Music 320
Autumn 2010–2011
Homework #7
LTI, Filters
140 points
Due in one week (11/12/2010)

Theory Problems

1. (30 points) [Linearity and Time-Invariance] Show the linearity and/or time-invariance (or not) of each filter given below.

   (a) $y(n) = x(n) + x(n-1)$ **Solution:** Let $x(n) = \alpha x_1(n) + \beta x_2(n)$ then

$$\begin{aligned} y(n) &= \alpha x_1(n) + \beta x_2(n) + \alpha x_1(n-1) + \beta x_2(n-1) \\ &= \alpha[x_1(n) + x_1(n-1)] + \beta[x_2(n) + x_2(n-1)] \\ &= \alpha y_1(n) + \beta y_2(n) \end{aligned}$$

   For a delayed input by m samples $x(n-m)$,

$$x(n-m) + x(n-m-1) = y(n-m)$$

   Therefore, the filter $y(n) = x(n) + x(n-1)$ is linear and time-invariant.

   (b) $y(n) = 2x(n) - x(n-2)$ **Solution:** Let $x(n) = \alpha x_1(n) + \beta x_2(n)$ then

$$\begin{aligned} y(n) &= 2[\alpha x_1(n) + \beta x_2(n)] - [\alpha x_1(n-2) + \beta x_2(n-2)) \\ &= \alpha[2x_1(n) - x_1(n-2)] + \beta[2x_2(n) - x_2(n-2)] \\ &= \alpha y_1(n) + \beta y_2(n) \end{aligned}$$

   For a delayed input by m samples $x(n-m)$,

$$2x(n-m) - x(n-m-2) = y(n-m)$$

   Therefore, the filter $y(n) = 2x(n) - x(n-2)$ is linear and time-invariant.

   (c) $y(n) = 3\sqrt{x(n)}$ **Solution:** Let $x(n) = \alpha x_1(n) + \beta x_2(n)$ then

$$\begin{aligned} y(n) &= 3\sqrt{\alpha x_1(n) + \beta x_2(n)} \\ &\neq \alpha y_1(n) + \beta y_2(n) = 3\alpha\sqrt{x_1(n)} + 3\beta\sqrt{x_2(n)} \end{aligned}$$

   For a delayed input by m samples $x(n-m)$,

$$3\sqrt{x(n-m)} = y(n-m)$$

   Therefore, $y(n) = 3\sqrt{x(n)}$ is nonlinear and time-invariant.

(d) $y(n) = x(n) - 0.5nx(n-1)$ **Solution:** Let $x(n) = \alpha x_1(n) + \beta x_2(n)$ then

$$\begin{aligned} y(n) &= \alpha x_1(n) + \beta x_2(n) - 0.5[\alpha n x_1(n-1) + \beta n x_2(n-1)] \\ &= \alpha[x_1(n) - 0.5nx_1(n-1)] + \beta[x_2(n) - 0.5nx_2(n-1)] \\ &= \alpha y_1(n) + \beta y_2(n) \end{aligned}$$

For a delayed input by m samples $x(n-m)$,

$$x(n-m) - 0.5nx(n-m-1) \neq y(n-m) = x(n-m) - 0.5(n-m)x(n-m-1)$$

Therefore, the filter $y(n) = x(n) - 0.5nx(n-1)$ is linear and time-varying.

(e) $y(n) = x(n) - y(n-1)$ **Solution:** For an input signal $x(n)$ starting at time zero, $y(0) = x(0)$ and thus is linear. Let's assume $y(k)$ is linear. Then $y(k+1) = x(k+1) - y(k)$ is linear because linear combinations of linear terms is linear. Therefore, $y(n) = x(n) - y(n-1)$ is linear for all n.
If the input signal is delayed to start at time m, we obtain $y(n) = 0$ for $n < m$, followed by

$$\begin{aligned} y(m) &= x(0) = y(0) \\ y(m+1) &= x(1) - y(m) = x(1) - x(0) = y(1) \\ y(m+2) &= x(2) - y(m+1) = x(2) - x(1) + x(0) = y(2) \\ &\cdots \end{aligned}$$

and so on. Therefore, the filter $y(n) = x(n) - y(n-1)$ is linear and time-invariant.

(f) $y(n) = \frac{x^2(n)}{2n}$ **Solution:** Let $x(n) = \alpha x_1(n) + \beta x_2(n)$ then

$$\begin{aligned} y(n) &= \frac{[\alpha x_1(n) + \beta x_2(n)]^2}{2n} \\ &= \frac{[\alpha^2 x_1^2(n) + 2\alpha\beta x_1(n)x_2(n) + \beta^2 x_2^2(n)]}{2n} \\ &\neq \alpha y_1(n) + \beta y_2(n) = \frac{\alpha x_1^2(n)}{2n} + \frac{\beta x_2^2(n)}{2n} \end{aligned}$$

For a delayed input by m samples $x(n-m)$,

$$\frac{x^2(n-m)}{2n} \neq y(n-m) = \frac{x^2(n-m)}{2(n-m)}$$

Therefore, the filter $y(n) = \frac{x^2(n)}{2n}$ is nonlinear and time-varying.

2. (15 points) [System Diagram]. Draw both Direct Form I and Direct Form II signal flow graphs for the filters determined by the difference equations below:

(a) (Allpole IIR case)

$$y(n) = \frac{1}{2}x(n) + \frac{1}{3}y(n-1) + \frac{1}{4}y(n-2)$$

Solution:

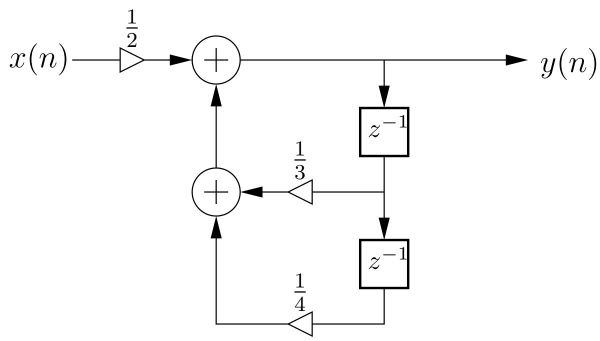

Figure 1: Direct Form I

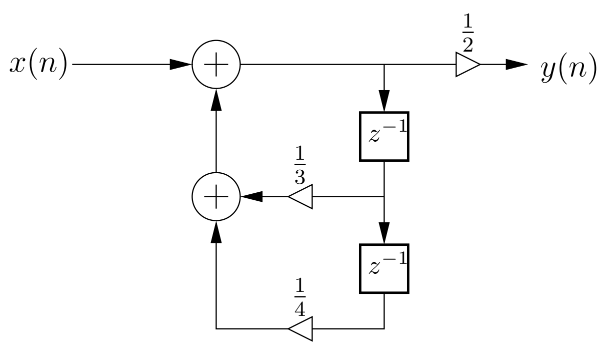

Figure 2: Direct Form II

(b) (Pole-zero IIR case)

$$y(n) = x(n) + 0.5x(n-1) - 0.8y(n-1) - 0.5y(n-2)$$

Solution:

(c) What is the *order* of each of the above two filters?
Solution: Both filters are second-order filters.

3. (20 points) [IIR Filters] Consider the filter

$$y(n) = 0.5x(n) + 0.5x(n-1) + 0.8y(n-1)$$

Find and sketch the first 10 samples of

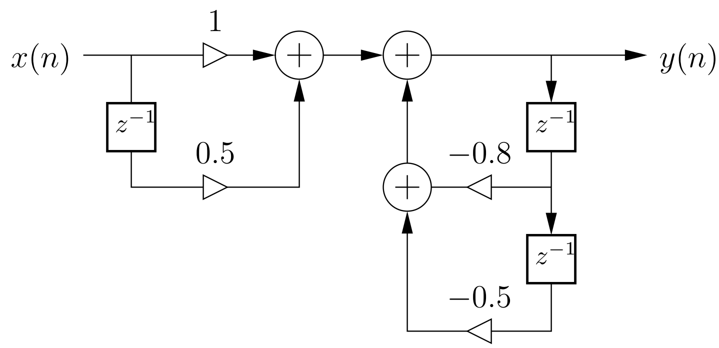

Figure 3: Direct Form I

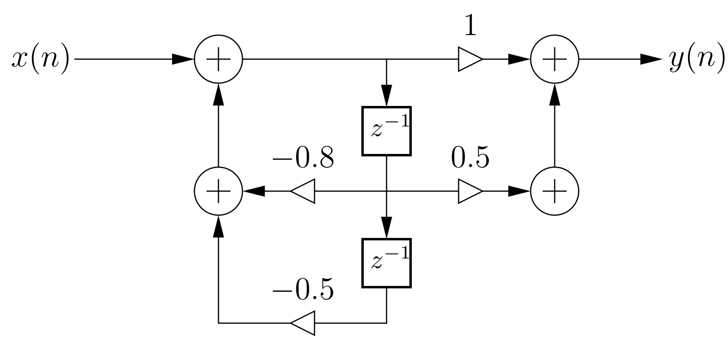

Figure 4: Direct Form II

(a) the impulse response.
Solution: $h(n) = [0.5, 0.9, 0.72, 0.576, 0.4608, 0.3686, 0.2949, 0.236, 0.1887, 0.151]$

(b) the output when the input is given by

$$x(n) = [1, -1, 1, 1, -1, -1, 0, 0, 0, 0]$$

Solution: $y(n) = [0.5, 0.4, 0.32, 1.256, 1.0048, -0.1962, -0.657, -0.5255, -0.4204, -0.3363]$

4. (35 points) [Frequency Response Analysis] For the filter defined by the difference equation

$$y(n) = 0.5x(n) + 0.5x(n-1) - 0.5y(n-1)$$

(a) (5 points) Find the transfer function.
Solution: Taking the z-transform on both sides of the equation yields

$$Y(z) = 0.5X(z) + 0.5z^{-1}X(z) - 0.5z^{-1}Y(z)$$

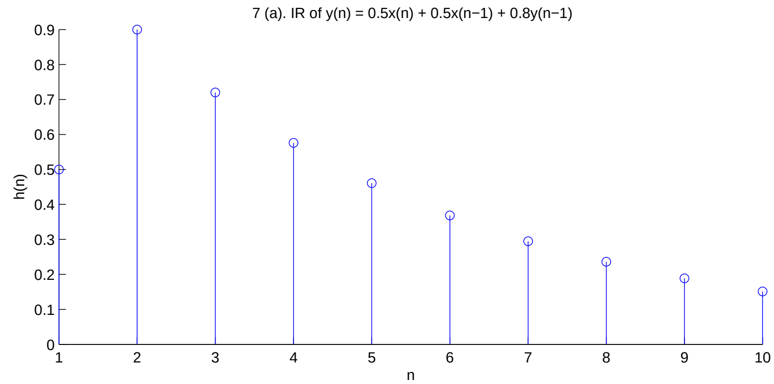

Figure 5: Impulse response

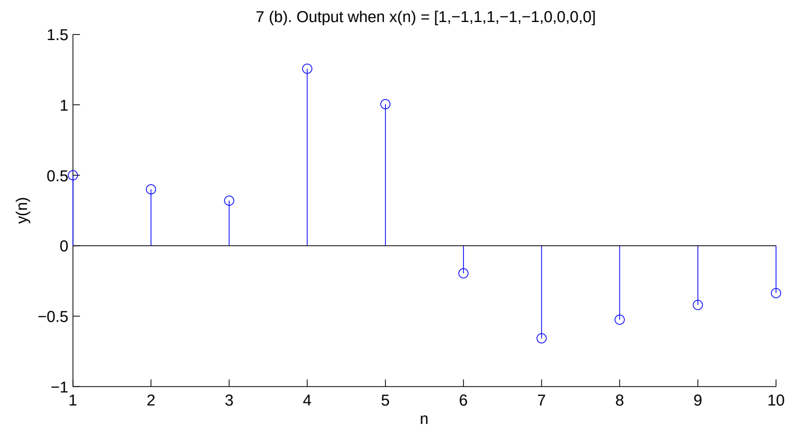

Figure 6: Output with given input

Solving for the ratio $Y(z)/X(z) = H(z)$ yields

$$H(z) = \frac{0.5 + 0.5z^{-1}}{1 + 0.5z^{-1}}$$

(b) (15 points) Find and sketch the amplitude response. Give specific values at $\omega T = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, and π.

Solution:

$$\begin{aligned} G(\omega) &\triangleq |H(e^{j\omega T})| \\ &= \frac{|0.5 + 0.5e^{-j\omega T}|}{|1 + 0.5e^{-j\omega T}|} \\ &= \frac{|0.5 + 0.5\cos(\omega T) - j0.5\sin(\omega T)|}{|1 + 0.5\cos(\omega T) - j0.5\sin(\omega T)|} \\ &= \sqrt{\frac{0.5(1 + \cos(\omega T))}{1.25 + \cos(\omega T)}} \end{aligned}$$

$G(0) \approx 0.67, G(\pi/(4T)) \approx 0.66, G(\pi/(2T)) \approx 0.63, G(3\pi/(4T)) \approx 0.52$, and $G(\pi/T) = 0$

(c) (15 points) Find and sketch the phase response. Give specific values at $\omega T = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, and π.
Solution:

$$\begin{aligned}
\Theta(\omega) &\triangleq \angle H(e^{j\omega T}) \\
&= \angle\left\{0.5 + 0.5e^{-j\omega T}\right\} - \angle\left\{1 + 0.5e^{-j\omega T}\right\} \\
&= \angle\left\{0.5 + 0.5\cos(\omega T) - j0.5\sin(\omega T)\right\} - \angle\left\{1 + 0.5\cos(\omega T) - j0.5\sin(\omega T)\right\} \\
&= \arctan\left\{\frac{\sin(\omega T)}{2 + \cos(\omega T)}\right\} - \arctan\left\{\frac{\sin(\omega T)}{1 + \cos(\omega T)}\right\}
\end{aligned}$$

$\Theta(0) = 0, \Theta(\pi/(4T)) \approx -0.137, \Theta(\pi/(4T)) \approx -0.322, \Theta(\pi/(4T)) \approx -0.677$, and $\Theta(\pi/T) = -\pi/2$

Lab Assignments

Follow the same file naming convention of the previous lab.

```
% Your Name / Lab #-Question #
```

For problems with question(s), include your answer(s) in the body of the script files as comments.

1. (20 points) Verify the *linearity* of the filter in problem 1(a) by the following steps:
   (a) Generate 60 samples of the sinusoid $x_0(n) = 0.5\cos(0.2\pi n)$
   (b) Zero-pad to length $N = 100$, forming the signal $x_1(n)$ using the Matlab statement "`x1 = [x0(:)',zeros(1,40)]`" or equivalent.
   (c) Using Matlab's `filter` function, filter the input signal $x_1(n)$ to obtain $y_1(n)$.
   (d) Plot x_1 in `subplot(2,1,1)` and y_1 in `subplot(2,1,2)`.
   (e) What kind of convolution has been performed (cyclic or acyclic)?
   (f) Would all output samples have been returned if we did not zero pad?

(g) Filter the input signal $x_2(n) = 2\cos(0.1\pi n)$ to obtain $y_2(n)$.

(h) Filter the input signal $x_3(n) = 0.5x_1(n) + 2x_2(n)$ to obtain $y_3(n)$.

(i) Compare $y_3(n)$ to $0.5y_1(n) + 2y_2(n)$ by plotting both signals on the same plot using `hold on` command in Matlab.

Turn in your Matlab code and answers.

Solution:

```
clear all
close all

x0=0.5*cos(0.2*pi*[0:59]);
x1=[x0 zeros(1,40)];
y1 = filter([1 1],1,x1);

figure
subplot(211)
plot([0:99],x1)
axis([0 100 -1 1])
grid
ylabel('amplitude')
title('x1')
subplot(212)
plot([0:99],y1)
grid
title('y1 (filtered fignal)')
xlabel('samples (n)')
ylabel('amplitude')

% ANSWER TO QUESTION
% ----------------------------
% What kind of convolution has been performed?
% Since we are applying the Difference Equation for computing the Output,
% this is equivalent to apply an A-Cyclic convolution.

% Would all output samples have been returned if we did not zero pad?
% No, since the filter is order 1, if we didn't have zero paded, we would
% have missed the last sampler of the output.


x2=[2*cos(0.1*pi*[0:59]) zeros(1,40)];
y2 = filter([1 1],[1 0],x2);
```

```
x3=0.5*x1+2*x2;
y3 = filter([1 1],[1 0],x3);

figure
plot([0:99],y3,'LineWidth',4)
hold
plot([0:99],0.5*y1+2*y2,'Color','y','LineWidth',2)
title('y3 (filtered fignal) BLUE and y3=0.5*y1+2*y2 YELLOW')
xlabel('samples (n)')
ylabel('amplitude')
```

2. (20 points) Verify the *time-invariance* of the filter in problem 1(b) using the following steps:

   (a) Filter the input signal $x(n) = \cos(0.25\pi n)$ to obtain $y(n)$.

   (b) Time shift (delay) the original input signal $x(n)$ by 3 samples to get $x_{s,3}(n) = \text{SHIFT}_3\{x\} = \cos(0.25\pi(n-3))$.

   (c) Filter $x_{s,3}(n)$ to obtain $y_{s,3}(n)$.

   (d) Compare $y_{s,3}(n)$ to $y(n)$ by plotting both signals on the same plot using `hold on` command in Matlab.

   (e) What would you have to do to make them the same?

   Turn in your Matlab code and answers.

   Solution:

```
clear all
close all

x=[cos(0.25*pi*[0:59]) zeros(1,40)];
y = filter([2 0 -1],1,x);

xs3=[cos(0.25*pi*([0:59]-3)) zeros(1,40)];
ys3 = filter([2 0 -1],1,xs3);

figure
plot([0:99],y)
hold
plot([0:99],ys3,'Color','r')
grid
title('y3 (filtered fignal) BLUE and y3=0.5*y1+2*y2 YELLOW')
xlabel('samples (n)')
ylabel('amplitude')
```

```
% ANSWER TO QUESTION
% ----------------------------
% What would you have to do make them the same?
% Since the filter is time invariant, to make them the same we only need to
% Time shift the output by -3 samplers.
```