Mean-Variance Analysis

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Portfolio Allocation (Two Assets)

- Assume an investor with CARA utility and $\tilde{R} \sim \mathcal{N}\left(\mu, \sigma^2\right)$:

$$-e^{-\alpha \tilde{w}} = -e^{-\alpha\left[w_0 R_f + \phi\left(\tilde{R} - R_f\right)\right]}$$

- Utility is log-normal as $-\alpha \tilde{w} \sim \mathcal{N}\left(-\alpha\left[w_0 R_f + \phi\left(\mu - R_f\right)\right], \alpha^2 \phi^2 \sigma^2\right)$
- Expected utility is equal to:

$$E\left[u\left(\tilde{w}\right)\right] = E\left[-e^{-\alpha \tilde{w}}\right] = -e^{-\alpha\left[w_0 R_f + \phi\left(\mu - R_f\right)\right] + \frac{1}{2}\alpha^2 \phi^2 \sigma^2}$$

- **Note that**:

$$-\alpha\left[w_0 R_f + \phi\left(\mu - R_f\right) - \frac{1}{2}\alpha \phi^2 \sigma^2\right]$$

  - The term in brackets is the Certainty Equivalent (CE)
  - The portfolio return $E\left[\tilde{w}\right] = w_0 R_f + \phi\left(\mu - R_f\right)$ is adjusted by risk
  - The (risk premium) insurance on the **wealth portfolio** is $-\frac{1}{2}\alpha \phi^2 \sigma^2$

Portfolio Allocation (Two Assets)

- The allocation problem is equivalent to maximizing CE:

$$\max_{\phi} \qquad w_0 R_f + \phi\left(\mu - R_f\right) - \frac{1}{2}\alpha\phi^2\sigma^2$$

- The optimal portfolio holding will be:

$$\phi^* = \left(\mu - R_f\right)/\alpha\sigma^2$$

- **Remarks**:
  - The allocation is increasing in the (equity) **risk premium**, decreasing in the **quantity of risk** (σ^2), and decreasing in the **risk aversion** (α)
  - Note CARA investor ignores wealth (w_0)
  - If DARA (e.g. CRRA utility), investor increases holdings ϕ with w_0
  - Because of normality, μ and σ^2 completely characterizes the risky asset

Portfolio Allocation (Multiple Assets)

- Assume an investor with CARA utility and $\tilde{R} \sim \mathscr{MN}(\mu, \Sigma)$
- Now, $w_0 = \phi_f + \sum_{i=1}^{n} \phi_i = \phi_f + \phi' \mathbf{1}$ (there are $n+1$ assets)
- As before, the allocation problem is equivalent to solving a mean-variance problem:

$$\max_{\phi} \qquad (\mu - R_f \mathbf{1})' \phi - \frac{1}{2} \alpha \phi' \Sigma \phi$$

- First order condition:

$$\mu - R_f \mathbf{1} - \alpha \Sigma \phi = 0$$

- Solving for ϕ:

$$\phi^* = \frac{1}{\alpha} \Sigma^{-1} (\mu - R_f \mathbf{1})$$

Mean-Variance Analysis

- Markowitz's mean variance (MV) analysis simplifies the single period portfolio choice problem by considering only the first two moments
- MV analysis is either:
  - Minimize the variance of the portfolio return subject to the constraint that the mean of the portfolio return equals a pre-determined level.
  - Maximize the mean of the portfolio return subject to the constraint that the variance of the portfolio return equals a pre-determined level.
- Under certain regularity conditions, we can write:

$$E\left[u\left(\tilde{w}\right)\right] = \sum_{n=0}^{\infty} \frac{1}{n!} u^{(n)}\left(E\left[\tilde{w}\right]\right) E\left[\left(\tilde{w} - E\left[\tilde{w}\right]\right)^{n}\right]$$

  It is natural to ask under what assumption about preferences and/or about return distribution the expected utility can be expressed as a function of only the first two moments, $E\left[u\left(\tilde{w}\right)\right] = f\left(E\left[\tilde{w}\right], V\left[\tilde{w}\right]\right)$

Mean-Variance Analysis

There are three ways to justify MV analysis:

1. Approximation

- Using a second-order expansion of the utility function around $\bar{w} = E[\tilde{w}]$:

$$u(w) \approx u(\bar{w}) + u'(\bar{w})(w - \bar{w}) + \frac{1}{2}u''(\bar{w})(w - \bar{w})^2$$

- and taking expectations:

$$E[u(\tilde{w})] \approx u(E[\tilde{w}]) + \frac{1}{2}u''(E[\tilde{w}])\,V[\tilde{w}] = f(E[\tilde{w}], V[\tilde{w}])$$

- This approximation ignores skewness and kurtosis (see Harvey and Siddique (2000, JoF) approximation that includes skewness)

Mean-Variance Analysis

2. Quadratic Utility

- Preferences are quadratic $u(w) = -\frac{1}{2}(w-a)^2$, so the objective function in the maximization problem is:

$$aE[\tilde{w}] - \frac{1}{2}E\left[\tilde{w}^2\right] = aE[\tilde{w}] - \frac{1}{2}\left(E[\tilde{w}]^2 + \frac{1}{2}var[\tilde{w}]\right)$$

- Quadratic utility is counter-intuitive because it implies increasing ARA
- Note that if $\tilde{y}$ is allowed, $\tilde{w} = w_0\tilde{R} + \tilde{y}$, and covariances will also matter:

$$var\left[w_0\tilde{R} + \tilde{y}\right] = w_0^2 var\left[\tilde{R}\right] + var[\tilde{y}] + 2w_0 cov\left[\tilde{R}, \tilde{y}\right]$$

Mean-Variance Analysis

3. Normally Distributed Returns

- If individual asset returns are normally distributed, then portfolio returns are also normally distributed

$$E\left[u\left(\tilde{w}\right)\right] = E\left[u\left(w_0 \tilde{R}_w\right)\right] = E\left[u\left(w_0\left(1+\mu_w+\sigma_w z\right)\right)\right] = f\left(\mu_w, \sigma_w^2\right)$$

  where $z \sim \mathcal{N}\left(0,1\right)$.
- In fact, this is also the case for all distributions characterized by location (mean) and scale (variance) parameters (i.e. elliptically distributed variables).
- Note that with normally distributed returns, return on wealth in unbounded, so wealth itself become negative. This contradicts the bounded liability of equities and ignores the possibility of bankruptcy.

Mean-Variance Calculus

- Assume there is no risk-free asset. The allocation problem is:

$$\begin{aligned} \min_{\pi} \quad & \frac{1}{2}\pi'\Sigma\pi \\ \text{s.t.} \quad & \mu'\pi = \mu_p \text{ and } \mathbf{1}'\pi = 1 \end{aligned}$$

  where $\mu = (\mu_1, ..., \mu_n)$

- To find the optimal allocation, solve the Lagrangean:

$$\mathscr{L} = \frac{1}{2}\pi'\Sigma\pi - \delta\left(\mu'\pi - \mu_p\right) - \gamma\left(\mathbf{1}'\pi - 1\right)$$

- FOC:

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial \pi} &= \Sigma\pi - \delta\mu - \gamma\mathbf{1} = 0 \\ \frac{\partial \mathscr{L}}{\partial \delta} &= \mu'\pi - \mu_p = 0 \\ \frac{\partial \mathscr{L}}{\partial \gamma} &= \mathbf{1}'\pi - 1 = 0 \end{aligned}$$

Mean-Variance Calculus

- Solving $\partial \mathscr{L} / \partial \pi$ for π:

$$\pi = \delta \Sigma^{-1} \mu + \gamma \Sigma^{-1} \mathbf{1}$$

- Substituting π in the other conditions:

$$\begin{aligned} \mu_p &= \delta \mu' \Sigma^{-1} \mu + \gamma \mu' \Sigma^{-1} \mathbf{1} \\ 1 &= \delta \mathbf{1}' \Sigma^{-1} \mu + \gamma \mathbf{1}' \Sigma^{-1} \mathbf{1} \end{aligned}$$

- Let $A \equiv \mu' \Sigma^{-1} \mathbf{1}$, $B \equiv \mu' \Sigma^{-1} \mu$, $C \equiv \mathbf{1}' \Sigma^{-1} \mathbf{1}$, so that:

$$\begin{bmatrix} \mu_p \\ 1 \end{bmatrix} = \begin{bmatrix} B & A \\ A' & C \end{bmatrix} \begin{bmatrix} \delta \\ \gamma \end{bmatrix}$$

- At the optimum, the Lagrange multipliers are:

$$\delta^* = \frac{C\mu_p - A}{D} \quad \text{and} \quad \gamma^* = \frac{B - A\mu_p}{D}$$

where $D = BC - A^2$

Mean-Variance Calculus

- If follows that the MV optimal portfolio weights are:

$$\pi_p = \delta^* \Sigma^{-1} \mu + \gamma^* \Sigma^{-1} \mathbf{1}$$

- The variance of the optimal portfolio is:

$$\begin{aligned}
\sigma_p^2 = \pi_p' \Sigma \pi_p &= \pi_p' \Sigma \left(\delta^* \Sigma^{-1} \mu + \gamma^* \Sigma^{-1} \mathbf{1} \right) \\
&= \delta^* \pi_p' \Sigma \Sigma^{-1} \mu + \gamma^* \pi_p' \Sigma \Sigma^{-1} \mathbf{1} \\
&= \delta^* \pi_p' \mu + \gamma^* \pi_p' \mathbf{1} \\
&= \delta^* \mu_p + \gamma^* \\
&= \frac{C}{D} \mu_p^2 - 2 \frac{A}{D} \mu_p + \frac{B}{D}
\end{aligned}$$

- This equation for σ_p^2 can be written as:

$$\frac{\sigma_p^2}{1/C} - \frac{(\mu_p - A/C)^2}{D/C^2} = 1$$

which corresponds to a hyperbola in $\{\mu_p, \sigma_p\}$ space

Global Minimum Variance Portfolio

- From the expression for the variance of any optimal portfolio on the MV frontier, the portfolio with minimum variance is:

$$\frac{\partial \sigma_p^2}{\partial \mu_p} = 2\frac{C}{D}\mu_p - 2\frac{A}{D} = 0 \Longrightarrow \mu_{GMV} = \frac{A}{C} \text{ and } \sigma_{GMV}^2 = \frac{1}{C}$$

- with the corresponding weights:

$$\pi_{GMV} = \frac{1}{C}\Sigma^{-1}\mathbf{1}$$

which are independent of the mean returns μ.

Global Minimum Variance Portfolio

- Any MV portfolio can be expressed as a convex combination of the MV optimal portfolio with $\mu = B/A$ and the GMV portfolio:

$$\pi_p = \lambda \pi_{BA} + (1-\lambda)\pi_{GMV}$$

  where $\lambda = \delta A$, $\pi_{BA} = \frac{1}{A}\Sigma^{-1}\mu$ and $\pi_{GMV} = \frac{1}{C}\Sigma^{-1}\mathbf{1}$.
- (BA is in the frontier) Given $\mu = B/A$, the corresponding optimal portfolio is:

$$\pi_{BA} = \left(\frac{C\frac{B}{A} - A}{D}\right)\Sigma^{-1}\mu + \left(\frac{B - A\frac{B}{A}}{D}\right)\Sigma^{-1}\mathbf{1} = \frac{1}{A}\Sigma^{-1}\mu$$

- (Confirm the claim) To see this, just check it:

$$\begin{aligned}
\pi_p &= \delta A \frac{1}{A}\Sigma^{-1}\mu + \left(1 - \left(\frac{C\mu_p - A}{D}\right)A\right)\frac{1}{C}\Sigma^{-1}\mathbf{1} \\
&= \delta\Sigma^{-1}\mu + \left(\frac{BC - A^2 - CA\mu_p + A^2}{D}\right)\frac{1}{C}\Sigma^{-1}\mathbf{1} \\
&= \delta\Sigma^{-1}\mu + \underbrace{\left(\frac{B - A\mu_p}{D}\right)}_{\gamma}\Sigma^{-1}\mathbf{1}
\end{aligned}$$

- In fact, this decomposition is true for **any two** frontier portfolios

Spanning of the MV Efficient Frontier

Theorem

Consider 2 MV frontier portfolios with $\mu_1 \neq \mu_2$, then any third MV frontier portfolio can be expressed as a weighted average of the first two portfolios.

Proof: Recall that the two optimal portfolio weights are

$$\pi_i = \delta_i \Sigma^{-1}\mu + \gamma_i \Sigma^{-1}\mathbf{1} \text{ for } i = 1, 2$$

Since $\mu_1 \neq \mu_2$, $\exists$ a unique α such that $\mu_3 = \alpha\mu_1 + (1-\alpha)\mu_2$. For the portfolio weights:

$$\begin{aligned}
\alpha\pi_1 + (1-\alpha)\pi_2 &= \alpha\left(\delta_1\Sigma^{-1}\mu + \gamma_1\Sigma^{-1}\mathbf{1}\right) + (1-\alpha)\left(\delta_2\Sigma^{-1}\mu + \gamma_2\Sigma^{-1}\mathbf{1}\right) \\
&= (\alpha\delta_1 + (1-\alpha)\delta_2)\Sigma^{-1}\mu + (\alpha\gamma_1 + (1-\alpha)\gamma_2)\Sigma^{-1}\mathbf{1} \\
&= \frac{C\mu_3 - A}{D}\Sigma^{-1}\mu + \frac{B\mu_3 - A}{D}\Sigma^{-1}\mathbf{1} \\
&= \delta_3\Sigma^{-1}\mu + \gamma_3\Sigma^{-1}\mathbf{1} \\
&= \pi_3
\end{aligned}$$

Covariance with Frontier Portfolios

Theorem

For any portfolio p

$$Cov\left[R_p, R_{GMV}\right] = \sigma^2_{GMV} = \frac{1}{C}$$

Proof: Consider a portfolio p and the GMV portfolio with return $\alpha R_p + (1-\alpha)\, R_{GMV}$. The variance of this portfolio is:

$$\alpha^2 \sigma_p^2 + 2\alpha\,(1-\alpha)\, Cov\left[R_p, R_{GMV}\right] + (1-\alpha)^2\, \sigma^2_{GMV}$$

The FOC for minimizing the variance of this portfolio wrt α is:

$$\alpha \sigma_p^2 + (1-2\alpha)\, Cov\left[R_p, R_{GMV}\right] - (1-\alpha)\, \sigma^2_{GMV} = 0$$

Since we must have $\alpha = 0$, this FOC implies a second condition: $Cov\left[R_p, R_{GMV}\right] = \sigma^2_{GMV}$

Covariance with Frontier Portfolios

Theorem

For any frontier portfolio *p, except for GMV portfolio, there exists a unique MV frontier portfolio $ZC(p)$ such that $Cov\left[R_p, R_{ZC(p)}\right] = 0$. This portfolio is called the zero-covariance (or zero-beta portfolio).*

Proof: The covariance between two MV frontier portfolios p and q is:

$$Cov\left[R_q, R_p\right] = \pi_q' \Sigma \pi_p = \pi_q' \Sigma \left(\delta_p \Sigma^{-1} \mu + \gamma_p \Sigma^{-1} \mathbf{1}\right) = \delta_p \mu_q + \gamma_p$$

With $\mu_q = -\gamma_p / \delta_p$ the covariance is zero
The mean of the zero-covariance (with p) portfolio is:

$$\mu_{ZC(p)} = \frac{A\mu_p - B}{C\mu_p - A} = \frac{A}{C} - \frac{D/C^2}{\mu_p - A/C}$$

The portfolio does not exist for the GMV portfolio with $\mu_p = A/C$

Beta Pricing Model

Theorem

***Let** p **be a frontier portfolio and let** q **be any other portfolio** then:*

$$\mu_q = \mu_{ZC(p)} + \beta_{pq}\left(\mu_p - \mu_{ZC(p)}\right)$$

where $\beta_{pq} = Cov\left[R_p, R_q\right] / Var\left[R_p\right]$

Proof: The covariance of p and q:

$$Cov\left[R_q, R_p\right] = \pi_q' \Sigma \pi_p = \pi_q' \Sigma \left(\delta_p \Sigma^{-1} \mu + \gamma_p \Sigma^{-1} \mathbf{1}\right) = \delta_p \mu_q + \gamma_p$$

implies

$$\frac{Cov\left[R_q, R_p\right]}{\delta_p} = \quad \mu_q + \frac{\gamma_p}{\delta_p} \quad = \mu_q - \mu_{ZC(p)}$$

since $\mu_{ZC(p)} = -\gamma_p / \delta_p$
The result follows from dividing above by $var\left[R_p\right] = Cov\left[R_p, R_p\right]$:

$$\frac{Var\left[R_p\right]}{\delta_p} = \quad \mu_p + \frac{\gamma_p}{\delta_p} \quad = \mu_p - \mu_{ZC(p)}$$

Mean-Variance Calculus with Risk-Free Asset

- Suppose we can also invest in a riskless asset with return R_f
- Let π be the weights in the risky assets; $1-\pi'\mathbf{1}$ is held in the riskless asset
- Total portfolio return is now $(1-\mathbf{1}'\pi)\,R_f+\mu'\pi=R_f+(\mu-R_f\mathbf{1})'\,\pi$
- A portfolio p is a MV frontier portfolio if and only if its weights π_p solve:

$$\min_{\pi} \qquad \frac{1}{2}\pi'\Sigma\pi \text{ s.t. } \mu_p=R_f+(\mu-R_f\mathbf{1})'\,\pi$$

- From the Lagrangean:

$$\mathscr{L}=\frac{1}{2}\pi'\Sigma\pi+\delta\left(\mu_p-R_f-(\mu-R_f\mathbf{1})'\,\pi\right)$$

the FOCs are:

$$\frac{\partial\mathscr{L}}{\partial\pi}=\Sigma\pi-\delta\,(\mu-R_f\mathbf{1})=0 \text{ and } \frac{\partial\mathscr{L}}{\partial\delta}=\mu_p-R_f-(\mu-R_f\mathbf{1})'\,\pi=0$$

- The solution is:

$$\pi_p=\frac{\mu_p-R_f}{(\mu-R_f\mathbf{1})'\,\Sigma^{-1}\,(\mu-R_f\mathbf{1})}\Sigma^{-1}\,(\mu-R_f\mathbf{1})$$

Mean-Variance Calculus with Risk-Free Asset

- The standard deviation of a frontier portfolio is then:

$$\sqrt{\pi_p' \Sigma \pi_p} = \frac{|\mu_p - R_f|}{\sqrt{(\mu - R_f \mathbf{1})' \Sigma^{-1} (\mu - R_f \mathbf{1})}}$$

- Let π_p^* be a optimal portfolio that satisfy $\mathbf{1}'\pi_p^* = 1$(invests only risky assets):

$$\pi_p^* = \frac{1}{\mathbf{1}'\Sigma^{-1}(\mu - R_f \mathbf{1})} \Sigma^{-1} (\mu - R_f \mathbf{1})$$

If $\mathbf{1}'\Sigma^{-1}(\mu - R_f \mathbf{1}) \neq 0$ (i.e. $R_f \neq R_{GMV}$)

- This portfolio is on both frontiers: it is called the **tangency portfolio**
- Two fund spanning: R_f and the **tangency portfolio** (if it exists) can span all the frontier
  - If $\mathbf{1}'\pi_p > (<) 1$ then short (long) the risk-free asset

Beta Pricing Model

Theorem

*Let p **be a MV portfolio** and let q **be any other portfolio**, then*

$$\mu_q = R_f + \beta_{pq} (\mu_p - R_f)$$

where $\beta_{pq} = Cov[R_p, R_q] / Var[R_p]$

Proof: The covariance of p and q is:

$$Cov[R_q, R_p] = \pi_q' \Sigma \pi_p = \frac{(\mu_p - R_f)(\mu_q - R_f)}{(\mu - R_f \mathbf{1})' \Sigma^{-1} (\mu - R_f \mathbf{1})}$$

and the variance of p is

$$var[R_p] = \frac{(\mu_p - R_f)^2}{(\mu - R_f \mathbf{1})' \Sigma^{-1} (\mu - R_f \mathbf{1})}$$

The result follows from dividing both sides

$$\beta_{pq} = \frac{\mu_q - R_f}{\mu_p - R_f}$$