi/8} \\ &= 2(0.9)^n e^{j(n\pi/8+\pi/4)} \end{aligned}$$

Amplitude (either is fine):

- Peak Amplitude = 2
- Amplitude Envelope = $2(.9)^n$

Phase (either is fine):

- Instantaneous Phase: $\frac{n\pi}{8} + \frac{\pi}{4}$ rad $= \frac{n}{16} + \frac{1}{8}$ cycles $= \frac{45n}{2} + 45$ degrees
- Initial: $\frac{\pi}{4}$ radians $= \frac{1}{8}$ cycles $= 45$ degrees

(b) (5 pts) What is the time constant τ of decay (in samples)?
Setting $(0.9)^n = e^{-n/\tau}$ gives $0.9 = e^{-1/\tau}$

$$\begin{aligned} -1/\tau &= \ln(0.9) \\ \tau &= -\frac{1}{\ln(0.9)} \\ &\approx 9.49 \text{ (samples)} \end{aligned}$$

(c) (5 pts) What is the 60 dB decay time T_{60} in time constants?

$$\begin{aligned} \frac{e^{-T_{60}/\tau}}{e^{-0/\tau}} &= 10^{-3} \\ \frac{-T_{60}}{\tau} &= \ln(10^{-3}) \\ T_{60} &= 3\ln(10)\cdot\tau \\ &\approx 6.91\tau \end{aligned}$$

(d) (5 pts) What is T_{60} in samples?

$$\begin{aligned} T_{60} &\approx 6.91\tau \\ &\approx 6.91 \cdot 9.49 \\ &\approx 65.58 \text{ (samples)} \end{aligned}$$

(e) (5 pts) What is the 80 dB decay time T_{80} in time constants?

$$\begin{aligned} \frac{e^{-T_{80}/\tau}}{e^{-0/\tau}} &= 10^{-4} \\ \frac{-T_{80}}{\tau} &= \ln(10^{-4}) \\ T_{60} &= 4\ln(10) \cdot \tau \\ &\approx 9.21\tau \end{aligned}$$

(f) (5 pts) What is T_{80} in samples?

$$\begin{aligned} T_{80} &\approx 9.21\tau \\ &\approx 9.21 \cdot 9.49 \\ &\approx 87.4 \text{ (samples)} \end{aligned}$$

(g) (5 pts) If the sampling rate is 800 Hz, what are τ and T_{60} in seconds, and what is the frequency of the sinusoid in Hz?
Since 1 (sample) $= T = 1/f_s = 1/800$ (seconds),

$$\begin{aligned} \tau &\approx 9.49/800 \approx 0.0119 \text{ (seconds)} \\ T_{60} &\approx 65.58/800 \approx 0.082 \text{ (seconds)} \end{aligned}$$

$$\begin{aligned} \omega n T &= 2\pi f n T = n\pi/8 \\ f &= 1/(16T) \\ &= f_s/16 \\ &= 50 \text{ (Hz)} \end{aligned}$$

Lab Assignments

For all lab assignments, submit your M-file scripts, functions, and figures in one zip file through coursework[1]. Within coursework, upload the zip file using the Drop Box menu.

The zip file should be named with your last name, first name and homework number. Each problem should be named with your last name, first name, homework number, and the problem number. So, for John Doe's zip file, the file should be titled `doe_john_hw2.zip`. For John Doe's answer to problem 2 on homework 2, the file would be titled `doe_john_hw2_q2.m`. Also, at the beginning of each script, include the following comment:

```
% Your Name / Lab # - Question #
```

For problems with question(s), include your answer(s) in the body of the script files as comments.

1. (20 pts) Define the discrete-time generalized sinusoid $x(n) = Xz_0^n$ for $n = 0, 1, 2, ...,$ where

$$X = 2e^{j\pi/4}$$

$$z_0 = 0.9e^{j\pi/8}$$

   (a) Plot $\mathrm{re}\{Xz_0^n\}$ and $\mathrm{im}\{Xz_0^n\}$ versus n.
   (b) Plot Xz_0^n as a collection of points in the complex plane (imaginary part versus real part).
   (c) Mark the time constant τ of decay on the plots.
   (d) Mark the 60 dB decay time T_{60} on the plots.

   Solution:

```
clear all
close all

X=2*exp(j*pi/4);
zo=0.9*exp(j*pi/8);
x=X*zo.^[0:99];
```

[1] http://coursework.stanford.edu

```
tau=-1/log(0.9);
T60=3*log(10)*tau;

figure;plot([0:99],real(x));grid;
xlabel('samples')
ylabel('amplitude')
title('Real[X zo^n]')
text(tau,real(X*zo.^tau),...
    '\bullet\leftarrow\fontname{times}\tau','FontSize',12)
text(T60,real(X*zo.^T60),...
    '\bullet\leftarrow\fontname{times}T_{60}','FontSize',12)

figure;plot([0:99],imag(x));grid;
xlabel('samples')
ylabel('amplitude')
title('Im[X zo^n]')
text(tau,imag(X*zo.^tau),...
    '\bullet\leftarrow\fontname{times}\tau','FontSize',12)
text(T60,imag(X*zo.^T60),...
    '\bullet\leftarrow\fontname{times}T_{60}','FontSize',12)

figure;plot(real(x),imag(x),'.');grid;
xlabel('Real[X zo^n]')
ylabel('Im[X zo^n]')
text(real(X*zo.^tau),imag(X*zo.^tau),...
    '\bullet\leftarrow\fontname{times}\tau','FontSize',12)
text(real(X*zo.^T60),imag(X*zo.^T60),...
    '\bullet\leftarrow\fontname{times}T_{60}','FontSize',12)
```

2. (25 pts) Additive synthesis, the sum of cosine waves, can be given by

$$y(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

where A_k, f_k, and ϕ_k mean the peak amplitude, frequency, and initial phase of kth sinusoidal component. Also, N is the number of frequency and amplitude components. Write a Matlab function that implements this synthesis method and saves the result as an audio file. The syntax of your function should be:

```
function y = additive(f, Z, fs, dur, name)

% function y = additive(f, Z, fs, dur, name)
% f: vector of frequencies in Hz
```

```
% Z: vector of complex amplitudes A*exp(j*phi)
% fs: sampling rate in Hz
% dur: total duration of the signal in seconds
% name: name of the output audio file
% f and Z must be of the same length:
% Z(1) corresponds to f(1) and so on.
% Your Name / Lab 2-2
```

Remember:

(a) Your function must be able to take any length of `f` and `Z`, as long as they are of the same length.

(b) Note that `Z` is a vector of complex amplitudes (that is, phasors), not real numbers.

(c) Try to make it run as fast as possible: can you implement this without using any loop in your code?

(Hints: Obtain the real sinusoid by taking the real part of a complex sinusoid. For a fast implementation, think about vector and matrix multiplication instead of loops)

Solution:

```
function y = additive(f, Z, fs, dur, name)

% function y = additive(f, Z, fs, dur, name)
% f: vector of frequencies in Hz
% Z: vector of complex amplitudes A*exp(j*phi)
% fs: sampling rate in Hz
% dur: total duration of the signal in seconds
% name: name of the output audio file
% f and Z must be of the same length:
% Z(1) corresponds to f(1) and so on.

t = 0:1/fs:dur;
zoz1 =  Z * exp(j*2*pi*f'*t); % assuming f,Z are row vectors
y=real(zoz1);

y = y/max(abs(y))*0.99;       % scaling to avoid clipping
wavwrite(y,fs,name);

return
```

3. (25 pts) Your task for this problem is to write a Matlab function to test the beating effect when adding two sinusoids. In this case, one of the sinusoidal signals will have a

fixed frequency f_c, while the other is a "chirp"—that is, its frequency sweeps linearly from $f_c + f_1$ to $f_c + f_2$. The sum can be expressed as

$$x(t) = A\cos(2\pi f_c t) + B\cos(2\pi(f_c + f_\Delta(t))t).$$

where f_Δ sweeps linearly from frequency f_1 to f_2. We will keep f_1 and f_2 much smaller than f_c, so that both sinusoids are always close together in frequency.

There are generally four separate sounding cases when considering the addition of two sinusoids:

(a) The two sinusoids are of the same frequency and sound identical

(b) The two frequencies are very close and beating occurs

(c) The two frequencies are create a rough sounding tone

(d) The two frequencies are far apart from one another and sound as two distinct tones

Write a Matlab function that implements this synthesis method and saves the result as an audio file. The syntax of your function should be:

```
function y_beat = beat_sweep(A, B, fc, f_delt, fs, dur, name)

% function y_beat = beat_sweep(A, B, fc, f_delt, fs, dur, name)
%
% A: amplitude of the center frequency cosine
% B: amplitude of the sweeping frequency cosine
% fc: center frequency in Hz
% f_delt: [f1 f2] vector for frequency sweeping
% fs: sampling rate in Hz
% dur: total duration of the signal in seconds
% name: name of the output audio file
% Your Name / Lab#-Q#
```

Test your function with f_c= 100Hz, 300Hz, 1000Hz and 3000Hz. For each of these frequencies,

(a) With $f_\Delta = [0, f_2]$, find the value of f_2 that is approximately just after the roughness zone.

(b) With $f_\Delta = [f_1, f_1]$, select f_1 so as you are hearing the transition from beats to roughness. Plot a few periods of your signal that let you clearly see these beats (or submit a script that creates the plot using your function).

Comment on your results.

Solution:

```
function y_beat = beat_sweep(A, B, fc, f_delt, fs, dur, name)
% function y_beat = beat_sweep(A, B, fc, f_delt, fs, dur, name)
%
% A: amplitude of the center frequency cosine
% B: amplitude of the sweeping frequency cosine
% fc: center frequency in Hz
% f_delt: [f1 f2] vector for frequency sweeping
% fs: sampling rate in Hz
% dur: total duration of the signal in seconds
% name: name of the output audio file
% Your Name / Lab#-Q#

dT=1/fs;
t=(0:dT:dur);

y_beat = A*cos(2*pi*fc*t) +...
    B*chirp(t, fc+f_delt(1), dur, fc+f_delt(2));

wavwrite(y_beat,fs,name);
return
```

OR

```
function y_beat = beat_sweep(A, B, fc, f_delt, fs, dur, name)
% A: amplitude of the center frquency cosine
% B: aplitude of the sweeping frequency cosine
% fc: center frequency in Hz
% f_delt: [f1 f2] vector for frequency sweeping
% fs: sampling rate in Hz
% dur: total duration of the signal in seconds
% name: name of the output audio file

% Sweep over time and frequency
t = (0:1/fs:(dur - 1/fs))';
f = linspace(f_delt(1), f_delt(2), length(t))';

%Get the instantaneous phase
phi = cumsum(2*pi*(f + fc))/fs;

%Compute the final mix
y_beat = A*cos(2*pi*fc*t) + B*cos(phi);

wavwrite(y_beat, fs, name);
```